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Multi-rogue solutions to the focusing NLS equation

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Abstract

The study of rogue waves is a booming topic mainly in oceanography but also in other fields. In this thesis I construct via Darboux transform a multi-parametric family of smooth quasi-rational solutions of the nonlinear Schödinger equation that present a behavior of rogue waves. For a general choice of parameters the second-order solutions give a model of "three sisters" (three higher than expected waves in a row) while for a particular choice of parameters we obtain the solutions given by Akhmediev et al. in a serie of articles in 2009. Then these solutions allow me to construct rational solutions of the KP-I equation that describe waves in shallow water.

Key-words : rogue waves, three sisters, NLS equation, KP-I equation, rational solutions, Darboux transform.

Résumé

L'étude des ondes scélérates est un sujet en plein essor principalement en océanographie mais également dans d'autres domaines. Dans cette thèse, je construit par transformation de Darboux une famille multi-paramétrique de solutions quasi-rationnelles lisses de l'équation de Schödinger non linéaire qui présentent un comportement d'ondes scélérates. Pour un choix générique de paramètres les solutions de deuxième ordre donnent un modèle de "trois sœurs" (une succession de trois vagues plus hautes que prévues) alors que pour un choix particulier de paramètres on obtient les solutions présentées par Akhmediev et al. dans une série d'articles de 2009. Ces solutions me permettent ensuite de construire des solutions rationnelles de l'équation KP-I qui décrit le mouvement des vagues dans une eau peu profonde.

Mots-clés : ondes scélérates, trois sœurs, équation NLS, équation KPI, solutions rationnelles, transformation de Darboux.

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1 Introduction

1.1 Notion of rogue waves in oceanography and other fields

Sailing is and always has been a dangerous activity because sea is an unpredictable element. For example here is how Philippe Lijour, first mate of the oil tanker Esso Languedoc describes the wave that hit the ship near Durban in South Africa in 1980 :

We were in a storm and the tanker was running before the sea. This amazing wave came from the aft and broke over the deck. I didn't see it until it was alongside the vessel but it was special, much bigger than the others. It took us by surprise. I never saw one again.

Another surprising and devastating phenomenon that can occur is the so called "Three sisters" wave. It consists of one huge wave followed by two others. All these waves have the particularity to be unexpectedly high and to appear suddenly before disappearing as suddenly. They are called rogue or freak. Testimonies and a list of occurrences can be found at [27]. Some descriptions are available in [22, 23].

For a long time, sailors who survived rogue waves and told their stories weren't taken seriously. Nowadays, even if some losses of ships can be blamed on rogue waves when a human mistake is in fact the cause of the sinking, it makes no doubt that rogue waves are a reality. Lots of pictures are available on the Internet. A Google Image search for "rogue waves" returns over three millions hits including pictures in Figure 1.

But these pictures can't be the basis of scientific works. To be able to analyze and produce modelings of rogue waves we need official measurement by instruments. A good source for such recordings is oil platforms. For example Figure 2 shows the New Year wave that hit the Draupner platform in 1995 in the North sea. After that, the study of rogue waves took off.

The researches on rogue waves are very active. Several mechanisms are believed to be possible sources of appearance of rogue waves. We can cite

- Interaction with a current : Waves from one current are driven into a current of opposing direction. The wave train compresses into a rogue wave. For example in the Indian ocean the Agulhas current goes against the westerlies.
- Spatial focusing : Small waves coming from different directions (open sea, refraction from coast, etc.) interfere and energy concentration happens.
- Spatio-temporal focusing : Waves with a large group velocity overtake waves with a smaller group velocity located in front of them and a unexpectedly big wave is produced.
- Nonlinear focusing : A wave concentrates the energy from other smaller waves and becomes huge and unstable before collapsing.

This list is not exhaustive and we could add the effects of the wind or of the Benjamin-Feir instability. Current "state of art" is well reflected in the volume [6] and several books especially [23].

As we pointed out, the classification of rogue waves doesn't depend as much on the height of the wave than its singularity with respect to other waves. To define rogue waves properly, we need parameters to describe the waves and the sea. First, the height of a wave H_w is defined as the larger distance between the top of the considered crest and the bottom of the troughs before and after. Then, we can define the significant wave height H_s of the sea as the average waves height over the higher third of the waves in a given time interval, usually between ten and thirty minutes. The considered wave is classified as rogue if

$$\frac{H_w}{H_s} > 2$$

Even if this is the standard definition of rogue waves it's not completely satisfactory. First it doesn't cover every occurrence of what we would like to call a rogue wave. For example the three sisters wave that hit the Louis Majesty in March 2010 in the Mediterranean didn't satisfy this criterion but killed two persons. Second it doesn't take into account what many specialists consider as a fundamental property of rogue waves : a short lifetime.

If rogue waves are especially studied in oceanography, they have recently appeared in other fields like non-linear optics [14, 36, 40], matter physics [11], physics of fluids [19] and even in financial markets [17, 41].

Figure 1: Some pictures of rogue waves

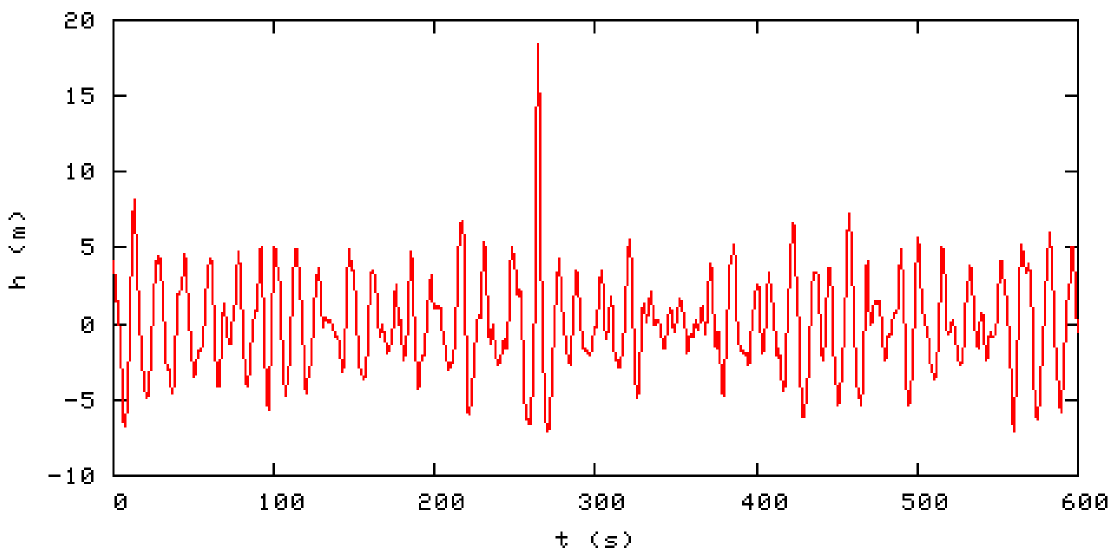

Figure 2: The Draupner wave

1.2 Deep water model : the NLS equation

The simplest model for deep water is the focusing non-linear Schrödinger equation. It was obtained in 1968 by Zakharov in [42] and it reads

$$ir_t + r_{xx} + 2|r|^2 r = 0. \tag{1}$$

Let denote by σ_1, σ_2, σ_3, σ_+, σ_- the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let define $U := U_0 + \lambda U_1$ and $V := V_0 + \lambda V_1 + \lambda^2 V_2$ where U_i and V_j satisfy

$$\begin{array}{rcl} U_0 & := & i(\overline{r}\sigma_+ + r\sigma_-), \\ U_1 & := & \frac{1}{2}\sigma_3, \\ V_0 & := & -i|r|^2\sigma_3 + (\overline{r}_x\sigma_+ - r_x\sigma_-), \\ V_1 & := & -iU_0, \\ V_2 & := & -iU_1. \end{array}$$

Using these two matrices U and V we can write a zero curvature representation for the NLS equation. Equation (1) is equivalent to

$$U_t - V_x + (UV - VU) = 0. \tag{2}$$

Equation (2) can be seen as the compatibility condition for the following overdetermined system of vectorial equations

$$\left\{ \begin{array}{rcl} F_x & = & UF \\ F_t & = & VF \end{array} \right. \tag{3}$$

Here F is a two-components vector function

$$F := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

The following theorem can be found in [34]

Theorem 1. If $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a solution of (3) for a real λ and matrices $U^{(0)}$ and $V^{(0)}$ constructed from a solution r_0 of (1) then r_1 defined by

$$r_1 := r_0 + 2i\lambda \frac{f_2\overline{f_1}}{|f_1|^2 + |f_2|^2}$$

is also a solution of (1).

For example, if we set $r_0 := e^{2it}$ then f_1 and f_2 defined by

$$\begin{aligned} f_1 &:= -i\left(\sqrt{2}x + \tfrac{\sqrt{2}}{2} - 2i\sqrt{2}t\right)e^{-it}, \\ f_2 &:= \left(\sqrt{2}x - \tfrac{\sqrt{2}}{2} - 2i\sqrt{2}t\right)e^{it} \end{aligned}$$

give a solution of (3) for $\lambda = 2$. If we compute r_1 given in the previous theorem we get back, up to a sign, the famous Peregrine solution obtained in [39] which is the first example of non-trivial quasi-rational solution of NLS. Theorem 1 can be used recursively. Once we obtain a new solution r_1 we are theoretically able to solve the associated system (3) with new matrices $U^{(1)}$ and $V^{(1)}$. Then we can generalize it by a determinant formula (see [34]) or we can use iterative application of this theorem "by hand" as Akhmediev et al. did in their articles [2, 3, 8]. In these works the machinery of Darboux transformations was used to obtain some very remarkable but isolated solutions having a rogue waves behavior. In Section 2 we present a multi-parametric class of solutions showing the existence of multiple Peregrine breather. Solutions in [2] can be obtained back from those with the right choice of parameters.

This method allows us obtain other solutions of NLS that are not quasi-rational. For example, from the same r_0 than previously and by solving the system for $\lambda < 2$ we can obtain a rational fraction in $\sinh(\omega t)\cos(\omega x)$, $\cosh(\omega t)\cos(\omega x)$, $\sinh(\omega t)\sin(\omega x)$ and

$\cosh(\omega t)\sin(\omega x)$ multiplied by an exponential. It's the kind of solutions presented in [4] where the second order rational solutions were written down for the first time. It was then obtained by taking the rational limit of these "trigonometric-hyperbolic" solutions often called Akhmediev breather.

Darboux transform is not the only possible approach to construct solutions of NLS. Its integrability by the inverse scattering method was proved in the famous work [43] by Zakharov and Shabat in 1971. A review of this topic and the Riemann-Hilbert problem approach can be found in [16, 30]. It allowed to construct multi-soliton solutions. An another successful technics to obtain solutions of NLS is the algebro-geometric approach where solutions are expressed in terms of Riemann theta-functions of algebraic curves. Exact formulas are given in [10, 29].

1.3 Shallow water model : the KdV and the KP equations

In [25], Korteweg and de Vries derived the KdV equation to represent the traveling of a wave in a channel of shallow water

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4}$$

This equation can be seen at the compatibility condition of the following system

$$\left\{ \begin{array}{rcl} \lambda\psi & = & -\psi_{xx} - u\psi \\ \psi_t & = & -4\psi_{xxx} - 6u\psi_x - 3u_x\psi \end{array} \right. \tag{5}$$

In [21], Kadomtsev and Petviashvili wrote down a 2+1 generalization of this equation the so-called KP equation

$$(u_t + 6uu_x + u_{xxx})_x = -3\alpha^2 u_{yy}. \tag{6}$$

Once again, we can associate to this equation a system of linear differential equations

$$\left\{ \begin{array}{rcl} \alpha\psi_y & = & -\psi_{xx} - u\psi \\ \psi_t & = & -4\psi_{xxx} - 6u\psi_x - 3(u_x - \alpha v)\psi \end{array} \right. \tag{7}$$

where v satisfies $v_x = u_y$. If $\alpha = i$ (resp. $\alpha = 1$) the equation is called KP-I (resp. KP-II). The KP-I model describes waves in a very shallow water when surface tension is strong and KP-II is used for the case of weak surface tension. From now we restrict ourselves to the case $\alpha = i$.

As in the case of NLS, these systems allow us to construct new solutions of (4) and (6) from simpler ones with the help of the following theorem

Theorem 2. Let u be a solution of (4) (resp. (6)) and let ψ and ψ_1 two solutions of (5) (resp. (7)) for this u for different λ. Let denote by $\sigma := \partial_x \log \psi_1$. Then $\widetilde{u}$ defined by

$$\widetilde{u} := u + 2\partial_x \sigma$$

is also a solution of (4) (resp. (6)) and a solution of the associated system is given by

$$\widetilde{\psi} := \psi_x - \sigma\psi.$$

This theorem can easily be generalized in the following way

Theorem 3. Let u be a solution of (4) (resp. (6)) and let $\psi, \psi_1, \ldots, \psi_n$ be solutions of (5) (resp. (7)) for this u. Then $\widetilde{u}$ defined by

$$\widetilde{u} := u + 2\partial_x^2 \log W(\psi_1, \ldots, \psi_n)$$

is also a solution of (4) (resp. (6)) and a solution of the associated system is given by

$$\widetilde{\psi} := \frac{W(\psi_1, \ldots, \psi_n, \psi)}{W(\psi_1, \ldots, \psi_n)}.$$

Here $W(g_1, \ldots, g_n)$ denotes the Wronski determinant of functions $g_1, \ldots, g_n$ with respect to x i.e. the determinant of the matrix A with entries

$$A_{ij} := \partial_x^{i-1} g_j.$$

A proof of this theorem can be found in [31]. In Section 4 we will show that it allows to generate a family of smooth real rational solutions of the KP-I equation from the quasi-rational solutions of the focusing NLS equation.

KdV is one of the most, if not the most, studied equation in the literature. It's the first equation where soliton solutions have been found and the first equation that have been successfully integrated via the inverse scattering method. After that, it was found that other equations admit similar results including KP. Different axis of development emerged in the study of these two solitons equations like construction of algebro-geometric solutions [10], study of solitons and rational solutions [1, 12, 26, 33, 37, 38] or theory of Darboux transforms [28, 31, 34, 35].

2 Construction of solutions

2.1 Darboux transform for the non-stationary Schrödinger equation

Let consider the 1+1 non-stationary Schrödinger equation with potential $u(x,t)$

$$i\psi_t + \psi_{xx} + u(x,t)\psi = 0. \tag{8}$$

As stated in [34], this equation satisfies the following remarkable property

Theorem 4. If $f_1, \dots, f_n, f$ are solutions of (8) for a base potential $u_0(x,t)$ then

$$\psi := \frac{W(f_1, \dots, f_n, f)}{W(f_1, \dots, f_n)}$$

is also a solution of (8) for the potential

$$u(x,t) := u_0(x,t) + 2\partial_x^2 \log W(f_1, \dots, f_n).$$

All these Darboux transform theorems are particular cases of a more general covariance theorem for several integrable systems obtained by Matveev.For completeness this theorem is presented in appendix A. Its use with a base potential $u_0(x,t) := 0$ already gives us interesting solutions.

Let $q_{2n}(k)$ be the polynomial defined by

$$q_{2n}(k) := \prod_{j=1}^{n} \left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1} B^2 \right), \quad \omega := \exp\left(\frac{i\pi}{2n+1}\right) \tag{9}$$

where B is a free real parameter and m_j are integers satisfying

$$0 \leq m_j \leq 2n-1, \quad m_l \neq 2n - m_j \tag{10}$$

for all l and j. Then, for any k,

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2n}(k)} \tag{11}$$

where $\Phi(k)$ is of the form

$$\Phi(k) := i \sum_{l=1}^{2n} \varphi_l (ik)^l, \quad \varphi_j \in \mathbf{R}, \tag{12}$$

is a solution of (8) with $u_0 = 0$. We can construct other solutions by applying a differential operator in k to f or by evaluating in different values of k. For example, let f_j be defined by

$$f_j(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=B}\,, \quad j = 1\ldots,n, \tag{13}$$

$$f_{n+j}(x,t) := D_k^{2j-1} f(k,x,t)\,|_{k=-B}\,, \quad j = 1\ldots,n \tag{14}$$

where

$$D_k := \frac{k^2}{k^2+B^2}\partial_k.$$

According to the previous theorem

$$\psi(x,t,k) = \frac{W(f_1,\ldots,f_{2n},f)}{W(f_1,\ldots,f_{2n})}$$

is a solution of (8) for the potential $u(x,t) = 2\partial_x^2 \log W(f_1,\ldots,f_{2n})$. Furthermore, for any constant C, s_C defined by

$$s_C(x,t) := C\psi(x,t,0)$$

is a rational solution of the non-stationary Schrödinger equation for the potential $u(x,t)$.

2.2 Reduction to the NLS equation

Let assume that, for a conveniently chosen C, the solution s_C satisfies

$$u(x,t) = 2\left(|s_C(x,t)|^2 - B^2\right). \tag{15}$$

Then s_C, now denoted by s, is a rational solution of the modified non-linear Schrödinger equation

$$is_t + s_{xx} + 2(|s|^2 - B^2)s = 0. \tag{16}$$

From there, we can easily construct a non-singular quasi-rational solution of (1) given by the formula

$$r(x,t) := s(x,t)e^{2iB^2t}.$$

Theorem 5. If $C = \pm q_{2n}(0)B^{1-2n}$ then s is a smooth rational solution of (16).

2.3 Proof of the reduction relation

The proof presented here is a modified and simplified version of the article [15] written by Eleonskii, Krichever and Kulagin . It consists in a residue analysis of a couple of meromorphic differentials constructed from the dual Baker-Akhiezer function introduced by Cherednik in [13]. Before tackling the proof of Theorem 5 itself, we need an easy algebraic lemma.

Lemma 6. The polynomial q_{2n} defined by (9) satisfies

$$2k^2 q_{2n}(k)\overline{q_{2n}(-\overline{k})} = (k^2+B^2)^{2n+1} + (k^2-B^2)^{2n+1}. \tag{17}$$

Proof
w satisfies
$$\overline{w}^{2m+1} = w^{4n+2-2m-1} = w^{2(2n-m)+1}$$
which leads to
$$\begin{aligned}2k^2 q_{2n}(k)\overline{q_{2n}(-\overline{k})} &= 2k^2 \prod_{j=1}^{n}\left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1}B^2\right)\prod_{j=1}^{n}\left(k^2 - \frac{\omega^{2(2n-m_j)+1}+1}{\omega^{2(2n-m_j)+1}-1}B^2\right)\\ &= 2\prod_{l=1}^{2n+1}\left(k^2 - \frac{\omega^{2l+1}+1}{\omega^{2l+1}-1}B^2\right).\end{aligned}$$
The roots of $(X+B^2)^{2n+1} + (X-B^2)^{2n+1}$ satisfy
$$\left(\frac{x_l+B^2}{x_l-B^2}\right)^{2n+1} = w^{2n+1}, \quad l = 1, \ldots, 2n+1$$
which yields
$$\left(\frac{x_l+B^2}{x_l-B^2}\right) = w^{2l+1}$$
or equivalently
$$x_l = \frac{w^{2l+1}+1}{w^{2l+1}-1}B^2.$$
Hence
$$(k^2+B^2)^{2n+1}+(k^2-B^2)^{2n+1} = 2\prod_{l=1}^{2n+1}\left(k^2 - \frac{\omega^{2l+1}+1}{\omega^{2l+1}-1}B^2\right) = 2k^2 q_{2n}(k)\overline{q_{2n}(-\overline{k})}$$

Let $\mathrm{d}\Omega$ be the meromorphic differential defined by
$$\mathrm{d}\Omega := \frac{q_{2n}(k)\overline{q_{2n}(-\overline{k})}}{(k^2-B^2)^{2n}}\,\mathrm{d}k.$$
Taking into account the previous lemma, we can re-write

$$\mathrm{d}\Omega = \frac{(k^2+B^2)^{2n+1}+(k^2-B^2)^{2n+1}}{2k^2(k^2-B^2)^{2n}}\,\mathrm{d}k.$$

In the neighborhoods of each of the points $k = \pm B$, we use two local parameters:

$$E(k) := k + \frac{B^2}{k}, \quad z(k) = k - \frac{B^2}{k}$$

satisfying the relation $E^2 = z^2 + 4B^2$. It yields

$$\mathrm{d}\Omega = \left(\frac{E^{2n+1}}{z^{2n+1}} + 1\right)\frac{\mathrm{d}E}{2}.$$

or equivalently:

$$\mathrm{d}\Omega = \left(\frac{(z^2+4B^2)^n}{z^{2n}} + \frac{z}{2B}\left(1+\frac{z^2}{4B^2}\right)^{-\frac{1}{2}}\right)\frac{\mathrm{d}z}{2}.$$

This leads to the following asymptotic expansion when $z \to 0$ for $\mathrm{d}\Omega$, near the points $k = \pm B$:

$$\mathrm{d}\Omega = \left(\frac{\alpha_0^{\pm}}{z^{2n}} + \frac{\alpha_1^{\pm}}{z^{2n-2}} + \ldots + \frac{\alpha_{n-1}^{\pm}}{z^2} + O(1)\right)\mathrm{d}z \qquad (18)$$

It is clear that $D_k = \partial_z$ and the derivatives of ψ with respect to z are given by

$$\partial_z^m \psi = \frac{W(f_1,\ldots,f_{2n},D_k^m f)}{W(f_1,\ldots,f_{2n})}$$

which gives us

$$\partial_z^{2j-1}\psi|_{k=\pm B} = 0 \quad j = 1,\ldots,n.$$

Therefore an expansion of ψ in the neighborhood of $k = \pm B$ when $z \to 0$ has the form:

$$\psi = \beta_0^{\pm} + \beta_1^{\pm} z^2 + \ldots + \beta_{n-1}^{\pm} z^{2n-2} + O(z^{2n}). \qquad (19)$$

Equations (12), (18) and (19) guarantee that $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_2$ defined by the formulas

$$\mathrm{d}\Omega_1 := \psi(x,t,k)\overline{\psi(x,t,-\overline{k})}\,\mathrm{d}\Omega$$

and

$$\mathrm{d}\Omega_2 := (k + \frac{B^2}{k})\,\mathrm{d}\Omega_1$$

are meromorphic differentials with vanishing residues in B and $-B$. If we consider these differentials defined on the compact sphere $\mathbf{C}P^1$ then the sum of residues equals 0 hence

$$\mathrm{res}_\infty\,\mathrm{d}\Omega_1 = 0$$

and

$$\mathrm{res}_0\,\mathrm{d}\Omega_2 = -\mathrm{res}_\infty\,\mathrm{d}\Omega_2.$$

When k belongs to a neighborhood of ∞, ψ and $\mathrm{d}\Omega$ admit the following expansions:

$$\psi(x,t,k) = \left(1 + \frac{\xi_1(x,t)}{k} + \frac{\xi_2(x,t)}{k^2} + \ldots\right) e^{kx+ik^2t+\phi(k)}, \quad (20)$$

$$\mathrm{d}\Omega = \left(1 + \frac{2nB^2}{k^2} + \ldots\right)\mathrm{d}k$$

such that

$$\mathrm{d}\Omega_1 = \left(1 + \frac{\xi_1 - \overline{\xi_1}}{k} + \frac{\xi_2 + \overline{\xi_2} - |\xi_1|^2 + 2nB^2}{k^2} + \ldots\right)\mathrm{d}k.$$

From these expansions we easily see that

$$\mathrm{res}_\infty\,\mathrm{d}\Omega_1 = \overline{\xi_1} - \xi_1$$

which means that ξ_1 is real and

$$-\mathrm{res}_\infty\,\mathrm{d}\Omega_2 = \xi_2 + \overline{\xi_2} - \xi_1^2 + (2n+1)B^2.$$

Substituting (20) into (8) we obtain the formulas

$$u = -2\partial_x \xi_1$$

and

$$i\partial_t \xi_1 + 2\partial_x \xi_2 + \partial_x^2 \xi_1 - 2\partial_x \xi_1 \xi_1 = 0. \tag{21}$$

The real part of (21) combined with the reality of ξ_1 yields

$$\partial_x \left(\xi_2 + \overline{\xi_2} - \frac{u}{2} - \xi_1^2 \right) = 0$$

or, equivalently,

$$\xi_2 + \overline{\xi_2} - \xi_1^2 = \frac{u}{2} + A(t).$$

Therefore

$$-\mathrm{res}_\infty \, \mathrm{d}\Omega_2 = \frac{u}{2} + A(t) + (2n+1)B^2.$$

Since $\mathrm{res}_0 \, \mathrm{d}\Omega_2 = B^2 |\psi(x,t,0)|^2 |q_{2n}(0)|^2 B^{-4n} = |s(x,t)|^2$, we have

$$\frac{u}{2} + A(t) + (2n+1)B^2 = |s(x,t)|^2.$$

Comparing the behaviors when $|x| \to \infty$ in this identity, we get $A(t) = -2nB^2$ which completes the proof of the crucial relation (15). Now, we can use the remark in [29] that any meromorphic solution of the focusing NLS equation can't have real poles $x_j(t)$.

2.4 Further remarks on condition (10)

In this subsection we denote by $\mathfrak{F}_n$ the set

$$\mathfrak{F}_n := \left\{ \{m_1 < \cdots < m_n\} \subset \{0, \ldots, 2n-1\}, \forall i, j,\ m_i + m_j \neq 2n \right\},$$

by ϕ_n the bijection

$$\begin{array}{rccc} \phi_n : & \{1, \ldots, 2n-1\} & \longrightarrow & \{1, \ldots, 2n-1\} \\ & m & \longmapsto & 2n - m \end{array}$$

and by $\widetilde{\phi}_n$ its restriction to $\{2, \ldots, 2n-2\}$.

Lemma 7. If $\{m_1 < \cdots < m_n\} \in \mathfrak{F}_n$ then $m_1 = 0$ and either $m_2 = 1$ or $m_n = 2n - 1$.

Proof
Let assume $m_1 > 0$. Then $\{m_1, \ldots, m_n\}$ and $\{\phi_n(m_1), \ldots, \phi_n(m_n)\}$ are disjoint hence their union is a subset of $\{1, \ldots, 2n-1\}$ of cardinality $2n$ which is absurd.

Now let assume $m_2 > 1$ and $m_n < 2n-1$. Then $\{m_2, \ldots, m_n\}$ and $\left\{\widetilde{\phi}_n(m_2), \ldots, \widetilde{\phi}_n(m_n)\right\}$ are disjoint hence their union is a subset of $\{2, \ldots, 2n-2\}$ of cardinality $2n-2$ which is absurd. We can remark that we can't have $m_2 = 1$ and $m_n = 2n-1$ simultaneously.

Proposition 8. ψ defined by

$$\begin{array}{cccc} \psi: & \{1, 2n+1\} \times \mathfrak{F}_n & \longrightarrow & \mathfrak{F}_{n+1} \\ & (m, \{0, m_2, \ldots, m_n\}) & \longmapsto & \{0, m_2+1, \ldots, m_n+1, m\} \end{array}$$

is a bijection.

Proof
Let assume that $(m, \{0, m_2, \ldots, m_n\})$ and $(M, \{0, M_2, \ldots, M_n\})$ have the same image by ψ. If $m \neq M$ then 1 and $2n+1$ are both in the set which is impossible. Then $m = M$ and it's straightforward that $m_i = M_i$.

Let $\{0 < p_2 < \cdots < p_{n+1}\}$ be an element of $\mathfrak{F}_{n+1}$. If $p_2 = 1$ then it's the image of $(1, \{0, p_2 - 1, \ldots, p_{n+1} - 1\})$ otherwise it's the image of $(2n+1, \{0, p_2 - 1, \ldots, p_n - 1\})$.

Corollary 9. $\operatorname{card}(\mathfrak{F}_n) = 2^{n-1}$

Proof
Proposition 8 yields $\operatorname{card}(\mathfrak{F}_{n+1}) = 2 \operatorname{card}(\mathfrak{F}_n)$ and $\operatorname{card}(\mathfrak{F}_1) = 1$

In the rest of this thesis, we set $m_j = j - 1$.

3 Analysis of these solutions

These solutions apparently depend on $2n+1$ parameters but only $2n-2$ of them are important. B is the well-known re-scaling parameter. If $s(x,t)$ is a solution then

$$\widetilde{s}(x,t) := Bs(Bx, B^2 t)$$

is also a solution. Then, we can set $B = 1$ without loss of generality. Using phases φ_1 and φ_2 means replacing x and t with $(x-\varphi_1)$ and $(t-\varphi_2)$ which is irrelevant for the derivatives. φ_1 and φ_2 are just translation parameters that don't change the profile of the solution.

3.1 Case n=1

Here, we obtain essentially one solution, the Peregrine solution given which reads

$$s(x,t) = \frac{-3 + 4x^2 + 16t^2 - 16it}{1 + 4x^2 + 16t^2}$$

for $\varphi_1 = 0$ and $\varphi_2 = \frac{\sqrt{3}}{4}$. A plot is given in Figure 3.

The computation of the first derivatives gives us

$$\begin{aligned} \partial_x(|s|^2) &= \tfrac{64x(-4x^2+48t^2+3}{(1+4x^2+16t^2)^3} \\ \partial_t(|s|^2) &= \tfrac{256t(-12x^2+16t^2+1}{(1+4x^2+16t^2)^3} \end{aligned}$$

which leads to three critical points in $(0,0)$,$(\frac{\sqrt{3}}{2},0)$ and $(-\frac{\sqrt{3}}{2},0)$. A study of the Hessian matrix in these points shows that $(0,0)$

is a local maximum where $|s| = 3$ and the other two are local minima where $s = 0$.

Now, we will study the energy of this solution. Of course, as the base level is 1, the energy in its strict meaning is infinite. But we can adapt the definition of energy to obtain two integral relations :

$$\int_{-\infty}^{+\infty} \left(|s(x,t|^2 - 1\right) \,\mathrm{d}x = 0 \tag{22}$$

and

$$\int_{-\infty}^{+\infty} \left(|s(x,t) - 1|^2\right) \,\mathrm{d}x = \frac{4\pi}{\sqrt{1+16t^2}} \tag{23}$$

The first relation shows that the excitation below the surface compensate exactly the excitation above, and that at every moment of time. The second is the first of a list of moments that can be obtained by integration by parts.

3.2 Case n=2

The solutions obtained here depend on two parameters φ_3 and φ_4. For convenience, we choose

$$\varphi_1 := 3\varphi_3, \quad \varphi_2 := 2\varphi_4 + \frac{3+\sqrt{5}}{16}\sqrt{10-2\sqrt{5}}.$$

We obtain

$$s(x,t) = \left(1 - 12\frac{G(2x,4t) + iH(2x,4t)}{Q(2x,4t)}\right) \tag{24}$$

where

$$\begin{aligned}
G(x,t) &:= x^4 + 6g_2(t)x^2 + 2\beta x + g_0(t) \\
H(x,t) &:= tx^4 + 2h_2(t)x^2 + 2\beta tx + h_0(t) \\
Q(x,t) &:= x^6 + 3g_2(t)x^4 - 2\beta x^3 + 3q_2(t)x^2 + 6\beta g_2(t)x + q_0(t)
\end{aligned}$$

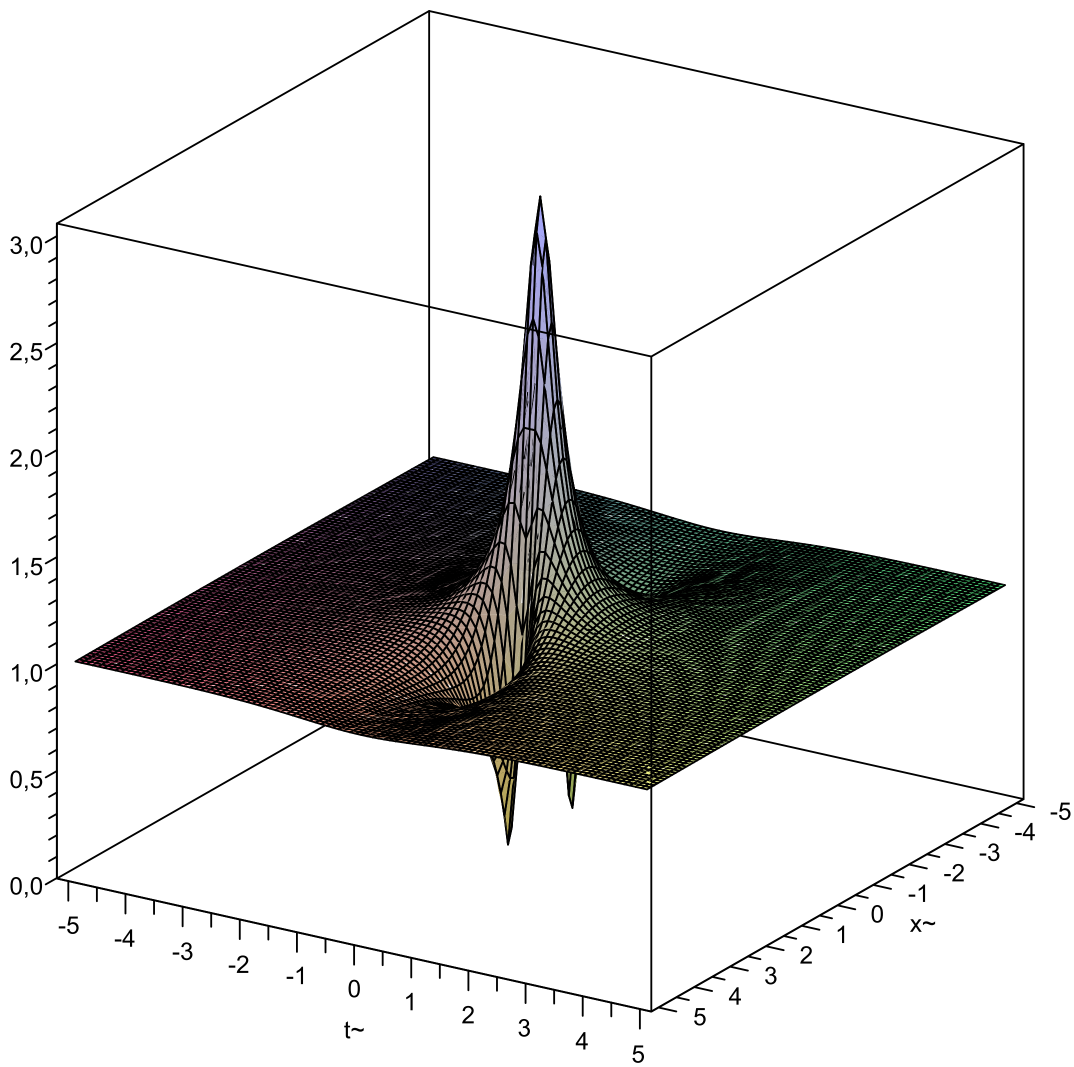

Figure 3: The Peregrine solution

with

$$\begin{aligned}
g_2(t) &:= t^2 + 1 \\
g_0(t) &:= 5t^4 + 18t^2 - 4\alpha t - 3 \\
h_2(t) &:= t^3 - 3t + \alpha \\
h_0(t) &:= t^5 + 2t^3 - 2\alpha t^2 - 15t + 2\alpha \\
q_2(t) &:= t^4 - 6t^2 + 4\alpha t + 9 \\
q_0(t) &:= t^6 + 27t^4 - 4\alpha t^3 + 99t^2 - 36\alpha t + \beta^2 + 4\alpha^2 + 9
\end{aligned}$$

In these formulas parameters α and β are given by

$$\alpha := (5 + \sqrt{5})\sqrt{10 - 2\sqrt{5}} - 96\varphi_4, \quad \beta := 96\varphi_3.$$

Figures 4 to 6 show this solution for different value of φ_3 and φ_4. We can remark that the profile shown in Figure 6 is different of the other two. The system of three peaks of similar height is replaced by one giant peak surrounded by four peaks of smaller amplitude. In this case the solution (24) is the second order solution obtained by Akhmediev et al. in [2].

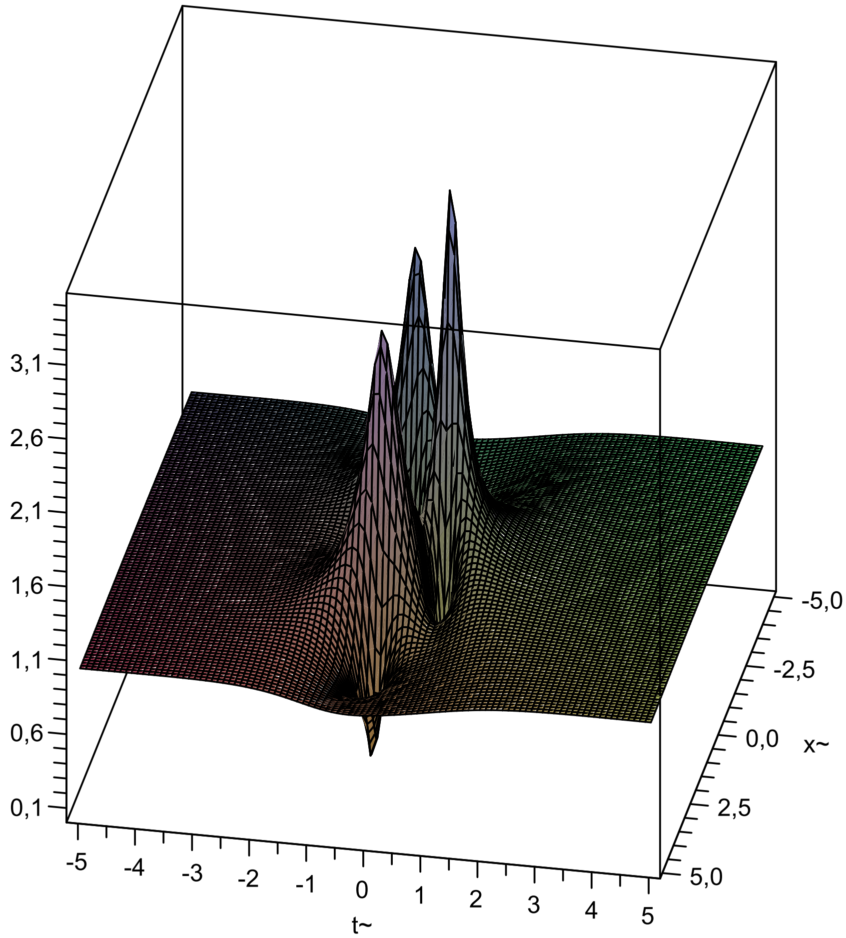

Figure 4: $\varphi_3 = 0$ and $\varphi_4 = 0$

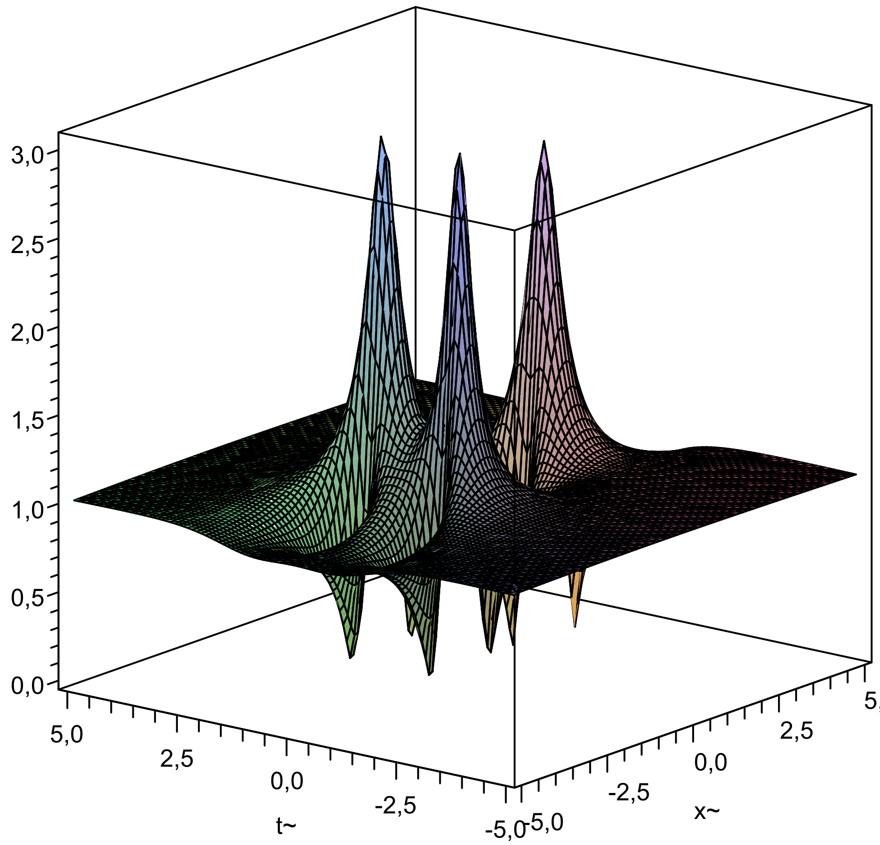

Figure 5: $\varphi_3 = 1$ and $\varphi_4 = \frac{5+\sqrt{5}}{96}\sqrt{10 - 2\sqrt{5}}$

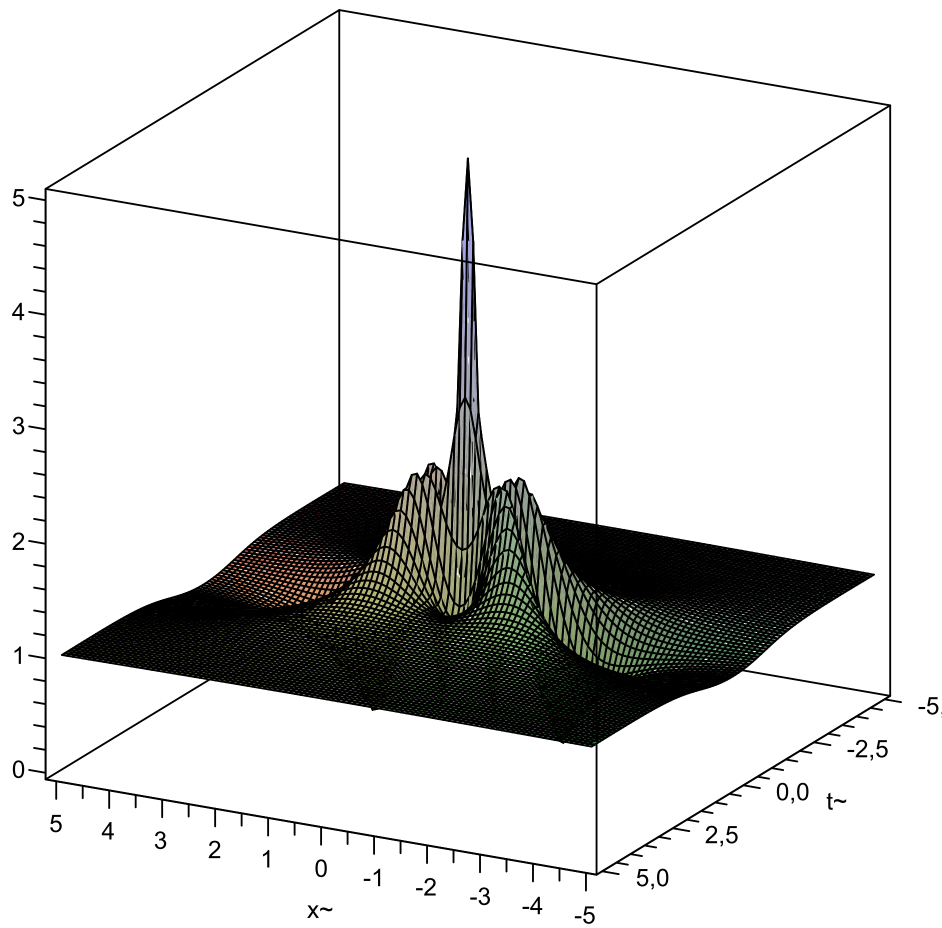

Figure 6: $\varphi_3 = 0$ and $\varphi_4 = \frac{5+\sqrt{5}}{96}\sqrt{10-2\sqrt{5}}$

3.3 Higher order solutions

In this subsection conjectures about solutions obtained for $n > 2$ are presented. We have to work from plots of solutions. Two of them can be found in figures 7 and 8. It represents third and fourth order solutions when all phases are equal to zero. It's believed to be the general profile. The solutions seem to have in general $\frac{n(n+1)}{2}$ maxima of comparable amplitude and $n(n+1)$ minima.

But once again for exceptional solutions we should observe the appearance of a "super-peak" surrounded by $n(n+1) - 2$ peaks of significantly smaller amplitudes.

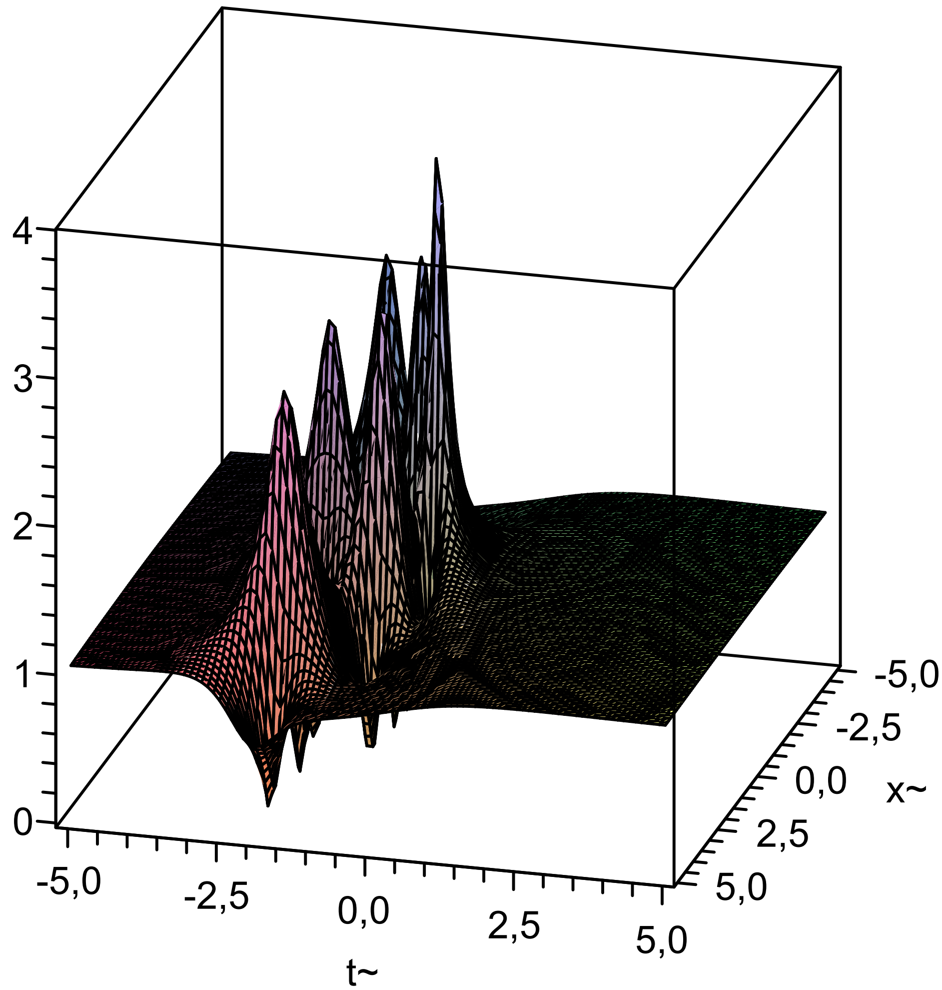

Figure 7: Solution of order 3 with all phases equal to 0

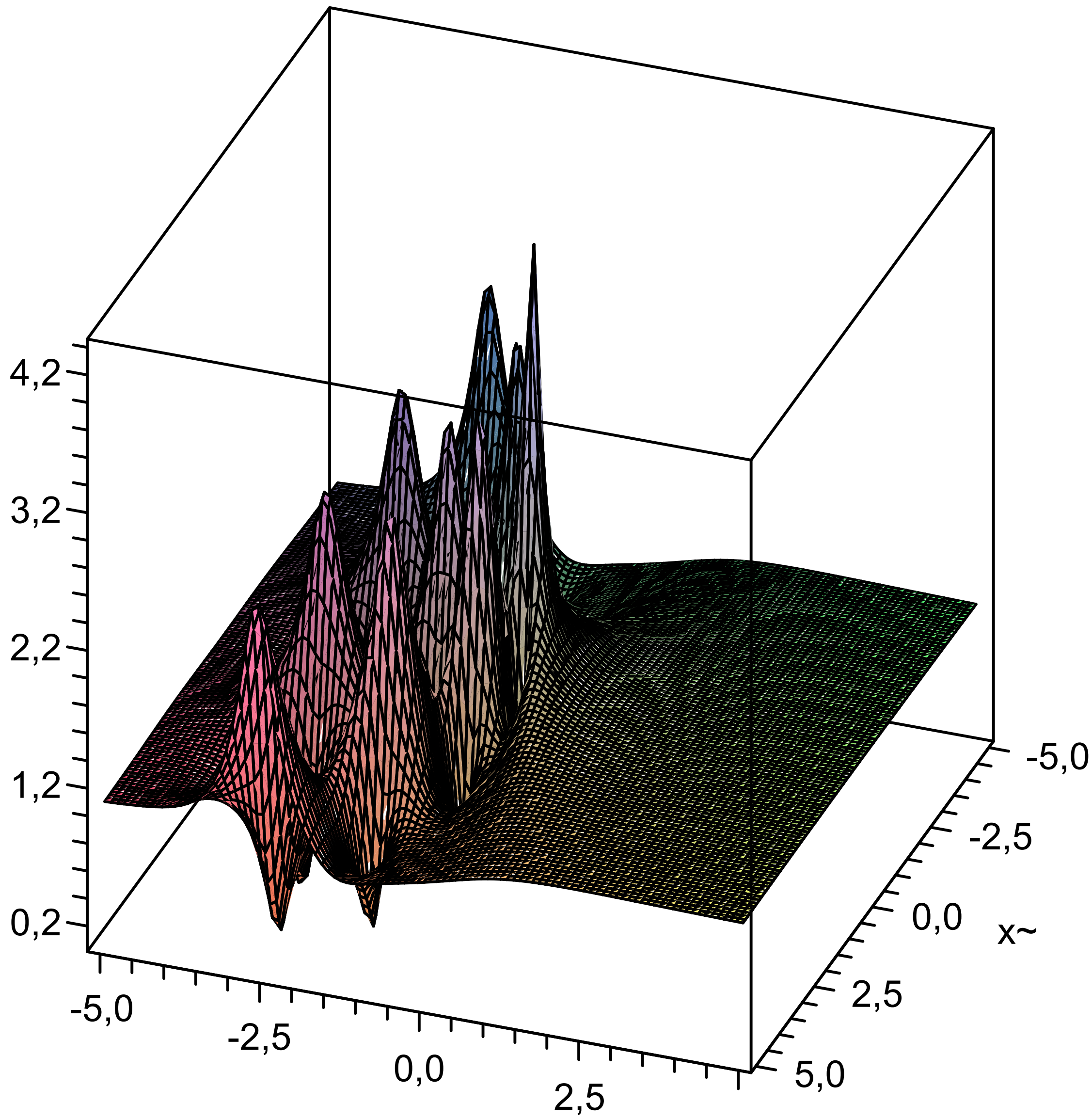

Figure 8: Solution of order 4 with all phases equal to 0

4 Solutions of the KP-I and the CKP equations

4.1 Solutions of the KP-I equation

With this family of multi-parametric solutions of NLS we can associate a family of solutions of the KP-I equation. If we replace

t with y and φ_3 with $-4t$ then f defined by (11) satisfy

$$\left\{ \begin{array}{rcl} if_y & = & -f_{xx} \\ f_t & = & -4f_{xxx} \end{array} \right.$$

and so do the functions f_j defined by (13) and (14).

Theorem 10. $u(x, y, t) := 2\left(|s|^2 - B^2\right)$ is a smooth real solution of KP-I that satisfies

$$\forall\, t \quad u(x, y, t) \longrightarrow 0 \quad \text{when} \quad x^2 + y^2 \longrightarrow +\infty$$

and

$$\forall\, y,\, t \quad \int_{-\infty}^{+\infty} u(x, y, t)\, \mathrm{d}x = 0.$$

Proof

The fact that u is a solution comes from Theorem 3 and condition (15). The integral relation is derived from the Wronskian representation

$$u(x, y, t) = 2\partial_x^2 \log W(f_1, \ldots, f_{2n}).$$

It means that relation (22) holds for every solution of NLS constructed in Section 2.

These solutions are clusters of peaks traveling to infinity, possibly on a line. Two series of plots for different values of φ_4 are presented in figures 9 and 10. We set $\varphi_1 = \varphi_2 = 0$.

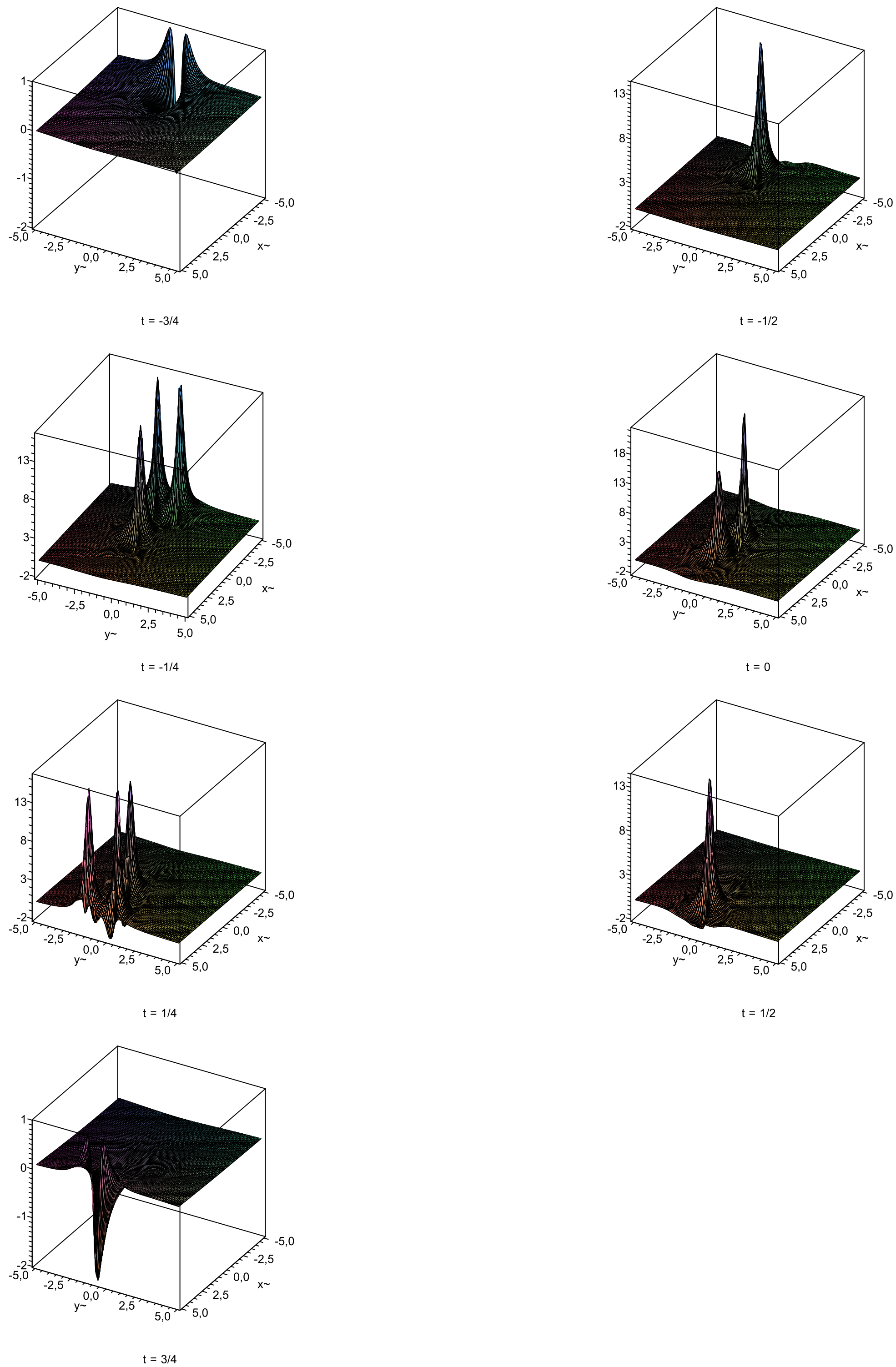

Figure 9: Solution of KP-I with $\varphi_4 = 0$

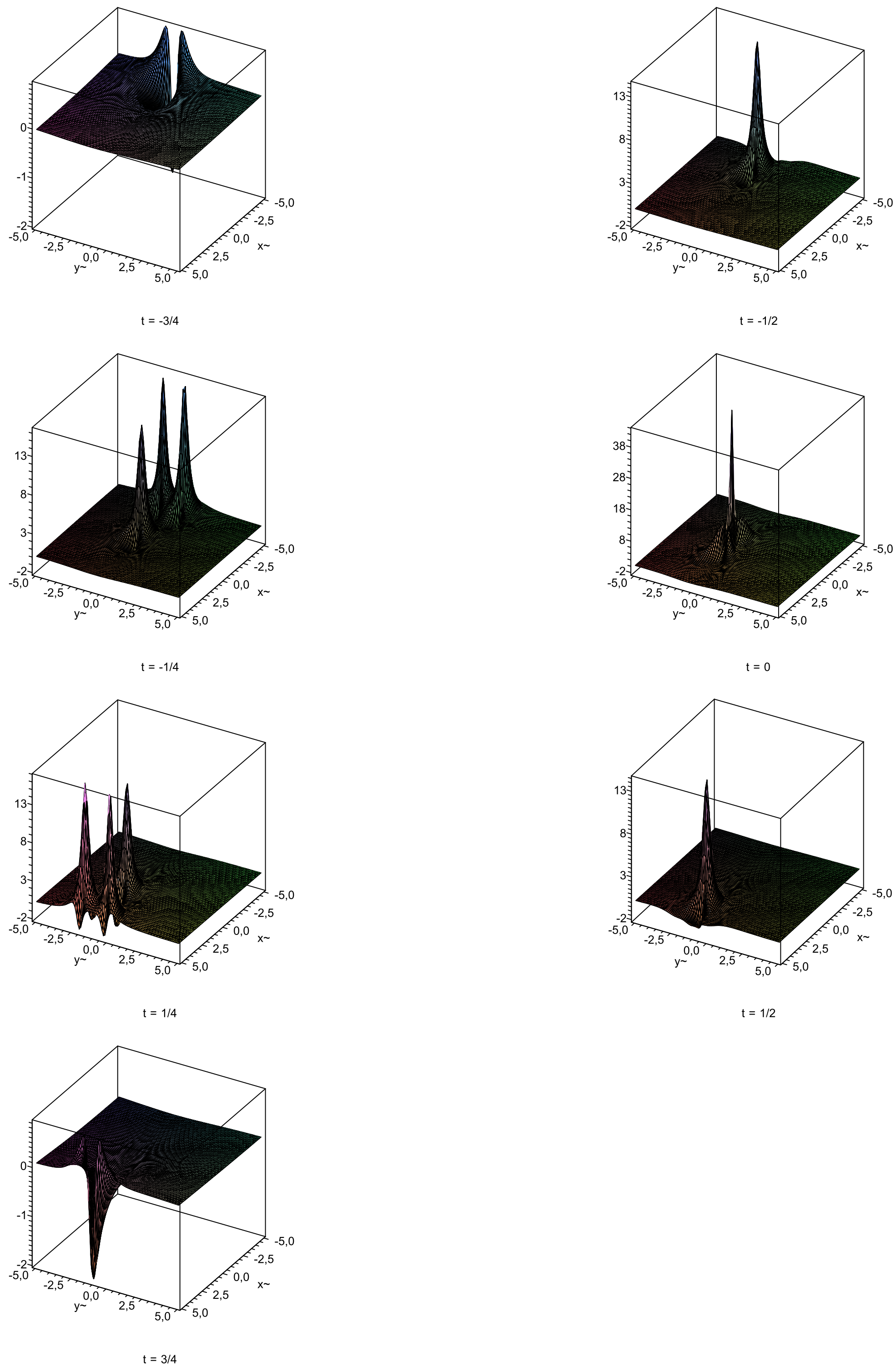

Figure 10: Solution of KP-I with $\varphi_4 = \frac{5+\sqrt{5}}{96}\sqrt{10-2\sqrt{5}}$

4.2 Solutions of the Johnson equation

In [20] the author introduces a two dimensional generalization of the cylindrical KdV equation called the Johnson equation (or cylindrical KP equation) that reads

$$\left(v_t + v_{xxx} + 6vv_x + \frac{v}{2t}\right)_x = \frac{3v_{yy}}{t^2}. \tag{25}$$

As stated in [24], the Johnson equation is closely related to the KP-I equation.

Proposition 11. If u is a solution of (6) then v defined by

$$v(x, y, t) := u(x + \frac{y^2 t}{12}, yt, t)$$

is a solution of (25).

We can use this change of variables to construct solutions of (25) from the solutions of KP-I presented above. Some of these solutions are represented in figures 11 and 12 for different values of φ_4 and $\varphi_1 = \varphi_2 = 0$. We can remark that this time the influence of φ_2 on these solutions is more complicated than a translation.

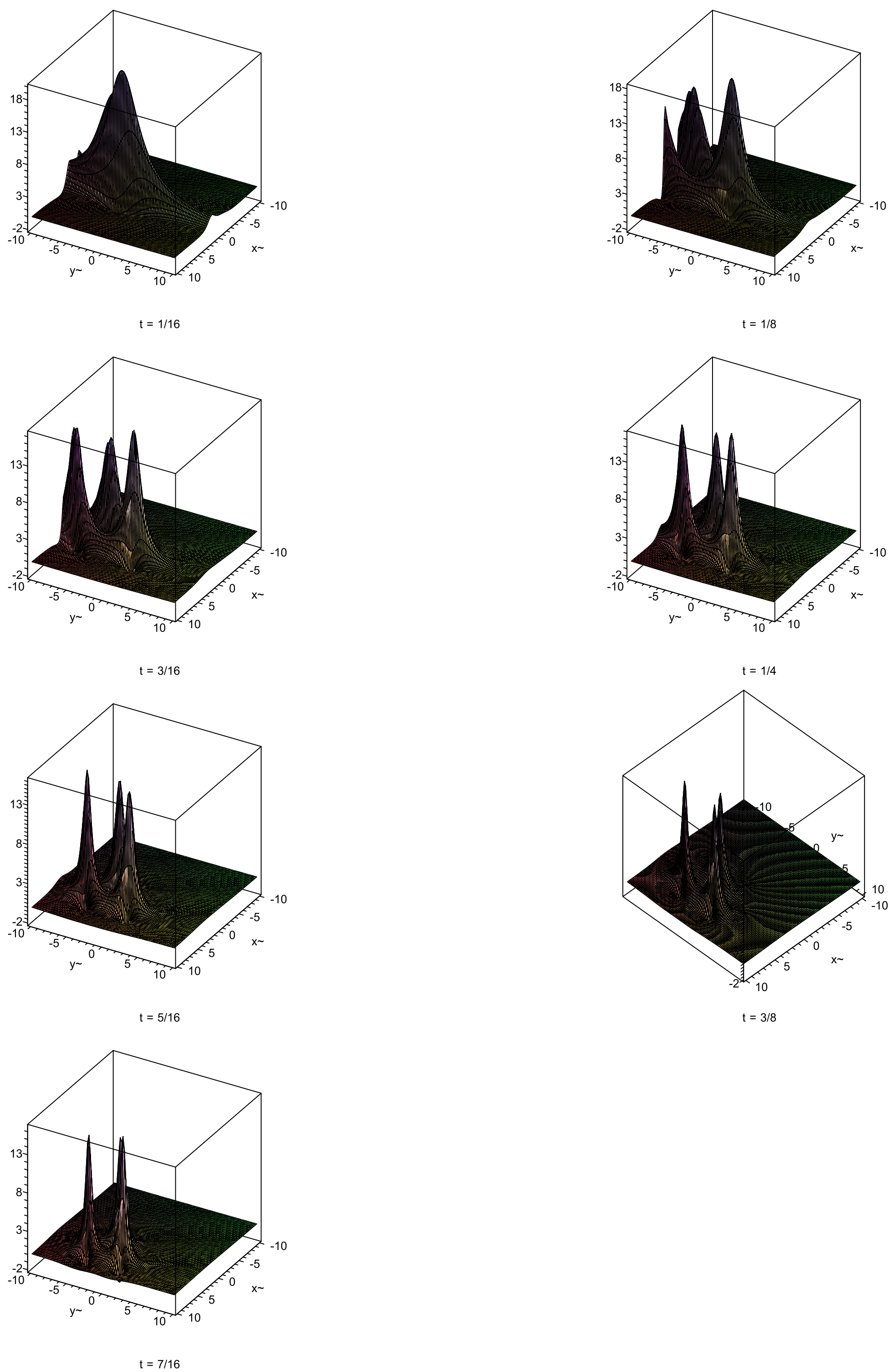

Figure 11: Solution of CKP with $\varphi_4 = 0$

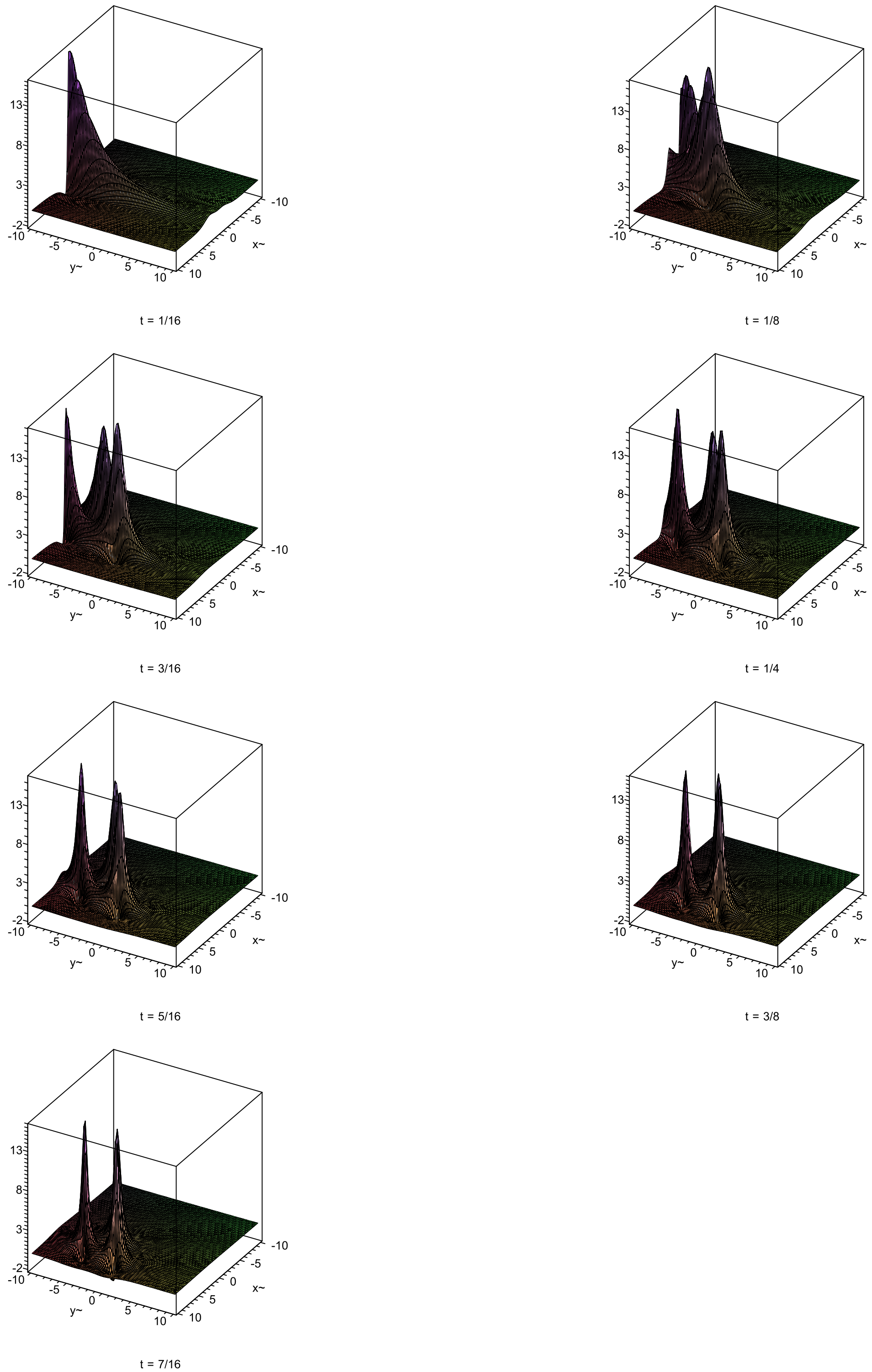

Figure 12: Solution of CKP with $\varphi_4 = \frac{5+\sqrt{5}}{96}\sqrt{10-2\sqrt{5}}$

5 Conclusion and open questions

The solutions presented here were obtained for the first time in the article [15] by Eleonskii, Krichever and Kulagin in 1986 up to some inaccuracies. Surprisingly this work has been overlooked in the integrable systems community and the rogue waves community until now. I'm inclined to believe that the work presented here have several advantages over the original exposition :

- The quasi-rational solutions of the NLS equation are described in a simpler and more elegant way using Wronski determinants.
- Using a Darboux transformation approach for the non-stationary Schrödinger equation gives a simplified version of the proof.
- It also allows us to construct for "free" a large family of real non-singular rational localized solution of the KP-I equation which might be associated with shallow water rogue waves.
- Maybe more than everything else it contains a qualitative study of these solutions making their multi-rogue waves behavior obvious.
- Finally the approach used here can be extended to many other equations of physical interest, for example the vectorial NLS equation. This work is in progress.

Nonetheless, several questions stay open.

First, a better understanding of the location and the dynamics of peaks and holes would be interesting. It would also provide a better understanding of the propagation of the peaks of the associated solution of KP-I.

Then, we can tackle the problem of higher values of n. If, theoretically, the same scheme can be applied for solutions obtained with $n \geq 3$, the amount of calculations would be overwhelming and the results not very satisfactory in the long term. To obtain similar results for generic n, we need to work from the Wronskian formula. Maybe, a re-writing of this formula in terms of theta functions or Fredholm determinants would help. See [10, 18, 29] for more information.

We can also wonder about the generality of these solutions. Can we choose the different parameters in the definition of f_j more generally and still obtain rational solutions with a system of peaks ? What is the position of these solutions among all rational solutions of NLS ?

Finally, other well-known equations admit rogue rational solutions. For example, some special rational solutions of the Hirota equation having the behavior of higher order Peregrine breathers were obtained in [9]. It could be interesting to use the same method as here to obtain the more general family of rational solutions.

A Covariance theorem

In this section the subscript denotes the derivative with respect to the evolution variable z and the prime or power notation is used for the derivative with respect to x.

Theorem 12. Let f be a solution of

$$f_z = \sum_{m=0}^{n} u_m(x,z) f^{(m)} \tag{26}$$

and let denote by ϕ another solution of (26) and $\sigma = \frac{\phi'}{\phi}$. Then ψ defined by

$$\psi = (\partial_x - \sigma) f \tag{27}$$

satisfies

$$\psi_z = \sum_{m=0}^{n} \widetilde{u_m}(x,z) \psi^{(m)} \tag{28}$$

where $\widetilde{u_m}$ are defined by

$$\begin{aligned} \widetilde{u_n} &= u_n, \\ \widetilde{u_m} &= \sum_{k=m+1}^{n} \sum_{j=m+1}^{k} C_k^j C_{j-1}^m \left[(u_k' - \sigma u_k)\phi^{(k-j)} + u_k \phi^{(k-j+1)} \right] \left(\phi^{-1}\right)^{(j-1-m)} \\ &\quad + \sum_{k=m}^{n} \sum_{j=m}^{k} C_k^j C_j^m u_k \phi^{(k-j)} \left(\phi^{-1}\right)^{(j-m)} \qquad 0 \le m \le n-1. \end{aligned}$$

Proof

Considering (27) as a differential equation for f we can write

$$f = \phi \int_{x_0}^{x} \phi^{-1}(\tau,z) \psi(\tau,z)\, d\tau. \tag{29}$$

Substituting (29) in equation (26) we get

$$\phi_z \int_{x_0}^{x} \phi^{-1}\psi\, d\tau + \phi \int_{x_0}^{x} (\phi^{-1}\psi)_z\, d\tau = \sum_{k=0}^{n} u_k \sum_{j=0}^{k} C_k^j \phi^{(k-j)} \left(\int_{x_0}^{x} \phi^{-1}\psi\, d\tau \right)^{(j)}.$$

As ϕ is solution of (26) the terms for $j = 0$ in the right-hand side of the last equation cancels out the term $\phi_z \int_{x_0}^{x} \phi^{-1}\psi \, d\tau$ in the left-hand side. Hence

$$\phi \int_{x_0}^{x} (\phi^{-1}\psi)_z \, d\tau = \sum_{k=1}^{n} u_k \sum_{j=1}^{k} C_k^j \phi^{(k-j)} \left(\phi^{-1}\psi\right)^{(j-1)}. \qquad (30)$$

Multiplying both sides of this relation by ϕ^{-1} and differentiating by x we get the following formula :

$$(\phi^{-1}\psi)_z = \left(\phi^{-1} \sum_{k=1}^{n} u_k \sum_{j=1}^{k} C_k^j \phi^{(k-j)} (\phi^{-1}\psi)^{(j-1)}\right)_x. \qquad (31)$$

We can easily check that (31) yields, up to a factor ϕ^{-1}, equation (28) with the $\widetilde{u_m}$ defined in theorem 12.

This result can be generalized in the case of a matricial equation (see [32]). It can also be used iteratively. In this case the formula is simpler than expected.

Theorem 13. Let $\phi_1, \ldots, \phi_N$ be linearly independent solutions of (26). The result of a N-step Darboux transformation applied to f is given by

$$\psi_N := D_N f = \frac{W(\phi_1, \ldots, \phi_N, f)}{W(\phi_1, \ldots, \phi_N)}$$

Proof

We prove this by induction on N. For $N = 1$ it's the previous theorem. Let assume that

$$\psi_{N-1} = \frac{W(\phi_1, \ldots, \phi_{N-1}, f)}{W(\phi_1, \ldots, \phi_{N-1})}.$$

Then

$$\phi[N] = \frac{W(\phi_1, \ldots, \phi_{N-1}, \phi_N)}{W(\phi_1, \ldots, \phi_{N-1})}$$

is a solution of the equation satisfied by ψ_{N-1}. If we denote by $\sigma[N]$ the logarithmic derivative of $\phi[N]$ then

$$\psi_N = \psi'_{N-1} - \sigma[N]\psi_{N-1}.$$

D_N is a differential operator of order N of the form

$$D_N = \partial_x^N + \sum_{m=0}^{N-1} a_m \partial_x^m \tag{32}$$

such that $D_N f = 0$ when $f = \phi_i$. The coefficients a_m are solutions of a system of N equations and can be written down with Cramer's formula. Then we can see that the right-hand side of (32) is the development along the last column of

$$\frac{W(\phi_1, \ldots, \phi_N, .)}{W(\phi_1, \ldots, \phi_N)}.$$

We can also generalize the formulas for the coefficients of the equation. For example

$$u_{n-2}[N] = u_{n-2} + nu_n \sum_{m=1}^{N} \sigma[m]' = u_{n-2} + nu_n \partial_x^2 \log(W(\phi_1, \ldots, \phi_N)).$$

B Maple code for plots

Here an example of the code used to produce the plots. This corresponds to Figure 4.

```
> restart;
> with(linalg):with(LinearAlgebra):
> assume(x,real,t,real);
> phi[3]:=0;phi[4]:=0;phi[1]:=3*phi[3];phi[2]:=2*phi[4]+(3+sqrt(5)
  )*sqrt(10-2*sqrt(5))/16;
```

$$\phi_3 := 0$$

$$\phi_4 := 0$$

$$\phi_1 := 0$$

$$\phi_2 := \frac{1}{16}\left(3+\sqrt{5}\right)\sqrt{10-2\sqrt{5}} \tag{1}$$

```
> w:=(sqrt(5)+1)/4+I*sqrt(10-2*sqrt(5))/4:
> f:=unapply(exp(k*(x-phi[1])+I*k^2*(t-phi[2])+k^3*phi[3]+I*k^4*
  phi[4])/((k^2-(w+1)/(w-1))*(k^2-(w^3+1)/(w^3-1))),x,t,k):f[1]:=
  unapply(k^2*diff(f(x,t,k),k)/(k^2+1),x,t,k):f[2]:=unapply(k^2*
  diff(f[1](x,t,k),k)/(k^2+1),x,t,k):f[3]:=unapply(k^2*diff(f[2]
  (x,t,k),k)/(k^2+1),x,t,k):
> a[1]:=unapply(f[1](x,t,1),x,t):a[2]:=unapply(f[1](x,t,-1),x,t):b
  [1]:=unapply(f[3](x,t,1),x,t):b[2]:=unapply(f[3](x,t,-1),x,t):
> p[1]:=unapply(diff(a[1](x,t),x),x,t):p[2]:=unapply(diff(a[2](x,
  t),x),x,t):q[1]:=unapply(diff(b[1](x,t),x),x,t):q[2]:=unapply
  (diff(b[2](x,t),x),x,t):
> W1:=unapply(det(wronskian([a[1](x,t),a[2](x,t),b[1](x,t),b[2](x,
  t)],x)),x,t):W2:=unapply(det(wronskian([p[1](x,t),p[2](x,t),q[1]
  (x,t),q[2](x,t)],x)),x,t):
> s:=simplify(-W2(x,t)/W1(x,t)):
> with(plots):
> plot3d(abs(s),x=-5..5,t=-5..5,numpoints=10000,axes=boxed):
```

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