**UNIVERSITÉ DE BOURGOGNE**
U.F.R. Sciences et Techniques
Institut de Mathématiques de Bourgogne
UMR 5584 du CNRS

# THÈSE

Pour l'obtention du grade de

DOCTEUR DE UNIVERSITÉ DE BOURGOGNE

Discipline : MATHÉMATIQUES

Présentée par

**Aymen BRAGHTHA**

le 20 Juin 2013

# Les zéros des intégrales pseudo-abéliennes: un cas non générique

Membres du Jury :

| | | |
|---|---|---|
| Emmanuel PAUL | Université de Paul Sabatier, Toulouse | Rapporteur |
| Daniel PANAZZOLO | Université de de Haute-Alsace, Mulhouse | Co-directeur de Thèse |
| Dimitry NOVIKOV | Institut Weizmann des Sciences, Israel | Rapporteur |
| Marcin BOBIEŃSKI | Université de Varsovie, Pologne | Examinateur |
| Pavao MARDESIĆ | Université de Bourgogne, IMB | Directeur de Thèse |
| Jean-Philippe ROLIN | Université de Bourgogne, IMB | Examinateur |

# Résumé


Les integrales abéliennes $I(h) = \int_{\gamma(h)} \omega$ sont les integrals d'une 1-forme polynomiale ou rationelle le long d'une famille de cycles $\gamma(h) \subset H^{-1}(h)$, $H \in \mathbb{C}[x,y]$. Les integrales abéliennes apparaissent comme la partie principale de la fonction déplacement de Poincaré de la perturbation $dH + \varepsilon\omega$ le long des cycles $\gamma(h)$. Pour une valeur regulière $h$ les zéros de cette fonction correspondent au cycles limites naissant dans la perturbation.

Varchenko et Khovanskii prouvent l'existence d'une borne uniforme, par rapport aux degrés de $H$ et $\omega$, du nombre de zéros des intégrales abéliennes associées à la perturbation.

Arnold pose le problème de l'existence d'une borne uniforme dans un cadre plus général c'est à dire pour les perturbations polynomiales des systèmes integrables. En particulier, les systèmes intégrables qui ont des intégrales premières de la forme générale $H = \prod_{i=0}^{k} P_i^{\alpha_i}, P_i \in \mathbb{R}[x,y], \alpha_i > 0$. Dans ce cas, on parle des intégrales pseudo-abéliennes à la place des intégrales abéliennes. Bobieński, Mardešić et Novikov ont prouvé, sous des conditions génériques, l'existence d'une borne locale uniforme du nombre de zéros des intégrales pseudo-abéliennes.

Dans ma thèse j'ai demontré un resultat concernant l'existence d'une borne uniforme locale pour un cas non-générique. Plus précisement, considérons une fonction multivaluée de la forme $H_0 = \prod_{i=0}^{k} P_i^{\epsilon_i}$, où $P_i \in \mathbb{R}[x,y], \epsilon_i > 0$ qui est a une intégrale première de la 1-forme polynomiale $\omega_0 = M_0 \frac{\mathrm{d}H_0}{H_0} = 0$, où $M_0 = \prod_{i=0}^{k} P_i$ (facteur d'integration).

Supposons que les courbes de niveaux $P_1 = 0, P_2 = 0$ et $P_3 = 0$ intersectent transversalement deux à deux en un point commun qui est le seul point triple dans le polycycle $\{H_0(x,y) = 0\}$ et $P_i(0,0) \neq 0, i = 3, \ldots, k$.

Maintenant considérons le déploiement $H_\lambda = P_\lambda^\epsilon P_1^{\epsilon_1} P_2^{\epsilon_2} \prod_{i=3}^{k} P_i^{\epsilon_i}, \epsilon > 0, \epsilon_i > 0, i = 1, \ldots, k$ de l'integrale première $H_0$.

Le feuilletage $\omega_\lambda = M_\lambda \frac{\mathrm{d}H_\lambda}{H_\lambda} = 0$ a des cycles $\gamma(\lambda, h) \subseteq \{H_\lambda(x,y) = h\}$ remplissent une composante connexe de $\mathbb{R}^2 \setminus \{P_\lambda \prod_{i=1}^{k} P_i = 0\}$. On considère une perturbation polynomiale $\omega_{\lambda,\kappa} = \omega_\lambda + \kappa\eta, \kappa > 0$ de système $\omega_\lambda$, où $\eta = R\mathrm{d}x + S\mathrm{d}y, R, S \in \mathbb{R}[x,y]$.

A chaque forme polynomiale $\eta$ on peut associer l'intégrale pseudo-Abelienne

$$J(\lambda, h) = \int_{\gamma(\lambda,h)} \frac{\eta}{M_\lambda},$$

qui est la partie principale de la fonction de déplacement de Poincaré $\Delta$

$$\Delta(\kappa, h) = \kappa h \int_{\gamma(\lambda,h)} \frac{\eta}{M_\lambda} + O(\kappa)$$

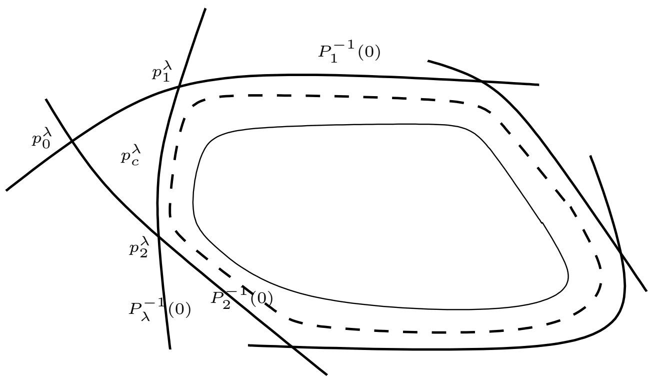
de la perturbation $\omega_{\lambda,\kappa}$ le long de cycle $\gamma(\lambda, h)$.

On impose les conditions suivantes

1. $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$.

2. $P_1^{-1}(0), P_2^{-1}(0)$ et $P_0^{-1}(0)$ s'intersectent transversalement deux à deux à l'origine qui est le seul point triple. Les courbes de niveaux $P_i^{-1}(0), i = 0, \ldots, k$ s'intersectent transversalement deux à deux.

3. $\eta = O((x,y)^4)$ à l'origine.




The cycle $\gamma_{(\lambda,h)}$

Le résultat principale de ma thèse est le suivant:

**Theorème** *Sous les conditions (1), (2) et (3), il existe une borne pour le nombre de zéros isolés de l'intégrale pseudo-Abelienne* $J(\lambda, h) = \int_{\gamma(\lambda,h)} \frac{\eta}{M_\lambda}$ *pour* $\lambda$ *proche de 0. La borne dépend seulement de* $n_i = \deg P_i, n = \deg(R, S)$ *et est uniforme par rapport au coefficients de polynomes* $P_\lambda, P_i, R, S$, *exposants* $\epsilon, \epsilon_i, i = 1, \ldots, k$.

Ce résultat est similaire au résultat de Bobieński [1] sauf que dans notre cas

1. L'intégrale première de Darboux $H_\lambda$ a une forme plus générale c'est à dire les exposants $\epsilon_1 \neq \epsilon_2$ (dans [1] $\epsilon_1 = \epsilon_2 = 1$).

2. La preuve est basée sur de techniques géométriques (éclatement en famille).

3. L'existence d'une condition technique (condition 3) sur la forme perturbative $\eta$.

**Les ingredients de la preuve**
La difficulté de la preuve est que le centre $p_c^\lambda$ genère d'autres points de ramification qui bifurquent de 0 de la fonction $J(\lambda, h)$ localisés sur une cercle de rayon d'ordre $|\lambda|^{\epsilon + \epsilon_+ + \epsilon_-}$.

Brièvement les ingredients de la preuve sont

(a) *Réctification de l'intégrale première* $H_\lambda$*:* Sous les conditions (1),(2) il existe un système de coordonnées analytiques locales $(x, y, \lambda)$ prés de l'origine $(0,0,0)$ telle que $H_\lambda$ est de la forme

$$H_\lambda = (x-\lambda)^\epsilon (y-x)^{\epsilon_+}(y+x)^{\epsilon_-}U, \lambda > 0$$

où $U$ est une fonction analytique et $U(0,0,0) \neq 0$.

(b) *L'éclatement en famille:* Le feuilletage $\mathcal{F}$ de dimension 1 dans $\mathbb{C}^3_{(x,y,\lambda)}$ qui est defini par l'intersection des courbes de nivaux $\{H(x,y,\lambda) = h\}$ et $\{\lambda = s\}$ possède une singularité à l'origine $(0,0,0)$ (point triple). Pour reduire cette singularité on éclate l'origine dans l'espace totale $\mathbb{C}^3_{(x,y,\lambda)}$ et le remplace par $\mathbb{CP}^2 = \sigma^{-1}((0,0,0))$ comme un diviseur exceptionel. Le feuilletage resultant $\sigma^*\mathcal{F}$ est defini par l'intersection de courbes de nivaux $\{\sigma^* H(x,y,\lambda) = h\}$ et $\{\sigma^*\lambda = s\}$. Les polycycles $\{\sigma^* H(x,y,\lambda) = 0, \sigma^*\lambda = s\}$ possèdent des singularités hyperboliques. On definit l'intégrale dans l'espace eclaté $J(s,h) = \int_{\delta(s,h)} \sigma^* \frac{\eta}{M_\lambda}$. La preuve de théorème est analogue au celle pour l'intégrale $\int_{\delta(s,h)} \sigma^* \frac{\eta}{M_\lambda}$.

(c) *Relation de la variation:* On définit la variation de la fonction $J$:

$$\mathcal{V}ar_{(h,\alpha)} J(\lambda, h) = J(, he^{i\pi\alpha}) - J(\lambda, he^{-i\pi\alpha}).$$

L'opérateur de variation $\mathcal{V}ar$ modifie le nombre de zéros de l'intégrale $J$ par une constante locale: la méthode de Petrov et le théorème de préparation pour les fonctions logarithmico-exponentielles [10] nous donnent une estimation du nombre de zéros de $J(\lambda,h)$ en terme de nombre de zéros de la fonction $\mathcal{V}ar_{(h,\alpha)} J(\lambda,h)$.

# Remerciements

Je tiens tout d'abord a remercier mon directeur de thèse Pavao Mardešić et mon co-directeur de thèse Daniel Panazzalo, pour m'avoir encadré tout au long de cette periode. Je leur en suis vraiment reconnaissant. Je voudrais également les remercier pour leur patience ainsi que pour le grande liberté et l'autonomie qu'ils m'ont laissées durant ce travail de thèse.

Je remercie ensuite Dimitry Novikov et Emmanuel Paul d'avoir accepté d'être rapporteurs de cette thèse. Je les remercie d'avoir consacré une partie de leur temps à la lecture de ce manuscrit. Je voudrais également remercier les autres membre de jury Jean-Philippe Rolin et Marcin Bobieński.

Merci également au personnel administratif du laboratoire, notamment Caroline Gérin. Ce travail n'aurait pas été possible sans l'aide financière de l'institut de Mathématiques de Bourgogne, de l'école doctorale Carnot Pasteur.

Je remercie le service de relations internationales de l'université de Bourgogne pour m'avoir facilité l'accés au logement pendant mes séjours à Dijon et au Crous.

Pour finir, je remercie de tout coeur ma famille: mes chers parents, mon frère et mes soeurs. Un très grand merci aussi à tous les amis que j'ai rencontré depuis mon arrivée à Dijon.

# Contents

# Introduction

Abelian integrals are integrals $I(h) = \int_{\gamma(h)}$ of polynomial or rational one-forms along cycles $\gamma(h) \subset H^{-1}(h), H \in \mathbb{C}[x,y]$. Abelian integrals appear as principal part of Poincaré displacement function of the perturbation $dH + \varepsilon\omega$ along $\gamma(h)$. Their zeros correspond to limit cycles born in the perturbation.

Varchenko and Khovanskii prove the existence of a bound, uniform with respect to the degree of $H$ and $\omega$, for the number of these zeros.

Arnold posed with insistence the analogous problem for more general polynomial perturbations of integrable systems. In particular for perturbations of systems having a Darboux first integral $H = \prod P_i^{\alpha_i}, P_i \in \mathbb{R}[x,y]$. Then, instead of Abelian integrals, one encouters pseudo-Abelian integrals.

My thesis is a continuation of the program of Bobieński, Mardešić and Novikov to extend the Varchenko-Khovanskii's Theorem from abelian integrals to pseudo-abelian integrals. We study, under some conditions, a non generic case unfolding a singularity of codimension one.

Precisely, we consider a multivalued function of the form $H_0 = \prod_{i=0}^{k} P_i^{\epsilon_i}$, with $P_i \in \mathbb{R}[x,y], \epsilon_i > 0$, which is a Darboux first integral of the polynomial one-form $\omega_0 = M_0 \frac{\mathrm{d}H_0}{H_0} = 0$ with integrating factor $M_0 = \prod_{i=0}^{k} P_i$, having a center whose bassin is bounded by a polycycle $\{H_0(x,y) = 0\}$. Assume $\{P_0 = 0\}, \{P_1 = 0\}$ and $\{P_2 = 0\}$ intersect in a common point which is the only triple point.

Consider an unfolding $\omega_\lambda$ of the form $\omega_0$, where $\lambda$ is a small parameter and $\omega_\lambda$ is a family of analytic one-forms $\omega_\lambda = M_\lambda \frac{dH}{H_\lambda}$, with Darboux first integral $H_\lambda = P_\lambda^{\epsilon} \prod_{i=1}^{k} P_i^{\epsilon_i}$, where $\epsilon, \epsilon_i > 0, P_\lambda, P_i \in \mathbb{R}[x,y]$ and integrating factor $M_\lambda = P_\lambda \prod_{i=1}^{k} P_i$. Let $\gamma(\lambda,h) \subset \{H_\lambda(x,y) = h\}$.

Consider pseudo-Abelian integral of the form

$$I(\lambda,h) = \int_{\gamma(\lambda,h)} \frac{\eta}{M_\lambda}, \quad \gamma(\lambda,h) \subset H_\lambda^{-1}(h) \quad M = P_\lambda \prod_{i=1}^{k} P_i$$

and $\eta$ is a polynomial one-form of degree at most $n$.

We impose the following conditions

(a) $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$,

(b) the levels curves $P_1 = 0, P_2 = 0$ and $P_0 = 0$ intersect transversally two by two at the origin which is the only triple point and for $i = 3, \ldots, k$ the level curves $P_i = 0$ intersect transversally and two by two,

(c) $\eta$ vanishes to the order order $\geq 4$ at $(x, y) = (0, 0)$ (technical condition).

**The principal result.** *Under above conditions (1), (2), (3), there exists a bound for the number of isolated zeros of the pseudo-Abelian integral* $I(\lambda, h) = \int_{\gamma(\lambda,h)} \frac{\eta}{M_\lambda}$ *for* $\lambda$ *close to 0. The bound is locally uniform with respect to all parameters in particular with respect to* $\lambda$.

Our result is similar to Bobieński's result [1]. The differences between our work and Bobieński's work [1] relies in the fact that:

(a) In our work the first integral $H_\lambda$ is more general in the sense that the exponents $\epsilon_1$ and $\epsilon_2$ are different and in [1] we have $\epsilon_1 = \epsilon_2 = 1$.

(b) Our approach is purely geometric which is based on the blow-up in family. This approach gives directly uniform validity of our study of the pseudo-Abelian integrals.

(c) On the other hand, in our work the polynomial one-form $\eta$ of the deformation $\omega_{\lambda,\kappa}$ vanishes to the order $\geq 4$ at $(0, 0)$.

# Chapter 1

# Tangential 16-th Hilbert Problem

The second part of Hilbert's 16th problem, asking for a bound $H(n)$ for the numbers of limit cycles and their relative positions for all planar polynomial differential systems of degree $n$, is still open even for the quadratic case ($n = 2$).

A weak form of this problem, proposed by Arnold, asks for the maximum $Z(m, n)$ of the numbers of isolated zeros of Abelian integrals of all polynomails 1-forms of degree $n$ over algebraic ovals of degree $m$.

## 1.1 Abelian integrals

Consider planar differential systems

$$\frac{\partial x}{\partial t} = P_n(x, y), \quad \frac{\partial y}{\partial t} = Q_n(x, y), \tag{1.1}$$

where $P_n, Q_n$ are real polynomials in $x, y$ and the maximum degree of $P$ and $Q$ is $n$. The second half of the famous Hilbert's 16th problem, proposed in 1900, can be stated as follows:

*For a given integer $n$, what is the maximum number of limit cycles of system (1.1) for all possible $P_n$ and $Q_n$?*

Usually, the maximum of the number of limit cycles is denoted by $H(n)$. Recall that a *limit cycle* of system (1.1) is an isolated closed orbit. Note that the problem is trivial for $n = 1$: a linear system may have periodic orbits but have no limit cycle .i.e, $H(1) = 0$. This problem is still open even for the case $n = 2$.

Let $H = H(x,y)$ be a polynomial in $x, y$ of degree $m \geq 2$, and the level curves $\gamma(h) \subset \{(x,y) : H(x,y) = h\}$ form a continuous family of ovals $\{\gamma(h)\}$ for $h_1 < h < h_2$. Consider a polynomial 1-form $\omega = f(x,y)\mathrm{d}x - g(x,y)\mathrm{d}y$, where $\max(\deg(f), \deg(g)) = n \geq 2$. Arnold proposed the following problem:

*For fixed integers $m$ and $n$ find the maximum $Z(m,n)$ of the numbers of isolated zeros of the Abelian integrals*

$$I(h) = \int_{\gamma(h)} \omega. \tag{1.2}$$

Abelian integral is the integral of a rational 1-form along an algebraic oval. We observe that the function $I(h)$ may be multivalued. The function $I(h)$, given the Abelian integral (1.2), is the first order term in $\varepsilon$ of the displacement function of the Poincaré map (see the next subsection) on a segment transversal to $\gamma(h)$ for the system

$$\frac{\partial x}{\partial t} = -\frac{\partial H(x,y)}{\partial y} + \varepsilon R(x,y), \quad \frac{\partial y}{\partial t} = \frac{\partial H(x,y)}{\partial x} + \varepsilon S(x,y), \tag{1.3}$$

where $H, R, S$ are the same as above when defining the Abelian integral $I(h)$.

**Results.** *This problem is also not solved completely, but there are many nice results concerning it.*

***Theorem 1*** *[8,9]. For given $m$ and $n$ the number $Z(m,n)$ is uniformly bounded with respect to the choice of the polynomial $H$, the family of ovals $\{\gamma(h)\}$ and the 1-form $\omega$.*

$$Z(m,n) < \infty.$$

*The proof of Varchenko is based on the asymptotic expansions of integrals along cycles in complex algebraic curves and some finiteness results from real analytic geometry. Khovanski observed that Abelian integrals belong to his class of Pfaff functions and applied his theory of fewnomials [10].*

***Theorem 2*** *[5]. $Z(m,n)$ is no greater than $2^{2^{Poly(n)}}$, where $Poly(n)=O(n^p)$.*

## 1.2 Displacement function

We consider a polynomial $H(x,y)$ of degree $m$ as in the previous subsection, the corresponding Hamiltonian vector field $X_H$

$$X_H = \frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}, \tag{1.4}$$

and the perturbed system $X_H + \varepsilon Y$ where $Y = R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}$, where $\deg(R,S) \leq n$ and $\varepsilon$ is a small parameter.

Suppose that there is a family of ovals, $\gamma(h) \subset H^{-1}(h)$, continuously depending on a parameter $h \in (a,b), a,b \in \mathbb{R}$. Then we may define the Abelian integral as before

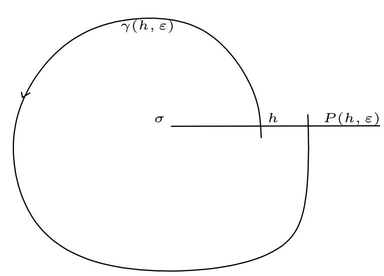
$$I(h) = \int_{\gamma(h)} \omega = \int_{\gamma(h)} -S\mathrm{d}x + R\mathrm{d}y. \tag{1.5}$$

All $\gamma(h)$ filling up a annulus for $h \in (a,b)$, are periodic orbits of the Hamiltonian system $X_H$.

Consider the question: How many orbits keep being unbroken and become the periodic orbits of the perturbed system $X_{H,\varepsilon} := X_H + \varepsilon Y$ for small $\varepsilon$ ?

This question can be proposed in the following way: Is it possible to find value $h \in (a,b)$, and some periodic orbits $\Gamma_\varepsilon$ of the perturbed systems $X_{H,\varepsilon}$, such that $\Gamma_\varepsilon$ tends to $\gamma(h)$, in the sense of Hausdorff distance, as $\varepsilon \to 0$ ? And how many such $\Gamma_\varepsilon$ for some $h$ ?

To answer this question, we take a segment $\sigma$, transversal to each oval $\gamma(h)$. We choose the values of the function $H$ to parametrize $\sigma$, and denote by $\gamma(h,\varepsilon)$ a piece of the orbit of the perturbed system $X_{H,\varepsilon} := X_H + \varepsilon Y$ between the starting point $h$ and the next intersection point $P(h,\varepsilon)$ with the transversal segment , see Figure below. The next intersection is possible for sufficiently small $\varepsilon$, since $\gamma(h,\varepsilon)$ is close to $\gamma(h)$. As usual, the difference $d(h,\varepsilon) = P(h,\varepsilon) - h$ is called the *displacement function*.




Displacement function.

**Theorem 3.**(Poincaré-Pontryagin). *We have that*

$$d(h,\varepsilon) = \varepsilon(I(h) + \varepsilon\psi(h,\varepsilon)), \quad as \quad \varepsilon \to 0, \tag{1.6}$$

*where $\psi(h,\varepsilon)$ is analytic and uniformly bounded for $(h,\varepsilon)$ in a compact region near $(h,0), h \in (a,b)$.*

## 1.3 Deformations of elliptic and hyperelliptic Hamiltonians

Let $X_H$ be the hamiltonian (1.4) and $X_{H,\varepsilon}$ be its perturbation. Concrete estimates are given with some restrictions on the Hamiltonian function $H$ to the following form:

$$H(x,y) = \frac{y^2}{2} + P_m(x), \tag{1.7}$$

where $P_m$ is a polynomial in $x$ of degree $m$. The level curves of $H$ are rational for $m = 1, 2$, elliptic for $m = 3, 4$ and hyperelliptic for $m \geq 5$. We assume $m \geq 2$ since the level curves have no oval if $m = 1$.

**Lemma 1.** *Suppose that for the function $H$ defined in (1.7) there is a family of ovals $\gamma(h) \subset H^{-1}(h)$, and $\omega$ is an arbitrary polynomial 1-form of degree $n$, then*

$$\int_{\gamma(h)} \omega = \begin{cases} \int_{\gamma(h)} p_1(x) y \mathrm{d}x, & n = 2, \\ \int_{\gamma(h)} p_k(x,h) y \mathrm{d}x, & n \geq 3, \end{cases} \tag{1.8}$$

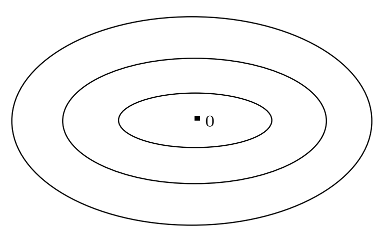
*where $p_1$ is a linear function in $x$, and $p_k(x,h)$ is a polynomial in $x$ and $h$ of degree $k = \frac{m(n-1)}{2}$ if $n$ is odd and $k = \frac{m(n-2)}{2}$ if $n$ is even.*

### 1.3.1 Elliptic Hamiltonians of degree $m = 2$

We choose $H = \frac{1}{2}x^2 + \frac{1}{2}y^2$. The ovals are circles $\{x^2 + y^2 = h^2\}$. The 1-form $\omega$ is of degree $n$, then by using the polar coordinates one finds that

$$\int_{\gamma(h^2)} \omega = h^2 Q_{n-1}(h), \tag{1.9}$$

where $Q_{n-1}(h)$ is a polynomial in $h$ of degree $(n-1)$, but depends only on $h^2$ by symmetry. $I(h)$ has at most $[\frac{(n-1)}{2}]$ zeros except the trivial zero at $h = 0$, which corresponds to the singularity at the origin.




The families of ovals $\gamma(h^2)$ for $m = 2$.

### 1.3.2 Elliptic Hamiltonians of degree $m = 3$

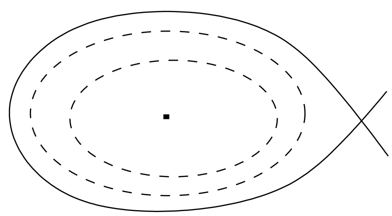
In this case if we suppose that the level curve $H$ contains a continuous family of ovals $\{\gamma(h)\}$, then the two singularities of the corresponding vector field $X_H$ must be a center and a saddle, which is chosen at $(-1, 0)$ and $(1, 0)$ respectively, and the elliptic Hamiltonian reads

$$H(x, y) = \frac{y^2}{2} - \frac{x^3}{3} + x, \tag{1.10}$$

with the continuous family of ovals

$$\{\gamma(h)\} = \{(x, y) : H(x, y) = h, -\frac{2}{3} \leq h \leq \frac{2}{3}\}, \tag{1.11}$$

shown in Figure below.



The families of ovals $\{\gamma(h)\}$ for the case $m = 3$.

The Abelian integral $I(h)$ can be expressed in the form $\int_{\gamma(h)} p_k(x, h)y\mathrm{d}x$, where $p_k$ is a polynomial in $x$ and $h$. We observe that along $\gamma(h)$

$$0 \equiv \frac{\partial H}{\partial x}\mathrm{d}x + \frac{\partial H}{\partial y}\mathrm{d}y = (1 - x^2)\mathrm{d}x + y\mathrm{d}y,$$

which implies $(1 - x^2)y\mathrm{d}x + y^2\mathrm{d}y \equiv 0$. Hence $I_2(h) = I_0(h)$, where we define $I_j(h) = \int_{\gamma(h)} x^i y\mathrm{d}x$. Similarly, we have

$$\int_{\gamma(h)} x^k(x^2 - 1)y\mathrm{d}x = \int_{\gamma(h)} x^k y^2\mathrm{d}y = \int_{\gamma(h)} x^k(2h + \frac{2x^3}{3} - 2x)\mathrm{d}y.$$

Using integration by parts on the right-hand side, we find the following induction formula,

$$(2k + 9)I_{k+2}(h) - 3(2k + 3)I_k(h) + 6khI_{k-1}(h) = 0,$$

where $k \geq 1$. Hence, it is not hard to prove the following result.

**Lemma 2.** *Suppose that $I(h)$ is the Abelian integral of the polynomial 1-form $\omega$ of degree at most $n$ over tha ovals $\gamma(h)$ defined in (1.11), then*

$$I(h) = Q_0(h)I_0(h) + Q_1(h)I_1(h), \tag{1.12}$$

*where $Q_0$ and $Q_1$ are polynomials,* $\deg(Q_0) \le [\frac{n-1}{2}], \deg(Q_1) \le [\frac{n}{2}] - 1$.

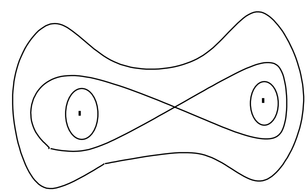
**Theorem 4** [12]. *The space of functions $\{I(h)\}$, defined in Lemma 2, has the Chebychev property on $h \in (\frac{-2}{3}, \frac{2}{3})$. This means that any nontrivial $I(h)$ has at most $n-1$ zeros, and there exists a 1-form $\omega$, such that $I(h)$ has exactly $n-1$ zeros.*

### 1.3.3 Elliptic Hamiltonians of degree $m = 4$

In this case we may take the function $H$ in the form

$$H(x, y) = \frac{y^2}{2} + a\frac{x^4}{4} + b\frac{x^3}{3} + c\frac{x^2}{2} \tag{1.13}$$

where $a > 0, b < 0$ and $H(x, 0)$ has only three real different critical values. The families of ovals $\{\gamma(h)\}$ on the level curves $H$, shown in Figure below, called the figure-eight loop. **Theorem 5** [12]. *Let $H$ be as (1.13)*



The family of ovals $\{\gamma(h)\}$ for $m = 4$.

*with the figure-eight loop. Then the space of elliptic integral $I(h)$ of a 1-form of degree $n$ over cycles vanishing at one of the two singularities of $X_H$ surrounded by the figure-eight loop has the Chebyshev property on the corresponding interval of $h$. This is means that the number of zeros of nontrivial $I(h)$ is less than the dimension of the sapce. This dimension is $n + [\frac{(n-1)}{2}]$.*

### 1.3.4 Hyperelliptic case $m \ge 5$

In this case the polynomial $P(x)$ in (1.7) has degree at least 5. The general result was proved by D. Novikov and S. Yakovenko as follows

**Theorem 6** [12]. *For any real polynomial $P(x) \in \mathbb{R}[x]$ of degree $m$ and any polynomial 1-form $\omega$ of degree $n$, the number of real ovals $\gamma \subset \{y^2 + P(x) = h\}$ yielding an isolated zero of the integral $I(h) = \int_\gamma \omega$, is bounded by a primitive recursive function $B(m; n)$ of integer variables $m$ and $n$, provided that all critical values of $P$ are real.*

## 1.4 Pseudo-Abelian integrals

In this section we introduce some results in the program of Bobieński, Mardešić and Novikov to extend the Varchenko-Khovanskii theorem from abelian integrals to pseudo-abelian integrals.

### 1.4.1 Pseudo-Abelian integrals: generic case

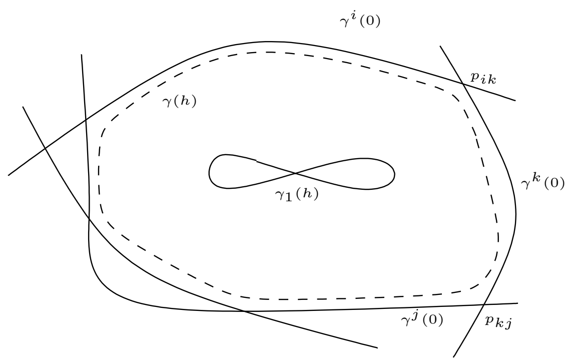
Consider system $\omega_1$ with first integral of Darboux type $H$

$$H(x,y) = \prod_{i=1}^{k} P_i^{\epsilon_i}, \qquad P_i \in \mathbb{R}[x,y], \quad \deg(P_i) \le n_i, \quad \epsilon_i \in \mathbb{R}_+ \tag{1.14}$$

More precisely, the polynomial integrable one-form $\omega_1$ be given by

$$\omega_1 = M\frac{dH}{H}, \tag{1.15}$$

where $M = \prod_{i=1}^{k} P_i$ is the integrating factor. The phase portrait of the one-form $\omega_1$ can be as in Figure below. Let $\mathscr{D}$ be the open period annulus with its closure intersecting the zero level curve $H^{-1}(0)$. Let the polycycle $\gamma(0) \subset H^{-1}(0)$ be a corresponding part of the boundary of $\mathscr{D}$.




The phase portrait of $\omega_1$. The period annulus $\mathscr{D}$ bounded by polycycles $\gamma(h)$ and $\gamma_1(h)$.

Consider a small polynomial deformation

$$\omega_1 + \varepsilon\omega_2, \qquad \omega_2 = Rdx + Sdy, \quad R, S \in \mathbb{R}[x,y]. \tag{1.16}$$

The integrable Darbouxian foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega_1 = 0$ is ramified over the level curve $H^{-1}(0)$.

Taking a transversal to the real trajectories of $\omega_1$ and parametrizing it with the values of $H$, it is known since Poincaré that the displacement function $d$ (see section above) is given by

$$d(h,\varepsilon) = \varepsilon h \int_{\gamma(h)} \frac{\omega_2}{M} + \psi(h,\varepsilon). \tag{1.17}$$

where $\psi(h,\varepsilon) = O(\varepsilon)$ is uniformly bounded for $(h,\varepsilon)$ in a compact region near $(h,0), h \in (h_{\min}, h_{\max})$.

We assume the genericity assumptions

1. The polycycle $\gamma(0)$ consists of edges $\gamma^i(0)$ which meet transversally -see Figure above. Any vertex $p_{ij}$ corresponds to the transversal intersection of level curves $P_i^{-1}(0)$ and $P_j^{-1}(0)$.

2. The first integral $H$ is regular at infinity.

Under the generic assumptions above the result was proved by D. Novikov in [10] and M. Bobieński and P. Mardešić in [2] as follows

**Theorem 7.** *Let $H, M, \omega_2$ as above. Then there exists a uniform bound for the number of isolated real zeros of pseudo-Abelian integrals associated to Darboux integrable one-forms close to $\omega_1$.*

In [2] the proof of Theorem 5 is based on many geometric technics. The essentiel ingredients for the proof are

**Proposition 1.** *Let*

$$H(x,y) = \prod_{i=1}^{k} P_i^{\epsilon_i}, \quad I(h) = \int_{\gamma(h)} \frac{\omega_2}{M}, \gamma(h) \in H^{-1}(h).$$

*We define the rescaled variation operator by $\mathcal{V}ar_{(h,\epsilon)}I(h) = I(he^{i\epsilon\pi}) - I(he^{-i\epsilon\pi})$. Then, we have*

$$\mathcal{V}ar_{(h,\epsilon_1)} \circ \ldots \circ \mathcal{V}ar_{(h,\epsilon_k)} I(h) \equiv 0.$$

*Proof.* To prove the proposition we use a partition of unity multiplying the form $\frac{\omega_2}{M}$ we can consider semilocal problem with a relative cycle $\gamma^i(h)$ close to one edge ($i$-th edge) $\gamma^i(0)$ of the polycycle $\gamma(0)$. Let $(\gamma^i(0))^{\mathbb{C}}$ be the complexification of the edge $\gamma^i(0)$ joining the singular point $p_{ii-1}, p_{ii+1}$. We have

$$\mathcal{V}ar_{(h,\epsilon_{i+1})} \circ \mathcal{V}ar_{(h,\epsilon_{i-1})} \gamma^i(h) = \widetilde{[\delta_{i-1}, \delta_{i+1}]},$$

where $\widetilde{[\delta_{i-1}, \delta_{i+1}]}$ is a complex closed cycle obtained as a lift of the commutator $[\delta_{i-1}(h), \delta_{i+1}(h)]$, where $\delta_{i-1}(h)$ and $\delta_{i+1}(h)$ are paths in $(\gamma^i(0))^{\mathbb{C}} \setminus \{p_{ii-1}, p_{ii+1}\}$ turning once counterclockwise around $p_{ii-1}$ and $p_{ii+1}$. On the other hand, we can keep $\widetilde{[\delta_{i-1}, \delta_{i+1}]}$ away from separatrices other than the edge $\gamma^i(0)$. Hence, locally we can put the first integral to the form $H = y^{\epsilon_i}$, where $y$ is a coordinate on $(\gamma^i(0))^{\mathbb{C}}$. Now, using that the variations commute, so we obtain

$$\mathcal{V}ar_{(h,\epsilon_1)} \circ \ldots \circ \mathcal{V}ar_{(h,\epsilon_k)} \gamma(h) \equiv 0.$$

□

To finish the proof of the theorem we consider $C_R = \{|h| = R, |\arg h| \leq \alpha\pi\}, C_\pm = \{r < |h| < R, |\arg h| = \pm\alpha\pi\}$ and $C_r = \{|h| = r, |\arg h| \leq \alpha\pi\}$. Let $D_{r,R}$ be slit annulus with boundary $\partial D_{r,R} = C_R \cup C_\pm \cup C_r$ Petrov's method allows to estimate the number of zeros of $I(h)$

$$\#Z_{D_{r,R}} I(h) \leq \frac{1}{2\pi}\Delta \arg_{\partial D_{r,R}} = \frac{1}{2\pi}\Big(\Delta \arg_{C_R} I(h) + \Delta \arg_{C_r} I(h) + \Delta \arg_{C_\pm} I(h)\Big).$$

1. The increment argument $\Delta \arg_{C_R} I(h)$ of $I(h)$ along $C_R$ is bounded by Gabrielov's theorem [12].
2. To estimate the limit $\lim_{r\to 0} \Delta \arg_{C_r} I(h)$, we investigate the leading term see Lemma 4.8 of [2].
3. The increment argument $\Delta \arg_{C_\pm} I(h)$ is locally bounded by the number of zeros $\#Z(\mathcal{V}ar_{(h,\alpha)} I(h))$.

one concludes by induction on $k$.

### 1.4.2 Pseudo-Abelian integrals: some non generic cases

In this subsection we introduce some results for non generic cases

**Pseudo-Abelian integrals associated to deformations of slow-fast Darboux integrable systems**

Consider Darboux integrable system $\omega_0 = M\frac{dH_0}{H_0}$, where $H_0 = \prod_{i=1}^k P_i^{\epsilon_i}, \epsilon_i > 0, P_i \in \mathbb{R}[x, y]$ and $M = \prod_{i=1}^k P_i$.

We consider one forms $\omega_\varepsilon$ given by

$$\omega_\varepsilon = P_0 M\frac{dH_0}{H_0} + \varepsilon M dP_0, P_0 \in \mathbb{R}[x, y].$$

with first integral $H_\varepsilon = P_0^\varepsilon H_0$. The Darboux integrable system $\omega_\varepsilon$ is slow-fast and $P_0 = 0$ is the slow manifold.

Consider the polynomial deformation of the system $\omega_\varepsilon$,

$$\omega_{\varepsilon,\kappa} = \omega_\varepsilon + \kappa\eta, \quad \kappa > 0,$$

where $\eta = Rdx + Sdy, R, S \in \mathbb{R}[x, y]$. The pseudo-abelian integral is given by

$$I_\varepsilon(h) = \int_{\gamma_\varepsilon(h)} \frac{\eta}{P_0 M}, \quad \gamma_\varepsilon(h) \subset H_\varepsilon^{-1}(h).$$

1. Let $D$ be a compact region bounded by $P_0 = 0$ and some separatrices $P_i = 0, i = 1, \ldots, k$. Assume that the functions $P_i, i = 0, \ldots, k$ are smooth and intersect transversally in $D$ and the foliation $\omega_0 = 0$ has no singularities on $\mathrm{Int}D$.

2. Assume that $P_0 = 0$ is tansversal to the foliation $\omega_0 = 0$ in all points of $D \cap \{P_0 = 0\}$ except one point $p_0$, where the contact is quadratic. Then for $\varepsilon \neq 0$ a singular point $p_\varepsilon$ bifurcates from $p_0$. It corresponds to a real value $h_\varepsilon = H_\varepsilon(p_\varepsilon)$. Let $\gamma_\varepsilon(h)$ be the family of cycles in the basin of the center bifurcating from $p_0$.

**Theorem 8**[4]. *Under the genericity conditions (1), (2), there exists a local bound for the number of isolated zeros of the pseudo-abelian integrals* $I(h,\varepsilon) = \int_{\gamma_\varepsilon(h)} \frac{\eta}{P_0 M}$, *for* $\varepsilon > 0$ *and* $h \in (0, h_\varepsilon)$. *This bound is uniform with respect to all parameters, in particular with respect to* $\varepsilon$.

### Unfolding generic exponential case

Let $M\frac{dH_0}{H_0} = 0$ be a polynomial integrable system having a Darboux first integrals of the form

$$H_0 = e^{\frac{R}{Q}} \prod_{i=1}^{k} P_i^{a_i}, \quad P_i, R, Q \in \mathbb{R}[x,y].$$

To each polynomial form $\eta$ one can associate the pseudo-abelian integrals $I(h)$ of $\frac{\eta}{M}$ along $\gamma(h) \subset H_0^{-1}(0)$ of real cycles in a region bounded by a polycycle.

Consider a real rational closed meromorphic one-form $\theta_0$ having a generalized Darboux first integral of the form

$$H_0 = e^{\frac{R}{Q}} \prod_{i=1}^{k} P_{a_i}, \quad \theta_0 = \frac{dH_0}{H_0}.$$

We assume that the following properties are satisfied by $\theta_0$ in same neighborhood of the polycycle $D \subset H_0^{-1}(0)$:

1. The curves $P_j^{-1}(0), Q^{-1}(0)$ are smooth and reduced.

2. $P_i^{-1}(0)$ and $P_j^{-1}(0)$, as well as $Q^{-1}(0)$ and $P_j^{-1}(0)$ intersect transversally.

Consider an unfolding $\theta_{\varepsilon,\alpha} = \frac{dH_{\varepsilon,\alpha}}{H_{\varepsilon,\alpha}}$ of the form $\theta_0$ with the Darboux first integral

$$H_{\varepsilon,\alpha} = Q^{\frac{\alpha-1}{\varepsilon}} (Q + \varepsilon R)^{\frac{1}{\varepsilon}} \prod_{i=1}^{k} P_i^{i}.$$

Consider pseudo-abelian integrals of the form

$$I_{\varepsilon,\alpha}(h) = \int_{\gamma_{\varepsilon,\alpha}(h)} \frac{\eta}{M}, \quad M = Q(Q + \varepsilon R) \prod_{i=1}^{k} P_i,$$

where $\gamma_{\varepsilon,\alpha}(h) \subset \{H_{\varepsilon,\alpha} = h\}, h \in (0, b(\varepsilon,\alpha))$ and $\eta$ is a polynomial one-form of degree at a most $n$.

**Theorem 9**[3]. *Under the genericity assumptions (1), (2) we have that the number of isolated zeros of pseudo-abelian integrals* $I_{\varepsilon,\alpha}(h) = \int_{\gamma_{\varepsilon,\alpha}(t)} \frac{\eta}{M}$ *in their maximal interval of definition* $(0, b(\varepsilon,\alpha))$ *is locally uniformly bounded.*

**Degenerate codimension one case**

Consider a polynomial one-form $\omega = M_\epsilon \frac{dH_\epsilon}{H_\epsilon}$ having a Darboux type first integral:

$$H_\epsilon = (x-\epsilon)^\alpha P \prod_{i=1}^{k} P_i^{\alpha_i}, \quad M_\epsilon = (x-\epsilon) P \prod_{i=1}^{k} P_i,$$

where $P, P_i \in \mathbb{R}[x,y], \alpha, \alpha_i \in \mathbb{R}_+, \epsilon$ is a sufficiently small parameter.

Let $D$ be the open period annulus whose closure intersects the zero level curve $H_\epsilon^{-1}(0)$. Let the polycycle $\gamma(0) \subset H_\epsilon^{-1}(0)$ be the corresponding part of boundary of $D$. The polycycle $\gamma(0)$ consists of edges $\gamma^i(0)$ contained in a smooth part of the level curve $P_i^{-1}(0)$.

We consider a small polynomial deformation of $\omega$

$$\omega + \kappa\eta, \quad \eta = Rdx + Sdy, \quad R, S \in \mathbb{R}[x,y].$$

Consider pseudo-Abelian integral of the form

$$I(\epsilon, h) = \int_{\gamma(h)} \frac{\eta}{M_\epsilon}, \quad \gamma(h) \subset H_\epsilon^{-1}(h).$$

This integral appears as the linear term with repsect to $\kappa$ of the displacement function $\Delta(\kappa, \epsilon, h)$

$$\Delta(\kappa, \epsilon, h) = \kappa h \int_{\gamma(h)} \frac{\eta}{M_\epsilon} + O(\kappa).$$

The limit cycles bifurcating in a compact domain $K \subset D$ are given by zeros of the pseudo-Abelian integral $I(\epsilon, h)$.

We impose the following genericity assumptions

1. The edges $\gamma^i(0), i = 1, \ldots, k$ intersect transversally two by two.
2. The polynomial $P$ has a critical point of Morse type $p = (0,0)$. Other polynomials $P_i, i = 1, \ldots, k$ satisfy $P_i(0,0) \neq 0$.

**Theorem 10**[1]. *Under the genericity assumptions there exists a bound of the number of isolated zeros of pseudo-Abelian integrals $I(\epsilon, h) = \int_{\gamma(h)} \frac{\eta}{M_\epsilon}$. The bound is locally uniform with respect to all parameters, in particular with respect $\epsilon$.*

Our principal result is similar to Bobieński's result [1]. The differences between our work and Bobieński's work [1] is:

1. In our work the first integral $H_\lambda$ is more general in the sense that the exponents $\epsilon_1$ and $\epsilon_2$ are different but in [1] we have $\epsilon_1 = \epsilon_2 = 1$.
2. Our approach is purely geometric which is based on the blow-up in family. This approach gives directly uniform validity of our study of the pseudo-Abelian integrals.
3. On the other hand, we assume in our work that the one polynomial form $\eta$ of the deformation $\omega_{\lambda,\kappa}$ vanishes to the order $\geq 4$ at $(0,0)$.

# Chapter 2

# Zeros of pseudo-abelian integrals: a codimension one case

## 2.1 Introduction and main result

In this chapter, we present a result which is part of a program of Bobieński, Mardešić and Novikov to extend the Varchenko-Khovanskii's theorem [8,13] from abelian integrals to pseudo-abelian integrals and prove the existence of a bound for the number of their zeros in function of the degree of the polynomial system only. In [2,11] they proved the local boundedness of the number of pseudo- abelian integrals under some generic conditions. Some non-generic cases have been studied in [1,3,4]. In this work we study one of non-generic case where an unfolding of a singularity of codimension one appears in the polycycle $\gamma_{(0,0)} \subset \{H_0(x,y)=0\}$.

More precisely, consider an unfolding $\omega_\lambda$ of the one-form $\omega_0$, where $\lambda$ is a small parameter and $\omega_\lambda$ is a family of analytic one-forms

$$\omega_\lambda = M_\lambda \frac{\mathrm{d}H_\lambda}{H_\lambda}, \tag{2.1}$$

with the Darboux first integral

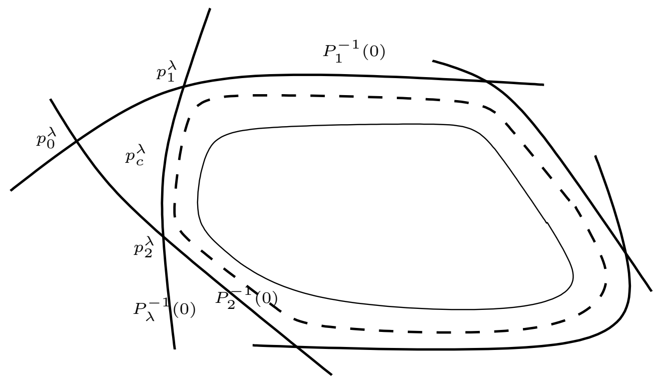
$$H_\lambda := P_\lambda^{\epsilon} \prod_{i=1}^{k} P_i^{\epsilon_i} = P_\lambda^{\epsilon} P_1^{\epsilon_1} P_2^{\epsilon_2} \prod_{i=3}^{k} P_i^{\epsilon_i} \tag{2.2}$$

with $\epsilon, \epsilon_i > 0, P_0, P_\lambda, P_j \in \mathbb{R}[x,y]$ and integrating factor $M_\lambda = P_\lambda \prod_{i=1}^{k} P_i$.

We assume that $P_0(0,0) = P_1(0,0) = P_2(0,0) = 0$ and $P_i(0,0) \neq 0$ for $i = 3, \ldots, k$.

Generically, the triple point unfolds into three saddle-type singular points $p_0^\lambda, p_1^\lambda, p_2^\lambda$ correspond to the transversal intersections of level curves $P_1^{-1}(0)$ and

$P_\lambda^{-1}(0)$, $P_1^{-1}(0)$ and $P_1^{-1}(0)$, and $P_2^{-1}(0)$ and $P_\lambda^{-1}(0)$. Here also appears a center $p_c^\lambda$ in the triangular region bounded by these levels curves




$D_{(\lambda,h)}, \gamma_{(\lambda,0)}$ and $\gamma_{(\lambda,h)}$

The foliation $\omega_\lambda = 0$ has a maximal nest of cycles $\gamma_{(\lambda,h)} \subseteq \{H_\lambda(x,y) = h\}, h \in (0, n(\lambda))$ filling a connected component of $\mathbb{R}^2 \setminus \{P_\lambda \prod_{i=1}^k P_i = 0\}$, which we denote $D_{(\lambda,h)}$, whose boundary is a polycycle $\gamma_{(\lambda,0)}$.

Consider a polynomial deformation $\omega_{\lambda,\kappa} = \omega_\lambda + \kappa\eta, \quad \kappa > 0$ of the system $\omega_\lambda = M_\lambda \frac{\mathrm{d}H_\lambda}{H_\lambda}$, where

$$\eta = R\mathrm{d}x + S\mathrm{d}y,$$

and $R, S \in \mathbb{R}[x,y]$ are a polynomials of degree $n$.

To such deformation one can associate the pseudo-Abelian integral

$$I(\lambda, h) = \int_{\gamma_{(\lambda,h)}} \Omega, \quad \Omega = \frac{\eta}{M_\lambda}, \tag{2.3}$$

which is the principal part of the Poincaré displacement function $\Delta$

$$\Delta(\kappa, \lambda, h) = \kappa h \int_{\gamma_{(\lambda,h)}} \frac{\eta}{M_\lambda} + O(\kappa)$$

of the deformation $\omega_{\lambda,\kappa}$ along $\gamma_{(\lambda,h)}$.

Let us impose the following assumptions:

1. $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$.

2. $P_1^{-1}(0), P_2^{-1}(0)$ and $P_0^{-1}(0)$ intersect transversally two by two at the origin which is the only triple point. The level curves $P_i^{-1}(0), i = 3, \cdots, k$ intersect transversally and two by two.

3. $\eta$ vanishes to the order $\geq 4$ at $(x,y) = (0,0)$.

Under above assumptions, we prove local uniform boundedness of the number of isolated zeros of pseudo-abelain integrals $I(\lambda, h)$ along cycles $\gamma_{(\lambda,h)}$.

**Theorem 1.** *Let $I(\lambda, h)$ be the family of pseudo-Abelian integrals as defined above. Under assumptions (1),(2), (3) there exists a bound for the number of isolated zeros of $I(\lambda, h)$. The bound depends only on $n_i = \deg P_i, n = \max(\deg R, \deg S)$ and is uniform in the coefficients of the polynomials $P_\lambda, P_i, R$ and $S$ and the exponents $\epsilon, \epsilon_i, i = 1, \dots, k$.*

**Remark 1.** *The differences between our work and Bobieński's work [1] are mentioned in Introduction.*

## 2.2 Local normal form and overview of the proof

In this section we obtain a local normal form near the triple point and discuss essential ingredients of the proof of Theorem 1.

### 2.2.1 Rectifying of the First Integral

Let us a establish a local normal form near the triple point for the unfolding of the degenerate polycyle $H_0$.

**Proposition 1.** *Under above assumptions (1),(2). There exists a local analytic coordinate system $(x, y, \lambda)$ at $(0, 0, 0)$ such that $H_\lambda$ takes the form*

$$H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} U, \quad \lambda > 0 \tag{2.4}$$

*where $U$ is an analytic unity function $U(0, 0, 0) \neq 0$.*

*Proof.* There exists an analytic coordinate system $(x, y)$ at $(0, 0)$ such that $P_1(x, y) = yU_1, P_2(x, y) = xU_2$ and $\prod_{i=3}^k P_i^{\epsilon_i} = V$, where $U_1, U_2, V$ are unities. In these coordinate and by Weierstrass preparation theorem we have $P_\lambda = (x - f(y, \lambda))U_0$, where $U_0$ is a unity, $\frac{\partial f}{\partial \lambda}(0, 0) \neq 0$ and $\frac{\partial f}{\partial y}(0, 0) \neq 0$. A second application of Weierstrass preparation theorem allows us to write $f(y, \lambda) = (y + g_0(\lambda))W$, where $W$ is a unity and $\frac{\partial g_0}{\partial \lambda}(0) \neq 0$. Now we put $\tilde{x} = \frac{x}{W}$. Then

$$\tilde{x}W + (y + g_0(\lambda))W = (\tilde{x} - y - g_0(\lambda))W$$

Finally, $P_\lambda = (\tilde{x} - y - \tilde{\lambda})WU_0$, where $\tilde{\lambda} = g_0(\lambda)$. The normal form (2.4) can be obtained by a linear rotation on $(\tilde{x}, y)$. □

### 2.2.2 Ingredients of the proof of Theorem 1

The ingredients of proof are

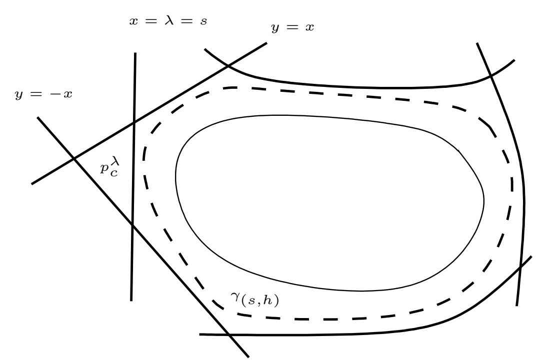
1. *Blow-up in families.* To simplify the singularity at the origin of the one-dimensional foliation in three dimension space, which is defined by the

intersection of level curves $\{H(x,y,\lambda)=h\}$ and $\{\lambda=s\}$ , we blow it up. We define the integral $J(s,t)=\int_{\delta_{(s,t)}}\sigma_1^*\Omega$ of the blown-up one-form $\sigma_1^*\Omega$ along a connected real smooth manifold of dimension one, where $t=\frac{s^a}{h}$. The proof of Theorem 1 is reduced to its analogue for the integral in the blown-up coordinates $J(s,t)=\int_{\delta_{(s,t)}}\sigma_1^*\Omega$ i.e. the proof of the boundedness of the number of zeros of the integral $J$. For more details see Section 3.

2. *Variation relations.* The operator $\mathcal{V}ar_{(h,\alpha)}$ (see Definition 1) modifies the number of zeros of $I(\lambda,h)$ by a locally bounded constant: Petrov's argument and preparation theorem for logarithmico-exponential functions [10] allows to estimate the number of real zeros of $I(\lambda,h)$ in terms of the number of zeros of $\mathcal{V}ar_{(h,\alpha)}I(\lambda,h)$.

Let $\gamma_{(\lambda,0)}$ be a polycycle in three-dimensional space equipped will two foliations $\{H(x,y,\lambda)=h,\lambda=s\}$. Assume that the center $p_c^\lambda$ cooresponds to a small basin bounded by $y=x, y=-x$ and $x=\lambda$ outside $\gamma_{(\lambda,0)}$.

We want to prove the uniform boundedness of the numbers of zeros of pseudo-abelian integral $I(\lambda,h)$ taken along the cycle $\gamma_{(\lambda,h)}$ (dashed cycle -see Figure below).




The cycle $\gamma_{(\lambda,h)}$.

The difficulty of the proof lies in the fact the center $p_c^\lambda$ generates possible ramification points, bifurcating from 0, of $I(\lambda,h)$ located on a circle whose radius is of order $|\lambda|^{\epsilon+\epsilon_++\epsilon_-}$. For the last reason it is difficult to get directly an $\lambda$-independent estimation. To overcome this difficulty in an $\lambda$-uniform way, we perform a blow-up in the family $\omega_\lambda$. Note that blow-up in a family was introduced in [5], see also [6].

## 2.3 Blowing-up of a codimension two singular foliation in dimension three

### 2.3.1 Desingularisation in family

The family of one-forms $\omega_\lambda$ in $\mathbb{C}_{(x,y)}$ given by (1.1) may be considered as a single form $\omega \in \Omega^1(\mathbb{C}_{(x,y,\lambda)})$ on the total space $\mathbb{C}^3_{(x,y,\lambda)}$ of the fibration $\pi : \mathbb{C}^3_{(x,y,\lambda)} \longrightarrow \mathbb{C}, \pi(x,y,\lambda) = \lambda$. Denote by $\mathcal{F}_\lambda$ the family of foliations of codimension one in $\mathbb{C}^2_{(x,y)}$ which are given by the equation $\omega_\lambda = 0, \omega_\lambda \in \Omega^1(\mathbb{C}^2_{(x,y)})$. The family of foliations $\mathcal{F}_\lambda$ glue to a single foliation $\mathcal{F}$ of codimension one in $\mathbb{C}^3_{(x,y,\lambda)}$ which is given by the equation $\omega = 0, \omega \in \Omega^1(\mathbb{C}^3_{(x,y,\lambda)})$. Let $\Pi$ be the foliation of codimension one in $\mathbb{C}^3_{(x,y,\lambda)}$ which is given by $\mathrm{d}\lambda = 0$. The intersection of the leaves $\{\lambda = s\}$ (complex planes) of the foliation $\Pi$ and the leaves of the foliation $\mathcal{F}$ define a singular foliation $\widetilde{\mathcal{F}}$ of codimension two in $\mathbb{C}^3_{(x,y,\lambda)}$. The foliation $\widetilde{\mathcal{F}}$ has a complicated singularity at the origin.

#### Blow-up

The idea to simplify the singularity at the origin is to blow it up in the total space $\mathbb{C}^3_{(x,y,\lambda)}$ of the fibration $\pi$. After this procedure the total space will not be a fibration but rather a singular foliation, whose leaves have codimension two. The blow-up of $\mathbb{C}^3_{(x,y,\lambda)}$ at the origin is defined as the incidence three dimensional manifold $W = \{(\xi, X) \in \mathbb{CP}^2 \times \mathbb{C}^3_{(x,y;\lambda)} : X \in \xi\}$. The blow down $\sigma : W \to \mathbb{C}^3_{(x,y,\lambda)}$ is just the restriction to $W$ of the projection $\mathbb{CP}^2 \times \mathbb{C}^3_{(x,y,\lambda)}$. The inverse map $\sigma^{-1} : \mathbb{C}^3_{(x,y;\lambda)} \to W$ is called blow-up and $\sigma^{-1}(0) = \mathbb{CP}^2 = \mathscr{D}$ is called exceptional divisor. The projective space $\mathbb{CP}^2$ is covered by three canonical charts: $W_1 = \{x \neq 0\}$ with coordinates $(Y_1, E_1)$, $W_2 = \{y \neq 0\}$ with coordinates $(X_2, E_2)$ and $W_3 = \{\lambda \neq 0\}$ with coordinates $(X_3, Y_3)$.

**Remark 2.** *The transition formulae follow from the requirement that the projective points* $(1 : Y_1 : E_1), (X_2 : 1 : E_2)$ *and* $(X_3 : Y_3 : 1)$ *coincide.*

$W_1, W_2$ and $W_3$ define canonical charts on $W$, with coordinates $(X_1, Y_1, E_1), (X_2, Y_2, E_2)$ and
$(X_3, Y_3, E_3)$ respectively. The blow-up $\sigma$ is written as:

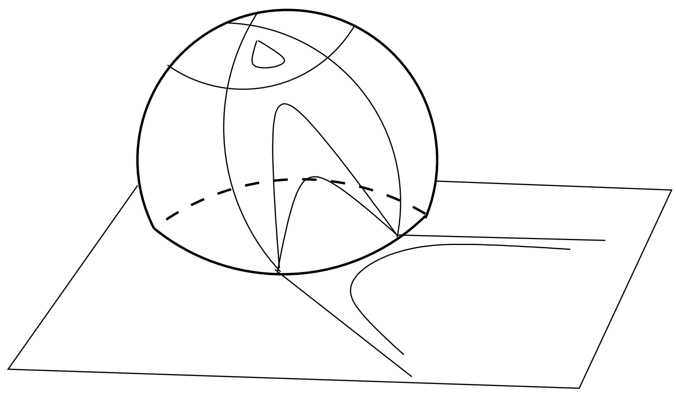
$$\begin{cases} \sigma_1 = \sigma|_{W_1} : x = X_1 \quad y = X_1 Y_1 \quad \lambda = E_1 X_1 \\ \sigma_2 = \sigma|_{W_2} : x = X_2 Y_2 \quad y = Y_2 \quad \lambda = E_2 Y_2 \\ \sigma_3 = \sigma|_{W_3} : x = X_3 E_3 \quad y = Y_3 E_3 \quad \lambda = E_3 \end{cases} \tag{2.5}$$

#### Blow-up of the foliation $\widetilde{\mathcal{F}}$

The blow-up of the codimension one singular foliation $\mathcal{F}$ produces a singular foliation $\sigma^*\mathcal{F}$ of codimension one in the ambient space $W$ which is given by

$\{\sigma^* H = h\}$. The blow-up of the codimension one singular foliation $\Pi$ produces a singular foliation $\sigma^*\Pi$ of codimension one in the ambient space $W$ which is given by $\{\sigma^*\pi = s\}$. In particular, the singular leaf of $\sigma^*\Pi$ is given by $\{\sigma^*\pi = 0\} = \mathscr{C} \cup \mathscr{D}$. The leaves of the blown up foliation $\sigma^*\mathcal{F}$ are transverse to generic leaves of the codimension one blown-up foliation $\sigma^*\Pi$. The exceptional divisor $\mathscr{D}$ intersects transversally $\mathscr{C}$ along the equatorial loop $\mathscr{L} \cong \mathbb{CP}^1$.

Let $\sigma^{-1}\widetilde{\mathcal{F}}$ be the lift of the foliation $\widetilde{\mathcal{F}}$ to the complement of the exceptional divisor $\mathscr{D}$. The foliation $\sigma^{-1}\widetilde{\mathcal{F}}$ is regular outside of the preimage of the hypersurface $\{H_\lambda = 0, \lambda = 0\}$. This foliation extends in a unique way to a holomorphic singular foliation $\sigma^*\widetilde{\mathcal{F}}$ on $W$ which we call the *blow-up* of the original codimension two foliation $\widetilde{\mathcal{F}}$ by the map $\sigma$.



The restriction of foliation $\sigma^*\widetilde{\mathcal{F}}$ near $\mathscr{D}$ to the real space.

We prove this result by explicitly computing $\sigma^*\widetilde{\mathcal{F}}$ in different charts. Starting from a foliation $\widetilde{\mathcal{F}}$ of codimension two where its leaves $L_{(s,h)}$ are defined by the system:

$$\begin{cases} H(x,y,\lambda) = h \\ \pi(x,y,\lambda) = \lambda = s. \end{cases} \tag{2.6}$$

The resulting foliation $\sigma^*\widetilde{\mathcal{F}}$ is obtained from (2.6) simply by pulling back the functions $H$ and $\pi$.

In the local chart $(x, y, \lambda)$ of Proposition 1, we have

$$\begin{cases} H(x,y,\lambda) = (x-\lambda)^{\epsilon}(y-x)^{\epsilon_+}(y+x)^{\epsilon_+}U \\ \pi(x,y,\lambda) = \lambda \quad . \end{cases}$$

In the chart $W_1$ we obtain

$$\begin{cases} (\sigma_1^* H)(X_1, Y_1, E_1) = H(X_1, X_1Y_1, X_1E_1) = X_1^a(1-E_1)^{\epsilon}(Y_1-1)^{\epsilon_+}(Y_1+1)^{\epsilon_-}U \\ (\sigma_1^*\pi)(X_1, Y_1, E_1) = \pi(X_1, X_1Y_1, X_1E_1) = X_1E_1, \end{cases}$$

where $a = \epsilon + \epsilon_+ + \epsilon_-$ and $U(0,0,0) \neq 0$.

Let $\sigma_1^*\widetilde{\mathcal{F}}$ be the foliation defined by the intersection of the levels of $\{\sigma_1^* H = h\}$ and $\{\sigma_1^*\pi = s\}$.

**Proposition 2.**

1. *The singularities of $\sigma_1^*\widetilde{\mathcal{F}}$ are located at the points $p_+ = (0,1,0), p_- = (0,-1,0), q_+ = (0,1,1)$ and $q_- = (0,-1,1)$.*

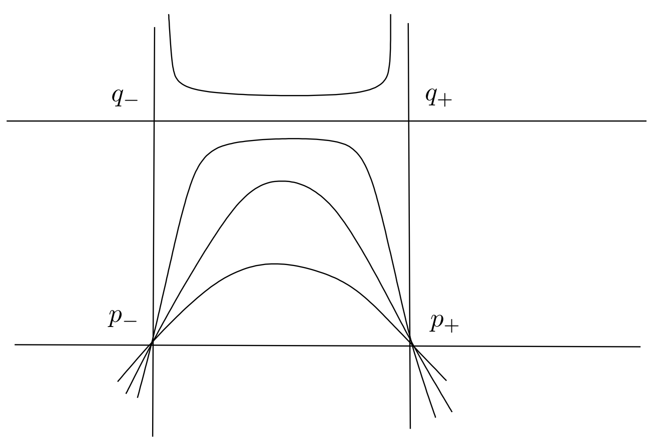
2. *All these singular points are linearisable saddles, with eigenvalues $\mu_+ = (\epsilon_+, -a, -\epsilon_-), \mu_- = (-\epsilon_-, a, \epsilon_-), \nu_+ = (0, -\epsilon, \epsilon_+)$ and $\nu_- = (0, -\epsilon, \epsilon_-)$ respectively.*

*Proof.* 1. Since $\sigma : W \to \mathbb{C}^3_{(x,y,\lambda)}$ is a biholomorphism outside $\mathscr{D}$, all singularities of $\sigma_1^*\widetilde{\mathcal{F}}$ on $W_1 \setminus \{X_1 = 0\}$ correspond to singularities of $\widetilde{\mathcal{F}}$.

Thus, it suffices to compute the singularities of $\sigma_1^*\widetilde{\mathcal{F}}$ on the exceptional divisor $\{X_1 = 0\}$. On the exceptional divisor, the foliation is given by the levels of

$$G := \frac{(\sigma_1^*\pi)^a}{\sigma_1^* H} = \frac{E_1^a}{(1-E_1)^\epsilon (Y_1 - 1)^{\epsilon_+}(Y_1+1)^{\epsilon_-} V},$$

where $V$ is a unity of the form $c + X_1 f$ and $f$ is a holomorphic function. The levels of $G$ are as pictured in figure below




Real picture of the levels of $G$

2. Let us compute the eigenvalues at $p_+, p_-, q_+$ and $q_-$. Near the exceptional divisor $\{X_1 = 0\}$, the foliation $\sigma_1^*\widetilde{\mathcal{F}}$ is given by

$$\begin{cases} (\sigma_1^* H)(X_1, Y_1, E_1) = X_1^a (1-E_1)^\epsilon (Y_1 - 1)^{\epsilon_+}(Y_1+1)^{\epsilon_-} V = h, \\ (\sigma_1^*\pi)(X_1, Y_1, E_1) = X_1 E_1 = s. \end{cases}$$

Near $p_\pm$ and after the respective changes of variable $Y_\pm = (Y_1 \mp 1)(Y_1 \pm 1)^{\frac{\epsilon_\mp}{\epsilon_\pm}}(1-E_1)^{\frac{\epsilon}{\epsilon_\pm}}V^{\frac{1}{\epsilon_\pm}}$, the blown-up foliation $\sigma_1^*\widetilde{\mathcal{F}}$ is given by the two first integrals $X_1^a Y_\pm^{\epsilon_\pm} = h$ and $X_1E_1 = s$. Then, near this point the vector field generating the foliation $\sigma_1^*\widetilde{\mathcal{F}}$ is given by

$$V_\pm(X_1, Y_\pm, E_1) = v_1^\pm X_1 \frac{\partial}{\partial X_1} + v_2^\pm Y_\pm \frac{\partial}{\partial Y_\pm} + v_3^\pm E_1 \frac{\partial}{\partial E_1},$$

where the vector $\mu_\pm = (v_1^\pm, v_2^\pm, v_3^\pm)$ satisfies the following equations

$$\langle (v_1^\pm, v_2^\pm, v_3^\pm), (a, \epsilon_\pm, 0)\rangle = 0, \quad \langle (v_1^\pm, v_2^\pm, v_3^\pm), (1,0,1)\rangle = 0.$$

Here $\langle , \rangle$ is the usual scalar product on $\mathbb{C}^3$. By simple calculations, we obtain

$$V_\pm(X_1, Y_\pm, E_1) = \pm\epsilon_\pm X_1 \frac{\partial}{\partial X_1} \mp aY_\pm \frac{\partial}{\partial Y_\pm} \mp \epsilon_\pm E_1 \frac{\partial}{\partial E_1}.$$

Similar computation shows that there are local coordinates near $q_\pm$ in which the vector field generating the foliation is given by

$$V_\pm(X_1, Y_\pm, E_\pm) = -\epsilon Y_\pm \frac{\partial}{Y_\pm} + \epsilon_\pm E_\pm \frac{\partial}{\partial E_\pm}.$$

□

## 2.4 Normal form coordinates near the polycyles

For reader's convenience, the index "*div*" in all notations refers to the word divisor.

Let $t := \frac{s^a}{h}, \tilde{t} = t^{-1}$. The blown-up foliation $\sigma_1^*\widetilde{\mathcal{F}}$ is given by the two first integrals

$$G = \frac{(\sigma_1^*\pi)^a}{\sigma_1^* H} = \frac{E_1^a}{(1-E_1)^\epsilon (Y_1-1)^{\epsilon_+}(Y_1+1)^{\epsilon_-}V} = t, \quad L = X_1E_1 = s.$$

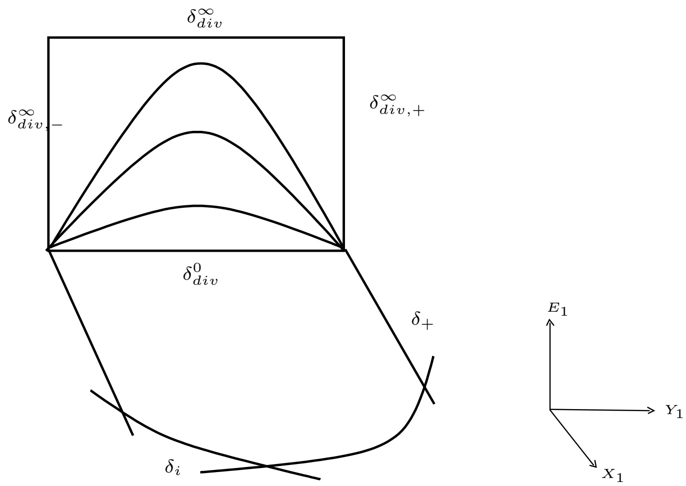
### 2.4.1 Polycycles

Consider the two-dimensional square $Q \subset \mathscr{D}$ with vertices $p_+, p_-, q_+$ and $q_-$. All levels curves $\{G = t\}$ inside $Q$ correspond to values of $t \in [0, +\infty]$ (see Figure below). We consider the family of polycycles (see Figure below),

$$\delta^t = \left(\sigma_1^{-1}\left(\gamma_{(0,0)} \setminus (0,0,0)\right) \cup (Q \cap \{G = t\})\right)^{\mathbb{R}}, t \in [0, +\infty],$$

where $(\dots)^{\mathbb{R}}$ denotes the real part of a complex analytic set.

Let $0 < m < M$. Consider the complex curves $C_0 = \{X_1 = 0, G = 0\}, C_{t_0} = \{X_1 = 0, G = t_0\}, t_0 \in [\frac{m}{2}, 2M], C^\infty_{div,\pm} = \{X_1 = 0, Y_1 = \pm 1\}, C^\infty_{div} = \{X_1 = 0, E_1 = 1\}, C_\pm = \{E_1 = 0, Y_1 = \pm 1\}$ and $C_i = \{E_1 = 0, \sigma_1^* P_i = 0\}, i = 3, \dots, k$.




The polycycles $\delta^t, t \in [0, +\infty]$.

Let $\delta_i, i = 3, \ldots, k$, be the edges contained in a smooth part of the complex curve $C_i, i = 3, \ldots, k$, $\delta_{\pm}$ be the edges contained in the complex curve $C_{\pm}$, $\delta^{\star}_{div}, \star \in \{0, t_0\}$ be the edge contained in a smooth part of the complex curve $C_{\star}$, $\delta^{\infty}_{div,\pm}$ be the edges contained in the complex curve $C^{\infty}_{div,\pm}$, $\delta^{\infty}_{div}$ be the edge contained in the complex curve $C^{\infty}_{div}$.

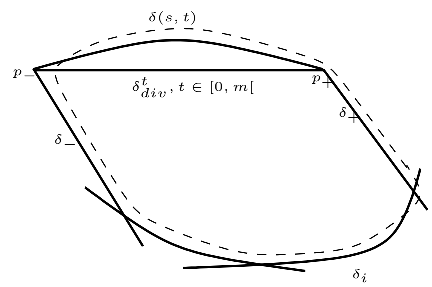
Let $p_{ij}, i, j = 3, \ldots, k$ be the vertex corresponding to the transversal of the edges $\delta_i$ and $\delta_j$, $p_{i\pm}, i = 3, k$ be the vertices corresponding to the transversal intersection of $\delta_{\pm}$ and $\delta_i$, $q_{\pm}$ be the vertex corresponding to the transversal intersection of $\delta^{\infty}_{div,\pm}$ and $\delta^{\infty}_{div}$.

Let $\delta(s,t) = \sigma^{-1}(\gamma_{(\lambda,h)}) \subset W$ be the pull-back of the cycle $\gamma_{(s,h)}$ by the blowing-up. Let $m, M$ be such that $0 \leq m < M$. Let $\delta^t$ be the polycycle corresponding to the cycle of integration $\delta(s,t)$, which is indicated by the dashed lines -see Figures below. We decompose these polycycles $\delta^t, t \in [0, +\infty]$ as follows:

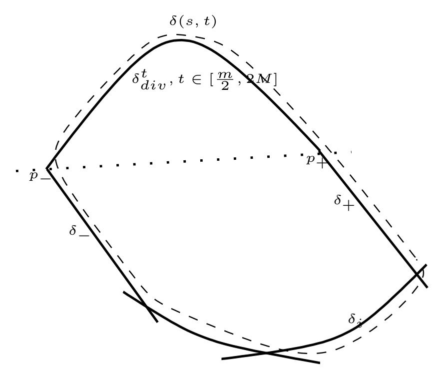
1. For $t \in [0, m[$, we decompose the polycycle $\delta^t$ as follows $\delta^t = \delta^t_{div} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$ close to the polycyle $\delta^0 = \delta^0_{div} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$.

2. For $t \in [\frac{m}{2}, 2M]$, we decompose the polycycle $\delta^t$ as follows $\delta^t = \delta^t_{div} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$ close to the polycycles $\delta^{t_0} = \delta^{t_0}_{div} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$.

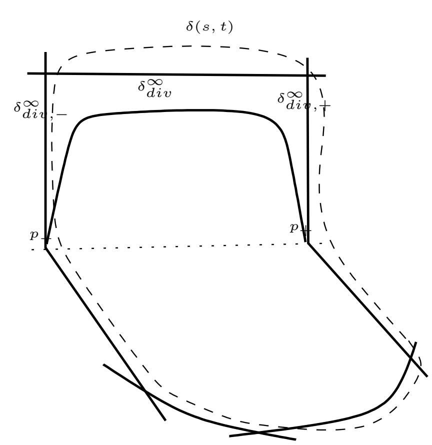
3. For $t \in [M, +\infty]$, a polycycles $\delta^t_{div} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$ close to the polycycles $\delta^{\infty} = \delta^{\infty}_{div} \sqcup \delta^{\infty}_{div,+} \sqcup \delta^{\infty}_{div,-} \sqcup \delta_+ \sqcup \delta_- \sqcup \delta_3 \sqcup \ldots \sqcup \delta_k$




The cycle $\delta(s,t)$ corresponds to the polycycles $\delta^t, t \in [0, m[$.




The cycle $\delta(s,t)$ corresponds to the polycycles $\delta^t, t \in [\frac{m}{2}, 2M]$.




The cycle $\delta(s,t)$ corresponds to the polycycles $\delta^t, t \in [M, +\infty]$.

### 2.4.2 Normal form coordinates near the polycyles

Now we obtain normal forms in the neighborhood of each separatrix of polycycles $\delta^t, t \in [0, +\infty]$.

**Proposition 3.**

1. *For $t \in [0, +\infty[$, there exists a local chart $(U^t_{div}, (X, Y, Z))$ defined in a neighborhood $U^t_{div} \subset W$ of the speratrix $\delta^t_{div}$ such that the foliation $\sigma_1^* \widetilde{\mathfrak{F}}$ is given by two first integrals*

$$K = (Y-1)^{-\epsilon_+}(Y+1)^{-\epsilon_-} Z^a = t, \quad L = XZ = s.$$

2. *There exsits a local chart $(U^\infty_{div}, (X, Y, Z))$ defined in a neighborhood $U^\infty_{div} \subset W$ of $\delta^\infty_{div}$ such that the foliation $\sigma_1^* \widetilde{\mathfrak{F}}$ is given by two first integrals*

$$\widetilde{K} = Z^\epsilon (Y-1)^{\epsilon_+}(Y+1)^{\epsilon_+} = \tilde{t}, \quad L = X = s$$

3. *There exists a local chart $\left(U^\infty_{div,\pm}, (X, Y, Z)\right)$ defined in a neighborhood $U^\infty_{div,\pm} \subset W$ of $\delta^\infty_{div,\pm}$ such that the foliation $\sigma_1^* \widetilde{\mathfrak{F}}$ is given by*

$$\widetilde{K} = Z^{-a}(1-Z)^\epsilon Y^{\epsilon_\pm} = \tilde{t}, \quad L = XZ = s.$$

4. *There exists a local chart $(U_\pm, (X, Y, Z))$ defined in a neighborhood $U_\pm \subset W$ of $\delta_\pm$ such that the foliation $\sigma_1^* \widetilde{\mathfrak{F}}$ is given by two first integrals $\widetilde{K}_\kappa, \kappa \in \{3, k\}$ and $L$*

$$\widetilde{K}_\kappa = (1-X)^{\epsilon_\kappa} Y^{\epsilon_\pm} Z^{-a} = \tilde{t}, \quad L = XZ = s.$$

5. *There exists a local chart $(U_i, (X, Y, Z)), i = 3, \ldots, k$ defined in a neighborhood $U_i \subset W$ of the separatrix $\delta_i$ such that the foliation $\sigma^* \widetilde{\mathfrak{F}}$ is given by two first integrals*

$$\widetilde{K} = X^{\epsilon_{i-1}}(1-X)^{\epsilon_{i+1}} Y^{\epsilon_i} = \tilde{t}, \quad L = Z = s.$$

*Proof.* 1. It suffices to define

$$X = X_1\left((1-E_1)^{-\epsilon} V^{-1}\right)^{-\frac{1}{a}}, \quad Y = Y_1, \quad Z = E_1\left((1-E_1)^{-\epsilon} V^{-1}\right)^{\frac{1}{a}}.$$

Hence $(X_1, Y_1, E_1) \to (X, Y, Z)$ is a diffeomorphism defined on a neighborhood $U^t_{div} \subset W$ of $\delta^t_{div}$.

2. It suffices to defines

$$X = X_1 E_1, \quad Y = Y_1, \quad Z = (1-E_1) E_1^{-\frac{a}{\epsilon}} V^{\frac{1}{\epsilon}}.$$

Hence $(X_1, Y_1, E_1) \to (X, Y, Z)$ is a diffeomorphism defined on a neighborhood $U^\infty_{div} \subset W$ of $\delta^\infty_{div}$.

3. It suffices to define

$$X = X_1, \quad Y = (Y_1 \mp 1)(Y_1 \pm 1)^{\frac{\epsilon_\mp}{\epsilon_\pm}} V^{\frac{1}{\epsilon_\pm}}, \quad Z = E_1.$$

   Hence $(X_1, Y_1, E_1) \to (X, Y, Z)$ is a diffeomorphism defined on a neighborhood $U^{\infty}_{div,\pm} \subset W$ of $\delta^{\infty}_{div,\pm}$.

4. It suffices to defines

$$X = X_1, \quad Y = (Y_1 \mp 1)(Y_1 \pm 1)^{\frac{\epsilon_\mp}{\epsilon_\pm}} (1 - E_1)^{\frac{\epsilon}{\epsilon_\pm}} \widetilde{V}^{\frac{1}{\epsilon}}, \quad Z = E_1$$

   Hence $(X_1, Y_1, E_1) \to (X, Y, Z)$ is diffeomorphism defined on a neighborhood $U_\pm \subset W$ of $\delta_\pm$.

□

### 2.4.3 Transversal sections

All transversal sections are of complex dimension two in the three-dimensional space.

Near $p_\pm$, we consider the transversal sections $\Sigma_{div}, \Sigma_{div,\pm}$ and $\Sigma_\pm$ to separatrices $\delta^t_{div}, \delta^{\infty}_{div,\pm}$ and $\delta_\pm$, respectively such that

1. In the local chart $(U^t_{div}, (X, Y, Z))$ and $\left(U^{\infty}_{div,\pm}, (X, Y, Z)\right)$ of Proposition 3 the transversal section $\Sigma_\pm$ is given by $\Sigma_\pm := \{X = 1\}$.

2. In the local chart $(U_\pm, (X, Y, Z))$ of Proposition 3 the transversal section $\Sigma_{div,\pm}$ is given by $\Sigma_{div,\pm} = \{Z = 1\}$ and the transversal section $\Sigma_{div}$ is given by $\Sigma_{div} := \{Y = 1\}$.

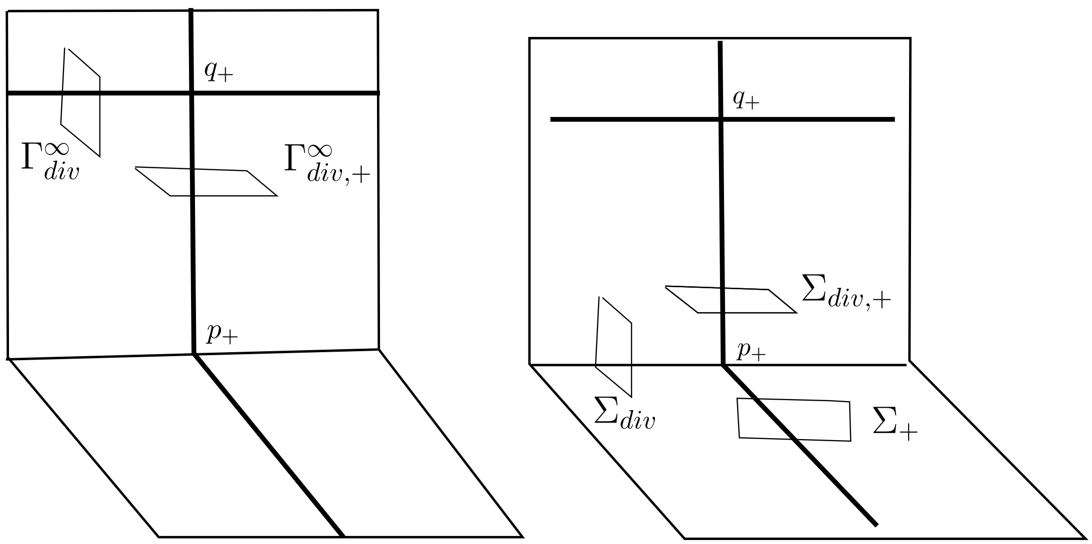
Near $q_\pm$, we consider the transversal sections $\Gamma^{\infty}_{div}$ and $\Gamma_{div,\pm}$ to separatrices $\delta^{\infty}_{div}$ and $\delta^{\infty}_{div,\pm}$, respectively such that

1. In the local chart $(U^{\infty}_{div}, (X, Y, Z))$ of Proposition 3 the transversal section $\Gamma^{\infty}_{div,\pm}$ is given by $\Gamma^{\infty}_{div,\pm} := \{Z = 1\}$.

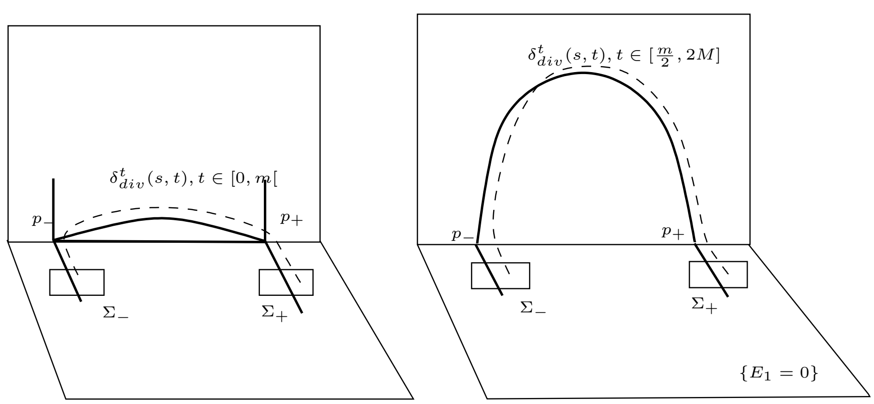
2. In the local chart $\left(U^{\infty}_{div,\pm}, (X, Y, Z)\right)$ of Proposition 3 the transversal section $\Gamma^{\infty}_{div}$ is given by $\Gamma^{\infty}_{div} := \{Y = 1\}$.

### 2.4.4 Relative cycles

Now, we consider relatives cycles obtained by intersecting the cycles $\delta(s, t)$ with the transversal sections defined in subsection 4.3.




The transversal sections

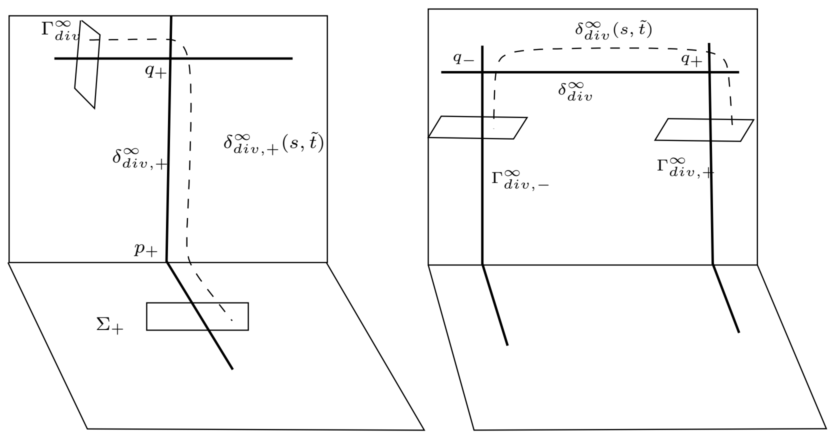



The relative cycles $\delta^t_{div}(s,t)$.

1. $\delta^t_{div}(s,t) := \delta(s,t) \cap U^t_{div}$ going from $\Sigma_-$ to $\Sigma_+$.

2. $\delta^\infty_{div}(s,\tilde{t}) := \delta(s,t) \cap U^\infty_{div}$ going from $\Gamma^\infty_{div,-}$ to $\Gamma^\infty_{div,+}$.

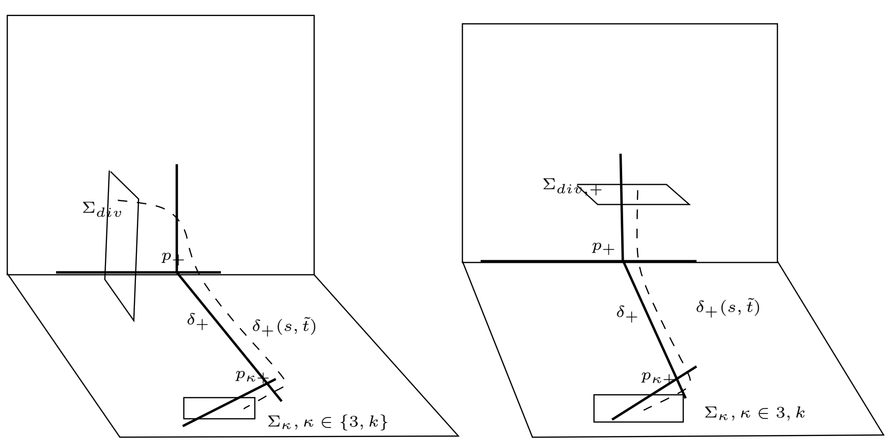
3. $\delta^\infty_{div,\pm}(s,\tilde{t}) := \delta(s,t) \cap U^\infty_{div,\pm}$ going from $\Sigma_\pm$ to $\Gamma^\infty_{div}$.




The relative cycles $\delta^\infty_{div}(s,\tilde{t}), \delta^\infty_{div,+}(s,\tilde{t})$.

4. (a) $\delta_\pm(s,\tilde{t}) := \delta(s,t) \cap U_\pm$ going from $\Sigma_\kappa, \kappa \in \{3,k\}$ to $\Sigma_{div,\pm}$.

   (b) $\delta_\pm(s,\tilde{t}) := \delta(s,t) \cap U_\pm$ going from $\Sigma_\kappa, \kappa \in \{3,k\}$ to $\Sigma_{div}$.




The relative cycle $\delta_+(s,\tilde{t})$.

5. $\delta_i(s,\tilde{t}) := \delta(s,t) \cap U_i, i = 3, \ldots, k$ going from $\Sigma_{i-1}$ to $\Sigma_{i+1}$.

The integral of the blown-up one form $\sigma_1^*\Omega$ along the cycle $\delta(s,t)$ will be denoted by

$$J(s,t) = \int_{\delta(s,t)} \sigma_1^*\Omega.$$

Let $V_{div}^0, V_{div}^{t_0}, t_0 \in [\frac{m}{2}, 2M[, V_{div,\pm}^\infty, V_{div}^\infty, V_\pm$ and $V_i, i = 3, \ldots, k$ are open neighborhoods of the separatrices $\delta_{div}^0, \delta_{div}^{t_0}, \delta_{div,\pm}^\infty, \delta_{div}^\infty, \delta_\pm$ and $\delta_i, i = 3, \ldots, k$ in the complex curves $C_0, C_{t_0} =, t_0 \in [\frac{m}{2}, 2M], C_{div,\pm}^\infty, C_{div}^\infty, C_\pm =$ and $C_i, i = 3, \ldots, k$.

Let $\varrho^\star, \star \in \{0, t_0\}, \varrho^\infty$ be a partition of unity subordinate to the following covers respectively

$$\{V_{div}^\star, V_+, V_-, V_3, \ldots, V_k\}, \quad \{V_{div}^\infty, V_{div,+}^\infty, V_{div,-}^\infty, V_+, V_-, V_3, \ldots, V_k\},$$

where $\varrho_{div}^\star, \varrho_{div,\pm}^\infty, \varrho_{div}^\infty, \varrho_\pm$ and $\varrho_i, i = 3, \ldots, k$ are respective partition of unity $\varrho^t$, that is,

$$\varrho_{div}^\star + \varrho_+ + \varrho_- + \sum_{i=3}^k \varrho_i = 1, \varrho_{div}^\infty + \varrho_{div,+}^\infty + \varrho_{div,-}^\infty + \sum_{i=3}^k \varrho_i = 1$$

which satisfy the additional property that $\varrho_{div}^\star = 1$ on some neighborhood $\widetilde{V}_{div}^\star \subset V_{div}^\star$ of the separatrix $\delta_{div}^\star$, $\varrho_{div}^\infty = 1$ on some neighborhood $\widetilde{V}_{div}^\infty \subset V_{div}^\infty$ of separatrix $\delta_{div}^\infty$, $\varrho_{div,\pm}^\infty = 1$ on some neighborhood $\widetilde{V}_{div,\pm}^\infty \subset V_{div,\pm}^\infty$ of separatrix $\delta_{div,\pm}^\infty$, $\varrho_\pm = 1$ on some neighborhood $\delta_\pm \subset \widetilde{V}_\pm \subset V_\pm$ and $\varrho_i = 1, i = 3, \ldots, k$ in a some neighborhood $\widetilde{V}_i \subset V_i$ of the separatrix $\delta_i$.

Let

$$\omega_{div}^t = (\sigma_1^*\Omega)\, \varrho_{div}^\star, \quad \omega_{div}^\infty = (\sigma_1^*\Omega)\, \varrho_{div}^\infty, \quad \omega_{div,+}^\infty = (\sigma_1^*\Omega)\, \varrho_{div,+}^\infty, \quad \omega_{div,-}^\infty = (\sigma_1^*\Omega)\, \varrho_{div,-}^\infty,$$
$$\omega_+ = (\sigma_1^*\Omega)\, \varrho_+ \omega_- = (\sigma_1^*\Omega)\, \varrho_-, \quad \omega_3 = (\sigma_1^*\Omega)\, \varrho_3, \ldots, \omega_k = (\sigma_1^*\Omega)\, \varrho_k.$$

**Proposition 4.** *The integral $J(s,t) = \int_{\delta(s,t)} \sigma_1^*\Omega$ has the following representations*

1. *If $\delta(s,t) \subset (V_{div}^\star \cup V_+ \cup V_- \cup V_3 \cup \ldots \cup V_k), \star \in \{0, t_0\}$, then we can write*

$$J(s,t) = J_{div}^\star(s,t) + J_-(s,t) + J_+(s,t) + \sum_{i=3}^k J_i(s,t),$$

*where*

$$J_{div}^t(s,t) = \int_{\delta_{div}^\star(s,t)} \omega_{div}^\star, J_\pm(s,t) = \int_{\delta_\pm(s,\tilde{t})} \omega_-, J_i(s,t) = \int_{\delta_i(s,\tilde{t})} \omega_i$$

2. *If* $\delta(s,t) \subset \left(V_{div}^{\infty} \cup V_{div,+}^{\infty} \cup V_{div,-}^{\infty} V_+ \cup V_- \cup V_3 \cup \ldots \cup V_k\right)$ *then we can write*

$$J(s,t) = J_{div}^{\infty}(s,\tilde{t}) + J_{div,+}^{\infty}(s,\tilde{t}) + J_{div,-}^{\infty}(s,\tilde{t}) + \sum_{i=3}^{k} J_i(s,t),$$

*where*

$$J_{div}^{\infty}(s,\tilde{t}) = \int_{\delta_{div}^{\infty}(s,\tilde{t})} \omega_{div}^{\infty},\, J_{div,\pm}^{\infty}(s,\tilde{t}) = \int_{\delta_{div,\pm}^{\infty}(s,\tilde{t})} \omega_{div,\pm}^{\infty}$$

## 2.5 Analytic continuation

In this section we show that the function $J(s,t)$ can be analytically continued to the universal cover of $\mathbb{C}_s^* \times \mathbb{C}_t^*$.

### 2.5.1 Transport of relative cycles

As in [2], we show that each relative cycle can be chosen as a lift of a base path to some Riemann surface of some multivalued function. Let $(U_{div}^t, (X,Y,Z))$,

$$(U_{div}^{\infty}, (X,Y,Z)), \left(U_{div,\pm}^{\infty}, (X,Y,Z)\right), (U_{\pm}, (X,Y,Z)), (U_i, (X,Y,Z)), i = 3, \ldots, k$$

be the charts of Proposition 3. Let us be more precise

1. In the local chart $(U_{div}^t, (X,Y,Z))$, the relative cycle $\delta_{div}^t(s,t)$ can be chosen as a lift to the Riemann surface $R_{div}^t$ of the multivalued function $\mathcal{G}_{\Phi_{div}^t}(Y)$ of some path in $C_\star$, where

   $$\mathcal{G}_{\Phi_{div}^t} := \left\{\left(Y, \Phi_{div}^t(Y)\right) : \Phi_{div}^t(Y) = (X,Z) = (\Phi_{div,1}^t(Y,s,t), \Phi_{div,2}^t(Y,s,t))\right\}$$

   and

   $$\Phi_{div,1}^t(Y,s,t) = \frac{s}{\Phi_{div,2}^t(Y,s,t)}, \quad \Phi_{div,2}^t(Y,s,t) = t^{\frac{1}{a}}(Y+1)^{\frac{\epsilon_-}{a}}(Y-1)^{\frac{\epsilon_+}{a}}.$$

2. In the local chart $(U_{div}^{\infty}, (X,Y,Z))$, the relative cycle $\delta_{div}^{\infty}(s,\tilde{t})$ can be chosen as a lift to the Riemann surface $R_{div}^{\infty}$ of the multivalued function $\mathcal{G}_{div}^{\infty}(Y)$ of some path in $C_{div}^{\infty}$, where

   $$\mathcal{G}_{div}^{\infty} := \left\{(Y, \Phi_{div}^{\infty}(Y)) : \Phi_{div}^{\infty}(Y) = (X,Z) = \left(\Phi_{div,1}^{\infty}(Y,s,\tilde{t}), \Phi_{div,2}^{\infty}(Y,s,\tilde{t})\right)\right\}$$

   and

   $$\Phi_{div,1}^{\infty}(Y,s,\tilde{t}) = s, \quad \Phi_{div,2}^{\infty}(Y,s,\tilde{t}) = \frac{\tilde{t}^{\frac{1}{\epsilon}}}{(Y-1)^{\frac{\epsilon_+}{\epsilon}}(Y+1)^{\frac{\epsilon_-}{\epsilon}}}.$$

3. In the local chart $\left(U^{\infty}_{div,\pm},(X,Y,Z)\right)$, the relative cycle $\delta^{\infty}_{div,\pm}(s,\tilde t)$ can be chosen as a lift to the Riemann surface $R^{\infty}_{div,\pm}$ of the multivalued function $\mathcal{G}^{\infty}_{div,\pm}(Z)$ of some path in $C^{\infty}_{div,\pm}$, where

$$\mathcal{G}^{\infty}_{div,\pm} := \left\{\left(Z,\Phi^{\infty}_{div,\pm}(Z)\right) : \Phi^{\infty}_{div,\pm}(Z) = \left(\Phi^{\infty}_{div,\pm,1}(Z,s,\tilde t),\Phi^{\infty}_{div,\pm,2}(Z,s,\tilde t)\right)\right\}$$

and

$$\Phi^{\infty}_{div,\pm,1}(Z,s,\tilde t) = \frac{s}{Z},\quad \Phi^{\infty}_{div,\pm,2}(Z,s,\tilde t) = \frac{\tilde t^{\frac{1}{\epsilon_\pm}} Z^{\frac{a}{\epsilon_\pm}}}{(1-Z)^{\frac{\epsilon}{\epsilon_\pm}}}.$$

4. In the local chart $(U_\pm,(X,Y,Z))$, the relative cycle $\delta_\pm(s,\tilde t)$ can be chosen as a lift to the local Riemann surface $R_\pm$ of the multivalued function $\mathcal{G}_\pm(X)$ of some path in $C_\pm$, where

$$\mathcal{G}_\pm := \left\{(X,\Phi_\pm(X)) : \Phi_\pm(X) = (Y,Z) = \left(\Phi_{\pm,1}(X,s,\tilde t),\Phi_{\pm,2}(X,s,\tilde t)\right)\right\}$$

and

$$\Phi_{\pm,1}(X,s,\tilde t) = \frac{\tilde t^{\frac{1}{\epsilon_\pm}} s^{\frac{a}{\epsilon_\pm}}}{X^{\frac{a}{\epsilon_\pm}}(1-X)^{\frac{\epsilon_K}{\epsilon_\pm}}},\quad \Phi_{\pm,2}(X,s,\tilde t) = \frac{s}{X}.$$

5. In the local chart $(U_i,(X,Y,Z))\,, i=3,\ldots,k$, the relative cycle $\delta_i(s,\tilde t)$ can be chosen as a lift to the Riemann surface $R_i$ of the multivalued function $\mathcal{G}_i(X)$ of some path in $C_i$, where

$$\mathcal{G}_i := \left\{(X,\Phi_i(X)) : \Phi_i(X) = (Y,Z) = \left(\Phi_{i,1}(X,s,\tilde t),\Phi_{i,2}(X,s,\tilde t)\right)\right\}$$

and

$$\Phi_{i,1}(X,s,\tilde t) = \frac{\tilde t^{\frac{1}{\epsilon_i}}}{X^{\frac{\epsilon_{i-1}}{\epsilon_i}}(1-X)^{\frac{\epsilon_{i+1}}{\epsilon_i}}},\quad \Phi_{i,2}(X,s,\tilde t) = 0.$$

## 2.6 Variation relations

In this section we calculate the variation of the integrals $J^t_{div}, J^{\infty}_{div}, J^{\infty}_{div,\pm}, J_\pm$ and $J_i, i=3,\ldots,k$, of Proposition 4. The calculation of variation is similar to [2]. The difference is that here we are in three-dimensional space on which a codimension two foliation is defined and we need to consider the variation with respect to both transversal variables.

**Definition 1.** *Given a multivalued function $\psi$ at $0\in\mathbb{C}$ i.e. a holomorphic function defined on $\widetilde{\mathbb{C}\setminus\{0\}}$, where $\widetilde{M}$ denotes the universal covering of $M$. We define the rescaled monodromy as*

$$\mathcal{M}on_{(t,\pm\alpha)}\psi(s,t) = \mathcal{M}on_{t^{\pm\frac{1}{\alpha}}}\psi(s,t) = \psi(s,te^{\pm i\pi\alpha}) \tag{2.7}$$

*and the variation as the difference between counterclockwise and clockwise continuation:*

$$\mathcal{V}ar_{(t,\alpha)}\psi(s,t) := \mathcal{M}on_{(t,+\alpha)}\psi(s,t) - \mathcal{M}on_{(t,-\alpha)}\psi(s,t). \tag{2.8}$$

*Iterated variations are defined as*

$$\mathcal{V}ar_{(t,\alpha_1),\cdots,(t,\alpha_k)} := \mathcal{V}ar_{(t,\alpha_1)} \circ \mathcal{V}ar_{(t,\alpha_2)} \circ \cdots \circ \mathcal{V}ar_{(t,\alpha_k)}. \tag{2.9}$$

*In particular*

$$\mathcal{V}ar^{\circ m}_{(t,\alpha)} := \underbrace{\mathcal{V}ar_{(t,\alpha)} \circ \cdots \circ \mathcal{V}ar_{(t,\alpha)}}_{m \quad times}.$$

*Let now $\psi$ be a multivalued function in two variables defined in $\widetilde{\mathbb{C}^2 \setminus \{st = 0\}}$. The mixed variation is defined as*

$$\begin{aligned}\mathcal{V}ar_{(s,t),(\alpha,\beta)}(\psi(s,t)) &:= \mathcal{V}ar_{(s,\alpha)} \circ \mathcal{V}ar_{(t,\beta)}(\psi(s,t))\\ &= \mathcal{V}ar_{(s,\alpha)}\left(\psi(s, te^{i\beta\pi}) - \psi(s, te^{-i\beta\pi})\right)\\ &= \psi(se^{i\alpha\pi}, te^{i\beta\pi}) - \psi(se^{-i\alpha\pi}, te^{i\beta\pi})\\ &- \psi(se^{i\alpha\pi}, te^{-i\beta\pi}) + \psi(se^{-i\alpha\pi}, te^{-i\beta\pi}).\end{aligned}$$

**Lemma 1.** *Let $\psi$ be a multivalued function in two variables defined in $\widetilde{\mathbb{C}^2 \setminus \{st = 0\}}$. The variations $\mathcal{V}ar_{(s,\alpha)}$ and $\mathcal{V}ar_{(t,\beta)}$ commute:*

$$\mathcal{V}ar_{(s,\alpha)} \circ \mathcal{V}ar_{(t,\beta)}\psi = \mathcal{V}ar_{(t,\beta)} \circ \mathcal{V}ar_{(s,\alpha)}\psi.$$

*Proof.* The proof is a consequence of the monodromy theorem which says that: If $\gamma_1, \gamma_2$ be a homotopic paths in $\mathbb{C}^2 \setminus \{st = 0\}$, then $\psi_{\gamma_1} = \psi_{\gamma_2}$ where $\psi_{\gamma_1} = \mathcal{M}on_{\gamma_1}\psi$ and $\psi_{\gamma_2} = \mathcal{M}on_{\gamma_2}\psi$. We consider

$$\gamma_1(\theta,\phi) = (s(\theta,\phi), t(\theta,\phi)) = \left(s, te^{i\theta}\right)_{\theta\in[0,\alpha]} \sqcup \left(se^{i\phi}, te^{i\alpha}\right)_{\phi\in[0,\beta]},$$

$$\gamma_2(\theta,\phi) = (s(\theta,\phi), t(\theta,\phi)) = \left(se^{i\phi}, t\right)_{\phi\in[0,\beta]} \sqcup \left(se^{i\beta}, te^{i\theta}\right)_{\theta\in[0,\alpha]}.$$

The paths $\gamma_1$ and $\gamma_2$ are homotopic and this implies that $\psi(se^{i\alpha\pi}, te^{i\beta\pi})$ can be defined either as $\psi_{\gamma_1}$ or $\psi_{\gamma_2}$. The same argument holds for the other germs $\psi(se^{-i\alpha\pi}, te^{i\beta\pi}), \psi(se^{i\alpha\pi}, te^{-i\beta\pi})$ and $\psi(se^{-i\alpha\pi}, te^{-i\beta\pi})$. □

### 2.6.1 Variation of the function $J^t_{div}$

In this subsection we study the analytic properties of the function

$$J^t_{div}(s,t) = \int_{\delta^t_{div}(s,t)} \omega^t_{div}.$$

In the local chart $(U^t_{div}, (X,Y,Z))$ of Proposition 3, the blown-up one-form $\omega^t_{div} = (\sigma_1^*\Omega)\varrho^\star_{div}, \star \in \{0, t_0\}$ is given by

$$\omega^t_{div} = F^t_{div,1}\mathrm{d}X + F^t_{div,2}\mathrm{d}Y + F^t_{div,3}\mathrm{d}Z.$$

To study the analytic properties of the function $J^t_{div}$, we distinguish two cases

1. For $t \in [0, m[$, the relative cycle $\delta^t_{div}(s,t)$ can be chosen as a lift of path in $C_0$.

2. For $t \in [\frac{m}{2}, 2M]$, the relative cycle $\delta^t_{div}(s,t)$ can be chosen as a lift of path in $C_{t_0}$.

On the chart $(U^t_{div}, (X,Y,Z))$ of Proposition 3, the linear projection $\Pi^t_{div}(X,Y,Z) = Y$ is every where transverse to the levels of the foliation $\sigma_1^* \widetilde{\mathcal{F}}$, which corresponds simply to the graphs of the multivalued functions

$$Y \longmapsto \Phi^t_{div}(Y) = (\Phi^t_{div,1}(s,t,Y), \Phi^t_{div,2}(s,t,Y)).$$

The relative cycle $\delta^t_{div}(s,t)$ is defined by two data:

1. An initial condition $\left(t^{\frac{-1}{\epsilon_-}}, s\right) = \left(Y^0_-, Z^0\right) \in \Sigma_- \backslash \{ZY_- = 0\}$ (starting point of $\delta^t_{div}(s,t)$).

2. A path $\delta^t_{div}(u) \subseteq \check{C}_\star$

$$\begin{aligned} \delta^t_{div} :\quad & \mathbb{R}^+ \longrightarrow \check{C}_\star \\ & u \longmapsto \delta^t_{div}(u) \end{aligned}$$

such that the starting point $\delta^t_{div}(0) = \Pi^\star_{div}(Y^0_-, Z^0)$, $\lim_{u\to+\infty} \delta^t_{div}(u) = p_+$ and $(Y^1_+, Z^1) = \delta^t_{div}(s,t) \cap (\Sigma_+ \setminus \{ZY_+ = 0\})$ (end point of $\delta^t_{div}(s,t,u)$). The path $\delta^t_{div}(u)$ is homotopic to a straight-line segment

$$L^\star_{div} = \left(\Pi^\star_{div}(Y^0_-, Z^0), \Pi^\star_{div}(Y^1_+, Z^1)\right)$$

joining $\Pi^\star_{div}(Y^0_-, Z^0)$ and $\Pi^\star_{div}(Y^1_+, Z^1)$. Then the relative cycle $\delta^t_{div}(s,t)$ is obtained by lifting the path $\delta^t_{div}(u)$ above $C_\star$ to the Riemann surface $R^t_{div}$.

The function $J^t_{div}$ can be rewritten as

$$J^t_{div}(s,t) = \int_{\delta^t_{div}(s,t)} \omega^t_{div} = \int_{\delta^t_{div}(u)} F^t_{div} \mathrm{d}Y,$$

where $F^t_{div}$ is the pull-back

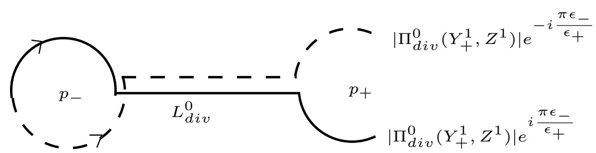
$$F^t_{div} = \left( F^t_{div,1} \circ \Phi^t_{div} \left( \frac{\partial \Phi^t_{div,1}}{\partial Y} \right) + F^t_{div,2} \circ \Phi^t_{div} + F^t_{div,3} \circ \Phi^t_{div} \left( \frac{\partial \Phi_{div,2}}{\partial Y} \right) \right).$$

**Variation of the cycle $\delta^t_{div}(s,t) \subset V^0_{div}$**

We assume the relative cycle $\delta^t_{div}(s,t)$ going from $\Sigma_-$ to $\Sigma_+$ inside the neighborhood $V^0_{div}$. We fix $s$ and we study the analytic properties of the function $J^t_{div}$ with respect to $t$.

1. *Clokwise monodromy $\mathcal{M}on_{(t,-\epsilon_-)}$ of the function $J^t_{div}$:* Now we vary continuously the starting point $\Pi^0_{div}(Y^0_-, Z^0)$ along a sufficiently small circular arc $\alpha_- = \{|\Pi^0_{div}(Y^0_-, Z^0)|e^{ir}, r \in [-\pi, 0]\}$ around $p_-$ lying on the upper half-plane and a circular arc $\alpha_+ = \{|\Pi^0_{div}((Y^1_+, Z^1))|e^{ir}, r \in [0, \frac{\pi\epsilon_-}{\epsilon_+}]\}$ around $p_+$ lying on the lower half-plane. Finally, the monodromy $\mathcal{M}on_{(t,-\epsilon_-)}$ of the relative cycle $\delta^t_{div}(s,t)$, denoted by $\alpha_- \cup \widetilde{L^0_{div}} \cup \alpha_+$, is the lifting of the path $\alpha_- \cup L^0_{div} \cup \alpha_+$, which indicated by solid part in Figure below, with initial condition $(Y^0_- e^{-i\pi}, Z^0)$ on the transversal $\Sigma_-$ and
$$\mathcal{M}on_{(t,-\epsilon_-)} J^t_{div}(s,t) = \int_{\alpha_- \cup \widetilde{L^0_{div}} \cup \alpha_+} \omega^t_{div}.$$

2. *Counterclokwise monodromy $\mathcal{M}on_{(t,\epsilon_-)}$ of the function $J^t_{div}$:* Now we vary continuously the starting point $\Pi^0_{div}(Y^0_-, Z^0)$ along a sufficiently small circular arc $\bar{\alpha}_+ = \{|\Pi^0_{div}(Y^0_-, Z^0)|e^{ir}, r \in [0, \pi]\}$ around $p_-$ lying on the upper half-plane and a circular arc $\bar{\alpha}_- = \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{ir}, r \in [-\frac{\pi\epsilon_-}{\epsilon_+}, 0]\}$ around $p_+$. Finally, the monodromy $\mathcal{M}on_{(t,\epsilon_-)}$ of the relative cycle $\delta^t_{div}(s,t)$, denoted by $\bar{\alpha}_+ \cup \widetilde{L^0_{div}} \cup \bar{\alpha}_-$, is the lifting of the path $\bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$, which is indicated by dashed part in Figure below, with initial condition $(Y^0_- e^{i\pi}, Z^0)$ on the transversal $\Sigma_-$ and
$$\mathcal{M}on_{(t,\epsilon_-)} J^t_{div}(s,t) = \int_{\bar{\alpha}_+ \cup \widetilde{L^0_{div}} \cup \bar{\alpha}_-} \omega^0_{div}.$$

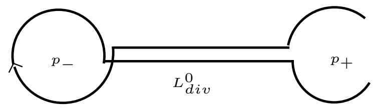


The paths $\alpha_- \cup L^0_{div} \cup \alpha_+$ and $\bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$.

**Conclusion.** The $t$-variation of the function $J^t_{div}$ is given by
$$\mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) = \int_{\alpha_- \cup L^0_{div} \cup \widetilde{\alpha_+ - \bar{\alpha}_+} \cup L^0_{div} \cup \bar{\alpha}_-} \omega^t_{div},$$
where $\alpha_- \cup L^0_{div} \cup \widetilde{\alpha_+ - \bar{\alpha}_+} \cup L^0_{div} \cup \bar{\alpha}_- \equiv \mathcal{V}ar_{(t,-\epsilon_-)} \delta^t_{div}(s,t)$ modulo homotopy is the lifting of the path $\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$ -see Figure below.

**Remark 3.** *The paths $\alpha_- \cup L^0_{div} \cup \alpha_+$ and $\bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$ are symmetric with respect to the real axes of the comlex curve $C_0$.*

Observe that the path $\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$ is not necessairly closed around $p_+$, if the quotient $\frac{\epsilon_-}{\epsilon_+}$ is not equal to 1. To complete it we




The path $\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$.

apply the monodromy operator $\mathcal{M}on_{(t,-\epsilon_+)}$ which moves its extremities $|\Pi^0_{div}(Y^1_+, Z^1)|e^{\pm i\pi\frac{\epsilon_-}{\epsilon_+}}$ to create two circular arcs

$$\alpha^+_+ := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{i\pi\frac{\epsilon_-}{\epsilon_+}+ir}, r \in [0,\pi]\},$$

$$\bar{\alpha}^-_- := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{-i\pi\frac{\epsilon_-}{\epsilon_+}+ir}, r \in [-\pi,0]\}$$

which are indicated in the Figure below by dashed lines. Shematically the resulting base path is

$$\mathcal{M}on_{(t,-\epsilon_+)}\left(\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-\right) = \alpha_- \cup L^0_{div} \cup \beta_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_-$$

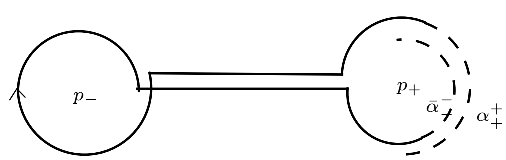
where

$$\beta_+ := \{|\Pi^0_{div}((Y^1_+, Z^1))|e^{ir}, r \in [0, \frac{\pi\epsilon_-}{\epsilon_+} + \pi]\},$$

$$\bar{\beta}_- := \{|\Pi^0_{div}((Y^1_+, Z^1))|e^{ir}, r \in [-\frac{\pi\epsilon_-}{\epsilon_+} - \pi, 0]\},$$

which is indicated in Figure below modulo homotopy. Then

$$\mathcal{M}on_{(t,-\epsilon_+)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) = \int_{\alpha_- \cup L^0_{div} \cup \widetilde{\beta_+ - \bar{\alpha}_+} \cup L^0_{div} \cup \bar{\beta}_-} \omega^0_{div}.$$

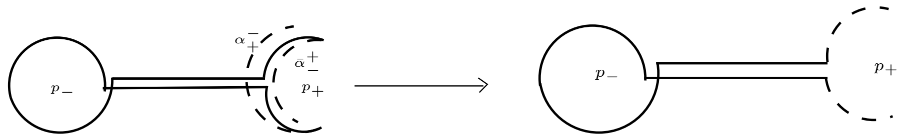



The path $\alpha_- \cup L^0_{div} \cup \beta_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_-$.

Conversely, the application of monodromy operator $\mathcal{M}on_{(t,\epsilon_+)}$ to the path $\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$ moves its extremities $|\Pi^0_{div}(Y^1_+, Z^1)|e^{\pm i\pi\frac{\epsilon_-}{\epsilon_+}}$ to create two circular arcs

$$\alpha^-_+ := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{i\pi\frac{\epsilon_-}{\epsilon_+}+ir}, r \in [-\pi,0]\},$$

$$\bar{\alpha}^+_- := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{-i\pi\frac{\epsilon_-}{\epsilon_+}+ir}, r \in [0,\pi]\},$$




The path $\alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+$.

which are indicated by dashed lines in first picture of Figure below. Hence we conclude that

$$\mathcal{M}on_{(t,\epsilon_+)}\left(\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-\right) = \alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+.$$

Here

$$\bar{\beta}_+ := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{ir}, r \in [0, \frac{-\pi\epsilon_-}{\epsilon_+} + \pi]\},$$

$$\beta_- := \{|\Pi^0_{div}(Y^1_+, Z^1)|e^{ir}, r \in [\frac{\pi\epsilon_-}{\epsilon_+} - \pi, 0]\},$$

which is illustrated, after simplification, in the second picture of Figure above. Then, we have

$$\mathcal{M}on_{(t,\epsilon_+)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) = \int_{\widetilde{\alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+}} \omega^t_{div},$$

where $\widetilde{\alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+}$ is the lift of the base path $\alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+$ to the leaves of $\sigma_1^*\widetilde{\mathcal{F}}$.

It should be remarked that after glueing two paths $\alpha_- \cup L^0_{div} \cup \beta_+ - \bar{\alpha}_+ \cup L \cup \bar{\beta}_-$ and $\alpha_- \cup L^0_{div} \cup \beta_- - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+$ together by making the difference between them, we obtain a closed loop.

In particular, we have one or two closed loops around each singular point, depending whether the quotient $\frac{\epsilon_-}{\epsilon_+}$ is equal to 1 or not. More concretly, let

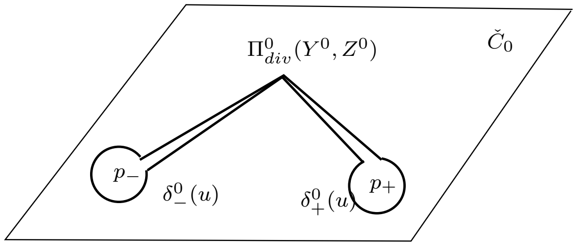
$$\begin{aligned} \delta^0_\star : \quad & [0,1] \longrightarrow \check{C}_0 \\ & u \longmapsto \delta^0_\star(u) \end{aligned},$$

be a continuous map, where $\delta^0_\star(0) = \delta^0_\star(1) = \Pi^0_{div}(Y^0_-, Z^0)$ and $\delta^0_\star(u)$ is a small path turning once counterclockwise around $p_\star$ (see Figure below).

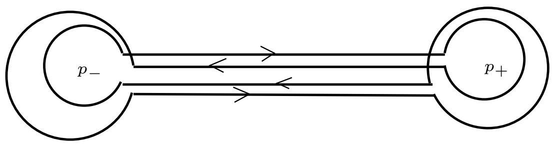
Finally, to calculate $\mathcal{V}ar_{(t,-\epsilon_-)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} J^t_{div}(s,t)$, we distinguish two cases:

(a) In the generic case $\epsilon_+ \neq \epsilon_-$, we have

$$\mathcal{V}ar_{(t,-\epsilon_-)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} J^t_{div}(s,t) = \int_{\alpha_- \cup L^0_{div} \cup \beta_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \widetilde{\bar{\beta}_- - \alpha_-} \cup L^0_{div} \cup \beta_- + \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+} \omega^0_{div}, \tag{2.10}$$




The paths $\delta^0_-(u), \delta^0_+(u)$.



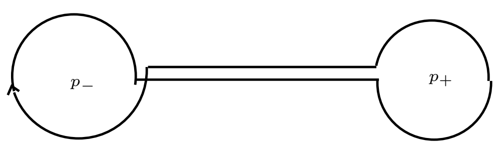

The commutator $[\delta^0_-(u), \delta^0_+(u)]$.

where the path $\alpha_- \cup L^0_{div} \cup \beta_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_- - \alpha_- \cup L^0_{div} \cup \beta_- + \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\beta}_+$ is homotopic to the commutator loop $[\delta^0_-(u), \delta^0_+(u)]$ (see Figure above)

(b) In the resonant case $\epsilon_+ = \epsilon_-$, after one variation $\mathcal{V}ar_{(t,-\epsilon_-)}$ of the function $J_t$ we obtain

$$\mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) = \int_{\alpha_- \cup L^0_{div} \cup \widetilde{\alpha_+ - \bar{\alpha}_+} \cup L^0_{div} \cup \bar{\alpha}_-} \omega^t_{div}, \tag{2.11}$$

where the path $\alpha_- \cup L^0_{div} \cup \alpha_+ - \bar{\alpha}_+ \cup L^0_{div} \cup \bar{\alpha}_-$ is homotopic to eight figure loop $\delta^0_+(u) \left(\delta^0_-(u)\right)^{-1}$ (see Figure below).




The figure eight loop.

Hence in this case we have a closed loop around each singular point $p_\pm$.

**Remark 4.** *$\delta^0_-(u), \delta^0_+(u)$ are elements of the first homotopy group of $C_0$ with base point $\Pi^0_{div}(Y^0_-, Z^0)$.*

Finally, we conclude as in the proof of Lemma 2.7 of [2] that:

(a) In the generic case $\epsilon_+ \neq \epsilon_-$, the function $J^t_{div}$ satisfies the variation equation

$$\mathcal{V}ar_{(t,a)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) \equiv 0. \tag{2.12}$$

(b) In the resonant case $\epsilon_+ = \epsilon_-$, the function $J^t_{div}$ satisfies the variation equation

$$\mathcal{V}ar_{(t,a)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) \equiv 0. \tag{2.13}$$

Now we assume that $t$ is fixed. A similar computation allows us to compute the variation of $J$ with respect to $s$

$$\mathcal{V}ar_{(s,1)} J^t_{div}(s,t) = J^t_{div}(se^{i\pi},t) - J^t_{div}(se^{-i\pi},t) = \int_{\text{figure eight loop}} \omega^t_{div} \tag{2.14}$$

and consequently we have

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(s,1)} J^t_{div}(s,t) \equiv 0. \tag{2.15}$$

### 2.6.2 Variation of the cycle $\delta^t_{div}(s,t) \subset V^{t_0}_{div}$

Now, we assume that the relative cycle $\delta^t_{div}(s,t)$ is going from $\Sigma_-$ to $\Sigma_+$ inside the neighborhood $V^{t_0}_{div}$.

1. The variation of the function $J^t_{div}(s,t)$ with respect to $s$ gives us

$$\mathcal{V}ar_{(s,1)} J^t_{div}(s,t) = \int_{\widetilde{\delta^{t_0}_+(u)(\delta^{t_0}_-(u))^{-1}}} \omega_{div},$$

where $\widetilde{\delta^{t_0}_+(u)(\delta^{t_0}_-(u))^{-1}}$ is the lifting of the figure eight loop $\delta^{t_0}_+(u)(\delta^{t_0}_-(u))^{-1} \subset C_0$.

2. The variation of $J^t_{div}(s,t)$ with respect to $t$ gives us

(a) If $\epsilon_- \neq \epsilon_+$, we have

$$\mathcal{V}ar_{(t,a)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} J^t_{div}(s,t) \equiv 0.$$

(b) If $\epsilon_- = \epsilon_+$, we have

$$\mathcal{V}ar_{(t,a)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} J^t_{div}(s,t) \equiv 0.$$

### 2.6.3 Variation of the function $J^\infty_{div,\pm}$

In the local chart $\left(U^\infty_{div,\pm}, (X,Y,Z)\right)$ of Proposition 3, the blown-up one-form $\omega^\infty_{div,\pm} = (\sigma_1^*\Omega)\varrho^\infty_{div,\pm}$ is given by

$$\omega^\infty_{div,\pm} = F^\infty_{div,\pm,1}\mathrm{d}X + F^\infty_{div,\pm,2}\mathrm{d}Y + F^\infty_{div,\pm,3}\mathrm{d}Z$$

and the linear projection $\Pi^\infty_{div,\pm}(X,Y,Z) = Z$ is everywhere transverse to the levels of the foliation $\sigma_1^*\widetilde{\mathcal{F}}$, which correspond simply to the graphs of the multi-valued function

$$Z \longmapsto \Phi^\infty_{div,\pm}(Z) = (\Phi^\infty_{div,\pm,1}(Z,s,\tilde{t}), \Phi^\infty_{div,\pm,2}(Z,s,\tilde{t})).$$

**Characterization of the relative cycle $\delta^{\infty}_{div,\pm}(s,\tilde{t})$**

The relative cycle $\delta^{\infty}_{div,\pm}(s,\tilde{t})$ is going from $\Gamma^{\infty}_{div}=\{Y=1\}$ to $\Sigma_{\pm}=\{X=1\}$. The relative cycle $\delta^{\infty}_{div,\pm}(s,\tilde{t})$ is defined by two data:

1. The initial condition (starting point) $(X^0,Z^0_{\pm})=(s,\tilde{t}^{\frac{1}{\epsilon}}):=\delta^{\infty}_{div,\pm}(s,\tilde{t})\cap(\Gamma^{\infty}_{div}\backslash\{XZ_{\pm}=0\})$. Let $(Y^1_{\pm},Z^1)=\left(\tilde{t}^{\frac{1}{\epsilon_{\pm}}},s\right):=\delta^{\infty}_{div,\pm}(s,\tilde{t})\cap(\Sigma_{\pm}\backslash\{ZY_{\pm}=0\})$ be the end point of the relative cycle $\delta^{\infty}_{div,\pm}(s,\tilde{t})$.

2. A path $\delta^{\infty}_{div,\pm}(u)\subseteq C^{\infty}_{div,\pm}$ such that

$$\begin{aligned}\delta^{\infty}_{div,\pm}:\quad & \mathbb{R}_{\geq 0}\longrightarrow C^{\infty}_{div,\pm}\\ & u\longmapsto \delta^{\infty}_{div,\pm}(u),\end{aligned}$$

   where $\delta^{\infty}_{div,\pm}(0)=\Pi^{\infty}_{div,\pm}(X^0,Z^0_{\pm}),\lim_{u\to+\infty}\delta^{\infty}_{div,\pm}(u)=p_{\pm}$. The path $\delta^{\infty}_{div,\pm}(u)$ is homotopic to a straight-line segment $L^{\infty}_{div,\pm}$ joining the starting point $\Pi^{\infty}_{div,\pm}(X^0,Z^0_{\pm})$ and the end point $\Pi^{\infty}_{div,\pm}(Y^1_{\pm},Z^1)$.

The function $J^{\infty}_{div,\pm}$ can be rewritten as

$$J^{\infty}_{div,\pm}(s,\tilde{t})=\int_{\delta^{\infty}_{div,\pm}(s,\tilde{t})}\omega^{\infty}_{div,\pm}=\int_{\delta^{\infty}_{div,\pm}(u)}F^{\infty}_{div,\pm}\mathrm{d}Z,$$

where $F^{\infty}_{div,\pm}$ is given by

$$F^{\infty}_{div,\pm}=F^{\infty}_{div,\pm,1}\circ\Phi^{\infty}_{div,\pm}\left(\frac{\partial\Phi^{\pm}_{div,\pm,1}}{\partial Z}\right)+F^{\infty}_{div,\pm,2}\circ\Phi^{\infty}_{div,\pm}\left(\frac{\partial\Phi^{\infty}_{div,\pm,2}}{\partial Z}\right)+F^{\infty}_{div,\pm,3}\circ\Phi^{\infty}_{div,\pm}.$$

**The monodromy of $\delta^{\infty}_{div,\pm}(s,\tilde{t})$**

Let us fix $s$. We consider the counterclockwise continuous deformation $\tilde{t}\to\tilde{t}e^{-ir},r\in[0,\pi\epsilon]$. A base path obtained by application of monodromy operator to the path $\delta^{\infty}_{div,\pm}(u)$ consists, modulo homotopy, of the straight-line segment $L^{\infty}_{div,\pm}$ and two circular arcs around $q_{\pm}$ and $p_{\pm}$ respectively

$$\alpha_-=\{|\Pi^{\infty}_{div,\pm}(X^0,Z^0_{\pm})|e^{ir},r\in[-\pi,0]\},\quad \alpha_+=\{|\Pi^{\infty}_{div,\pm}(Y^1_{\pm},Z^1)|e^{ir},r\in[0,\frac{\pi\epsilon}{\epsilon_{\pm}}]\}.$$

Denote it by $\mathcal{M}on_{(\tilde{t},\epsilon)}\delta^{\infty}_{div,\pm}(u)\equiv\alpha_-\cup L^{\infty}_{div,\pm}\cup\alpha_+$ modulo homotopy. Hence one obtains

$$\mathcal{M}on_{(\tilde{t},\epsilon)}J^{\infty}_{div,\pm}(s,\tilde{t})=\int_{\widetilde{\alpha_-\cup L^{\infty}_{div,\pm}\cup\alpha_+}}\omega^{\infty}_{div,\pm},$$

where $\widetilde{\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+}$ is the lift of the base path $\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+$ to a Riemann surface. Consequently, the variation is given by

$$\mathcal{V}ar_{(\tilde{t},\epsilon)} J^{\infty}_{div,\pm}(s,\tilde{t}) = \int_{\widetilde{\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+ - \bar{\alpha}_+ \cup L^{\infty}_{div,\pm} \cup \bar{\alpha}_-}} \omega^{\infty}_{div,\pm},$$

where the path $\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+$ is symmetric to $\bar{\alpha}_+ \cup L^{\infty}_{div,\pm} \cup \bar{\alpha}_-$ with respect to the real line of the complex plane $C^{\infty}_{div}$ and $\widetilde{\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+ - \bar{\alpha}_+ \cup L^{\infty}_{div,\pm} \cup \bar{\alpha}_-}$ is the lift of $\alpha_- \cup L^{\infty}_{div,\pm} \cup \alpha_+ - \bar{\alpha}_+ \cup L^{\infty}_{div,\pm} \cup \bar{\alpha}_-$.

Let $\Pi_1\left(C^{\infty}_{div,\pm}, \Pi^{\infty}_{div,\pm}(X^0, Z^0_{\pm})\right) = \langle \ell^{\pm}_+(u), \ell^{\pm}_-(u) \rangle$ be the first homotopy group of the punctured complexe plane $C^{\infty}_{div,\pm}$ with base point $\Pi^{\infty}_{div,\pm}(X^0, Z^0_{\pm})$, where $\ell^{\pm}_+(u)$ and $\ell^{\pm}_-(u)$ are smalls paths turning once counterclockwise around $q_{\pm}$ and $p_{\pm}$ respectively. Finally, we distinguish two cases

1. If $\epsilon \neq \epsilon_{\pm}$, we have

$$\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_{\pm})} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^{\infty}_{div,\pm}(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},-a)} \int_{\widetilde{[\ell^{\pm}_-(u),\ell^{\pm}_+(u)]}} \omega^{\infty}_{div,\pm} \equiv 0, \tag{2.16}$$

where $\widetilde{[\ell^{\pm}_-(u), \ell^{\pm}_+(u)]}$ is the lift of the commutator loop $[\ell^{\pm}_-(u), \ell^{\pm}_+(u)]$.

2. If $\epsilon = \epsilon_{\pm}$, we have

$$\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^{\infty}_{div,\pm}(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},-a)} \int_{\widetilde{\ell^{\pm}_-(u)(\ell^{\pm}_+(u))^{-1}}} \omega^{\infty}_{div,\pm} \equiv 0, \tag{2.17}$$

where $\widetilde{\ell^{\pm}_-(u)(\ell^{\pm}_+(u))^{-1}}$ is the lift of the figure eight loop $\ell^{\pm}_-(u)(\ell^{\pm}_+(u))^{-1}$.

On the other hand we have

1. If $\epsilon \neq \epsilon_{\pm}$

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_{\pm})} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^{\infty}_{div,\pm}(s,\tilde{t}) = \mathcal{V}ar_{(s,1)} \int_{\widetilde{[\ell^{\pm}_-(u),\ell^{\pm}_+(u)]}} \omega^{\infty}_{div,\pm} \equiv 0. \tag{2.18}$$

2. If $\epsilon = \epsilon_{\pm}$

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^{\infty}_{div,\pm}(s,\tilde{t}) = \mathcal{V}ar_{(s,1)} \int_{\widetilde{\ell^{\pm}_-(u)(\ell^{\pm}_+(u))^{-1}}} \omega^{\infty}_{div,\pm} \equiv 0. \tag{2.19}$$

### 2.6.4 Variation of the function $J^{\infty}_{div}$

In the local chart $(U^{\infty}_{div}, (X, Y, Z))$, the blown-up one-form $\omega^{\infty}_{div} = (\sigma_1^* \Omega)\varrho^{\infty}_{div}$ is given by

$$\omega^{\infty}_{div} = F^{\infty}_{div,1}\mathrm{d}X + F^{\infty}_{div,2}\mathrm{d}Y + F^{\infty}_{div,3}\mathrm{d}Z$$

and the linear projection $\Pi^{\infty}_{div}(X, Y, Z) = Y$ is everywhere transverse to the levels of the foliation $\sigma_1^*\widetilde{\mathcal{F}}$, which correspond simply to the graphs of the multi-valued function

$$Y \longmapsto \Phi^{\infty}_{div}(Y) = (\Phi^{\infty}_{div,1}(Y), \Phi^{\infty}_{div,2}(Y)).$$

**Characterization of the relative cycle $\delta^{\infty}_{div}(s, \tilde{t})$**

The relative cycle $\delta^{\infty}_{div}(s, \tilde{t})$ can be characterized by two data:

1. The initial condition $(X^0, Y^0_-) = (s, \tilde{t}^{\frac{1}{\epsilon_-}}) := \delta^{\infty}_{div}(s, \tilde{t}) \cap (\Gamma^{\infty}_{div,-} \setminus \{XY_- = 0\})$.

2. A path $\delta^{\infty}_{div}(u) \subset C^{\infty}_{div}$

$$\begin{aligned} \delta^{\infty}_{div} : \quad & \mathbb{R}_{\geq 0} \longrightarrow C^{\infty}_{div} \\ & u \longmapsto \delta^{\infty}_{div}(u), \end{aligned}$$

   such that $\delta^{\infty}_{div}(0) = \Pi^{\infty}_{div}(X^0, Y^0_-), \lim_{u\to+\infty} \delta^{\infty}_{div}(u) = q_+$ and $(X^1, Y^1_+) = (s, \tilde{t}^{\frac{1}{\epsilon_+}}) = \delta^{\infty}_{div}(s, \tilde{t}) \cap (\Gamma^{\infty}_{div,+} \setminus \{XY_+ = 0\})$. The path $\delta^{\infty}_{div}(u)$ is homotopic to a straight-line segment $L^{\infty}_{div}$ joining the point $\Pi^{\infty}_{div}(X^0, Y^0_-)$ and $\Pi^{\infty}_{div}(X^1, Y^1_+)$.

The function $J^{\infty}_{div}$ can be rewritten as

$$J^{\infty}_{div}(s, \tilde{t}) = \int_{\delta^{\infty}_{div}(s,\tilde{t})} \omega^{\infty}_{div} = \int_{\delta^{\infty}_{div}(u)} F^{\infty}_{div}\mathrm{d}Y,$$

where

$$F^{\infty}_{div} = F^{\infty}_{div,1} \circ \Phi^{\infty}_{div} \left( \frac{\partial \Phi^{\infty}_{div,1}}{\partial Y} \right) + F^{\infty}_{div,2} \circ \Phi^{\infty}_{div} + F^{\infty}_{div,3} \circ \Phi^{\infty}_{div} \left( \frac{\partial \Phi^{\infty}_{div,2}}{\partial Y} \right).$$

**The variation of $J^{\infty}_{div}$ near $\tilde{t} = 0$**

The path is obtained by application of the monodromy operator to a path $\delta^{\infty}_{div}(u)$. It consists, modulo homotopy, of the straight-line segment $L^{\infty}_{div}$ and two circular arcs around $q_-$ and $q_+$ respectively

$$\alpha_- = \{|\Pi^{\infty}_{div}(X^0, Y^0_-)|e^{ir}, r \in [-\pi, 0]\}, \quad \alpha_+ = \{|\Pi^{\infty}_{div}(X^1, Y^1_+)|e^{ir}.r \in [0, \pi\epsilon_- \diagup \epsilon_+]\}.$$

Denote it by $\mathcal{M}on_{(\tilde{t},\epsilon_-)}\delta^{\infty}_{div}(u) = \alpha_- \cup L^{\infty}_{div} \cup \alpha_+$ modulo homotopy. Similarly, the path $\mathcal{M}on_{(\tilde{t},-\epsilon_-)}\delta^{\infty}_{div}(u)$ is homotopic to $\bar{\alpha}_- \cup L^{\infty}_{div} \cup \bar{\alpha}_+$ which is symmetric to $\alpha_- \cup L^{\infty}_{div} \cup \alpha_+$. One obtains

$$\mathcal{V}ar_{(\tilde{t},\epsilon_-)} J^{\infty}_{div}(s,\tilde{t}) = \int_{\alpha_- \cup L^{\infty}_{div} \cup \widetilde{\alpha_+ - \bar{\alpha}_-} \cup L^{\infty}_{div} \cup \bar{\alpha}_+} \omega^{\infty}_{div}.$$

Let $\Pi_1(\check{Y}^{\mathbb{C}}, \Pi^{\infty}_{div}(X^0, Y^0_-)) := \langle \ell_-(u), \ell_+(u) \rangle$ be the first homotopy group of the punctured complex plane $C^{\infty}_{div}$ with base point $\Pi^{\infty}_{div}(X^0, Y^0_-)$ such that $\ell_-(u)$ and $\ell_+(u)$ are paths turning once counterclockwise around $q_-$ and $q_+$ respectively. Finally, we have

1. If $\epsilon_- = \epsilon_+$ (resonant case), the function $J^{\infty}_{div}$ satisfies the following iterated variations equation

$$\mathcal{V}ar_{(\tilde{t},\epsilon)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_-)} J^{\infty}_{div}(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},\epsilon)} \int_{\ell_-(u)\widetilde{(\ell_+(u))^{-1}}} \omega^{\infty}_{div} \equiv 0. \tag{2.20}$$

2. If $\epsilon_- \neq \epsilon_+$ (generic case), the function $J^{\infty}_{div}$ satisfies the following iterated variations equation

$$\mathcal{V}ar_{(\tilde{t},\epsilon)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_-)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_+)} J^{\infty}_{div}(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},\epsilon)} \int_{[\ell_-\widetilde{(u),\ell_+}(u)]} \omega^{\infty}_{div} \equiv 0. \tag{2.21}$$

On the other hand we should remark that the restriction of the foliation $\sigma_1^*\widetilde{\mathcal{F}}$ near the separatrix $\delta^{\infty}_{div}$ is given by an analytic function on $X$. Hence, we have

$$\mathcal{V}ar_{(s,1)} J^{\infty}_{div}(s,\tilde{t}) \equiv 0. \tag{2.22}$$

### 2.6.5 Variation of the function $J_\pm$

In the local chart $(U_\pm, (X,Y,Z))$, the blown-up one form $\omega_\pm = (\sigma_1^*\Omega)\varrho_\pm$ is given by

$$\omega_\pm = F_{\pm,1}\mathrm{d}X + F_{\pm,2}\mathrm{d}Y + F_{\pm,3}\mathrm{d}Z$$

and the linear projection $\Pi_\pm(X,Y,Z) = X$ is everywhere transverse to the levels of the foliation $\sigma_1^*\widetilde{\mathcal{F}}$, which correspond simply to the graphs of the multivalued function

$$X \longmapsto \Phi_\pm(X) = (\Phi_{\pm,1}(X), \Phi_{\pm,2}(X)).$$

**First case**

We assume that the relative cycle $\delta_\pm(s,\tilde{t})$ going from $\Sigma_{div,\pm}$ to $\Sigma_\kappa$ inside the neighborhood $U_\pm$ is near the separatrix $\delta_\pm$.

The relative cycle $\delta_\pm(s,\tilde{t})$ can be characterized by two data:

1. The initial condition $(s, \tilde{t}^{\frac{1}{\epsilon_\pm}}) = (X^0, Y^0_\pm) = \delta_\pm(s,\tilde{t}) \cap (\Sigma_{div,\pm} \setminus \{XY_\pm = 0\})$ (starting point of $\delta_\pm(s,\tilde{t})$). Let $(\tilde{t}^{\frac{1}{\epsilon_\kappa}}, s) = (X^1_\kappa, Z^1) = \delta_\pm(s,\tilde{t}) \cap (\Sigma_\kappa \setminus \{ZX_\kappa = 0\})$ be the end point of $\delta_\pm(s,\tilde{t})$.

2. A path $\delta_\pm(u) \subset C_\pm$, which is homotopic to a straight-line segment $L_\pm$ joining $\Pi_\pm(X^0, Y^0_\pm)$ and $\Pi_\pm(X^1_\kappa, Z^1)$.

The function $J_\pm$ can be rewritten as follows

$$J_\pm(s,\tilde{t}) = \int_{\delta_\pm(s,\tilde{t})} \omega_\pm = \int_{\delta_\pm(u)} F_\pm \mathrm{d}X.$$

where the multivalued function $F_\pm$ is given by

$$F_\pm = F_{\pm,1} \circ \Phi_\pm + F_{\pm,2} \circ \Phi_\pm \left(\frac{\partial \Phi_{\pm,1}}{\partial X}\right) + F_{\pm,3} \circ \Phi_\pm \left(\frac{\partial \Phi_{\pm,2}}{\partial X}\right).$$

Let $\Pi_1(\check{X}^{\mathbb{C}}, \Pi_\pm(X^0_1, Y^0_\pm)) = \langle \delta^\pm(u), \delta^\kappa(u) \rangle$ be the first homotopy group of $C_\pm$ with base point $\Pi_\pm(X^0_1, E^0_1)$, where $\delta^\pm(u), \delta^\kappa(t)$ are two paths turning once counterclockwise around $p_\pm, p_{\kappa\pm}$ respectively.

*Step 1:* Let us fix $s$. The computation of variation of the function $J_\pm$ near $\tilde{t} = 0$ gives us two different equations, depending whether the quotient $\frac{\epsilon_\kappa}{\epsilon_\pm}$ is equal to 1 or not i.e.

1. If $\epsilon_\pm \neq \epsilon_\kappa$ (generic case), we have

$$\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},a)} \int_{[\delta^\kappa \widetilde{(u), \delta^\pm}(u)]} \omega_\pm \equiv 0. \tag{2.23}$$

2. If $\epsilon_\pm = \epsilon_\kappa$ (resonant case), we have

$$\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} J_\pm(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},a)} \int_{\delta^\pm \widetilde{(u)(\delta^\kappa}(u))^{-1}} \omega_\pm \equiv 0. \tag{2.24}$$

On the other hand we have

1. If $\epsilon_\pm \neq \epsilon_\kappa$

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t}) = \mathcal{V}ar_{(s,1)} \int_{[\delta^\kappa \widetilde{(u), \delta^\pm}(u)]} \omega_\pm \equiv 0. \tag{2.25}$$

2. If $\epsilon_\pm = \epsilon_\kappa$

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} J_\pm(s,\tilde{t}) = \mathcal{V}ar_{(s,1)} \int_{\delta^\pm \widetilde{(u)(\delta^\kappa}(u))^{-1}} \omega_\pm \equiv 0. \tag{2.26}$$

**Second case**

We assume that the relative cycle $\delta_{\pm}(s,\tilde{t})$ going from $\Sigma_{div}$ to $\Sigma_{\kappa}$ inside the neighborhood $U_{\pm}$ near the separatrix $\delta_{\pm}$.

The relative cycle $\delta^{\pm}(s,\tilde{t})$ is defined by:

1. An initial condition (starting point) $(X^0,Z^0) = (s\tilde{t}^{\frac{1}{a}},\tilde{t}^{-\frac{1}{a}}) = \delta_{\pm}(s,\tilde{t}) \cap (\Sigma_{div} \setminus \{XZ=0\})$. Let $(X^1_{\kappa},Z^1) = (\tilde{t}^{\frac{1}{\epsilon_{\kappa}}},s) = (\Sigma_{\kappa} \setminus \{ZX_{\kappa}=0\})$ be the end point of $\delta_{\pm}(s,\tilde{t})$.

2. A loop $\delta_{\pm}(u) \subset C_{\pm}$ which is defined above and homotopic to the straight line segment $L = \big(\Pi_{\pm}(X^0,Z^0),\Pi_{\pm}(X^1,Z^1)\big)$.

We can write the function $J_{\pm}$ as follows

$$J_{\pm}(s,\tilde{t}) = \int_{\delta^{\pm}(s,\tilde{t})} \omega_{\pm} = \int_{\delta_{\pm}(u)} F_{\pm}\mathrm{d}X.$$

If we fix $s$, the function $J_{\pm}$ satisfies the following iterated variations with respect to $\tilde{t}$

$$\mathcal{V}ar_{(\tilde{t},\epsilon_{\pm})} \circ \mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_{\kappa})} J_{\pm}(s,\tilde{t}) = \mathcal{V}ar_{(\tilde{t},\epsilon_{\pm})} \int_{[\delta^{\kappa}\widetilde{(u),\delta^{\pm}}(u)]} \omega_{\pm} \equiv 0. \quad (2.27)$$

On the other hand $J_{\pm}$ satisfies the mixed iterated variations equation

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_{\kappa})} J_{\pm}(s,\tilde{t}) = \mathcal{V}ar_{(s,1)} \int_{[\delta^{\kappa}\widetilde{(u),\delta^{\pm}}(u)]} \omega_{\pm} \equiv 0. \quad (2.28)$$

### 2.6.6 Variation of the function $J_i$

In the chart $(U_i,(X,Y,Z))$ the blown-up one-form $\omega_i$ is given by

$$\omega_i = F_{i,1}\mathrm{d}X + F_{i,2}\mathrm{d}Y + F_{i,3}\mathrm{d}Z,$$

and the linear projection $\Pi_i(X,Y,Z)=X$ is everywhere transverse to the levels of the foliation $\sigma_1^*\widetilde{\mathcal{F}}$, which correspond simply to the graphs of the multivalued function

$$X \longmapsto \Phi_i(X) = (\Phi_{i,1}(X),\Phi_{i,2}(X)).$$

Let $(X_{i-1},Y,Z)$ and $(X_{i+1},Y,Z)$ be local holomorphic coordinates in which $p_{ii-1}=0, p_{ii+1}=0, \Sigma_{i-1}=\{Y=1\}$ and $\Sigma_{i+1}=\{Y=1\}$ respectively. As previously, we can characterize the relative cycle $\delta_i(s,\tilde{t})$ by :

1. An initial condition $(X^0_{i-1},Z^0) = (\tilde{t}^{\frac{1}{\epsilon_{i-1}}},s) = \delta_i(s,\tilde{t}) \cap (\Sigma_{i-1} \setminus \{ZX_{i-1}=0\})$. Let $(X^1_{i+1},Z^1) = (\tilde{t}^{\frac{1}{\epsilon_{i+1}}},s) = \delta_i(s,\tilde{t}) \cap (\Sigma_{i+1} \setminus \{ZX_{i+1}=0\})$.

2. A path $\delta_i(u) \subset C_i$ which is homotopic to a straight-line segment joining $\Pi_i(X^0_{i-1},Z^0)$ and $\Pi_i(X^1_{i+1},Z^1)$. The relative cycle $\delta_i(s,\tilde{t})$ is a lift of $\delta_i(u)$ to the Riemann surface $R_i$.

The function $J_i$ can be writen as follows

$$J_i(s,\tilde{t}) = \int_{\delta_i(s,\tilde{t})} \omega_i = \int_{\delta_i(u)} F_i \mathrm{d}X,$$

where

$$F_i = F_{i,1}\circ\Phi_i + F_{i,2}\circ\Phi_i\left(\frac{\partial\Phi_{i,1}}{\partial X}\right) + F_{i,3}\circ\Phi_i\left(\frac{\partial\Phi_{i,2}}{\partial X}\right).$$

By a similar computation of the variation as in [2], we have

1. If $\epsilon_{i-1}\neq\epsilon_{i+1}$, we have

$$\mathcal{V}ar_{(\tilde{t},\epsilon_{i-1})}\circ\mathcal{V}ar_{(\tilde{t},\epsilon_{i+1})}J_i(s,\tilde{t}) = \int_{[\delta_{i-1}\widetilde{(u),\delta_{i+1}}(u)]}\omega_i,$$

   where $[\delta_{i-1}\widetilde{(u),\delta_{i+1}}(u)]$ is a cycle obtained as a lift of the commutator $[\delta_{i-1}(u),\delta_{i+1}(u)]$, where $\delta_{i-1}(u)$ and $\delta_{i+1}(u)$ are paths in $C_i$ turning once counterclockwise around $p_{ii-1}$ and $p_{ii+1}$.

2. If $\epsilon_{i-1}=\epsilon_{i+1}$, we have

$$\mathcal{V}ar_{(\tilde{t},\epsilon_{i-1})}J_i(s,\tilde{t}) = \int_{\delta_{i-1}\widetilde{(u)(\delta_{i+1}}(u))^{-1}}\omega_i,$$

   where $\delta_{i-1}\widetilde{(u)(\delta_{i+1}}(u))^{-1}$ is a cycle obtained as a lift of the figure eight loop $\delta_{i-1}(u)(\delta_{i+1}(u))^{-1}\subset C_i$.

On the other hand the foliation $\sigma_1^*\widetilde{\mathcal{F}}$ is given by an analytic function on $Z$ i.e. $\mathcal{V}ar_{(s,1)}J_i(s,\tilde{t})=0$.

### 2.6.7 Conclusion

We conclude in this section by summarizing the analytic properties of $J^t_{div}$, $J^\infty_{div}$, $J^\infty_{div,\pm}$, $J_\pm$ and $J_i, i=3,\ldots,k$ studied above:

1. The function $J^t_{div}, t\in[0,+\infty[$ satisfies the following variations equations

   (a)
$$\mathcal{V}ar^2_{(s,1)}J^t_{div}(s,t)\equiv 0. \tag{2.29}$$

   (b) If $\epsilon_-\neq\epsilon_+$
$$\mathcal{V}ar_{(t,a)}\left(\mathcal{V}ar_{(t,-\epsilon_+)}\circ\mathcal{V}ar_{(t,-\epsilon_-)}J^t_{div}(s,t)\right)\equiv 0. \tag{2.30}$$

   (c) If $\epsilon_+=\epsilon_-$
$$\mathcal{V}ar_{(t,a)}\left(\mathcal{V}ar_{(t,-\epsilon_-)}J^t_{div}(s,t)\right)\equiv 0. \tag{2.31}$$

2. We assume that $\delta_\pm(s,\tilde{t})$ is going from $\Sigma_{div,\pm}$ to $\Sigma_\kappa$. The function $J_\pm$ satisfies the variation equations

   (a) If $\epsilon_\pm \neq \epsilon_\kappa$, we have

$$\mathcal{V}ar_{(s,1)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t})\right) \equiv 0, \tag{2.32}$$

$$\mathcal{V}ar_{(\tilde{t},-a)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t})\right) \equiv 0 \tag{2.33}$$

   (b) If $\epsilon_\pm = \epsilon_\kappa$, we have

$$\mathcal{V}ar_{(s,1)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} J_\pm(s,\tilde{t})\right) \equiv 0, \tag{2.34}$$

$$\mathcal{V}ar_{(\tilde{t},-a)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} J_\pm(s,\tilde{t})\right) \equiv 0. \tag{2.35}$$

3. We assume that $\delta_\pm(s,\tilde{t})$ is going from $\Sigma_{div}$ to $\Sigma_\kappa$. The function $J_\pm$ satisfies the variation equations

$$\mathcal{V}ar_{(s,1)}\left(\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t})\right) \equiv 0, \tag{2.36}$$

$$\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)}\left(\mathcal{V}ar_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_\kappa)} J_\pm(s,\tilde{t})\right) \equiv 0. \tag{2.37}$$

4. The function $J^\infty_{div}$ satisfies the following equations

   (a)

$$\mathcal{V}ar_{(s,1)} J^\infty_{div}(s,\tilde{t}) \equiv 0. \tag{2.38}$$

   (b) If $\epsilon_+ \neq \epsilon_-$, we have

$$\mathcal{V}ar_{(\tilde{t},\epsilon)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_+)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_-)} J^\infty_{div}(s,\tilde{t})\right) \equiv 0. \tag{2.39}$$

   (c) If $\epsilon_+ = \epsilon_-$, we have

$$\mathcal{V}ar_{(\tilde{t},\epsilon)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_+)} J^\infty_{div}(s,\tilde{t})\right) \equiv 0. \tag{2.40}$$

5. The function $J^\infty_{div,\pm}$ satisfies the following variation equations

   (a) If $\epsilon_\pm \neq \epsilon$, we have

$$\mathcal{V}ar_{(s,1)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^\infty_{div,\pm}(s,\tilde{t})\right) \equiv 0, \tag{2.41}$$

$$\mathcal{V}ar_{(\tilde{t},-a)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_\pm)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} J^\infty_{div,\pm}(s,\tilde{t})\right) \equiv 0. \tag{2.42}$$

(b) If $\epsilon_\pm = \epsilon$, we have

$$\mathcal{V}ar_{(s,1)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon)} J^\infty_{div,\pm}(s,\tilde{t})\right) \equiv 0, \tag{2.43}$$

$$\mathcal{V}ar_{(\tilde{t},-a)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon)} J^\infty_{div,\pm}(s,\tilde{t})\right) \equiv 0. \tag{2.44}$$

6. The function $J_i, i = 3, \ldots, k$, satisfies the following equations

(a)

$$\mathcal{V}ar_{(s,1)} J_i(s,\tilde{t}) \equiv 0. \tag{2.45}$$

(b) If $\epsilon_{i-1} \neq \epsilon_{i+1}$, then

$$\mathcal{V}ar_{(\tilde{t},i)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_{i-1})} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_{i+1})} J_i(s,\tilde{t})\right) \equiv 0. \tag{2.46}$$

(c) If $\epsilon_{i-1} = \epsilon_{i+1}$, then

$$\mathcal{V}ar_{(\tilde{t},\epsilon_i)}\left(\mathcal{V}ar_{(\tilde{t},\epsilon_{i+1})} J_i(s,\tilde{t})\right) \equiv 0. \tag{2.47}$$

## 2.7 Proof of Theorem 1

By blowing-up, the proof of the local boundedness of the number of zeros of the integral $I(\lambda, h) = \int_{\gamma(\lambda,h)} \Omega$ is reduced to proving the local boundedness of the number of zeros of the integral in the blown-up coordinates $J(s,t) = \int_{\delta(s,t)} \sigma_1^* \Omega$.

**Corollary 1.** *The $s$-variation of the integral $J(s,t)$ is an integral of the form $\sigma_1^*\Omega = \sigma_1^*\left(\frac{\eta}{M_\lambda}\right)$ along the figure eight loop*

$$\mathcal{V}ar_{(s,1)} J(s,t) = \int_{figure\ eight\ loop} \sigma_1^* \Omega.$$

*Proof.* 1. Let us fix $\tilde{t} = \frac{1}{t}$. The functions $J^\infty_{div,\pm}(s,\tilde{t})$ are a meromorphic in $s$, then we have

$$\mathcal{V}ar_{(s,1)} J^\infty_{div,\pm}(s,t) = \int_{\tilde{\gamma}_\pm} \omega^\infty_{div,\pm},$$

where $\tilde{\gamma}_\pm$ are a lift of a closed loops $\gamma_\pm \subset C^\infty_{div,\pm}$ which consist of line segments connecting $q_\pm$ with singular points $p_\pm$, encircling the latter along a small counterclockwise circular arc and then returning along the same segment in the opposite direction. On the other hand, the function $J^\infty_{div}(s,\tilde{t})$ is analytic in $s$ i.e.

$$\mathcal{V}ar_{(s,1)} J^\infty_{div}(s,\tilde{t}) = \int_{\widetilde{\ell\ell^{-1}}} \omega^\infty_{div} = 0,$$

where $\tilde{\ell}$ is the lifts of a segment $\ell \subset C^{\infty}_{div}$. Then we conclude that $\mathcal{V}ar_{(s,1)}J(s,t)$ is the integral $\sigma_1^*\Omega$ over the lift of the eight figure $\gamma_+\ell\gamma_-\ell^{-1}$ on a small neignborhood on the complex curve $\{X_1 = 0, G = t\}, t \in [M, +\infty]$.

2. For $t \in [0, 2M]$ fixed, we have the function $\mathcal{V}ar_{(s,1)}J_{div}(s,t) = \int_{\text{figure eight loop}} \omega_{div}$. ☐

**Proposition 5.** *The function $s \mapsto \mathcal{V}ar_{(s,1)}J(s,t)$ is $O(s^{\mu})$ uniformly in $t$, for some constant $\mu > 0$.*

*Proof.* As $\eta$ vanishes to the order $\geq 4$ at $(x,y) = (0,0)$ we have $\sigma_1^*\Omega = \sigma_1^*\left(\frac{\eta}{M_\lambda}\right)$ is $O(X_1)$. We conclude that, for all closed paths of finit length contained in sufficiently small neighborhood of the exceptional divisor $\{X_1 = 0\}$. Since $\mathcal{V}ar_{(s,1)}J(s,t)$ is the integral of $\sigma_1^*\Omega$ over the lift of the eight figure on $\{X_1 = 0, G = t\}, t \in [0, +\infty]$. We conclude that $X_1 = O(s)$ in this lift. ☐

**Lemma 2.** *The function $J(s,t)$ satisfies the following variation equations*

1. *For $\delta(s,t) \subset (V^{\star}_{div} \cup V_+ \cup V_- \cup V_3 \cup \ldots \cup V_k), \star \in \{0, t_0\}$, we have*

$$\mathcal{V}ar_{(t,a)} \circ \mathcal{V}ar_{(t,-\epsilon_-)} \circ \mathcal{V}ar_{(t,-\epsilon_+)} \circ \mathcal{V}ar_{(t,-\epsilon_3)} \circ \ldots \circ \mathcal{V}ar_{(t,-\epsilon_k)} J(s,t) \equiv 0, \tag{2.48}$$

2. *For $\delta(s,t) \subset \left(V^{\infty}_{div} \cup V^{\infty}_{div,+} \cup V^{\infty}_{div,-} \cup V_+ \cup V_- \cup V_3 \cup \ldots \cup V_k\right)$, we have*

$$\mathcal{V}ar^{\circ 2}_{(\tilde{t},-a)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_+)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_-)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon)} \circ \mathcal{V}ar_{(\tilde{t},\epsilon_3)} \circ \ldots \circ \mathcal{V}ar_{(\tilde{t},\epsilon_k)} J(s,t) = 0. \tag{2.49}$$

3. *If $t$ is fixed, we have*

$$\mathcal{V}ar^{\circ 2}_{(s,1)} J(s,t) = 0. \tag{2.50}$$

*Proof.* 1. We obtain equations (2.48) and (2.49) by using Theorem 1.3 [2].

2. The equation (2.50) is a consequence of Proposition 5.

☐

Let $\Lambda$ be the parameters space it is formed of coefficients of the polynomials $P_i, R, S$, exponents $\epsilon_i$ and degrees $n_i = \deg P_i, n = \max(\deg R, \deg S)$. Consider the following finite-dimensional functional space $\mathcal{P}$

$$\mathcal{P}(v, V; \alpha_1, \ldots, \alpha_{k+1}) = \left\{ \sum_{j=1}^{k} \sum_{n,l} c_{jln}(s) t^{\alpha_j n} \log^n(t) : c_{jln} \in \mathbb{C}, v \leq \alpha_j n \leq V, 0 \leq l \leq k \right\}.$$

As a consequence of the equation $\mathcal{V}ar^{\circ 2}_{(s,1)} J(s,t) = 0$, we can write the function $J(s,t)$ as follows

$$J(s,t) = J_1(s,t) + \log s J_2(s,t).$$

By Proposition 5, we have $J_2(s,t) = \mathcal{V}ar_{(s,1)}J(s,t)$ is $O(s^\mu), \mu > 0$.

**Lemma 3.** *The functions $J_1(s,.), J_2(s,.)$ are two meromorphic families in $s$ and satisfy following variation equation with respect to $t$*

$$\mathcal{V}ar_{(t,\alpha_1)} \circ \ldots \circ \mathcal{V}ar_{(t,\alpha_{k+1})} J_i(s,t) = 0.$$

*Then, there exists a family of meromorphic functions $P_1(s,.), P_2(s,.)$ in $\mathcal{P}(\ldots)$ such that $|t|^{-M}|J_i(s,t) - P_i(s,t)| \overset{t\to 0}{\longrightarrow} 0$ uniformly in $s, i = 1,2$ and $J_2(s,t) - P_2(s,t) = O(s^\mu), \mu > 0$ uniformly in $t$ and $(J_2(s,t) - P_2(s,t))\log s = O(s^\mu \log s)$. Moreover $J(s,t) \neq 0$. Then for sufficiently big $V$: $P_1(s,t) + P_2(s,t)\log s \neq 0$.*

**Theorem 2.** *For $s$ sufficiently small and $t \in [0, +\infty]$, the number of zeros $\#\{t : J(s,t) = 0\}$ is locally bounded.*

*Proof.* Let $C_R = \{|t| = R, |\arg t| \leq \alpha\pi\}, C_\pm = \{r < |t| < R, |\arg t| = \pm\alpha\pi\}$ and $C_r := \{|t| = r, |\arg t| \leq \alpha\pi\}$. To count the number of zeros of $J(s,t)$ in the sector $C_{r,R} = C_R \cup C_r \cup C_\pm$ apply Petrov's method which gives us

$$\#Z(J(s,t)|_{C_{r,R}}) \leq \frac{1}{2\pi}\Big(\Delta \arg_{C_R} J(s,t) + \Delta \arg_{C_r} J(s,t) + \Delta \arg_{C_\pm} J(s,t)\Big).$$

1. The increment of argument $\Delta \arg_{C_R} J(s,t)$ of $J(s,t)$ on the counterclockwise arc $C_R$ is uniformly bounded from above by Gabrielov's theorem [7].
2. The increment of argument $\Delta \arg_{C_\pm} J(s,t)$ along the segment $C_\pm$ of $J(s,t)$ is bounded from above by the number of zeros of $\mathcal{V}ar_\alpha J(s,t)$. On the other hand, using the $t$-variation equation (or $\tilde{t}$-variation equation)

$$\mathcal{V}ar_{(t,\alpha)} \circ \mathcal{V}ar_{(t,\alpha_1)} \circ \ldots \circ \mathcal{V}ar_{(t,\alpha_k)} J(s,t) = 0,$$

near the ramification point $t = 0$ (or $\tilde{t} = 0$), the function $\mathcal{V}ar_{(t,\alpha)}J(s,t)$ has the form

$$\begin{aligned}\mathcal{V}ar_{(t,\alpha)}J(s,t) &= \mathcal{V}ar_{(t,\alpha)}\left(J_1(s,t) + J_2(s,t)\log s\right)\\ &= F(e^{\frac{\alpha_1}{\alpha}\log t}, \ldots, e^{\frac{\alpha_k}{\alpha}\log t}, e^{\log s}) = G(s,t)\end{aligned}$$

where $F$ is a meromorphic function. The function $G$ is a logarithmico-analytic of type 1 in the variable $s$ -see [10]. Lion-Rolin's theorem [10] allows to write

$$G(s,t) = y_0^{q_0} y_1^{q_1} G(t) U(t, y_0, y_0)$$

with $y_0 = s - \theta_0(t), y_1 = \log y_0 - \theta_1(t)$, where $\theta_0, \theta_1, G$ are logarithmico-expenential functions and $U$ is a logarithmico-expenential unity function. As the number of zeros of a logarithmico-exponential function is bounded, the number of zeros $\#Z\left(G(s,t)\right)$ is bounded.

3. To estimate the limit of the increment of argument $\Delta \arg_{C_r} J(s,t)$ along the small arc $C_r$ $\lim_{r\to 0} \Delta \arg_{C_r} J(s,t)$, we investigate the leading term of $J(s,t)$ at $t=0$. By Lemma 3 we have $J_1(s,t) + J_2(s,t)\log s - (P_1(s,t) + P_2(s,t)\log s)$ is $O(t^M)$ uniformly in $s$. For each $\beta \in \Lambda$, we can choose the leading term $P$ of $P_1(s,t) + P_2(s,t)\log s$. The increment argument of $P$ is bounded.

□

## 2.8 Open problems

In this section we propose some open problems

1. Elimination of the technical condition that $\eta$ vanishes to order $\geq 4$ at $(x,y) = (0,0)$.

2. In the spirit to use blowing-up in families, we consider unfolding of non-generic singularities case. Let us be more precise. We consider a function of the form $H_0 = \prod_{i=1}^k P_i^{a_i}(x,y,0)$, which are Darboux first integrals of the analytic system $\omega_0 = M_0 \frac{\mathrm{d}H_0}{H_0} = 0$, with $P_i \in \mathbb{C}[x,y,\lambda]$ analytic functions, $a_i \in \mathbb{C}$ and $M_0 = \prod_{i=1}^k P_i(x,y,0)$. We assume that the polycycle has only nongeneric singularities.

   Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n, |\lambda_i| \leq \epsilon_i, \epsilon_i$ be sufficiently small. Consider an unfolding $\omega_\lambda = M_\lambda \frac{\mathrm{d}H_\lambda}{H_\lambda}$, of the form $\omega_0$, where $\omega_\lambda$ are one-forms with the Darboux first integral

$$H_\lambda := H = \prod_{i=1}^k P_i^{a_i}(x,y,\lambda). \tag{2.51}$$

   The foliation of codimension one in $n+2$ dimensional space $\omega_\lambda = 0$ has a maximal nest of cycle $\delta(\lambda,h) \subseteq \{H_\lambda = h\}$ filling a connected component of $\mathbb{R}^2 \setminus \{M_\lambda = 0\}$ will be denoted $D_{(\lambda,h)}$ and $\partial D_{(\lambda,h)}$ its border. Consider the polynomial deformation of the system $\omega_\lambda$

$$\omega_\lambda + \varepsilon\eta = 0, \quad \varepsilon > 0 \tag{2.52}$$

   where $\eta = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ be a polynomial form. Consider pseudo-Abelian integrals

$$I(h,\lambda) = \int_{\delta(\lambda,\lambda)} M_\lambda^{-1}\eta, \quad M_\lambda = \prod_{i=1}^k P_i(x,y,\lambda) \tag{2.53}$$

   which appears as the linear term with respect to $\varepsilon$ of the displacement function of the polynomial deformation $\omega_\lambda + \varepsilon\eta$.

   *Question:* We want to prove, under some generic conditions, local uniform boundedness of the number of isolated zeros of $I(h,\lambda)$, for $h \in (0, h_0(\lambda))$. Here the bound is locally uniform with respect to all parameters: the coefficients of the polynomials, exponents and $\lambda$.

*Acknowledgments* I am grateful to my P.hD advisor Pavao Mardešić who interested me in the subject and for useful discussions and hints. I would like also to thank my co-advisor professor Daniel Panazzolo (université Haute-Alsace Mulhouse) for fruitful discussions.

# Bibliography


[1] Bobieński, Marcin Pseudo-Abelian integrals along Darboux cycles *a codimension one case. J. Differential Equations 246 (2009), no. 3, 1264-1273.*

[2] Bobieński, Marcin; Mardešić, Pavao *Pseudo-Abelian integrals along Darboux cycles. Proc. Lond. Math. Soc. (3) 97 (2008), no. 3, 669-688.*

[3] Bobieński, Marcin; Mardešić, Pavao; Novikov, Dmitry *Pseudo-Abelian integrals: unfolding generic exponential. J. Differential Equations 247 (2009), no. 12, 3357-3376.*

[4] Bobieński, Marcin; Mardešić, Pavao; Novikov, Dmitry *Pseudo-Abelian integrals on slow-fast Darboux systems. Ann. Inst. Fourier (Grenoble), 2013, to appear.*

[5] Binyamini, Gal; Novikov, Dmitry; Yakovenko, Sergei On the number of zeros of Abelian integrals. Invent. Math. 181 (2010), no. 2, 227-289.

[6] Denkowska, Zofia; Roussarie, Robert *A method of desingularization for analytic two-dimensional vector field families. Bol. Soc. Brasil. Mat. (N.S.) 22 (1991), no. 1, 93-126.*

[7] Dumortier, Freddy; Roussarie, Robert; Sotomayor, Jorge *Bifurcations of cuspidal loops. Nonlinearity 10 (1997), no. 6, 1369-1408.*

[8] Gabrièlov, A. M. *Projections of semianalytic sets. (Russian) Funkcional. Anal. i Priložen. 2 1968 no. 4, 18-30.*

[9] Khovanskii, A. G. *Real analytic manifolds with the property of finiteness, and complex abelian integrals. (Russian) Funktsional. Anal. i Prilozhen. 18 (1984), no. 2, 40-50.*

[10] Khovanskii, A. G. *Fewnomials. Translations of Mathematical Monographs, 88. American Mathematical Society, Providence, RI, 1991. 139 pp.*

[11] Novikov, Dmitry *On limit cycles appearing by polynomial perturbation of Darbouxian integrable systems. Geom. Funct. Anal. 18 (2009), no. 5, 1750-1773.*

[12] Novikov, D.; Yakovenko, S. *Simple exponential estimate for the number of real zeros of complete Abelian integrals. Ann. Inst. Fourier (Grenoble) 45 (1995), no. 4, 897-927.*

[13] Petrov, G. S. *On the nonoscillation of elliptic integrals. (Russian) Funktsional. Anal. i Prilozhen. 31 (1997), no. 4, 47–51, 95; translation in Funct. Anal. Appl. 31 (1997), no. 4, 262-265 (1998).*

[14] Roussarie, Robert *Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998. 204 pp.*

[15] Varchenko, A. N. *Estimation of the number of zeros of an abelian integral depending on a parameter, and limit cycles. (Russian) Funktsional. Anal. i Prilozhen. 18 (1984), no. 2, 14-25.*