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Par

M. CARNEVALE Joao

Groups acting on the line and the circle with at most N fixed points

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ECOLE DOCTORALE
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Titre: Action de groupes sur la droite et le cercle avec au plus N points fixes.

Mots clés: action de groupe, groupe projectif linéaire, Théorème de Hölder, Théorème de Solodov, groupes de convergence.

Résume: Un thème classique dans les systèmes dynamiques est que la première information fondamentale provient de la compréhension des orbites périodiques. Lorsque l'on étudie les actions de groupe, cela signifie que l'on veut comprendre les points fixes des éléments du groupe, et une question naturelle qui en ressort est: Quels groupes d'homéomorphismes peuvent agir sur une variété de dimension 1 ayant tous les éléments non triviaux avec au plus de N points fixes? Notre objectif principal dans ce travail est d'aborder cette question et de comprendre quelles propriétés une telle hypothèse dynamique peut induire sur le groupe.

Pour le cas N=0, un résultat classique de O. Hölder implique qu'un tel groupe d'homéomorphismes agissant sur la droite est toujours semi-conjugué à un sous-groupe de translations et qu'un tel groupe d'homéomorphismes agissant sur le cercle est toujours semi-conjugué à un sous-groupe de rotations. Pour N>0 il y a deux exemples classiques pour cette question: l'action du groupe affine sur la droite, avec N=1, et l'action du groupe linéaire projectif sur le cercle, avec N=2.

Un résultat de V. V. Solodov montre qu'il y a une classification similaire pour les actions de groupe sur la droite: si N=1, le groupe est ou bien abélien ou bien semi-conjugué à un sous-groupe du groupe affine. Par contre, N. Kovačević; a présenté de nouveaux exemples d'actions de groupe sur le cercle avec N=2 qui ne sont semi-conjugués à aucun sous-groupe du groupe linéaire projectif, ce qui prouve qu'une affirmation similaire n'est pas vrai pour les actions de groupe sur le cercle.

Dans ce travail, nous montrons que le résultat de Solodov est valable même pour N=2. De plus, sous l'hypothèse additionnelle de non-discrétion, il y a une classification similaire pour les actions de groupe sur le cercle avec N=2. De plus, inspirés par certaines des idées de Kovačević;, nous avons introduit le concept de produit amalgamé d'actions du cercle en considérant le blow-up de deux actions de groupes distincts et en les réarrangeant de sorte que l'ensemble invariant minimal d'une action de groupe soit inclus dans le complément de l'ensemble invariant minimal de l'autre. Ce concept s'avère être un excellent outil pour créer de nouveaux exemples d'actions de groupe sur le cercle qui ne sont semi-conjuguées à aucun sous-groupe du groupe linéaire projectif, et telles que chaque élément non trivial a au plus N points fixes. Il conduit également à la construction d'une deuxième famille d'exemples d'actions de groupe où tout élément non trivial a au plus N points fixes, qui sont des extensions HNN d'actions.

Enfin, nous présentons des exemples de haute régularité, qui ne peuvent être obtenus directement par le produit amalgamé d'actions, de groupes de type fini de difféomorphismes du cercle où tout élément non trivial fixe au plus 2 points et qui ne sont pas semi-conjugués (et même pas isomorphe) à n'importe quel sous-groupe du groupe linéaire projectif. Par conséquent, nous pouvons conclure que la seule augmentation de la régularité ne nous donne pas un théorème de classification.

ECOLE DOCTORALE CARNOT - PASTEUR

Title: Groups acting on the line and the circle with at most N fixed points.

Keywords: group action, projective linear group, Holder's Theorem, Solodov's Theorem, convergence groups.

Abstract: A classical theme in dynamical systems is that the first fundamental information comes from the understanding of periodic orbits. When studying group actions, this means that we want to understand the fixed points of elements of the group, and a natural question that emerges from that is: Which groups of homeomorphisms can act on a 1-manifold having all non-trivial elements with at most N fixed points? Our main objective in this work is to approach that question and understand what properties can such dynamical hypothesis induces to the group.

For the case $N = 0$, a classical result from O. Hölder implies that such group of homeomorphisms acting on the line is always semi-conjugate to a subgroup of translations and that such group of homeomorphisms acting on the circle is always is semi-conjugate to a subgroup of rotations. Now, for $N > 0$ there are two classical examples for that question, the action of the affine group on the line, with N=1, and the action of the protective linear group on the circle, with N=2, and if by one hand a result from V. V. Solodov shows that we have a similar classification for group actions on the line which states that if $N = 1$ then the group is either elementary or semi-conjugate to a subgroup of the affine group, by the other hand, N. Kovačević presented new examples of group actions on the circle with $N = 2$ which are not semi-conjugate to any subgroup of the protective linear group, which proves that a similar statement doesn't hold for group actions on the circle. In this work we show that Solodov's result holds even for $N = 2$ and that once included the hypothesis of non-discreteness a similar classification also holds for group action on the circle with $N = 2$. Moreover, inspired by some of the ideas of Kovačević we introduced the concept of amalgamated product of actions of the circle by considering the blow-up of two distinct groups actions and rearranging them so that the minimal invariant set of one group action is included in the complement of the minimal invariant set of the other. This concept proves to be a great tool to create new examples of group actions on the circle which are not semi-conjugate to any subgroup of the protective linear group, and such that every non-trivial element has at most N fixed points, and it also leads to the construction of a second family of examples of group actions where every non-trivial element has at most N fixed points, which are HNN-extensions of actions.

Finally, we present examples with high regularity, that cannot be obtained directly by the amalgamated product of actions, of finitely generated groups of diffeomorphisms of the circle where every non-trivial element fixes at most 2 points and which are not semi-conjugate (and even not isomorphic) to any subgroup of the protective linear group. Therefore, we can conclude that only increase the regularity doesn't give us a classification theorem.

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Contents

1 Introduction

1.1 General setting and classical results

A classical theme in dynamical systems is that the first fundamental information comes from the understanding of periodic orbits. When studying group actions, this means that we want to understand the fixed points of elements of the group.

Definition 1.1. Let X be a topological space, and $G \leq \mathrm{Homeo}(X)$ a subgroup of homeomorphisms of X. Given $N \in \mathbb{N}$, we say that G *has at most N fixed points* if every non-trivial element of G has at most N fixed points. When $N = 0$, we simply say that the action of G on X is *free*.

The main purpose of this paper is to study the following question.

Question 1.2. *Given $N \in \mathbb{N}$, which subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ have at most N fixed points?*

Here $\mathrm{Homeo}_+(\mathbb{S}^1)$ denotes the group of orientation-preserving circle homeomorphisms. In this work, we will always assume that homeomorphisms do preserve the orientation, without explicitly saying this.

Let us first discuss Question 1.2 for the case of actions on the real line, and for (very) small values of N. In this case, we have two examples of geometric nature:

- the group of translations $\mathrm{Isom}_+(\mathbb{R})$, which acts freely on $\mathbb{R}$ (i.e. it has at most 0 fixed points);
- the group of affine transformations $\mathrm{Aff}_+(\mathbb{R})$, which has at most 1 fixed point.

For $N = 0$, the answer to Question 1.2 goes back to Hölder [13] (see also [11, Theorem 6.10] for a more recent exposition), who proved that any free action on the real line can be reduced to an action by translations.

Theorem 1.3 (Hölder)**.** *Every subgroup $G \leq \mathrm{Homeo}_+(\mathbb{R})$ acting freely on $\mathbb{R}$ is abelian and moreover its action is semi-conjugate to an action by translations.*

See Chapter 2 for the notion of semi-conjugacy. It is natural to ask whether actions with at most 1 fixed point are always semi-conjugate to actions by affine transformations. The answer, by Solodov [23] (see also the discussion in Ghys [11, Theorem 6.12], and an alternative proof by Kovačević in [16]), says that this is almost the case.

Theorem 1.4 (Solodov)**.** *Let $G \leq \mathrm{Homeo}_+(\mathbb{R})$ be a subgroup with at most 1 fixed point. Then*

- *either the action of G admits a unique fixed point and G is abelian, or*
- *the actions of G is semi-conjugate to an action by affine transformations.*

In Appendix A.2, we obtain a natural extension to the first case which is not covered by the previous theorems, that is, when $N = 2$.

Theorem A. *Consider a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{R})$ with at most 2 fixed points. Then we have two possibilities:*

- *G is abelian, and every fixed point of a non-trivial element is a global fixed point of G, or*
- *the action of G is semi-conjugate to an action by affine transformations.*

In other words, there is no new interesting group with at most two fixed points. In fact, the group of affine transformations $\mathrm{Aff}_+(\mathbb{R})$ is the only known example of group with at most N fixed points, for any $N \geq 1$, which acts minimally on the real line. This suggests the following conjecture.

Conjecture 1.5. *Let $N \in \mathbb{N}$ and let $G \leq \mathrm{Homeo}_+(\mathbb{R})$ be a subgroup with at most N fixed points. Let $I \subset \mathbb{R}$ be a maximal interval without global fixed points for G. Then the restriction of the action of G to I is semi-conjugate to the action by affine transformations.*

Remark 1.6. The above conjecture holds true under higher regularity assumptions. Indeed, a result by Akhmedov [1] and [2] gives that any subgroup of $\mathrm{Diff}^r_+([0,1])$, with $r > 1$, with at most N fixed points is solvable. On the other hand, a classical result of Plante [22] gives that if a solvable group acts on the line such that the set of fixed point of any element acting non-trivial is discrete, then the action is semi-conjugate to an affine action. Using results from Bonatti, Monteverde, Navas, and Rivas [5], it is not difficult to conclude that if the affine action is non-abelian, then the semi-conjugacy is actually a conjugacy.

For actions on the circle, the canonical example of group acting freely is the group of rigid rotations $\mathrm{SO}(2)$. Somehow, this is the only example, after the following result which is a direct consequence of Theorem 1.3 (see Ghys [11, Theorem 6.10]).

Theorem 1.7. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup whose action on the circle is free. Then G is abelian and semi-conjugate to a group of rotations.*

It is not difficult to see that if a group G acts on the circle with at most 1 fixed point, then this must be a global fixed point and then the action reduces to a free action on the real line. There is a classical group acting with at most 2 fixed points: the Möbius group $\mathrm{PSL}(2,\mathbb{R})$. This group may be seen in two ways:

- it is the group of projective transformations on the projective line $\mathbb{RP}^1$;
- it is also the group of isometries of the hyperbolic disc $\mathbb{D}$ on its circle at infinity.

In the following, by *Möbius action*, we will mean the standard action of a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which will be described in Section 2.4. Note that lifting $\mathrm{PSL}(2,\mathbb{R})$ to the N-fold cover of the circle provides an example of group with at most $2N$ fixed points. We denote this lift by $\mathrm{PSL}^{(N)}(2,\mathbb{R})$.

Question 1.2 for $N = 2$ consists in deciding whether groups with at most 2 fixed points are in some sense comparable to subgroups of $\mathrm{PSL}(2,\mathbb{R})$. For this, we say that a group G is *Möbius-Like* if every element is individually conjugate into $\mathrm{PSL}(2,\mathbb{R})$. In practice, this means that any element in G is either periodic, or admits exactly one fixed point, or admits exactly two fixed points, one attracting and one repelling.

Remark 1.8. The cyclic group generated by a homeomorphism of $\mathbb{S}^1$ with two parabolic fixed points is never conjugate into $\mathrm{PSL}(2,\mathbb{R})$, although it admits at most 2 fixed points.

Many works tried to prove that Möbius-Like groups are indeed (semi)-conjugate to subgroups of $\mathrm{PSL}(2,\mathbb{R})$ in particular through the notion of *convergence groups* (detailed in Section 2.4). However, in the opposite direction, Kovačević [17] showed that being Möbius-Like is not enough to determine the conjugacy class of subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

Theorem 1.9 (Kovačević)**.** *There is a finitely presented Möbius-Like subgroup $K \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ whose action is minimal (every orbit is dense) but not conjugate into $\mathrm{PSL}(2,\mathbb{R})$.*

The main results of our work are inspired by the construction of Kovačević. They can be separated in two families of results of different nature.

1.2 Results on dynamical characterization of subgroups of $\mathrm{PSL}(2,\mathbb{R})$

With our first results, we want to understand which additional conditions allow to conclude that a Möbius-Like subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Following this principle, Kovačević showed in [16] that a Möbius-Like subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with a global fixed point is always conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and for the first statement, we will generalize this result to *elementary* subgroups, that is, subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ preserving a Borel probability measure on $\mathbb{S}^1$.

Theorem B. *If $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is a finitely generated elementary subgroup, whose action is Möbius-Like, then G is semi-conjugate to an elementary subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and, moreover, the corresponding morphism $G \to \mathrm{PSL}(2,\mathbb{R})$ is injective.*

For the second statement, we will assume the following topological condition.

Definition 1.10. A non-elementary subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ of circle homeomorphisms is *locally discrete* if for every interval $I \subset \mathbb{S}^1$ which intersects the minimal invariant subset, the identity is isolated among the subset of restrictions $\{g|_I : g \in G\} \subset C^0(I;\mathbb{S}^1)$, with respect to the C^0 topology.

Equivalently, we say that G is *non-locally discrete* if there exists a non-wandering interval $I \subset \mathbb{S}^1$ such that the action of G restricted to I is non-discrete.

Theorem C. *Let G be a non-elementary subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, which is non-locally discrete. Then G is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.*

Question 1.11. *Let G be a non-elementary subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most N fixed points, whose action is non-locally discrete. Is it true that G is conjugate to a subgroup of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$, for some $k \in \mathbb{N}$?*

1.3 A more structured theory of examples of groups with at most N fixed points

The next series of results goes in the direction of giving a structured theory of examples as those by Kovačević. Let us briefly sketch here how these examples are obtained. Let F be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and $x, y \in \mathbb{S}^1$ two points with disjoint and free orbits. Denote by $\widetilde{F}$ the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ given by a blow-up of the action of F on the orbits of x and y including respectively the intervals I and J (see Definition 2.14 for a formal definition of blow-up). Now, let $g \in \mathrm{PSL}(2,\mathbb{R})$ be a Möbius transformation mapping I to $\mathbb{S}^1 \smallsetminus \mathrm{int}(J)$ and define K to be the subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ generated by F and g. From the choices of $\widetilde{F}$ and g one can notice that, for every $k \in \mathbb{Z} \smallsetminus \{0\}$ and every $f \in \widetilde{F} \smallsetminus \{\mathrm{id}\}$ it follows that

$$f(\mathrm{int}(I \cup J)) \subset \mathbb{S}^1 \smallsetminus (I \cup J) \quad \text{and} \quad g^k(\mathbb{S}^1 \smallsetminus (I \cup J)) \subset \mathrm{int}(I \cup J).$$

Therefore, the subgroup K is a free amalgamated product of $\widetilde{F}$ and $\langle g \rangle$ and as we show in Lemma 4.4, its action can always be semi-conjugate to a minimal action. Let us denote by $\widetilde{K}$ the resulting group acting minimally. In [17], Kovačević proceeds by proving that every element in $\widetilde{K}$ can be conjugate to a Möbius transformation, moreover every element generated by the product (which is not conjugate to any element of $\widetilde{F}$ or $\langle g \rangle$) is actually hyperbolic, being conjugate to a Möbius transformation with one fixed point in $I \cup J$ and the other fixed point in the complement of $I \cup J$. We will show in Lemma 5.3 that is always the case when the subgroup admits a proper ping-pong partition with non-crossing intervals (in Kovačević example, the two intervals in $\mathbb{S}^1 \smallsetminus (I \cup J)$ are non-crossing for the action of $\langle g \rangle$). A sufficient condition to have the subgroup $\widetilde{K}$ not conjugate to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$, is to choose F non-discrete. In that case, there is a sequence of elements in $\widetilde{F}$ that converges to a non-decreasing function which is constant on every interval of the orbits of I and J, hence it does not satisfies the criterion for convergence groups (see 2.16).

The more general result given by Lemma 5.3 lead us to the (oral) conjecture by Bonatti, suggesting that the only way for a minimal subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points not to be conjugate to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is by having an amalgamated product of subgroups with at most 2 fixed points each.

Conjecture 1.12 (Bonatti)**.** *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup with at most 2 fixed points and whose action is minimal. Assume that the action of G is not semi-conjugate to a Möbius action. Then G is the amalgamated product over an abelian subgroup.*

The spirit of the conjecture is that the action of H should split as an amalgamated product of groups G and F admitting some wandering intervals and each of F and G acts, roughly speaking, in the orbit of the wandering intervals of the other.

Let us also remark that the groups built by Kovačević are however isomorphic to subgroups of $\mathrm{PSL}(2,\mathbb{R})$. This suggests the following problem.

Question 1.13. *Is there a Möbius-Like subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ which is not isomorphic to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$?*

1.4 Mechanisms for building group actions on $\mathbb{S}^1$ with at most N fixed points

The constructions of Kovačević are obtained by taking the blow-up of two distinct orbits in a group action on the circle, and adding to the group a hyperbolic element that maps the complement of an open interval into an open interval of the other orbit, this way the group generated will be isomorphic to the free product of the previous group with $\mathbb{Z}$. Inspired by this example, in Chapter 4 we consider the blow-up of two group actions on the circle, so that the resulting minimal invariant set for one group action is contained in the complement of the minimal invariant set for the other, so that the generated group is isomorphic to a (amalgamated) free product of the two previous groups. In Definition 4.5 we formalize this construction, by introducing the *amalgamated product* of two subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$, denoted by $(F,\overline{x}) \star_{\theta,\sigma} (G,\overline{y})$ (here $\overline{x}$, $\overline{y}$, θ, and σ are for keeping track of required additional choices). Indeed, in this notation, the example of Kovačević is described by $(F,\overline{x}) \star_{\mathrm{id},\mathrm{id}} (\mathbb{Z},\overline{y})$. After, with Theorem 4.7 we determine conditions which guarantee that an amalgamated product is well defined, and also unique up to conjugacy. Then we explore when an amalgamated product produces an example of group acting with at most 2 (or N) fixed points and also when it is Möbius-Like. In particular, the free product of any two given subgroups with at most 2 fixed points also acts with at most 2 fixed points (see Corollary 6.3).

Theorem D. *Consider two countable subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most $2n$ fixed points, and two collections of n points $\overline{x} = \{x_1, \dots, x_n\} \subset \mathbb{S}^1$ and $\overline{y} = \{y_1, \dots, y_n\} \subset \mathbb{S}^1$ such that, for all distinct $i, j \in \{1, \dots, n\}$, the following conditions are satisfied:*

- *$x_i \notin F.x_j$ and $y_i \notin G.y_j$,*
- *the stabilizers $\mathrm{Stab}(F, x_i) = S_F$ and $\mathrm{Stab}(G, y_i) = S_G$ are abstractly isomorphic, with isomorphism $\theta : S_F \xrightarrow{\sim} S_G$,*
- *$\mathrm{Fix}(s_f) = \{x_1, \dots, x_n\}$ and $\mathrm{Fix}(s_g) = \{y_1, \dots, y_n\}$, for all $s_f \in S_F$ and all $s_g \in S_G$,*
- *$S_F \subsetneq F$, $S_G \subsetneq G$ and at least one of the indexes $[F : S_F]$ and $[G : S_G]$ is greater than 2.*

Then, the amalgamated product $(F,\overline{x}) \star_{\theta,\sigma} (G,\overline{y})$ *has at most* $2n$ *fixed points (here* σ *is any order-preserving permutation of* n *elements).*

With Theorem 5.4 we will see that if additionally we require the one of the starting subgroups F and G is non-discrete in $\mathrm{Homeo}_+(\mathbb{S}^1)$, then the resulting amalgamated product $(F,\overline{x}) \star_{\theta,\sigma} (G,\overline{y})$ will not be conjugate to any subgroup of any lift $\mathrm{PSL}^{(N)}(2,\mathbb{R})$, thus producing systematically the desired examples of groups with this property.

In the case of subgroups with at most two fixed points, we will give with Theorems 6.1 and 6.2 two more precise results. Particularly, Theorem 6.2 is very similar to the situation considered by Kovačević. In Theorem 6.18 we will describe a different construction using HNN extensions of actions instead of amalgamated products. As we will show, these examples can still be seen as very particular amalgamated product, thus they do not give a solution for the Conjecture 1.12. However, for the goal of characterize the examples with at most 2 fixed points, the HNN extensions will be considered as a distinct example from the ordinary amalgamated product, given its particularities.

The very general procedures above are then used to give very specific examples of groups with remarkable properties.

Theorem E. *There exists a finitely generated group of real-analytic circle diffeomorphisms, acting minimally and with at most* 2 *fixed points, whose action is not semi-conjugate into* $\mathrm{PSL}(2,\mathbb{R})$.

Theorem F. *There exists a finitely generated group of smooth (*C^∞*) circle diffeomorphisms, with at most 2 fixed points, and which is not isomorphic to any subgroup of* $\mathrm{PSL}(2,\mathbb{R})$.

Let us however point out that the examples obtained in the last two theorems are not Möbius-Like and we remark that Navas in [21] has presented a contruction of a group of real-analytic circle diffeomorphisms which is Möbius-Like and whose action is not semi-conjugate into $\mathrm{PSL}(2,\mathbb{R})$, but it is not finitely generated. This suggests the following.

Question 1.14. *Is there any finitely generated group of real-analytic circle diffeomorphisms which is Möbius-Like but not semi-conjugate into* $\mathrm{PSL}(2,\mathbb{R})$*?*

Question 1.15. *Is there any finitely generated group of circle homeomorphisms which is Möbius-Like but not isomorphic to any subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*?*

1.5 Structure of this work

In Chapter 2, we start by recalling fundamental facts and terminology for groups acting on one-manifolds. At the end of the chapter we will prove our first main result, namely Theorem B, about elementary Möbius-Like groups. In Chapter 3, we will discuss results on non-locally discrete subgroups of circle homeomorphisms with at most N fixed points. The main result discussed here is Theorem C. The notion of amalgamated product of actions on the circle is discussed in Section 4. Then we move in Chapter 5 to the main technical results which give bound on the number of fixed points when making an amalgamated product of actions, and thus presenting a structured theory for the amalgamated product of group actions. In Chapters 6 and 7 we use the technical results to produce examples of groups actions with new remarkable properties (notably Theorems E and F). Finally, in Appendix A, we present the results on group actions on the line, which we have obtained in our Master thesis.

2 Preliminaries

2.1 Group actions: definitions and first properties

In this work, we will limit ourselves to countable groups, usually finitely generated and finitely presented. Now, an action of a group G on a topological space X is a homomorphism ϕ from G to the group $\mathrm{Homeo}(X)$ of homeomorphisms of X. An element $g \in G$ and a point $x \in X$ produces the point $g \cdot x = \phi(g)(x)$. Choosing a topology in G, if ϕ is a continuous homomorphism then the action is called a continuous action. If ϕ is differentiable or analytical, then the action will be called according to its regularity.

An action ϕ is *faithful* if it is injective, that is, if non trivial elements in the group act non trivially on the space.

Note that if ϕ is any action on X, it defines canonically an action of the quotient group $G/\ker(\phi)$ of X and this action is faithful.

Example 2.1. If G is a subgroup of $\mathrm{Homeo}(X)$ (or Diff(X) in case of higher regularity), then there is a natural faithful continuous action (called the *tautological action*) induced by the inclusion morphism. In this case we will denote $g \cdot x = g(x)$.

For simplicity, except in the case of a different action ϕ being already described, we will always consider that any subgroup of $\mathrm{Homeo}(X)$ is acting faithfully and continuously on X by the tautological action.

Let us set up some further notation. Given a point $x \in X$, we write $G.x = \{\phi(g)(x) \mid g \in G\}$ for its orbit, and $\mathrm{Stab}(G,x) = \{g \in G \mid \phi(g)(x) = x\}$ for its stabilizer in G. We say that $x \in X$ is a *fixed point of* g if $\phi(g)(x) = x$ and a *global fixed point* if $G.x = \{x\}$, or alternatively $\mathrm{Stab}(G,x) = G$. We denote by $\mathrm{Fix}(g) = \{x \in X \mid \phi(g)(x) = x\}$ the set of fixed points and $\mathrm{supp}(g) = \{x \in X \mid \phi(g)(x) \neq x\}$ the support of g. A subset $Y \subset X$ is *invariant* under the action ϕ (or shortly ϕ*-invariant*) if it contains the orbit of any point $y \in Y$, that is, $G.Y \subset Y$.

Concretely, we will basically consider actions on the real line $\mathbb{R}$ and on the circle $\mathbb{S}^1$, which we consider as the 1-dimensional compact manifold which is the quotient of the real line $\mathbb{R}$ by the subgroup of integers $\mathbb{Z}$, thus for the rest of this section X will be $\mathbb{R}$ or $\mathbb{S}^1$.

Let f be an element of $\mathrm{Homeo}_+(X)$, with an isolated fixed point x. Then there are three mutually exclusive possibilities for the local dynamics imposed by f:

1. either the image by f of every sufficiently small neighborhood $\mathcal{U}$ of x is a proper subset of $\mathcal{U}$, in which case we say that x is a *attracting fixed point*, or

2. the pre-image by f of every sufficiently small neighborhood $\mathcal{U}$ of x is a proper subset of $\mathcal{U}$, in which case we say that x is a *repelling fixed point*, or

3. both image and pre-image by f of every sufficiently small neighborhood $\mathcal{U}$ of x are not contained in $\mathcal{U}$, in which case we say that x is a *parabolic fixed point.*

We also denote by *hyperbolic fixed point* an isolated fixed point which attracting or repelling. And one may notice that for f with only isolated fixed points, it follows that between two attracting fixed points there is always a repelling fixed point, and similarly between two repelling fixed points there is always an attracting fixed point. Then, for $X = \mathbb{S}^1$, there is an equal amount of repelling and attracting fixed points.

Given two elements f and g of $\mathrm{Homeo}_+(X)$ such that $g^{-1}f$ fixes a point x, then the the graphs of f and g are crossing at the point $(x, f(x))$ and we say that f and f are *crossing.* Moreover, if the fixed point x is hyperbolic then we say that f and g have a *hyperbolic crossing.*

In this paper we will mainly work with the uniform norm over the metric space $\mathrm{Homeo}_+(X)$, which is defined as

$$\|f - g\| = \sup_{x \in X} |f(x) - g(x)|.$$

But, for the case of $X = \mathbb{S}^1$, we may also use the $\mathcal{C}^0$ distance over the metric space $\mathrm{Homeo}_+(\mathbb{S}^1)$, which we define as

$$\mathrm{dist}_{\mathcal{C}^0}(f, g) = \|f - g\| + \|f^{-1} - g^{-1}\|.$$

As one can notice both define the same topology in $\mathrm{Homeo}_+(\mathbb{S}^1)$, but the space is also complete for the $\mathcal{C}^0$ distance.

We say that an element $f \in \mathrm{Homeo}_+(X)$ is called *ε-close to the identity* if $\|f - \mathrm{id}\| \leq \varepsilon$. And for ε smaller than $\frac{1}{2}$, we say that an element $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ without fixed points and ε-close to the identity is *positive* (or *above the identity*) if

$$f(x) \in (x, x + \varepsilon] \quad \text{for all } x \in \mathbb{S}^1.$$

For $X = \mathbb{R}$, we say that an element $f \in \mathrm{Homeo}_+(\mathbb{R})$ without fixed points is *positive* (or *above the identity*) if

$$f(x) > x \quad \text{for all } x \in \mathbb{R}.$$

Given an element $f \in \mathrm{Homeo}_+(X)$, we say that a connected component (a, b) of the $\mathrm{supp}(f)$ is *above the identity* if

$$f(x) \in (x, b) \quad \text{for all } x \in (a, b).$$

An element $f \in \mathrm{Homeo}_+(X)$ is *negative* (or *below the identity*) if f^{-1} is positive. And a connected component (a, b) of the $\mathrm{supp}(f)$ is *below the identity* if it is above the identity for f^{-1}.

We may also say that two connected components of the support *are on the same side of the identity*, if both are above the identity or both are below the identity.

2.2 Minimal invariant subsets

Given a group action, we define the *minimal invariant subset* as the smallest (by inclusion) closed non-empty set such that the orbit of every point is contained in the set. Such sets always exist for actions on compact topological spaces. We have the following classification for minimal sets for actions on $\mathbb{S}^1$: either it is the whole space, or a Cantor set, or a finite set. The first case happens when all the points have dense orbits, and in this case the action is called *minimal.* See [11].

Lemma 2.2. *Let G be a subgroup of $\mathrm{Homeo}_+(X)$, X being $\mathbb{R}$ or $\mathbb{S}^1$, with at most N fixed points, then X contains a minimal invariant subset for the action of G.*

Let us restrict ourselves to the case where $X = \mathbb{R}$, since $\mathbb{S}^1$ is a compact topological space. Then the proof of this result will be a direct application of the following stronger lemma (see for instance [20, Proposition 2.1.12]).

Lemma 2.3. *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ and I a bounded open interval of $\mathbb{R}$ such that the orbit of any point $x \in \mathbb{R}$ intersects I, then the closure of any orbit $\overline{G.x}$ contains a minimal invariant subset for the action of G.*

Proof. Given $x \in \mathbb{R}$, consider the set $\mathcal{A}$ of all compact subsets A contained in $\overline{I} \cap \overline{G.x}$ such that $G.A \cap \overline{I} = A$, equipped with the inclusion $\subset$. One can observe that $(\mathcal{A}, \subset)$ is a partially ordered set satisfying the hypothesis of Zorn's Lemma. Therefore, there exists a minimal element $Y \in \mathcal{A}$ and we define the invariant subset M as

$$M := G.Y = \bigcup_{y \in Y \subset \mathbb{R}} G.y.$$

We claim that M is a closed subset. Indeed, for any $m_\infty \in \overline{M}$ there exists a sequence $(m_n)_{n \in \mathbb{N}} \subset M$ converging to M_∞ and, by hypothesis, $G.m_\infty$ intersects the open interval I, so we choose $g \in G$ such that $g(m_\infty) \in I$. Now, for sufficiently large n, we have

$$g(m_n) \in M \cap \overline{I} = G.Y \cap \overline{I} = Y.$$

But, since Y is closed, Y also contains the limit $g(m_\infty)$ and therefore $m_\infty = g^{-1}g(m_\infty) \in G.Y = M$.

Now, by the minimality of Y, for any closed invariant subset $M' \subset M$ we have

$$M' \cap \overline{I} = Y$$

which implies that $Y \subset M'$ and since its invariant by the action of G, we have $M = G.Y \subset M'$ and therefore, $M = M'$. So we conclude that M is a minimal invariant subset for the action of G. ☐

Now we can continue to the proof of Lemma 2.2 for the case $X = \mathbb{R}$.

Proof of Lemma 2.2. We will assume that there are no global fixed points for the action of G, otherwise such point would be a minimal invariant subset. Now, since every element has at most N fixed points and there are no global fixed points, we can choose $N+1$ elements $\{g_1, \dots, g_{N+1}\} \in G$ such that no point is fixed by all g_n and we define the function $F : \mathbb{R} \to \mathbb{R}$ as:

$$F(x) := \max_{n \in \{1,\dots,N+1\}} g_n^{\pm 1}(x).$$

One can observe that F is increasing, continuous and $F(x) > x$, for all $x \in \mathbb{R}$. Now, let I be any bounded open interval containing $[0, F(0)]$ and we claim that every orbit intersects I. Indeed, for all $x \in \mathbb{R}$ there exists $k \in \mathbb{Z}$ such that $F^k(x) < 0$ and $F^{k+1}(x) \geq 0$, so we have

$$F^{k+1}(x) = F(F^k(x)) \in [0, F(0)) \subset I.$$

Therefore, the orbit of every point intersects the interval I and, by Lemma 2.3, we conclude that there exists a minimal invariant subset for the action of G. ☐

Lemma 2.4. *Let G be an abelian subgroup of* $\mathrm{Homeo}_+(X)$*, X being $\mathbb{R}$ or $\mathbb{S}^1$, with at most N fixed points. If there exists an element $f \in G$ with* $\mathrm{Fix}(f) \neq \emptyset$*, then the minimal invariant subset of the action of G is finite. Moreover, there exists a minimal invariant subset contained in* $\mathrm{Fix}(f)$*, for every $f \in G$.*

Proof. Let $x \in X$ be a fixed point of $f \in G$, then for every $g \in G$ we have

$$fg(x) = gf(x) = g(x).$$

Therefore, $G \cdot x \subset \mathrm{Fix}(f)$ for every $x \in \mathrm{Fix}(f)$, which follows that $\mathrm{Fix}(f)$ is a closed and invariant subset of X. Now, by lemmas 2.2 and 2.3, the closure of the orbit $\overline{G.x}$ contains a minimal invariant subset for the action of G and since it is contained in $\mathrm{Fix}(f)$, we conclude that there exists a minimal invariant subset contained in $\mathrm{Fix}(f)$. ☐

Corollary 2.5. *Let G be an abelian subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$*, with at most N fixed points, then every fixed point is globally fixed.*

This corollary comes from the fact that the only finite minimal invariant subsets on $\mathbb{R}$ are single points which are globally fixed. The complete classification for the minimal invariant subsets of group actions on $\mathbb{R}$ is given in the following classical theorem.

Theorem 2.6. *Let G be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *with a minimal invariant subset M. Then, the closure of any orbit contains a minimal invariant subset for the action of G and moreover there are four mutually exclusive possibilities:*

1. *All the orbits are dense and $M = \mathbb{R}$ is the only minimal invariant subset.*
2. *All minimal invariant subsets are globally fixed points.*
3. *All minimal invariant subsets are closed orbits of G, which are discrete and unbounded. Moreover, there exists an element $g \in G$ without fixed points such that all minimal invariant subsets are orbits of g.*
4. *There exists a unique minimal invariant subset M which is an unbounded Cantor set, i.e. perfect, totally disconnected and unbounded.*

Remark 2.7. Such classification is, in fact, analogous to the classification for minimal invariant subsets of the circle, since every finite invariant subset of the line is globally fixed and every closed discrete subset of the circle is finite. The ideas presented in [11] for the proof of the classification of minimal sets of the circle can be used to prove Theorem 2.6.

2.3 Semi-conjugacy and blow-up

We start this part of the chapter with the notion of monotone 1-degree map, which is fundamental when studying group actions on the circle. For further discussion, we recommend the monographs by Calegari [6] and Kim, Koberda, and Mj [15], from which we borrow some terminology.

A continuous map $h : \mathbb{R} \to \mathbb{R}$ is called a *monotone 1-degree map* if h is non-decreasing and if

$$h(x+1) = h(x) + 1, \text{ for all } x \in \mathbb{R}.$$

We also say that h is a *monotone proper map* if h is non-decreasing and $\lim_{x\to\pm\infty} h(x) = \pm\infty$.

Furthermore, we say that a continuous map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a *monotone 1-degree map* if h has a lift to $\mathbb{R}$ which is a monotone 1-degree map. We also denote by $\mathrm{Gap}(h)$ the set of locally constant points of a monotone 1-degree map $h : \mathbb{S}^1 \to \mathbb{S}^1$, and set $\mathrm{Core}(h) = \mathbb{S}^1 \smallsetminus \mathrm{Gap}(h)$.

Now we are ready to present the concept of semi-conjugacy for subgroups of $\mathrm{Homeo}_+(X)$, with X being $\mathbb{R}$ or $\mathbb{S}^1$, and its definition will follow.

Let G and F be two subgroups of $\mathrm{Homeo}_+(\mathbb{R})$, we say that the action of G is *semi-conjugate* to F if there exist a surjective morphism $\theta : G \to F$ and a continuous, monotone proper map $h : \mathbb{R} \to \mathbb{R}$, which is equivariant, in other words

$$\theta(g)h = hg, \quad \text{for every } g \in G.$$

Similarly, let G and F be two subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$, we say that the action of G is *semi-conjugate* to F if there exist a surjective morphism $\theta : G \to F$ and a continuous, monotone 1-degree map $h : \mathbb{S}^1 \to \mathbb{S}^1$, which is equivariant.

In both situations, the map h is called a *semi-conjugacy*, and when h is a homeomorphism, we say that it is a *conjugacy*, in which case we also say that G and F are *conjugate*.

There exists a natural equivalence relation for elements of $\mathrm{Homeo}_+(X)$ given by

$$G \sim F \quad \text{if, and only if,} \quad G \text{ and } F \text{ are conjugate.}$$

However, we remark that such equivalence relation does not hold for semi-conjugacies.

The notion of conjugacy and semi-conjugacy can be extended to homeomorphisms by stating that f and g in $\mathrm{Homeo}_+(X)$ are (semi-)conjugate by the (semi-)conjugacy h if the subgroups $\langle f \rangle$ and $\langle g \rangle$ are (semi-)conjugate by the (semi-)conjugacy h.

In this case, the surjective morphism $\theta : \langle f \rangle \cong \mathbb{Z} \to \langle g \rangle \cong \mathbb{Z}$ is either the identity or the inversion, which implies that $fh = hg$ or $f^{-1}h = hg$. If we are in the second case, we replace our notation by saying that f^{-1} and g are (semi-)conjugate by the (semi-)conjugacy h.

Lemma 2.8. *If two elements f and g of $\mathrm{Homeo}_+(X)$ are conjugate, then $|\mathrm{Fix}(f)| = |\mathrm{Fix}(g)|$ and the local dynamics imposed by each isolated fixed point is invariant by conjugacy.*

More precisely, for each attracting, repelling or parabolic fixed point of g, f contains a fixed point with the same behavior.

Proof. Let h be the conjugacy, we then have, for every $x \in \mathrm{Fix}(g)$, that

$$fh(x) = hg(x) = h(x),$$

then $h(\mathrm{Fix}(g)) \subset \mathrm{Fix}(f)$, which implies that $|\mathrm{Fix}(f)| \geq |\mathrm{Fix}(g)|$.

On the other hand, for every $x \in \mathrm{Fix}(f)$, we have that

$$gh^{-1}(x) = h^{-1}f(x) = h^{-1}(x),$$

then $h^{-1}(\mathrm{Fix}(f)) \subset \mathrm{Fix}(g)$, which implies that $|\mathrm{Fix}(f)| \leq |\mathrm{Fix}(g)|$.

Now, let us suppose that $x \in \mathrm{Fix}(g)$ is an attracting fixed point, then for every sufficiently small neighborhood $\mathcal{U}$ of x, we have that $g(\mathcal{U}) \subsetneq \mathcal{U}$, and since h is a homeomorphism it follows that

$$fh(\mathcal{U}) = hg(\mathcal{U}) \subsetneq h(\mathcal{U})$$

then the neighborhood $h(\mathcal{U})$ (which is as small as we wish) of the fixed point $h(x)$ is sent into a proper subset of itself by f. Then $h(x)$ is an attracting fixed point of f.

A similarly argument can be made for repelling and parabolic fixed points, which concludes the proof. □

Next, we proceed with two classical but very useful results which characterize some of the equivalence classes implied by the conjugacy of elements in $\mathrm{Homeo}_+(\mathbb{R})$.

Lemma 2.9. *Any two positive homeomorphisms of the line with no fixed points are conjugate.*

Proof. Since the conjugacy defines an equivalence relation, it is enough to prove that any given positive $f \in \mathrm{Homeo}_+(\mathbb{R})$ is conjugate to the translation $T \in \mathrm{Homeo}_+(\mathbb{R})$, such that

$$T : x \longmapsto x + 1.$$

For such, we will construct an element $h \in \mathrm{Homeo}_+(\mathbb{R})$ such that

$$h(x+1) = fh(x) \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

We start by fixing $h(0) = 0$, then

$$h(1) = fh(0) = f(0).$$

So, we take any homeomorphism $\varphi : [0, 1] \to [0, f(0)]$ and since f is positive, we have $f(0) > 0$, and φ is order-preserving.

Now, observe that, in general, if h and f respects (2.1), them for any $k \in \mathbb{Z}$ it also follows that

$$h(x + k) = f^k h(x) \quad \text{for all } x \in \mathbb{R}.$$

For $x \in [0, 1]$, we define $h(x) = \varphi(x)$ and, for any $x \in \mathbb{R}$ we have that there exists $k \in \mathbb{Z}$ such that $x - k \in [0, 1]$, then the only consistent way to define $h(x)$ is given by

$$h(x) = h((x - k) + k) = f^k h(x - k) := f^k \varphi(x - k),$$

which implies that h is defined as an order-preserving homeomorphism of $\mathbb{R}$. Let us now show the equality (2.1).

Take any $x \in \mathbb{R}$, and let $k \in \mathbb{Z}$ be the integer such that $x - k \in [0, 1]$, then for $x + 1$ such integer will be $k + 1$ and we have

$$h(x + 1) = f^{k+1}\varphi(x + 1 - (k + 1)) = f^{k+1}\varphi(x - k) = ff^k\varphi(x - k) = fh(x)$$

as we wanted to prove. □

Lemma 2.10. *Let f and g be two order-preserving homeomorphisms of X with* $\mathrm{Fix}(f) = \{z_1, \ldots, z_n\}$ *and* $\mathrm{Fix}(g) = \{y_1, \ldots, y_n\}$*, such that the collections $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_n\}$ are ordered, and the fixed points z_i and y_i have the same behavior (attracting, repelling or parabolic), for all $i \in \{1, \ldots, n\}$.*

Then, f and g are conjugate in $\mathrm{Homeo}_+(\mathbb{S}^1)$.

Proof. Define the intervals $I_i = [z_i, z_{i+1}]$ and $J_i = [y_i, y_{i+1}]$ for $i = 0, \ldots, n$ where, for the case of $X = \mathbb{S}^1$, consider $0 \sim n$ and $1 \sim n + 1$, and for the case of $X = \mathbb{R}$, consider $z_0 = y_0 = -\infty$ and $z_{n+1} = y_{n+1} = +\infty$.

Since the collections of fixed points are ordered, these intervals form a partition of X, with

$$\bigcup_{i=0}^{n} I_i = \bigcup_{i=0}^{n} J_i = X \quad \text{and, for any } i \neq j, \quad \mathring{I}_i \cap \mathring{I}_j = \mathring{J}_i \cap \mathring{J}_j = \emptyset.$$

Then, as the fixed points z_i and y_i have the same behavior it is easy to verify that, for every $i \in \{1, \ldots, n\}$, the connected components (z_i, z_{i+1}) and (y_i, y_{i+1}) are on the same side of the identity, which implies, by Lemma 2.9, that f and g are conjugate when restricted to each component I_i and J_i. In other words, for each i, there exists an order-preserving homeomorphism $h : I_i \to J_i$ satisfying

$$h_i f(x) = gh_i(x) \quad \text{for all } x \in I_i.$$

Now, consider the order-preserving homeomorphism $h : X \to X$ defined as

$$h(x) = h_i(x) \quad \text{for all } x \in I_i \text{ and all } i \in \{1, \ldots, n\}.$$

It is easy to check that f and g are conjugate with conjugacy h. □

Now, one may notice that an analogous of Lemma 2.9 for the case of circle homeomorphisms is not true. In fact, given two rotations R_α and R_β, with $\alpha \neq \beta \pmod 1$, then R_α and R_β are not conjugate by circle homeomorphisms. However, we do have an invariant by semi-conjugacies for circle homeomorphisms without fixed points, which is the *rotation number*.

Let f be an element of $\mathrm{Homeo}_+(\mathbb{S}^1)$, x be any point of $\mathbb{S}^1$ and $\widetilde{f}$ any lift of f into $\mathrm{Homeo}_+(\mathbb{R})$. After the work of Poincaré, we define the *rotation number* of f by

$$\mathrm{rot}(f) = \lim_{n\to+\infty} \frac{1}{n}[F^n(x) - x] \qquad \bmod 1.$$

The rotation number is an invariant by semi-conjugacies, as if f is semi-conjugate to g then $\mathrm{rot}(f) = \mathrm{rot}(g)$. Moreover, if $\mathrm{rot}(f) = \frac{p}{q}$ (we assume that the rational number $\frac{p}{q}$ is in the lowest terms), then f has a periodic orbit, every periodic orbit has period q and f is conjugate to $R_{\frac{p}{q}}$ if and only if f has finite order.

Theorem 2.11. *If the rotation number* $\mathrm{rot}(f)$ *of* $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ *is irrational, then* f *is semi-conjugate to the rotation of angle* $\mathrm{rot}(f)$*. The semiconjugacy is a conjugacy if and only if all the orbits of* f *are dense.*

These results are the combined work of Poincaré and Denjoy. For a proof of this theorem and a detailed construction of the rotation number, see [20].

We follow the text with a result about the topological properties of the $\mathrm{Core}(h)$ for a given semi-conjugacy h, which will point to a characterization for the core.

Lemma 2.12. *For two subgroups* G *and* F *of* $\mathrm{Homeo}_+(\mathbb{S}^1)$*, such that* G *is semi-conjugate to* F *with semi-conjugacy* $h : \mathbb{S}^1 \to \mathbb{S}^1$*, the subset* $\mathrm{Core}(h)$ *(hence* $\mathrm{Gap}(h)$*) is* G*-invariant and has no isolated points.*

In particular, when $\mathrm{Gap}(h) \subset \mathbb{S}^1$ *is dense, then* $\mathrm{Core}(h)$ *is a Cantor set.*

Proof. Invariance of $\mathrm{Core}(h)$ follows from equivariance. The fact that $\mathrm{Core}(h)$ has no isolated points corresponds to [6, Lemma 2.14]. □

Lemma 2.13. *Let* G *and* F *be two subgroups of* $\mathrm{Homeo}_+(\mathbb{S}^1)$*, such that* F *is semi-conjugate to* G *with semi-conjugacy* $h : \mathbb{S}^1 \to \mathbb{S}^1$*. Then* $\mathrm{Core}(h)$ *is the only closed subset* $X \subset \mathbb{S}^1$ *such that for every* $x \in \mathbb{S}^1$*, we have* $\#(h^{-1}(x) \cap X) \in \{1, 2\}$*.*

Proof. It is clear from the definition of $\mathrm{Core}(h)$ that it satisfies the required condition. Conversely, let $X \subset \mathbb{S}^1$ be as in the statement. Observe that for every $x \in \mathbb{S}^1 \smallsetminus h(\mathrm{Gap}(h))$ we have $h^{-1}(x) \in X$, otherwise we would have a point $x \in \mathbb{S}^1$ with $h^{-1}(x) \cap X = \emptyset$ which contradicts the assumption on X. This implies $\mathrm{Core}(h) \subset X$. If $\mathrm{Core}(h) = \mathbb{S}^1$, then we get the desired conclusion. Otherwise, $\mathrm{Gap}(h)$ is non-empty. Now, if $\mathrm{Core}(h) \neq X$, then there exists a point $y \in \mathrm{Gap}(h) \cap X$, but this would imply

$$\#(h^{-1}(h(y)) \cap X) \geq 3,$$

which is an absurd. □

This gives us a much more concrete idea of what is a conjugacy and it also points us to the concept of blow-up of group actions, which is presented below and will be precisely stated in Definition 2.14.

Let G and F be two subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$, such that G is semi-conjugate to F with semi-conjugacy $h : \mathbb{S}^1 \to \mathbb{S}^1$. Write $A = h(\mathrm{Gap}(h)) \subset \mathbb{S}^1$, and we say that G is a *blow-up* of F on the subset A.

This will be a very useful definition trough the text and a fundamental tool to construct semi-conjugate actions. A blow-up is typically not unique, however it is unique up to conjugacy, as the next theorem says (Theorem 2.15). So that the dynamical properties will be invariant to any chosen blow-up. See also [14, Section 2.1].

Definition 2.14. Let F be a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ and let $A \subset \mathbb{S}^1$ be an F-invariant, at most countable subset. Let $\{a_k\}_{k\in\Omega} \subset A$ be a choice of representatives for every F-orbit in A.

For each $k \in \Omega$ write $S_k = \mathrm{Stab}(F, a_k)$ and choose a group action $\phi_k : S_k \to \mathrm{Homeo}_+([0,1])$. Then, a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is an *isomorphic blow-up of F at $\{a_k\}_{k\in\Omega}$ and including $\{\phi_k\}_{k\in\Omega}$ on the intervals*, if there exist a continuous, monotone 1-degree map $h : \mathbb{S}^1 \to \mathbb{S}^1$ and an isomorphim $\theta : G \xrightarrow{\sim} F$, satisfying the following properties.

- G is semi-conjugate to F with semi-conjugacy h and homomorphism θ.
- $h(\mathrm{Gap}(h)) = A$.
- For each $k \in \Omega$, the subgroup $\theta^{-1}(S_k) \leq G$ is the stabilizer of the interval $h^{-1}(a_k)$ and the restriction $\theta^{-1}(S_k)|_{h^{-1}(a_k)}$ is conjugate to the action ϕ_k. That is, there exists an order-preserving homeomorphism $t_k : [0,1] \to h^{-1}(a_k)$ such that for all $x \in h^{-1}(a_k)$ and all $s \in S_k$, it follows that $\theta(s)(x) = t_k\phi_k(s)t_k^{-1}(x)$.

Theorem 2.15. *Let F be a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ and let $A \subset \mathbb{S}^1$ be an F-invariant, at most countable subset, with choice of representatives $\{a_k\}_{k\in\Omega} \subset A$ for the F-orbits in A. For each $k \in \Omega$ write $S_k = \mathrm{Stab}(F, a_k)$, and consider any family of group actions $\phi_k : S_k \to \mathrm{Homeo}_+([0,1])$.*

Then, there exists an isomorphic blow-up of F at $\{a_k\}_{k\in\Omega}$ and including $\{\phi_k\}_{k\in\Omega}$ on the intervals, which is unique up to conjugacy.

The statement above is certainly well-known to experts, but we include a proof in Appendix A for completeness, as it is difficult to find a detailed proof in the literature.

2.4 Möbius action on the circle

We start with the linear group $\mathrm{GL}(2,\mathbb{R})$ which consists of 2×2 real invertible matrices. The center of such group is the subgroup of scalar matrices and the quotient of the linear group by its center is the projective group, which we denote as $\mathrm{PGL}(2,\mathbb{R})$. The elements of such group may be represented as 2×2 real invertible matrices with determinant equal to 1 or -1.

There is a natural action of $\mathrm{PGL}(2,\mathbb{R})$ on the circle (seen as the real projective line $\mathbb{RP}^1$). Indeed, $\mathrm{GL}(2,\mathbb{R})$ acts linearly on the vector space $\mathbb{R}^2$ which induces a linear action on the set of lines containing the origin, which is $\mathbb{RP}^1$ by definition. Such group action $\mathrm{PGL}(2,\mathbb{R}) \times \mathbb{RP}^1 \to \mathbb{RP}^1$ is described by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x \longmapsto \frac{ax+b}{cx+d} \tag{2.2}$$

Now, the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$, also known as the Möbius group, is isomorphic to the subgroup of $\mathrm{PGL}(2,\mathbb{R})$ of matrices with positive determinant. And for the rest of the text, by *Möbius action on the circle*, we will mean the restriction of the standard action described in (2.2) to the subgroup $\mathrm{PSL}(2,\mathbb{R})$.

A well known feature of this action is that it can be extended to the disc as the group of isometries of the hyperbolic disc $\mathbb{D}$.

Now, for a given element $\mathcal{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{PSL}(2,\mathbb{R})$ there exists three mutually exclusive possibilities:

1. $\mathcal{M}$ has no real eigenvalue, in which case there exists $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]/_{(-\frac{\pi}{2}\sim\frac{\pi}{2})}$ such that after a change of basis we can write

$$\mathcal{M} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

and by considering $x = \tan(\phi)$, the standard action given by (2.2) leads to

$$\mathcal{M} \cdot \tan(\phi) = \frac{\cos(\theta)\tan(\phi) + \sin(\theta)}{-\sin(\theta)\tan(\phi) + \cos(\theta)} = \frac{\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)}{-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi)} = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \tan(\theta+\phi)$$

which defines a rotation on $\mathbb{RP}^1$. Such element will be called *elliptic*.

2. $\mathcal{M}$ has one real eigenvalue with multiplicity 2, in which case after a change of basis we can write

$$\mathcal{M} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

then, the standard action given by (2.2) leads to

$$\mathcal{M} \cdot x = x + \alpha$$

which fixes only the $\infty \in \mathbb{RP}^1$ which is a parabolic fixed point and acts as a translation at $\mathbb{RP}^1 \setminus \{\infty\}$. Such element will be called *parabolic*.

3. $\mathcal{M}$ has two real eigenvalues, in which case there exists $a > 1$ such that after a change of basis we can write

$$\mathcal{M} = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$$

then, the standard action given by (2.2) leads to

$$\mathcal{M} \cdot x = a^2 x$$

which fixes the points 0 and $\infty \in \mathbb{RP}^1$, where 0 is a repelling fixed point and ∞ is an attracting. Such element will be called *hyperbolic*.

To summarize, each element of the Möbius group action on the circle is either elliptic (have no fixed points and acts as a rotation), or parabolic (have exactly one parabolic fixed point), or hyperbolic (have one attracting fixed point and one repelling fixed point). From this, one can notice that the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ acts on the circle with at most 2 fixed points.

We say that an element f of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is *Möbius-Like* if f is conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$, and a subgroup F of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is *Möbius-Like* if every element of F is conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$.

Note that a Möbius-Like subgroup F may not be conjugate to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$, since the conjugacy doesn't need to be the same for every element. Indeed, after the work of Kovačević in [17] we do have examples Möbius-Like subgroups that are not conjugate into any subgroup of Möbius. This topic will be raised again in Chapter 6, where will be present new families of examples.

Now, since the number of fixed points and its behaviors (attracting, repelling or parabolic) characterizes equivalence classes for conjugacy (see Lemma 2.10), it is easy to verify that:

– no circle homeomorphism with 3 or more fixed points is Möbius-Like,

– a circle homeomorphism with 2 fixed points is Möbius-Like if and only if both points are hyperbolic,

– all circle homeomorphisms with 1 fixed point are Möbius-Like, and

– a circle homeomorphism without fixed points is Möbius-Like if and only if it is conjugate to a rotation.

Note that a circle homeomorphism with 2 fixed points which is not Möbius-Like has 2 parabolic fixed points. Such homeomorphism will be called *bi-parabolic*.

Now, for the last line of the characterization of Möbius-Like homeomorphisms, we remark that not every circle homeomorphism without periodic points is conjugate to a rotation. Indeed, by Poincaré's work, if the rotation number of such homeomorphism is irrational then it is conjugate to a rotation only if it has a dense orbit. In a celebrated work, Denjoy exhibited the first examples of aperiodic circle homeomorphisms without dense orbits. For the case of rational rotation number, in order to be conjugate to a rotation, the homeomorphism needs to have finite order.

Therefore, one may replace the last line of the characterization of Möbius-Like homeomorphisms by these two next lines

– a circle homeomorphism with rational rotation number is Möbius-Like iff it has a finite order,

– a circle homeomorphism with irrational rotation number is Möbius-Like iff it has a dense orbit.

Many works tried to prove that Möbius-Like groups were indeed (semi)-conjugate to subgroups of $\mathrm{PSL}(2,\mathbb{R})$ in particular through the notion of *convergence groups*. A major result in this context has been obtained by Tukia [25] (in the torsion-free case), and then independently extended by Gabai [10] and Casson and Jungreis [8]. They determine the topological conjugacy class of subgroups of $\mathrm{PSL}(2,\mathbb{R})$ by a dynamical condition, which is called the *convergence property*. In the specific case of the circle, this can be defined as follows (see Kovačević [16, Observation 1.4]).

Definition 2.16. For a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, we say that G is a *convergence group* if every sequence $\{g_n\}_{n\in\mathbb{N}}$ in G has a subsequence $\{g_{n_i}\}_{i\in\mathbb{N}}$, such that either:

- there exist points x and $y \in \mathbb{S}^1$, satisfying

  $g_{n_i}(t) \to y$ for every $t \in \mathbb{S}^1 \setminus \{x\}$, as $i \to +\infty$,

  $g_{n_i}^{-1}(t) \to x$ for every $t \in \mathbb{S}^1 \setminus \{y\}$, as $i \to +\infty$, or

- there exists a homeomorphism $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$, satisfying

  $g_{n_i}(t) \to g(t)$ for every $t \in \mathbb{S}^1$, as $i \to +\infty$,

  $g_{n_i}^{-1}(t) \to g^{-1}(t)$ for every $t \in \mathbb{S}^1$, as $i \to +\infty$.

Theorem 2.17 (Tukia; Casson–Jungreis; Gabai)**.** *A subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is conjugate into $\mathrm{PSL}(2,\mathbb{R})$ if and only if G is a convergence subgroup.*

Let us also mention the recent work of Baik [4], which studies a possible characterization of subgroups of $\mathrm{PSL}(2,\mathbb{R})$ in terms of invariant laminations on the circle.

2.5 Elementary and non-elementary subgroups

A subgroup G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is *elementary* if its action preserves a Borel probability measure. Concretely, a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is elementary, if either it admits a finite orbit or it is semi-conjugate to a subgroup of rotations (see [20, Proposition 1.1.1]), so that the action can be basically reduced to an action on the line.

Indeed, given any point $x \in \mathbb{S}^1$, we can compute the rotation number $\mathrm{rot}(g)$ for every element g in an elementary subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ using the invariant measure μ, and the result does not depend on the choices of the point x and the invariant measure μ: the map $\mathrm{rot} : G \to \mathbb{S}^1$ defined by $\mathrm{rot}(g) = \mu[x, g(x))$, is in fact a homomorphism. As a consequence we have a homomorphism $G \to \mathrm{SO}(2)$ defined by $g \mapsto R_{\mathrm{rot}(g)}$, which semi-conjugates G to a subgroup of rotations. We have the following basic result.

Lemma 2.18. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup with invariant Borel probability measure μ on $\mathbb{S}^1$. Then, the kernel of* rot *fixes* $\mathrm{supp}(\mu)$ *pointwise.*

Proof. Take $x \in \mathrm{supp}(\mu)$, and consider the morphism $g \mapsto \mu[x, g(x))$. If $g(x) \neq x$, then $\mu[x, g(x)) \neq 0$ and thus $\mathrm{rot}(g) \neq 0$. □

Let us give a more precise statement in the case when the elementary subgroup preserves an atomless Borel probability measure, and it has at most N fixed points.

Lemma 2.19. *If a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ with at most N fixed points, preserves an atomless Borel probability measure μ, then G is semi-conjugate to a group of rotations and, moreover, the corresponding morphism $G \to \mathrm{SO}(2)$ is injective.*

In particular, G is isomorphic to a subgroup of $\mathrm{SO}(2)$.

Proof. As explained before, the morphism $g \mapsto R_{\mathrm{rot}(g)}$ gives a semi-conjugacy to an action by rotations. If there was an element in the kernel, it will fix the support of μ, which is infinite. As we are assuming that G acts with at most N fixed points, this gives that the kernel is trivial. This gives the desired conclusion. □

Let us remark that for a non-elementary group with at most 2 fixed points, the action has a very good dynamical property which will be stated in Theorem 2.20. But before that, we need to define when a group action of the circle is *proximal*.

Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup. We say that the action of G on the circle is *proximal* if for every non-empty open intervals $I, J \subset \mathbb{S}^1$, there exists an element $g \in G$ such that $g(I) \subset J$.

We also say that the action is proximal *in restriction to the minimal invariant subset* if the previous statement holds only for intervals $J \subset \mathbb{S}^1$ which are non-wandering.

The following fundamental result can be deduced from the work of Antonov [3] (see also Ghys [11]).

Theorem 2.20. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a non-elementary subgroup. Then there exists a finite order element $\gamma \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which commutes with every $g \in G$, and such that the induced action of G on the quotient $\mathbb{S}^1/\langle\gamma\rangle$ is proximal in restriction to the minimal invariant subset.*

We immediately deduce the following.

Proposition 2.21. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a non-elementary subgroup with at most 2 fixed points. Then the action of G is proximal in restriction to the minimal invariant subset.*

Moreover, there exists an element $g \in G$ conjugate to a hyperbolic element of $\mathrm{PSL}(2, \mathbb{R})$.

Proof. Assume that the action of G is not proximal in restriction to the minimal invariant subset. After Theorem 2.20 there exists a non-trivial finite order element $\gamma \in \mathrm{Homeo}_+(\mathbb{S}^1)$ centralizing G, such that the induced action on the circle $\mathbb{S}^1/\langle\gamma\rangle$ is proximal in restriction to the minimal invariant subset. As the action on $\mathbb{S}^1/\langle\gamma\rangle$ is proximal in restriction to the minimal invariant subset, the group G contains elements with at least one attracting fixed point. Thus, in the original action on $\mathbb{S}^1$, G contains elements with at least n fixed attracting fixed points, where n is the order of γ. But the assumption that the action has at most 2 fixed points, forces $n = 1$, which contradicts the non-triviality of γ. □

Remark 2.22. Note that a straightforward consequence of the results above is Hölder's theorem in the case of the circle (Theorem 1.7). By an analogous approach for actions on the real line, one can also obtain the classical Hölder's theorem (Theorem 1.3). See for instance Malyutin [18].

2.6 Möbius-Like elementary groups

Here we discuss the first new contribution of this work, namely Theorem B, which states that finitely generated, Möbius-Like elementary subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ are semi-conjugate into $\mathrm{PSL}(2,\mathbb{R})$.

Proof of Theorem B. Let ν be a Borel probability measure preserved by the action of G on the circle. Consider the homomorphism $\mathrm{rot} : G \to \mathrm{SO}(2)$ defined by $g \mapsto R_{\mathrm{rot}(g)}$. Now, since each non-trivial element of G fixes at most 2 points, by Lemmas 2.18 and 2.19, one can notice that if the support $\mathrm{supp}(\nu)$ has more than 2 points then the kernel of rot is trivial and $\mathrm{rot} : G \to \mathrm{SO}(2)$ defines an isomorphism between G and a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Thus, from now on, we can assume that the action of G has a finite orbit, which is either a fixed point or a pair of points.

Assume there is a unique global fixed point for G. Then, by Theorem 1.3, G is semi-conjugate to a subgroup of affine transformations, and the corresponding morphism $G \to \mathrm{Aff}_+(\mathbb{R}) \leq \mathrm{PSL}(2,\mathbb{R})$ is injective.

Assume next that G has two global fixed points $p, q \in \mathbb{S}^1$, and let I_λ and I_ρ be the connected components of $\mathbb{S}^1 \smallsetminus \{p,q\}$. Denote by $\lambda : G \to \mathrm{Homeo}_+(\mathbb{R})$ and $\rho : G \to \mathrm{Homeo}_+(\mathbb{R})$ the actions of G restricted to I_λ and I_ρ, respectively, which are semi-conjugate to actions by translations after Theorem 1.3, and moreover the corresponding induced homomorphisms $G/\ker\lambda \to \mathbb{R}$ and $G/\ker\rho \to \mathbb{R}$ are injective. Now, since the number of fixed points of a non-trivial $g \in G$ on the whole circle is at most 2, it follows that if $\lambda(g) = \mathrm{id}$ or $\rho(g) = \mathrm{id}$ then $g = \mathrm{id}$. By consequence, each of these actions are faithful, and we have $\lambda(G) \simeq \rho(G) \simeq G$, in particular G is abelian.

If $G \simeq \mathbb{Z}$ then we choose a generator $g \in G$ and on each side of the circle we can conjugate $\lambda(g)$ and $\rho(g)$ to one of the translations $x \mapsto x+1$ or $x \mapsto x-1$ depending if its graph is above or below the identity. But since G is Möbius-Like, one side of its graph should be above the identity and the other side below it, so g is conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$ and therefore G is also conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

For $G \simeq \mathbb{Z}^{d+1}$ with $d \geq 1$, we choose a free basis of generators $\{f, g_1, \ldots, g_d\}$ and replacing f with f^{-1} if necessary, we assume that $f(x) > x$ for every $x \in I_\rho$ and $f(x) < x$ for every $x \in I_\lambda$. By Hölder's theorem (Theorem 1.3), both $\lambda(G)$ and $\rho(G)$ are semi-conjugate to actions of translations and we can choose a semi-conjugacy such that $\lambda(f)$ is sent to the translation by $x \mapsto x-1$ and $\rho(f)$ is sent to the translation by $x \mapsto x+1$. Now, for every other element $g \in G$, we have $\lambda(g)$ and $\rho(g)$ are sent to translations $x \mapsto x+\alpha$ and $x \mapsto x+\beta$. We claim that $\alpha + \beta = 0$.

Indeed, if $\alpha+\beta\neq 0$ then there exists an integer $N\in\mathbb{Z}$ such that $N(\alpha+\beta)>1$ and so, there exists a second integer $M\in\mathbb{Z}$ such that

$$N\alpha > M > N(-\beta).$$

Therefore, $N\alpha - M > 0$ and $N\beta + M > 0$ and so the element $g^N f^M \in G$ is sent, by the semi-conjugacy, to translations larger than the identity on both sides of the circle. We conclude that the element $g^N f^M$ is sent by the semi-conjugacy to a circle homeomorphism with two parabolic fixed points which is an absurd, because G is Möbius-Like.

Now we can write $\beta=-\alpha$ and we have that for every other element $g\in G$, $\lambda(g)$ and $\rho(g)$ are sent to translations $x\mapsto x+\alpha$ and $x\mapsto x-\alpha$, so it is an element of $\mathrm{PSL}(2,\mathbb{R})$ and therefore the group G is sent, by semi-conjugacy, to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

For the last case, we will assume that G has a finite orbit of order 2, and denote by ν the corresponding invariant probability measure. After the previous discussion, we have a short exact sequence

$$1\to \ker(\mathrm{rot})\to G\to \mathbb{Z}_2\to 1.$$

Let $a\in G$ be an element of the group with $\mathrm{rot}(a)=1/2$. Now, observe that if $a^2\neq \mathrm{id}$ then a^2 fixes the 2 atoms of ν and no other points, so they are both parabolic fixed points. This contradicts the Möbius-Like assumption. Thus, we can conclude that $a^2=\mathrm{id}$. Therefore the exact sequence splits and G can be written as $G=\ker(\mathrm{rot})\rtimes_A \mathbb{Z}_2$, where A is the involution defined by the conjugacy by a. Since $\ker(\mathrm{rot})$ is also a free abelian group of finite rank, we have that for some integer n, G is isomorphic to $\mathbb{Z}^n\rtimes_A\mathbb{Z}_2$.

Claim. *We have $A=-\mathrm{id}$, thus $G\cong\mathbb{Z}^n\rtimes_{-\mathrm{id}}\mathbb{Z}_2$ is semi-conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, and the corresponding homomorphism $G\to\mathrm{PSL}(2,\mathbb{R})$ is injective.*

Proof of claim. Note that $\mathrm{Out}(\mathbb{Z}^n)=\mathrm{Aut}(\mathbb{Z}^n)=\mathrm{GL}(n,\mathbb{Z})$, and order 2 elements in $\mathrm{GL}(n,\mathbb{Z})$ can be always conjugated, in $\mathrm{GL}(n,\mathbb{Z})$, to a matrix of the form

$$A=\begin{pmatrix}1 &&&&&&\\ &\ddots&&&&&\\ &&-1&&&&\\ &&&\ddots&&&\\ &&&&0&1&\\ &&&&1&0&\\ &&&&&&\ddots\end{pmatrix}.$$

See for instance Casselman [7, Section 4]. Thus, up to change of basis of $\mathbb{Z}^n$, we have an A-invariant direct sum decomposition $\mathbb{Z}^n=\mathbb{Z}^{n_1}\oplus\mathbb{Z}^{n_2}\oplus(\mathbb{Z}^2)^{n_3}$, such that A acts on $\mathbb{Z}^{n_1}$ as the identity, on $\mathbb{Z}^{n_2}$ as $-\mathrm{id}$ and on every $\mathbb{Z}^2$-factor of $(\mathbb{Z}^2)^{n_3}$ as the permutation matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

We have to prove that $n_1=n_3=0$. The fact that $n_1=0$ follows from the fact that if an element with two fixed points is centralized by an element which exchanges the two points, then both fixed points must be of the same dynamical nature, hence parabolic, contradicting the Möbius-Like assumption. For n_3, assume there exists $f,g\in\ker(\mathrm{rot})$ such that $afa^{-1}=g$. In particular the subgroup $H=\langle f,g\rangle\simeq\mathbb{Z}^2$ is fixing the two atoms of ν, and by the previous case for global fixed points, we can assume f acts (up to semi-conjugacy) by translations $x\mapsto x-1$ and $x\mapsto x+1$ respectively on I_λ and I_ρ, and similarly g acts (up to the same semi-conjugacy) by translations

$x \mapsto x - \alpha$ and $x \mapsto x + \alpha$, respectively, on the same intervals. However, f and g are conjugate one to the other by a, so we must have $\alpha = 1$, contradicting that they generate a rank 2 free abelian group. $\square$

The claim concludes the proof. $\square$

3 The non-locally-discrete case

Here we will be interested in groups acting with at most two fixed points, satisfying certain topological conditions in $\mathrm{Homeo}_+(\mathbb{S}^1)$, with respect to the C^0 topology. We will start this section with an interesting example that will motivate our definition of non-local discreteness.

Lemma 3.1. *There exists a finitely generated non-elementary subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ with at most two fixed points, but not Möbius-Like, and a non-empty wandering interval $I \subset \mathbb{S}^1$ such that the action of G restricted to I is non-discrete.*

Proof. First, we will construct an example of a non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with the stabilizer of a point $p \in \mathbb{S}^1$ being parabolic and isomorphic to $\mathbb{Z}^2$. For this, let T_α, T_β be two parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$ fixing the same point $p \in \mathbb{S}^1$, such that the subgroup $T = \langle T_\alpha, T_\beta \rangle$ is free abelian of rank 2. Using [15, Lemma 3.4], we can find a countable subset $D \subset \mathrm{SO}(2)$, such that, for every rotation $R_\rho \in \mathrm{SO}(2) \smallsetminus D$, it follows that $\langle T, R_\rho \rangle \simeq T * R_\rho$. So, we take a rotation $R_\rho \in \mathrm{SO}(2) \smallsetminus D$, and set $F = \langle T, R_\rho \rangle$. Observe that the stabilizer of the point p has not changed, that is $\mathrm{Stab}(F,p) = \mathrm{Stab}(T,p) = T$. Indeed, for every element $g \in \mathrm{PSL}(2,\mathbb{R})$ that fixes the point p, we have that $gT_\alpha g^{-1}$ commutes with T_α, so that if $\mathrm{Stab}(F,p) \neq T$, we could find an element $g \in F \smallsetminus T$ such that $[gT_\alpha g^{-1}, T_\alpha] = \mathrm{id}$. This is not possible after our choice of $R_\rho \notin D$. Finally, note that F is a non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with a parabolic stabilizer T of the point p isomorphic to $\mathbb{Z}^2$.

Now, let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be the isomorphic blow-up of F at the point p including a minimal action by translations of $\mathrm{Stab}(F,p)$ on an interval I (see Definition 2.14). We can choose such an action so that there are elements of $\mathrm{Stab}(F,p)$ with two parabolic fixed points. Since every element of $\mathrm{Stab}(F,p)$ has at most 1 fixed point, every non-trivial element of G will have at most 2 fixed points. Therefore, G is a non-elementary group action of the circle, with at most 2 fixed points whose action restricted to I is non-discrete. □

There is a natural strengthening of the notion of discreteness, which is more appropriate when one is interested in the dynamics of the group. After Lemma 3.1, a conventional definition for non-locally discrete actions of the circle will not be enough for the results that we are willing to present in this section. Therefore, we rather consider the notion of local discreteness introduced in Definition 1.10, which only takes into account the behavior in restriction to the minimal invariant subset. Now we are ready to discuss Theorem C, which states that non-locally discrete non-elementary subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points are conjugate into $\mathrm{PSL}(2,\mathbb{R})$. This will require some preliminary lemmas.

Lemma 3.2. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup acting with no global fixed point, and with at most 2 fixed points. Then, the minimal invariant subset of G contains every fixed point of any non-trivial element.*

Proof. If the action of G is free, there is nothing to prove. Therefore we can assume there exists a non-trivial element with at least one fixed point. Since $\mathbb{S}^1$ is a compact topological space, the action of G on $\mathbb{S}^1$ admits a minimal invariant subset Λ, and since we are assuming G acts with at most 2 fixed points and it has no global fixed point, this must be unique, and it contains at least two points. Take now any non-trivial element f and a point $x \in \mathrm{Fix}(f)$. If $x \in \Lambda$, we are done, otherwise, take a point $z \in \Lambda$. Then there are two cases: either $z \in \mathrm{Fix}(f)$ or one of the sequences $f^n(z)$ and $f^{-n}(z)$ converges to x. If the latter occurs, then $x \in \overline{\Lambda} = \Lambda$. In the other case, we can find a distinct $z' \in \Lambda \smallsetminus \{z\}$, which is not fixed by f, and so the previous argument applies. □

Lemma 3.3. *Every non-discrete non-elementary subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ whose action on the circle has at most N fixed points, is minimal.*

Proof. Assume the action admits an invariant Cantor set Λ. For given ε, every element $g \in G$ which is ε-close to the identity must fix every gap of Λ whose size exceeds ε. For sufficiently small ε, this gives that g fixes more than N points, and thus $g = \mathrm{id}$. □

Lemma 3.4. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a non-elementary subgroup with at most 2 fixed points and non-locally discrete, then its action is minimal and non-discrete.*

Moreover, for every interval $J \subset \mathbb{S}^1$ and every $\varepsilon > 0$, there exists an element $g \in G$ which is ε-close to the identity and has no fixed points in the interval J.

Proof. Since G is non-locally discrete, there exists a non-empty open interval $I \subset \mathbb{S}^1$ which intersects the minimal set K of G, and a sequence of elements $(g_n)_{n\in\mathbb{N}} \subset G$, such that $g_n|_I \to \mathrm{id}|_I$. Moreover, since G is non-elementary, by Lemma 2.21, there exists an element $f \in G$ conjugate to a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ and, by Lemma 3.2, both points fixed by f are in K, and their orbits are dense in K. Therefore, there exists a hyperbolic element in G, that we keep denoting by f (actually this is only a conjugate of the previous f), such that its repelling fixed point is in the interior of the interval I. Note that after the choice of f, we have $|f^m(I)| \to 1$ as $m \to \infty$.

Now, to show that the action of G is non-discrete, we will consider the following family of sequences, indexed by $m \in \mathbb{N}$,

$$(h_{m,n})_{n\in\mathbb{N}} := (f^m g_n f^{-m})_{n\in\mathbb{N}} \subset G.$$

Note that for a fixed $m \in \mathbb{N}$, the sequence $h_{m,n}$ converges to id in restriction to $f^m(I)$. Indeed, for every $\xi > 0$, choose $m \in \mathbb{N}$ such that $|f^m(I)| > 1 - \frac{\xi}{2}$, and then choose $n \in \mathbb{N}$ such that $|h_{m,n}(x) - x| < \frac{\xi}{2}$ for every $x \in f^m(I)$. We claim that $|h_{m,n}(x) - x| < \xi$, for every point $x \in \mathbb{S}^1$. For this, denote the endpoints of the interval $f^m(I)$ by a and $b \in \mathbb{S}^1$, that is $f^m(I) = (a,b)$, then for every $x \in \mathbb{S}^1$ such that $b < x < a$, it follows that

$$|h_{m,n}(x) - x| < |h_{m,n}(a) - b| < \left|a + \frac{\xi}{2} - b\right| < \left|a + \frac{\xi}{2} - \left(a - \frac{\xi}{2}\right)\right| = \xi.$$

We conclude that the action of G is non-discrete, and by Lemma 3.3 it is minimal.

Now, for the second part of the lemma, if $(h_{m,n})_{n\in\mathbb{N}} \subset G$ contains a subsequence without fixed points, there is nothing to do. Otherwise, by taking a subsequence, we can assume that the fixed points of $h_{m,n}$ are converging to p and $q \in \mathbb{S}^1$ (possibly $p = q$). Let $J \subset \mathbb{S}^1$ be any interval and since G is non-elementary and minimal, by Theorem 2.21 there exists an element $f \in G$ which sends p and q to the complement of J. Now, by choosing $h_{m,n}$ close enough to the identity, we can assume that $f h_{m,n} f^{-1} \in G$ is ε-close to the identity and this element only fixes $f(p)$ and $f(q)$, which are in the complement of the interval J. □

Lemma 3.5. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a non-discrete subgroup with at most 2 fixed points whose action is minimal, then there exists a sequence of elements in G converging to the identity and without fixed points.*

Proof. If for every $\varepsilon > 0$ there exists an element in G without fixed points, and ε-close to the identity then there is nothing to prove. Therefore we will assume that for a fixed (small) $\varepsilon > 0$, every element ε-close to the identity has fixed points. By Lemma 3.4, we can define recursively a sequence of nested intervals $J_{n+1} \subset J_n \subset \mathbb{S}^1$ with the following properties.

1. For any $n \in \mathbb{N}$, there exists an element $g_n \in G$ such that $\|g_n - \mathrm{id}\| \leq \frac{\varepsilon}{2}$ and the graph of g_n is strictly above the identity on the complement of J_n, and equal or below the identity on J_n.

2. The sequence J_n is shrinking to a point $p \in \mathbb{S}^1$, namely $\bigcap_n J_n = \{p\}$.

Note that, when n is sufficiently large so that $|J_n| < 1 - \frac{\varepsilon}{2}$, replacing g_n by an appropriate power g_n^m, guarantees also the condition that $\frac{\varepsilon}{4} < \|g_n - \mathrm{id}\| \leq \frac{\varepsilon}{2}$. Indeed, if we assume that g_n is $\frac{\varepsilon}{4}$-close to the identity (otherwise $m = 1$ works), since g_n has fixed points only in J_n, for a sufficiently large power $m \in \mathbb{N}$, the distance of g_n^m to the identity will be larger than $\frac{\varepsilon}{2}$. Therefore there exists $m_0 \in \mathbb{N}$ such that $g_n^{m_0}$ is not $\frac{\varepsilon}{2}$-close to the identity, but g_n^m is $\frac{\varepsilon}{2}$-close to the identity for every $0 \leq m < m_0$. Now one can observe that $g_n^{m_0-1}$ is $\frac{\varepsilon}{2}$-close to the identity, but it is not $\frac{\varepsilon}{4}$-close. Indeed, there exists $x \in \mathbb{S}^1$ such that

$$\frac{\varepsilon}{2} < |g_n^{m_0}(x) - x| < \left|g_n\left(g_n^{m_0-1}(x)\right) - g_n^{m_0-1}(x)\right| + |g_n^{m_0-1}(x) - x| < \frac{\varepsilon}{4} + |g_n^{m_0-1}(x) - x|.$$

From now on we will assume that $\frac{\varepsilon}{4} < \|g_n - \mathrm{id}\| \leq \frac{\varepsilon}{2}$, and that n is sufficiently large so that we can find a point $x_n \in \mathbb{S}^1 \smallsetminus J_n$ such that $g_n(x_n) \in (x_n, x_n + \frac{\varepsilon}{4})$ and consider the interval $I_n \subset \mathbb{S}^1$ defined by $I_n := (x_n + \frac{\varepsilon}{12}, x_n + \frac{\varepsilon}{6})$. By taking a subsequence we can assume that x_n converges to a point $x \in \mathbb{S}^1$, and for $n_0 \in \mathbb{N}$ large enough we have that $\bigcap_{n \geq n_0} I_n$ is a non-trivial interval, which we will denote by I. With such choices, for every $y \in I$ and $n \geq n_0$ we have $g_n(y) > y + \frac{\varepsilon}{12}$. On the other hand, since for every $n \in \mathbb{N}$ we have that $g_n(x_n) > x_n + \frac{\varepsilon}{4}$, g_n is above the identity on $[x_n, x_n + \frac{\varepsilon}{4}]$ and therefore J_n does not intersect $[x_n, x_n + \frac{\varepsilon}{4}]$. Thus, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$ one has $|x_n - x| < \frac{\varepsilon}{48}$, which implies that the interval J_n is at distance at least $\frac{\varepsilon}{24}$ to the interval I. We will write J for the union $\bigcup_{n \geq n_1} J_n \subset \mathbb{S}^1 \smallsetminus I$.

Now, using Lemma 3.4, we take an element $f \in G$ which is $\frac{\varepsilon}{2}$-close to the identity and strictly below the identity in the complement of I, therefore there exists $\delta > 0$ such that $f(y) < y - \delta$ for every $y \in J$. Take $m > \max\{n_0, n_1\}$ sufficiently large, such that $|J_m| < \delta$, and we claim that the element $f^{-1}g_m \in G$ is ε-close to the identity and it has no fixed points in the circle.

Indeed, since f is strictly below the identity on the complement of I and g_m is strictly above the identity on the complement of J_m, it is clear that g_m does not cross f in the complement of $J_m \cup I$. Now, one can notice that the size of the interval I is smaller than $\frac{\varepsilon}{12}$ and $g_m(y) > y + \frac{\varepsilon}{12}$ for every $y \in I$, which implies that g_m does not cross f in I. Similarly, the size of the interval J_m is smaller than δ and $f(y) < y - \delta$ for every $y \in J \supset J_m$, which implies that g_m does not cross f in J_m. Therefore, g_m does not cross f in the whole circle $\mathbb{S}^1$, which implies that the element $f^{-1}g_m$ has no fixed points in $\mathbb{S}^1$, but $f^{-1}g_m$ is a composition of two elements $\frac{\varepsilon}{2}$-close to the identity, therefore it is also ε-close to the identity. □

Given a subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, we denote by $\overline{G}$ its *closure* in $\mathrm{Homeo}_+(\mathbb{S}^1)$ with respect to the C^0 topology, which is still a subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$. Clearly $G = \overline{G}$ if and only if G is discrete with respect to the C^0 topology.

Lemma 3.6. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a subgroup with at most 2 fixed points acting minimally, then its closure $\overline{G}$ has at most 2 fixed points. Moreover, if G is non-discrete then $\overline{G}$ is Möbius-Like.*

Proof. We will assume that the group G is non-discrete, otherwise $\overline{G} = G$ which has at most 2 fixed points. One can notice that an element of $\overline{G}$ can cross the identity at most twice, indeed if an element $g \in \overline{G}$ crosses hyperbolically the identity 3 or more times, then any element sufficiently C^0-close to g will also cross the identity at least 3 times and therefore it will have 3 or more fixed points.

Now, we claim that for every element $f \in \overline{G}$, the support of f cannot have two connected components on the same side of the identity. Indeed, by Lemma 3.5, we take an element $g \in G$,

ε-close to the identity without fixed points, one can notice that for ε sufficiently small, g or g^{-1} crosses the element f at least 4 times, which contradicts the hypothesis of at most 2 fixed points.

The last argument implies that for every element $f \in \overline{G}$, the complement of $\mathrm{Fix}(f)$ can contain at most 2 intervals, so that we are reduced to three cases: either f has at most 2 fixed points, or f fixes one non-trivial interval, or f fixes two intervals where at least one of them is non-trivial and the two complementary intervals are on opposite sides of the identity.

In the first case there is nothing else to prove, therefore we will focus on the two other cases. First, let us assume that there exists an element $f \in \overline{G}$ such that f fixes two intervals with at least one of them being non-trivial. Let $\varepsilon > 0$ be smaller than the size of both non-trivial fixed intervals and than the size of both complementary intervals then, by Lemma 3.5, choose an element $g \in G$ which is ε-close to the identity without fixed points. We claim that the support of the element $f^{-1}gfg^{-1} \in \overline{G}$ contains at least 3 connected components, which contradicts our hypothesis of G being a subgroup with at most 2 fixed points. For this, we conclude that no element in $\overline{G}$ fixes two intervals with one of them being non-trivial.

Let us assume that $f \in \overline{G}$ fixes only one non-trivial interval and no other point. Similarly as for the last argument, we take $\varepsilon > 0$ smaller than the size of the fixed interval and its complement then, by Lemma 3.5, we choose an element $g \in G$ ε-close to the identity without fixed points. We claim that the element $f^{-1}gfg^{-1} \in \overline{G}$ fixes two intervals with at least one of them being non-trivial, which by the last argument contradicts our hypothesis of G being a subgroup with at most 2 fixed points. For this, we conclude that no element in $\overline{G}$ fixes a non-trivial interval.

Therefore, we conclude that for every element of $\overline{G}$ has at most 2 fixed points, and since no element can have two connected components of the support on the same side of the identity there is no element with two parabolic fixed points, which implies that $\overline{G}$ is also Möbius-Like. ☐

In the next lemma we will mostly discuss circle homeomorphisms close to the identity without fixed points, such that it makes sense to say that they are *above the identity* or *below the identity.* For that reason, we will state the next definition.

Definition 3.7. Let $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ be a circle homeomorphism $\frac{1}{3}$-close to the identity free of fixed points. We say that f is *positive* (or *above the identity*) if for every $x \in \mathbb{S}^1$, we have $f(x) \in (x, x+\frac{1}{3}]$. If f is not positive, we say that f is *negative* (or *below the identity*).

Remark 3.8. Every circle homeomorphism $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which is $\frac{1}{3}$-close to the identity and without fixed points is positive or negative. Moreover, f and f^{-1} have opposite signs and the composition of two positive circle homeomorphisms f_1 and f_2 is positive if and only if $f_1 f_2$ is $\frac{1}{3}$-close to the identity.

Lemma 3.9. *Let $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ be a circle homeomorphism of order $q \geq 2$, with rotation number $\mathrm{rot}(f) = \frac{p}{q}$, where $p, q \in \mathbb{Z} \setminus \{0\}$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every positive δ-close to the identity circle homeomorphism $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$, it follows that:*

$$\mathrm{dist}_{\mathcal{C}^0}(f, gf) < \varepsilon \quad \textit{and} \quad \mathrm{rot}(gf) \in \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{q^3}\right].$$

Proof. We fix $\varepsilon > 0$ and we will find $\delta_1 > 0$ which satisfies that $\mathrm{dist}_{\mathcal{C}^0}(f, gf) < \varepsilon$ for every circle homeomorphism $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which is δ_1-close to the identity. We recall that $\|f - gf\| = \|\mathrm{id} - g\|$, therefore if g is $\frac{\varepsilon}{2}$-close to the identity we can assure that $\|f - gf\| \leq \frac{\varepsilon}{2}$.

On the other hand, since f^{-1} is uniformly continuous on $\mathbb{S}^1$, there exists $\delta_0 > 0$ such that if $|x - y| \leq \delta_0$ then $|f^{-1}(x) - f^{-1}(y)| < \frac{\varepsilon}{2}$. So, if we choose $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ δ_0-close to the identity (so that $\delta_0 \geq \|\mathrm{id} - g\| = \|g^{-1} - \mathrm{id}\|$), we then have

$$\|f^{-1}g^{-1} - f^{-1}\| < \frac{\varepsilon}{2}.$$

Therefore, we take $\delta_1 = \min\{\delta_0, \frac{\varepsilon}{2}\}$ and we conclude that

$$\mathrm{dist}_{\mathcal{C}^0}(f, gf) = \|f - gf\| + \|f^{-1} - f^{-1}g^{-1}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Without loss of generality we will assume that $\gcd(p, q) = 1$ and let $\widetilde{f} \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ be the lift of f such that $\widetilde{f}(0) \in (0, 1)$. Since the composition is continuous in the metric space of homeomorphisms of the line, there exists $\delta_2 > 0$ such that for every positive homeomorphism $\widetilde{g} \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ which is δ_2-close to the identity (on $\mathbb{R}$) we have that

$$\left(\widetilde{g}\widetilde{f}\right)^q(x) \in \left(\widetilde{f}^q(x), \widetilde{f}^q(x) + \frac{1}{q^2}\right) \qquad \text{for every } x \in \mathbb{R}.$$

Now, from the assumption, we have that $\widetilde{f}^q(x) = x + p$ for any $x \in \mathbb{R}$, therefore we have

$$\left(\widetilde{g}\widetilde{f}\right)^q(x) \in \left(x + p, x + p + \frac{1}{q^2}\right).$$

Thus, for any $n \geq 2$ one has

$$\left(\widetilde{g}\widetilde{f}\right)^{nq}(x) \in \left(\left(\widetilde{g}\widetilde{f}\right)^{(n-1)q}(x) + p, \left(\widetilde{g}\widetilde{f}\right)^{(n-1)q}(x) + p + \frac{1}{q^2}\right)$$

On the other hand, one has

$$\left(\left(\widetilde{g}\widetilde{f}\right)^{(n-1)q}(x) + p, \left(\widetilde{g}\widetilde{f}\right)^{(n-1)q}(x) + p + \frac{1}{q^2}\right) \subset \cdots \subset \left(x + np, x + np + \frac{n}{q^2}\right).$$

Therefore, we have that the rotation number of $gf \in \mathrm{Homeo}_+(\mathbb{S}^1)$, where $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is the projection of $\widetilde{g}$, is given by

$$\mathrm{rot}(gf) = \lim_{n\to+\infty} \frac{\left(\widetilde{g}\widetilde{f}\right)^{nq}(0)}{nq} \in \left[\frac{np}{nq}, \frac{np + \frac{n}{q^2}}{nq}\right] = \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{q^3}\right].$$

To finish the proof, we just need to observe that $\mathrm{rot}(gf) \neq \mathrm{rot}(f)$. Indeed, if $\mathrm{rot}(gf) = \frac{p}{q}$ then there exists a point $x \in \mathbb{S}^1$ such that $(gf)^q(x) = x$, which implies that $\left(\widetilde{g}\widetilde{f}\right)^q(x) = x + k$ for some integer $k \in \mathbb{Z}$. On the other hand, we know that

$$k = \left(\widetilde{g}\widetilde{f}\right)^q(x) - x \in \left(p, p + \frac{1}{q^2}\right),$$

which however does not contain any integer number. So, we conclude that $\mathrm{rot}(gf) \in \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{q^3}\right]$ for every positive circle homeomorphism $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which is δ_2-close to the identity. Taking $\delta = \min\{\delta_1, \delta_2\}$, we have the statement of the lemma. □

Lemma 3.10. *With the assumptions as in Lemma 3.5, we have that $\overline{G}$ contains an element with irrational rotation number.*

Proof. If the subgroup $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ has an element with an irrational rotation number there is nothing to prove, because G is, by definition, a subgroup of $\overline{G}$. Therefore, we will suppose that G has no element with irrational rotation number.

Take a sequence of elements $(f_n)_{n\in\mathbb{N}} \subset G$ without fixed points and converging to the identity, whose existence is ensured by Lemma 3.5. By changing f_n for f_n^{-1} when necessary and taking a subsequence we can assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of positive circle homeomorphisms whose distance to the identity decreases.

Now we are going to contruct a sequence $(h_n)_{n\in\mathbb{N}} \subset G$ converging in the space of circle homeomorphisms to $h \in \overline{G}$ with irrational rotation number. We start by choosing a numeric sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\sum \varepsilon_n < \frac{1}{3}$ and we define $h_0 \in G$ as the first element f_{m_0} such that $\mathrm{dist}_{\mathcal{C}^0}(f_{m_0}, \mathrm{id}) < \varepsilon_0$.

Now, let us assume by induction that h_n is a positive circle homeomorphism with rotation number equal to $\frac{p_n}{q_n}$ and $\mathrm{dist}_{\mathcal{C}^0}(h_n, \mathrm{id}) < \sum_{k=0}^{n} \varepsilon_k$ then, by Lemma 3.9, there exists $\delta > 0$ such that for every for every positive circle homeomorphism $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ which is δ-close to the identity, we have that $\mathrm{dist}_{\mathcal{C}^0}(h_n, gh_n) < \varepsilon_{n+1}$ and $\mathrm{rot}(gh_n) \in \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3}\right]$. Therefore, for every $n \in \mathbb{N}$, we define h_{n+1} as $f_{m_n} h_n \in G$, where f_{m_n} is the first element of the sequence $(f_n)_{n\in\mathbb{N}}$ such that $\|f_{m_n} - \mathrm{id}\| < \delta$. Now, one can notice that

$$\mathrm{dist}_{\mathcal{C}^0}(h_{n+1}, \mathrm{id}) \leq \mathrm{dist}_{\mathcal{C}^0}(h_{n+1}, h_n) + \mathrm{dist}_{\mathcal{C}^0}(h_n, \mathrm{id}) < \mathrm{dist}_{\mathcal{C}^0}(f_{m_n} h_n, h_n) + \sum_{k=0}^{n} \varepsilon_k < \sum_{k=0}^{n+1} \varepsilon_k.$$

Since $\sum_{k=0}^{n+1} \varepsilon_k < \frac{1}{3}$, one can notice that $h_{n+1} \in G$ is also a positive circle homeomorphism with rational rotation number that we will denote by $\frac{p_{n+1}}{q_{n+1}}$, which is contained in the interval $\left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3}\right]$.

We claim that the sequence $(h_n)_{n\in\mathbb{N}} \subset G$ converges into $h \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and that $\mathrm{rot}(h) \notin \mathbb{Q}$. Indeed, for every n and $m \in \mathbb{N}$ it follows that

$$\mathrm{dist}_{\mathcal{C}^0}(h_{n+m}, h_n) \leq \sum_{k=n}^{n+m} \mathrm{dist}_{\mathcal{C}^0}(h_{k+1}, h_k) < \sum_{k=n}^{n+m} \varepsilon_k < \sum_{k=n}^{+\infty} \varepsilon_k \xrightarrow{n\to+\infty} 0.$$

Then, $(h_n)_{n\in\mathbb{N}} \subset G$ is a Cauchy sequence with respect to the $\mathcal{C}^0$ topology and by completeness of $\mathrm{Homeo}_+(\mathbb{S}^1)$ under this topology, we have that h_n converges to a circle homeomorphism $h \in \overline{G}$.

Now we define, for every $n \in \mathbb{N}$, the intervals $I_n = \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^2}\right)$. By the classical Dirichlet's approximation theorem, we have that $\bigcap I_n$ is an irrational number. Let us detail this for completeness. First, we recall that

$$\frac{p_{n+1}}{q_{n+1}} \in \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^3}\right].$$

So that we only have to prove that $\frac{p_{n+1}}{q_{n+1}} + \frac{1}{q_{n+1}^2} < \frac{p_n}{q_n} + \frac{1}{q_n^2}$. We will first show that $q_{n+1} > q_n$. Indeed, if $q_{n+1} \leq q_n$ then

$$p_{n+1} \in \left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{q_{n+1}}{q_n^3}\right] \subset \left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{1}{q_n^2}\right]$$

which is an absurd, because $\left(\frac{p_n q_{n+1}}{q_n}, \frac{p_n q_{n+1}}{q_n} + \frac{1}{q_n^2}\right]$ does not contain any integer. Now, we have the following inequality

$$\frac{p_{n+1}}{q_{n+1}} + \frac{1}{q_{n+1}^2} \leq \frac{p_n}{q_n} + \frac{1}{q_n^3} + \frac{1}{q_{n+1}^2} \leq \frac{p_n}{q_n} + \frac{1}{q_n^3} + \frac{1}{(q_n+1)^2} = \frac{p_n}{q_n} + \frac{q_n^3 + (q_n+1)^2}{q_n^3(q_n+1)^2}$$

$$= \frac{p_n}{q_n} + \frac{1}{q_n^2}\frac{q_n^3 + q_n^2 + 2q_n + 1}{q_n^3 + 2q_n^2 + q_n} < \frac{p_n}{q_n} + \frac{1}{q_n^2}\frac{q_n^3 + q_n^2 + 2q_n + 1 + (q_n^2 - q_n - 1)}{q_n^3 + 2q_n^2 + q_n} = \frac{p_n}{q_n} + \frac{1}{q_n^2}$$

So, we conclude that $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$ and we also remark, by the definition of I_n, that $\mathrm{rot}(h_{n+k}) \in I_n$ for every $k > 0$, then by the continuity of $\mathrm{rot} : \mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{S}^1$, it follows that $\mathrm{rot}(h) \in I_n$, for every $n \in \mathbb{N}$.

On the other hand, $(q_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of integers, which implies that $q_n \to +\infty$ and that $|I_n| = \frac{1}{q_n^2} \longrightarrow 0$, so $\bigcap_{n\in\mathbb{N}} I_n$ converges to a point $\alpha \in \mathbb{R}$, which is the only possible value of $\mathrm{rot}(h)$. We claim that α is not a rational number, otherwise $\alpha = \frac{p}{q}$ with p and $q \in \mathbb{Z}$ and $q \geq 2$. Which implies that $\frac{p}{q} \in I_n = \left(\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{1}{q_n^2}\right)$, for every $n \in \mathbb{N}$. Therefore, for n sufficiently large, we have $2q < q_n$ and then

$$p \in \left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{q}{q_n^2}\right) \subset \left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{1}{2q_n}\right)$$

which is an absurd, because $\left(\frac{p_n q}{q_n}, \frac{p_n q}{q_n} + \frac{1}{2q_n}\right)$ does not contain any integer.

We conclude that $h \in \overline{G}$ has an irrational rotation number. □

Lemma 3.11. *With the assumptions as in Lemma 3.5, we have that $\overline{G}$ contains an element which is conjugate to an irrational rotation. Therefore, up to conjugacy, $\overline{G}$ contains* $\mathrm{SO}(2)$.

Proof. Let $f \in \overline{G}$ be an element with irrational rotation number, and let us suppose it is not conjugate to a rotation. Let μ be an ergodic f-invariant probability measure, and denote by Λ the minimal f-invariant Cantor set. Take a point $x \in \Lambda$ which is not in the closure of any non-wandering interval, so that we can find an increasing sequence $\{n_j\}$ such that the intervals $[x, f^{n_j}(x))$ are nested, and such that $\mu([x, f^{n_j}(x)))$ tends to 0 as $j \to \infty$. Under such assumptions, the distance between x and $f^{n_j}(x)$ tends to 0. Indeed, if the sequence of intervals $[x, f^{n_j}(x))$ does not converge to the point x, then there exists $y \in \mathbb{S}^1 \setminus \{x\}$ such that $[x, y] \subset [x, f^{n_j}(x))$, for every $j \in \mathbb{N}$. On the other hand, the measure of $\mu([x, f^{n_j}(x)))$ goes to 0 and it implies that $\mu([x, y]) = 0$, which is not possible since $[x, y]$ would be a non-wandering interval.

Now, we remark that for every gap $I_a = [a, b]$ (that is, the closure of a connected component of $\mathbb{S}^1 \setminus \Lambda$), we have that $g(a) \notin (a, b)$ for every $g \in \overline{G}$. Therefore, $f^{n_j}(x) \to x$ for every $x \in \Lambda$ which is not in the closure of a gap and $f^{n_j}(a) > a + |I_a|$ for every $a \in \mathbb{S}^1$ which is the leftmost point of a gap.

Take $\varepsilon > 0$ smaller than the size of the two largest gaps. By Lemma 3.5, we can choose a positive element $g \in G$ which is ε-close to the identity. Note that for $j \in \mathbb{N}$ sufficiently large, g crosses f^{n_j} at least 4 times, which is an absurd since $g^{-1} f^{n_j} \in \overline{G}$ has at most 2 fixed points. So we conclude that $f \in \overline{G}$ with $\mathrm{rot}(f) \notin \mathbb{Q}$ is conjugate to an irrational rotation and therefore $\overline{G}$ contains a subgroup which conjugate to $\mathrm{SO}(2)$. □

The following result can be obtained as a direct consequence of a theorem by Giblin and Markovic [12, Theorem 1.2]. For sake of completeness, we will provide a direct proof.

Lemma 3.12. *Let $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ be a non-elementary closed subgroup with at most 2 fixed points. Assume G contains the subgroup of rotations $\mathrm{SO}(2)$. Then, $G = \mathrm{PSL}(2, \mathbb{R})$.*

Proof. We denote by $H = H_y$ the stabilizer in G of a point $y \in \mathbb{S}^1$, which is a closed subgroup of G. As $\mathrm{SO}(2)$ acts transitively on $\mathbb{S}^1$, we have that all point stabilizers are conjugate (by a rotation) in G, and $\mathbb{S}^1 \cong G/H$. Moreover, as H fixes a point and $\mathrm{SO}(2)$ acts freely, we can write $G = \mathrm{SO}(2)H$, and $H \cap \mathrm{SO}(2) = \{\mathrm{id}\}$. Thus, in order to understand what is G, we are reduced to understand what H is.

As G is non-elementary and with at most 2 fixed points, there exists a hyperbolic element $f \in G$ (Lemma 2.21); by transitivity, we can assume that y is the repelling fixed point of f, and we denote by z the other fixed point. We also denote by $I = (y, z)$ the oriented interval between y and z.

Claim 1. *For any $\delta \in (0, 1)$, there exists an element $h \in H$ such that $|h(I)| = \delta$.*

Proof. For any $k \in \mathbb{Z}$, let us consider the continuous function $d_k : \mathbb{S}^1 \to (0,1)$, defined by $d_k(t) := |f^k R_t(I)|$. As we are assuming that y is the repelling fixed point for f, for any sufficiently small $t > 0$ we have $R_t(I) \subset \mathbb{S}^1 \setminus \{y\}$ and thus $d_k(t) \to 0$ as $k \to \infty$; similarly, for any sufficiently small $t < 0$ we have $d_k(t) \to 1$ as $k \to \infty$. Therefore, by continuity of d_k, for k large enough, there exists $t \in \mathbb{S}^1$ such that $d_k(t) = \delta$. The element $g = f^k R_t$ satisfies $|g(I)| = \delta$. Take then a rotation $R_s \in \mathrm{SO}(2)$ such that $R_s g(y) = y$. Thus $h := R_s g \in H$ and $|h(I)| = \delta$, as desired. □

As G acts with at most 2 fixed points, we have that H acts with at most 1 fixed point on the open interval $\mathbb{S}^1 \setminus \{y\}$. Moreover, by Claim 1, such action is transitive. It follows from Solodov's theorem (Theorem 1.4) H is conjugate to closed a subgroup of the affine group $A \leq \mathrm{Aff}_+(\mathbb{R})$, which acts transitively and contains hyperbolic elements.

We now want to argue that H is actually conjugate to the whole group $\mathrm{Aff}_+(\mathbb{R})$. Note first that as H is closed and acts transitively on $\mathbb{S}^1 \setminus \{y\}$, we have that H contains a normal subgroup $T \cong \mathbb{R}$ isomorphic to the subgroup of translations in $\mathrm{Aff}_+(\mathbb{R})$, and $H \cong T \rtimes A$, where $A \leq H$ is isomorphic to a closed subgroup of the multiplicative group $\mathbb{R}_+^*$, given by the stabilizer in H of the point z. It is enough then to prove the following

Claim 2. *The subgroup A is non-discrete.*

Proof. Fix $\varepsilon \in (0, 1/2)$. For any $s \in \mathbb{S}^1 \setminus \{0\}$, the element $g_s = f R_s f^{-1}$ is not a rotation, and we have that $\|g_s - \mathrm{id}\|_\infty \to 0$ as $s \to 0$. In particular, we can choose $s > 0$ such that $\|g_s - \mathrm{id}\|_\infty < \varepsilon$.

Now, for $t \in (0, \varepsilon)$, the composition $R_t^{-1} g_s$ has at most 2 fixed points and satisfies $\|R_t^{-1} g_s - \mathrm{id}\| < \varepsilon$. Moreover, by the choice of s, when $t = 0$ the graph of the composition $R_0^{-1} g_s = g_s$ is above the diagonal, whereas when $t = \varepsilon$ we have that the graph of $R_\varepsilon^{-1} g_s$ is below the diagonal. Therefore, by continuity, the subset $E \subset (0, \varepsilon)$ such that $R_t^{-1} g_s$ has 2 fixed points, denoted by y_t, z_t, is non-empty, and the length of the interval (y_t, z_t) for $t \in E$ varies continuously from 0 to 1. Thus, we can choose $t_0 \in E$ such that $|(y_{t_0}, z_{t_0})| = |(y, z)|$.

We can now take a rotation $S \in \mathrm{SO}(2) \leq G$ such that $k = S R_t^{-1} g_s S^{-1}$ is in H, and actually, as $|(y_{t_0}, z_{t_0})| = |(y, z)|$ we get that $k \in A$. As $\varepsilon > 0$ was arbitrary, we deduce that A is non-discrete. □

At this point, one can deduce that $G \cong \mathrm{SO}(2)\mathrm{Aff}_+(\mathbb{R})$ is a connected non-compact Lie group acting transitively on $\mathbb{S}^1$ with at most 2 fixed points, and use this to conclude that G is actually equal to $\mathrm{PSL}(2, \mathbb{R})$, relying on some classic Lie theory [11, §4.1]. We prefer however to follow more elementary arguments of dynamical nature.

As H is conjugate to the affine group $\mathrm{Aff}_+(\mathbb{R})$, there exists a homeomorphism $\phi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ such that $\phi H \phi^{-1} = \mathrm{Stab}(\mathrm{PSL}(2, \mathbb{R}), y)$. This conjugation ϕ is unique up to an affine rescaling, which we will fix in the proof of Claim 3 below. Let us also consider the homeomorphism $\phi_2 \in \mathrm{Homeo}_+(\mathbb{S}^1)$ given by $\phi_2 := R_{\frac{1}{2}}\, \phi\, R_{\frac{1}{2}}$, and the point $y_2 := R_{\frac{1}{2}}(y)$. Note that the element ϕ_2 conjugates $\mathrm{Stab}(G, y_2) = H_{y_2}$ and $\mathrm{Stab}(PSL(2, \mathbb{R}), y_2)$. We also have that $\mathrm{Stab}(G, y, y_2) = H \cap H_{y_2}$ is isomorphic to $\mathbb{R}_+^*$. So, the map

$$\psi : g \in \mathrm{Stab}(G, y, y_2) \mapsto R_{\frac{1}{2}}\, g\, R_{\frac{1}{2}} \in \mathrm{Stab}(G, y, y_2)$$

defines a non-trivial automorphism of $\mathbb{R}_+^*$, with $\psi^2 = \mathrm{id}$, so that $\psi(g) = g^{-1}$. We deduce that for every $g \in \mathrm{Stab}(G, y, y_2)$, we have

$$\begin{aligned} \phi_2\, g\, \phi_2^{-1} &= R_{\frac{1}{2}}\, \phi\, R_{\frac{1}{2}}\, g\, R_{\frac{1}{2}}\, \phi^{-1}\, R_{\frac{1}{2}} = (\phi\, R_{\frac{1}{2}}\, g\, R_{\frac{1}{2}}\, \phi^{-1})^{-1} \\ &= \phi\, R_{\frac{1}{2}}\, g^{-1} R_{\frac{1}{2}}\, \phi^{-1} = \phi\ g\ \phi^{-1}. \end{aligned}$$

Therefore $\gamma := \phi_2^{-1}\phi$ is in the centralizer of $\mathrm{Stab}(G, y, y_2) \cong \mathbb{R}_+^*$, so there are two elements $g_+, g_- \in \mathrm{Stab}(G, y, y_2)$ such that $\gamma = g_+$ on $[y, y_2]$, and $\gamma = g_-$ on $[y_2, y]$. As ψ takes g_+ and g_- to their inverses, we deduce that $R_{\frac{1}{2}}\,\gamma\,R_{\frac{1}{2}} = \gamma^{-1}$, and thus $\gamma = g_+ = g_-$ on $\mathbb{S}^1$, and therefore $\gamma \in \mathrm{Stab}(G, y, y_2)$. Similarly, we get that $\phi\,\phi_2^{-1} = \phi_2\,\gamma\,\phi_2^{-1} \in \mathrm{Stab}(\mathrm{PSL}(2,\mathbb{R}), y, y_2)$.

Claim 3. *There exists* $\Phi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ *that conjugate* $\mathrm{stab}(\overline{G}, x)$ *with* $\mathrm{stab}(PSL(2,\mathbb{R}), x)$*, and also conjugate* $\mathrm{stab}(\overline{G}, x_2)$ *with* $\mathrm{stab}(PSL(2,\mathbb{R}), x_2)$*.*

Proof. Take $\rho \in \mathrm{Stab}(G, y, y_2) \cong \mathbb{R}_+^*$ such that, $\rho^{-2} = \gamma$. Then, the element $\Phi := \rho\,\phi_2$ conjugates $\mathrm{Stab}(G, y)$ to $\mathrm{Stab}(\mathrm{PSL}(2,\mathbb{R}), y)$, and similarly does $\Phi_2 := R_{\frac{1}{2}}\,\Phi\,R_{\frac{1}{2}}$ with $\mathrm{Stab}(G, y_2)$ and $\mathrm{Stab}(\mathrm{PSL}(2,\mathbb{R}), y_2)$. Note that

$$\Phi_2 = R_{\frac{1}{2}}\,\rho\,\phi\,R_{\frac{1}{2}} = R_{\frac{1}{2}}\,\rho\,R_{\frac{1}{2}}\,R_{\frac{1}{2}}\,\phi\,R_{\frac{1}{2}} = \rho^{-1}R_{\frac{1}{2}}\,\phi\,R_{\frac{1}{2}} = \rho^{-1}\phi_2$$

Since $\rho^{-2} = \phi\,\phi_2^{-1}$, we have that $\mathrm{id} = \rho\,\phi\,\phi_2^{-1}\rho = \Phi\,\Phi_2^{-1}$. So we have that $\Phi = \Phi_2$. ☐

Now, observe that $G = \mathrm{Stab}(G, y_2)\,\mathrm{Stab}(G, y)$. Indeed, take any $g \in G$. If $g \in \mathrm{Stab}(G, y)$ we are done, otherwise write $z = g(y) \neq y$, and take an element $f \in \mathrm{Stab}(G, y_2)$ such that $f(z) = y$. Thus $fg \in \mathrm{Stab}(G, y)$, as desired. Similarly, one gets $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Stab}(\mathrm{PSL}(2,\mathbb{R}), y_2)\,\mathrm{Stab}(\mathrm{PSL}(2,\mathbb{R}), y)$. We conclude that the homeomorphism Φ from Claim 3 conjugates G to $\mathrm{PSL}(2,\mathbb{R})$, as desired. ☐

We can now prove the main result of this section.

Proof of Theorem C. Let G be a non-elementary subgroup with at most 2 fixed points. If G is non-locally discrete, then after Lemma 3.4, G is non-discrete, so that by Lemma 3.11 its closure $\overline{G}$ contains a conjugate copy of the subgroup of rotations $\mathrm{SO}(2)$. From Lemma 3.12, we conclude that $\overline{G}$ is conjugate to $\mathrm{PSL}(2,\mathbb{R})$, so that G is conjugate into $\mathrm{PSL}(2,\mathbb{R})$. ☐

4 Amalgamated products of group actions

In this chapter we present the main tool used to build examples for the rest of this work, the definition of amalgamated product of group actions (stated in Definition 4.5). As we show in Theorem 4.7, such product has a direct construction which is unique up to conjugacy and the dynamics imposed by the amalgamated product will be very convenient once we try to track a boundary for the number of fixed points of non-trivial elements in future chapters.

The first step leading to such definition is the concept of proper ping-pong partition, which is given below.

Definition 4.1. Let F and G be non-trivial groups of homeomorphisms of a topological space X, and assume that $S = F \cap G$ is a proper subgroup in both F and G. A *proper ping-pong partition* $(\mathcal{U}_F, \mathcal{U}_G)$ *for* F *and* G is a pair of non-empty disjoint open subsets $\mathcal{U}_F$ and $\mathcal{U}_G \subset X$ with finitely many connected components, such that:

i. $(F \setminus S)(\mathcal{U}_G) \subset \mathcal{U}_F$ and $(G \setminus S)(\mathcal{U}_F) \subset \mathcal{U}_G$;

ii. $S(\mathcal{U}_F) = \mathcal{U}_F$ and $S(\mathcal{U}_G) = \mathcal{U}_G$;

iii. $S \lneq F$ and $S \lneq G$ and the index of S in one of them is greater than 2.

Lemma 4.2 (Ping-Pong Lemma)**.** *With the notations as in Definition 4.1, if* $(\mathcal{U}_F, \mathcal{U}_G)$ *is a proper ping-pong partition for* F *and* G*, then the subgroup* $\langle F, G\rangle \leq \mathrm{Homeo}(X)$ *is isomorphic to the amalgamated free product* $F *_S G$*.*

This is in fact a particular case of the ping-pong lemma for amalgamated free products (which holds more generally for bijections of a set), originally due to Fenchel and Nielsen [9]. A detailed proof can be found in Maskit [19, Section VII.A].

For the next result we need the assumption that the space X be compact.

Lemma 4.3. *Assume that* X *is a compact topological space. With notation as in Definition 4.1, if* $(\mathcal{U}_F, \mathcal{U}_G)$ *is a proper ping-pong partition for* F *and* G*, then the closure of the partition* $\overline{\mathcal{U}_F \cup \mathcal{U}_G}$ *contains an invariant closed subset for the action of* $H = \langle F, G\rangle$ *on* X*.*

In particular, a necessary condition for the action of H *being minimal is that* $\overline{\mathcal{U}_F \cup \mathcal{U}_G} = X$*.*

Proof. This is also classical, but we sketch a proof as it is quite elementary. Set $F^* = F \setminus S$ and $G^* = G \setminus S$, then, using the fact that $(\mathcal{U}_F, \mathcal{U}_G)$ is a ping-pong partition, we see that the family of subsets $\{\Lambda_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{aligned}
\Lambda_0 &= \overline{\mathcal{U}_F} \cup \overline{\mathcal{U}_G},\\
\Lambda_1 &= \overline{F^*(\mathcal{U}_G)} \cup \overline{G^*(\mathcal{U}_F)},\\
\Lambda_2 &= \overline{F^*G^*(\mathcal{U}_F)} \cup \overline{G^*F^*(\mathcal{U}_G)},\\
\Lambda_3 &= \overline{F^*G^*F^*(\mathcal{U}_G)} \cup \overline{G^*F^*G^*(\mathcal{U}_F)}, \ldots
\end{aligned}$$

has the finite intersection properties (more precisely, they are nested) and therefore $\Lambda = \bigcap_{i\in\mathbb{N}} \Lambda_i$ is closed, non-empty. It is not difficult to see that Λ is invariant by F and G, and hence by H. □

Lemma 4.4. *With notation as in Lemma 4.3, assume that* $X = \mathbb{S}^1$ *is the circle and that* $H \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ *preserves the orientation. Then, the minimal set of the action of* H *on the circle is unique and infinite.*

Proof. By Lemma 4.3, it follows that $\overline{\mathcal{U}_F \cup \mathcal{U}_G}$ contains a minimal set, that we will denote by K. Let us suppose by contradiction that K is finite. Without loss of generality, we assume that the index of S in G is greater than 2.

Define $N_G := \#\{x \in K \cap \mathcal{U}_G\}$ and $N_F := \#\{x \in K \cap \mathcal{U}_F\}$ and observe that $N_F \geq N_G$, since there exists $f \in F \setminus S$ and for each point $x \in K \cap \mathcal{U}_G$, we have $f(x) \in K \cap \mathcal{U}_F$. On the other hand, there are two elements $g_1 \neq g_2 \in G \setminus S$ from different left cosets of S in G, and then, for each two points $x_i, x_j \in K \cap \mathcal{U}_F$ we have $g_1(x_i) \neq g_2(x_j)$, otherwise, $g_1^{-1} g_2(x_j) = x_i$ that is $g_1^{-1} g_2 \in G$ sends a point of $\mathcal{U}_F$ into $\mathcal{U}_F$, then should have $g_1^{-1} g_2 \in S$, that is, there exists $s \in S$ with $g_2 = g_1 s$, an absurd since g_1 and g_2 are from different left cosets. This implies in $2N_G \geq N_F$.

We conclude that the only possibility is $N_F = N_G = 0$, that is, $K \cap (\mathcal{U}_F \cup \mathcal{U}_G) = \emptyset$. The same argument can be done to $\overline{\mathcal{U}_F}$ and $\overline{\mathcal{U}_G}$ by continuity and taking in consideration the orientation-preserving property, and that implies in $K = \emptyset$ which is an absurd.

Now, since K is a minimal non-finite on the circle, then it is unique. ☐

We now introduce the notion of *amalgamated product of actions*, this will be a very important concept that will be used in the constructions of the mains examples of this paper.

Definition 4.5. Given two countable subgroups F and $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ and two collections of n circularly-ordered points $\overline{x} = \{x_1, \ldots, x_n\} \subset \mathbb{S}^1$ and $\overline{y} = \{y_1, \ldots, y_n\} \subset \mathbb{S}^1$ such that, for all distinct $i, j \in \{1, \ldots, n\}$, the following conditions are satisfied:

- $x_i \notin F.x_j$ and $y_i \notin G.y_j$,
- $\mathrm{Stab}(F, x_i) = S_F$ and $\mathrm{Stab}(G, y_i) = S_G$, with $\theta : S_F \xrightarrow{\sim} S_G$.

Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a cyclical permutation of n points. We say that a subgroup $H \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is an *amalgamated product of the subgroups F and G on $\overline{x}$ and $\overline{y}$ by the morphism $\theta : S_F \to S_G$ and the permutation σ*, if H contains two subgroups Γ_F and $\Gamma_G \leq H$, satisfying the following properties.

i. There exists an isomorphism $\Psi : F *_\theta G \xrightarrow{\sim} H$, with $\Psi(F) = \Gamma_F$ and $\Psi(G) = \Gamma_G$.

ii. The action of Γ_F on the circle is semi-conjugate to the action of F, with conjugacy $h_F : \mathbb{S}^1 \to \mathbb{S}^1$ and $\mathrm{Core}(h_F) = X_F \subset \mathbb{S}^1$. More precisely, for all $f \in F$, let $\gamma_f = \Psi(f)$, then $h_F \gamma_f = f h_F$. Similarly, the action of Γ_G on the circle is semi-conjugate to the action of G, with conjugacy $h_G : \mathbb{S}^1 \to \mathbb{S}^1$ and $\mathrm{Core}(h_G) = X_G \subset \mathbb{S}^1$.

iii. $X_F \subset \bigcup_{i=1}^n h_G^{-1}(y_i)$ and $X_F \cap h_G^{-1}(y_i) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$.

iii'. $X_G \subset \bigcup_{i=1}^n h_F^{-1}(x_i)$ and $X_G \cap h_F^{-1}(x_i) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$.

iv. $(h_F^{-1}(x_1), h_G^{-1}(y_{\sigma(1)}), \ldots, h_F^{-1}(x_n), h_G^{-1}(y_{\sigma(n)}))$ is an ordered partition of the circle. Moreover, $\left(\bigcup_{i=1}^n h_F^{-1}(x_i), \bigcup_{i=1}^n h_G^{-1}(y_{\sigma(i)})\right)$ is a proper ping-pong partition for $\Psi(F)$ and $\Psi(G)$.

We will use the abbreviated notation $(F, \overline{x}) \star_{\theta,\sigma} (G, \overline{y})$ for the *amalgamated product of the subgroups F and G on $\overline{x}$ and $\overline{y}$ by the isomorphism $\theta : S_F \to S_G$ and permutation σ.*

For the case where $\overline{x} = \{x\}$ and $\overline{y} = \{y\}$ are both singletons, we will omit the trivial permutation of one element σ and write $(F, \overline{x}) \star_\theta (G, \overline{y})$.

Remark 4.6. Since both X_F and X_G have no isolated points, items iii and iii' in Definition 4.5 imply that $h_F^{-1}(x_i)$ and $h_G^{-1}(y_i)$ are non-trivial intervals, for every $i \in \{1, \ldots, n\}$. Furthermore, by definition of the core, X_F contains the limit points of $h_F^{-1}(x_i)$ which implies in $\partial\{h_F^{-1}(x_i)\} \subset X_F \subset \bigcup_{i=1}^n h_G^{-1}(y_i)$ and similarly, $\partial\{h_G^{-1}(y_i)\} \subset \bigcup_{i=1}^n h_F^{-1}(x_i)$.

Therefore, the assumption in item iv of Definition 4.5 is not so strong. In fact, if the subsets $(h_F^{-1}(x_1), h_G^{-1}(y_{\sigma(1)}), \dots, h_F^{-1}(x_n), h_G^{-1}(y_{\sigma(n)}))$ have pairwise disjoints interiors and they are cyclically ordered, then the limit points of $h_F^{-1}(x_i)$ and $h_G^{-1}(y_i)$ coincide and these subsets form an ordered partition of the circle.

In the next theorem we show that the concept of amalgamated product of group actions is well defined, for such, we present a construction of the product given the subgroups F and G, the collection of points $\overline{x}$ and $\overline{y}$, the isomorphism $\theta : \mathrm{Stab}(F, \overline{x}) \to \mathrm{Stab}(G, \overline{y})$ and the cyclical permutation σ, then we prove that such construction is in fact unique up to conjugacy.

Theorem 4.7. *Consider two countable subgroups F and $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, and two collections of n points $\overline{x} = \{x_1, \dots, x_n\} \subset \mathbb{S}^1$ and $\overline{y} = \{y_1, \dots, y_n\} \subset \mathbb{S}^1$ such that, for all distinct $i, j \in \{1, \dots, n\}$, the following conditions are satisfied:*

- $x_j \notin F.x_i$ *and* $y_j \notin G.y_i$,
- $\mathrm{Stab}(F, x_i) = S_F$ *and* $\mathrm{Stab}(G, y_i) = S_G$, *with* $\theta : S_F \xrightarrow{\sim} S_G$,
- $S_F \lneqq F$, $S_G \lneqq G$ *and at least one of the indices* $[F : S_F]$ *and* $[G : S_G]$ *is greater than* 2.

Let $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ be any cyclical permutation of n points. Then, there exists a subgroup $H \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, which is minimal and $H = (F, \overline{x}) \star_{\theta,\sigma} (G, \overline{y})$. Furthermore, the subgroup H is unique up to conjugacy.

Proof.
Summary of first constructions − First, we are going to blow-up the actions of F and G on the orbits of $x_1, \dots, x_n$ and $y_1, \dots, y_n$ respectively, in such a way that the included intervals in the action of F are exactly the complement of the included intervals in G. Then, we are going to construct new group actions $\Psi : F \times \mathbb{S}^1 \to \mathbb{S}^1$ and $\Psi : G \times \mathbb{S}^1 \to \mathbb{S}^1$, in such a way that the action Ψ restricted to the stabilizers S_F and S_G coincides and the included intervals form a proper ping-pong partition for $\Psi(F)$ and $\Psi(G)$. We start by the subsets where the actions are naturally defined and then extend it inductively for the whole circle. The subgroup of homeomorphisms H in the statement of Theorem 4.7 will be semi-conjugate to the generated group by $\Psi(F)$ and $\Psi(G)$.
Blow-up − For the blow-up of the actions of F and G, we begin choosing a partition of the circle $(a_1, b_1, \dots, a_n, b_n)$ with $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n < a_1$. The interval $[a_i, b_i]$ will be set as the pre-image of the point x_i for the orbital opening of F and the interval $[b_i, a_{i+1}]$ the pre-image of the point $y_{\sigma(i)}$ is for the orbital opening of G. For this, we take two functions $h_F, h_G : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying the following properties:

- h_F and h_G are weakly monotone increasing and continuous functions,
- $h_F^{-1}(x_i) = [a_i, b_i]$ and $I_\xi := h_F^{-1}(\xi)$ is a non-trivial closed interval for any $\xi \in \bigcup_{i=1}^n F.x_i$,
- $h_G^{-1}(y_{\sigma(i)}) = [b_i, a_{i+1}]$ and $J_\eta := h_G^{-1}(\eta)$ is a non-trivial closed interval for any $\eta \in \bigcup_{i=1}^n G.y_i$,
- $h_F^{-1}(z)$ is a point for any $z \notin \bigcup_{i=1}^n F.x_i$,
- $h_G^{-1}(z)$ is a point for any $z \notin \bigcup_{i=1}^n G.y_i$.

Subsets where the action will be defined – In the following write $\mathcal{I} = \bigcup_{i=1}^n \bigcup_{\xi \in F.x_i} I_\xi$ and $\mathcal{J} = \bigcup_{i=1}^n \bigcup_{\eta \in G.y_i} J_\eta$. For every $i \in \{1, \dots, n\}$ and $\xi \in F.x_i$, let $A_\xi : I_{x_i} \to I_\xi$ be the unique linear orientation-preserving homeomorphism, and similarly write $B_\eta : J_{y_i} \to J_\eta$. Now, let us define by induction the following subsets:

$$K_F^m = \begin{cases} \overline{\mathbb{S}^1 \smallsetminus \mathcal{I}} & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\xi \in F.x_i} A_\xi (K_G^{m-1} \cap I_{x_i}) & \text{if } m \geq 1, \end{cases}$$

$$K_G^m = \begin{cases} \overline{\mathbb{S}^1 \smallsetminus \mathcal{J}} & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\eta \in G.y_i} B_\eta (K_F^{m-1} \cap J_{y_i}) & \text{if } m \geq 1. \end{cases}$$

Then we denote $K^m = K_F^m \cup K_G^m$, for any $m \in \mathbb{N}$, and $K = \bigcup_{m \in \mathbb{N}} K^m$ and remark that

$$\bigcup_{m \in \mathbb{N}} K_F^m = K \cap \bigcup_{i=1}^n J_{y_i} \quad \text{and} \quad \bigcup_{m \in \mathbb{N}} K_G^m = K \cap \bigcup_{i=1}^n I_{x_i}.$$

For any choice of $i \in \{1, \dots, n\}$, let $\xi \in F.x_i$ and $\eta \in G.y_i$ be points in the orbits of x_i and y_i by the subgroup F and G respectively. Then, select any two elements $f_\xi \in F$ and $g_\eta \in G$, such that $f_\xi(x_i) = \xi$ and $g_\eta(y_i) = \eta$, and define

$$\Psi(f_\xi)|_{K \cap I_i} := A_\xi, \quad \Psi(g_\eta)|_{K \cap J_i} := B_\eta.$$

Summary of what follows – We want to define an action of the amalgamated free product $F *_S G$ on the circle. For this, we will first define an action of such group on the subset K and then take an extension to $\mathbb{S}^1$. By the universal property of amalgamated products, it is enough to define the action for F and G, in such a way that definitions coincide for the stabilizers S_F and S_G. For this, we will first define the action of the stabilizer, and as for the subset K above, the construction will follow an iterative scheme. The same will be for the actions of F and G.

Note first that from the tautological actions of F and G, we can define actions $\psi_F : F \to \mathrm{Homeo}_+(\mathbb{S}^1 \smallsetminus \mathcal{I})$ and $\psi_G : G \to \mathrm{Homeo}_+(\mathbb{S}^1 \smallsetminus \mathcal{J})$ by setting

$$\psi_F(f)(z) = h_F^{-1}\, f\, h_F(z) \quad \text{for } f \in F \text{ and } z \in \mathbb{S}^1 \smallsetminus \mathcal{I},$$
$$\psi_G(g)(z) = h_G^{-1}\, g\, h_G(z) \quad \text{for } g \in G \text{ and } z \in \mathbb{S}^1 \smallsetminus \mathcal{J},$$

and the definition can be extended by continuity to actions on the closures K_F^0 and K_G^0, that we still denote (with abuse of notation) by ψ_F and ψ_G respectively.

Definition of the action for the stabilizers – Now we will define the action of the stabilizers $\Psi : S_F \to \mathrm{Homeo}_+(K)$ and $\Psi : S_G \to \mathrm{Homeo}_+(K)$.

Since $S_F \cong S_G$, we can choose an isomorphism $\theta : S_F \to S_G$ and define, for all $s \in S_F$,

$$\Psi(s)|_{K_F^0} := \overline{\psi_F}(s) : K_F^0 \to K_F^0 \quad \text{and} \quad \Psi(s)|_{K_G^0} := \overline{\psi_G}(\theta(s)) : K_G^0 \to K_G^0.$$

Take $s \in S_F$ and let $\xi_1, \xi_2 \in F.x_i$ and $\eta_1, \eta_2 \in G.y_i$ be such that $s(\xi_1) = \xi_2$ and $\theta(s)(\eta_1) = \eta_2$. By induction, we define for every $i \in \{1, \dots, n\}$

$$\Psi(s)|_{A_{\xi_1}(K_G^m \cap I_{x_i})} := A_{\xi_2}\, \Psi(f_{\xi_2}^{-1}\, s\, f_{\xi_1})|_{K_G^m}\, A_{\xi_1}^{-1},$$
$$\Psi(s)|_{B_{\eta_1}(K_F^n \cap J_{y_i})} := B_{\eta_2}\, \Psi(\theta^{-1}(g_{\eta_2}^{-1}\, \theta(s)\, g_{\eta_1}))|_{K_F^m}\, B_{\eta_1}^{-1}.$$

This way we have, for all $s \in S_F$, that $\Psi(s)$ is a well-defined homeomorphism in $\mathrm{Homeo}_+(K)$. And similarly, for $s \in S_G$, we define $\Psi(s) := \Psi(\theta^{-1}(s))$ and we have, for all $s \in S_G$, that $\Psi(s)$ is a well-defined homeomorphism in $\mathrm{Homeo}_+(K)$. Moreover, we have an equivalence between the image of the stabilizers $\Psi(S_F) = \Psi(S_G)$.

Definition of the action for F and G – Now we will construct the group actions $\Psi : F \to \mathrm{Homeo}_+(K)$ and $\Psi : G \to \mathrm{Homeo}_+(K)$. For this, remember that the representatives $f_\xi \in F$ and $g_\eta \in G$, are already defined by

$$\Psi(f_\xi)|_{K\cap I_{x_i}} := A_\xi, \quad \Psi(g_\eta)|_{K\cap J_{y_i}} := B_\eta.$$

Note that, for all $f \in F$, $\Psi(f)|_{K_F^0}$ can be defined by the homeomorphism $\overline{\psi_F}(f)$ and, similarly, for all $g \in G$, $\Psi(g)|_{K_G^0}$ can be defined by the homeomorphism $\overline{\psi_G}(g)$.

To extend this action for $\Psi(f)|_K$, one should notice that, if $f(x_i) = \xi$, we have $f_\xi^{-1} f \in S_F$. So, as $\Psi(f_\xi^{-1} f)$ is already defined in the subset K and $\Psi(f_\xi)$ is already defined in $K \cap I_{x_i}$ then the only consistent way to extend the action is given by

$$\Psi(f)|_{K\cap I_{x_i}} := \Psi(f_\xi)\, \Psi(f_\xi^{-1} f).$$

This way we have that $\Psi(f)|_{K\cap I_{x_i}}$ is well defined for every $i \in \{1, \ldots, n\}$ and every $f \in F$.

Now, observe that the points $z \in K$ such that $\Psi(f)(z)$ is still not defined are exactly

$$z \in K \setminus \left(\left(K \cap \bigcup_{i=1}^{n} I_{x_i} \right) \cup K_F^0 \right) = \left(K \cap \bigcup_{i=1}^{n} J_{y_i} \right) \setminus K_F^0 = \bigcup_{m\geq 1} K_F^m,$$

and for all $m \geq 1$ and all $z \in K_F^m$, it follows that there exists $A_{\xi'}$ such that, $A_{\xi'}^{-1}(z) \in K_G^{m-1} \subset K \cap \bigcup_{i=1}^{n} I_{x_i}$. Then, to ensure the properties of a group action, we define $\Psi(f)(z)$ by

$$\Psi(f)(z) := \Psi(f)\, A_{\xi'}\, A_{\xi'}^{-1}(z) = \Psi(f)\, \Psi(f_{\xi'})(A_{\xi'}^{-1}(z)) = \Psi(f\, f_{\xi'})(A_{\xi'}^{-1}(z)).$$

Since $f\, f_{\xi'} \in F$ is already defined at the point $A_{\xi'}^{-1}(z)$, we have that $\Psi(f)(z)$ is defined for $z \in K_F^m$ and furthermore $\Psi : F \times K \to K$ is a continuous group action which can be extended to $\overline{K}$.

An analogous construction can be done to define the continuous group action $\Psi : G \times \overline{K} \to \overline{K}$.

Getting an action on the circle – Since $\overline{K} \subset \mathbb{S}^1$ is a closed subset of $\mathbb{S}^1$, we can extend the action Ψ to the whole circle $\mathbb{S}^1$ in a continuous and orientation-preserving way, which we still denote (with abuse of notation) by $\Psi : F \times \mathbb{S}^1 \to \mathbb{S}^1$ and $\Psi : G \times \mathbb{S}^1 \to \mathbb{S}^1$.

Proofs of the items – Let H be the subgroup of circle homeomorphisms generated by $\langle \Psi(F), \Psi(G) \rangle$ and we will prove the conclusions in the statement of Theorem 4.7.

Let $\Gamma_F = \Psi(F)$ and $\Gamma_G = \Psi(G)$, and notice that the action of Γ_F on $\mathbb{S}^1$ is semi-conjugate to the action of F on $\mathbb{S}^1$ with semi-conjugacy h_F and $\mathrm{Core}(h_F) = X_F = \mathbb{S}^1 \setminus \mathring{\mathcal{I}}$.

Similarly, the action of Γ_G on $\mathbb{S}^1$ is semi-conjugate to the action of G on $\mathbb{S}^1$ with semi-conjugacy h_G and $\mathrm{Core}(h_G) = X_G = \mathbb{S}^1 \setminus \mathring{\mathcal{J}}$. This way, the items ii, iii and iii' of Definition 4.5 are satisfied.

For the items 4.5.i and iv, remember that $h_F^{-1}(x_i) = [a_i, b_i]$ and $h_G^{-1}(y_i) = [b_i, a_{i+1}]$ which, by the choices of a_i and b_i, implies that $(h_F^{-1}(x_1), h_G^{-1}(y_{\sigma(1)}), \ldots, h_F^{-1}(x_n), h_G^{-1}(y_{\sigma(n)}))$ is an ordered partition of the circle. Now, with the previous notation of $h_F^{-1}(x_i) = I_{x_i}$ and $h_G^{-1}(y_i) = J_{y_{\sigma(i)}}$, observe that for all $f \in F \setminus S_F$ and all $i \in \{1, \ldots, n\}$ there exists a element f_ξ with $\xi \in F^*.x_i$ such that $f_\xi^{-1} f \in S_F$. Then, it follows that

$$\Psi(f)(I_{x_i}) = \Psi(f_\xi)\Psi(f_\xi^{-1} f)(I_{x_i}) = \Psi(f_\xi)(I_{x_i}) = I_\xi$$

From the blow-up, it is clear that $I_\xi \cap I_{x_i} = \emptyset$ for all $i \in \{1, \ldots, n\}$, then we conclude that $\Psi(f)(I_{x_i})$ is contained at the interior of $J_{\sigma(i)}$ for some $i \in \{1, \ldots, n\}$. That is

$$\Psi(f)\left(\bigcup_{i=1}^{n} I_{x_i}\right) \subset \bigcup_{i=1}^{n} \mathring{J}_{y_{\sigma(i)}},$$

and similarly, for all $g \in G \setminus S_G$, we have $\Psi(g)\left(\bigcup_{i=1}^{n} J_{y_i}\right) \subset \bigcup_{i=1}^{n} \mathring{I}_{x_i}$. Then, $\left(\bigcup_{i=1}^{n} I_{x_i}, \bigcup_{i=1}^{n} J_{y_i}\right)$ is a proper ping-pong partition for the actions of $\Psi(F)$ and $\Psi(G)$. And finely, let us denote $S := \Psi(S_F) = \Psi(S_G)$, then by the Lemma 4.2,

$$H = \langle \Psi(F), \Psi(G) \rangle = \Psi(F) *_S \Psi(G) \cong F *_\theta G.$$

Now, by Lemma 4.4, if the action of H on the circle is not minimal then it can be semi-conjugated to a minimal action by collapsing some intervals, in which case we redefine the continuous group action Ψ by the collapse of these intervals and conclude that the new subgroup of homeomorphisms $H = \langle \Psi(F), \Psi(G) \rangle$ acts minimally over $\mathbb{S}^1$. We remark that none of the proofs of the previous items are impacted by this change, since we are changing the previous H for a minimal representative in the semi-conjugacy class of the subgroup generated by $\Psi(F)$ and $\Psi(G)$.

Uniqueness of H will be given by the following Lemma 4.8, and with that we conclude the proof of the theorem. □

Lemma 4.8. *Given two subgroups H_1 and $H_2 \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ obtained by the amalgamated product of the subgroups F and G on the points $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\} \subset \mathbb{S}^1$ by the isomorphism $\theta\colon S_F \to S_G$ and permutation σ. Then, it follows that the actions of H_1 and H_2 are conjugate.*

Proof. For the proof of this theorem, we will explicitly construct a continuous, monotone and 1-degree conjugation map, that we will denote by $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$.

For $q \in \{1, 2\}$, we will use notation as in Definition 4.5, that is, Γ_F^q and $\Gamma_G^q \leq H_q$ will be the subgroups with actions on $\mathbb{S}^1$ isomorphic and semi-conjugate to the actions of F and G on $\mathbb{S}^1$, with semi-conjugacies h_{F_q} and h_{G_q} and isomorphisms γ_{F_q} and γ_{G_q}, and the cores for each semi-conjugation will be denoted as X_F^q and X_G^q. We also define the subsets $\widetilde{X}_F^q \subset X_F^q$ and $\widetilde{X}_G^q \subset X_G^q$ by the set of the points where each semi-conjugacy is bijective, that is, they are the cores without the two preimages of each blown-up point. Observe that, the semi-conjugacies satisfy, for every $f \in F$ and $g \in G$, the following:

- $h_{F_q}^{-1} f h_{F_q}(x) = \gamma_{F_q}(f)(x)$ for every $x \in \widetilde{X}_F^q$,
- $h_{G_q}^{-1} g h_{G_q}(x) = \gamma_{G_q}(g)(x)$ for every $x \in \widetilde{X}_G^q$.

Now, we define the maps $\alpha_F : \widetilde{X}_F^1 \to \widetilde{X}_F^2$ and $\alpha_G : \widetilde{X}_G^1 \to \widetilde{X}_G^2$, by

$$\alpha_F(x) := h_{F_2}^{-1} h_{F_1}(x) \text{ for } x \in \widetilde{X}_F^1 \text{ and } \alpha_G(x) := h_{G_2}^{-1} h_{G_1}(x) \text{ for } x \in \widetilde{X}_G^1.$$

Then, we extend the maps α_F and α_G by continuity to the closures, which are exactly the cores, $\overline{\widetilde{X}_F^1} = X_F^1$ and $\overline{\widetilde{X}_G^1} = X_G^1$. Moreover, since h_{F_q} and h_{G_q} are semi-conjugacies, this implies that α_F and α_G are both continuous and order-preserving.

Observe that the subsets $\widetilde{X}_F^q$ and $\widetilde{X}_G^q$ are disjoints, and if $x \in X_F^1 \cap X_G^1$, then by items iii and iii' in Definition 4.5, there exists $i, j \in \{1, \ldots, n\}$ with $x \in h_{F_1}^{-1}(x_i) \cap h_{G_1}^{-1}(y_j)$. Then, such x is a limit point of the intervals $h_{F_1}^{-1}(x_i)$ and $h_{G_1}^{-1}(y_j)$, since this intersection has empty interior. Moreover, by item iv in Definition 4.5, we have that $j = \sigma(i)$ or $j = \sigma(i-1)$. We assume, without loss of generality, that $j = \sigma(i)$, which implies that

$$x = \sup h_{F_1}^{-1}(x_i) = \inf h_{G_1}^{-1}(y_{\sigma(i)}).$$

Finally, notice that by the definition, $\alpha_F(x) = \sup h_{F_2}^{-1}(x_i)$ and $\alpha_G(x) = \inf h_{G_2}^{-1}(y_{\sigma(i)})$, and since H_2 is also an amalgamated product by the order-preserving permutation σ, we have

$$\alpha_F(x) = \sup h_{F_2}^{-1}(x_i) = \inf h_{G_2}^{-1}(y_{\sigma(i)}) = \alpha_G(x).$$

Therefore we can define the map $\alpha : X_F^1 \cup X_G^1 \to X_F^2 \cup X_G^2$ as the natural continuous and order-preserving extension of both $\alpha_F : X_F^1 \to X_F^2$ and $\alpha_G : X_G^1 \to X_G^2$.

Now, denote the stabilizers by $S_1 := \Gamma_F^1 \cap \Gamma_G^1$ and $S_2 := \Gamma_F^2 \cap \Gamma_G^2$. We will prove that, for every $s \in S_1$, there exists $s' \in S_2$ such that: $\alpha\, s(x) = s'\, \alpha(x)$, for all $x \in X_F^1 \cup X_G^1$.

First, let us fix $s \in S_1$ and, by the properties of h_{F_q} and h_{G_q} itemized previously, we have $\gamma_{F_1}^{-1}(s) =: s_f \in S_F$ with $h_{F_1}^{-1} s_f h_{F_1}(x) = s(x)$ for all $x \in \widetilde{X}_F^1$ and, after fixing $s_f \in S_F$, there exists $\gamma_{F_2}(s_f) =: s' \in S_2$ with $h_{F_2}^{-1} s_f h_{F_2}(x) = s'(x)$ for all $x \in \widetilde{X}_F^2$. Therefore, we have

$$\alpha\, s\, \alpha^{-1}(x) = s'(x), \text{ for all } x \in \widetilde{X}_F^2.$$

On the other hand, for the same $s \in S_1$ and $s' \in S_2$, there exists $\gamma_{G_1}^{-1}(s) =: s_g \in S_G$ such that $h_{G_1}^{-1} s_g h_{G_1}(x) = s(x)$ for all $x \in \widetilde{X}_G^1$ and, similarly, $\gamma_{G_2}^{-1}(s') =: s'_g \in S_G$ such that $h_{G_1}^{-1} s'_g h_{G_1}(x) = s'(x)$ for all $x \in \widetilde{X}_G^2$. So we conclude that

$$\alpha\, s\, \alpha^{-1}(x) = s'(x), \text{ for all } x \in \widetilde{X}_G^2 \quad \text{if, and only if,} \quad s_g = s'_g.$$

But H_1 and H_2 are both amalgamated product by the same isomorphism θ, then we have that $s_g = \theta(s_f) = s'_g$, which implies that $\alpha\, s\, \alpha^{-1}(x) = s'(x)$, for all $x \in \widetilde{X}_F^2 \cup \widetilde{X}_G^2$. And, as α is the continuously extension to $X_F^2 \cup X_G^2$, this property remains, and we conclude that for every $s \in S_1$, there exists $s' \in S_2$ such that

$$\alpha\, s(x) = s'\, \alpha(x), \text{ for all } x \in X_F^1 \cup X_G^1.$$

We now proceed to extend, by induction, the map α to the whole circle. For such, we will define four families of subsets of $\mathbb{S}^1$ that will be necessary in this argument.

For every $i \in \{1, \dots, n\}$ and every $\xi \in F.x_i$, choose an element $f_\xi \in F$, such that $f_\xi(x_i) = \xi$. We do the same for every $\eta \in G.y_i$, choosing $g_\eta \in G$, such that $g_\eta(y_i) = \eta$ and denote by $H_q(f_\xi)$ and $H_q(g_\eta)$ the elements associated to f_ξ and g_η on the group action of H_q, for $q = 1, 2$. We define the following subsets by induction, in a very similar way as in the proof of Theorem 4.7.

$$Y_F^m = \begin{cases} X_F^1 & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\xi \in F.x_i} H_1(f_\xi)(Y_G^{m-1} \cap I_{x_i}^1) & \text{if } m \geq 1, \end{cases}$$

$$Y_G^m = \begin{cases} X_G^1 & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\eta \in G.y_i} H_1(g_\eta)(Y_F^{m-1} \cap J_{y_i}^1) & \text{if } m \geq 1, \end{cases}$$

$$Z_F^m = \begin{cases} X_F^2 & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\xi \in F.x_i} H_2(f_\xi)(Z_G^{m-1} \cap I_{x_i}^2) & \text{if } m \geq 1, \end{cases}$$

$$Z_G^m = \begin{cases} X_G^2 & \text{if } m = 0, \\ \bigcup_{i=1}^n \bigcup_{\eta \in G.y_i} H_2(g_\eta)(Z_F^{m-1} \cap J_{y_i}^2) & \text{if } m \geq 1, \end{cases}$$

where, similarly to the proof of Theorem 4.7, $I^q_{x_i} := h^{-1}_{F_q}(x_i)$ and $J^q_{y_i} := h^{-1}_{G_q}(y_i)$, for $q \in \{1, 2\}$.

Now, define

$$\alpha|_{H_1(f_\xi)(Y^m_G \cap I^1_{x_i})} := H_2(f_\xi)\,\Gamma|_{(Y^m_G \cap I^1_{x_i})}\, H_1(f_\xi)^{-1},$$
$$\alpha|_{H_1(g_\eta)(Y^m_F \cap J^1_{y_i})} := H_2(g_\eta)\,\Gamma|_{(Y^m_F \cap J^1_{y_i})}\, H_1(g_\eta)^{-1},$$

and denote by $Y := \bigcup_m (Y^m_F \cup Y^m_G)$ and $Z := \bigcup_m (Z^m_F \cup Z^m_G)$.

By the induction and extending naturally by continuity, we have a well defined continuous, monotone and 1-degree map $\alpha : \overline{Y} \to \overline{Z}$. Moreover, one can observe that both $\overline{Y}$ and $\overline{Z}$ are closed and invariant subsets for the actions of H_1 and H_2, therefore, by minimality we have that $\overline{Y} = \overline{Z} = \mathbb{S}^1$, which implies that we have $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$.

Furthermore, such map α satisfies that, for every $\gamma_1 \in \Gamma^1_F \cup \Gamma^1_G$, there exists $\gamma_2 \in \Gamma^2_F \cup \Gamma^2_G$ with

$$\alpha\, \gamma_1\, \alpha^{-1}(x) = \gamma_2(x) \text{ for all } x \in \mathbb{S}^1.$$

Now, since $H_1 = \Gamma^1_F *_{S_1} \Gamma^1_G$ and $H_2 = \Gamma^2_F *_{S_2} \Gamma^2_G$, the map α conjugates the action of H_1 on $\mathbb{S}^1$ to the action of H_2 on $\mathbb{S}^1$, as we wanted to prove. □

5 Tracking the number of fixed points

In this section, we will be paying attention to the number of fixed points each element of the action have. Our goal will be to apply previous results on amalgamated product of group actions to construct a family of actions on the circle with a limited number of fixed points. We will divide this section into two parts, in the first one we will present Theorem D, which is a mechanism to construct examples of group actions on $\mathbb{S}^1$ with at most $2n$ fixed points, and in the second part we will show the conditions so that these examples are not conjugates into any subgroup of $\mathrm{PSL}_n(2,\mathbb{R})$.

5.1 Constructing group actions with at most $2n$ fixed points

The proof of Theorem D will be a direct application of the following Lemma 5.3, more precisely the item 5.3.i. For its statement, we will use the notation $b_0(\mathcal{U})$ for the number of connected components of a non-empty subset $\mathcal{U}$ of a topological space X. We will also need the following notion.

Definition 5.1. For a subset $G \subset \mathrm{Homeo}_+(\mathbb{S}^1)$ and two distinct points x and y of $\mathbb{S}^1$, we say that the (partial) orbits of x and y are *cross free for the action of* G, if for all $g \in G$ we have that either $\{g(x), g(y)\} \subset [x, y]$ or $\{g(x), g(y)\} \subset [y, x]$.

A collection of circle ordered points $(x_1, x_2, \dots, x_n)$ is *proper cross free for the action of* G if for any pair $\{x_i, x_j\}$ the orbits of x_i and x_j are *cross free for the action of* G and for every $g \in G$ with $\mathrm{Fix}(g) \cap \{x_1, \dots, x_n\} = \emptyset$ there exists at least one pair of consecutive points $\{x_i, x_{i+1}\}$ such that $\{g(x_i), g(x_{i+1})\} \subset [x_i, x_{i+1}]$.

If for two intervals I and $J \subset \mathbb{S}^1$, the orbits of any points $x \in I$ and $y \in J$ are cross free, we say that the orbits of the intervals I and J are *cross free for the action of* G and, similarly, we say that a collection of circle ordered intervals $(I_1, I_2, \dots, I_n)$ is *proper cross free for the action of* G if $(x_1, x_2, \dots, x_n)$ is proper cross free for any choice of distinct points $x_i \in I_i$.

We remark that the collection of one single point (x_1) is always proper cross free and we have the following result.

Lemma 5.2. *Let $(x_1, x_2, \dots, x_n)$ be a collection of circle ordered points which is proper cross free for the action of a subset $G \subset \mathrm{Homeo}_+(\mathbb{S}^1)$, then for every $g \in G$ with $\mathrm{Fix}(g) \cap \{x_1, \dots, x_n\} = \emptyset$ there exists a pair of consecutive points $\{x_i, x_{i+1}\}$ such that $\{g(x_1), \dots, g(x_n)\} \subset [x_i, x_{i+1}]$.*

Proof. Fix $g \in G$ with $\mathrm{Fix}(g) \cap \{x_1, \dots, x_n\} = \emptyset$ and let $k \in \{1, \dots, n\}$ be such that $g(x_k)$ and $g(x_{k+1})$ are contained in $[x_k, x_{k+1}]$. For any index $t \in \{1, \dots, n\}$ other than k and $k+1$, it follows that the pairs $\{x_t, x_k\}$ and $\{x_t, x_{k+1}\}$ are cross free. Thus, since $g(x_k) \in [x_k, x_{k+1}] \subset [x_k, x_t]$ and $g(x_{k+1}) \in [x_k, x_{k+1}] \subset [x_t, x_{k+1}]$, it implies that $g(x_t)$ is contained in both intervals $[x_k, x_t]$ and $[x_t, x_{k+1}]$, therefore $g(x_t) \in [x_k, x_t] \cap [x_t, x_{k+1}] = [x_k, x_{k+1}]$. □

Lemma 5.3. *Consider two subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with $H = \langle F, G\rangle$ and $F \cap G = S$. Let $(\mathcal{U}_F, \mathcal{U}_G)$ be a proper ping-pong partition of F and G. Then, if H acts minimally, for every element $h \in H$ which is not conjugate in H into $F \cup G$, the following statements hold.*

i. *The number of fixed points of h does not exceed $2\min\{b_0(\mathcal{U}_F), b_0(\mathcal{U}_G)\}$.*

ii. *Moreover, if the collection of intervals $\mathcal{U}_G$ is proper cross free for the action of $F^* = F \setminus S$, then h is Möbius-Like.*

Proof. Since $h \in H$ is not conjugate into $F \cup G$ and $H \cong F *_S G$, then h or h^{-1} is conjugate to an element $\tilde{h}$ that can be written as $\tilde{h} = g_n f_n \cdots g_1 f_1$, with $f_i \in F^*$ and $g_i \in G^*$. Clearly $\#(\mathrm{Fix}(h)) = \#(\mathrm{Fix}(\tilde{h}))$.

Now, observe that after Definition 4.1, we have the inclusion $\tilde{h}(\mathcal{U}_G) \subset \mathcal{U}_G$. Moreover, one may notice that such inclusion is proper, otherwise if $\tilde{h}(\mathcal{U}_G) = \mathcal{U}_G$ we also have $\tilde{h}(\mathcal{U}_F) = \mathcal{U}_F$ and $\tilde{h} \in S$. This implies that for each connected component I of $\mathcal{U}_G$, either $\mathrm{Fix}(\tilde{h}) \cap I = \emptyset$ or there exists a non-empty attracting interval A (possibly degenerate) defined as $A = \bigcap_{n\in\mathbb{N}} \tilde{h}^n(\overline{I})$. We will see next that the interval A is indeed always degenerate, for this let us consider the family of closed subsets Λ_n as defined in the proof of Lemma 4.3. Notice that, after Definition 4.1.i, the interval $\tilde{h}^n(I)$ is contained in the interior of Λ_i for all $i \le n$ and therefore the interior $\mathrm{int}(A)$ is contained in the intersection $\bigcap_{n\in\mathbb{N}} \mathrm{int}(\Lambda_n)$.

Note that this intersection is H-invariant, its complement is non-empty (it contains for instance $\partial\mathcal{U}_F$), and, as we are assuming that the action is minimal, the complement must be dense. We deduce that the interval A cannot have interior points, and therefore it is reduced to a single point.

We conclude that for each closed component $\overline{I}$, the images of $\tilde{h}^n(\overline{I})$ converges to a point in $\overline{I}$, which is the only fixed point of $\tilde{h}$ in $\overline{I}$. Moreover, if such point is in the interior of I, then it is an attracting fixed point of $\tilde{h}$. By repeating the argument for $\tilde{h}^{-1}$ and $\mathcal{U}_F$, we have that for each closed component $\overline{J}$, the images of $\tilde{h}^{-n}(\overline{J})$ converges to a point in $\overline{J}$, which is the only fixed point of $\tilde{h}^{-1}$ (and of $\tilde{h}$) in $\overline{J}$ and, if such point is in the interior of J, then it is a reppeling fixed point of $\tilde{h}$.

Now, we have that $\tilde{h}$ has at most 1 fixed point for each closed component $\overline{I}$ of $\overline{\mathcal{U}_G}$ and $\overline{J}$ of $\overline{\mathcal{U}_F}$, and more precisely $\#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_G}) \le b_0(\mathcal{U}_G)$ and $\#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_F}) \le b_0(\mathcal{U}_F)$. Since the action is minimal, by Lemma 4.3 we have $\overline{\mathcal{U}_G \cup \mathcal{U}_F} = \mathbb{S}^1$, which implies

$$\#(\mathrm{Fix}(h)) = \#(\mathrm{Fix}(\tilde{h})) \le b_0(\mathcal{U}_G) + b_0(\mathcal{U}_F).$$

To finish the proof of the lemma, suppose that $\mathcal{U}_G$ and $\mathcal{U}_F$ have a different number of connected components $b_0(\mathcal{U}_G) \neq b_0(\mathcal{U}_F)$. Then, either $\mathcal{U}_F$ or $\mathcal{U}_G$ has at least two consecutive components, and we can define a new ping-pong partition replacing these two components by the smallest open interval of $\mathbb{S}^1$ containing them. After repeating this process finitely many times, we find a new proper ping-pong partition $(\widetilde{\mathcal{U}_F}, \widetilde{\mathcal{U}_G})$ with $b_0(\widetilde{\mathcal{U}_F}) = b_0(\widetilde{\mathcal{U}_G}) = \min\{b_0(\mathcal{U}_F), b_0(\mathcal{U}_G)\}$. Applying the previous arguments for this new partition, we find the upper bound

$$\#(\mathrm{Fix}(h)) \le b_0(\widetilde{\mathcal{U}_G}) + b_0(\widetilde{\mathcal{U}_F}) = 2\min\{b_0(\mathcal{U}_F), b_0(\mathcal{U}_G)\}.$$

This proves statement i. For statement ii, after the argument of the last paragraph, we can assume that the ping-pong partition $(\mathcal{U}_F, \mathcal{U}_G)$ has no two consecutive intervals which are connected components of the same open subset $\mathcal{U}_F$ or $\mathcal{U}_G$. By minimality of the action (Lemma 4.3), we can write $\partial\mathcal{U}_F = \partial\mathcal{U}_G = \{x_k\}_{k=1}^{2n}$, so that the sequence of points x_k is circularly ordered, and $I_k := (x_{2k-1}, x_{2k})$ is a connected component of $\mathcal{U}_G$ and $J_k := (x_{2k}, x_{2k+1})$ is a connected component of $\mathcal{U}_F$, for every $k \in \{1, \ldots, n\}$. See Figure 5.1. Observe that after this modification, the hypothesis of $\mathcal{U}_G$ being proper cross free for the action of F^* in the statement ii still holds for the intervals $\{J_k\}_{k=1}^{n}$ and therefore, the collections $(x_1, x_3, \ldots, x_{2n-1})$ and $(x_2, x_4, \ldots, x_{2n})$ are proper cross free for the action of F^*.

Claim. *For any $f \in F^*$, the image $f(\mathcal{U}_G)$ is contained in a single connected component of $\mathcal{U}_F$.*

Proof of claim. We will consider the case $n \ge 1$, since for $n = 1$ the claim is trivial. Now, take any $f \in F^*$ and observe that $\mathrm{Fix}(f) \cap \{x_1, \ldots, x_{2n}\} = \emptyset$. As the collection $(x_1, x_3, \ldots, x_{2n-1})$ is proper cross free, by Lemma 5.2, we have that there exists an index k such that $f(x_{2j-1}) \in [x_{2k-1}, x_{2k+1}]$,

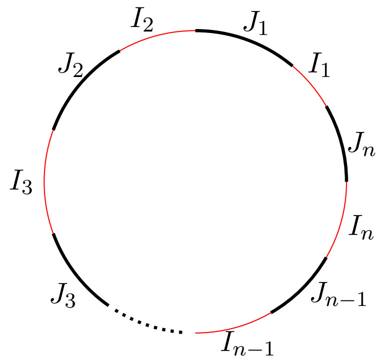

Figure 5.1: Alternating ping-pong partition for the proof of Lemma 5.3.ii.

for all $j \in \{1, \dots, n\}$. Moreover, since x_{2j-1} is an endpoint of a connected component of $\mathcal{U}_G$, we conclude that $f(x_{2j-1}) \in [x_{2k-1}, x_{2k+1}] \cap \overline{\mathcal{U}_F} = [x_{2k}, x_{2k+1}]$, for all $j \in \{1, \dots, n\}$.

Similarly, the collection $(x_2, x_4, \dots, x_{2n})$ is also proper cross free and, by Lemma 5.2, we have that there exists an index k_2 such that $f(x_{2j}) \in [x_{2k_2}, x_{2k_2+2}] \cap \overline{\mathcal{U}_F} = [x_{2k_2}, x_{2k_2+1}]$, for all $j \in \{1, \dots, n\}$. On the other hand, between any two even indexes there is at least one odd index and, as f is an order-preserving homoemorphism, it follows that there exists $j \in \{1, \dots, n\}$ such that $f(x_{2j-1}) \in [x_{2k_2}, x_{2k_2+1}]$, which implies that $[x_{2k_2}, x_{2k_2+1}] = [x_{2k}, x_{2k+1}]$ and then for all $j \in \{1, \dots, n\}$ we have $f(x_{2j-1}) \in [x_{2k_2}, x_{2k_2+1}]$. Therefore, $f(\mathcal{U}_G) \subset \mathcal{U}_F \cap [x_{2k_2}, x_{2k_2+1}] = J_{k_2}$. ☐

Now, as for the first statement, take a conjugate element $\tilde{h}$ which can be written as $\tilde{h} = g_n f_n \cdots g_1 f_1$, with $f_i \in F^*$ and $g_i \in G^*$. After the Claim, the image $f_1(\mathcal{U}_G)$ is contained in a single connected component of $\mathcal{U}_F$ and then by continuity, $\tilde{h}(\mathcal{U}_G)$ is also contained in a single component I of $\mathcal{U}_G$. This implies that all other connected components of $\mathcal{U}_G$ contain no fixed point of $\tilde{h}$ and after Lemma 5.3, we have the upper bound $\#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_G}) \leq 1$. We observe that the same holds for $\tilde{h}^{-1}$ and $\mathcal{U}_F$, arguing with f_n^{-1} instead of f_1. Hence we obtain

$$\#\mathrm{Fix}(h) = \#(\mathrm{Fix}(\tilde{h}) \cap (\overline{\mathcal{U}_G} \cup \overline{\mathcal{U}_F})) \leq \#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_G}) + \#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_F}) \leq 2.$$

To prove the h is Möbius-Like it is enough to show that h cannot contain more than one parabolic fixed point. For such, let us suppose that $\tilde{h}$ does not contain any attracting fixed point, otherwise there would be no parabolic fixed points. Then, the sequence $\tilde{h}^n(\mathcal{U}_G)$ converges to an endpoint x of $\overline{\mathcal{U}_G}$, which will be a fixed point of $\tilde{h}$ in $\overline{\mathcal{U}_G}$ and, since every endpoint of $\overline{\mathcal{U}_G}$ is also contained in $\overline{\mathcal{U}_F}$, we have that x is also a fixed point of $\tilde{h}$ in $\overline{\mathcal{U}_F}$. Now, from the previous argument we have that $\#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_G}) \leq 1$ and $\#(\mathrm{Fix}(\tilde{h}) \cap \overline{\mathcal{U}_F}) \leq 1$, therefore x is the only fixed point of $\tilde{h}$ in both $\overline{\mathcal{U}_G}$ and $\overline{\mathcal{U}_G}$, which implies that x is the only fixed point of $\tilde{h}$ in $\mathbb{S}^1$. Thus we conclude that if $\tilde{h}$ has no attracting fixed points then it has only one fixed point. ☐

Proof of Theorem D. Using the notations as in Definition 4.5, one can notice that the elements of $(F, \overline{x}) \star_{\theta,\sigma} (G, \overline{y})$ which are conjugate into $\Psi(F)$ and $\Psi(G)$ (the blow-ups of F and G) fixes at most $2n$ points, since the only elements affected by the blow-up were in the stabilizers S_F and S_G and they fixes either $\overline{x}$ or $\overline{y}$, so after the blow-up they have exactly $2n$ fixed points.

Now, since $\Psi(F)$ and $\Psi(G)$ satisfies a proper ping-pong partition with $2n$ partitions, by Lemma 5.3.i every other element generated by the amalgamated product of $\Psi(F)$ and $\Psi(G)$ also fixes at most $2n$ points. Therefore, $(F, \overline{x}) \star_{\theta,\sigma} (G, \overline{y})$ has at most $2n$ fixed points. ☐

5.2 Conditions for being non-conjugate into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$

In the next theorem we will present a sufficient condition to have the amalgamated product of group actions not conjugate to any subgroup of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. The idea behind this result is inspired by the one presented by Kovačević in [17] for the case $k=1$, which uses a non-discrete sequence of elements in one of the original subgroups to conclude that blow-up of this sequence by the amalgamated product cannot be contained in a convergence group, and therefore it cannot be conjugate to a subgroup of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$.

We remark that the condition described below is sufficient, but it is not necessary.

Theorem 5.4. *Consider two countable subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most $2n$ fixed points, and two collections of n points $\overline{x}=\{x_1,\dots,x_n\}\subset\mathbb{S}^1$ and $\overline{y}=\{y_1,\dots,y_n\}\subset\mathbb{S}^1$ such that, for all distinct $i,j\in\{1,\dots,n\}$, the following conditions are satisfied:*

- *$x_j\notin F.x_i$ and $y_j\notin G.y_i$,*
- *$\mathrm{Stab}(F,x_i)=S_F$ and $\mathrm{Stab}(G,y_i)=S_G$, with $\theta: S_F\xrightarrow{\sim} S_G$,*
- *$\mathrm{Fix}(s_f)=\{x_1,\dots,x_n\}$ and $\mathrm{Fix}(s_g)=\{y_1,\dots,y_n\}$, for all $s_f\in S_F\smallsetminus\{\mathrm{id}\}$ and $s_g\in S_G\smallsetminus\{\mathrm{id}\}$,*
- *$S_F\subsetneqq F$, $S_G\subsetneqq G$ and at least one of the indexes $[F:S_F]$ and $[G:S_G]$ is greater than 2.*

Let σ be any circular-order-preserving permutation of $\{1\dots,n\}$ and assume that the action of F or G on $\mathbb{S}^1$ is non-discrete. Then, any minimal group action of H, with $H=(F,\overline{x})\star_{\theta,\sigma}(G,\overline{y})$, is not conjugate into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$, for every $k\geq 1$.

Proof. Let us assume, without loss of generality, that the action of F on $\mathbb{S}^1$ is non-discrete and that the sequence of distinct elements $\{f_i\}_{i\in\mathbb{N}}\subset F$ converges to the identity on $\mathbb{S}^1$. Now, let us show that we can always assume, upon extracting a subsequence, that $f_i\notin S_F$ for every $i\in\mathbb{N}$.

Indeed, if it is not the case, there would be a sequence for which every element is contained in the stabilizer $\{s_i\}_{i\in\mathbb{N}}\subset S_F$ with $s_i\to\mathrm{id}$ on $\mathbb{S}^1$, but then, take any element $f\in F\smallsetminus S_F$. Since the points x_i have disjoint orbits, it is clear that $f(x_1)\neq x_i$ for all $i\in\{1,\dots,n\}$. Then, consider the sequence $\{fs_if^{-1}\}_{i\in\mathbb{N}}\subset F$. That sequence converges to the identity on $\mathbb{S}^1$ and observe that $fs_if^{-1}(x_1)\neq f(f^{-1}(x_1))=x_1$, therefore fs_if^{-1} does not fix the point x_1 and the sequence has no element in the stabilizer S.

Now, fix a sequence of distinct elements $\{f_i\}_{i\in\mathbb{N}}\subset F\smallsetminus S$ converging to the identity on $\mathbb{S}^1$, and as in Definition 4.5 write $\Psi: F*_\theta G\xrightarrow{\sim} H$ for the isomorphism, and $\mathrm{Core}(\Psi)=X_F$ for the core for the semi-conjugation of $\Psi(F)\leq H$ to F. Notice that, the sequence of distinct elements $\{\Psi(f_i)\}_{i\in\mathbb{N}}\subset H$ converges to the identity exactly in X_F and to a constant value in every connected component of the complementary of X_F. In particular, $\{\Psi(f_i)\}_{i\in\mathbb{N}}$ does not converge to a homeomorphism.

We now argue that H is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$. For this, note that for any $M\geq 1$, choosing M distinct points $y_1,y_2,\dots,y_M\in X_F$, we have $\{\Psi(f_i)(y_i)\}_{i\in\mathbb{N}}$ converges to M different points. Taking $M\geq 2$, then Theorem 2.17 implies that the group H is conjugate into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ only if the sequence $\{\Psi(f_i)\}_{i\in\mathbb{N}}$ converges to a homeomorphism, which is not the case.

Finally, fix $k\geq 2$ and take $M>k$ for the previous choice of points. If H is conjugate into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$, then there exists a periodic homeomorphism τ of order k which commutes with H, so that the induced action of H on the quotient circle $\mathbb{S}^1/\langle\tau\rangle$ is conjugate to the action of a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. However, the image of the sequence $\{\Psi(f_i)(y_i)\}_{i\in\mathbb{N}}$ in this quotient, would converge to at least $M/k>1$ different points, and as before, this is not possible. ☐

6 Examples with at most 2 fixed points

In the previous section, we presented a way to construct examples of group actions with at most $2n$ fixed points that are not conjugate into $\mathrm{PSL}_n(2,\mathbb{R})$ by introducing the Theorem D. Now, we will be interested in the case where the group action has at most 2 fixed points and it is not conjugate into the Möbius group. One can notice that a first example for this is given by Theorem D for the case $n = 1$ (this will be stated by Theorem 6.1).

In the first section, we will also present the construction of amalgamated product of group actions such that the proper ping-pong partitions has more than 2 intervals, but still the action has no more than 2 fixed points (see Theorem 6.2) and we will discuss about the conditions to have such group action being Möbius-Like.

For the second part, Section 6.2, we will present a construction of HNN-extension of group actions, which will also act with at most 2 fixed points and, as we will argue, it should be considered as a different family of examples.

6.1 Amalgamated product of group actions with at most 2 fixed points

This result is the first natural way to construct amalgamated product of group actions with at most 2 fixed points, which is by considering the restriction of Theorem D for the case $n = 1$.

Theorem 6.1. *Consider a countable subgroup F of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and a point $x \in \mathbb{S}^1$ with stabilizer $S := \mathrm{Stab}(F, x)$, satisfying that:*

- $\mathrm{Fix}(s) = \{x\}$, *for all $s \in S$,*
- *$S \lneq F$ and the index $[F : S]$ is greater than 2.*

Then, for all $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points such that $\mathrm{Stab}(G, x) = S_G$ is isomorphic to S through an isomorphism $\theta : S \xrightarrow{\sim} S_G$, there exists a minimal group action $H \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, with $H = (F, x) \star_\theta (G, x)$, which is unique up to conjugations, satisfying that:

1. *H has at most 2 fixed points.*
2. *If F is non-discrete, then H is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$.*

Proof. Notice that we are in the hypothesis of Theorem D for the case $n = 1$. Therefore the first conclusion is a direct application of it and, if F is non-discrete, we are also in the hypothesis of Theorem 5.4 which implies in the second conclusion of H not being conjugate into $\mathrm{PSL}(2,\mathbb{R})$. □

Now, we will present an interesting way to construct amalgamated products of group actions with at most 2 fixed points, even though we may have more than 2 intervals in the proper ping-pong partition. Although the statement of this theorem being quite surprising, the proof will be a direct application of Theorem D and Lemma 5.3 from previous chapters.

Theorem 6.2. *Consider two countable subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and two collections of n points $\overline{x} = \{x_1, \ldots, x_n\} \subset \mathbb{S}^1$ and $\overline{y} = \{y_1, \ldots, y_n\} \subset \mathbb{S}^1$ such that, for all distinct $i, j \in \{1, \ldots, n\}$, the following conditions are satisfied:*

- *$x_i \notin F.x_j$ and $y_i \notin G.y_j$,*
- $\mathrm{Stab}(F, x_i) = \mathrm{Stab}(G, y_j) = \mathrm{id}$,
- *F and G are non-trivial and at least one of them has more than 2 elements.*

- *the collection of points* $(x_1, x_2, \dots, x_n)$ *is proper cross free for the action of* F.

Let σ *be any order-preserving permutation of* n *elements. Then, any minimal group action* H*, with* $H = (F, \overline{x}) \star_{\theta,\sigma} (G, \overline{y})$ *has at most* 2 *fixed points.*

Proof. Observe that the subgroups F and $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ act with at most 2 fixed points, then they also act with at most $2n$ fixed points, which implies, by Theorem D, that the group H acts with at most $2n$ fixed points. Now, since the stabilizers are trivial, we have $S_F = S_G = \{\mathrm{id}\}$, no element of F or G fixes any blown-up point, so the blow-up of the actions of F and G in the construction of H does not add any fixed point to the elements semi-conjugate into $F \cup G$. So, as both F and G act with at most 2 fixed points, the only elements of H that can have more than 2 fixed points are the elements that cannot be semi-conjugate into $F \cup G$.

Now, with notations as in Definition 4.5, the blown-up collection of points $\overline{x}$ and $\overline{y}$ will form the ordered partition of the circle $(h_F^{-1}(x_1), h_G^{-1}(y_{\sigma(1)}), \dots, h_F^{-1}(x_n), h_G^{-1}(y_{\sigma(n)}))$ which is a proper ping-pong partition for the actions of $\Psi(F)$ and $\Psi(G)$. Moreover, since $(x_1, x_2, \dots, x_n)$ is proper cross free for the action of F, we have that $(h_F^{-1}(x_1), \dots, h_F^{-1}(x_n))$ is proper cross free for the action of $\Psi(F)$.

To summary, we have that:

- $H = \langle \Psi(F), \Psi(G) \rangle$.
- $\left(\bigcup_{i=1}^n h_F^{-1}(x_i), \bigcup_{i=1}^n h_G^{-1}(y_{\sigma(i)})\right)$ is a proper ping-pong partition for $\Psi(F)$ and $\Psi(G)$.
- $(h_F^{-1}(x_1), \dots, h_F^{-1}(x_n))$ is proper cross free for the action of $\Psi(F)$.

Therefore, by Lemma 5.3, more precisely the item 5.3.ii, it follows that the number of fixed points of any $h \in H$ which is not conjugate into $\Psi(F) \cup \Psi(G)$ is at most 2, so every non-trivial element of H has at most 2 fixed points. □

We remark that, even for $n \geq 2$, Theorem 6.2 does not contradicts Theorem D, since an action with at most 2 fixed points is also an action with at most $2n$ fixed points, for every positive n. The notation of an *action with at most* N *fixed points* in Definition 1.1 does not assume the existence of an element with exactly N fixed points.

Next we present a simple corollary from an algebraic perspective that best summarizes what was shown by both previous theorems.

Corollary 6.3. *Given any two countable subgroups* F *and* G *acting on the circle with at most* 2 *fixed points, there exists an action of the free product* $F * G$ *on the circle with at most* 2 *fixed points.*

Proof. Since both subgroups are countable, one can find x and $y \in \mathbb{S}^1$ such that the stabilizers $\mathrm{Stab}(F, x)$ and $\mathrm{Stab}(G, y)$ are trivial. Then, by applying the Theorem 6.2 for $n = 1$ we have that the subgroup $H = (F, x) \star_{\mathrm{id},\mathrm{id}} (G, y)$ has at most 2 fixed points, but H is also isomorphic to the free product $F * G$, which implies in the conclusion, that $F * G$ can act on the circle with at most 2 fixed points. □

We follow the text by describing the necessary conditions for an amalgamated product of actions being Möbius-Like.

We recall from Section 2.4 that a subgroup H of $\mathrm{Homeo}_+(\mathbb{S}^1)$ is Möbius-Like, if every element of the subgroup is conjugate into an element of the Möbius group $\mathrm{PSL}(2, \mathbb{R})$, which is equivalent to say that H acts with at most 2 fixed points, contains no bi-parabolic element and every $h \in H$ without fixed points is conjugate to a rotation.

Now, one may notice that, with notation as in Definition 4.5, if the subgroups F and G are Möbius-Like then any bi-parabolic element of the subgroup $(F,\overline{x}) \star_{\mathrm{id},\sigma} (G,\overline{y})$ is contained in the stabilizers $\Psi(S_F) = \Psi(S_G)$. Indeed, we have the following lemma.

Lemma 6.4. *With notation as in Definition 4.5, let the subgroups F and G be Möbius-Like with $\overline{x}$ proper cross free for the action of F and let H be the amalgamated product $H = (F,\overline{x}) \star_{\theta,\sigma} (G,\overline{y})$.*

Then, if H contains a bi-parabolic element, the stabilizer $\Psi(S_F)$ is not trivial and it contains the bi-parabolic element.

Proof. Remark that all the elements in H that are not conjugate into $\Psi(F) \cup \Psi(G)$ are Möbius-Like (see Lemma 5.3), and therefore, such bi-parabolic element would be conjugate into $\Psi(F) \cup \Psi(G)$.

Let us assume that such bi-parabolic element $p \in H$ exists and that $p \in \Psi(F)$. As F and G are Möbius-Like, they didn't contain any bi-parabolic element before the blow-up for the amalgamated product, which implies that the blow-up have interfered in the amount of fixed points of p and therefore $\Psi^{-1}(p) \in S_F$, as we wanted to prove. □

Now, before we state the next lemma, we remark that since the stabilizers S_F and S_G are abelians subgroups acting on $\mathbb{S}^1$ with global fixed points, we have that S_F and S_G are both isomorphic into subgroups of $\mathbb{R}$ and the isomorphism $\theta : S_F \to S_G$ can be extended to an automorphism of the ordered abelian group $(\mathbb{R}, <)$. So, θ can preserve or invert the orientation of each element $s \in S_F$.

Lemma 6.5. *With notation as in Definition 4.5, let the subgroups F and G be Möbius-Like with $\overline{x}$ proper cross free for the action of F and let us suppose that the amalgamated product $H = (F,\overline{x}) \star_{\theta,\sigma} (G,\overline{y})$ has at most 2 fixed points.*

Then H contains a bi-parabolic element, if and only if, the isomorphism $\theta : S_F \to S_G$ preserves the order of at least one element $s \in S_F$.

Proof. From Lemma 6.4, we have that the stabilizers $S_F \cong S_G$ are non-trivial, which implies that the number of intervals in the proper ping-pong partition is at most 2. Otherwise, $\Psi(s)$ would fixes more than 2 points, for any $s \in S_F$. Therefore, the collections of points $\overline{x}$ and $\overline{y}$ are single points, and we will denote by $\overline{x} = \{x_1\}$ and $\overline{y} = \{y_1\}$, and remark that the stabilizers S_F and S_G act without fixed points on $\mathbb{S}^1 \setminus \{x_1\}$ and $\mathbb{S}^1 \setminus \{y_1\}$, respectively.

Now, for any element $\gamma_s \in \Psi(S_F) = \Psi(S_G)$, observe that the signal of $\gamma_s(z) - z$ on X_F extend to the whole interval $h_G^{-1}(y_1)$ and similarly, the signal of $\gamma_s(z) - z$ on X_G extend to the interval $h_F^{-1}(x_1)$. Now, as we have the action of γ_s on X_F given by $h_F\, s\, h_F^{-1}$ and the action of γ_s on X_G given by $h_G\, \theta(s)\, h_G^{-1}$, then the signal of $\gamma_s(z) - z$ on X_F is the opposite to the signal $\gamma_s(z) - z$ on X_G if and only if θ inverts the orientation of γ_s.

These signals are extended to the partition of the circle $(h_F^{-1}(x_1), h_G^{-1}(y_1))$, which implies in γ_s fixing exactly the 2 limit points of the partition with both fixed points being parabolic if and only if θ preserves the orientation of γ_s. □

Lemma 6.6. *If a subgroup $F \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is non Möbius-Like, then any blow-up of F is also non Möbius-Like.*

Proof. Let $f \in F$ be a non Möbius-Like homeomorphism. There are two possibilities: either f is bi-parabolic, or f is a non conjugate blow-up of a rotation. In the latter case, a blow-up of the group action F would contain the blow-up of a non conjugate blow-up of rotation, which is also a non conjugate blow-up and then non Möbius-Like.

Now, for f bi-parabolic, a blow-up of the group action F would still contain a bi-parabolic element or (in the case of the blow-up be on the orbit of those parabolic fixed points) an element with at least 3 fixed points, which is non Möbius-Like. □

We are now ready for the next two theorems which will describe the necessary and sufficient conditions to have an amalgamated product of actions being Möbius-Like for the cases with trivial and non-trivial stabilizers.

Theorem 6.7. *Consider two countable subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and two collections of n points $\overline{x} = \{x_1, \dots, x_n\} \subset \mathbb{S}^1$ and $\overline{y} = \{y_1, \dots, y_n\} \subset \mathbb{S}^1$ such that, for all distinct $i, j \in \{1, \dots, n\}$, the following conditions are satisfied:*

- $x_i \notin F.x_j$ *and* $y_i \notin G.y_j$,
- $\mathrm{Stab}(F, x_i) = \mathrm{Stab}(G, y_j) = \mathrm{id}$,
- *the collection of points $(x_1, x_2, \dots, x_n)$ is proper cross free for the action of F.*

Let σ be any order-preserving permutation of n elements. Then, the minimal amalgamated product of group actions $H = (F, \overline{x}) \star_{\mathrm{id},\sigma} (G, \overline{y})$ is Möbius-Like if, and only if, F and G are Möbius-Like and contains no elements with irrational rotation number.

Proof. With notation as in Definition 4.5, if F or G is non Möbius-Like, then by Lemma 6.6 the subgroup $\Psi(F) \cup \Psi(G)$ is also non Möbius-Like. Furthermore, if F or G contains an element with irrational rotation number, then $\Psi(F)$ or $\Psi(G)$ will also contain an element with irrational rotation number, but since $\Psi(F)$ and $\Psi(G)$ preserves a proper ping-pong partition, such element has no dense orbits. Therefore, the conditions presented by the theorem are indeed necessary and we will now show that they are also sufficient.

First, from Theorem 6.2, we know that H has at most 2 fixed points and since the stabilizer $\Psi(S)$ is trivial, from Lemma 6.4, it does not contain any bi-parabolic element. Therefore, the only possible non Möbius-Like elements in H are the blow-up of rotations.

Now, Lemma 5.3 implies that any element of H without fixed points is contained in $\Psi(F) \cup \Psi(G)$, and since F and G are Möbius-Like and contains no elements with irrational rotation number, such element would be isomorphic to a rational rotation. Therefore, such element is also conjugate to a rotation. Then, we conclude that H is indeed Möbius-Like. □

Theorem 6.8. *Consider two countable subgroups F and G of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, two points x and $y \in \mathbb{S}^1$ and with stabilizers $S_F := \mathrm{Stab}(F, x)$ and $S_G := \mathrm{Stab}(G, y)$ such that, the following conditions are satisfied:*

- $\mathrm{Fix}(s_f) = \{x\}$, *for all* $s_f \in S_F$ *and* $\mathrm{Fix}(s_g) = \{y\}$, *for all* $s_g \in S_g$,
- $S_F \lneqq F$ *and* $S_G \lneqq G$, *with the index* $[F : S_F] > 2$,
- *there exists a isomorphism* $\theta : \mathrm{Stab}(F, x) \xrightarrow{\sim} \mathrm{Stab}(G, y)$,
- *the stabilizers* $\mathrm{Stab}(F, x) \cong \mathrm{Stab}(G, y)$ *are non trivial.*

Then, any minimal group action H, with $H = (F, x) \star_\theta (G, y)$ Möbius-Like if, and only if, F and G are Möbius-Like, they contains no elements with irrational rotation number and θ is order-inverting.

Proof. Note that the arguments in the proof of Theorem 6.7 still hold for any element which is not in the stabilizers. Therefore, it is enough to prove that both $\Psi(S_G)$ and $\Psi(S_F)$ are Möbius-Like (here Ψ is the one given by Definition 4.5).

On the other hand, all elements in $\Psi(S_G)$ and $\Psi(S_F)$ have 2 fixed points, so the only non Möbius-Like element that can possibly be in $\Psi(S_G)$ or $\Psi(S_F)$ are bi-parabolic elements, which by Lemma 6.5, exists if and only if θ is not order-inverting, as we wanted to prove. □

An interesting question that arises from the Theorem 6.8 is for which stabilizers S_F and S_G there exists an order-inverting $\theta : S_F \xrightarrow{\sim} S_G$ so that we can make a Möbius-Like amalgamated product of actions. And, as we have seen in the proof of Lemma 6.5, for the subgroup $(F, x) \star_\theta (G, y)$ be Möbius-Like, the subgroups S_F and S_G should act on $\mathbb{R}$ without fixed points, then by Theorem 1.3 they are abelian and semi-conjugate into $\mathrm{Isom}_+(\mathbb{R}) \cong \mathbb{R}$. Therefore, let $S_F \cong A_F \subset \mathbb{R}$ and $S_G \cong A_G \subset \mathbb{R}$, for every isomorphism $\theta : S_F \to S_G$ there is an associated isomorphism $\alpha_\theta : A_F \to A_G$, and α_θ inverts (or preserves) the order if, and only if, θ does it. Since there is a natural bijection between order-preserving and inverting isomorphisms, to prove the existence of an order-inverting θ is the same as to prove the existence of an order-preserving action.

So, to find stabilizers such that the amalgamated product of actions can be Möbius-Like is, in fact, to find two abelian subgroups A and B of $\mathbb{R}$ which are isomorphic by an order-preserving isomorphism.

6.2 HNN-extension of group actions with at most 2 fixed points

The next result will be the direct construction of an example that has as inspiration the HNN-extension of groups. It will be part of the family of examples given by Theorem 6.18, but we recommend reading it, since the ideas used here will be fundamental to the more general construction that follows.

The idea is to construct a group action respecting a proper ping-pong partition (just as in the case of usual amalgamated products), but with the inclusion of a finite order element that permutes the intervals of the partition. For simplicity, in this first example such element will be the order 2 rotation.

Theorem 6.9. *Consider a countable subgroup F of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and a point $x \in \mathbb{S}^1$ with stabilizer $S := \mathrm{Stab}(F, x)$, satisfying that:*

- $\mathrm{Fix}(s) = \{x\}$, *for all $s \in S$,*
- $S \lneqq F$ *and the index $[F : S]$ is greater than 2.*

Then there exists a minimal subgroup $H \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ which contains two subgroups Γ_S and Γ_F, with $\Gamma_S < \Gamma_F < H$, and satisfying that:

i. *There exists an injective morphism $\Psi : F \to H$, with $\Psi(F) = \Gamma_F$ and $\Psi(S) = \Gamma_S$;*

ii. *Γ_F and Γ_S are semi-conjugate to F and S, with the same semi-conjugacy;*

iii. *H is isomorphic to $\Gamma_F *_{\Gamma_S} \langle R_{\frac{1}{2}}, \Gamma_S \rangle$;*

iv. *H has at most 2 fixed points.*

Proof. **Blow-up** – First, we are going to blow-up the action of F on the orbit of x. For this, we take a function $h : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying the following properties:

- $h^{-1}(x) = [0, \frac{1}{2}] =: I$ and $I_\xi := h^{-1}(\xi)$ is a non-trivial closed interval for any $\xi \in F.x$,
- $h^{-1}(z)$ is a point for any $z \notin F.x$.

Subsets where the action will be defined – Let $\mathcal{I} = \bigcup_{\xi \in F.x} I_\xi$ and $A_\xi : I \to I_\xi$ be the unique linear orientation-preserving homeomorphism. We define by induction the following subsets

$$K^n = \begin{cases} \overline{\mathbb{S}^1 \setminus \mathcal{I}} & \text{if } n = 0, \\ \bigcup_{\xi \in F.x} A_\xi(R_{\frac{1}{2}}(K^{n-1})) & \text{if } n \geq 1. \end{cases}$$

We denote $K^n_{\frac{1}{2}} := R_{\frac{1}{2}}(K^n)$ the mirror set of K^n for any $n \in \mathbb{N}$, and $K = \bigcup_{n \in \mathbb{N}} K^n \cup K^n_{\frac{1}{2}}$. Then, for any choices of $\xi \in F.x$, select elements $f_\xi \in F$ such that $f_\xi(x) = \xi$ and define $\Psi(f_\xi)|_{K \cap I} := A_\xi$.

Definition of the action for the stabilizers – Now, notice that from the tautological action of F we can define the group action $\psi : F \to \mathrm{Homeo}_+(K^0)$ by setting

$$\psi(f)(z) = h^{-1} f h(z) \text{ for every } f \in F \text{ and } z \in \mathbb{S}^1 \setminus \mathcal{I},$$

and then, extend it by continuity for every $z \in K^0$.

Now, take any $s \in S$ and let ξ_1 and $\xi_2 \in F.x$ be such that $s(\xi_1) = \xi_2$. By induction, define

$$\Psi(s)|_{K^n_{\frac{1}{2}}} := R_{\frac{1}{2}}\, \Psi(s^{-1})|_{K^n}\, R_{\frac{1}{2}}, \tag{6.1}$$

$$\Psi(s)|_{A_{\xi_1}(K^n_{\frac{1}{2}})} := A_{\xi_2}\, \Psi(f_{\xi_2}^{-1}\, s\, f_{\xi_1})|_{K^n_{\frac{1}{2}}}\, A_{\xi_1}^{-1}. \tag{6.2}$$

Observe that, this way we have for all $s \in S$, the homeomorphism $\Psi(s) \in \mathrm{Homeo}_+(K)$ is well defined. And notice that, for every $s \in S$ and every $z \in K$, the following property is satisfied

$$\Psi(s^{-1})(z) = R_{\frac{1}{2}}\, \Psi(s)\, R_{\frac{1}{2}}(z). \tag{6.3}$$

Indeed, for $z \in K^n_{\frac{1}{2}}$, this is a clear application of (6.1) after changing s for s^{-1}, and for $z \in K^n$, we have $R_{\frac{1}{2}}(z) \in K^n_{\frac{1}{2}}$, then by (6.1) it follows that

$$R_{\frac{1}{2}}\, \Psi(s)\, R_{\frac{1}{2}}(z) = R_{\frac{1}{2}}\, R_{\frac{1}{2}}\, \Psi(s^{-1})\, R_{\frac{1}{2}}\, R_{\frac{1}{2}}(z) = \Psi(s^{-1})(z).$$

Definition of the action for F – Now we will define an action of the groups $\Psi : F \to \mathrm{Homeo}_+(K)$. For this, remember that the representatives $f_\xi \in F$, are already defined as

$$\Psi(f_\xi)|_{K \cap I} := A_\xi$$

and, for all $f \in F$, the element $\Psi(f)|_{K^0}$ is already well defined by $\psi(f) : K^0 \to K^0$.

To define $\Psi(f)|_K$, one should note that, if $f(x) = \xi$, then $f_\xi^{-1} f$ is contained in the stabilizer S. Therefore, $\Psi(f_\xi^{-1} f)(z)$ is already defined for all $z \in K$ and $\Psi(f_\xi)(z')$ is already defined for all $z' \in K \cap I$. Then the only consistent way to define $\Psi(f)$ over $K \cap I$ is given by

$$\Psi(f)|_{K \cap I} := \Psi(f_\xi)\, \Psi(f_\xi^{-1} f) = A_\xi\, \Psi(f_\xi^{-1} f).$$

Now, observe that the points $z \in K$ such that $\Psi(f)(z)$ is still not defined are exactly

$$z \in K \setminus \Big((K \cap I) \cup K^0\Big) = \Big(K \cap [\tfrac{1}{2}, 0]\Big) \setminus K^0 = \bigcup_{n \geq 1} K^n.$$

For all $n \geq 1$ and all $z \in K^n$, it follows that there exists $A_{\xi'}$ such that $A_{\xi'}^{-1}(z) \in K_{\frac{1}{2}}^{n-1} \subset K \cap I$. Then, to ensure the properties of a group action, $\Psi(f)(z)$ is uniquely defined by

$$\Psi(f)(z) := \Psi(f)\, A_{\xi'}\, A_{\xi'}^{-1}(z) = \Psi(f)\, \Psi(f_{\xi'})(A_{\xi'}^{-1}(z)) = \Psi(f\, f_{\xi'})(A_{\xi'}^{-1}(z))$$

and as $f\, f_{\xi'} \in F$ is already defined at the point $A_{\xi'}^{-1}(z)$, we have that $\Psi(f)(z)$ is defined for all $z \in K^n$. Furthermore, $\Psi : F \times K \to K$ is a continuous group action, which can be extend naturally to a continuous action on $\overline{K}$.

Getting an action on the circle – Since $\overline{K} \subset \mathbb{S}^1$ is a closed subset of $\mathbb{S}^1$, we can extend the action Ψ to the whole circle $\mathbb{S}^1$ in a continuous and orientation-preserving way, which we still denote (with abuse of notation) by $\Psi : F \times \mathbb{S}^1 \to \mathbb{S}^1$.

Proofs of the items – Let H be the subgroup of circle homeomorphisms generated by $\langle \Psi(F), R_{\frac{1}{2}} \rangle$ and we will prove the conclusions in the statement of Theorem 6.9.

Let $\Gamma_F = \Psi(F)$ and $\Gamma_S = \Psi(S)$, and notice that both actions of Γ_F and Γ_S on $\mathbb{S}^1$ are semi-conjugate to the action of F and S on $\mathbb{S}^1$ with semi-conjugacy h given at the beginning of the construction. Therefore, the items 6.9.i and ii are satisfied.

Now, observe that $([0, \frac{1}{2}], [\frac{1}{2}, 0])$ is a proper ping-pong partition for the subgroups $\Psi(F)$ and $\langle R_{\frac{1}{2}}, \Psi(S) \rangle$. Indeed, for all $f \in F \setminus S$ it follows that $f(x) = \xi \in F.x \setminus \{x\}$, then

$$\Psi(f)\left([0, \tfrac{1}{2}]\right) = \Psi(f)(I) = I_\xi \subset [\tfrac{1}{2}, 0].$$

For the subgroup $\langle R_{\frac{1}{2}}, \Psi(S) \rangle$, notice that, by the property stated in (6.3), we can describe all the elements of the subgroup as

$$\langle R_{\frac{1}{2}}, \Psi(S) \rangle = \{\Psi(s), \Psi(s)\, R_{\frac{1}{2}} \mid s \in S\}.$$

Then, for all $\omega \in \langle R_{\frac{1}{2}}, \Psi(S) \rangle \setminus \Psi(S)$, we have $\omega = \Psi(s)\, R_{\frac{1}{2}}$ for some $s \in S$, and it follows that

$$\omega\left([\tfrac{1}{2}, 0]\right) = \Psi(s)\, R_{\frac{1}{2}}\left([\tfrac{1}{2}, 0]\right) = \Psi(s)\left([0, \tfrac{1}{2}]\right) = [0, \tfrac{1}{2}].$$

After applying Lemma 4.2, it follows that

$$\langle \Psi(F), R_{\frac{1}{2}} \rangle = \langle \Psi(F), \langle R_{\frac{1}{2}}, \Psi(S) \rangle \rangle = \Psi(F) *_{\Psi(S)} \langle R_{\frac{1}{2}}, \Psi(S) \rangle \cong F *_S \langle \widetilde{R}_{\frac{1}{2}}, S \rangle.$$

Therefore, the item 6.9.iii is proved.

Finally, by Lemma 4.4, if the action of H on the circle is not minimal then it can be semi-conjugated to a minimal action by collapsing some intervals, in which case we redefine the continuous group action Ψ by the collapse of these intervals and conclude that the new subgroup of homeomorphisms $H = \langle \Psi(F), R_{\frac{1}{2}} \rangle$ acts minimally on $\mathbb{S}^1$. We remark that, after minimilizing the action of H the subgroup may not caontain the actual rotation of order 2, but none of the proofs of the previous items are impacted by this change, since we are changing the previous H for a minimal representative in the semi-conjugacy class of the subgroup generated by $\Psi(F)$ and $R_{\frac{1}{2}}$.

Then, the proof of 6.9.iv is a direct application of the Lemma 5.3, since every element which is conjugate into $\Psi(F) \cup \langle R_{\frac{1}{2}}, \Psi(S) \rangle$ has at most 2 fixed points. □

Remark 6.10. One should notice that the subgroup H constructed in the proof of Theorem 6.9 is generated by $\Psi(F)$ and $R_{\frac{1}{2}}$, where the only new relation is given by the isomorphism α, where $\alpha : \Psi(S) \to \Psi(S)$ is defined by $\alpha(\Psi(s)) := R_{\frac{1}{2}} \Psi(s) R_{\frac{1}{2}} = \Psi(s^{-1})$.

So, we have that H is a HNN-extension of $\Psi(F)$ relative to the automorphism α and we may say that H is isomorphic to $\Psi(F)*_\alpha$, which is also isomorphic to $F*_a$, where $a : S \to S$ is the inversion isomorphism. In fact, after Definition 6.20, one can easily conclude that the subgroup H constructed in the proof of Theorem 6.9 is conjugate to the HNN-extension of group actions given by $(F, x)\star_a$.

Now we will follow with more general constructions and we begin with a definition of what can be considered as the trivial amalgamated product of a stabilizer with itself.

Definition 6.11. Let S be a countable subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with a global fixed point $x \in \mathbb{S}^1$ and fix two distinct points $a, b \in \mathbb{S}^1$. Now, take the following:

- an automorphism $\varphi \in \mathrm{Aut}(S)$,
- an order-preserving homeomorphism $t : \mathbb{S}^1 \setminus \{x\} \to (a, b)$,
- an order-preserving homeomorphism $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ with $\rho^2 = \mathrm{id}$ and $\rho(a) = b$.

Then, we define the group homomorphism $H_{\varphi,\rho,t} : S \to \mathrm{Homeo}_+(\mathbb{S}^1)$ as the following:

$$H_{\varphi,\rho,t}(s)(z) = \begin{cases} tst^{-1}(z) & \text{for every } z \in (a, b), \\ \rho t\varphi(s)t^{-1}\rho(z) & \text{for every } z \in (b, a), \end{cases}$$
$$H_{\varphi,\rho,t}(s)(a) = a \text{ and } H_{\varphi,\rho,t}(s)(b) = b, \text{ for every } s \in S.$$

Remark 6.12 (Uniqueness). The subgroup $H_{\varphi,\rho,t}(S) \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ is unique up to conjugacy, for any choice of points $a, b \in \mathbb{S}^1$ and any choices of order-preserving homeomorphisms t and ρ satisfying the hypothesis of the statement. Therefore, we will denote the subgroup only by $H_\varphi(S)$, whenever the choice of the homeomorphisms ρ and t are irrelevant.

Remark 6.13. From Definition 4.5, one can observe that the subgroup $H_\varphi(S)$ is a trivial amalgamated product of actions of the group S with itself on the point x, that is, $H_\varphi(S) = (S, x) \star_\varphi (S, x)$.

The next lemma shows that for any countable subgroup F having S as the stabilizer for the amalgamated product $(F, x) \star_\varphi (F, x)$, the blow-up of S by the amalgamated product is always semi-conjugate to the subgroup $H_\varphi(S)$.

Lemma 6.14. *Let F be a countable subgroup of* $\mathrm{Homeo}_+(\mathbb{S}^1)$ *and let $x \in \mathbb{S}^1$ be a point whose stabilizer $S := \mathrm{Stab}(F, x)$ has index $[F : S] > 2$. Consider the following:*

- *an automorphism $\varphi \in \mathrm{Aut}(S)$,*
- *an amalgamated product of actions $H = (F, x) \star_\varphi (F, x)$,*
- *the isomorphism $\Psi : F *_\varphi F \xrightarrow{\sim} H$ as in Definition 4.5.*

Then the group action $\Psi(S)$ is semi-conjugate to $H_\varphi(S)$.

Proof. For the proof of this lemma, we are going to construct the semi-conjugacy $\beta : \mathbb{S}^1 \to \mathbb{S}^1$ from the group action $\Psi(S)$ to $H_\varphi(S)$.

First, since we have the amalgamated product of two copies of the subgroup F, we will instead denote them by F_1 and F_2, respectively, so that we write $H = (F_1, x) \star_\varphi (F_2, x)$. Now, as in Definition 4.5, for $i \in \{1, 2\}$, we will write $\Psi(F_i) = \Gamma_{F_i}$ and we have that Γ_{F_i} is semi-conjugate to F_i with semi-conjugacy h_{F_i} and $\mathrm{Core}(h_{F_i}) = X_{F_i} \subset \mathbb{S}^1$.

Now, let us define the continuous order-preserving map $\beta : \mathbb{S}^1 \to \mathbb{S}^1$ as the following:

$$\beta(z) = \begin{cases} t\, h_{F_1}(z) & \text{for every } z \in h_{F_2}^{-1}(x), \\ \rho\, t\, h_{F_2}(z) & \text{for every } z \in h_{F_1}^{-1}(x). \end{cases}$$

Notice that the union $h_{F_2}^{-1}(x) \cup h_{F_1}^{-1}(x)$ covers the circle, so the map β is defined for all points $z \in \mathbb{S}^1$. Also observe that for every $s \in S$ and every $z \in \mathbb{S}^1$ we have:

$$H_\varphi(s)\, \beta(z) = \beta\, \Psi(s)(z).$$

In particular, for $z \in h_{F_2}^{-1}(x)$ we have:

$$\begin{aligned} H_\varphi(s)\, \beta(z) &= H_\varphi(s)\, t\, h_{F_1}(z) = t\, s\, t^{-1} t\, h_{F_1}(z) \\ &= t\, s\, h_{F_1}(z) = t\, h_{F_1}\, \Psi(s)(z) = \beta\, \Psi(s)(z). \end{aligned}$$

Similarly, for $z \in h_{F_1}^{-1}(x)$ we have:

$$\begin{aligned} H_\varphi(s)\, \beta(z) &= H_\varphi(s)\, \rho\, t\, h_{F_2}(z) = \rho\, t\, \varphi(s)\, t^{-1} \rho^2\, t\, h_{F_2}(z) \\ &= \rho\, t\, \varphi(s)\, h_{F_2}(z) = \rho\, t\, h_{F_2}\, \Psi(s)(z) = \beta\, \Psi(s)(z). \end{aligned}$$

□

Remark 6.15. One can argue that $\Psi(S)$ is the natural $(S, x) \star_\varphi (S, x)$ when it is considered as a subgroup of $(F, x) \star_\varphi (F, x)$. Indeed, as in Remark 6.13, one can observe that the subgroup $\Psi(S)$ in Lemma 6.14 is also a trivial amalgamated product of actions of the group S with itself on the point x, that is, $\Psi(S) = (S, x) \star_\varphi (S, x)$.

Corollary 6.16. *Let F_1 and F_2 be two countable subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$, both satisfying the assumptions in Lemma 6.14 with $\mathrm{Stab}(F_1, x) = \mathrm{Stab}(F_2, x) = S$, and suppose that the subsets $F_1.x$ and $F_2.x \subset \mathbb{S}^1$ are homeomorphic through an order-preserving homeomorphism.*

Consider, for $i \in \{1, 2\}$, the amalgamated product of actions $H_i = (F_i, x) \star_\varphi (F_i, x)$ and the isomorphism $\Psi_i : F_i *_\varphi F_i \xrightarrow{\sim} H_i$ given by Definition 4.5. Then the subgroups $\Psi_1(S)$ and $\Psi_2(S)$ are conjugate.

Proof. This is a direct application of uniqueness of the blow-up, see Theorem 2.15. □

Lemma 6.17. *With notation as in Definition 6.11, assume in addition that S acts freely on $\mathbb{S}^1 \smallsetminus \{x\}$. Then, different choices of ρ and t give conjugate subgroups $\langle H_{\varphi,\rho,t}(S), \rho\rangle$ in $\mathrm{Homeo}_+(\mathbb{S}^1)$. Moreover, we have the following.*

1. *$H_{\varphi,\rho,t}(S)$ is a normal subgroup of $\langle H_{\varphi,\rho,t}(S), \rho\rangle$ if and only if $\varphi^2 = \mathrm{id}$, in which case $\langle H_{\varphi,\rho,t}(S), \rho\rangle$ is isomorphic to $S \rtimes_\varphi \mathbb{Z}_2$.*

2. *The action of $\langle H_{\varphi,\rho,t}(S), \rho\rangle$ has at most 2 fixed points if and only if $\varphi^2 = \mathrm{id}$.*

Proof. First, let us show that, for every $s \in S$ we have:

$$\rho\, H_{\varphi,\rho,t}(s)\, \rho(z) = \begin{cases} H_{\varphi,\rho,t}(\varphi(s))(z) & \text{for every } z \in (a, b), \\ H_{\varphi,\rho,t}(\varphi^{-1}(s))(z) & \text{for every } z \in (b, a). \end{cases} \tag{6.4}$$

Indeed, for $z \in (a, b)$ we have $\rho(z) \in (b, a)$ and it follows that

$$\rho\, H_{\varphi,\rho,t}(s)\, \rho(z) = \rho^2\, t\, \varphi(s)\, t^{-1} \rho^2(z) = t\, \varphi(s)\, t^{-1}(z) = H_{\varphi,\rho,t}(\varphi(s))(z).$$

Also, for $z \in (b, a)$ we have $\rho(z) \in (a, b)$ and it follows that

$$\rho\, H_{\varphi,\rho,t}(s)\, \rho(z) = \rho\, t\, s\, t^{-1} \rho(z) = H_{\varphi,\rho,t}(\varphi^{-1}(s))(z).$$

Now, let us suppose that $\varphi^2 \neq \mathrm{id}$ and choose $s \in S$ such that $\varphi^2(s) \neq s$. It follows that

$$H_{\varphi,\rho,t}(s)^{-1} \rho\, H_{\varphi,\rho,t}(\varphi(s))\, \rho(z) = \begin{cases} H_{\varphi,\rho,t}(s^{-1}\varphi^2(s))(z) & \text{for every } z \in (a, b), \\ z & \text{for every } z \in (b, a). \end{cases}$$

Therefore, there exists an element in $\langle H_{\varphi,\rho,t}(S), \rho \rangle$ which fixes the interval (b, a) pointwise, but it is not trivial.

For the case where $\varphi^2 = \mathrm{id}$ we have, for all $s \in S$ and all $z \in \mathbb{S}^1$, that

$$\rho\, H_{\varphi,\rho,t}(s)\, \rho(z) = H_{\varphi,\rho,t}(\varphi(s))(z),$$

which implies that $H_{\varphi,\rho,t}(S)$ is a normal subgroup of $\langle H_{\varphi,\rho,t}(S), \rho \rangle$. Furthermore, since $\langle \rho \rangle = \{\mathrm{id}, \rho\}$, it follows that for any element $\omega \in \langle H_{\varphi,\rho,t}(S), \rho \rangle = H_{\varphi,\rho,t}(S) \rtimes \langle \rho \rangle$, we have $\omega \in H_{\varphi,\rho,t}(S)$ or $\omega \in H_{\varphi,\rho,t}(S)\, \rho$.

Now, our assumptions give that $\mathrm{Fix}(H_{\varphi,\rho,t}(s)) = \{a, b\}$ and $\mathrm{Fix}(H_{\varphi,\rho,t}(s)\, \rho) = \emptyset$, for any non-trivial $s \in S$. Therefore, the generated group $\langle H_{\varphi,\rho,t}(S), \rho \rangle$ has at most 2 fixed points. □

The next theorem is the main result of this section. It shows the existence of an HNN-extension of a given countable subgroup F, also acting with at most 2 fixed points. This will be reference for the definition of HNN-extensions of group actions given in Definition 6.20.

Theorem 6.18. *Let F be a countable subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and $x \in \mathbb{S}^1$ a point whose stabilizer $S := \mathrm{Stab}(F, x)$ acts freely on $\mathbb{S}^1 \setminus \{x\}$ and has index $[F : S] > 2$. Consider the following:*

- *an automorphism $\varphi \in \mathrm{Aut}(S)$, with $\varphi^2 = \mathrm{id}$,*
- *the minimal amalgamated product of actions $H = (F, x) \star_\varphi (F, x)$,*
- *the isomorphism $\Psi : F *_\varphi F \xrightarrow{\sim} H$ as in Definition 4.5.*

Then, there exists a non-trivial homeomorphism $\widetilde{\rho} \in \mathrm{Homeo}_+(\mathbb{S}^1)$ with $\widetilde{\rho}^2 = \mathrm{id}$, such that the generated group $\langle H, \widetilde{\rho} \rangle$ has at most 2 fixed points and has H as a normal subgroup.

Furthermore, the group $\langle H, \widetilde{\rho} \rangle$ is isomorphic to the HNN-extension of F relative to the isomorphism $\widetilde{\varphi}$, where $\widetilde{\varphi}$ is an extension of φ.

Proof of Theorem 6.18. Fix two distinct points a and $b \in \mathbb{S}^1$, an order-preserving homeomorphism $t : \mathbb{S}^1 \setminus \{x\} \to (a, b)$ and an order-preserving homeomorphism $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ with $\rho^2 = \mathrm{id}$ and $\rho(a) = b$, as in Definition 6.11. By Lemma 6.14, for any choice of ρ and t, we have that $\Psi(S)$ is semi-conjugate to $H_{\varphi,\rho,t}(S)$, more precisely, one can observe that $\Psi(S)$ is a blow-up of $H_{\varphi,\rho,t}(S)$ on the points $t(F^*.x)$ and $\rho\, t(F^*.x)$, where $F^* = F \setminus S$.

Now, let us denote by $\beta : \mathbb{S}^1 \to \mathbb{S}^1$ the semi-conjugacy of $\Psi(S)$ to $H_{\varphi,\rho,t}(S)$, $I_\eta := \beta^{-1}(\eta)$ the intervals for every $\eta \in t(F^*.x) \cup \rho\, t(F^*.x)$ and $\mathcal{I} = \bigcup_\eta I_\eta$ the union of the opened intervals. Now we define the homeomorphism $\widetilde{\rho} : \mathbb{S}^1 \to \mathbb{S}^1$ as the following:

$$\widetilde{\rho}(z) = \begin{cases} \beta^{-1} \rho\, \beta(z) & \text{for every } z \in \mathbb{S}^1 \setminus \mathcal{I}, \\ A_\eta(z) & \text{for every } z \in I_\eta, \end{cases}$$

where $A_\eta : I_\eta \to I_{\rho(\eta)}$ is a orientation-preserving homeomorphism from I_η to $I_{\rho(\eta)}$ that will be fixed in the future.

It is clear that $\langle \Psi(S), \widetilde{\rho} \rangle$ is also a blow-up of $\langle H_{\varphi,\rho,t}(S), \rho \rangle$ on the points $t(F^*.x)$ and $\rho\, t(F^*.x)$, with semi-conjugacy β.

On the other hand, since for any $z \in t(F^*.x) \cup \rho\, t(F^*.x)$ the stabilizers $\mathrm{Stab}(H_{\varphi,\rho,t}(S), z)$ are trivial, this blow-up creates no new fixed point, and from Lemma 6.17, we conclude that $\langle \Psi(S), \widetilde{\rho} \rangle$ has at most 2 fixed points.

Now, as we are under the assumptions of Lemma 6.17, from (6.4) we also know that

$$\rho\, H_{\varphi,\rho,t}(s)\, \rho = H_{\varphi,\rho,t}(\varphi(s)) \quad \text{for every } s \in S,$$

therefore, by the semi-conjugacy, we have

$$\beta\, \widetilde{\rho}\, \Psi(s)\, \widetilde{\rho} = \beta\, \Psi(\varphi(s)) \quad \text{for every } s \in S. \tag{6.5}$$

From (6.5), it follows that $\widetilde{\rho}\, \Psi(s)\, \widetilde{\rho}(z) = \Psi(\varphi(s))(z)$ for every $z \in \mathbb{S}^1 \setminus \mathcal{I}$, but since every non-trivial element has at most two fixed points, it implies that $\widetilde{\rho}\, \Psi(s)\, \widetilde{\rho} = \Psi(\varphi(s))$. By this argument, we conclude that $\Psi(S)$ is a normal subgroup of $\langle \Psi(S), \widetilde{\rho} \rangle$, and since $\langle \widetilde{\rho} \rangle = \{\mathrm{id}, \widetilde{\rho}\}$, we have the following description for the generated group

$$\langle \Psi(S), \widetilde{\rho} \rangle = \Psi(S) \rtimes_\varphi \widetilde{\rho} = \{\Psi(S), \widetilde{\rho}\, \Psi(S)\}.$$

Now, one can observe that both subgroups $\Psi(S)$ and $\langle \tilde{\rho} \rangle$ have at most 2 fixed points, in fact, $\langle \tilde{\rho} \rangle$ is free of fixed points and the points $\beta^{-1}(a)$ and $\beta^{-1}(b)$ are the only fixed points of non-trivial elements of $\Psi(S)$. So, every non-trivial element of $\langle \Psi(S), \widetilde{\rho} \rangle$ which is conjugate into $\Psi(S) \cup \langle \tilde{\rho} \rangle$ also has at most 2 fixed points. On the other hand, for every $\Psi(s) \in \Psi(S)$ we have $\Psi(s)(\beta^{-1}([a,b])) = \beta^{-1}([a,b])$ and for every $\omega \in \widetilde{\rho}\Psi(S)$ we have $\omega(\beta^{-1}([a,b])) = \beta^{-1}([b,a])$. So, $(\beta^{-1}([a,b]), \beta^{-1}([b,a]))$ is a proper ping-pong partition for the action of $\langle \Psi(S), \widetilde{\rho} \rangle$ and, by Lemma 5.3, every element of $\langle \Psi(S), \widetilde{\rho} \rangle$ which is not conjugate into $\Psi(S) \cup \langle \tilde{\rho} \rangle$ also has at most 2 fixed points, therefore the subgroup $\langle \Psi(S), \widetilde{\rho} \rangle$ acts with at most 2 fixed points.

Notice that we proved both conclusions of the theorem for the generated group $\langle \Psi(S), \widetilde{\rho} \rangle$, in effect, $\langle \Psi(S), \widetilde{\rho} \rangle$ has at most 2 fixed points and has $\Psi(S)$ as a normal subgroup. Now, we need to extend this result to the subgroup H and the key point here is to extend, in the correct way, the automorphism $\varphi \in \mathrm{Aut}(S)$ to an isomorphism $\varphi : F_1 \to F_2$, where F_1 and F_2 are the two copies of F in the amalgamated product.

First, observe that the subset $\mathcal{I} = \bigcup_\eta I_\eta$ for every $\eta \in t(F^*.x) \cup \rho\, t(F^*.x)$ is dense over $\mathbb{S}^1$. Indeed, with notation as in Definition 4.5 and denoting by (F_1, x_1) and (F_2, x_2) the two copies of (F, x) and $(\mathcal{U}_1, \mathcal{U}_2)$ the proper ping-pong partition, we have that $\mathcal{I}$ is the union of the images of the blown-up intervals $\overline{\mathcal{U}_1}$ and $\overline{\mathcal{U}_2}$ by F_2^* and F_1^*, respectively.

$$\mathcal{I} = F_1^*(\overline{\mathcal{U}_2}) \cup F_2^*(\overline{\mathcal{U}_1}).$$

Moreover, as we shown in the proof of Lemma 4.3, the closure of $\mathcal{I}$ contains a closed H-invariant subset, but since the action of H is minimal (by Theorem 4.7, the minimality is given by the hypothesis of index $[F : S]$ larger than 2), then the only closed H-invariant set is the full circle $\mathbb{S}^1$, which implies that the closure of $\mathcal{I}$ contains $\mathbb{S}^1$ and we conclude that $\mathcal{I}$ is dense.

Now, like in the construction of the amalgamated product of actions (see Theorem 4.7), choose representatives elements $f_\alpha \in F$, for every $\alpha \in F.x$, such that $f_\alpha(x) = \alpha$ and we denote by $f_{1,\alpha}$ and $f_{2,\alpha}$ the two copies of these elements in F_1 and F_2, and we argue that it is possible to choose a family of orientation-preserving homeomorphisms $A_\eta : I_\eta \to I_{\rho(\eta)}$, for every $\eta \in t(F^*.x) \cup \rho\, t(F^*.x)$, such that, for any $\gamma \in F.x$ and $\alpha \in F^*.x$ we have

$$\begin{cases} A_{\rho t(f_\alpha(\gamma))}\Psi(f_{2,\alpha})A_{t(\gamma)}(z) = \Psi(f_{1,\alpha})(z) & \text{for every } z \in I_{t(\gamma)}, \\ A_{t(f_\alpha(\gamma))}\Psi(f_{1,\alpha})A_{\rho t(\gamma)}(z) = \Psi(f_{2,\alpha})(z) & \text{for every } z \in I_{\rho t(\gamma)}. \end{cases} \tag{6.6}$$

In fact, by the uniqueness of the amalgamated product of group actions, H is conjugate to the amalgamated product where in the construction (see proof of Theorem 4.7) it is chosen the same representative elements as here and the homoeomorphisms defined by the representatives are the orientation-preserving linear homeomorphisms (as it was done for the proof of Theorem 4.7). So, after changing H by its conjugate, (6.6) is satisfied by choosing $A_\eta : I_\eta \to I_{\rho(\eta)}$ as the orientation-preserving linear homeomorphisms between these intervals, for every $\eta \in t(F.x) \cup \rho\, t(F.x)$ (notice that we are also defining A_η for $I_\eta = I_{t(x)}$ and $I_{\rho t(x)}$, which are the partition intervals).

Therefore, for every $z \in \mathcal{I}$ and every $\alpha \in F^*.x$, it follows that

$$\widetilde{\rho}\,\Psi(f_{1,\alpha})\,\widetilde{\rho}(z) = \Psi(f_{2,\alpha})(z) \quad \text{and} \quad \widetilde{\rho}\,\Psi(f_{2,\alpha})\,\widetilde{\rho}(z) = \Psi(f_{1,\alpha})(z).$$

Since $\mathcal{I}$ is dense over $\mathbb{S}^1$, by continuity we have that $\widetilde{\rho}\,\Psi(f_{1,\alpha})\,\widetilde{\rho} = \Psi(f_{2,\alpha})$ and $\widetilde{\rho}\,\Psi(f_{2,\alpha})\,\widetilde{\rho} = \Psi(f_{1,\alpha})$. Then, we extend the automorphism $\varphi \in \mathrm{Aut}(S)$ to an isomorphism $\varphi : F_1 \to F_2$ by defining, for every $\alpha \in F^*.x$,

$$\varphi(f_{1,\alpha}) := f_{2,\alpha}.$$

Observe that φ is a well-defined isomorphism. Indeed, for every $f \in F_1^*$, there exists $\alpha \in F_1^*.x$ such that $f f_{1,\alpha}^{-1} = s \in S$, therefore $\varphi(f) = \varphi(s)\varphi(f_{1,\alpha}) = \varphi(s)f_{2,\alpha} \in F_2$ is uniquely defined. From the definition, it is clear that we still have $\varphi^2 = \mathrm{id}$ and it follows that, for every $f \in F_1$ with $f = sf_{1,\alpha}$, we have:

$$\widetilde{\rho}\,\Psi(f)\,\widetilde{\rho} = \widetilde{\rho}\,\Psi(s)\Psi(f_{1,\alpha})\,\widetilde{\rho} = \widetilde{\rho}\,\Psi(s)\,\widetilde{\rho}\widetilde{\rho}\Psi(f_{1,\alpha})\,\widetilde{\rho} = \Psi(\varphi(s))\Psi(\varphi(f_{1,\alpha})) = \Psi(\varphi(f)) \in \Psi(F_2).$$

And similarly, for every $f \in F_2$, we also have

$$\widetilde{\rho}\,\Psi(f)\,\widetilde{\rho} = \Psi(\varphi(f)) \in \Psi(F_1).$$

So, since $H = \langle \Psi(F_1), \Psi(F_2) \rangle$ and, for every $h \in \Psi(F_1) \cup \Psi(F_2)$, we have $\widetilde{\rho}\,h\,\widetilde{\rho} \in H$ we conclude that H is a normal subgroup of $\langle H, \widetilde{\rho} \rangle$, as we wanted to prove.

Now, we will show that the subgroup $\langle H, \widetilde{\rho} \rangle$ has at most 2 fixed points. For such, notice that $\widetilde{\rho}\,\Psi(f)\,\widetilde{\rho} = \Psi(F_2)$, therefore

$$\langle H, \widetilde{\rho} \rangle = \langle \Psi(F_1), \Psi(F_2), \widetilde{\rho} \rangle = \langle \Psi(F_1), \widetilde{\rho} \rangle = \langle \Psi(F_1), \langle \Psi(S), \widetilde{\rho} \rangle \rangle.$$

Since H has at most 2 fixed points, the subgroup $\Psi(F_1) \leq H$ also has at most 2 fixed points and we have already shown that the subgroup $\langle \Psi(S), \widetilde{\rho} \rangle = \{\Psi(S), \widetilde{\rho}\,\Psi(S)\}$ also acts with at most 2 fixed points, so every non-trivial element of $\langle \Psi(F_1), \langle \Psi(S), \widetilde{\rho} \rangle \rangle$ which is conjugate into $\Psi(F_1) \cup \langle \Psi(S), \widetilde{\rho} \rangle$ also have at most 2 fixed points.

We claim that $(\mathcal{U}_1, \mathcal{U}_2)$ is a proper ping-pong partition for the subgroups $\Psi(F_1)$ and $\langle \Psi(S), \widetilde{\rho} \rangle$. For such, remark that $(\mathcal{U}_1, \mathcal{U}_2)$ and $(\beta^{-1}[a, b], \beta^{-1}[b, a])$ are the same partitions and they are the proper ping-pong partition for $\Psi(F_1)$ and $\Psi(F_2)$, so $\Psi(S)(\mathcal{U}_i) = \mathcal{U}_i$, for $i \in \{1, 2\}$, but $\widetilde{\rho}(\mathcal{U}_1) = \mathcal{U}_2$. Now, by definition, for every $\Psi(f) \in \Psi(F_1^*) = \Psi(F_1) \smallsetminus \psi(S)$, we have $\Psi(f)(\mathcal{U}_2) \subset \mathcal{U}_1$. On the other hand, we have that $\langle \Psi(S), \widetilde{\rho} \rangle \smallsetminus \Psi(S) = \widetilde{\rho}\,\Psi(S)$, and therefore, for every $\omega \in \langle \Psi(S), \widetilde{\rho} \rangle \smallsetminus \Psi(S)$ it follows that $\omega(\mathcal{U}_1) = \mathcal{U}_2$, which proves our claim.

Therefore, applying Lemma 4.2, we have

$$\langle H, \widetilde{\rho} \rangle = \langle \Psi(F_1), \langle \Psi(S), \widetilde{\rho} \rangle \rangle = \Psi(F_1) *_{\Psi(S)} \langle \Psi(S), \widetilde{\rho} \rangle,$$

and, by Lemma 5.3, one can conclude that every element of $\langle \Psi(F_1), \langle \Psi(S), \widetilde{\rho} \rangle \rangle$ which is not conjugate into $\Psi(F_1) \cup \langle \Psi(S), \widetilde{\rho} \rangle$ have at most 2 fixed points, which implies that $\langle H, \widetilde{\rho} \rangle$ has at most 2 fixed points, as we wanted to prove.

Now, for the last conclusion, it is enough to observe that the subgroup $\langle H, \widetilde{\rho} \rangle$ is generated by $\Psi(F_1)$ and $\widetilde{\rho}$ and that the only relation between these two subgroups is given by the isomorphism $\Psi(f_1) \mapsto \widetilde{\rho}\Psi(f_1)\widetilde{\rho} = \Psi(\varphi(f_1))$, which is isomorphic to φ. Therefore, since $\Psi(F_1) \cong F$, we conclude that $\langle H, \widetilde{\rho} \rangle$ is isomorphic to the HNN-extension $F*_{\varphi}$. □

Remark 6.19. One can notice that the homeomorphism $\widetilde{\rho}$ given by the Theorem 6.18 is unique up to conjugacy. Indeed, it is uniquely defined by the choice of the amalgamated product of actions $H = (F, x) \star_{\varphi} (F, x)$ (see (6.7)) and, by Theorem 4.7, H itself is uniquely defined up to conjugacy. So the only fundamental information is given by the action of the subgroup F, the chosen point x and the automorphism $\varphi \in \mathrm{Aut}(S)$.

Furthermore, the extension $\widetilde{\varphi}$ stated at the Theorem 6.18 is also uniquely defined by the automorphism φ.

The next statement will take the Remark 6.19 in consideration to define the HNN-extension of group actions, after Theorem 6.18, by using only the fundamental information needed.

Definition 6.20. Let F be a countable subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and $x \in \mathbb{S}^1$ a point whose stabilizer $S := \mathrm{Stab}(F, x)$ acts freely on $\mathbb{S}^1 \smallsetminus \{x\}$ and has index $[F : S] > 2$. Consider the following:

- an automorphism $\varphi \in \mathrm{Aut}(S)$, with $\varphi^2 = \mathrm{id}$,
- the minimal amalgamated product of actions $H = (F, x) \star_{\varphi} (F, x)$,
- the isomorphism $\Psi : F *_{\varphi} F \xrightarrow{\sim} H$ as in Definition 4.5.

Let $\widetilde{\rho} \in \mathrm{Homeo}_+(\mathbb{S}^1)$ be the non-trivial homeomorphism given by Theorem 6.18, such that, the subgroup $\langle H, \widetilde{\rho} \rangle$ is isomorphic to $F*_{\widetilde{\varphi}}$, with $\widetilde{\varphi}$ being an extension of φ.

Then, we say that the subgroup defined as $\langle H, \widetilde{\rho} \rangle$ is a *HNN-extension of the subgroup F on the point x by the isomorphism φ* and we use the abbreviated notation $(F, x)\star_{\varphi}$.

Remark 6.21. After Theorem 6.18 and Remark 6.19, the HNN-extension of the subgroup F on the point x by the isomorphism φ is well defined and it is unique up to conjugacy.

We clarify that the definition of $(F, \overline{x})\star_{\varphi}$ for $\overline{x} = \{x_1, \ldots, x_n\}$ can be made. We chose to do only the case where $n = 1$ for simplicity and because it is the most interesting case for group actions with at most 2 fixed points. We recall that for $n > 1$, if the stabilizer S is non-trivial, then there exists elements with 4 or more fixed points.

The next theorem will be a direct application of Theorem 6.1 and it will give a condition to have $(F, x)\star_{\varphi}$ non conjugate to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Theorem 6.22. *Let F be a countable subgroup of* $\mathrm{Homeo}_+(\mathbb{S}^1)$ *with at most* 2 *fixed points, and $x \in \mathbb{S}^1$ a point whose stabilizer $S := \mathrm{Stab}(F, x)$ acts freely on $\mathbb{S}^1 \smallsetminus \{x\}$ and has index $[F : S] > 2$. Let $\varphi \in \mathrm{Aut}(S)$ be an automorphism, with $\varphi^2 = \mathrm{id}$.*

Then, if F is non-discrete, $(F, x)\star_{\varphi}$ is not conjugate into $\mathrm{PSL}(2, \mathbb{R})$.

Proof. Since F is non discrete, by Theorem 6.1, the amalgamated product of actions $(F,x)\star_\varphi(F,x)$ is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$. On the other hand, by Definition 6.20, $(F,x)\star_\varphi(F,x)$ is conjugate to a subgroup of $F\star_\varphi$, which implies that $(F,x)\star_\varphi$ cannot be conjugate into $\mathrm{PSL}(2,\mathbb{R})$. □

The HNN-extension of group actions present a very distinct dynamics when compared with others amalgamated products of actions, mainly because of the existence of an order 2 element which permutes the intervals of the proper ping-pong partition. And, in last theorem of this chapter, we will show that, for many choices of $\mathrm{Stab}(F,x)$ and φ, only trivial amalgamated products are conjugate to HNN-extensions.

This fact encourages us to characterize the HNN-extension of group actions as a different family of examples, since the dynamics of theses examples have much more similarities between themselves than with other usual amalgamated products.

Theorem 6.23. *Let F be a countable subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ with at most 2 fixed points, and $x\in\mathbb{S}^1$ a point whose stabilizer $S:=\mathrm{Stab}(F,x)$ acts freely on $\mathbb{S}^1\setminus\{x\}$ and has index $[F:S]>2$. Consider the following:*

- *an automorphism $\varphi\in\mathrm{Aut}(S)$, with $\varphi^2=\mathrm{id}$,*
- *the minimal amalgamated product of actions $H=(F,x)\star_\varphi(F,x)$,*
- *the isomorphism $\Psi: F*_\varphi F\xrightarrow{\sim}H$ as in Definition 4.5.*
- *the homeomorphism $\widetilde{\rho}$, with $\widetilde{\rho}^2=\mathrm{id}$, as in Theorem 6.18.*

Let $\langle H,\widetilde{\rho}\rangle=(F,x)\star_\varphi$ be the HNN-extension of the subgroup F on the point x by the isomorphism φ, as in Definition 6.20.

Then, if $\varphi=\mathrm{id}$ and S is not trivial nor isomorphic to $\mathbb{Z}$, the subgroup $\langle H,\widetilde{\rho}\rangle$ is an amalgamated product of group actions with index larger than 2 over the stabilizer.

Moreover, the only amalgamated product of group actions conjugate to $(F,x)\star_\varphi$ is given by $(F,x)\star_\varphi(\Gamma,x)$, where $[\Gamma:\Psi(S)]=2$.

Proof. Let us suppose that $\langle H,\widetilde{\rho}\rangle=(G_1,\overline{y}_1)\star_{\theta,\sigma}(G_2,\overline{y}_2)$ with a proper ping-pong partition $(\mathcal{U}_1,\mathcal{U}_2)$ and denote by $S_g=G_1\cap G_2$ the stabilizer of the amalgamated product and $\Psi_g: G_1*_\theta G_2\to\langle H,\widetilde{\rho}\rangle$ the ping-pong isomorphism. Now, since $\varphi=\mathrm{id}$, we have that $\Psi(s)$ has fixed points and it commutes with $\widetilde{\rho}$ (an order 2 element), then, for every non-trivial $s\in S$, the homeomorphism $\Psi(s)$ is bi-parabolic and it cannot be conjugate into $\mathrm{PSL}(2,\mathbb{R})$.

If there exists a non-trivial element $s\in S$ such that $\Psi(s)\notin S_g$, than all elements $\Psi(s)$ should be contained in $\Psi_g(G_1)$ or in $\Psi_g(G_2)$, since none of them would fix the same points as the elements of S_g and the new elements generated by an amalgamated product of group actions are all Möbius-Like. Therefore, we argue that either $\Psi(G_1)^*=\Psi_g(G_1)\setminus S_g$ or $\Psi(G_2)^*=\Psi_g(G_2)\setminus S_g$ contains every non-trivial element of $\Psi(S)$.

Indeed, for any non-trivial $s\in S$, the two fixed points of $\Psi(s)$ should be in the interior of intervals of only one between $\mathcal{U}_1$ and $\mathcal{U}_2$, after all, the non-trivial elements with fixed points at the exterior of these intervals are the elements of the stabilizer S_g, and, by taking a sufficiently large n, the element $\Psi(s^n)$ should send all the others intervals close to the fixed points, therefore to the interior of $\mathcal{U}_1$ or $\mathcal{U}_2$. Since this partition is a proper ping-pong partition for the subgroups $\Psi_g(G_1)$ and $\Psi_g(G_2)$, that is enough to determine whether $\Psi(s)$ is contained in $\Psi_g(G_1)$ or in $\Psi_g(G_2)$. Moreover, for all $s\in S$, the fixed points of $\Psi(s)$ are the same, which implies that all of them should be contained at the same $\Psi_g(G_i)$.

Now, let us assume that the subgroup $\Psi(S)$ is contained in $\Psi_g(G_1)$ and its fixed points are in the interior of $\mathcal{U}_1$. One should notice that for at least one interval I, defined by the two fixed points of $\Psi(S)$, there is an interval of $\mathcal{U}_2$, but as S is not trivial nor isomorphic to $\mathbb{Z}$, its action on I is non-discrete, which implies that there exists an element $s \in S$ such that $\Psi(s)(\mathcal{U}_2) \cap \mathcal{U}_2 \neq \emptyset$, which contradicts the hypothesis of $(\mathcal{U}_1, \mathcal{U}_2)$ being a proper ping-pong partition for $\Psi_g(G_1)$ and $\Psi_g(G_2)$. So, we conclude that S_g contains the subgroup $\Psi(S)$. Furthermore, since S_g is abelian by construction and no other non-trivial element of $\langle H, \widetilde{\rho}\rangle$ have the fixed points of $\mathrm{Fix}(\Psi(S \smallsetminus \mathrm{id}))$, we have that $S_g = \Psi(S)$.

In particular, as the stabilizer S_g is not trivial, the amalgamated product $(G_1, \overline{y}_1) \star_{\theta,\sigma} (G_2, \overline{y}_2)$ has a proper ping-pong partition of only 2 intervals, that we will denote by (I_1, I_2), otherwise the elements of S_g would have more than 2 fixed points. Moreover, both $(G_1, \overline{y}_1) \star_{\theta,\sigma} (G_2, \overline{y}_2)$ and $H = (F_1, x) \star_\varphi (F_2, x)$ have the same ping-pong partition (I_1, I_2), and we have (after changing the index, if necessary) that:

$$\begin{array}{ll} \Psi_g(g_1)(I_2) \subset I_1, & \text{for every } g_1 \in G_1^* = G_1 \smallsetminus S_g, \\ \Psi_g(g_2)(I_1) \subset I_2, & \text{for every } g_2 \in G_2^* = G_2 \smallsetminus S_g, \\ \Psi(f_1)(I_2) \subset I_1, & \text{for every } f_1 \in F_1^* = F_1 \smallsetminus \Psi(S), \\ \Psi(f_2)(I_1) \subset I_2, & \text{for every } f_2 \in F_2^* = F_2 \smallsetminus \Psi(S). \end{array} \tag{6.7}$$

Now, since $\widetilde{\rho}$ has no fixed point, it should be contained in $\Psi_g(G_1)$ or $\Psi_g(G_2)$. Let us suppose, without loss, that $\widetilde{\rho} \in \Psi_g(G_2)$, then for any $g_2 \in G_2^*$ it follows that

$$\widetilde{\rho}\Psi_g(g_2)(I_1) \subset \widetilde{\rho}(I_2) = I_1, \quad \text{with } \widetilde{\rho}\Psi_g(g_2) \in \Psi(G_2).$$

Therefore, the only way to not contradict (6.7) is to have $\widetilde{\rho}\Psi_g(G_1^*) \in S_g$ which implies that the index of $[G_2 : S_g]$ is equal to 2, and $\Psi_g(G_2) = \langle \widetilde{\rho}, S_g\rangle = \langle \widetilde{\rho}, \Psi(S)\rangle$.

Now, we claim that the subgroup $\Psi(F_1)$ is contained in $\Psi_g(G_1)$. Indeed, no independent generator of $\Psi(F_1)$ can be created from the combination of others generators and $\widetilde{\rho}$. But, from the proof of Theorem 6.18, we have that

$$\langle \Psi(F_1), \langle \widetilde{\rho}, \Psi(S)\rangle\rangle = \langle H, \widetilde{\rho}\rangle = \langle \Psi_g(G_1), \langle \widetilde{\rho}, \Psi(S)\rangle\rangle.$$

So, $\Psi_g(G_1)$ is also contained in $\Psi(F_1)$, which shows that $(G_1, \overline{y}_1) \star_{\theta,\sigma} (G_2, \overline{y}_2) = (F, x) \star_\varphi (\Gamma, x)$, where $\Gamma \cong \langle \widetilde{\rho}, \Psi(S)\rangle$ and has index $[\Gamma : S] = 2$, as we wanted to prove. ☐

Remark 6.24. On the hypothesis of Theorem 6.23, if S is isomorphic to $\mathbb{Z}$, one can conclude that the subgroup $\langle H, \widetilde{\rho}\rangle$ is not an amalgamated product of Möbius-Like group actions with index larger than 2 over the stabilizer. Indeed, if we suppose G_1 and G_2 Möbius-Like, this implies that every bi-parabolic element should be in contained in the stabilizer S_g, then we have $\Psi(S)$ contained in S_g and the same results as in Theorem 6.23 follow.

Corollary 6.25. *On the hypothesis of Theorem 6.23, φ can be chosen as an automorphism of S which does not invert the orientation of at least one element $s \in S$, and the same results follow.*

Proof. Let us fix an orientation for $\mathbb{S}^1 \smallsetminus \{x\}$ with S acting on it and suppose that the automorphism $\varphi \in \mathrm{Aut}(S)$ does not invert the orientation of at least one non-trivial $s \in S$, then for such $s \in S$ we have that s and $\varphi(s)$ with the same orientation on $\mathbb{S}^1 \smallsetminus \{x\}$, which implies that after the blow-up of the point x, $\Psi(s)$ will have the same orientation on both sides of the circle and it will not be conjugate into $\mathrm{PSL}(2, \mathbb{R})$. In fact, it will have two parabolic fixed points.

Now, following the proof of Theorem 6.23, since $\Psi(s)$ is non Möbius-Like, than it is contained in the stabilizer S_g and therefore $\Psi(s) \in S_g$ for all $s \in S$ (they share the same fixed points). And as no other element fixes the same points we conclude that $\Psi(S) = s_G$, and the same results as in Theorem 6.23 follow. ☐

7 Real-analytical and smooth examples

In this chapter we will present two examples of finitely generated subgroups of diffeomorphisms acting with at most 2 fixed points which are not conjugate into $\mathrm{PSL}(2,\mathbb{R})$. One may notice that both of these examples are not Möbius-Like and we remark that a finitely generated Möbius-Like subgroup of diffeomorphisms which is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$ is yet to be found.

For the second part of this chapter we will present an example which is not even isomorphic to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$, however such example does not act minimally. And, although we show in Remark 7.1 a way to minimilize this action, we lose the regularity by doing it. We remark that a finitely generated subgroup of diffeomorphisms acting minimally with at most 2 fixed points such that it is not isomorphic to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is yet to be found.

Note also that none of the methods presented in previous chapters can be used here, since none of these methods has been shown to be able to maintain a regularity than higher C^0.

7.1 A group of real-analytic diffeomorphisms which is not conjugate into $\mathrm{PSL}(2,\mathbb{R})$

Let us discuss here Theorem E, that is let us give a concrete example of group of real-analytic diffeomorphisms with at most 2 fixed points whose action is minimal and which is not conjugate to any subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

For this, recall that the central extension of degree 2 of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ is naturally identified with the special linear group $\mathrm{SL}(2,\mathbb{R})$, by considering the action on half-lines in $\mathbb{R}^2$. Both groups contain the group of rigid rotations $\mathrm{SO}(2)$ as a subgroup (which canonically determines the actions on the circle), so that there is a homomorphism from the amalgamated product $\mathrm{PSL}(2,\mathbb{R}) *_{\mathrm{SO}(2)} \mathrm{SL}(2,\mathbb{R})$ into the group of real-analytic diffeomorphisms $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ (the question whether such map is injective is mentioned by Tsuboi in [24] and it is still open up to our knowledge). Keeping the notation as in [24], we will denote by $G^{(2)}$ the image of such homomorphism. Also, we will use square brackets for homographies in $\mathrm{PSL}(2,\mathbb{R})$ and parentheses for matrices in $\mathrm{SL}(2,\mathbb{R})$, and we will write them with respect to the canonical basis of $\mathbb{R}^2$.

Proof of Theorem E. Let $f \in \mathrm{SL}(2,\mathbb{R})$ be the linear transformation given by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and let $g \in \mathrm{PSL}(2,\mathbb{R})$ be the homography represented by the matrix $\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}$. We consider them as elements of $G^{(2)}$, and denote by $R_{\frac{1}{2}} \in G^{(2)}$ the order-two rigid rotation. We then write $\overline{g} = R_{\frac{1}{2}} g R_{\frac{1}{2}}$. Note that the element f commutes with $R_{\frac{1}{2}}$. It is not difficult to verify that the subgroup in $G^{(2)}$ generated by $f, g, \overline{g}$ acts on the circle with the ping-pong partition as in Figure 7.1, and thus is isomorphic to the free group F_3.

Indeed, using projective coordinates, we have eight intervals

$$
\begin{array}{llll}
I_1 = (\infty, -2), & I_2 = (-2,-1), & I_3 = (-1, -\frac{1}{2}), & I_4 = (-\frac{1}{2}, 0) \\
I_5 = (0, \frac{1}{2}), & I_6 = (\frac{1}{2}, 1), & I_7 = (1,2), & I_8 = (2, \infty),
\end{array}
$$

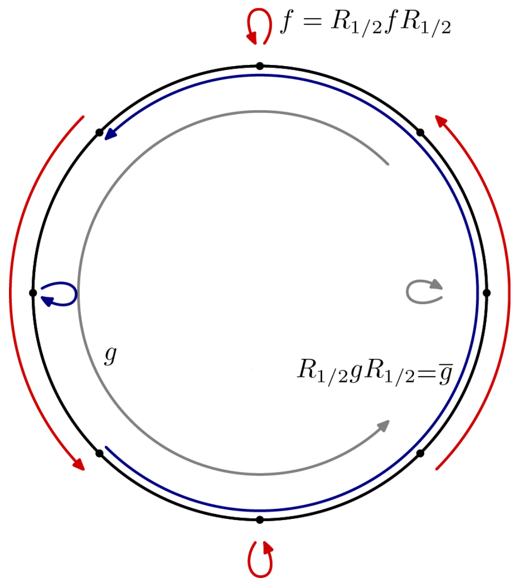

Figure 7.1: Example of finitely generated subgroup of $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ with at most 2 fixed points and whose action is minimal (see Theorem E).

and the elements satisfy

$$\begin{aligned}
f(I_1) &= I_1 \cup \{-2\} \cup I_2 \cup \{-1\} \cup I_3, \\
f^{-1}(I_4) &= I_2 \cup \{-1\} \cup I_3 \cup \left\{-\tfrac{1}{2}\right\} \cup I_4, \\
g(I_7) &= I_7 \cup \{2\} \cup I_8 \cup \{\infty\} \cup I_1 \cup \{-2\} \cup I_2 \cup \{-1\} \cup I_3 \cup \left\{-\tfrac{1}{2}\right\} \cup I_4 \cup \{0\} \cup I_5, \\
g^{-1}(I_6) &= I_8 \cup \{\infty\} \cup I_1 \cup \{-2\} \cup I_2 \cup \{-1\} \cup I_3 \cup \left\{-\tfrac{1}{2}\right\} \cup I_4 \cup \{0\} \cup I_5 \cup \left\{\tfrac{1}{2}\right\} \cup I_6, \\
R_{\frac{1}{2}}(I_i) &= I_{i+4}, \quad \text{for every } i \in \mathbb{Z}_8.
\end{aligned}$$

A short computation gives that derivatives of the generators on the intervals of the partitions which are "expanded" (that is, sent to unions of more than one interval) are always greater than 1 on the corresponding interval.

To see this, we compute derivatives with respect to the coordinate $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]/_{(-\frac{\pi}{2} \sim \frac{\pi}{2})}$ given by $\tan\theta = x$, where x is the projective coordinate. Observe that when $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a projective transformation the corresponding map

$$H = \tan^{-1} \circ h \circ \tan$$

has derivative

$$H'(\theta) = \frac{h'(x)}{1 + h(x)^2}(1 + x^2) = \frac{1 + x^2}{(ax+b)^2 + (cx+d)^2}.$$

Using the expressions for f and g, and using that the action induced by f on the quotient of the circle by the rotation $R_{\frac{1}{2}}$ is a projective transformation, one gets the desired estimates for the derivatives.

Moreover, from this, we can check that the composition fg has a unique fixed point (which will correspond to $\frac{1}{2}$ in projective coordinates, see Figure 7.1). This gives that every endpoint of element of the partition $\{I_1, \ldots, I_8\}$ is fixed by at least one element with 2 or less fixed points, which implies that every endpoint of the partition is contained in the minimal set. From this, we deduce that the action of $\langle f, g, \overline{g} \rangle$ is minimal. Indeed, if there was a wandering interval, it must be contained in one interval of the partition. Take a wandering interval J of maximal size, and apply the generator that expands such interval. As the derivative of such generator is greater than 1 on J, we must have that its image is strictly larger, which is a contradiction.

Now, we will show that the action of $\langle f, g, \overline{g} \rangle$ has at most two fixed points. For this, we consider the segments $A = [2, \frac{1}{2}]$ and $B = [\frac{1}{2}, 2]$, and observe that (A, B) is a ping-pong partition for the actions of $\langle f, R_{\frac{1}{2}} \rangle$ and $\langle g \rangle$ (see Lemma 4.2), and then, as both subgroups have ate most 2 fixed points we conclude that every element in $\langle f, g, R_{\frac{1}{2}} \rangle$ conjugate into $\langle f, R_{\frac{1}{2}} \rangle \cup \langle g \rangle$ also fixes 2 points or less. For the elements that are not conjugate into $\langle f, R_{\frac{1}{2}} \rangle \cup \langle g \rangle$, we apply Lemma 5.3 to show that the number of fixed points of such element is bounded by the number of intervals in the proper ping-pong partition, which is 2. Therefore, the generated subgroup $\langle f, g, R_{\frac{1}{2}} \rangle$ has at most two fixed points, and we conclude that the action of $\langle f, g, \overline{g} \rangle$ also has at most two fixed points.

To conclude the proof of Theorem E, we remark that f has two parabolic fixed points, so the group $\langle f, g, R_{\frac{1}{2}} \rangle$ is not conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. □

7.2 A group of smooth diffeomorphisms which is not isomorphic into $\mathrm{PSL}(2, \mathbb{R})$

Here we will discuss Theorem F, that is, we give an example of group of C^∞ circle diffeomorphisms, with at most two fixed points, but which is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Proof of Theorem F. For the construction of this example we will consider the following maps with respect to the projective coordinates of the circle: fix λ and $\mu > 1$, such that, $\log \lambda$ and $\log \mu$ are linearly independent over $\mathbb{Q}$, and set

$$f(x) = \begin{cases} \lambda x & \text{for } x \in [0, \infty], \\ \mu x & \text{for } x \in [\infty, 0]. \end{cases}$$

For convenience set $g = R_{\frac{1}{2}} f R_{\frac{1}{2}}$. It is clear that f and g generate a rank 2 abelian free group. Moreover, conjugation by the rotation $R_{\frac{1}{2}}$ defines an action of $\mathbb{Z}_2$ on such $\mathbb{Z}^2$ given by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(with respect to the basis f, g). In other terms, the group $G = \langle f, R_{\frac{1}{2}} \rangle$ is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}_2$. By construction, one can observe that G acts with at most 2 fixed points, and by the proof of Theorem B, G is not Möbius-Like, and in particular G is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Finally, conjugating G by a suitable C^∞ homeomorphism which is infinitely flat at 0 and ∞, we can embed G into $\mathrm{Diff}^\infty(\mathbb{S}^1)$. This proves Theorem F. □

Remark 7.1. An example of a *minimal* finitely generated group of circle homeomorphisms, with at most 2 fixed points, and which is not isomorphic to any subgroup of $\mathrm{PSL}(2, \mathbb{R})$, can be build by considering any amalgamated product of this subgroup over a trivial stabilizer.

A Appendix

A.1 Proof of Theorem 2.15

Existence – We will prove the existence by constructing the blow-up. First, we take any continuous 1-degree map $h : \mathbb{S}^1 \to \mathbb{S}^1$, satisfying that:

- $h^{-1}(x)$ is a singleton, for all $x \in \mathbb{S}^1 \smallsetminus A$,
- $h^{-1}(a) =: I_a$ is a non-trivial compact interval, for all $a \in A$.

Such a continuous map h clearly exists. Now, for each $a \in A$ denote by $t_a : [0,1] \to I_a$ the unique linear orientation-preserving homeomorphism mapping the interval $[0,1]$ onto $I_a \subset \mathbb{S}^1$. The next step is to choose for every $k \in \Omega$ and every $\xi \in F.a_k$, an element $g_\xi \in F$ such that $g_\xi(a_k) = \xi$ and define $\vartheta(g_\xi) : I_{a_k} \to I_\xi$ as $\vartheta(g_\xi) := t_\xi t_{a_k}^{-1}$.

For every a_k and every element $s \in S_k$ of the stabilizer, we define the homeomorphism $\vartheta(s)\colon I_{a_k} \to I_{a_k}$ as

$$\vartheta(s)(x) = t_{a_k}\phi_k(s)t_{a_k}^{-1}(x), \quad x \in I_{a_k}.$$

Now, for each element $g \in F$, we define $\vartheta(g)(x)$ for all $x \in \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} I_a$ as $\vartheta(g)(x) = h^{-1}gh(x)$. It is well defined since h is a bijection in restriction to this subset.

The next step is to define $\vartheta(g)(x)$ on every I_a, for $a \in A$. Fix $a \in A$, and let $k \in \Omega$ be such that $a \in F.a_k$; write also $a' := g(a)$, which is also a point on the orbit $F.a_k$. Observe that $g_{a'}^{-1}gg_a(a_k) = a_k$, therefore $g_{a'}^{-1}gg_a \in S_k$, hence we can define

$$\vartheta(g)(x) = \vartheta(g_{a'})\vartheta(g_{a'}^{-1}gg_a)\vartheta(g_a)^{-1}(x), \quad \text{for all } x \in I_a.$$

This gives the desired homeomorphism $\vartheta(g) : \mathbb{S}^1 \to \mathbb{S}^1$. One can observe that, by construction, the application $\vartheta : F \to \mathrm{Homeo}_+(\mathbb{S}^1)$ satisfies the conditions to be a group homomorphism. Furthermore, the homomorphism $\vartheta : F \to \mathrm{Homeo}_+(\mathbb{S}^1)$ is injective, indeed taking two elements $g_1, g_2 \in F$ with $g_1 \neq g_2$, there exists a point $x \in \mathbb{S}^1 \smallsetminus A$ such that $g_1(x) \neq g_2(x)$ (this is because A is at most countable), then both $g_1(x)$ and $g_2(x)$ are contained in $\mathbb{S}^1 \smallsetminus A$ and the preimages $h^{-1}(x), h^{-1}g_1(x)$ and $h^{-1}g_2(x)$ are all singletons with $h^{-1}g_1(x) \neq h^{-1}g_2(x)$. Therefore

$$\vartheta(g_1)(h^{-1}(x)) = h^{-1}g_1h(h^{-1}(x)) = h^{-1}g_1(x) \neq h^{-1}g_2(x) = h^{-1}g_2h(h^{-1}(x)) = \vartheta(g_2)(h^{-1}(x)).$$

Consider the subgroup $G := \vartheta(F) \leq \mathrm{Homeo}_+(\mathbb{S}^1)$ and the isomorphism $\theta = \vartheta^{-1} : G \to F$. It is clear that the subgroup G is an isomorphic blow-up of F in $\{a_k\}_{k\in\Omega}$ and including $\{\phi_k\}_{k\in\Omega}$ on the intervals.

Uniqueness – For the uniqueness, we are going to construct the conjugacy between any two blow-ups. Let G_1 and G_2 be two subgroups of $\mathrm{Homeo}_+(\mathbb{S}^1)$, obtained by an isomorphic blow-up of F in $\{a_k\}_{k\in\Omega}$ and including $\{\phi_k\}_{k\in\Omega}$ on the intervals. For $i \in \{1,2\}$, denote by h_i the semi-conjugacy for G_i to F, and the isomorphism $\theta_i : G_i \to F$, and write $X_i = \mathrm{Core}(h_i)$.

First, observe that

$$\text{for all} \quad x_1 \in \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_1^{-1}(a) \quad \text{and all} \quad x_2 \in \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_2^{-1}(a)$$

we have for all $g \in F$

$$\theta_1(g)(x_1) = h_1^{-1}gh_1(x_1) \quad \text{and} \quad \theta_2(g)(x_2) = h_2^{-1}gh_2(x_2).$$

Then, we can define an application $\beta : \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_1^{-1}(a) \to \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_2^{-1}(a)$ as

$$\beta(x) := h_2^{-1}h_1(x) \quad \text{for all } x \in \mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_1^{-1}(a).$$

Observe that β is an order-preserving homeomorphism and we can extend it to a homeomorphisms of the closures of $\mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_1^{-1}(a)$ and $\mathbb{S}^1 \smallsetminus \bigcup_{a\in A} h_2^{-1}(a)$, which are the cores X_1 and X_2, respectively. We still denote such map by $\beta : X_1 \to X_2$, and we observe that for every $g \in G$ it satisfies that

$$\beta\theta_1(g)\beta^{-1}(x) = \theta_2(g)(x) \qquad \text{for all } x \in X_1.$$

Now, for all $k \in \Omega$ and all $s \in S_k$ there exist two order-preserving homeomorphisms

$$t_{1,k} \colon [0,1] \to h_1^{-1}(a_k) \quad \text{and} \quad t_{2,k} : [0,1] \to h_2^{-1}(a_k),$$

such that

$$\theta_1(s) = t_{1,k}\phi_k(s)t_{1,k}^{-1} \qquad \text{and} \qquad \theta_2(s) = t_{2,k}\phi_k(s)t_{2,k}^{-1}.$$

Then we can extend β such that it gives a homeomorphism from $h_1^{-1}(a_k) \to h_2^{-1}(a_k)$, by the expression

$$\beta(x) := t_{2,k}t_{1,k}^{-1}(x) \quad \text{for all } x \in h_1^{-1}(a_k).$$

Observe that this extension of β is an order-preserving homeomorphism and for every a_k and every $s \in S_k$ it follows that

$$\beta\theta_1(s)\beta^{-1}(x) = \theta_2(s)(x) \qquad \text{for all } x \in h_1^{-1}(a_k).$$

Now, we are going to conclude the definition of β to the whole circle by extending it to the preimages $h^{-1}(a)$ for all $a \in A$, in an equivariant way: for every $k \in \Omega$ and every $\xi \in F.a_k$ choose an element $g_\xi \in F$ such that $g_\xi(a_k) = \xi$, therefore for any element $s' \in \mathrm{Stab}_F(\xi)$ we have

$$\theta_1(s') = \theta_1(g_\xi g_\xi^{-1}s'g_\xi g_\xi^{-1}) = \theta_1(g_\xi)\theta_1(g_\xi^{-1}s'g_\xi)\theta_1(g_\xi^{-1}).$$

Notice that, $g_\xi^{-1}s'g_\xi(a_k) = a_k$, then this element is in S_k and it follows that

$$\theta_1(g_\xi^{-1}s'g_\xi) = t_{1,k}\phi_k(g_\xi^{-1}s'g_\xi)t_{1,k}^{-1}.$$

Then, we have that:

$$\theta_1(s') = \theta_1(g_\xi)t_{1,k}\phi_k(g_\xi^{-1}s'g_\xi)t_{1,k}^{-1}\theta(g_\xi^{-1}), \text{ for all } s' \in \mathrm{Stab}_F(\xi)$$

Similarly,

$$\theta_2(s') = \theta_2(g_\xi)t_{2,k}\phi_k(g_\xi^{-1}s'g_\xi)t_{2,k}^{-1}\theta(g_\xi^{-1}).$$

Now, we define $\beta \colon h_1^{-1}(\xi) \to h_2^{-1}(\xi)$ as

$$\beta(x) := \theta_1(g_\xi)t_{2,k}t_{1,k}^{-1}(\theta_2(g_\xi))^{-1}(x) \quad \text{for all } x \in h_1^{-1}(\xi).$$

Observe that it is an order-preserving homeomorphism and for every $s \in \mathrm{Stab}_F(\xi)$ it follows that

$$\beta\theta_1(s)\beta^{-1}(x) = \theta_2(s)(x) \qquad \text{for all } x \in h_1^{-1}(\xi).$$

As a summary of the construction, we have an order-preserving homeomorphism $\beta : \mathbb{S}^1 \to \mathbb{S}^1$ defined as:

$$\beta(x) = \begin{cases} h_2^{-1}h_1(x) & \text{for all } x \in X_1, \\ t_{2,k}t_{1,k}^{-1}(x) & \text{for any } a_k \text{ and all } x \in h_1^{-1}(a_k), \\ \theta_1(g_\xi)t_{2,k}t_{1,k}^{-1}(\theta_2(g_\xi))^{-1}(x) & \text{for any } \xi \in F.a_k \text{ and all } x \in h_1^{-1}(\xi), \end{cases}$$

where g_ξ is any fixed element of F such that $g_\xi(a_k) = \xi$.

With this definition, β satisfies, for every $k \in \Omega$ and any $s \in S_k$, the relation $\beta\theta_1(s)\beta^{-1} = \theta_2(s)$.

Now, to conclude the proof, observe that for all $x \in \mathbb{S}^1$ and every $g \in F$ we have that

$$\beta\theta_1(g)\beta^{-1} = \theta_2(g).$$

Indeed, for $x \in X_1$ it is clear, and for any $x \in \mathbb{S}^1 \smallsetminus X_1$ there exist one $k \in \Omega$ and one point $\xi \in F.a_k$ with $x \in h_1^{-1}(\xi)$ (possibly $\xi = a_k$), and then denoting $g(\xi) = \eta \in F.a_k$, we have that $g_\eta^{-1} g g_\xi \in S_k$, which implies:

$$\beta\theta_1(g)\beta^{-1} = \beta\theta_1(g_\eta g_\eta^{-1} g g_\xi g_\xi^{-1})\beta^{-1} = \beta\theta_1(g_\eta)\theta_1(g_\eta^{-1} g g_\xi)\theta_1(g_\xi^{-1})\beta^{-1}$$

$$= \beta\theta_1(g_\eta)\beta^{-1}\beta\theta_1(g_\eta^{-1} g g_\xi)\beta^{-1}\beta\theta_1(g_\xi^{-1})\beta^{-1} = \theta_2(g_\eta)\theta_2(g_\eta^{-1} g g_\xi)(\theta_2(g_\xi))^{-1} = \theta_2(g). \qquad \square$$

Remark A.1. Observe that a similar construction to the one in the proof of uniqueness in Theorem 2.15 can be used to prove that any two isomorphic blow-ups of F, the first in $\{a_k\}_{k\in\Omega}$ and including $\{\phi_k\}_{k\in\Omega}$ on the intervals and the second in $\{b_k\}_{k\in\Omega}$ and including $\{\varphi_k\}_{k\in\Omega}$ on the intervals, are conjugate if there exists an order-preserving homeomorphism $\tau : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\tau(\bigsqcup G.a_k) = \bigsqcup G.b_k$ and for every $k \in \Omega$ the homeomorphisms ϕ_k and φ_k are conjugate.

A.2 Groups of homeomorphisms of the line with at most 2 fixed points

The main purpose of this section is to prove the following result on group actions of the line.

Theorem A.2. *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ with at most 2 fixed points, then*

- *either G is abelian, or*
- *the action of G is semi-conjugate to an action by affine transformations.*

We first analyse the situation when the action admits a global fixed point, and this is essentially based on Solodov's theorem (Theorem 1.4).

Lemma A.3. *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ with at most 2 fixed points such that there exists a global fixed point $x \in \mathbb{R}$, then G is abelian and any point which is fixed by a non-trivial element is globally fixed.*

Proof. We let $G_- \leq \mathrm{Homeo}_+((-\infty, x))$ and $G_+ \leq \mathrm{Homeo}_+((x, +\infty))$ denote the subgroups obtained by considering the restriction of the action of G to the two G-invariant half-lines, respectively. Note that the morphisms $G \to G_\pm$ are both isomorphisms, as elements in one of the kernels fix a half-line and thus are globally trivial. Notice that if either G_- or G_+ is abelian, this implies that G is also abelian and so there is nothing to prove. So we will assume that none of them is abelian, and look for a contradiction. Since every element of G has at most one fixed point other than x, it follows that both G_- and G_+ acts with at most 1 fixed point and therefore, by Solodov's theorem (Theorem 1.4), both G_- and G_+ are semi-conjugate to non-abelians subgroups of $\mathrm{Aff}_+(\mathbb{R})$. Now, any element $g \in G$ which is non-trivial in the abelianization $G/[G,G]$, gives in G_- and G_+ elements which are semi-conjugate to homotheties, and thus g admits at least three fixed points, giving the desired contradiction. $\square$

We next move to the case where the action has no global fixed points. We first introduce some terminology which will provide the combinatorial set-up for the core of the proof of Theorem A.2: for homeomorphisms of the line it is natural to consider whenever its graph is above or below the identity, and since we are restricting ourselves to homeomorphisms with at most N fixed points, this information can be encoded in a finite string as in the following definition.

Definition A.4. Let $h \in \mathrm{Homeo}_+(\mathbb{R})$ be an orientation-preserving homeomorphism with $\mathrm{Fix}(h) = \{x_1, \ldots, x_n\}$, where $x_1 < \cdots < x_n$. We say that h *is of type* $(\varepsilon_0, , \ldots, \varepsilon_n)$ for $\varepsilon_k \in \{+, -\}$ if the sign of $h - \mathrm{id}$ restricted to (x_k, x_{k+1}) is ε_k for every $k \in \{0, \ldots, n\}$ (here we write $x_0 = -\infty$ and $x_{n+1} = +\infty$).

Remark A.5. The notion presented in Definition A.4 is invariant by conjugation and for every $h \in \mathrm{Homeo}_+(\mathbb{R})$ and $n \in \mathbb{N}^*$, h and h^n are of the same type, but h and h^{-1} are of opposite types (every coordinate has the opposite sign).

We can now discuss the main technical lemma, which describes the case when the group contains an element with two fixed points.

Lemma A.6. *Let G be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *with at most* 2 *fixed points, whose action admits no global fixed point. Assume there exists an element $g \in G$ such that* $\mathrm{Fix}(g) = \{x, y\}$*, with $x < y$. Then, the orbit of x does not intersect the interval (x, y).*

Proof. We will structure the proof by considering all the possible types for g. To start with, we argue that it is enough to restrict to the cases where the type of g is $(+, +, +), (-, +, +)$ and $(+, -, +)$, since all the 2^3 possible types are can be reduced to this case by considering g^{-1} or the conjugate of g by an orientation-reversing isometry.

Arguing by contradiction, let us suppose that there exists $f \in G$ with $f(x) \in (x, y)$. In this case $f(y) \neq y$, otherwise the subgroup defined by $\langle g, f \rangle \leq G$ would act with at most 2 fixed points with one of them being globally fixed, and by Lemma A.3, this would imply that x is also globally fixed, which is not the case.

Case 1.(a) g is of type $(+, +, +)$ and $f(y) < y$. Consider $h = fgf^{-1} \in G$, it is of type $(+, +, +)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < f(y) < y$, then one can observe that $h(x) \in (x, y)$ but $h(y) > y$, so we take $f' = h$ and we are reduced to the next case 1.(b).

Case 1.(b) g is of type $(+, +, +)$ and $y < f(y)$. Consider $h = fgf^{-1} \in G$, it is of type $(+, +, +)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < y < f(y)$, therefore:

- $hg^{-1}(x) = h(x) > x$
- $hg^{-1}(f(x)) < h(f(x)) = f(x)$
- $hg^{-1}(y) = h(y) > y$
- $hg^{-1}(f(y)) < h(f(y)) = f(y)$

which implies that hg^{-1} has a fixed point in each of the intervals $(x, f(x))$, $(f(x), y)$ and $(y, f(y))$. This contradicts the assumption that G has at most 2 fixed points.

Case 2.(a) g is of type $(-, +, +)$ and $f(y) < y$. Consider $h = fgf^{-1} \in G$, it is of type $(-, +, +)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < f(y) < y$, then take $n \in \mathbb{N}$ sufficiently large such that $g^n(x-1) < h(x-1)$ and $g^n(y+1) > h(y+1)$. Since the type of g^n and its fixed points are constant for every positive n, it follows that:

- $hg^{-n}(h(x-1)) > h(x-1)$
- $hg^{-n}(x) = h(x) < x$
- $hg^{-n}(y) = h(y) > y$
- $hg^{-n}(h(y+1)) < h(y+1)$

which implies that hg^{-n} has a fixed point in each of the intervals $(h(x-1), x)$, (x, y) and $(y, h(y+1))$, which gives the desired contradiction.

Case 2.(b) g **is of type** $(-,+,+)$ **and** $y < f(y)$**.** Consider $h = fgf^{-1} \in G$, it is of type $(-,+,+)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < y < f(y)$. Then one can observe that $h^{-1}(x) \in (x, y)$ but $h^{-1}(y) < y$, so we take $f' = h$ and we are back to the case 2.(a).

Case 3.(a) g **is of type** $(+,+,-)$ **and** $f(y) < y$**.** Consider $h = fgf^{-1} \in G$, it is of type $(+,+,-)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < f(y) < y$. Take $n \in \mathbb{N}$ sufficiently large such that $g^{-n}(x-1) < h^{-1}(x-1)$ and $g^{-n}(y+1) > h^{-1}(y+1)$ then we have:

- $h^{-1}g^n(h^{-1}(x-1)) > h^{-1}(x-1)$
- $h^{-1}g^n(x) = h^{-1}(x) < x$
- $h^{-1}g^n(y) = h^{-1}(y) > y$
- $h^{-1}g^n(h^{-1}(y+1)) < h^{-1}(y+1)$

which implies that $h^{-1}g^n$ has a fixed point in each of the intervals $(h^{-1}(x-1), x)$, (x, y) and $(y, h^{-1}(y+1))$, which again gives the desired contradiction.

Case 3.(b) g **is of type** $(+,+,-)$ **and** $y < f(y)$**.** Consider $h = fgf^{-1} \in G$, it is of type $(+,+,-)$ with $\mathrm{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < y < f(y)$. Then one can observe that $g(x) \in (f(x), f(y))$ but $g(f(y)) < f(y)$, so we take $g' = h$ and $f' = g$ and we are back to the case 3.(a).

Case 4.(a) g **is of type** $(+,-,+)$ **and** $f(y) < y$**.** This is by far the most complex case, and its proof will be given by a long construction.

First, we remark that as the action of G has no global fixed point, we can find two elements $u, v \in G$ such that $u(y) < x$ and $v(x) > y$. Define the following elements of G:

- $h = fgf^{-1}$ of type $(+,-,+)$, with $\mathrm{Fix}(h) = \{f(x), f(y)\} = \{\tilde{x}, \tilde{y}\}$
- $f_u = ugu^{-1}$ of type $(+,-,+)$ with $\mathrm{Fix}(f_u) = \{u(x), u(y)\}$
- $f_v = vgv^{-1}$ of type $(+,-,+)$ with $\mathrm{Fix}(f_v) = \{v(x), v(y)\}$

These points satisfy the order relation

$$u(x) < u(y) < x < \tilde{x} < \tilde{y} < y < v(x) < v(y).$$

Now, observe that there exists $n_1 \in \mathbb{N}$, such that for every $n \geq n_1$ one has $h^{-n}(x) < f_v^{-1}(x)$, and thus

$$h^n f_v h^{-n}(x) < h^n f_v(f_v^{-1}(x)) = h^n(x) < \tilde{x}.$$

On the other hand, there exists $n_2 \in \mathbb{N}$ such that for every $n \geq n_2$ one has $h^n f_v(\tilde{y}) > y + 2$, and thus

$$h^n f_v h^{-n}(y) > h^n f_v h^{-n}(\tilde{y}) = h^n f_v(\tilde{y}) > y + 2.$$

Let $n = \max\{n_1, n_2\}$ and define $s = h^n f_v h^{-n} \in G_+$, so we have $s(x) < \tilde{x}$ and $y + 2 < s(y)$.

Similarly, one can observe that there exists $m_1 \in \mathbb{N}$, such that for every $m \geq m_1$ one has $g^{-m}(f_u(x)) > \tilde{y}$, and thus

$$g^{-m} f_u g^m(x) = g^{-m} f_u(x) > \tilde{y}.$$

On the other hand, there exists $m_2 \in \mathbb{N}$ such that for every $m \geq m_2$ one has $g^{-m}(f_u(y)) < y+1$, and thus
$$g^{-m} f_u g^m(y) = g^{-m} f_u(y) < y+1.$$
Let $m = \max\{m_1, m_2\}$ and define $t = g^{-m} f_u g^m \in G$, so we have $\tilde{y} < t(x)$ and $t(y) < y+1$.

We also observe that $\operatorname{Fix}(s) = \{h^n(v(x)), h^n(v(y))\}$ and $\operatorname{Fix}(t) = \{g^{-m}(u(x)), g^{-m}(u(y))\}$, and we have
$$g^{-m}(u(x)) < g^{-m}(u(y)) < u(y) < x < y < v(x) < h^n(v(x)) < h^n(v(y)).$$
Now, let $z_1 := g^{-m}(u(x)) \in \mathbb{R}$, so that z_1 is a fixed point of t, which is smaller than both fixed points of s, which is of type $(+,-,+)$. Therefore, $t(z_1) = z_1$ and $s(z_1) > z_1$. Similarly, let $z_2 := h^n(v(y)) \in \mathbb{R}$, so that z_2 is a fixed point of s, which is larger than both fixed points of t, which is of type $(+,-,+)$. Therefore, $s(z_2) = z_2$ and $t(z_2) > z_2$. This leads to the following inequalities:

- $t(z_1) = z_1 < s(z_1)$
- $t(x) > \tilde{y} > \tilde{x} > s(x)$
- $t(y) < y+1 < y+2 < s(y)$
- $t(z_2) > z_2 = s(z_2)$

which implies that $ts^{-1} \in G$ is a non-trivial element with at least one fixed point in each of the intervals (z_1, x), (x, y) and (y, z_2), adding up to at least 3 fixed points. This contradicts the hypothesis that G has at most 2 fixed points.

Case 4.(b) g is of type $(+,-,+)$ and $y < f(y)$. Consider $h = fgf^{-1} \in G$, it is of type $(+,-,+)$ with $\operatorname{Fix}(h) = \{f(x), f(y)\}$ satisfying that $x < f(x) < y < f(y)$. Then one can observe that $h(x) \in (x, y)$ but $h(y) < y$, so we take $f' = h$ and we are reduced to the previous case 4.(a).

As we have covered all possible situations, we conclude that if there exists an element $g \in G$ such that $\operatorname{Fix}(g) = \{x, y\}$, then the orbit of x by G does not intersect the interval (x, y). □

Proof of Theorem A.2. If the action of G admits a global fixed point, then Lemma A.3 implies that G is abelian. So we can assume that the action of G has no global fixed point. If G has at most 1 fixed point, then we conclude by Solodov's theorem (Theorem 1.4) that the action is semi-conjugate to an affine action. When G contains non-trivial elements with exactly two fixed points, Lemma A.6 implies that any interval of the form (x, y), where x, y are fixed by some non-trivial element, is wandering. Now, since there are elements with non-global fixed points we have that the complement $\mathbb{R} \smallsetminus G.(x, y)$ contains the minimal invariant subset which is uncountable, so the non-decreasing map $h : \mathbb{R} \to \mathbb{R}$ that collapses any such interval to a single point defines a semi-conjugacy of the action of G to an action with at most 1 fixed point, so that we are reduced to the previous case. □

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