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Algebraic torus actions over characteristic zero fields

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Composition du Jury

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Contents

Foreword

In this thesis, some results are classical but difficult to find in the literature. The original ones have given rise to two articles.

- Real torus actions on real affine algebraic varieties, *Math. Z.*, 301(2):1507–1536, 2022.
- Torus actions on affine varieties over characteristic zero fields, `https://arxiv.org/abs/2208.01495`, July 2022.

The first one focuses on the $\mathbb{C}/\mathbb{R}$ case. Its main results correspond to the content of Section 4.1.

The second one (preprint) extends the first article by treating the case of arbitrary Galois extension (in characteristic zero). The main results correspond to the content of Sections 3.2 and 4.2.

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[1] On ne prononce pas le 'z' !

[2] Non, je n'ai pas redoublé !

[3] J'ai sans doute eu le meilleur bureau !

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Cette thèse compte 176 pages, soit autant que la longueur en mètres du *Danielle Casanova*. La version précédente en comptait 172, la longueur du *Napoléon Bonaparte*...

Introduction

Aims and scope

In this thesis, we study affine varieties endowed with an effective torus action over a characteristic zero field. The aim is to describe these varieties with a geometrico-combinatorial datum. Over an algebraically closed field of characteristic zero, Altmann and Hausen gave a presentation of such varieties in terms of *polyhedral divisors* on a certain *rational quotient* for the torus action (see [AH06]). Their presentation was partially extended over a non-algebraically closed field by Langlois (see [Lan15, Lan21]), and by Dubouloz, Liendo and Petitjean (see [DL18, DP20]). Using Galois descent tools, we extend the *Altmann-Hausen presentation* of affine varieties endowed with an effective torus action to arbitrary fields of characteristic zero. Thus we get a complete description of such varieties. We recover as a particular case of our presentation the one given in [DL18, DP20], and also the one given in [Lan15, Lan21] restricted to characteristic zero fields.

The reader only interested in the $\mathbb{C}/\mathbb{R}$ case may skip Chapter 3 and may go directly to the next one (see also [Gil22a] for a self contained version of the Altmann-Hausen presentation over $\mathbb{R}$).

Example

We start by illustrating the Altmann-Hausen presentation on an example over the field of complex numbers $\mathbb{C}$. A one-dimensional torus is isomorphic to the multiplicative affine group $\mathbb{G}_{m,\mathbb{C}} := \operatorname{Spec}(\mathbb{C}[M]) \cong \mathbb{C}^*$, where $M = \mathbb{Z}$. Consider the affine plane $\mathbb{A}^2_{\mathbb{C}}$ endowed with the effective torus action

$$\mu(t,(x,y)) := \left(tx, t^{-1}y\right).$$

Then, the coordinate ring $A := \mathbb{C}[x,y]$ of $\mathbb{A}^2_{\mathbb{C}}$ is M-graded

$$A = \bigoplus_{m\in M} A_m,$$

where $A_0 = \mathbb{C}[xy]$, $A_m = A_0 x^m$ and $A_{-m} = A_0 y^m$ with $m \in \mathbb{N}^*$. Therefore, the weight cone of the action is $\omega_M = \mathbb{Q}$, and its dual cone ω_N is trivial.

In this situation, the *Altmann-Hausen quotient* corresponds to the categorical quotient $Y := \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}(A_0)$. Consider the polyhedral divisor

$$\mathcal{D} = [0;1] \otimes \{0\}$$

on Y, where $[0;1] \subset (M_{\mathbb{Q}})^\vee := (M \otimes_{\mathbb{Z}} \mathbb{Q})^\vee$, and $0 \in Y$. We denote by $\langle .|. \rangle$ the corresponding pairing. We can evaluate $\mathcal{D}$ in $m \in M$ by letting

$$\mathcal{D}(m) = \min(\langle m|[0;1]\rangle) \otimes \{0\} = \begin{cases} 0 & \text{if } m \geq 0; \\ m \otimes \{0\} & \text{if } m \leq 0. \end{cases}$$

We get a Weil $\mathbb{Q}$-divisor. From the triple $(\omega_N, Y, \mathcal{D})$, Altmann and Hausen define an M-graded algebra

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \omega_N^\vee \cap M} \mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m))\right) \mathfrak{X}_m.$$

In our example, we get

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \mathbb{Z}_{<0}} A_0 (xy)^{-m} \mathfrak{X}_m \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} A_0 \mathfrak{X}_m.$$

Identifying $\mathfrak{X}_m$ with x^m, we get an isomorphism of M-graded algebras

$$A \cong A[Y, \mathcal{D}].$$

Therefore, we get an isomorphism of $\mathbb{G}_{m,\mathbb{C}}$-varieties

$$X \cong X[Y, \mathcal{D}],$$

where $X[Y, \mathcal{D}] := \mathrm{Spec}(A[Y, \mathcal{D}])$. In other words, from the triple $(\omega_N, Y, \mathcal{D})$, we recover the affine variety $\mathbb{A}^2_{\mathbb{C}}$ endowed with the effective torus action μ of $\mathbb{G}_{m,\mathbb{C}}$ (see Example E.2.3).

Altmann-Hausen presentation

Let $n \in \mathbb{N}^*$, let M and N be dual lattices of rank n, and let $\mathbb{T} := \mathrm{Spec}(\Bbbk[M]) \cong \mathbb{G}^n_{m,\Bbbk}$ be a $\Bbbk$-torus. From a triple $(\omega_N, Y, \mathcal{D})$, where ω_N is a pointed cone in $N_{\mathbb{Q}}$, and where $\mathcal{D}$ is a *certain* polyhedral divisor on a normal *semi-projective* variety Y, Altmann and Hausen constructed an affine variety $X[Y, \mathcal{D}]$ endowed with an effective torus action. The triple $(\omega_N, Y, \mathcal{D})$ is called an *AH-datum*. The main results obtained in [AH06] can be summarized as follows.

Theorem AH (See Theorem 3.1.12)**.** *Let $\Bbbk$ be an algebraically closed field of characteristic zero, and let $\mathbb{T} := \mathrm{Spec}(\Bbbk[M])$ be a $\Bbbk$-torus.*

(i) Let $(\omega_N, Y, \mathcal{D})$ be an AH-datum. The affine scheme $X[Y, \mathcal{D}]$ is a normal $\Bbbk$-variety endowed with an effective $\mathbb{T}$-action.

(ii) Conversely, let X be an affine normal variety endowed with an effective $\mathbb{T}$-action. There exists an AH-datum $(\omega_N, Y, \mathcal{D})$ such that there is an isomorphism of $\mathbb{T}$-varieties $X \cong X[Y, \mathcal{D}]$, i.e. the graded $\Bbbk$-algebras $\Bbbk[X]$ and $A[Y, \mathcal{D}]$ are isomorphic.

As it is mentioned above, the main goal of this thesis is to extend this presentation to arbitrary fields of characteristic zero. To do this, this thesis is divided in several chapters as follows.

Overview of the thesis

Chapter 1. In this chapter, we write down some well known results on the Galois descent theory, but they are difficult to find in the literature.

Since we are working with varieties over non-closed fields of characteristic zero, it is convenient to deal with the Galois descent language, and more exactly with the $\Bbbk$-*structure* language. Let $\Bbbk$ be a characteristic zero field, then the separable closure $\Bbbk_s$ of $\Bbbk$ in a fixed algebraic closure $\overline{\Bbbk}$ satisfies $\Bbbk_s = \overline{\Bbbk}$. Therefore, the field extension $\overline{\Bbbk}/\Bbbk$ is Galois, and we denote by Γ its profinite Galois group. Roughly, a $\Bbbk$-*structure* on an algebraic $\overline{\Bbbk}$-variety X is a continuous action $\Gamma \times X \to X$ such that the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{CD} X @>{\sigma_\gamma}>> X \\ @VVV @VVV \\ \operatorname{Spec}(\overline{\Bbbk}) @>{\operatorname{Spec}(\gamma)}>> \operatorname{Spec}(\overline{\Bbbk}). \end{CD}$$

The main result of Chapter 1 is the well known equivalence of categories that is summarized in the next theorem.

Theorem A (See Theorem 1.4.11)**.** *There is an equivalence of categories between the category of pairs (X, σ) consisting of a quasi-projective $\overline{\Bbbk}$-variety X endowed with a $\Bbbk$-structure σ, and the category of quasi-projective $\Bbbk$-varieties.*

Once we know the existence of a $\Bbbk$-structure σ on a quasi-projective $\Bbbk'$-variety X, the Galois cohomology set

$$\mathrm{H}^1_{cont}\left(\Gamma, \operatorname{Aut}(X)\right)$$

parametrizes the set of $\Bbbk$-forms of X, that is the set of $\Bbbk$-varieties X_0 such that $X_0 \times_{\operatorname{Spec}(\Bbbk)} \operatorname{Spec}(\overline{\Bbbk})$ is isomorphic to X.

We have a similar result concerning algebraic groups. In this case, if G is a $\overline{\Bbbk}$-group, we consider $\Bbbk$-structures that are compatible with the group structure on G. These $\Bbbk$-structures are called $\Bbbk$-*group structures*.

Corollary B (See Corollary 1.4.12)**.** *There is an equivalence of categories between the category of pairs (G, τ) consisting of a $\overline{\Bbbk}$-group G endowed with a $\Bbbk$-group structure τ, and the category of $\Bbbk$-groups.*

Furthermore, in the context of quasi-projective varieties endowed with a group action, we get the next result.

Corollary C (See Corollary 1.4.13)**.** *Let G_0 be a $\Bbbk$-group. There is a one-to-one correspondence between quasi-projective $\Bbbk$-varieties endowed with a G_0-action and tuples $(G, \tau, X, \sigma, \mu)$ consisting of:*

(i) a $\overline{\Bbbk}$-group G endowed with a $\Bbbk$-group structure τ such that $G/\Gamma \cong G_0$;
(ii) a quasi-projective $\overline{\Bbbk}$-variety X endowed with a $\Bbbk$-structure σ;
(iii) an action $\mu : G \times X \to X$ such that the following diagram commutes for all $\gamma \in \Gamma$:

$$\begin{CD} G \times X @>{\mu}>> X \\ @V{\tau_\gamma \times \sigma_\gamma}VV @VV{\sigma_\gamma}V \\ G \times X @>>{\mu}> X. \end{CD}$$

Chapter 2. In this chapter we study tori, tori actions, and toric varieties over arbitrary fields of characteristic zero. This chapter is a *warm-up* to the Altmann-Hausen theory; there are no new results.

By definition, an n-dimensional torus over an algebraically closed field $\Bbbk$ is isomorphic to $\mathbb{G}^n_{m,\Bbbk} := \operatorname{Spec}(\Bbbk[M])$, where $M = \mathbb{Z}^n$. The group of $\Bbbk$-points of $\mathbb{G}^n_{m,\Bbbk}$ is the multiplicative group $(\Bbbk^*)^n$. However, over a non-closed field $\Bbbk$ of characteristic zero, a $\Bbbk$-torus is an affine algebraic group T such that $T \times_{\operatorname{Spec}(\Bbbk)} \operatorname{Spec}(\overline{\Bbbk})$ is isomorphic to $\mathbb{G}^n_{m,\overline{\Bbbk}}$ for some $n \in \mathbb{N}^*$. The main difference with the algebraically closed case is that there exist non-split $\Bbbk$-tori, that is tori that are not isomorphic to $\mathbb{G}_{m,\Bbbk}^n$ with $n = \dim(T)$. A classical example of non-split torus is the real circle

$$\mathbb{S}^1 := \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1)).$$

Using the equivalence of categories of Theorem A, one can shows that $\mathbb{S}^1$ corresponds to the pair $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$, where τ_1 is the $\mathbb{R}$-group structure on $\mathbb{G}_{m,\mathbb{C}}$ defined by $z \mapsto \bar{z}^{-1}$.

We pursue the example started at the beginning of the introduction. Consider the action μ of $\mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^2_{\mathbb{C}}$ defined above by $\mu(t,(x,y)) = (tx, t^{-1}y)$, and let $\sigma : (x,y) \mapsto (\bar{y}, \bar{x})$ be an $\mathbb{R}$-structure on $\mathbb{A}^2_{\mathbb{C}}$. The pair $(\mathbb{A}^2_{\mathbb{C}}, \sigma)$ corresponds to the Weil restriction $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}}) \cong \mathbb{A}^2_{\mathbb{R}}$ of $\mathbb{A}^1_{\mathbb{C}}$. We can construct an $\mathbb{S}^1$-action on the affine $\mathbb{R}$-variety $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}})$ since the following diagram commutes (see Exemple 4.1.15)

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} & \xrightarrow{\ \mu\ } & \mathbb{A}^2_{\mathbb{C}} \\ {\scriptstyle \tau \times \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbb{G}_{m,\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}} & \xrightarrow[\ \mu\ ]{} & \mathbb{A}^2_{\mathbb{C}}. \end{array}$$

Below in the introduction, we will give the Altmann-Hausen presentation of the $\mathbb{S}^1$-variety $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}})$.

The notion of torsor will play a prominent role in the Altmann-Hausen presentation over arbitrary fields of characteristic zero. Let $\Bbbk$ be a field and let T be a $\Bbbk$-torus. A T-torsor is a $\Bbbk$-variety V endowed with a T-action μ such that $T_{\overline{\Bbbk}} \cong V_{\overline{\Bbbk}}$, and such that $\mu_{\overline{\Bbbk}} : T_{\overline{\Bbbk}} \times V_{\overline{\Bbbk}} \to V_{\overline{\Bbbk}}$ corresponds to the natural action by translation on $T_{\overline{\Bbbk}}$. For instance T itself is a T-torsor, called *the trivial T-torsor* (up to appropriate isomorphisms). By Hilbert's 90 Theorem, a split torus has no non-trivial torsor. Therefore, over algebraically closed fields, since all tori are split, there is no non-trivial torsor. However, a non-split torus may have non-trivial torsors. For instance, the real circle $\mathbb{S}^1$ has a non-trivial torsor given by

$$\operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1)).$$

This observation will affect the Altmann-Hausen presentation over arbitrary fields of characteristic zero. Indeed, at first glance, one may think that the Altmann-Hausen presentation described in [AH06] (see Theorem AH) extends over arbitrary characteristic zero fields by using some Γ-equivariant AH-datum. In fact, since the acting torus may have a certain non-trivial torsor, we may need an additional datum that encodes this torsor. Namely, for $\mathbb{S}^1$-actions studied in [DL18, DP20], the authors observed that an additional datum is needed in the presentation of affine $\mathbb{R}$-varieties endowed with an effective $\mathbb{S}^1$-action with respect to the *expected* Γ-equivariant AH-datum.

Torsors appear for instance in the theory of toric varieties over arbitrary fields. Indeed, the dense open orbit of a T-toric $\Bbbk$-variety is a T-torsors. In this context, the *warm-up result* to the Altmann-Hausen presentation over a characteristic zero field is the following one.

Proposition D (See Proposition 2.4.6)**.** *Let $\Bbbk$ be a field, and let $\Gamma := \operatorname{Gal}(\overline{\Bbbk}/\Bbbk)$. Let $\mathbb{T} := \operatorname{Spec}(\overline{\Bbbk}[M])$ be a $\overline{\Bbbk}$-torus, and let τ be a $\Bbbk$-group structure on $\mathbb{T}$ (we get a $\Bbbk$-torus $(\mathbb{T}, \tau)$).*

(i) Let ω_N be a pointed cone in $N_{\mathbb{Q}}$ that is stable under the Γ-action induced by τ, and let h be a cocycle that encodes a $(\mathbb{T}, \tau)$-torsor. Then, the affine $\mathbb{T}$-toric $\overline{\Bbbk}$-variety X_{ω_N} admits a $\Bbbk$-structure $\sigma_{X_{\omega_N},h}$ such that $(X_{\omega_N}, \sigma_{X_{\omega_N},h})$ is a $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety that contains the torsor encoded by h as a $(\mathbb{T}, \tau)$-stable dense open subset.

(ii) Let (X, σ) be an affine $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety. There exists a pointed cone ω_N in $N_{\mathbb{Q}}$ that is stable under the Γ-action induced by τ, and a cocycle h such that the $(\mathbb{T}, \tau)$-toric $\Bbbk$-varieties (X, σ) and $(X_{\omega_N}, \sigma_{X_{\omega_N},h})$ are isomorphic.

Based on a result of Huruguen (see [Hur11, Proposition 1.19]), one of the main result of this chapter can be summarized as follows.

Proposition E (See Corollary 2.3.16 and Proposition 2.4.9). *Let $\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$ be a split $\overline{\mathbb{k}}$-torus and let Λ be a fan in $N_{\mathbb{Q}}$. Let X_Λ be the associated split toric $\overline{\mathbb{k}}$-variety and let σ be a $\mathbb{k}$-structure on the dense open orbit $(X_\Lambda)_{\mathbb{T}} \cong \mathbb{T}$.*

- *There exists a $\mathbb{k}$-group structure τ^σ on $\mathbb{T}$ such that $(\mathbb{T}, \sigma)$ is a $(\mathbb{T}, \tau^\sigma)$-torsor.*
- *The $\mathbb{k}$-structure σ extends to a $\mathbb{k}$-structure σ_Λ on X_Λ if and only if the fan Λ is stable under the Γ-action $\hat{\tau}^\sigma$ on $N_{\mathbb{Q}}$ (i.e., for any cone $\lambda \in \Lambda$ and for any $\gamma \in \Gamma$, $\hat{\tau}^\sigma_\gamma(\lambda) \in \Lambda$).*
- *Furthermore, if σ_Λ is such a $\mathbb{k}$-structure, then*

$$\mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X_\Lambda)\right) \cong \mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right).$$

  In other words, the set of $\mathbb{k}$-forms of X_Λ in the category of $(\mathbb{T}, \tau^\sigma)$-toric varieties is parametrized by the set of $(\mathbb{T}, \tau^\sigma)$-torsors (see Proposition 2.4.9 for a precise statement).

Using this proposition, this chapter ends on a classification of smooth toric Del Pezzo $\mathbb{R}$-surfaces (see Proposition 2.5.4).

Chapter 3. This chapter, which is the main core of the thesis, is divided in two parts.

The first part is devoted to introduce and define the combinatorial material used in the Altmann-Hausen presentation over an algebraically closed field of characteristic zero. After stating the main results of [AH06], we detail and prove a method mentioned in [AH06], called *toric downgrading*, used to determine an AH-datum from an affine variety endowed with an effective torus action. Since we will extend the Altmann-Hausen presentation using this method, we prove the second item of Theorem AH using it. Let X be an affine $\mathbb{k}$-variety endowed with an action of the $\mathbb{k}$-torus $\mathbb{G}^d_{m,\mathbb{k}}$. The general idea of the toric downgrading method is as follows. We embed X is some affine space $\mathbb{A}^n_{\mathbb{k}}$ as in Proposition 3.1.15. An AH-datum for the $\mathbb{G}^d_{m,\mathbb{k}}$-action on $\mathbb{A}^n_{\mathbb{k}}$ is quite simple to determine, and from this we can deduce an AH-datum for the $\mathbb{G}^d_{m,\mathbb{k}}$-action on X.

The second part is devoted to the main results of this thesis. All necessary tools to extend the Altmann-Hausen presentation are given in the previous chapters. As it is mentioned above, we use a Galois equivariant version of the toric downgrading method introduced in [AH06]. This method is based on the next proposition.

Proposition F (See Proposition 3.2.6). *Let X be an affine $\overline{\mathbb{k}}$-variety endowed with an action of the d-dimensional $\overline{\mathbb{k}}$-torus $\mathbb{G}^d_{m,\mathbb{k}} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$. Let σ be a $\mathbb{k}$-structure on X, and let τ be a $\mathbb{k}$-group structure on $\mathbb{T}$. If the $\mathbb{k}$-torus $(\mathbb{G}^d_{m,\overline{\mathbb{k}}}, \tau)$ acts effectively on (X, σ), then there exists $n \in \mathbb{N}, n \geq d$ such that:*

(i) *There is a $\mathbb{k}$-group structure τ' on $\mathbb{G}^n_{m,\overline{\mathbb{k}}}$ that extends to a $\mathbb{k}$-structure σ' on $\mathbb{A}^n_{\overline{\mathbb{k}}}$;*

(ii) *$(\mathbb{G}^d_{m,\overline{\mathbb{k}}}, \tau)$ is a closed subgroup of $(\mathbb{G}^n_{m,\overline{\mathbb{k}}}, \tau')$; and*

(iii) *(X, σ) is a closed subvariety of $(\mathbb{A}^n_{\overline{\mathbb{k}}}, \sigma')$ and $(X, \sigma) \hookrightarrow (\mathbb{A}^n_{\overline{\mathbb{k}}}, \sigma')$ is $(\mathbb{G}^d_{m,\overline{\mathbb{k}}}, \tau)$-equivariant. Moreover, X intersects the dense open orbit of $\mathbb{A}^n_{\overline{\mathbb{k}}}$ for the natural $\mathbb{G}^n_{m,\overline{\mathbb{k}}}$-action.*

Since the acting torus may be non split and may have non trivial torsors, we need an additional datum in the Altmann-Hausen presentation over arbitrary fields of characteristic zero. Namely, a cocycle h that encodes a certain torsor appears in the description of affine varieties endowed with an effective torus action. A generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ is an AH-datum $(\omega_N, Y, \mathcal{D})$ such that ω_N is Γ-stable, together with a $\mathbb{k}$-structure σ_Y on Y and with a certain cocycle h. The main result of this thesis can be summarized as follows (Compare with Proposition D).

Theorem G (See Theorem 3.2.3). *Let $\Bbbk$ be a field, and let $\Gamma := \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$. Let $\mathbb{T}$ be a $\overline{\Bbbk}$-torus, and let τ be a $\Bbbk$-group structure on $\mathbb{T}$.*

(i) Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum over $\overline{\Bbbk}$. The affine $\mathbb{T}$-variety $X[Y, \mathcal{D}]$ admits a $\Bbbk$-structure $\sigma_{X[Y,\mathcal{D}]}$ such that $(\mathbb{T}, \tau)$ acts effectively on $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

(ii) Let (X, σ) be a normal affine $\Bbbk$-variety endowed with an effective $(\mathbb{T}, \tau)$-action. There exists a generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ such that $(X, \sigma) \cong (X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$ as $(\mathbb{T}, \tau)$-varieties.

If the Galois cohomology set that classifies these T-torsors is trivial, then as we will see in Section 3.2.3, we get a simpler presentation of affine varieties endowed with an effective T-action.

Chapter 4. This chapter is based on the above with simplifying assumptions. First, we consider the finite Galois extension $\mathbb{C}/\mathbb{R}$, and then, we consider two-dimensional torus actions over arbitrary fields of characteristic zero.

In a first part, we specify Theorem G to the real setting and we illustrate the generalized Altmann-Hausen presentation with examples. We recall and we prove that an $\mathbb{R}$-torus is isomorphic to a product of $\mathbb{G}_{m,\mathbb{R}}$, $\mathbb{S}^1$, and $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Then, specifying Theorem G to $\mathbb{S}^1$-actions, we recover the presentation given in [DL18].

As promised, we pursue the example started above. We have constructed an $\mathbb{S}^1$-action on $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}}) \cong \mathbb{A}^2_{\mathbb{R}}$ from a $\mathbb{G}_{m,\mathbb{C}}$-action on $\mathbb{A}^2_{\mathbb{C}}$. Then, the generalized AH-datum is $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$, where $(\omega_N, Y, \mathcal{D})$ is the AH-datum for the action of $\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[\mathbb{Z}])$ on $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x, y])$, where σ_Y is the $\mathbb{R}$-structure on $Y = \mathrm{Spec}(\mathbb{C}[z])$ defined by $z \mapsto \bar{z}$, and where $h \in \mathrm{Hom}_{gr}(\mathbb{Z}, \mathbb{C}(z)^*)$ is defined by $h : m \mapsto z^{-m}$. Indeed, using the datum (σ_Y, h) we can construct a $\Bbbk$-structure $\sigma_{X[Y,\mathcal{D}]}$ on $X[Y, \mathcal{D}]$ such that $\mathbb{S}^1$ acts on $(X[Y, D], \sigma_{X[Y,\mathcal{D}]})$, and such that the $\mathbb{S}^1$-varieties $(\mathbb{A}^2_{\mathbb{C}}, \sigma)$ and $(X[Y, D], \sigma_{X[Y,\mathcal{D}]})$ are isomorphic (see Exemple 4.1.15).

In a second part, we focus on two-dimensional $\Bbbk$-torus actions because there is a complete description of the Galois cohomology set that classifies T-torsors in [ELFST14]. Recall that if this set is trivial, then the Altmann-Hausen presentation over arbitrary fields simplifies. We recall the classification of two-dimensional $\Bbbk$-torus given in [Vos65], and we give a proof. Using birational geometry tools, we relate the torsor encoded by the cocycle h to a certain Del Pezzo surface. Our main result about two-dimensional $\Bbbk$-torus actions is summarized as follows.

Proposition H. *Let $(\mathbb{T}, \tau)$ be a two-dimensional $\Bbbk$-torus and let $(\mathbb{T}, \sigma)$ be a $(\mathbb{T}, \tau)$-torsor. There is a minimal toric Del Pezzo $\Bbbk$-surface X that contains $(\mathbb{T}, \sigma)$ as a $(\mathbb{T}, \tau)$-stable dense open subset. Moreover, we have the following possibilities for X:*

(i) X is a $\Bbbk$-form of $\mathbb{P}^2_{\overline{\Bbbk}}$; or

(ii) X is a $\Bbbk$-form of the Del Pezzo $\overline{\Bbbk}$-surface of degree 6; or

(iii) X is a $\Bbbk$-form of $\mathbb{P}^1_{\overline{\Bbbk}} \times \mathbb{P}^1_{\overline{\Bbbk}}$.

Chapter 5. This thesis ends by some lines of study that should be explored further.

Appendices The appendices are devoted to describe some technical details needed in this thesis; namely on scheme theory, on Galois theory, on convex and on birational geometry. In Appendix E, we illustrate the Altmann-Hausen presentation of surfaces endowed with a $\mathbb{G}_{m,\mathbb{C}}$-action.

Chapter 1

Galois descent

The general idea of the descent theory is based on the following observation (this paragraph is based on [Poo17, Chapter 4, §4.1]). Let S be a topological space, and let $\{S_i\}_{i\in I}$ be an open covering of S. Suppose we are given a sheaf $\mathscr{F}_i$ on S_i for each i. Under what conditions is there a sheaf $\mathscr{F}$ on S such that $\mathscr{F}|_{S_i} \cong \mathscr{F}_i$? If $\mathscr{F}$ exists, then the restriction of $\mathscr{F}|_{S_i}$ and $\mathscr{F}|_{S_j}$ to $S_{i,j} := S_i \cap S_j$ must be isomorphic, and there exists isomorphisms

$$\phi_{i,j} : \mathscr{F}|_{S_{i,j}} \to \mathscr{F}|_{S_{i,j}}. \tag{1.1}$$

On the triple intersection, these isomorphisms satisfy the cocycle condition

$$\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j},\ \forall i, j, k \in I. \tag{1.2}$$

Consider the disjoint union $S' := \bigsqcup S_i$, and let $p : S' \to S$ be the " open covering morphism ". To give the $\mathscr{F}_i$ on all the S_i is equivalent to give a single sheaf $\mathscr{F}'$ on S'. So, under what conditions is there a sheaf $\mathscr{F}$ on S such that $p^{-1}\mathscr{F} \cong \mathscr{F}'$? Note that $S_{i,j} := S_i \cap S_j = S_i \times_S S_j$, and $S' \times_S S' = \bigsqcup S_{i,j}$. If p_1, p_2 denote the two projection $S' \times_S S' \to S'$, the condition (1.1) is equivalent to give an isomorphism

$$\psi : p_1^{-1}\mathscr{F}' \to p_2^{-1}\mathscr{F}'$$

of sheaves on $S \times_S S'$. Let $p_{1,3} : S' \times_S S' \times_S S'$ be the projection onto the first and third coordinates, and so on. Then, $p_{1,3}^{-1}\psi$ is an isomorphism of sheaves on $S' \times_S S' \times_S S'$. The cocycle condition (1.2) can now be rewritten as

$$p_{1,3}^{-1}\psi = p_{2,3}^{-1}\psi \circ p_{1,2}^{-1}\psi.$$

Similarly, let us consider the case of schemes. In this chapter, $p : S' \to S$ is a scheme morphism, and we will often have $S := \mathrm{Spec}(\Bbbk)$ and $S' := \mathrm{Spec}(\Bbbk')$, where $\Bbbk'/\Bbbk$ is a Galois extension (see Section 1.2). Let X' be an S'-scheme. The aim of this chapter is to answer to the following question: under what condition is X' isomorphic to an S-scheme of the form $p^*X := X \times_S S'$ for some S-scheme X ? A descent datum on an S'-scheme X' is an isomorphism

$$\psi : p_1^*X' := X' \times_S S' \to p_2^*X' := S' \times_S X'$$

satisfying the cocycle condition

$$p_{1,3}^*\psi = p_{2,3}^*\psi \circ p_{1,2}^*\psi.$$

A descent datum is called effective if there exists an S-scheme X such that $p^*X \cong X'$.

In the context of Galois descent $\mathrm{Spec}(\Bbbk') \to \mathrm{Spec}(\Bbbk)$, where $\Bbbk'/\Bbbk$ is a Galois extension, there is an equivalent language that is more convenient, namely the language of $\Bbbk$-structures on $\Bbbk'$-varieties. Let $\Gamma := \mathrm{Gal}(\Bbbk'/\Bbbk)$ be endowed with the Krull topology (see Appendix B). Roughly, a $\Bbbk$-*structure* on an algebraic $\Bbbk'$-variety X is a continuous action $\Gamma \times X \to X$ such that the following diagram commutes for all $\gamma \in \Gamma$ (see Definition 1.2.1 and Lemma 1.2.6):

$$\begin{array}{ccc} X & \xrightarrow{\sigma_\gamma} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\Bbbk') & \xrightarrow{\mathrm{Spec}(\gamma)} & \mathrm{Spec}(\Bbbk'). \end{array}$$

A result on the effectiveness of Galois descent is the following. There is an equivalence of categories between the category of quasi-projective algebraic $\Bbbk$-varieties and the category of quasi-projective algebraic $\Bbbk'$-varieties endowed with a $\Bbbk$-structure; see Theorem 1.4.11 for a precise statement.

1.1 Descent data

To well understand the general idea of the descent data construction, the reader is referred to [Poo17, Chapter 4]. This section is based on [BLR90, Chapter 6]. See also [Sta, Section 03O6].

Let $p : S' \to S$ be a scheme morphism. In the following, we will often use $S := \mathrm{Spec}(\Bbbk)$ and $S' := \mathrm{Spec}(\Bbbk')$, where $\Bbbk'/\Bbbk$ is a non-necessarily finite Galois extension (see Section 1.2). Let X' be an S'-scheme. The aim of this section is to answer to the following question: under what condition is X' isomorphic to an S-scheme of the form $X \times_S S'$ for some S-scheme X ?

Let X be an S-scheme, and let $X_{S'} := X \times_S S'$. Then, we get a canonical $S' \times_S S'$-isomorphism $\psi_{can} : X_{S'} \times_S S' \to S' \times_S X_{S'}$, and there are three associated $S' \times_S S' \times_S S'$-isomorphisms

$$\begin{aligned} \psi_1 &: S' \times_S X_{S'} \times_S S' \to S' \times_S S' \times_S X_{S'} \\ \psi_2 &: X_{S'} \times_S S' \times_S S' \to S' \times_S S' \times_S X_{S'} \\ \psi_3 &: X_{S'} \times_S S' \times_S S' \to S' \times_S X_{S'} \times_S S' \end{aligned}$$

by inserting the identity in the first, second and third position, respectively. These isomorphisms satisfy the cocycle condition

$$\psi_2 = \psi_1 \circ \psi_3.$$

This observation leads to the next definition.

Definition 1.1.1. Let $\mathrm{Sch}_{S'/S}$ be the category of S'-schemes with descent data defined as follows.

- An object is a pair (X', ψ), where X' is an S'-scheme and

  $$\psi : X' \times_S S' \to S' \times_S X'$$

  is an $S' \times_S S'$-isomorphism satisfying the cocycle condition:

  $$\psi_2 = \psi_1 \circ \psi_3.$$

  The morphism ψ is called a *descent datum* on X'.
- An arrow $\varphi : (X', \psi_X) \to (Y', \psi_Y)$ is an S'-morphism $\varphi : X' \to Y'$ such that the next diagram commutes

  $$\begin{array}{ccc} X' \times_S S' & \xrightarrow{\psi_X} & S' \times_\Bbbk X' \\ {\scriptstyle \varphi \times id}\downarrow & & \downarrow{\scriptstyle id \times \varphi} \\ Y' \times_S S' & \xrightarrow{\psi_Y} & S' \times_\Bbbk Y'. \end{array}$$

A descent datum ψ on an S'-scheme X' is called *effective* if there exists an S-scheme X such that

$$(X', \psi) \cong (X \times_S S', \psi_{can}),$$

where ψ_{can} is the canonical $S' \times_S S'$-isomorphism $\psi_{can} : X_{S'} \times_S S' \to S' \times_S X_{S'}$, and with the obvious notion of isomorphism of descent data.

If the morphism $p : S' \to S$ satisfies certain properties, then we get some results concerning the functor $X \to X_{S'}$, and on the effectiveness of the descent with respect to p. In this thesis, we focus on field extensions. We will see that a non-necessarily finite Galois extension defines a faithfully flat and quasi-compact (fpqc) morphism (see [Sta, Definitions 0253 and 01K3]).

Lemma 1.1.2. *Let $\Bbbk'/\Bbbk$ be a field extension. Then,* $\mathrm{Spec}(\Bbbk') \to \mathrm{Spec}(\Bbbk)$ *is a faithfully flat and quasi-compact morphism.*

Proof. Let $\Bbbk'/\Bbbk$ be a field extension.

- The morphism $\mathrm{Spec}(\Bbbk') \to \mathrm{Spec}(\Bbbk)$ is trivially quasi-compact (see Definition A.6.6)
- We show that the inclusion $\Bbbk \hookrightarrow \Bbbk'$ is faithfully flat (see [Sta, Section 00H9]). We have to prove that any sequence $N' \to N \to N''$ of $\Bbbk$-modules is exact if and only if the tensored sequence $N' \otimes_{\Bbbk} \Bbbk' \to N \otimes_{\Bbbk} \Bbbk' \to N'' \otimes_{\Bbbk} \Bbbk'$ is exact. We denote $f : N' \to N$ and $g : N \to N''$ and $\{b_i\}$ a basis of the $\Bbbk$-vector space $\Bbbk'$.

Suppose that $N' \to N \to N''$ is exact, i.e $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Show that $\mathrm{Im}(f \otimes Id) = \mathrm{Ker}(g \otimes Id)$. Let $y \in \mathrm{Im}(f \otimes Id)$, there exists a finite number of elements $x_i \in N'$ such that $\sum f(x_i) \otimes b_i = y$. Then, composing with $g \otimes Id$, we obtain $\sum g(f(x_i)) \otimes b_i = 0$, hence $y \in \mathrm{Ker}(g \otimes Id)$. Conversely, let $y \in \mathrm{Ker}(g \otimes Id)$, there exists a finite number of elements $y_i \in N$ such that $y = \sum y_i \otimes b_i$. Then, composing with $g \otimes Id$, we get $\sum g(y_i) \otimes b_i = 0$. Let $\{v_j\}$ be a basis of N'', then there exists $\lambda_{ij} \in \Bbbk$ such that $g(y_i) = \sum \lambda_{ij} v_j$ for all i. Then, $\sum \lambda_{ij} v_j \otimes b_i = 0$, hence $\lambda_{ij} = 0$ for all i and j, so $g(y_i) = 0$ for all i, and since $\mathrm{Im}(f) = \mathrm{Ker}(g)$, there exists $x_i \in N'$ such that $f(x_i) = y_i$. We conclude that $y = (f \otimes Id)(\sum x_i \otimes b_i) \in \mathrm{Im}(f \otimes Id)$.

Suppose that $N' \otimes_{\Bbbk} \Bbbk' \to N \otimes_{\Bbbk} \Bbbk' \to N'' \otimes_{\Bbbk} \Bbbk'$ is exact, i.e $\mathrm{Im}(f \otimes Id) = \mathrm{Ker}(g \otimes Id)$. Show that $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Let $y \in \mathrm{Im}(f)$, there exists $x \in N'$ such that $f(x) = y$. Note that $f(x) \otimes b_1 \in \mathrm{Im}(f \otimes Id)$, hence $(g \otimes Id)(f(x) \otimes b_1) = g(f(x)) \otimes b_1 = 0$. Then, $g(f(x)) = 0$, so $y \in \mathrm{Ker}(g)$. Conversely, let $y \in \mathrm{Ker}(g)$, then $y \otimes b_1 \in \mathrm{Ker}(g \otimes Id) = \mathrm{Im}(f \otimes Id)$. Hence, there exists a finite number of elements $x_i \in N'$ such that $y \otimes b_1 = \sum f(x_i) \otimes b_i$. Hence, $y = f(x_1) \in \mathrm{Im}(f)$. □

Theorem 1.1.3 ([SGA03, Corollaire 7.9], see [BLR90, Theorem 6], see also [GW20, Theorem 14.72] and [Sta, Lemma 0247])**.** *Let $p : S' \to S$ be an fpqc morphism of schemes.*

(i) The functor $X \to X \times_S S'$ from S-schemes to S'-schemes with descent data is fully faithful (see [Sta, Definition 001C]).

(ii) The functor $X \to X \times_S S'$ from quasi-affine (resp. affine) S-schemes to quasi-affine (resp. affine) S'-schemes with descent data defines an equivalence of categories (see [Sta, Definitions 01P6 and 001J]).

(iii) Assume that S and S' are affine. Then, a descent datum ψ on an S'-scheme X' is effective if and only if X' can be covered by affine open subschemes that are invariant under ψ.

In the case where S and S' are affine, descent data can be described in terms of modules over rings (see for instance [Vis, §4.2], and [BLR90, Chapter 6]).

We now give some results on the behavior of morphisms with respect to descent.

Theorem 1.1.4 ([GW20, Propositions 14.50, 14.51, 14.53, 14.57], see also [Poo17, Theorem 4.3.7])**.** *Let $p : S' \to S$ be an fpqc morphism. Let $f : X \to Y$ be a morphism of S-schemes, and let $f' := f \times id : X \times_S S' \to Y \times_S S'$ its base change. If f' is has one of the following property,*

- *open;*
- *closed;*
- *quasi-compact;*
- *quasi-separated;*
- *separated;*
- *open;*
- *isomorphism;*
- *open (resp. closed) immersion;*
- *proper;*
- *affine;*
- *finite;*
- *...;*

then so has f. Furthermore, if $S = \mathrm{Spec}(\Bbbk)$ and $S' = \mathrm{Spec}(\Bbbk')$, where $\Bbbk'/\Bbbk$ is a field extension, then X is a quasi-projective (resp. projective) $\Bbbk$-scheme if and only if $X \times_S S'$ is a quasi-projective (resp. projective) $\Bbbk'$-scheme.

In the next proposition, we give an equivalent formulation of a descent datum.

Proposition 1.1.5 ([BLR90, Chapter 6])**.** *The following assertions are equivalent:*
(i) To have a descent datum (X', ψ).
(ii) To have a Cartesian diagram

$$\begin{array}{ccccc} X' \times_S S' \times_S S' & \xrightarrow{\tilde{p}_{12}, \tilde{p}_{13}, \tilde{p}_{23}} & X' \times_S S' & \xrightarrow{\tilde{p}_1, \tilde{p}_2} & X' \\ \downarrow & & \downarrow & & \downarrow \\ S' \times_S S' \times_S S' & \xrightarrow{p_{12}, p_{13}, p_{23}} & S' \times_S S' & \xrightarrow{p_1, p_2} & S' \end{array}$$

where the p_i and the p_{ij} are the natural projections, the $\tilde{p}_i$ are S'-scheme morphisms and $\tilde{p}_{ij}$ are $S' \times_S S'$-scheme morphisms such that

$$\begin{cases} \tilde{p}_1 \circ \tilde{p}_{12} &= \tilde{p}_1 \circ \tilde{p}_{13} \\ \tilde{p}_1 \circ \tilde{p}_{23} &= \tilde{p}_2 \circ \tilde{p}_{12} \\ \tilde{p}_2 \circ \tilde{p}_{13} &= \tilde{p}_2 \circ \tilde{p}_{23}. \end{cases} \tag{1.3}$$

Proof. We prove: $(i) \implies (ii)$. Let $\Bbbk'/\Bbbk$ be a field extension. Since we are interested in Galois descent of varieties admitting a Γ-invariant affine open covering, we prove the result in the case where $S = \mathrm{Spec}(\Bbbk)$, $S' = \mathrm{Spec}(\Bbbk')$, and X' is affine.

Let $X' = \mathrm{Spec}(A')$ be a $\Bbbk'$-scheme together with a descent datum $\psi : X' \times_S S' \to S' \times_S X'$. So, we get a $\Bbbk' \otimes_\Bbbk \Bbbk'$-algebra isomorphism

$$\psi^\sharp : \Bbbk' \otimes_\Bbbk A' \to A' \otimes_\Bbbk \Bbbk',$$

and three associated $\Bbbk' \otimes_\Bbbk \Bbbk' \otimes_\Bbbk \Bbbk'$-algebra isomorphisms

$$\begin{aligned} \psi_1^\sharp &: \Bbbk' \otimes_\Bbbk \Bbbk' \otimes_\Bbbk A' \to \Bbbk' \otimes_\Bbbk A' \otimes_\Bbbk \Bbbk' \\ \psi_2^\sharp &: \Bbbk' \otimes_\Bbbk \Bbbk' \otimes_\Bbbk A' \to A' \otimes_\Bbbk \Bbbk' \otimes_\Bbbk \Bbbk' \\ \psi_3^\sharp &: \Bbbk' \otimes_\Bbbk A' \otimes_\Bbbk \Bbbk' \to A' \otimes_\Bbbk \Bbbk' \otimes_\Bbbk \Bbbk', \end{aligned}$$

by inserting the identity in the first, second and third position, respectively. These isomorphisms satisfy the cocycle condition

$$\psi_2^\sharp = \psi_3^\sharp \circ \psi_1^\sharp.$$

Then, we can construct a co-Cartesian diagram

$$\begin{array}{ccccc} A' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{\hat{p}^\sharp_{12}, \hat{p}^\sharp_{13}, \hat{p}^\sharp_{23}} & A' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{\hat{p}^\sharp_{1}, \hat{p}^\sharp_{2}} & A' \\ \uparrow & & \uparrow & & \uparrow \\ \Bbbk' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p^\sharp_{12}, p^\sharp_{13}, p^\sharp_{23}} & \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p^\sharp_{1}, p^\sharp_{2}} & \Bbbk' \end{array}$$

where

$$\hat{p}^\sharp_1(a) = a \otimes 1 \qquad \hat{p}^\sharp_2(a) = \psi^\sharp(1 \otimes a)$$

$$\hat{p}^\sharp_{12}(a \otimes b) = a \otimes b \otimes 1 \qquad \hat{p}^\sharp_{13}(a \otimes b) = a \otimes 1 \otimes b \qquad \hat{p}^\sharp_{23}(a \otimes b) = \psi^\sharp(1 \otimes a) \otimes b,$$

and where the $p^\sharp_1$ and $p^\sharp_{ij}$ are the canonical inclusions. We have to check the following equalities.

$$\begin{cases} \hat{p}^\sharp_{12} \circ \hat{p}^\sharp_1 & = \hat{p}^\sharp_{13} \circ \hat{p}^\sharp_1 \\ \hat{p}^\sharp_{23} \circ \hat{p}^\sharp_1 & = \hat{p}^\sharp_{12} \circ \hat{p}^\sharp_2 \\ \hat{p}^\sharp_{13} \circ \hat{p}^\sharp_2 & = \hat{p}^\sharp_{23} \circ \hat{p}^\sharp_2. \end{cases}$$

The two first equalities are trivially satisfied. The last one is implied by the cocycle condition. □

1.2 𝕜-structures on 𝕜′-varieties

In this section, we introduce a language specific to the Galois descent setting. From now on, $\Bbbk'/\Bbbk$ is a non-necessarily finite Galois extension of profinite Galois group Γ (see Proposition B.1.8). We will see in Section 1.3 that this language is equivalent to the one of the fpqc descent $\mathrm{Spec}(\Bbbk') \to \mathrm{Spec}(\Bbbk)$. In the particular case of Galois descent $\mathbb{C}/\mathbb{R}$, we refer to [Ben16, Chapter 4] for a self contained description.

1.2.1 𝕜-structures

Every $\Bbbk'$-variety X (see Appendix A.6) can be viewed as a $\Bbbk$-scheme via the composition of its structure morphism $X \to \mathrm{Spec}(\Bbbk')$ with the morphism $\mathrm{Spec}(\Bbbk') \to \mathrm{Spec}(\Bbbk)$ induced by the inclusion $\Bbbk \hookrightarrow \Bbbk'$. We denote this scheme by $X_{/\Bbbk}$. The Galois group Γ acts on $\mathrm{Spec}(\Bbbk')$, where the action is induced by the field automorphisms $(\gamma : \Bbbk' \to \Bbbk') \in \Gamma$.

Definition 1.2.1. (i) A $\Bbbk$-*form* of a $\Bbbk'$-variety X is a $\Bbbk$-variety X_0 together with an isomorphism $(X_0)_{\Bbbk'} := X_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk') \cong X$ of $\Bbbk'$-varieties. By abuse of notation we will often write: X_0 is a $\Bbbk$-form of X instead of $(X_0, \cong)$.

(ii) A $\Bbbk$-*structure* $\sigma : \Gamma \to \mathrm{Aut}(X_{/\Bbbk})$ on a $\Bbbk'$-variety X is an *algebraic semilinear* Γ-action.

- *Semilinear* Γ-action means that σ satisfies

$$\sigma_{\gamma_1\gamma_2} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1}, \quad \forall\ \gamma_1, \gamma_2 \in \Gamma,$$

and for each $\gamma \in \Gamma$, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\sigma_\gamma} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\Bbbk') & \xrightarrow{\mathrm{Spec}(\gamma)} & \mathrm{Spec}(\Bbbk') \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\Bbbk) & \end{array}$$

- *Algebraic* means that there exists a finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$ and a $\Bbbk_1$-form X_1 of the $\Bbbk'$-variety X such that the restriction of σ to $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$ coincides with the natural $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$-action on $(X_1)_{\Bbbk'} \cong X$.

(iii) Two $\Bbbk$-structures σ and σ' on X are *equivalent* if there exists a $\Bbbk'$-automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\sigma'_\gamma = \varphi \circ \sigma_\gamma \circ \varphi^{-1}$ for all $\gamma \in \Gamma$.

(iv) A $\Bbbk$-*morphism* between two $\Bbbk'$-varieties X and X' with $\Bbbk$-structures σ and σ' is a morphism of $\Bbbk'$-varieties $f : X \to X'$ such that $\sigma'_\gamma \circ f = f \circ \sigma_\gamma$, as morphisms of $\Bbbk$-schemes, for all $\gamma \in \Gamma$.

Remark 1.2.2. Let X, Z be two $\Bbbk'$-varieties, and let σ_X (resp. σ_Z) be a $\Bbbk$-structure on X (resp. on Z). Then $\sigma_X \times \sigma_Z$ is a $\Bbbk$-structure on $X \times_{\mathrm{Spec}(\Bbbk)} Z$.

Remark 1.2.3 (Definition of an $\mathbb{R}$-structure)**.** Let $\Bbbk' = \mathbb{C}$ and $\Bbbk = \mathbb{R}$. Then, $\Gamma = \{id, \gamma\} \cong \mathbb{Z}/2\mathbb{Z}$. In this setting, we often make the following abuse of notation: we call an $\mathbb{R}$-structure σ the morphism of $\mathbb{R}$-schemes σ_γ. Therefore, an $\mathbb{R}$-structure on an algebraic $\mathbb{C}$-variety X is an involution of $\mathbb{R}$-schemes σ on X such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \bar{z})} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Example 1.2.4. The affine $\mathbb{R}$-variety $\mathbb{A}^1_\mathbb{R}$ is an $\mathbb{R}$-form of $\mathbb{A}^1_\mathbb{C}$ since $(\mathbb{A}^1_\mathbb{R})_\mathbb{C} \cong \mathbb{A}^1_\mathbb{C}$. There is a canonical $\mathbb{R}$-structure σ_0 on $\mathbb{A}^1_\mathbb{R} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) = \mathrm{Spec}(\mathbb{R}[x] \otimes_\mathbb{R} \mathbb{C})$ defined by $\sigma_0^\sharp(x^m \otimes z) = x^m \otimes \bar{z}$. This $\mathbb{R}$-structure induces an $\mathbb{R}$-structure σ on $\mathbb{A}^1_\mathbb{C}$ satisfying the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}[x] \otimes_\mathbb{R} \mathbb{C} & \xrightarrow{\cong} & \mathbb{C}[x] \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbb{R}[x] \otimes_\mathbb{R} \mathbb{C} & \xrightarrow{\cong} & \mathbb{C}[x] \end{array} \qquad \begin{array}{ccc} x^m \otimes z & \longmapsto & zx^m \\ \downarrow & & \downarrow \\ x^m \otimes \bar{z} & \longmapsto & \bar{z}x^m \end{array}$$

In other words, σ is the following $\mathbb{R}$-structure on $\mathbb{A}^1_\mathbb{C}$

$$\begin{aligned} \sigma : \mathbb{A}^1_\mathbb{C} &\to \mathbb{A}^1_\mathbb{C} \\ x &\mapsto \bar{x} \end{aligned} \qquad\qquad \begin{aligned} \sigma^\sharp : \mathbb{C}[x] &\to \mathbb{C}[x] \\ ax^n &\mapsto \bar{a}x^n \end{aligned}$$

Note that $\mathbb{A}^1_\mathbb{R} = \mathrm{Spec}(\mathbb{R}[x]) = \mathrm{Spec}(\mathbb{C}[x]^\Gamma)$.
If σ' is another $\mathbb{R}$-structure on $\mathbb{A}^1_\mathbb{C}$, then $\varphi := \sigma' \circ \sigma^{-1} \in \mathrm{Aut}(\mathbb{A}^1_\mathbb{C})$, hence $\sigma' = \varphi \circ \sigma$, where φ is a $\mathbb{C}$-automorphism satisfying certain condition. Therefore, to construct another $\mathbb{R}$-structure σ' on $\mathbb{A}^1_\mathbb{C}$, it is sufficient to know $\mathrm{Aut}(\mathbb{A}^1_\mathbb{C})$. Recall that

$$\mathrm{Aut}(\mathbb{A}^1_\mathbb{C}) = \{x \mapsto ax + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

One can show that σ is, up to equivalence, the only $\mathbb{R}$-structure on $\mathbb{A}^1_{\mathbb{C}}$. Furthermore, $\mathbb{A}^1_{\mathbb{R}}$ is, up to isomorphism, the only $\mathbb{R}$-form of $\mathbb{A}^1_{\mathbb{C}}$, and $\mathbb{A}^1_{\mathbb{R}}$ corresponds to the pair $(\mathbb{A}^1_{\mathbb{C}}, \sigma)$ (see Theorem 1.4.1)

Example 1.2.5. A $\mathbb{k}'$-variety may have several non-isomorphic $\mathbb{k}$-forms. For instance, let $\mathbb{k}' = \mathbb{C}$ and let $\mathbb{k} = \mathbb{R}$. Then, $C := \mathrm{Proj}(\mathbb{R}[x,y,z]/(x^2+y^2+z^2))$ and $\mathbb{P}^1_{\mathbb{R}}$ are two non-isomorphic $\mathbb{R}$-forms of $\mathbb{P}^1_{\mathbb{C}}$. Furthermore, C and $\mathbb{P}^1_{\mathbb{R}}$ corresponds respectively to the following non-equivalent $\mathbb{R}$-structures on $\mathbb{P}^1_{\mathbb{C}}$ (see Theorem 1.4.1, and [Ben16, Theorem 4.1])

$$\begin{aligned} \mathbb{P}^1_{\mathbb{C}} &\to \mathbb{P}^1_{\mathbb{C}} \\ [u:v] &\mapsto [\bar{v}:-\bar{u}] \end{aligned} \qquad \begin{aligned} \mathbb{P}^1_{\mathbb{C}} &\to \mathbb{P}^1_{\mathbb{C}} \\ [u:v] &\mapsto [\bar{u}:\bar{v}]. \end{aligned}$$

We give a connection between $\mathbb{k}$-structures and *continuous* semilinear Γ-actions.

Lemma 1.2.6. *Let $X = \mathrm{Spec}(A)$ be a $\mathbb{k}'$-variety and let σ be a semilinear Γ-action on X. The following assertions are equivalent.*

(i) The map

$$\begin{aligned} \Gamma \times A &\to A \\ (\gamma, a) &\mapsto \sigma^\sharp_\gamma(a) \end{aligned}$$

is continuous, where Γ is endowed with the Krull topology, and A is equipped with the discrete topology.

(ii) For all $a \in A$, the stabilizer

$$\mathrm{Stab}_\Gamma(a) := \{\gamma \in \Gamma \mid \sigma^\sharp_\gamma(a) = a\}$$

is an open subgroup of Γ.

(iii) We have

$$A = \bigcup_i A^{\mathrm{Gal}(\mathbb{k}'/\mathbb{k}_i)} =: \mathrm{colim}_i A^{\mathrm{Gal}(\mathbb{k}'/\mathbb{k}_i)},$$

where the colimit is taken over all finite Galois extension $\mathbb{k}_i/\mathbb{k}$.

Furthermore, if the map $\Gamma \times A \to A$ is continuous, then σ is a $\mathbb{k}$-structure. That is there exists a finite Galois extension $\mathbb{k}_1/\mathbb{k}$ in $\mathbb{k}'$ and a $\mathbb{k}_1$-form X_1 of the $\mathbb{k}'$-variety X such that the restriction of σ to $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}_1)$ coincides with the natural $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}_1)$-action on $(X_1)_{\mathbb{k}'} \cong X$.

Remark 1.2.7. We will see in Lemma 1.4.10 that if σ is a $\mathbb{k}$-structure on $X = \mathrm{Spec}(A)$, then the induced map $\Gamma \times A \to A$ is continuous.

Proof. The equivalence between the first and the second item is classic (See [Ber10, Definition II.3.1]).

- $(i) \Rightarrow (ii)$. We denote

$$\begin{aligned} \mu : \Gamma \times A &\to A \\ (\gamma, a) &\mapsto \sigma^\sharp_\gamma(a). \end{aligned}$$

Let $a \in A$, and let $\gamma_0 \in \mathrm{Stab}_\Gamma(a)$. By definition of the product topology, there exists an open subset $U \subset \Gamma$ containing (γ_0, a) such that $U \times \{a\} \subset \mu^{-1}(a)$. Therefore, $U \subset \mathrm{Stab}_\Gamma(a)$, and $\mathrm{Stab}_\Gamma(a)$ is open.

• $(ii) \Rightarrow (i)$. This paragraph is based on [Gue]. We denote $\mu : \Gamma \times A \to A, \quad (\gamma, a) \mapsto \sigma^\sharp_\gamma(a)$. Let $b \in A$. Since A is equipped with the discrete topology, it suffices to show that $\mu^{-1}(b)$ is open in $\Gamma \times A$ equipped with the product topology.

$$\begin{aligned}\mu^{-1}(b) &= \{(\gamma, a) \in \Gamma \times A \mid \sigma^\sharp_\gamma(a) = b\}\\ &= \bigcup_{a\in A}\{(\gamma, a) \mid \gamma \in \Gamma, \sigma^\sharp_\gamma(a) = b\}\\ &= \bigcup_{a\in \Gamma\cdot b}\{(\gamma, a) \mid \gamma \in \Gamma, \sigma^\sharp_\gamma(a) = b\}.\end{aligned}$$

For each $a \in \Gamma \cdot b$, take $\gamma_a \in \Gamma$ such that $\sigma^\sharp_{\gamma_a}(b) = a$. Then

$$\begin{aligned}\mu^{-1}(b) &= \bigcup_{a\in \Gamma\cdot b}\{(\gamma, a) \mid \gamma \in \Gamma, \sigma^\sharp_{\gamma\gamma_a}(b) = b\}\\ &= \bigcup_{a\in \Gamma\cdot b}(\mathrm{Stab}_\Gamma(b)\gamma_a^{-1}) \times \{a\}\end{aligned}$$

Therefore, $\mu^{-1}(b)$ is open in $\Gamma \times A$.

• $(ii) \Leftrightarrow (iii)$ see [Ber10, Lemma II.3.3].

• (inspired from [vDdB]) Assume that the map $\Gamma\times A \to A$, $(\gamma, a) \mapsto \sigma^\sharp_\gamma(a)$ is continuous. Since A is a finitely generated $\Bbbk'$-algebra, we can assume that

$$A = \Bbbk'[x_2, \dots, x_n]/I,$$

where I is a prime ideal. For all $i \in \{2, \dots, n\}$, let Γ_i be the stabilizer of x_i. Then, $\Gamma_1 := \cap_i\Gamma_i$ is an open subgroup of Γ. Then, any $\gamma \in \Gamma_1$ acts on A by

$$\sigma^\sharp_\gamma : \sum a_{i_2,\dots,i_n} x_2^{i_2}\cdots x_n^{i_n} \mapsto \sum \gamma(a_{i_2,\dots,i_n}) x_2^{i_2}\cdots x_n^{i_n}.$$

Shrinking Γ_1 if necessary, we may assume that Γ_1 is open and normal, and that I is defined over $\Bbbk_1 := \Bbbk'^{\Gamma_1}$. This means that there is an ideal $I_1 \subset \Bbbk_1[x_2, \dots, x_n]$ such that $A_1 := \Bbbk_1[x_2, \dots, x_n]/I_1$ satisfies $A_1 \otimes_{\Bbbk_1} \Bbbk' \cong A$. Then, we obtain a Γ_1-equivariant isomorphism

$$A_1 \otimes_{\Bbbk_1} \Bbbk' \cong A,$$

where the Γ_1-action on $A_1 \otimes_{\Bbbk_1} \Bbbk'$ is given by $\gamma \mapsto id \otimes \gamma$. □

1.2.2 $\Bbbk$-group structures

We have similar definitions for algebraic groups.

Definition 1.2.8. (i) Let G be an algebraic $\Bbbk'$-group. An algebraic $\Bbbk$-group G_0 together with an isomorphism $(G_0)_{\Bbbk'} := G_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}\,(\Bbbk') \cong G$ is called a $\Bbbk$-*form* of G.

(ii) A $\Bbbk$-*group structure* τ on an algebraic $\Bbbk'$-group G is a $\Bbbk$-structure $\tau : G \to G$ such that the multiplication $G \times G \to G$, the inverse $G \to G$ and the unity $\mathrm{Spec}(\mathbb{C}) \to G$ are $\Bbbk$-morphisms (see Remark 1.2.2).

(iii) Two $\Bbbk$-group structures τ and τ' on G are *equivalent* if there exists a $\Bbbk'$-group automorphism $\varphi \in \mathrm{Aut}_{gr}(G)$ such that $\tau'_\gamma = \varphi \circ \tau_\gamma \circ \varphi^{-1}$ for all $\gamma \in \Gamma$.

(iv) A $\Bbbk$-*morphism* between two algebraic $\Bbbk'$-groups G and G' with $\Bbbk$-structures τ and τ' is a morphism of algebraic $\Bbbk'$-groups $f : G \to G'$ such that $\tau'_\gamma \circ f = f \circ \tau_\gamma$ as morphisms of $\Bbbk$-schemes for all $\gamma \in \Gamma$.

Example 1.2.9. The following affine $\mathbb{R}$-varieties are algebraic $\mathbb{R}$-groups that are $\mathbb{R}$-forms of the algebraic $\mathbb{C}$-group (see Section 4.1.1)

$$\mathbb{G}_{m,\mathbb{C}} := \mathrm{Spec}(\mathbb{C}[t^{\pm 1}]) \cong \mathrm{Spec}(\mathbb{C}[x, y]/(xy - 1)).$$

1. Let $\mathbb{G}_{m,\mathbb{R}} := \mathrm{Spec}(\mathbb{R}[t^{\pm 1}]) \cong \mathrm{Spec}(\mathbb{R}[x, y]/(xy - 1))$. From the $\mathbb{C}$-isomorphism

$$(\mathbb{G}_{m,\mathbb{R}})_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}},$$

we construct an $\mathbb{R}$-group structure τ_0 on $\mathbb{G}_{m,\mathbb{C}}$ (see Example 1.2.4) defined by

$$\begin{aligned} \tau_0 : \mathbb{G}_{m,\mathbb{C}} &\to \mathbb{G}_{m,\mathbb{C}} \\ t &\mapsto \bar{t} \end{aligned} \qquad\qquad \begin{aligned} \tau_0^\sharp : \mathbb{C}[t^{\pm 1}] &\to \mathbb{C}[t^{\pm 1}] \\ at^n &\mapsto \bar{a}t^n \end{aligned}$$

Note that $\mathbb{G}_{m,\mathbb{R}} = \operatorname{Spec}(\mathbb{C}[t^{\pm 1}]^\Gamma)$. The $\mathbb{R}$-group $\mathbb{G}_{m,\mathbb{R}}$ corresponds to the pair $(\mathbb{G}_{m,\mathbb{C}}, \tau_0)$ (see Corollary 1.4.2).

2. Let $\mathbb{S}^1 := \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1))$. From the $\mathbb{C}$-isomorphisms

$$\mathbb{R}[x,y]/(x^2+y^2-1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x,y]/(x^2+y^2-1) \cong \mathbb{C}[x,y]/(xy-1) \cong \mathbb{C}[t^{\pm 1}],$$

and from the canonical $\mathbb{R}$-group structure on $(\mathbb{S}^1)_{\mathbb{C}}$, we construct an $\mathbb{R}$-group structure τ_1 on $\mathbb{G}_{m,\mathbb{C}}$ defined by

$$\begin{aligned} \tau_1 : \mathbb{G}_{m,\mathbb{C}} &\to \mathbb{G}_{m,\mathbb{C}} \\ t &\mapsto \bar{t}^{-1} \end{aligned} \qquad\qquad \begin{aligned} \tau_1^\sharp : \mathbb{C}[t^{\pm 1}] &\to \mathbb{C}[t^{\pm 1}] \\ at^n &\mapsto \bar{a}t^{-n} \end{aligned}$$

Note that $\mathbb{S}^1 \cong \operatorname{Spec}((\mathbb{C}[t^{\pm 1}])^\Gamma)$. The $\mathbb{R}$-group $\mathbb{S}^1$ corresponds to the pair $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$ (see Corollary 1.4.2).

Example 1.2.10. Consider the following $\mathbb{R}$-variety called the *Weil restriction* of $\mathbb{G}_{m,\mathbb{C}}$

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) = \operatorname{Spec}\left(\frac{\mathbb{R}[x_1,y_1,x_2,y_2]}{(x_1y_1 - x_2y_2 - 1, x_2y_1 + x_1y_2)}\right).$$

From the $\mathbb{C}$-isomorphism $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}))_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}^2$ defined by

$$\begin{aligned} \mathbb{C}[x,y,x',y'](xy-1, x'y'-1) &\to \mathbb{C}[x_1,y_1,x_2,y_2]/(x_1y_1 - x_2y_2 - 1, x_2y_1 + x_1y_2) \\ x &\mapsto x_1 + ix_2 \\ y &\mapsto y_1 + iy_2 \\ x' &\mapsto x_1 - ix_2 \\ y' &\mapsto y_1 - iy_2 \end{aligned}$$

we construct an $\mathbb{R}$-group structure τ_2 on $\mathbb{G}_{m,\mathbb{C}}^2$ defined by

$$\begin{aligned} \tau_2 : \mathbb{G}_{m,\mathbb{C}}^2 &\to \mathbb{G}_{m,\mathbb{C}}^2 \\ (s,t) &\mapsto (\bar{t}, \bar{s}) \end{aligned} \qquad\qquad \begin{aligned} \tau_1^\sharp : \mathbb{C}[t^{\pm 1}] &\to \mathbb{C}[t^{\pm 1}] \\ as^nt^m &\mapsto \bar{a}s^mt^n. \end{aligned}$$

The $\mathbb{R}$-group $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ corresponds to the pair $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ (see Corollary 1.4.2).

1.2.3 Equivariant $\mathbb{k}$-structures on G-varieties

We have similar definitions for varieties endowed with an algebraic group action. Let G be an algebraic $\mathbb{k}'$-group, and let X be a G-variety. A G-equivariant $\mathbb{k}'$-automorphism φ of X is an automorphism of X such that the following diagram commutes

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ {\scriptstyle id \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G \times X & \xrightarrow[\mu]{} & X. \end{array}$$

We denote by $\mathrm{Aut}^G(X)$ the group of G-equivariant $\Bbbk'$-automorphisms of X.

Definition 1.2.11. Let G be an algebraic $\Bbbk'$-group equipped with a $\Bbbk$-group structure τ, and let X be a G-variety (i.e X is a $\Bbbk'$-variety endowed with a G-action). We denote by μ the G-action on X.

(i) A (G,τ)-*equivariant* $\Bbbk$-*structure* σ on X is a $\Bbbk$-structure σ on X such that the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} G \times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle \tau_\gamma \times \sigma_\gamma}\downarrow & & \downarrow{\scriptstyle \sigma_\gamma} \\ G \times X & \xrightarrow{\ \mu\ } & X \end{array}$$

(ii) Two (G,τ)-equivariant $\Bbbk$-group structures σ and σ' on X are *equivalent* if there exists a G-equivariant $\Bbbk'$-automorphism $\varphi \in \mathrm{Aut}^G(X)$ such that $\sigma'_\gamma = \varphi \circ \sigma_\gamma \circ \varphi^{-1}$ for all $\gamma \in \Gamma$.

Example 1.2.12. Consider the action of $\mathbb{G}^2_{m,\mathbb{C}}$ on $\mathbb{A}^3_{\mathbb{C}}$ given by $(s,t)\cdot(x,y,z) = (sx, ty, stz)$. The Weil restriction $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$ acts on $(\mathbb{A}^3_{\mathbb{C}}, \sigma')$, where $\sigma'(x,y,z) = (\overline{y}, \overline{x}, \overline{z})$. Therefore, σ is a $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$-equivariant $\mathbb{R}$-structure on X.

1.3 Descent data and continuous semilinear Γ-actions

1.3.1 Finite Galois extensions

Let $\Bbbk'/\Bbbk$ be a finite Galois extension. The *equivalence* between descent data and $\Bbbk$-structures in the context of finite Galois descent is based on the next lemma.

Lemma 1.3.1. *Let $\Bbbk'/\Bbbk$ be a finite Galois extension of Galois group Γ. The natural map*

$$\begin{aligned} \Bbbk' \otimes_{\Bbbk} \Bbbk' &\to \prod_{\Gamma} \Bbbk' \\ a \otimes b &\mapsto (a\gamma(b))_{\gamma\in\Gamma} \end{aligned}$$

is an isomorphism of $\Bbbk'$-algebras, where the $\Bbbk'$-algebra structure on $\Bbbk' \otimes_{\Bbbk} \Bbbk'$ is defined by $\lambda\cdot(a\otimes b) = (\lambda a)\otimes b$, and the $\Bbbk'$-algebra structure on $\prod_\Gamma \Bbbk'$ is defined by $\lambda\cdot(c_\gamma)_{\gamma\in\Gamma} = (\lambda c_\gamma)_{\gamma\in\Gamma}$.

Proof. Since the extension $\Bbbk'/\Bbbk$ is separable, by the primitive element theorem ([Lan02, V Theorem 4.6]), there exists $u \in \Bbbk'$ such that $\Bbbk' = \Bbbk(u)$ Moreover, since the extension is algebraic, u is an algebraic element and $\Bbbk' = \Bbbk(u) = \Bbbk[u]$. Let $f \in \Bbbk[t]$ be the minimal polynomial of u (of degree $[\Bbbk' : \Bbbk]$). Hence, $\Bbbk' \cong \Bbbk[t]/(f)$. Since the extension is Galois and $f \in \Bbbk[t]$, we have $\gamma(f(u)) = f(\gamma(u)) = 0$ for all $\gamma \in \Gamma$. Hence $f = \prod_{\gamma\in\Gamma}(t-\gamma(u))$. By Chinese remainder theorem in $\Bbbk'[t]$ ([Lan02, Corollary 2.2]), we obtain an isomorphism of $\Bbbk'$-algebras

$$\Bbbk'[t]/(f) \to \prod_{\gamma\in\Gamma} \Bbbk'[t]/\left(t-\gamma(u)\right).$$

Therefore, we obtain the following $\Bbbk'$-algebra isomorphisms

$$\begin{aligned} \Bbbk'[u] &\to \prod_{\gamma\in\Gamma} \Bbbk'[\gamma(u)] \to \prod_{\gamma\in\Gamma} \Bbbk'[u] \\ a_i u^i &\mapsto \left(a_i \gamma(u)^i\right)_{\gamma\in\Gamma} \mapsto \left(\gamma^{-1}(a_i) u^i\right)_{\gamma\in\Gamma}. \end{aligned}$$

Then, since the extension $\mathbb{k}'/\mathbb{k}$ is faithfully flat, $\mathbb{k}' \otimes_{\mathbb{k}} \mathbb{k}' \cong \mathbb{k}' \otimes_{\mathbb{k}} \mathbb{k}[t]/(f) \cong \mathbb{k}'[t]/(f)$. Finally, we have constructed an isomorphism of $\mathbb{k}'$-algebras

$$\begin{aligned} \mathbb{k}' \otimes_{\mathbb{k}} \mathbb{k}' &\to \prod_{\gamma \in \Gamma} \mathbb{k}'_{\gamma^{-1}} \\ a \otimes b &\mapsto \left(\gamma^{-1}(a)b\right)_{\gamma \in \Gamma}, \end{aligned}$$

where the $\mathbb{k}'$-algebra structures are defined by $\lambda \cdot (a \otimes b) = (\lambda a) \otimes b$ and, if $c \in \mathbb{k}'_\gamma$, $\lambda \cdot c = \gamma(\lambda)c$. Therefore, we obtain a $\mathbb{k}'$-algebra isomorphism

$$\begin{aligned} \mathbb{k}' \otimes_{\mathbb{k}} \mathbb{k}' &\to \prod_{\gamma \in \Gamma} \mathbb{k}' \\ a \otimes b &\mapsto (a\gamma(b))_{\gamma \in \Gamma}. \end{aligned}$$

□

Remark 1.3.2 (Generalization of Lemma 1.3.1 [Rui20, Lemma 1.37])**.** Let $\mathbb{k}'/\mathbb{k}$ be an arbitrary finite separable[1] extension of degree d. Let $\mathbb{k}''$ be the normal closure of $\mathbb{k}'/\mathbb{k}$ (see [Sta, Definition 0BMF]). Then, we get an isomorphism

$$\mathbb{k}'' \otimes_{\mathbb{k}} \mathbb{k}' \cong (\mathbb{k}'')^d.$$

Indeed, as in the proof of Lemma 1.3.1, we get an isomorphism

$$\mathbb{k}' = \mathbb{k}'(u) \cong \mathbb{k}[t]/(f),$$

where f is of degree d. Then, since $\mathbb{k}''$ is the normal closure of $\mathbb{k}'$, f splits in $\mathbb{k}''[t]$, and by [Sta, Lemmas 0EXM and 09DT], $\mathbb{k}''/\mathbb{k}'$ is a finite extension and $\mathbb{k}''/\mathbb{k}$ is Galois. We get

$$f = \prod_{i=1}^{d} (t - x_i),$$

where the x_i are Galois conjugate of u, that is $x_i = \gamma_i(u)$ for some $\gamma_i \in \mathrm{Gal}(\mathbb{k}''/\mathbb{k})$. Then, as in the proof of Lemma 1.3.1, we get an isomorphism

$$\mathbb{k}'' \otimes_{\mathbb{k}} \mathbb{k}' \cong \mathbb{k}''[t]/(f) \cong \prod_{i=1}^{d} \mathbb{k}''[t]/(t - \gamma_i(u)) \cong (\mathbb{k}'')^d.$$

In the proof of the next proposition, we will see that from a descent datum on a certain $\mathbb{k}'$-variety X', we can construct a $\mathbb{k}$-structure on X'. Conversely, from a $\mathbb{k}$-structure on a certain $\mathbb{k}'$-variety X', we can construct a descent datum on X'.

Proposition 1.3.3. *Let $\mathbb{k}'/\mathbb{k}$ be a finite Galois extension of Galois group Γ. There is an equivalence of categories between:*
(i) the category consisting of pairs (X', ψ), where X' is a $\mathbb{k}'$-variety covered by ψ-invariant affine open subsets, and ψ is a descent datum on X'; and
(ii) the category consisting of pairs (X', σ), where X' is a $\mathbb{k}'$-variety covered by Γ-invariant[2] affine open subsets, and σ is a $\mathbb{k}$-structure on X'.

[1] This is the case if $\mathbb{k}$ is a perfect field, in particular if $\mathbb{k}$ is a characteristic zero field.
[2] An affine open subset U of X is Γ-invariant if for all $\gamma \in \Gamma$, $\sigma_\gamma(U) = U$.

Proof. See [BLR90, §6.2] and [Sta, §0CDQ]. Let $X' = \operatorname{Spec}(A')$ be an affine $\Bbbk'$-variety. From Lemma 1.3.1, we get $\Bbbk'$-algebra isomorphisms (see also [Sta, §0CDQ])

$$\begin{aligned} \Bbbk' \otimes_{\Bbbk} \Bbbk' &\to \prod_{\Gamma} \Bbbk' \\ a \otimes b &\mapsto (a\gamma(b))_{\gamma\in\Gamma} \end{aligned} \qquad\qquad \begin{aligned} \Bbbk' \otimes_{\Bbbk} \Bbbk' &\to \prod_{\Gamma} \Bbbk' \\ a \otimes b &\mapsto \left(\gamma^{-1}(a)b\right)_{\gamma\in\Gamma}. \end{aligned}$$

From these isomorphisms, we obtain identifications (as $\Bbbk'$-algebras)

$$A' \otimes_{\Bbbk} \Bbbk' = \prod_{\Gamma} A', \qquad \Bbbk' \otimes_{\Bbbk} A' = \prod_{\Gamma} A'. \tag{1.4}$$

Observe that we have a $\Bbbk'$-algebra isomorphism

$$\begin{array}{ccccc} \Bbbk' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' & \longrightarrow & \prod_{\Gamma\times\Gamma} \Bbbk' & \longrightarrow & \prod_{\Gamma\times\Gamma} \Bbbk' \\ a \otimes b \otimes c & \mapsto & (\gamma_1(a)\gamma_2(b)c) & \mapsto & (\gamma_1(a)\gamma_1(\gamma_2(b))c)\,. \end{array}$$

The reason for choosing $\gamma_1(a)\gamma_1(\gamma_2(b))c$ and not $\gamma_1(a)\gamma_2(b)c$ is that the formulas below simplify, but it is not strictly necessary. By the same reasoning as before, we obtain identifications (as $\Bbbk'$-algebras)

$$\Bbbk' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} A' = \prod_{\Gamma\times\Gamma} A', \quad \Bbbk' \otimes_{\Bbbk} A' \otimes_{\Bbbk} \Bbbk' = \prod_{\Gamma\times\Gamma} A', \quad \text{and} \quad A' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' = \prod_{\Gamma\times\Gamma} A'. \tag{1.5}$$

• Let $X' = \operatorname{Spec}(A')$ be an affine $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ. We get a $\Bbbk$-algebra isomorphism

$$\prod_{\gamma\in\Gamma} \sigma_\gamma^\sharp : \prod_{\Gamma} A' \to \prod_{\Gamma} A'.$$

Then, we can construct a co-Cartesian diagram

$$\begin{array}{ccccc} \prod_{\Gamma\times\Gamma} A' & \xleftarrow{\hat{p}_{12}^\sharp,\, \hat{p}_{13}^\sharp,\, \hat{p}_{23}^\sharp} & \prod_{\Gamma} A' & \xleftarrow{\hat{p}_1^\sharp,\, \hat{p}_2^\sharp} & A' \\ \uparrow & & \uparrow & & \uparrow \\ \Bbbk' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p_{12}^\sharp,\, p_{13}^\sharp,\, p_{23}^\sharp} & \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p_1^\sharp,\, p_2^\sharp} & \Bbbk' \end{array}$$

where

$$\begin{aligned} \hat{p}_1^\sharp(a) &= (a)_{\gamma\in\Gamma} \\ \hat{p}_2^\sharp(a) &= \left(\sigma_\gamma^\sharp(a)\right)_{\gamma\in\Gamma} \end{aligned} \qquad \begin{aligned} \hat{p}_{12}^\sharp\left((a_\gamma)_{\gamma\in\Gamma}\right) &= (a_{\gamma_1})_{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma} \\ \hat{p}_{13}^\sharp\left((a_\gamma)_{\gamma\in\Gamma}\right) &= (a_{\gamma_1\gamma_2})_{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma} \\ \hat{p}_{23}^\sharp\left((a_\gamma)_{\gamma\in\Gamma}\right) &= \left(\sigma_{\gamma_1}^\sharp(a_{\gamma_2})\right)_{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma} \end{aligned}$$

and the $p_i^\sharp$ and $p_{ij}^\sharp$ are the canonical inclusions. We have to check the following equalities

$$\begin{cases} \hat{p}_{12}^\sharp \circ \hat{p}_1^\sharp &= \hat{p}_{13}^\sharp \circ \hat{p}_1^\sharp \\ \hat{p}_{23}^\sharp \circ \hat{p}_1^\sharp &= \hat{p}_{12}^\sharp \circ \hat{p}_2^\sharp \\ \hat{p}_{13}^\sharp \circ \hat{p}_2^\sharp &= \hat{p}_{23}^\sharp \circ \hat{p}_2^\sharp. \end{cases}$$

The two first equalities are trivially satisfied. Since for all $\gamma_1, \gamma_2 \in \Gamma$, $\sigma^\sharp_{\gamma_1\gamma_2} = \sigma^\sharp_{\gamma_1} \circ \sigma^\sharp_{\gamma_2}$, the last condition is fulfilled. Therefore, using identifications (1.4) and Proposition 1.1.5, we obtain a descent datum on X'.

• Conversely, let $X' = \mathrm{Spec}(A')$ be an affine $\mathbb{k}'$-variety and let

$$\psi : X' \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}') \to \mathrm{Spec}(\mathbb{k}') \times_{\mathrm{Spec}(\mathbb{k})} X'$$

be a descent datum in X'. Then, we get a $\mathbb{k}' \otimes_{\mathbb{k}} \mathbb{k}'$-algebra isomorphism

$$\begin{aligned} \psi^\sharp : \mathbb{k}' \otimes_{\mathbb{k}} A' &\to A' \otimes_{\mathbb{k}} \mathbb{k}' \\ z \otimes a &\mapsto \sum_i z a_i \otimes z_i, \end{aligned}$$

where $\sum_i a_i \otimes z_i := \psi^\sharp(1 \otimes a)$. Using the above identifications (1.4), we can write:

$$\begin{aligned} \psi^\sharp : \prod_{\gamma\in\Gamma} A' &\to \prod_{\gamma\in\Gamma} A' \\ \left(\gamma^{-1}(z)a\right)_{\gamma\in\Gamma} &\mapsto \left(\sum_i z a_i \gamma(z_i)\right)_{\gamma\in\Gamma}. \end{aligned}$$

Let $\gamma_0 \in \Gamma$. Then we obtain a $\mathbb{k}$-morphism over γ_0, denoted $\sigma^\sharp_{\gamma_0}$,

$$\begin{aligned} \sigma^\sharp_{\gamma_0} : A' &\to A' \\ \gamma_0^{-1}(z)a &\mapsto \sum_i z a_i \gamma_0(z_i). \end{aligned}$$

Therefore, we get

$$\psi^\sharp = \prod_{\gamma\in\Gamma} \sigma^\sharp_\gamma : \prod_\Gamma A' \to \prod_\Gamma A'.$$

We get the above co-Cartesian diagram with

$$\hat{p}^\sharp_1(a) = a \otimes 1 \qquad \hat{p}^\sharp_2(a) = \psi^\sharp(1 \otimes a)$$

$$\hat{p}^\sharp_{12}(a \otimes b) = a \otimes b \otimes 1 \qquad \hat{p}^\sharp_{13}(a \otimes b) = a \otimes 1 \otimes b \qquad \hat{p}^\sharp_{23}(a \otimes b) = \psi^\sharp(1 \otimes a) \otimes b$$

Since $\hat{p}^\sharp_{13} \circ \hat{p}^\sharp_2 = \hat{p}^\sharp_{23} \circ \hat{p}^\sharp_2$ (it is a consequence of the cocycle condition $\psi_2 = \psi_1 \circ \psi_3$), then for all $\gamma_1, \gamma_2 \in \Gamma$, $\sigma^\sharp_{\gamma_1\gamma_2} = \sigma^\sharp_{\gamma_1} \circ \sigma^\sharp_{\gamma_2}$. Therefore, we get a semilinear Γ-action on X'. Furthermore, since the descent datum is effective (see Theorem 1.1.3), one can show that σ is a $\mathbb{k}$-structure on X'.

• Let $\varphi : (X' = \mathrm{Spec}(A'), \sigma_X) \to (Y' = \mathrm{Spec}(B'), \sigma_Y)$ be an arrow. That is, the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} B' & \xrightarrow{\ \sigma_Y{}^\sharp_\gamma\ } & B' \\ \varphi^\sharp \downarrow & & \downarrow \varphi^\sharp \\ A' & \xrightarrow{\ \sigma_X{}^\sharp_\gamma\ } & A'. \end{array}$$

Then, the next diagram is commutative

$$\begin{array}{ccc} \prod_{\gamma\in\Gamma} B' & \xrightarrow{\prod_{\gamma\in\Gamma} \sigma_{Y}{}_{\gamma}^{\sharp}} & \prod_{\gamma\in\Gamma} B' \\ {\scriptstyle \prod_{\gamma\in\Gamma} \varphi^{\sharp}}\downarrow & & \downarrow{\scriptstyle \prod_{\gamma\in\Gamma} \varphi^{\sharp}} \\ \prod_{\gamma\in\Gamma} A' & \xrightarrow{\prod_{\gamma\in\Gamma} \sigma_{X}{}_{\gamma}^{\sharp}} & \prod_{\gamma\in\Gamma} A'. \end{array}$$

Therefore, we get an arrow $\varphi : (X', \prod_{\gamma\in\Gamma} \sigma_{X\gamma}) \to (Y', \prod_{\gamma\in\Gamma} \sigma_{X\gamma})$ (see Definition 1.1.1).

• Let $\varphi : (X' = \mathrm{Spec}(A'), \psi_X) \to (Y' = \mathrm{Spec}(B'), \psi_Y)$ be an arrow. That is, the following diagram commutes

$$\begin{array}{ccc} \Bbbk' \otimes_{\Bbbk} B' & \xrightarrow{\psi_Y^{\sharp}} & B' \otimes_{\Bbbk} \Bbbk' \\ {\scriptstyle id\otimes\varphi^{\sharp}}\downarrow & & \downarrow{\scriptstyle \varphi^{\sharp}\otimes id} \\ \Bbbk' \otimes_{\Bbbk} A' & \xrightarrow{\psi_X^{\sharp}} & A' \otimes_{\Bbbk} \Bbbk'. \end{array}$$

Let σ_X and σ_Y be the $\Bbbk$-structures on X and Y respectively associated to ψ'_X and ψ'_Y. Using the above identifications (1.4), one can show that $\varphi : X' \to Y'$ is a $\Bbbk$-morphism such that for all $\gamma \in \Gamma$, $\sigma_{X\gamma} \circ \varphi = \varphi \circ \sigma_{Y\gamma}$. □

1.3.2 Infinite Galois extensions

Let $\Bbbk'/\Bbbk$ be an infinite Galois extension. In this context, the morphism of Lemma 1.3.1 is injective but not surjective. The image of this morphism will play an important role: it is the reason why $\Bbbk$-structures are defined as in Definition 1.2.1 (see the *algebraic* condition). This paragraph is based on [vDdB].

Lemma 1.3.4. *Let $\Bbbk'/\Bbbk$ be a non necessarily finite Galois extension. The natural map*

$$\Bbbk' \otimes_{\Bbbk} \Bbbk' \to \prod_{\gamma\in\Gamma} \Bbbk'$$
$$a \otimes b \mapsto (a\gamma(b))_{\gamma\in\Gamma}$$

is an injective $\Bbbk'$-algebra morphism (where the $\Bbbk'$-algebra structure on $\Bbbk' \otimes_{\Bbbk} \Bbbk'$ is defined by $\lambda \cdot (a \otimes b) = (\lambda a) \otimes b$), and its image is

$$R := \left\{ (a_\gamma)_{\gamma\in\Gamma} \in \prod_{\Gamma} \Bbbk' \,\middle|\, \exists H \subset \Gamma \text{ an open subgroup such that } a_{\gamma'\gamma} = a_\gamma \ \forall \gamma' \in H, \gamma \in \Gamma \right\}.$$

Proof. If $\Bbbk'/\Bbbk$ is a finite Galois extension, then this map is an isomorphism of $\Bbbk'$-algebras by Lemma 1.3.1. Assume that $\Bbbk'/\Bbbk$ is an infinite Galois extension with profinite group Γ. Let $\Bbbk_1/\Bbbk$ be a finite Galois extension in $\Bbbk'$. From Lemma 1.3.1, we get an isomorphism of $\Bbbk'$-algebras

$$\Bbbk' \otimes_{\Bbbk} \Bbbk_1 \to \prod_{\gamma\in \mathrm{Gal}(\Bbbk_1/\Bbbk)} \Bbbk'_\gamma$$
$$a \otimes b \mapsto (a\gamma(b))_{\gamma\in\Gamma},$$

where the $\Bbbk'$-algebra structure on $\Bbbk' \otimes_{\Bbbk} \Bbbk_1$ is defined by $\lambda \cdot (a \otimes b) = (\lambda a) \otimes b$, and the $\Bbbk'$-algebra structure on $\prod_{\Gamma} \Bbbk'$ is defined by $\lambda \cdot (c_\gamma)_{\gamma\in\Gamma} = (\lambda c_\gamma)_{\gamma\in\Gamma}$.

Let $\Bbbk \subset \Bbbk_1 \subset \Bbbk_2$ be a tower of finite Galois extensions in $\Bbbk'$. By the Galois correspondence (see Theorem B.3.1), there is a natural projection $\pi_{2,1} : \mathrm{Gal}(\Bbbk_2/\Bbbk) \to \mathrm{Gal}(\Bbbk_1/\Bbbk)$. Therefore, we obtain a transition morphism

$$\begin{aligned} q_{1,2} : \prod_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} \Bbbk'_\gamma &\to \prod_{\gamma \in \mathrm{Gal}(\Bbbk_2/\Bbbk)} \Bbbk'_\gamma \\ (a_\gamma)_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} &\mapsto \left(a_{\pi(\gamma)}\right)_{\gamma \in \mathrm{Gal}(\Bbbk_2/\Bbbk)}. \end{aligned}$$

Observe that we have constructed a directed inductive system (see Definitions A.5.1 and A.5.2). We have to prove that the colimit taken over all finite Galois extensions $\Bbbk_1/\Bbbk$ in $\Bbbk'$ is R (see Definition A.5.4). First, observe that for all finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$, there is a morphism:

$$\begin{aligned} \prod_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} \Bbbk'_\gamma &\to \prod_{\gamma \in \Gamma} \Bbbk'_\gamma \\ (a_\gamma)_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} &\mapsto \left(a_{\pi(\gamma)}\right)_{\gamma \in \Gamma}, \end{aligned}$$

where $\pi : \Gamma \to \mathrm{Gal}(\Bbbk_1/\Bbbk)$ is the natural projection. Hence, we get a morphism $s_1 : \prod_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} \Bbbk'_\gamma \to R$ (put $H := \mathrm{Gal}(\Bbbk'/\Bbbk_1)$ in the definition of R). Let $\Bbbk \subset \Bbbk_1 \subset \Bbbk_2$ be a tower of finite Galois extensions in $\Bbbk'$. Observe that $s_1 = s_2 \circ q_{1,2}$. Finally, one can show that R satisfies the universal property of the colimit (see Definition A.5.4). Therefore, the colimit taken over all finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$ is

$$\mathrm{colim} \left(\prod_{\gamma \in \mathrm{Gal}(\Bbbk_1/\Bbbk)} \Bbbk'_\gamma \right) = R.$$

On the other hand, since tensor products commute with colimits (see [Sta, Lemma 00DD]), we have

$$\Bbbk' \otimes_\Bbbk \Bbbk' \cong \mathrm{colim} \left(\Bbbk' \otimes_\Bbbk \Bbbk_1\right),$$

where the colimit is taken over all finite Galois extensions $\Bbbk_1/\Bbbk$ in $\Bbbk'$. Finally, we obtain the desired result. □

Remark 1.3.5. Let A' be a $\Bbbk'$-algebra. Then we get $\Bbbk'$-algebra isomorphism

$$\begin{aligned} A' \otimes_\Bbbk \Bbbk' &\to R_{A'} \\ a \otimes b &\mapsto (a\gamma(b))_{\gamma \in \Gamma} \end{aligned}$$

where $R_{A'} := A' \otimes_{\Bbbk'} R = R \otimes_{\Bbbk'} A'$ consists of

$$R_{A'} := \left\{ (a_\gamma)_\Gamma \in \prod_{\gamma \in \Gamma} A' \,\middle|\, \exists H \subset \Gamma \text{ an open subgroup such that } a_{\gamma'\gamma} = a_\gamma \ \forall \gamma' \in H, \gamma \in \Gamma \right\}.$$

Indeed, the construction of R is based on finite products and colimits, both of which commute with arbitrary direct sum (see [Ste]). Thus, the result for $A = \Bbbk'$ implies the general case.

Proposition 1.3.6. *Let $\Bbbk'/\Bbbk$ be a non necessarily finite Galois extension of profinite Galois group Γ. There is an equivalence of categories between:*

(i) the category consisting of pairs (X', ψ), where X' is a $\Bbbk'$-variety covered by ψ-invariant affine open subsets, and ψ is a descent datum on X'; and

(ii) the category consisting of pairs (X', σ), where X' is a $\Bbbk'$-variety covered by Γ-invariant affine open subsets, and σ is a semilinear Γ-action on X' such that the map $\Gamma \times A \to A$ is continuous for all Γ-invariant affine open subset $U = \mathrm{Spec}(A)$, where Γ is endowed with the Krull topology, and A is equipped with the discrete topology.

Remark 1.3.7. We will see in Lemma 1.4.10 that the continuous semilinear Γ-action mentioned in Proposition 1.3.6 corresponds to $\Bbbk$-structure.

Proof. See [BLR90, §6.2]. See [vDdB] for a shorter and easier proof using the fact that $X' = \mathrm{Spec}(A')$ is of finite type (see Definition A.6.7). We give here a powerful proof, see also [vDdB].

• Let $X' = \mathrm{Spec}(A')$ be an affine $\Bbbk'$-variety endowed with a semilinear Γ-action on X' such that the map $\Gamma \times A \to A$ is continuous. From Lemma 1.3.4, we get $\Bbbk'$-algebra isomorphisms (see the proof of Proposition 1.3.3)

$$\Bbbk' \otimes_{\Bbbk} \Bbbk' \to R \subset \prod_{\Gamma} \Bbbk' \qquad\qquad \Bbbk' \otimes_{\Bbbk} \Bbbk' \to R \subset \prod_{\Gamma} \Bbbk'.$$

From these isomorphisms, we obtain identifications (as $\Bbbk'$-algebras)

$$A' \otimes_{\Bbbk} \Bbbk' = R_{A'} \subset \prod_{\Gamma} A', \qquad \Bbbk' \otimes_{\Bbbk} A' = R_{A'} \subset \prod_{\Gamma} A'. \tag{1.6}$$

We get a well defined $\Bbbk$-algebra isomorphism

$$\prod_{\gamma \in \Gamma} \sigma_{\gamma}^{\sharp} : R_{A'} \to R_{A'}. \tag{1.7}$$

Indeed, let $(a_{\gamma})_{\gamma \in \Gamma} \in R_{A'}$. There exists an open subgroup $H \subset \Gamma$ such that $a_{\gamma'\gamma} = a_{\gamma}$ for all $\gamma' \in H$ and $\gamma \in \Gamma$. Since the action is continuous, for all $\gamma \in \Gamma$ the group $H_{\gamma} := \mathrm{Stab}_{\Gamma}(\sigma_{\gamma}^{\sharp}(a_{\gamma}))$ is an open subgroup of Γ (see Lemma 1.2.6). Note that $H' := \cap_{\Gamma} H_{\gamma} = \cap_{\Gamma/H} H_{\gamma}$. Thus, since Γ/H is finite (see Theorem B.3.1), H' is an open subgroup of Γ. Then, for all $\gamma' \in H \cap H'$ and $\gamma \in \Gamma$, we have:

$$\sigma_{\gamma'\gamma}^{\sharp}(a_{\gamma'\gamma}) = \sigma_{\gamma'}^{\sharp}\left(\sigma_{\gamma}^{\sharp}(a_{\gamma'\gamma})\right) = \sigma_{\gamma'}^{\sharp}\left(\sigma_{\gamma}^{\sharp}(a_{\gamma})\right) = \sigma_{\gamma}^{\sharp}(a_{\gamma}). \tag{1.8}$$

From the isomorphism (1.7) and the identifications (1.5) mentioned in the proof of Proposition 1.3.3, we can construct a co-Cartesian diagram

$$\begin{array}{ccccc}
R_{A'} \otimes_{\Bbbk} \Bbbk' & \xleftarrow{\hat{p}_{12}^{\sharp}, \hat{p}_{13}^{\sharp}, \hat{p}_{23}^{\sharp}} & R_{A'} & \xleftarrow{\hat{p}_{1}^{\sharp}, \hat{p}_{2}^{\sharp}} & A' \\
\uparrow & & \uparrow & & \uparrow \\
\Bbbk' \otimes_{\Bbbk} \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p_{12}^{\sharp}, p_{13}^{\sharp}, p_{23}^{\sharp}} & \Bbbk' \otimes_{\Bbbk} \Bbbk' & \xleftarrow{p_{1}^{\sharp}, p_{2}^{\sharp}} & \Bbbk'
\end{array}$$

We conclude as in the proof of Proposition 1.3.3.

• Conversely, let $X' = \mathrm{Spec}(A')$ be an affine $\Bbbk'$-variety and let $\psi : X' \times_S S' \to S' \times_S X'$ be a descent datum in X'. Then, we get $\psi^{\sharp} : \Bbbk' \otimes_{\Bbbk} A' \to A' \otimes_{\Bbbk} \Bbbk'$. Using the above identifications (1.6), we can write:

$$\psi^{\sharp} : R_{A'} \to R_{A'}.$$

As in the proof of Proposition 1.3.3, one can show that there exists a semilinear Γ-action $\sigma : \Gamma \to \mathrm{Aut}(X_{/\Bbbk})$. We will prove below that σ is a $\Bbbk$-structure.

• For the study of arrows, it is similar to the proof of Proposition 1.3.3.

• Finally, it remains to prove that the morphism $\Gamma \times A' \to A'$ obtained from the pair (X', ψ) is continuous. Since the descent datum is effective (See Theorem 1.1.3), there exists an affine $\Bbbk$-variety $X_0 = \mathrm{Spec}(A_0)$ such that

$$(X', \psi) \cong ((X_0)_{\Bbbk'}, \psi_{can}),$$

where

$$\psi^\sharp_{can} : \Bbbk' \otimes_\Bbbk (A_0 \otimes_\Bbbk \Bbbk') \to (A_0 \otimes_\Bbbk \Bbbk') \otimes_\Bbbk \Bbbk'$$
$$z \otimes a \otimes z_0 \mapsto a \otimes z \otimes z_0.$$

The semilinear Γ-action on $A_0 \otimes_\Bbbk \Bbbk'$ induced by ψ_{can} is defined by $\gamma \mapsto id \otimes \gamma$. Then, by Example A.5.5 and [Sta, Lemma 00DD] (tensor product commutes with colimits), we get

$$\bigcup_i A_0 \otimes_\Bbbk \Bbbk_i = \mathrm{colim}_i A_0 \otimes_\Bbbk \Bbbk_i \cong A_0 \otimes_\Bbbk \mathrm{colim}_i \Bbbk_i \cong A_0 \otimes_\Bbbk \Bbbk',$$

where $\Bbbk_i/\Bbbk$ are finite Galois extensions. By Lemma 1.2.6, the Γ-action on $A_0 \otimes_\Bbbk \Bbbk'$ is continuous. Therefore, since A' and $A_0 \otimes_\Bbbk \Bbbk'$ are Γ-equivariantly isomorphic, then σ is a $\Bbbk$-structure on X'. □

1.4 Galois descent theorems

In this section, we state the main results on the effectiveness of Galois descent using the language of $\Bbbk$-structures. With respect to Section 1.1, we also introduce an additional tool, that is, a Galois cohomology set. This set will parametrize $\Bbbk$-forms of a given $\Bbbk'$-variety.

First, we focus on finite Galois extensions. Let $\Bbbk'/\Bbbk$ be a finite Galois extension, we give a self contained proof of the effectiveness of Galois descent of quasi-projective $\Bbbk'$-varieties endowed with a $\Bbbk$-structure. Secondly, we consider a non-necessarily finite Galois-extension. The effectiveness of Galois descent of quasi-projective $\Bbbk'$-varieties endowed with a $\Bbbk$-structure is a consequence of the fpqc-descent described in Section 1.1.

1.4.1 Finite Galois extensions

In this section, $\Bbbk'/\Bbbk$ is a finite Galois extension of Galois group Γ. If X_0 is a $\Bbbk$-variety, then $(X_0)_{\Bbbk'} := X_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk')$ is endowed with the canonical $\Bbbk$-structure $\sigma_0 : \gamma \mapsto id \times \mathrm{Spec}(\gamma)$. Furthermore, if X_0' is another $\Bbbk$-variety and $f : X_0 \to X_0'$ is a morphism of $\Bbbk$-varieties, then $f \times id : (X_0)_{\Bbbk'} \to (X_0')_{\Bbbk'}$ is a morphism of $\Bbbk'$-varieties such that for all $\gamma \in \Gamma$, $\sigma'_{0\gamma} \circ (f \times id) = (f \times id) \circ \sigma_{0\gamma}$. Therefore, we get a functor $X_0 \mapsto ((X_0)_{\Bbbk'}, \sigma_0)$ from the category of quasi-projective $\Bbbk$-varieties to the category of pairs (X, σ), where X is a quasi-projective $\Bbbk'$-variety and σ is a $\Bbbk$-structure on X (see Theorem 1.1.4). The next theorem is a direct consequence of Theorem 1.1.3 combined with Proposition 1.3.3, but we give a proof using the language of $\Bbbk$-structure.

Theorem 1.4.1. *The functor $X_0 \mapsto ((X_0)_{\Bbbk'}, \sigma_0)$, where $\sigma_{0\gamma} := id \times \mathrm{Spec}(\gamma)$ for all $\gamma \in \Gamma$, induces an equivalence of categories between the category of pairs (X, σ) consisting of a quasi-projective $\Bbbk'$-variety X endowed with a $\Bbbk$-structure σ, and the category of quasi-projective $\Bbbk$-varieties. Moreover, equivalent $\Bbbk$-structures on X correspond to isomorphic $\Bbbk$-forms of X.*

Proof. This proof is based on [Ben16, §3.1.3].

• Let X be a quasi-projective $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ. Then, $X_{/\Bbbk}$ is a quasi-projective $\Bbbk$-scheme (see [Sta, Lemma 0B42]). Therefore, $X_0 := X_{/\Bbbk}/\Gamma$ is a $\Bbbk$-scheme (see Section A.4). Let $U = \mathrm{Spec}(A)$ be a Γ-invariant affine open subset of X (see Lemma A.4.5), and let $U_0 := U_{/\Bbbk}/\Gamma = \mathrm{Spec}(A^\Gamma)$. By Speiser Lemma, we get a Γ-equivariant $\Bbbk'$-isomorphism

$$A^\Gamma \otimes_\Bbbk \Bbbk' \cong A, \quad a \otimes z \mapsto az.$$

That is, the next diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} A^{\Gamma} \otimes_{\Bbbk} \Bbbk' & \xrightarrow{\cong} & A \\ {\scriptstyle id \otimes \gamma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{\gamma}^{\sharp}} \\ A^{\Gamma} \otimes_{\Bbbk} \Bbbk' & \xrightarrow{\cong} & A \end{array}$$

Hence, we get a Γ-equivariant $\Bbbk'$-isomorphism $(U_0)_{\Bbbk'} \cong U$. Observe that the morphism $U \to (U_0)_{\Bbbk'}$ corresponds to $(\pi|_U, f|_U)$, where $f : X \to \mathrm{Spec}(\Bbbk)$ is the structural morphism of X, and $\pi : X_{/\Bbbk} \to X_0$ is the quotient morphism. Therefore, if we cover X by Γ-invariant affine open subsets, the isomorphisms $(\pi|_U, f|_U)$ can be glued together and we get a Γ-equivariant $\Bbbk'$-isomorphism (π, f) (see Section A.4) such that the next diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} X & \xrightarrow{(\pi, f)} & (X_0)_{\Bbbk'} \\ {\scriptstyle \sigma_{\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{0\gamma} = id \times \mathrm{Spec}(\gamma)} \\ X & \xrightarrow{(\pi, f)} & (X_0)_{\Bbbk'} \end{array}$$

Since $X \cong (X_0)_{\Bbbk'}$, by Theorem 1.1.4, X_0 is a quasi-projective $\Bbbk$-variety. Furthermore, (X, σ) is isomorphic to $((X_0)_{\Bbbk'}, \sigma_0)$. Moreover, let X' be another quasi-projective $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ', and let $f : X \to X'$ be a $\Bbbk'$-morphism satisfying $\sigma'_{\gamma} \circ f = f \circ \sigma$ for all $\gamma \in \Gamma$. The morphism $\pi' \circ f$ is Γ-invariant, so by Definition A.4.1, there exists a unique $\Bbbk$-morphism $f_0 : X_0 \to X'_0$. Therefore, we obtain a functor from the category consisting of pairs (Z, σ_Z), where Z is a quasi-projective $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ_Z, to the category of quasi-projective $\Bbbk$-variety.

• Conversely, let X_0 be a $\Bbbk$-form of X, and let σ_0 be the natural $\Bbbk$-structure on $(X_0)_{\Bbbk'}$. By definition, there exists a $\Bbbk'$-isomorphism $\varphi : X \to (X_0)_{\Bbbk'}$. Then, σ_0 induces a $\Bbbk$-structure σ on X such that φ is Γ-equivariant, that is the next diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & (X_0)_{\Bbbk'} \\ {\scriptstyle \sigma_{\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{0\gamma} = id \times \mathrm{Spec}(\gamma)} \\ X & \xrightarrow{\varphi} & (X_0)_{\Bbbk'} \end{array}$$

The projection $pr_1 : X_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk') \to X_0$ is the categorical quotient of $(X_0)_{\Bbbk'}$ by Γ (see Proposition A.4.7). Therefore, the isomorphism $\varphi : (X_0)_{\Bbbk'} \to X$ descends to a $\Bbbk$-isomorphism $X_0 \cong X/\Gamma$ (see Section A.4).

Finally, we have the desired equivalence of categories (see [Sta, §0013]). □

Using this equivalence, by abuse of notation we write (X, σ) to refer to a quasi-projective $\Bbbk$-variety. We denote by $(X, \sigma) \mapsto X/\Gamma$ the inverse functor.

From Theorem 1.4.1, we get the next result. Recall that an algebraic group (see Section A.2) is a quasi-projective variety from Chevalley's theorem (see [Con02, Corollary 1.2]).

Corollary 1.4.2. *The functor $G_0 \mapsto ((G_0)_{\Bbbk'}, \tau_0)$, where $\tau_{0\gamma} := id \times \mathrm{Spec}(\gamma)$ for all $\gamma \in \Gamma$, induces an equivalence of categories between the category of pairs (G, τ) consisting of an algebraic $\Bbbk'$-group G endowed with a $\Bbbk$-group structure τ, and the category of algebraic $\Bbbk$-groups. Moreover, equivalent $\Bbbk$-group structures on G correspond to isomorphic $\Bbbk$-forms of G.*

Proof. Let G be an algebraic $\Bbbk'$-group endowed with a $\Bbbk$-group structure τ. Then, by Theorem 1.4.1, $G_0 := G/\Gamma$ is a quasi-projective $\Bbbk$-variety. By definition, the multiplication $m : G \times_{\mathrm{Spec}(\Bbbk')} G \to G$, the neutral element $e : \mathrm{Spec}(\Bbbk') \to G$, and the inverse $i : G \to G$, are

Γ-equivariant $\Bbbk'$-morphisms. Therefore, by Theorem 1.4.1, these morphisms descend respectively to $\Bbbk$-morphisms $m_0 : G_0 \times_{\mathrm{Spec}(\Bbbk)} G_0 \to G_0$, $e_0 : \mathrm{Spec}(\Bbbk) \to G_0$, and $i_0 : G_0 \to G_0$. One can show that G_0 is an algebraic $\Bbbk$-group. Conversely, if G_0 is a $\Bbbk$-form of G, then the natural $\Bbbk$-structure on $(G_0)_{\Bbbk'}$ is a $\Bbbk$-group structure, and G is endowed with a $\Bbbk$-group structure. □

Using this equivalence, by abuse of notation we write (G, τ) to refer to an algebraic $\Bbbk$-group.

The next result is a consequence of Theorem 1.4.1 combined with Corollary 1.4.2.

Corollary 1.4.3. *Let G_0 be a $\Bbbk$-group. There is a one-to-one correspondence between quasi-projective $\Bbbk$-varieties endowed with a G_0-action and tuples $(G, \tau, X, \sigma, \mu)$ consisting of:*
(i) a $\Bbbk'$-group G endowed with a $\Bbbk$-group structure τ such that $G/\Gamma \cong G_0$;
(ii) a quasi-projective $\Bbbk'$-variety X endowed with an $\Bbbk$-structure σ;
(iii) an action $\mu : G \times X \to X$ such that the following diagram commutes for all $\gamma \in \Gamma$:

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \tau_\gamma \times \sigma_\gamma}\downarrow & & \downarrow{\scriptstyle \sigma_\gamma} \\ G \times X & \xrightarrow[\mu]{} & X \end{array}$$

Once we know the existence of a $\Bbbk$-structure σ on a $\Bbbk'$-variety X, a Galois cohomology set can be used to parametrize the equivalence classes of $\Bbbk$-structures on X (see for instance [Ben16, §3.2]). First, note that $\mathrm{Aut}(X)$ is an abstract group equipped with a Γ-action

$$\begin{aligned} \Gamma \times \mathrm{Aut}(X) &\to \mathrm{Aut}(X) \\ (\gamma, \varphi) &\mapsto \gamma \cdot \varphi := \sigma_\gamma^{-1} \circ \varphi \circ \sigma_\gamma. \end{aligned}$$

Assume there exists another $\Bbbk$-structure σ' on X. The map $c : \Gamma \to \mathrm{Aut}(X)$ defined for all $\gamma \in \Gamma$ by $c_\gamma := \sigma'_\gamma \circ \sigma_\gamma^{-1}$ is a cocycle; that is a map such that for all $\gamma_1, \gamma_2 \in \Gamma$, $c_{\gamma_1\gamma_2} = c_{\gamma_1} \circ (\gamma_1 \cdot c_{\gamma_2})$.

Conversely, let $c : \Gamma \to \mathrm{Aut}(X)$ be a map. Then, the map $\gamma \mapsto c_\gamma \circ \sigma_\gamma$ is a $\Bbbk$-structure if and only if c is a cocycle.

Furthermore, if c and c' are two cocycles, then the associated $\Bbbk$-structures $\gamma \mapsto c_\gamma \circ \sigma_\gamma$ and $\gamma \mapsto c'_\gamma \circ \sigma_\gamma$ are equivalent if and only if the cocycles c and c' are *equivalent*, that is there exists $\varphi \in \mathrm{Aut}(X)$ such that for all $\gamma \in \Gamma$, $c'_\gamma = \varphi^{-1} \circ c_\gamma \circ (\gamma \cdot \varphi)$. The set of cocycles modulo this equivalence relation is the *first pointed set of Galois cohomology*

$$\mathrm{H}^1(\Gamma, \mathrm{Aut}(X)).$$

This pointed set has a distinguished point (see [Ser97, §I.5], or [Ber10, Definition II.3.13]), which is the class of the trivial cocycle. Then, we get the next proposition.

Proposition 1.4.4. *Let $\Bbbk'/\Bbbk$ be a finite Galois extension. Let X be a $\Bbbk'$-variety equipped with a $\Bbbk$-structure σ. There is a bijection*

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathrm{Aut}(X)) &\simeq \{\textit{equivalence classes of } \Bbbk\textit{-structures on } X\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \sigma_\gamma) \end{aligned}$$

that sends the trivial cocycle $\gamma \mapsto id$ to the equivalence class of σ.

By Proposition 1.4.4 combined with Theorem 1.4.1, we get the next result.

Proposition 1.4.5 ([Ser97, §III.1.3, Proposition 5], [GW20, Theorem 14.90])**.** *Let $\mathbb{k}'/\mathbb{k}$ be a finite Galois extension of Galois group Γ. Let X be a quasi-projective $\mathbb{k}'$-variety equipped with a $\mathbb{k}$-structure σ. There is a bijection*

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathrm{Aut}(X)) &\simeq \{\textit{isomorphism classes of } \mathbb{k}\textit{-forms of } X\} \\ (\gamma \mapsto c_\gamma) &\mapsto (X, (\gamma \mapsto c_\gamma \circ \sigma_\gamma)). \end{aligned}$$

As for $\mathbb{k}$-structures on a $\mathbb{k}'$-variety X, a Galois cohomology set can be used to parametrize the equivalence classes of $\mathbb{k}$-group structures on a $\mathbb{k}'$-group G. The next results are a straightforward consequences of Proposition 1.4.5.

Proposition 1.4.6 ([Ser97, §III.1.3, Corollary of Proposition 5])**.** *Let G be a $\mathbb{k}'$-group equipped with a $\mathbb{k}$-group structure τ. There is a bijection*

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathrm{Aut}_{gr}(G)) &\simeq \{\textit{equivalence classes of } \mathbb{k}\textit{-group structures on } G\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \tau_\gamma) \end{aligned}$$

that sends the trivial cocycle $\gamma \mapsto id$ to the equivalence class of τ ($\mathrm{Aut}_{gr}(G)$ consists of $\mathbb{k}'$-group automorphisms of G). Furthermore, there is a bijection

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathrm{Aut}_{gr}(G)) &\simeq \{\textit{isomorphism classes of } \mathbb{k}\textit{-forms, as group, of } G\} \\ (\gamma \mapsto c_\gamma) &\mapsto (G, (\gamma \mapsto c_\gamma \circ \tau_\gamma)). \end{aligned}$$

Proposition 1.4.7 ([Wed18, Corollary 10.1])**.** *Let (G,τ) be a $\mathbb{k}$-group. Let (X,σ) be a quasi-projective (G,τ)-variety. There is a bijection*

$$\begin{aligned} \mathrm{H}^1\left(\Gamma, \mathrm{Aut}^G(X)\right) &\simeq \{\textit{equivalence classes of } (G,\tau)\textit{-equivariant } \mathbb{k}\textit{-structures on } X\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \sigma_\gamma). \end{aligned}$$

Furthermore, there is a bijection

$$\begin{aligned} \mathrm{H}^1\left(\Gamma, \mathrm{Aut}^G(X)\right) &\simeq \{\textit{isomorphism classes of } \mathbb{k}\textit{-forms, as } G/\Gamma\textit{-variety, of } X\} \\ (\gamma \mapsto c_\gamma) &\mapsto (X, (\gamma \mapsto c_\gamma \circ \sigma_\gamma)). \end{aligned}$$

1.4.2 Connection between infinite and finite Galois descent

Let $\mathbb{k}'/\mathbb{k}$ be a non-necessarily finite Galois extension. In Propositions 1.4.8 and 1.4.9, we explain how are related $\mathbb{k}$-structures on a $\mathbb{k}'$-variety X and Galois subextensions $\mathbb{k} \subset \mathbb{k}_1 \subset \mathbb{k}'$. These results are consequences of the effectiveness of Galois descent mentioned in Theorem 1.1.3.

Let $\gamma \in \Gamma$. Since $\mathbb{k}_1$ is a normal subextension in $\mathbb{k}'$, $\gamma(\mathbb{k}_1) = \mathbb{k}_1$. We get an automorphism $\gamma|_{\mathbb{k}_1} : \mathbb{k}_1 \to \mathbb{k}_1$ (see more details in Section B.3). Therefore, we obtain a continuous surjective homomorphism

$$\begin{aligned} \Gamma &\to \mathrm{Gal}(\mathbb{k}_1/\mathbb{k}) \\ \gamma &\mapsto \gamma|_{\mathbb{k}_1}, \end{aligned}$$

with kernel $H := \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_1)$. In other words, we obtain a short exact sequence

$$1 \longrightarrow H := \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_1) \longrightarrow \Gamma := \mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \longrightarrow \mathrm{Gal}(\mathbb{k}_1/\mathbb{k}) \longrightarrow 1$$

of profinite topological groups. Hence, H is a normal subgroup of Γ and we have an isomorphism

$$\begin{aligned} \Gamma/H &\to \mathrm{Gal}(\mathbb{k}_1/\mathbb{k}) \\ \gamma H &\mapsto \gamma|_{\mathbb{k}_1}. \end{aligned}$$

Proposition 1.4.8 (Connection between infinite and finite Galois descent)**.** *Let X be a $\mathbb{k}'$-variety endowed with a $\mathbb{k}$-structure σ. Let $\mathbb{k}_1/\mathbb{k}$ be a finite Galois extension and let X_1 be a $\mathbb{k}_1$-form of X as in Definition 1.2.1. There exists a $\mathbb{k}$-structure $\sigma_1 : \mathrm{Gal}\,(\mathbb{k}_1/\mathbb{k}) \to \mathrm{Aut}\left(X_{1/\mathbb{k}}\right)$ such that the following diagram commutes for all $\gamma \in \Gamma$:*

$$\begin{array}{ccc} X_1 \times_{\mathrm{Spec}(\mathbb{k}_1)} \mathrm{Spec}\,(\mathbb{k}') & \xrightarrow{\;\cong\;} & X \\ {\scriptstyle {\sigma_1}_{\gamma|_{\mathbb{k}_1}} \times \mathrm{Spec}(\gamma)}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma_\gamma} \\ X_1 \times_{\mathrm{Spec}(\mathbb{k}_1)} \mathrm{Spec}\,(\mathbb{k}') & \xrightarrow{\;\cong\;} & X \end{array}$$

Furthermore, if X is quasi-projective (resp. affine), there exists a $\mathbb{k}$-form X_0 of X in the category of quasi-projective (resp. affine) varieties and a Γ-equivariant isomorphism $(X_0)_{\mathbb{k}'} \cong X$, where the $\mathbb{k}$-structure on X is σ, and the $\mathbb{k}$-structure on $(X_0)_{\mathbb{k}'}$ is given by $\gamma \in \Gamma \mapsto id \times \mathrm{Spec}(\gamma)$.

Proof. Let X be a $\mathbb{k}'$-variety endowed with a $\mathbb{k}$-structure σ, and let $f_X : X \to \mathrm{Spec}(\mathbb{k}')$ be its structural morphism. Let $\mathbb{k}_1/\mathbb{k}$ be a finite Galois extension in $\mathbb{k}'$, let $\iota : \mathbb{k}_1 \hookrightarrow \mathbb{k}'$ be the inclusion morphism, and let X_1 be a $\mathbb{k}_1$-form of X as in Definition 1.2.1. We denote $H := \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_1)$. By definition, we have an H-equivariant isomorphism $(X_1)_{\mathbb{k}'} \cong X$, where the $\mathbb{k}_1$-structure on $(X_1)_{\mathbb{k}'}$ is given by $\gamma \in H \mapsto id \times \mathrm{Spec}(\gamma)$, and the $\mathbb{k}_1$-structure on X is the restriction $\sigma|_H$.

Let $U_1 = \mathrm{Spec}(A_1)$ be an affine open subset of X_1, then $(U_1)_{\mathbb{k}'}$ is an H-invariant affine open subset of $(X_1)_{\mathbb{k}'}$ (see [Sta, Lemma 01JY]). Hence, $X \cong (X_1)_{\mathbb{k}'}$ is covered by H-invariant affine open subsets. Note that for all such H-invariant affine open subset $U = \mathrm{Spec}(A)$, the map $H \times A \to A$ is continuous. Indeed, $A \cong A_1 \otimes_{\mathbb{k}_1} \mathbb{k}'$, then by Example A.5.5 and [Sta, Lemma 00DD] (tensor product commutes with colimits), we get

$$\bigcup_i A_1 \otimes_{\mathbb{k}_1} \mathbb{k}_i = \mathrm{colim}_i A_1 \otimes_{\mathbb{k}_1} \mathbb{k}_i \cong A_1 \otimes_{\mathbb{k}_1} \mathrm{colim}_i \mathbb{k}_i \cong A_1 \otimes_{\mathbb{k}_1} \mathbb{k}',$$

where $\mathbb{k}_i/\mathbb{k}_1$ are finite Galois extensions. We conclude using Lemma 1.2.6.

The projection $pr_1 : (X_1)_{\mathbb{k}'} \to X_1$ is a categorical quotient for the H-action in the category of $\mathbb{k}_1$-schemes. Recall that $X \cong (X_1)_{\mathbb{k}'}$ is covered by $\sigma|_H$-invariant affine open subsets. Therefore, combining Theorem 1.1.3 together with Proposition 1.3.6, for any H-invariant morphism $\pi' : (X_1)_{\mathbb{k}'} \to Z$ where Z is a $\mathbb{k}_1$-scheme, there exists a unique $\mathbb{k}_1$-morphism $f : X_1 \to Z$ such that $f \circ pr_1 = \pi'$. Hence, we get a categorical quotient $\pi : X_{/\mathbb{k}_1} \to X_1$ in the category of $\mathbb{k}_1$-schemes (see Definition A.4.1).

Let $\gamma \in \Gamma$, and let $\gamma|_{\mathbb{k}_1}^* X_{/\mathbb{k}_1}$ be the $\mathbb{k}_1$-scheme with structural morphism $\mathrm{Spec}(\gamma|_{\mathbb{k}_1}) \circ \iota^\sharp \circ f_X$. Then the morphism of $\mathbb{k}_1$-schemes

$$\pi \circ \sigma_\gamma : \gamma|_{\mathbb{k}_1}^* X_{/\mathbb{k}_1} \to X_{/\mathbb{k}_1} \to X_1$$

is H-invariant (since H is a normal subgroup of Γ). By Definition A.4.1, there exists a unique morphism of $\mathbb{k}_1$-schemes ${\sigma_1}_\gamma : \gamma|_{\mathbb{k}_1}^* X_1 \to X_1$ satisfying $\pi \circ \sigma_\gamma = {\sigma_1}_\gamma \circ \pi$. Note that if $\gamma' \in \gamma H \in \Gamma/H \cong \mathrm{Gal}\,(\mathbb{k}_1/\mathbb{k})$, then ${\sigma_1}_\gamma = {\sigma_1}_{\gamma'}$ (since ${\sigma_1}_\gamma = id$ for all $\gamma \in H$). Therefore we obtain a $\mathbb{k}$-structure σ_1 on the $\mathbb{k}_1$-variety $X_1 = X/H$ and the following diagram is commutative for all $\gamma \in \Gamma$:

$$\begin{array}{ccccc}
X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk') & & \xrightarrow{\cong} & & X \\
& \searrow^{pr_1} & & \swarrow^{\pi} & \\
\sigma_{1\,\gamma|_{\Bbbk_1}} \times \mathrm{Spec}(\gamma) \downarrow & & X_1 & & \downarrow \sigma_\gamma \\
X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk') & & \downarrow \sigma_{1\,\gamma|_{\Bbbk_1}} & & X \\
& \searrow^{pr_1} & & \swarrow^{\pi} & \\
& & X_1 & &
\end{array}$$

If X is quasi-projective (resp. affine), then so is X_1 by Theorem 1.1.4. Furthermore, by Theorem 1.4.1, the categorical quotient $X_0 := X_1/\mathrm{Gal}\,(\Bbbk_1/\Bbbk)$ exists in the category of quasi-projective (resp. affine) $\Bbbk$-varieties and we have a $\mathrm{Gal}\,(\Bbbk_1/\Bbbk)$-equivariant isomorphism $(X_0)_{\Bbbk_1} \cong X_1$ where the $\Bbbk$-structure on X_1 is σ_1 and the $\Bbbk$-structure on $(X_0)_{\Bbbk_1}$ is given by $\gamma \in \mathrm{Gal}\,(\Bbbk_1/\Bbbk) \mapsto id \times \mathrm{Spec}(\gamma)$.
Finally, X_0 is a $\Bbbk$-form of X, and since the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc}
\Bbbk_1 \otimes_{\Bbbk_1} \Bbbk' & \xrightarrow{\cong} & \Bbbk' \\
\gamma|_{\Bbbk_1} \otimes \gamma \downarrow & & \downarrow \gamma \\
\Bbbk_1 \otimes_{\Bbbk_1} \Bbbk' & \xrightarrow{\cong} & \Bbbk'.
\end{array}$$

we obtain the following commutative diagram for all $\gamma \in \Gamma$

$$\begin{array}{ccccccc}
(X_0)_{\Bbbk'} & \xrightarrow{\cong} & X_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}\,(\Bbbk_1 \otimes_{\Bbbk_1} \Bbbk') & \xrightarrow{\cong} & (X_1)_{\Bbbk'} & \xrightarrow{\cong} & X \\
id \times \mathrm{Spec}(\gamma) \downarrow & & id \times \mathrm{Spec}\left(\gamma|_{\Bbbk_1} \otimes \gamma\right) \downarrow & & (\sigma_1)_{\gamma|_{\Bbbk_1}} \times \mathrm{Spec}(\gamma) \downarrow & & \downarrow \sigma_\gamma \\
(X_0)_{\Bbbk'} & \xrightarrow{\cong} & X_0 \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}\,(\Bbbk_1 \otimes_{\Bbbk_1} \Bbbk') & \xrightarrow{\cong} & (X_1)_{\Bbbk'} & \xrightarrow{\cong} & X.
\end{array}$$

□

Proposition 1.4.9. *If σ is a $\Bbbk$-structure on a quasi-projective $\Bbbk'$-variety X, then for all finite Galois extensions $\Bbbk_1/\Bbbk$ in $\Bbbk'$ there exists a $\Bbbk_1$-form X_1 of the $\Bbbk'$-variety X such that the restriction of σ to* $\mathrm{Gal}\,(\Bbbk'/\Bbbk_1)$ *coincides with the natural* $\mathrm{Gal}\,(\Bbbk'/\Bbbk_1)$*-action on $(X_1)_{\Bbbk'} \cong X$.*

Proof. Since σ is a $\Bbbk$-structure on X, there exists a finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$ and a $\Bbbk_1$-form X_1 of the $\Bbbk'$-variety X such that the restriction of σ to $H_1 := \mathrm{Gal}\,(\Bbbk'/\Bbbk_1)$ coincides with the natural H_1-action on $X \cong X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk')$. Let $\Bbbk_2/\Bbbk$ be another finite Galois extension in $\Bbbk'$. There exists a finite Galois extension $\Bbbk_3/\Bbbk$ that contains $\Bbbk_1$ and $\Bbbk_2$ [Sta, Lemmas 0EXM and 09DT], we have

$$\begin{array}{ccccc}
 & \Bbbk_1 & & & \\
\nearrow & & \searrow & & \\
\Bbbk & & & \Bbbk_3 & \longrightarrow \Bbbk'. \\
\searrow & & \nearrow & & \\
 & \Bbbk_2 & & &
\end{array}$$

.

Let $H_3 := \mathrm{Gal}\,(\Bbbk'/\Bbbk_3)$ and let $X_3 := X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk_3)$; it is a $\Bbbk_3$-form of X. From the H_1-equivariant isomorphism $(X_1)_{\Bbbk'} \cong X$, we obtain an H_3-equivariant isomorphism $(X_3)_{\Bbbk'} \cong X$ such that the following diagram commutes for all $\gamma \in H_3$

$$\begin{array}{ccccccc}
(X_3)_{\Bbbk'} & \xrightarrow{\cong} & X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk_3 \otimes_{\Bbbk_3} \Bbbk') & \xrightarrow{\cong} & (X_1)_{\Bbbk'} & \xrightarrow{\cong} & X \\
{\scriptstyle id \times \mathrm{Spec}(\gamma)} \downarrow & & {\scriptstyle id \times \mathrm{Spec}(id \otimes \gamma)} \downarrow & & {\scriptstyle id \times \mathrm{Spec}(\gamma)} \downarrow & & \downarrow {\scriptstyle \sigma|_{H_3 \gamma}} \\
(X_3)_{\Bbbk'} & \xrightarrow{\cong} & X_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\,(\Bbbk_3 \otimes_{\Bbbk_3} \Bbbk') & \xrightarrow{\cong} & (X_1)_{\Bbbk'} & \xrightarrow{\cong} & X.
\end{array}$$

By Proposition 1.4.8, there exists a $\Bbbk$-structure σ_3 on X_3 such that the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc}
X_3 \times_{\mathrm{Spec}(\Bbbk_3)} \mathrm{Spec}\,(\Bbbk') & \xrightarrow{\cong} & X \\
{\scriptstyle \sigma_{3\gamma|_{\Bbbk_3}} \times \mathrm{Spec}(\gamma)} \downarrow & & \downarrow {\scriptstyle \sigma_\gamma} \\
X_3 \times_{\mathrm{Spec}(\Bbbk_3)} \mathrm{Spec}\,(\Bbbk') & \xrightarrow{\cong} & X.
\end{array}$$

Since $\mathrm{Gal}\,(\Bbbk_3/\Bbbk_2)$ is a (normal) finite Galois subgroup of $\mathrm{Gal}\,(\Bbbk_3/\Bbbk)$, by Theorem 1.4.1, the categorical quotient $X_2 := X_3/\mathrm{Gal}\,(\Bbbk_3/\Bbbk_2)$ exists in the category of quasi-projective $\Bbbk_2$-varieties and we have a $\mathrm{Gal}\,(\Bbbk_3/\Bbbk_2)$-equivariant isomorphism $(X_2)_{\Bbbk_3} \cong X_3$ where the $\Bbbk_2$-structure on X_3 is $\sigma_3|_{\mathrm{Gal}(\Bbbk_3/\Bbbk_2)}$ and the $\Bbbk_2$-structure on $(X_2)_{\Bbbk_3}$ is given by $\gamma \in \mathrm{Gal}\,(\Bbbk_3/\Bbbk_2) \mapsto id \times \mathrm{Spec}(\gamma)$. By the proof of Proposition 1.4.8, we obtain the following commutative diagram for all $\gamma \in H_2 := \mathrm{Gal}\,(\Bbbk'/\Bbbk_2)$

$$\begin{array}{ccccccc}
(X_2)_{\Bbbk'} & \xrightarrow{\cong} & X_2 \times_{\mathrm{Spec}(\Bbbk_2)} \mathrm{Spec}\,(\Bbbk_3 \otimes_{\Bbbk_3} \Bbbk') & \xrightarrow{\cong} & (X_3)_{\Bbbk'} & \xrightarrow{\cong} & X \\
{\scriptstyle id \times \mathrm{Spec}(\gamma)} \downarrow & & {\scriptstyle id \times \mathrm{Spec}\,(\gamma|_{\Bbbk_3} \otimes \gamma)} \downarrow & & {\scriptstyle (\sigma_3)_{\gamma|_{\Bbbk_3}} \times \mathrm{Spec}(\gamma)} \downarrow & & \downarrow {\scriptstyle \sigma_\gamma} \\
(X_2)_{\Bbbk'} & \xrightarrow{\cong} & X_2 \times_{\mathrm{Spec}(\Bbbk_2)} \mathrm{Spec}\,(\Bbbk_3 \otimes_{\Bbbk_3} \Bbbk') & \xrightarrow{\cong} & (X_3)_{\Bbbk'} & \xrightarrow{\cong} & X.
\end{array}$$

Therefore, there exists a $\Bbbk_2$-form X_2 of the $\Bbbk'$-variety X such that the restriction of σ to $\mathrm{Gal}\,(\Bbbk'/\Bbbk_2)$ coincides with the natural $\mathrm{Gal}\,(\Bbbk'/\Bbbk_2)$-action on $(X_2)_{\Bbbk'} \cong X$. □

We have similar definitions and properties for algebraic groups, which are quasi-projective varieties.

We are now ready to make the connection between $\Bbbk$-structures and *continuous* semilinear Γ-actions on $\Bbbk'$-varieties covered by Γ-invariant affine open subsets (compare with Lemma 1.2.6).

Lemma 1.4.10. *Let $X = \mathrm{Spec}(A)$ be a $\Bbbk'$-variety and let σ be a semilinear Γ-action on X. The following assertions are equivalent.*

(i) The map

$$\begin{aligned} \Gamma \times A &\to A \\ (\gamma, a) &\mapsto \sigma_\gamma^\sharp(a) \end{aligned}$$

is continuous, where Γ is endowed with the Krull topology, and A is equipped with the discrete topology.

(ii) The map σ is a $\Bbbk$-structure. That is there exists a finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$ and a $\Bbbk_1$-form X_1 of the $\Bbbk'$-variety X such that the restriction of σ to $\mathrm{Gal}\,(\Bbbk'/\Bbbk_1)$ *coincides with the natural* $\mathrm{Gal}\,(\Bbbk'/\Bbbk_1)$*-action on $(X_1)_{\Bbbk'} \cong X$.*

Proof. • $(i) \Rightarrow (ii)$. Assume that the map $\Gamma \times A \to A$, $(\gamma, a) \mapsto \sigma^\sharp_\gamma(a)$ is continuous. Then, by Proposition 1.3.6 combined with Theorem 1.1.3, there exists $\Bbbk$-form X_0 of the $\Bbbk'$-variety X such that σ coincides with the natural Γ-action on $(X_0)_{\Bbbk'} \cong X$. (See also Lemma 1.2.6).

• $(ii) \Rightarrow (i)$. Let σ be a $\Bbbk$-structure on X. By Propositions 1.4.8 and 1.4.9, there exists a $\Bbbk$-form $X_0 = \mathrm{Spec}(A_0)$ of the $\Bbbk'$-variety X such that σ coincides with the natural Γ-action on $(X_0)_{\Bbbk'} \cong X$. Therefore the map $\Gamma \times A \to A$ is continuous (see the proof of Proposition 1.4.8) □

1.4.3 Infinite Galois extensions

Let $\Bbbk'/\Bbbk$ be a non-necessarily finite Galois extension of Galois group Γ. The next theorem is a direct consequence of Theorem 1.1.3 combined with Proposition 1.3.6. It is also a straightforward consequence of Proposition 1.4.9. In both cases, the next theorem is a consequence of the effectiveness of Galois descent mentioned in Theorem 1.1.3.

Theorem 1.4.11. *The functor $X_0 \mapsto ((X_0)_{\Bbbk'}, \sigma_0)$, where ${\sigma_0}_\gamma := id \times \mathrm{Spec}(\gamma)$ for all $\gamma \in \Gamma$, induces an equivalence of categories between the category of pairs (X, σ) consisting of a quasi-projective $\Bbbk'$-variety X endowed with a $\Bbbk$-structure σ, and the category of quasi-projective $\Bbbk$-varieties. Moreover, equivalent $\Bbbk$-structures on X correspond to isomorphic $\Bbbk$-forms of X.*

Using this equivalence, by abuse of notation we write (X, σ) to refer to a quasi-projective $\Bbbk$-variety. The inverse functor is denoted by $(X, \sigma) \mapsto X/\Gamma$.

As in Section 1.4.1, we get the next results.

Corollary 1.4.12. *The functor $G_0 \mapsto ((G_0)_{\Bbbk'}, \tau_0)$, where ${\tau_0}_\gamma := id \times \mathrm{Spec}(\gamma)$ for all $\gamma \in \Gamma$, induces an equivalence of categories between the category of pairs (G, τ) consisting of an algebraic $\Bbbk'$-group G endowed with a $\Bbbk$-group structure, and the category of algebraic $\Bbbk$-groups. Moreover, equivalent $\Bbbk$-group structures on G correspond to isomorphic $\Bbbk$-forms of G.*

Corollary 1.4.13. *Let G_0 be a $\Bbbk$-group. There is a one-to-one correspondence between quasi-projective $\Bbbk$-varieties endowed with a G_0-action and tuples $(G, \tau, X, \sigma, \mu)$ consisting of:*

(i) a $\Bbbk'$-group G endowed with a $\Bbbk$-group structure τ such that $G/\Gamma \cong G_0$;

(ii) a quasi-projective $\Bbbk'$-variety X endowed with an $\Bbbk$-structure σ;

(iii) an action $\mu : G \times X \to X$ such that the following diagram commutes for all $\gamma \in \Gamma$:

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \tau_\gamma \times \sigma_\gamma}\downarrow & & \downarrow{\scriptstyle \sigma_\gamma} \\ G \times X & \xrightarrow[\mu]{} & X \end{array}$$

As we will see below, Theorem 1.4.11 induces some results on the cohomology level. Let G be a discrete topological abstract group endowed with a continuous Γ-action (we will specify $G = \mathrm{Aut}(X)$, see the next lemma). A cocycle $c : \Gamma \to G$ is a *continuous* map such that for all $\gamma_1, \gamma_2 \in \Gamma$, $c_{\gamma_1\gamma_2} = c_{\gamma_1}(\gamma_1 \cdot c_{\gamma_2})$. Two cocycles c and c' are *equivalent* if there exists $g \in G$ such that for all $\gamma \in \Gamma$, $c'_\gamma = g c_\gamma(\gamma \cdot g)$. The set of cocycles modulo this equivalence relation is the *first pointed set of Galois cohomology*

$$\mathrm{H}^1_{cont}(\Gamma, G).$$

Lemma 1.4.14. *Let $\Bbbk'/\Bbbk$ be a Galois extension of Galois group Γ. Let X be $\Bbbk'$-variety and let σ be a $\Bbbk$-structure on X. The profinite Galois group Γ acts continuously on $\mathrm{Aut}(X)$ by*

$$\Gamma \times \mathrm{Aut}(X) \to \mathrm{Aut}(X)$$
$$(\gamma, \varphi) \mapsto \sigma_\gamma^{-1} \circ \varphi \circ \sigma_\gamma,$$

where $\mathrm{Aut}(X)$ is equipped with the discrete topology.

Proof. By [Ber10, Lemma II.3.3], it is equivalent to show that $\mathrm{Aut}(X) = \bigcup_i \mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_i)}$, where the union is taken over all finite Galois extensions $\Bbbk_i/\Bbbk$ in $\Bbbk'$. We have to show that $\mathrm{Aut}(X) \subset \bigcup_i \mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_i)}$. By definition of a $\Bbbk$-structure, there exists a finite Galois extension $\Bbbk_1/\Bbbk$ in $\Bbbk'$, and a $\Bbbk_1$-variety X_1 such that there is a $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$-isomorphism $(X_1)_{\Bbbk'} \cong X$. Let $X_1 = \bigcup_j U_j$ be a finite affine open cover of X_1. Then $\bigcup_j (U_j)_{\Bbbk'}$ is a $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$-invariant[3] affine open cover of $(X_1)_{\Bbbk'} \cong X$ (see [Sta, Lemma 01JY]). Let $\varphi \in \mathrm{Aut}((X_1)_{\Bbbk'}) \cong \mathrm{Aut}(X)$, then $\bigcup_j \varphi((U_j)_{\Bbbk'})$ is an affine open cover of $(X_1)_{\Bbbk'} \cong X$. Since varieties are finite type schemes, there exists a finite Galois extension $\Bbbk_2/\Bbbk$ containing $\Bbbk_1$ in $\Bbbk'$, and $\Bbbk_2$-algebras B_j such that, for all j, $\mathrm{Spec}(B_j \otimes_{\Bbbk_2} \Bbbk') \cong \varphi((U_j)_{\Bbbk'})$. Therefore, by Lemmas 1.4.8 and 1.4.9, we may assume that $\Bbbk_1 = \Bbbk_2$. We fix some j. We write $U_j = \mathrm{Spec}(A_j)$ and $V_j = \mathrm{Spec}(B_j)$, both are affine $\Bbbk_1$-varieties. Let $B_j := \Bbbk_1[f_1, \ldots, f_d]$. Then, for all $k \in \{1, \ldots, d\}$, there exist $z_{k,l} \in \Bbbk'$ and $a_{k,l} \in A_j$ (depending on j) such that

$$\begin{aligned} \varphi|_{U'_j}^{\sharp} : B_j \otimes_{\Bbbk_1} \Bbbk' &\to A_j \otimes_{\Bbbk_1} \Bbbk' \\ f_k \otimes 1 &\mapsto \sum_l a_{k,l} \otimes z_{k,l}. \end{aligned}$$

Let $\Bbbk_3/\Bbbk$ be a finite Galois extension in $\Bbbk'$ containing $\Bbbk_1$ and all of the z_k^l. As before, we may assume that $\Bbbk_1 = \Bbbk_3$. Then, there exist $\Bbbk_1$-isomorphisms ψ_j such that

$$\begin{aligned} \varphi|_{U'_j}^{\sharp} : B_j \otimes_{\Bbbk_1} \Bbbk' &\to A_j \otimes_{\Bbbk_1} \Bbbk' \\ b \otimes z &\mapsto \psi_j(b) \otimes z. \end{aligned}$$

Since $\psi_j(B_j) = A_j$, it follows that $\bigcup_j V_j$ is an affine open cover of X_1 such that $\varphi((U_j)_{\Bbbk'}) = (V_j)_{\Bbbk'}$ for all j. Hence, $\varphi \in \mathrm{Aut}((X_1)_{\Bbbk'})^{\mathrm{Gal}(\Bbbk'/\Bbbk_1)} \cong \mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_1)}$. ☐

The next result is a corollary of Theorem 1.4.11.

Corollary 1.4.15. *Let X be a quasi-projective $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ. We have an isomorphism of pointed set*

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}(X)\right) \cong \mathrm{colim}_i\ \mathrm{H}^1\left(\mathrm{Gal}(\Bbbk_i/\Bbbk), \mathrm{Aut}(X/\mathrm{Gal}(\Bbbk'/\Bbbk_i))\right),$$

where the colimit *is taken over all finite Galois extensions $\Bbbk_i/\Bbbk$ in $\Bbbk'$.*

Proof. Let $\Bbbk_i/\Bbbk$ be a Galois extension in $\Bbbk'$. Then, by Theorem 1.4.11,

$$\mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_i)} \cong \mathrm{Aut}(X/\mathrm{Gal}(\Bbbk'/\Bbbk_i)).$$

Since the Γ-action on $\mathrm{Aut}(X)$ is continuous (see [Ber10, Example II.3.2 (5)]) , by Lemma 1.2.6, we get

$$\mathrm{Aut}(X) = \bigcup_i \mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_i)} \cong \mathrm{colim}_i \mathrm{Aut}(X)^{\mathrm{Gal}(\Bbbk'/\Bbbk_i)},$$

where the colimit is taken over all finite Galois extensions $\Bbbk_i/\Bbbk$ in $\Bbbk'$. Moreover, by the Galois correspondence (see Theorem B.3.1),

$$\Gamma/\mathrm{Gal}(\Bbbk'/\Bbbk_i) \cong \mathrm{Gal}(\Bbbk_i/\Bbbk).$$

Finally, by [Ser97, §I.2.2, Proposition 8] (see also [Ber10, Theorem II.3.33]), we get

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}(X)\right) \cong \mathrm{colim}_i\ \mathrm{H}^1\left(\mathrm{Gal}(\Bbbk_i/\Bbbk), \mathrm{Aut}(X/\mathrm{Gal}(\Bbbk'/\Bbbk_i))\right).$$

☐

[3] An affine open subset U of X is $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$-invariant if for all $\gamma \in \mathrm{Gal}(\Bbbk'/\Bbbk_1)$, $\sigma_\gamma(U) = U$.

From Corollary 1.4.15, we get the next result, which generalizes Proposition 1.4.5.

Proposition 1.4.16 ([Ser97, §III.1.3, Proposition 5], also [GW20, Theorem 14.91])**.** *Let $\Bbbk'/\Bbbk$ be a Galois extension of Galois group Γ. Let X be a quasi-projective $\Bbbk'$-variety equipped with a $\Bbbk$-structure σ. There is a bijection*

$$\begin{aligned} \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}(X)) &\simeq \{\textit{equivalence classes of } \Bbbk\textit{-structures on } X\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \sigma_\gamma) \end{aligned}$$

that sends the trivial cocycle to the equivalence class of σ.Furthermore, there is a bijection

$$\begin{aligned} \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}(X)) &\simeq \{\textit{isomorphism classes of } \Bbbk\textit{-forms of } X\} \\ (\gamma \mapsto c_\gamma) &\mapsto (X, (\gamma \mapsto c_\gamma \circ \sigma_\gamma)) . \end{aligned}$$

Proof. Let $\Bbbk_i/\Bbbk$ be a finite Galois extension in $\Bbbk'$. By Propositions 1.4.8, 1.4.9 and Theorem 1.4.11, there exists a $\Bbbk_i$-form $X_i \cong X/\mathrm{Gal}(\Bbbk'/\Bbbk_i)$ of X and a $\Bbbk$-structure σ_i on X_i such that $X/\Gamma \cong X_i/\mathrm{Gal}(\Bbbk_i/\Bbbk)$. Furthermore, by Proposition 1.4.5, there is a bijection

$$\mathrm{H}^1(\mathrm{Gal}(\Bbbk_i/\Bbbk), \mathrm{Aut}(X_i)) \simeq \{\text{isomorphism classes of } \Bbbk\text{-forms of } X_i\}.$$

Therefore, we get an isomorphism

$$\mathrm{colim}_i\ \mathrm{H}^1(\mathrm{Gal}(\Bbbk_i/\Bbbk), \mathrm{Aut}(X_i)) \simeq \{\text{isomorphism classes of } \Bbbk\text{-forms of } X\}.$$

From Corollary 1.4.15 combined with Theorem 1.4.11, we get the desired result. □

Proposition 1.4.17 ([Ser97, §III.1.3, Proposition 5])**.** *Let G be a $\Bbbk'$-group equipped with a $\Bbbk$-group structure τ. There is a bijection*

$$\begin{aligned} \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}_{gr}(G)) &\simeq \{\textit{equivalence classes of } \Bbbk\textit{-group structures on } G\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \tau_\gamma) \end{aligned}$$

that sends the trivial cocycle to the equivalence class of τ. Furthermore, there is a bijection

$$\begin{aligned} \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}_{gr}(G)) &\simeq \{\textit{isomorphism classes of } \Bbbk\textit{-forms, as group, of } G\} \\ (\gamma \mapsto c_\gamma) &\mapsto (G, (\gamma \mapsto c_\gamma \circ \tau_\gamma)) . \end{aligned}$$

Proposition 1.4.18 ([Wed18, Corollary 10.1])**.** *Let (G, τ) be a $\Bbbk$-torus. Let (X, σ) be a quasi-projective (G, τ)-variety. There is a bijection*

$$\begin{aligned} \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^G(X)\right) &\simeq \{\textit{equivalence classes of } (G,\tau)\textit{-equivariant } \Bbbk\textit{-structures on } X\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\gamma \mapsto c_\gamma \circ \sigma_\gamma) . \end{aligned}$$

Furthermore, there is a bijection

$$\begin{aligned} \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^G(X)\right) &\simeq \{\textit{isomorphism classes of } \Bbbk\textit{-forms, as } G/\Gamma\textit{-variety, of } X\} \\ (\gamma \mapsto c_\gamma) &\mapsto (X, (\gamma \mapsto c_\gamma \circ \sigma_\gamma)) . \end{aligned}$$

Chapter 2

Toric varieties

The theory of toric varieties gives explicit relations between algebraic geometry and combinatorial geometry, which explains the prominent role of these varieties in various domains of mathematics. According to Fulton's preface of [Ful93], "toric varieties provide a [...] way to see many examples and phenomena in algebraic geometry", and they "have provided a remarkably fertile testing ground for general theories". Classical references for toric varieties are over algebraically closed fields of characteristic zero (see [Ful93], [KKMSD73], [Oda88], [Ewa96], [CLS11]), but the main results of this theory works for split toric varieties over non-algebraically closed fields (see [Dem70] for the smooth case, which is the starting point of toric geometry). There is also a notion of non-split toric varieties, see for instance [Hur11], [ELFST14] and [Dun16]. Namely, a toric $\mathbb{k}$-variety, where $\mathbb{k}$ is a field, is a normal $\mathbb{k}$-variety X such that there exists a $\mathbb{k}$-torus acting effectively on X with a dense open orbit. In this chapter, we first study $\mathbb{k}$-tori, then $\mathbb{k}$-tori actions, and finally toric $\mathbb{k}$-varieties.

Over an algebraically closed field, a $\mathbb{k}$-torus of dimension n is the algebraic group $\mathbb{G}_{m,\mathbb{k}}^n$. Let $\mathbb{k}$ be an arbitrary field. A $\mathbb{k}$-torus T is an affine algebraic $\mathbb{k}$-group such that $T_{\mathbb{k}_s} \cong \mathbb{G}_{m,\mathbb{k}_s}^n$, for some $n \in \mathbb{N}^*$, where $\mathbb{k}_s$ denotes the separable closure of $\mathbb{k}$ in $\overline{\mathbb{k}}$. Since we will (mainly) work over perfect fields (more exactly over characteristic zero fields), $\mathbb{k}_s = \overline{\mathbb{k}}$. Over $\overline{\mathbb{k}}$, all tori are *split*, i.e. isomorphic to $\mathbb{G}_{m,\overline{\mathbb{k}}}^n$ for some $n \in \mathbb{N}^*$, but over $\mathbb{k}$ a torus may be non-split. A well known non-split $\mathbb{R}$-torus is the circle

$$\mathbb{S}^1 := \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1)).$$

The theory of split toric $\mathbb{k}$-varieties over a field $\mathbb{k}$, that is, toric varieties with a split $\mathbb{k}$-torus action, is similar to the one over an algebraically closed field of characteristic zero. If the acting $\mathbb{k}$-torus T is non-split, then it is more convenient to use Galois descent tools as follows. Let X be a T-toric $\mathbb{k}$-variety, then $X_{\overline{\mathbb{k}}}$ is a split toric $\overline{\mathbb{k}}$-variety, and we get canonical $\mathbb{k}$-structures σ on $X_{\overline{\mathbb{k}}}$ and τ on $T_{\overline{\mathbb{k}}}$. Therefore, the toric variety X corresponds to the split toric variety $X_{\overline{\mathbb{k}}}$ equipped with additional Galois descent data (see Corollary 1.4.13).

The dense open orbit of a T-toric $\mathbb{k}$-variety is a T-torsor, that is a $\mathbb{k}$-variety V endowed with a T-action μ such that $T_{\overline{\mathbb{k}}} \cong V_{\overline{\mathbb{k}}}$, and such that $\mu_{\overline{\mathbb{k}}} : T_{\overline{\mathbb{k}}} \times V_{\overline{\mathbb{k}}} \to V_{\overline{\mathbb{k}}}$ corresponds to the natural action by translation on $T_{\overline{\mathbb{k}}}$. We will see in Proposition 2.4.9 that $\mathbb{k}$-forms of a $\mathbb{T}$-toric $\overline{\mathbb{k}}$-variety in the category of toric varieties are parametrized by a Galois cohomology set that classifies torsors. Finally, in Section 2.5, we will give a classification of smooth Del Pezzo $\mathbb{R}$-surfaces, which are toric $\mathbb{R}$-varieties.

As mentioned in the introduction, this chapter is a *warm-up* to the Altmann-Hausen theory. There is no new result; we give a compilation of known results.

2.1 Preliminaries in convex geometry

This section is based on [AH06, §1] and on [Gil22a, §1]. From here on, N denotes a lattice, i.e. a finitely generated free abelian group, and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denotes its dual lattice. The associated $\mathbb{Q}$-vector spaces are denoted by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ respectively, and the the corresponding pairing by:

$$M \times N \to \mathbb{Z}, \quad (u, v) \mapsto \langle u|v\rangle := u(v).$$

Let us recall some results of [Ful93, §1.2]. Let N' be a lattice, and let $f : N \to N'$ be a lattice homomorphism. It induces a unique $\mathbb{Q}$-linear map $N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$, also denoted by f. A subset $\omega_N \subset N_{\mathbb{Q}}$ is called a *convex polyhedral cone* if there exists a finite set $S \subset N_{\mathbb{Q}}$ such that

$$\omega_N = \mathrm{Cone}(S) := \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \in \mathbb{Q}_{\geq 0} \right\} \subset N_{\mathbb{Q}}.$$

A cone ω_N is *strongly convex*, or *pointed*, if $\omega_N \cap (-\omega_N) = \{0\}$. For us, a *cone* in $N_{\mathbb{Q}}$ is always a convex polyhedral cone. The dual cone of ω_N is defined by

$$\omega_N^{\vee} := \{u \in M_{\mathbb{Q}} \mid \forall v \in \omega_N,\ \langle u|v\rangle \geq 0\};$$

it is a cone in $M_{\mathbb{Q}}$. Furthermore, the dual of a pointed cone is a full dimensional cone. Let ω_N be a cone in $N_{\mathbb{Q}}$. A *face* τ_N of ω_N is given by $\tau_N = \omega_N \cap u^{\perp}$, for some $u \in \omega_N^{\vee}$, where $u^{\perp} := \{v \in \omega_N \mid \forall u \in \omega_N^{\vee},\ \langle u|v\rangle = 0\}$. Recall that a face of a cone is a cone. The relative interior $\mathrm{Relint}(\omega_N)$ of a cone ω_N is obtained by removing all proper faces from ω_N.

A *quasi-fan* Λ in $N_{\mathbb{Q}}$ (or in $M_{\mathbb{Q}}$) is a finite collection of cones in $N_{\mathbb{Q}}$ (or in $M_{\mathbb{Q}}$) such that, for any $\lambda \in \Lambda$, all the faces of λ belong to Λ, and for any $\lambda_1, \lambda_2 \in \Lambda$, the intersection $\lambda_1 \cap \lambda_2$ is a face of both λ_i. The *support* of a quasi-fan is the union of all its cones. A quasi-fan is called a *fan* if all its cones are strongly convex.

2.2 Tori

2.2.1 Definitions and first properties

In this section, $\Bbbk$ is an arbitrary field and $\Bbbk'$ is a non-necessarily finite Galois extension with profinite Galois group Γ. See for instance [Man86, §30.3.3] for a survey of the theory of tori and of torsors.

Let $n \in \mathbb{N}^*$, and let M be a rank n lattice. Then,

$$\Bbbk[M] := \left\{ \sum_{\text{finite}} a_m \chi^m \;\middle|\; a_m \in \Bbbk, m \in M \right\},$$

is a Hopf $\Bbbk$-algebra, where χ^m are indeterminates satisfying $\chi^{m+m'} = \chi^m \chi^{m'}$ for all $m, m' \in M$. The coproduct, the counity, and the antipode are respectively:

$$\begin{aligned} \Bbbk[M] &\to \Bbbk[M] \otimes_{\Bbbk} \Bbbk[M] \\ \chi^m &\mapsto \chi^m \otimes \chi^m \end{aligned} \qquad \begin{aligned} \Bbbk[M] &\to \Bbbk \\ a\chi^m &\mapsto a \end{aligned} \qquad \begin{aligned} \Bbbk[M] &\to \Bbbk[M] \\ \chi^m &\mapsto \chi^{-m}. \end{aligned}$$

Therefore, $\mathbb{T} := \mathrm{Spec}(\Bbbk[M])$ is an affine algebraic $\Bbbk$-group (see Section A.2). In particular, if $M = \mathbb{Z}^n$, then we define

$$\mathbb{G}^n_{m,\Bbbk} := \mathrm{Spec}(\Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}]),$$

where $t_1 := \chi^{(1,0,\dots,0)}$, ..., $t_n := \chi^{(0,\dots,0,1)}$. This affine algebraic $\mathbb{k}$-group can be defined as a representable functor. Namely, the functor from the category of $\mathbb{k}$-algebras to the category of abstract groups

$$\begin{aligned} \mathbb{G}_{m,\mathbb{k}}^n : \mathrm{Alg}_{\mathbb{k}} &\to \mathrm{Grp} \\ R &\mapsto (R^*)^n \end{aligned}$$

is representable by the affine algebraic group $\mathrm{Spec}(\mathbb{k}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$.

Fix a $\mathbb{Z}$-basis $\{e_1, \dots, e_n\}$ of M. We obtain an isomorphism $M \cong \mathbb{Z}^n$, and an isomorphism $\mathbb{k}[M] \cong \mathbb{k}[\mathbb{Z}^n]$. Hence, $\mathbb{T}$ is isomorphic to $\mathbb{G}_{m,\mathbb{k}}^n$. Conversely, if $\mathbb{T}$ is an affine algebraic $\mathbb{k}$-group isomorphic to $\mathbb{G}_{m,\mathbb{k}}^n$, for some $n \in \mathbb{N}^*$, then there exists a lattice M, the character lattice of $\mathbb{T}$, such that $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$.

Definition 2.2.1. A $\mathbb{k}$-*torus* T is an affine algebraic $\mathbb{k}$-group such that $T_{\mathbb{k}_s} \cong \mathbb{G}_{m,\mathbb{k}_s}^n$ for some integer n, where $\mathbb{k}_s$ is the separable closure of $\mathbb{k}$ in $\overline{\mathbb{k}}$. It is called a *split* $\mathbb{k}$-*torus* if $T \cong \mathbb{G}_{m,\mathbb{k}}^n$ for some integer n.

Example 2.2.2. The $\mathbb{R}$-groups of Examples 1.2.9 and 1.2.10 are $\mathbb{R}$-tori.

Since a $\mathbb{k}$-torus is finitely generated, we get the next result.

Proposition 2.2.3 ([Ono61, Proposition 1.2.1] [Sta, Lemmas 0EXM and 09DT])**.** *For any* $\mathbb{k}$-*torus* T*, there exists a finite Galois extension* $\mathbb{k}_1/\mathbb{k}$ *in* $\overline{\mathbb{k}}$ *such that* $T_{\mathbb{k}_1}$ *is a split torus.*

2.2.2 Tori and Galois cohomology

The next proposition, that classifies n-dimensional $\mathbb{k}$-tori, is a direct consequence of Proposition 1.4.17.

Proposition 2.2.4. *Let* $\mathbb{k}'/\mathbb{k}$ *be a Galois extension of Galois group* Γ*, and let* $n \in \mathbb{N}^*$*. There is a bijection*

$$\begin{aligned} \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}_{gr}\left(\mathbb{G}_{m,\mathbb{k}'}^n\right)\right) &\simeq \{\textit{isomorphism classes of } n\textit{-dimensional } \mathbb{k}\textit{-tori that splits over } \mathbb{k}'\} \\ (\gamma \mapsto c_\gamma) &\mapsto \left(\mathbb{G}_{m,\mathbb{k}'}^n, (\gamma \mapsto c_\gamma \circ \tau_\gamma)\right) \end{aligned}$$

that sends the trivial cocycle $\gamma \mapsto id$ *to a split* $\mathbb{k}$-*torus. Recall that the* Γ-*action on* $\mathrm{Aut}_{gr}(\mathbb{G}_{m,\mathbb{k}'}^n)$ *is defined by* $(\gamma, \varphi) \mapsto \sigma_\gamma^{-1} \circ \varphi \circ \sigma_\gamma$.

Based on the next lemma, we get in Proposition 2.2.6 an exact sequence of Galois cohomology sets that is used to relate $\mathbb{k}$-forms of a split $\mathbb{k}'$-torus $\mathbb{T}$ in the category of varieties to the $\mathbb{k}$-forms of $\mathbb{T}$ in the category of groups, and to the $\mathbb{k}$-forms of $\mathbb{T}$ in the category of $\mathbb{T}$-varieties (i.e $\mathbb{T}$-torsors, see Proposition 2.3.11).

Lemma 2.2.5. *Let* $\mathbb{k}$ *be a field. Let* $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$ *be a split* $\mathbb{k}$-*torus. Then,*

$$\mathrm{Aut}_{gr}(\mathbb{T}) \cong \mathrm{GL}(M) \quad \textit{and} \quad \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \cong \mathbb{T}(\mathbb{k})$$

Furthermore, the group $\mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$ *is a normal subgroup of* $\mathrm{Aut}(\mathbb{T})$*, and we have a split short exact sequence of groups*

$$1 \longrightarrow \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \longrightarrow \mathrm{Aut}(\mathbb{T}) \longrightarrow \mathrm{Aut}_{gr}(\mathbb{T}) \longrightarrow 1.$$

In other words, we have a semi-direct product $\mathrm{Aut}(\mathbb{T}) \cong \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \rtimes \mathrm{Aut}_{gr}(\mathbb{T})$.

Proof. Let $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$ be a split $\Bbbk$-torus. Let $f \in \mathrm{Aut}(\mathbb{T})$. Since $f^\sharp : \Bbbk[M] \to \Bbbk[M]$ is a $\Bbbk$-algebra automorphism, one can show that there exists $\tilde{f} \in \mathrm{GL}(M)$, and a group morphism $h : M \to \Bbbk^*$ such that $f^\sharp(\chi^m) = h(\tilde{f}(m))\chi^{\tilde{f}(m)}$ for all $m \in M$. Therefore, we get a surjective map

$$\begin{aligned} \mathrm{Aut}(\mathbb{T}) &\to \mathrm{GL}(M) \\ f &\mapsto \tilde{f} \end{aligned}$$

such that $\widetilde{f_1 f_2} = \tilde{f}_2 \tilde{f}_1$.

• We show that $\mathrm{Aut}_{gr}(\mathbb{T}) \cong \mathrm{GL}(M)$. Let $f \in \mathrm{Aut}_{gr}(\mathbb{T}) \subset \mathrm{Aut}(\mathbb{T})$. There exists $\tilde{f} \in \mathrm{GL}(M)$, and a group morphism $h : M \to \Bbbk^*$ such that $f^\sharp(\chi^m) = h(\tilde{f}(m))\chi^{\tilde{f}(m)}$ for all $m \in M$. Since $f^\sharp : \Bbbk[M] \to \Bbbk[M]$ is a Hopf $\Bbbk$-algebra automorphism, then $h(m) = 1$ for all $m \in M$. Therefore, we get a bijective map

$$\begin{aligned} \mathrm{Aut}_{gr}(\mathbb{T}) &\to \mathrm{GL}(M) \\ f &\mapsto \tilde{f} \end{aligned}$$

such that $\widetilde{f_1 f_2} = \tilde{f}_2 \tilde{f}_1$.

• We show that $\mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \cong \mathbb{T}(\Bbbk)$. Let $f \in \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \subset \mathrm{Aut}(\mathbb{T})$. There exists $\tilde{f} \in \mathrm{GL}(M)$, and a group morphism $h : M \to \Bbbk^*$ such that $f^\sharp(\chi^m) = h(\tilde{f}(m))\chi^{\tilde{f}(m)}$ for all $m \in M$. Then, using the definition of being $\mathbb{T}$-equivariant, $\tilde{f} = id$. Therefore, we get a group isomorphism

$$\mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \to \mathrm{Hom}_{gr}(M, \Bbbk^*) \cong \mathbb{T}(\Bbbk).$$

• We show that that $\mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$ is a normal subgroup of $\mathrm{Aut}(\mathbb{T})$. Let $f_1 \in \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$. There exists a group morphism $h_1 : M \to \Bbbk^*$ such that $f_1^\sharp(\chi^m) = h_1(m)\chi^m$ for all $m \in M$. Let $f_2 \in \mathrm{Aut}(\mathbb{T})$. There exists $\tilde{f}_2 \in \mathrm{GL}(M)$, and a group morphism $h_2 : M \to \Bbbk^*$ such that $f_2^\sharp(\chi^m) = h_2(\tilde{f}_2(m))\chi^{\tilde{f}_2(m)}$ for all $m \in M$. Then,

$$f_2^\sharp \circ f_1^\sharp \circ (f_2^{-1})^\sharp : \chi^m \mapsto (h_1 \circ \tilde{f}_2^{\,-1})(m)\chi^m.$$

Therefore, $f_2^{-1} \circ f_1 \circ f_2 \in \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$.

• Finally, we get an obvious short exact sequence of groups

$$1 \longrightarrow \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \longrightarrow \mathrm{Aut}(\mathbb{T}) \longrightarrow \mathrm{Aut}_{gr}(\mathbb{T}) \longrightarrow 1$$

admitting a section. □

Proposition 2.2.6. *Let $\Bbbk'/\Bbbk$ be a Galois extension. Let $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$ be a split $\Bbbk'$-torus and let τ be a $\Bbbk$-group structure on $\mathbb{T}$. Then we have the following exact sequence of pointed sets (see [Ber10, Definition II.3.13])*

$$1 \to \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})^\Gamma \to \mathrm{Aut}(\mathbb{T})^\Gamma \to \mathrm{Aut}_{gr}(\mathbb{T})^\Gamma \to \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right) \to \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}(\mathbb{T})) \to \mathrm{H}^1_{cont}(\Gamma, \mathrm{Aut}_{gr}(\mathbb{T})),$$

where the Γ-action on $\mathrm{Aut}(\mathbb{T})$ is given by $\gamma \cdot \varphi = \tau_\gamma^{-1} \circ \varphi \circ \tau_\gamma$ for all $\gamma \in \Gamma$ and $\varphi \in \mathrm{Aut}(\mathbb{T})$.

Proof. By Lemma 2.2.5, the group $\mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$ is a normal subgroup of $\mathrm{Aut}(\mathbb{T})$, and we have a short exact sequence of Γ-groups (see [Ber10, Definition II.3.1])

$$1 \longrightarrow \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}) \longrightarrow \mathrm{Aut}(\mathbb{T}) \longrightarrow \mathrm{Aut}_{gr}(\mathbb{T}) \longrightarrow 1.$$

Then, by [BS64, 1.17 Proposition] (or [Ber10, Proposition II.4.7]), we obtain the desired result. □

2.2.3 Tori and Γ-representations

Let $\Bbbk'/\Bbbk$ be a Galois extension. If σ is a $\Bbbk$-structure on $\mathbb{T} := \mathrm{Spec}(\Bbbk'[M])$, then there exists a map $h : \Gamma \to \mathrm{Hom}_{gr}(M, (\Bbbk')^*)$ (a cocycle), and a group morphism $\tilde{\sigma} : \Gamma \to \mathrm{GL}(M)$ such that

$$\sigma_\gamma^\sharp : \chi^m \mapsto h_\gamma(\tilde{\sigma}_\gamma(m))\chi^{\tilde{\sigma}_\gamma(m)}$$

for all $\gamma \in \Gamma$ (see the proof of Lemma 2.2.5, and see Section 2.3.3).

Remark 2.2.7 (Γ-action on the character lattice)**.** Let τ be a $\Bbbk$-group structure on $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$. Then τ induces a Γ-representation $\tilde{\tau} : \Gamma \to \mathrm{GL}(M)$, and a dual Γ-representation $\hat{\tau} : \Gamma \to \mathrm{GL}(N)$, such that $\tilde{\tau}_{\gamma_1\gamma_2} = \tilde{\tau}_{\gamma_1} \circ \tilde{\tau}_{\gamma_2}$ and $\hat{\tau}_{\gamma_1\gamma_2} = \hat{\tau}_{\gamma_2} \circ \hat{\tau}_{\gamma_1}$ for all $\gamma_1, \gamma_2 \in \Gamma$. Indeed, based on Lemma 2.2.5, we can write

$$\begin{aligned} \tau^\sharp : \Gamma &\to \mathrm{Aut}_\Bbbk\left(\Bbbk'[M]\right) \\ \gamma &\mapsto \left(a_m\chi^m \mapsto \gamma(a_m)\chi^{\tilde{\tau}_\gamma(m)}\right). \end{aligned}$$

Two Γ-representations ρ and ρ' on $\mathrm{GL}_n(\mathbb{Z})$ are equivalent if there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $\rho'(\gamma) = P \circ \rho(\gamma) \circ P^{-1}$ for all $\gamma \in \Gamma$. Then, we get the next result.

Lemma 2.2.8. *There is a one-to-one correspondence between $\Bbbk$-group structures on $\mathbb{G}^n_{m,\Bbbk'}$ and Γ-representations in $\mathrm{GL}_n(\mathbb{Z})$. Furthermore, equivalent classes of $\Bbbk$-group structures on $\mathbb{G}^n_{m,\Bbbk'}$ correspond to equivalent classes of Γ-representations in $\mathrm{GL}_n(\mathbb{Z})$.*

Example 2.2.9. The split $\Bbbk$-torus $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$ corresponds to $(\mathbb{T}_{\Bbbk'}, \tau_0)$, where τ_0 is the $\Bbbk$-group structure on $\mathbb{T}_{\Bbbk'} = \mathrm{Spec}\left(\Bbbk'[M]\right)$ such that

$$\begin{aligned} \tau_0^\sharp : \Gamma &\to \mathrm{Aut}_\Bbbk\left(\Bbbk'[M]\right) \\ \gamma &\mapsto \left(a_m\chi^m \mapsto \gamma(a_m)\chi^m\right). \end{aligned}$$

Therefore, the Γ-action on M is trivial.

Remark 2.2.10 (Factorization by a finite Galois extension that splits the $\Bbbk$-torus)**.** We pursue Remark 2.2.7. Let $\Bbbk_1/\Bbbk$ be a finite Galois extension that splits $(\mathbb{T}, \tau)$ and let $H := \mathrm{Gal}(\Bbbk'/\Bbbk_1)$ be the absolute Galois group of $\Bbbk'/\Bbbk_1$. By Propositions 1.4.8 and 1.4.9, there exists $\Bbbk_1$-form $\mathbb{T}_1$ of $\mathbb{T}$ and a $\Bbbk$-structure τ_1 on $\mathbb{T}_1$ such that the following diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} T_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\left(\Bbbk'\right) & \xrightarrow{\cong} & \mathbb{T} \\ {\scriptstyle \tau_{1\gamma|_{\Bbbk_1}} \times \mathrm{Spec}(\gamma)}\downarrow & & \downarrow{\scriptstyle \tau_\gamma} \\ T_1 \times_{\mathrm{Spec}(\Bbbk_1)} \mathrm{Spec}\left(\Bbbk'\right) & \xrightarrow{\cong} & \mathbb{T} \end{array}$$

Since $\Bbbk_1$ is a splitting field of $(\mathbb{T}, \tau)$, we can assume $\mathbb{T}_1 = \mathrm{Spec}\left(\Bbbk_1[M]\right)$. Therefore, the H-action on M is trivial, and the Γ-action on M factorizes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\tau}} & \mathrm{GL}(M) \\ \downarrow & \nearrow_{\tilde{\tau}_1} & \\ \Gamma/H \cong \mathrm{Gal}\left(\Bbbk_1/\Bbbk\right) & & \end{array}$$

This means that the profinite group Γ acts on M as the finite Galois group $\mathrm{Gal}(\Bbbk_1/\Bbbk)$, and this action comes from the $\mathrm{Gal}(\Bbbk_1/\Bbbk)$-action on $\mathbb{T}_1$. Furthermore, since the kernel of $\hat{\tau}_1$ is a normal subgroup of $\mathrm{Gal}(\Bbbk_1/\Bbbk)$, by the Galois correspondence (see Theorem B.3.1), there exists a finite Galois extension $\Bbbk_2/\Bbbk$ in $\Bbbk_1$ such that the induced map $\mathrm{Gal}(\Bbbk_2/\Bbbk) \cong \mathrm{Gal}(\Bbbk_1/\Bbbk)/\mathrm{Ker}(\hat{\tau}_1) \hookrightarrow \mathrm{GL}(M)$ is injective. Therefore, the corresponding $\Bbbk_2$-torus $\mathbb{T}_2$ is split and endowed with a $\Bbbk$-group structure τ_2 as in Proposition 1.4.9. The field $\Bbbk_2$ is called a *splitting field* of the $\Bbbk$-torus $(\mathbb{T}, \tau)$.

An inclusion of $\Bbbk$-tori corresponds to Γ-equivariant exact sequences of lattices.

Remark 2.2.11 (Inclusions of tori and character lattices)**.** Let $(\mathbb{T} = \mathrm{Spec}(\Bbbk'[M]), \tau)$ be a subtorus of the $\Bbbk$-torus $(\mathbb{G}^n_{m,\Bbbk'} = \mathrm{Spec}(\Bbbk'[M']), \tau')$. The inclusion $\mathbb{T} \hookrightarrow \mathbb{G}^n_{m,\Bbbk'}$ induces a surjective lattice homomorphism $M' \to M$. Let M_Y be the kernel of this homomorphism, it is a sublattice of M'. Moreover, the Γ-action $\tilde{\tau}'$ on M' induces a Γ-action $\tilde{\tau}_Y$ on M_Y. Let τ_Y be the induced $\Bbbk$-group structure on $\mathbb{T}_Y = \mathrm{Spec}(\Bbbk'[M_Y])$. The following diagram of complex algebraic groups commutes for all $\gamma \in \Gamma$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{G}^n_{m,\Bbbk'} & \longrightarrow & \mathbb{T}_Y & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tau_\gamma} & & \downarrow{\scriptstyle \tau'_\gamma} & & \downarrow{\scriptstyle \tau_{Y\gamma}} & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{G}^n_{m,\Bbbk'} & \longrightarrow & \mathbb{T}_Y & \longrightarrow & 1.
\end{array}$$

There exists an injective morphism $F : N \to N'$ and a surjective homomorphism $P : N' \to N_Y$, and the following diagrams of free $\mathbb{Z}$-modules commute for all $\gamma \in \Gamma$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\tau}_\gamma} & & \downarrow{\scriptstyle \hat{\tau}'_\gamma} & & \downarrow{\scriptstyle \hat{\tau_Y}_\gamma} & & \\
0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\tau_Y}_\gamma} & & \downarrow{\scriptstyle \tilde{\tau}'_\gamma} & & \downarrow{\scriptstyle \tilde{\tau}_\gamma} & & \\
0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M & \longrightarrow & 0.
\end{array}$$

There always exists a section $s^* : M \to M'$ (i.e. $F^* \circ s^* = id_M$) [BLR90, Proposition A.3.1], but not always a Γ-equivariant one. Therefore, we obtain a section $\mathbb{T}_Y \to \mathbb{G}^n_{m,\Bbbk'}$, but not always a Γ-equivariant one. In other words, $\mathbb{G}^n_{m,\Bbbk'} \cong \mathbb{T} \times \mathbb{T}_Y$, but this isomorphism is not always Γ-equivariant.

2.2.4 Examples of tori

In this subsection, we introduce some well known $\Bbbk$-tori: the $\Bbbk$-torus Weil restriction of a split $\Bbbk'$-torus, which is a quasi-trivial $\Bbbk$-torus, and the norm one $\Bbbk$-torus.

Quasi-trivial tori

Using the linear Γ-action on M (see Remark 2.2.7), we define a class of tori having nice properties (see Proposition 2.3.19).

Definition 2.2.12. Let T be $\mathbb{k}$-torus that splits over a finite Galois extension $\mathbb{k}_1/\mathbb{k}$. It is called a *quasi-trivial* $\mathbb{k}$*-torus* if there exits a basis of the character lattice M of $T_{\mathbb{k}_1} \cong \mathbb{G}^n_{m,\mathbb{k}_1}$ that is permuted by $\mathrm{Gal}\,(\mathbb{k}_1/\mathbb{k})$.

Example 2.2.13. Split $\mathbb{k}$-tori are quasi-trivial.

Counter-example 2.2.14. The real circle $\mathbb{S}^1 := \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1))$ (see Example 1.2.9), corresponding to $(\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}\,(\mathbb{C}[t^{\pm 1}])\,, \tau_1)$, where $\tau_1^\sharp(at^m) = \overline{a}t^{-m}$, is not a quasi-trivial $\mathbb{R}$-torus.

The Weil restriction of a quasi-projective $\mathbb{k}$-variety is a representable functor from the category of $\mathbb{k}$-schemes to the category of sets, see for instance [BLR90, §7.6] and [CGP10, §A.5], and also the notes [JMS]. See the notes [CT76] and the article [CTS77, §2] for details on quasi-trivial $\mathbb{k}$-tori, Weil restriction $\mathbb{k}$-tori, norm one $\mathbb{k}$-tori and the relations between these tori.

Proposition 2.2.15. *Let $\mathbb{k}'/\mathbb{k}$ be a finite separable field extension of degree d. The functor*

$$\begin{aligned} \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) : Alg_{\mathbb{k}} &\to Grp \\ R &\mapsto \mathbb{G}_{m,\mathbb{k}'}\left(R_{\mathbb{k}'}\right) = R^*_{\mathbb{k}'}, \end{aligned}$$

is representable by a $\mathbb{k}$-torus, called the Weil restriction of $\mathbb{G}_{m,\mathbb{k}'}$. Furthermore, if $\mathbb{k}'/\mathbb{k}$ is a finite Galois extension of Galois group Γ, the functor $\mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$ is representable by the $\mathbb{k}$-torus $(\mathbb{G}^d_{m,\mathbb{k}'}, \tau^R)$, where τ^R is the $\mathbb{k}$-group structure defined for any $\gamma_0 \in \Gamma$ by

$$\begin{aligned} \tilde{\tau}^R_{\gamma_0} : \mathbb{Z}^d &\to \mathbb{Z}^d \\ (m_\gamma)_{\gamma\in\Gamma} &\mapsto \left(m_{\gamma_0^{-1}\gamma}\right)_{\gamma\in\Gamma}, \end{aligned}$$

and where we have identified $\prod_{\gamma\in\Gamma}\mathbb{Z}$ with $\mathbb{Z}^d$.

Proof. Let $\mathbb{k}''$ be the normal closure of $\mathbb{k}'/\mathbb{k}$ (see [Sta, Definition 0BMF]). Then, $\mathbb{k}''/\mathbb{k}$ is a finite Galois extension ([Sta, Lemmas 0EXM and 09DT]). By Theorem A.7.2 and Proposition A.7.3, the functor $\mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$ is representable by an affine $\mathbb{k}$-group, and there is a natural isomorphism

$$\mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}'') \cong \mathrm{R}_{\mathbb{k}''\otimes_{\mathbb{k}}\mathbb{k}'/\mathbb{k}'}\left(\mathbb{G}_{m,\mathbb{k}''} \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}')\right).$$

Using Lemma 1.3.1 and Remark 1.3.2, we get

$$\left(\mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}'')\right)(R) \cong R^* \otimes_{\mathbb{k}} \mathbb{k}' \cong (R^*)^d = \mathbb{G}^d_{m,\mathbb{k}''}(R),$$

where R is a $\mathbb{k}''$-algebra. Therefore, $\mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)$ is a $\mathbb{k}$-torus of dimension d that splits over $\mathbb{k}''$ (see also [Rui20, Corollary 1.38]).

Assume that $\mathbb{k}'/\mathbb{k}$ is a finite Galois extension (so $\mathbb{k}'' = \mathbb{k}'$). Therefore,

$$\left(\mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}')\right)(R) \cong R^* \otimes_{\mathbb{k}} \mathbb{k}' \cong \prod_{\Gamma} R^* = \mathbb{G}^d_{m,\mathbb{k}'}(R),$$

where the last isomorphism is $a \otimes z \mapsto (a\gamma(z))_\Gamma$ (see Lemma 1.3.1), and where R is a $\mathbb{k}'$-algebra. Finally, note that the next diagram commutes for all $\gamma \in \Gamma$

$$\begin{array}{ccc} \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}') & \xrightarrow{\cong} & \mathbb{G}^d_{m,\mathbb{k}'} \\ \Big\downarrow{\scriptstyle \mathrm{Spec}(id\times\gamma)} & & \Big\downarrow{\scriptstyle \tau^R_\gamma} \\ \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right) \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\mathbb{k}') & \xrightarrow{\cong} & \mathbb{G}^d_{m,\mathbb{k}'} \end{array}$$

□

Definition 2.2.16. A $\Bbbk$-torus T is a *Weil restriction* $\Bbbk$*-torus* if there exists a finite (separable) extension $\Bbbk'/\Bbbk$ in $\Bbbk_s$ such that $T \cong \mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}^n_{m,\Bbbk'})$.

Note that a Weil restriction $\Bbbk$-torus is a quasi-trivial $\Bbbk$-torus. Indeed, the $\Bbbk$-group structure τ^R defined in Proposition 2.2.15 induces a Γ-action on M by permutation.

Example 2.2.17 (See example 1.2.10)**.** If $\Bbbk' = \mathbb{C}$ and $\Bbbk = \mathbb{R}$, the Weil restriction $\mathbb{R}$-torus is:

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) = \mathrm{Spec}\left(\frac{\mathbb{R}[x_1, y_1, x_2, y_2]}{(x_1y_1 - x_2y_2 - 1, x_2y_1 + x_1y_2)}\right).$$

It corresponds to $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$, where τ_2 is the $\mathbb{R}$-group structure defined by $\tau_2 : (z, w) \mapsto (\bar{w}, \bar{z})$. The corresponding Γ-representation is given by

$$\tilde{\tau}_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 2.2.18 ([CTS77, §2])**.** *A quasi-trivial* $\Bbbk$*-torus is isomorphic to a product of tori* $\mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}^n_{m,\Bbbk'}\right)$*, where* $\Bbbk'$ *are finite extensions of* $\Bbbk$ *in* $\Bbbk_s$*, where* $\Bbbk_s$ *the separable closure of* $\Bbbk$ *in* $\overline{\Bbbk}$

Remark 2.2.19. Any real torus is isomorphic to a torus of the form $\mathbb{G}^{n_0}_{m,\mathbb{R}} \times (\mathbb{S}^1)^{n_1} \times \mathrm{R}^{n_2}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, with $n_0, n_1, n_2 \in \mathbb{N}$ (see Proposition 4.1.2). An $\mathbb{R}$-torus is quasi-trivial if and only if it contains no $\mathbb{S}^1$-factors.

Norm one tori

Norm one tori appear for instance in the work of Colliot-Thélène, Sansuc and Wei. In particular, they study Brauer groups, torsors and rational points (see [CT14], [CTS77], [Wei14b], [Wei14a]).

Proposition 2.2.20 (See [CT76, III.3], [CTS77, §2])**.** *Let* $\Bbbk'/\Bbbk$ *be a finite separable field extension of degree d. There is a surjective algebraic* $\Bbbk$*-group homomorphism*

$$N_{\Bbbk'/\Bbbk} : \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \to \mathbb{G}_{m,\Bbbk}.$$

The kernel is a $\Bbbk$*-torus of dimension* $d-1$*, called the* norm one torus *and denoted by* $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})$*. In other words, we have the following exact sequence*

$$1 \longrightarrow \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \longrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \longrightarrow \mathbb{G}_{m,\Bbbk} \longrightarrow 1.$$

Furthermore, if $\Bbbk'/\Bbbk$ *is a finite Galois extension,* $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})$ *is* $\Bbbk$*-torus that splits over* $\Bbbk'$*.*

Proof. Let $\Bbbk'/\Bbbk$ be a finite field extension of degree d of Galois group Γ (see Definition B.1.1). Let R be a $\Bbbk$-algebra, and denote $R_{\Bbbk'} := R \otimes_{\Bbbk} \Bbbk'$. Then, $\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})(R) = \mathbb{G}_{m,\Bbbk'}(R_{\Bbbk'}) = R^*_{\Bbbk'}$. The map

$$\begin{aligned} N_{\Bbbk'/\Bbbk}(R) : R^*_{\Bbbk'} &\to R^* \\ v \otimes z &\mapsto \prod_{\Gamma} v\gamma(z) \end{aligned}$$

is a surjective homomorphism of abstract groups (since $\prod \gamma(z) \in \Bbbk$, see [Lan02, Chapter VI, Theorem 5.1]). Therefore, we get a surjective algebraic $\Bbbk$-group homomorphism

$$N_{\Bbbk'/\Bbbk} : \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \to \mathbb{G}_{m,\Bbbk}.$$

The kernel is a $\Bbbk$-group denoted $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)$, and we get

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) = \left\{ v \otimes z \in R^*_{\Bbbk'} \,\middle|\, \prod_{\Gamma} v\gamma(z) = 1 \right\}.$$

We have the following exact sequence of algebraic $\Bbbk$-groups

$$1 \longrightarrow \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \longrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) \longrightarrow \mathbb{G}_{m,\Bbbk} \longrightarrow 1.$$

Let $\Bbbk''$ be the normal closure of $\Bbbk'/\Bbbk$. Since we get

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}) \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk'') \cong \mathbb{G}^{d-1}_{m,\Bbbk''},$$

then $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})$ is a $\Bbbk$-torus of dimension $d-1$ that splits over $\Bbbk''$.

Furthermore, if $\Bbbk'/\Bbbk$ is a finite Galois extension, then $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})$ is a $\Bbbk$-torus of dimension $d-1$ that splits over $\Bbbk'$, and the above exact sequence corresponds to the following commutative diagram of algebraic $\Bbbk'$-groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}^{d-1}_{m,\Bbbk'} & \longrightarrow & \mathbb{G}^{d}_{m,\Bbbk'} & \longrightarrow & \mathbb{G}_{m,\Bbbk'} & \longrightarrow & 1 \\
 & & \downarrow \tau^{(1)}_{\gamma} & & \downarrow \tau^{R}_{\gamma} & & \downarrow \tau_{0\gamma} & & \\
1 & \longrightarrow & \mathbb{G}^{d-1}_{m,\Bbbk'} & \longrightarrow & \mathbb{G}^{d}_{m,\Bbbk'} & \longrightarrow & \mathbb{G}_{m,\Bbbk'} & \longrightarrow & 1,
\end{array}$$

where $\tau^{(1)}$ is the induced $\Bbbk$-group structure on $\mathbb{G}^{d-1}_{m,\Bbbk'}$ corresponding to the norm one torus. The above diagram corresponds to the following commutative diagram of lattices

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{P^*} & \mathbb{Z}^{d} & \xrightarrow{F^*} & \mathbb{Z}^{d-1} & \longrightarrow & 0 \\
 & & \downarrow \tilde{\tau}_{0\gamma} & & \downarrow \tilde{\tau}^{R}_{\gamma} & & \downarrow \tilde{\tau}^{(1)}_{\gamma} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{d} & \longrightarrow & \mathbb{Z}^{d-1} & \longrightarrow & 0,
\end{array}$$

where $P^*(m) = (m, \dots, m)$, and $F^*(m_1, \dots, m_d) = (m_1 - m_d, \dots, m_{d-1} - m_d)$. ☐

Definition 2.2.21. A $\Bbbk$-torus T is a *norm one* $\Bbbk$*-torus* if there exists a finite Galois extension $\Bbbk'/\Bbbk$ in $\overline{\Bbbk}$ such that $T \cong \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}^{n}_{m,\Bbbk'}\right)$.

Example 2.2.22. Let $\Bbbk'/\Bbbk$ be a finite Galois extension of degree d. Then

$$N_{\Bbbk'/\Bbbk}(\Bbbk) : (\Bbbk')^* \to \Bbbk^*$$

corresponds to the usual norm map of a field extension (see for instance [Lan02, Chapter VI §5]). As usual, if $\omega_1, \dots, \omega_d$ is $\Bbbk$-basis of $\Bbbk'$, we denote

$$\Xi := \omega_1 x_1 + \cdots + \omega_n x_d \in \Bbbk'[x_1, \dots, x_d]$$

and $N_{\Bbbk'/\Bbbk}(\Xi) \in \Bbbk[x_1, \dots, x_d]$ be the induced form of degree d (see [Fla53]). Then,

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) = \mathrm{Spec}\left(\Bbbk[x_1, \dots, x_d]/(N_{\Bbbk'/\Bbbk}(\Xi) - 1)\right).$$

For instance, the real circle $\mathbb{S}^1 := \mathrm{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 - 1))$ is the norm one $\mathbb{R}$-torus.

Example 2.2.23 (See Proposition 4.2.4)**.** Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension of degree 3. The Galois group is $\Gamma := \{id, \gamma, \gamma^2\}$. Let $(\mathbb{G}^3_{m,\mathbb{k}'}, \tau^R)$, and let $(\mathbb{G}^2_{m,\mathbb{k}'}, \tau^{(1)})$, where

$$\tilde{\tau}^R_\gamma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \tilde{\tau}^{(1)}_\gamma = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then, we get a short exact sequence of $\mathbb{k}$-groups

$$1 \longrightarrow \left(\mathbb{G}^2_{m,\mathbb{k}'}, \tau^{(1)}\right) \longrightarrow \left(\mathbb{G}^3_{m,\mathbb{k}'}, \tau^R\right) \longrightarrow (\mathbb{G}_{m,\mathbb{k}'}, \tau_0) \longrightarrow 1.$$

Therefore, $(\mathbb{G}^2_{m,\mathbb{k}'}, \tau^{(1)})$ corresponds to $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$.

More generally, we get the next example.

Example 2.2.24 (Norm one torus and cyclic field extension)**.** Let $\mathbb{k}'/\mathbb{k}$ be a cyclic Galois extension of degree d, and let $\Gamma = \mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \{id, \gamma, \gamma^2, \ldots, \gamma^{d-1}\}$, where $\gamma \in \Gamma$ satisfies $\gamma^d = id$. The exact sequence

$$1 \longrightarrow \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow \mathbb{G}_{m,\mathbb{k}} \longrightarrow 1$$

corresponds to the following commutative diagram of lattices

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{P^*} & \mathbb{Z}^d & \xrightarrow{F^*} & \mathbb{Z}^{d-1} & \longrightarrow & 0 \\ & & \downarrow \tilde{\tau}_{0\gamma} & & \downarrow \tilde{\tau}^R_\gamma & & \downarrow \tilde{\tau}^{(1)}_\gamma & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^d & \longrightarrow & \mathbb{Z}^{d-1} & \longrightarrow & 0, \end{array}$$

where

$$\tilde{\tau}^R_\gamma = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \quad \tilde{\tau}^{(1)}_\gamma = \begin{bmatrix} 0 & \cdots & 0 & -1 \\ 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{bmatrix} \quad F^* = \begin{bmatrix} 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{bmatrix} \quad P^* = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let $\mathbb{k}'/\mathbb{k}$ be a finite Galois extension of degree d. Recall that we have the short exact sequence of $\mathbb{k}$-tori

$$1 \longrightarrow \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow \mathbb{G}_{m,\mathbb{k}} \longrightarrow 1.$$

The next lemma has consequences on cohomology level, see for instance Proposition 2.3.21.

Lemma 2.2.25. *Let $\mathbb{k}'/\mathbb{k}$ be a cyclic Galois extension of degree d. There is a short exact sequence of $\mathbb{k}$-tori*

$$1 \longrightarrow \mathbb{G}_{m,\mathbb{k}} \longrightarrow \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}}) \longrightarrow 1.$$

Proof. Let $\mathbb{k}'/\mathbb{k}$ be a cyclic Galois extension of degree d of Galois group $\Gamma = \{id, \gamma, \ldots, \gamma^{d-1}\}$, where $\gamma \in \Gamma$ satisfies $\gamma^d = id$. By Proposition A.7.5, there is a closed immersion $\mathbb{G}_{m,\mathbb{k}} \hookrightarrow \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$. Therefore, we get the next short exact sequence of $\mathbb{k}$-tori

$$1 \longrightarrow \mathbb{G}_{m,\mathbb{k}} \longrightarrow \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \longrightarrow T \longrightarrow 1.$$

This sequence corresponds to the following commutative diagram of lattices

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{d-1} & \xrightarrow{P_T^*} & \mathbb{Z}^d & \xrightarrow{F_T^*} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \big\downarrow \tilde{\tau}_\gamma & & \big\downarrow \tilde{\tau}_\gamma^R & & \big\downarrow \tilde{\tau}_{0\gamma} & & \\
0 & \longrightarrow & \mathbb{Z}^{d-1} & \longrightarrow & \mathbb{Z}^d & \longrightarrow & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where

$$\tilde{\tau}_{0\gamma} = \begin{bmatrix} 1 \end{bmatrix} \quad \tilde{\tau}_\gamma^R = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \quad \tilde{\tau}_\gamma = \begin{bmatrix} 0 & \cdots & 0 & -1 \\ 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{bmatrix} \quad P_T^* = \begin{bmatrix} 1 & & 0 \\ -1 & \ddots & \\ & \ddots & 1 \\ 0 & & -1 \end{bmatrix}$$

$$F_T^* = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}.$$

Therefore, $\tilde{\tau}_\gamma = \tilde{\tau}_\gamma^{(1)}$, and $T \cong \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk})$. $\square$

See Examples 2.2.26, 4.1.9, and 4.1.11.

Explicit examples

Example 2.2.26. Let $\Bbbk' = \mathbb{C}$, and let $\Bbbk = \mathbb{R}$. The following exact sequence of algebraic $\Bbbk$-groups

$$1 \longrightarrow \mathbb{S}^1 = \mathrm{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow \mathbb{G}_{m,\mathbb{R}} \longrightarrow 1$$

corresponds to the following commutative diagrams.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_{m,\mathbb{C}} & \longrightarrow & \mathbb{G}_{m,\mathbb{C}}^2 & \longrightarrow & \mathbb{G}_{m,\mathbb{C}}^2 & \longrightarrow & 1 \\
 & & \big\downarrow \tau_1 & & \big\downarrow \tau_2 & & \big\downarrow \tau_0 & & \\
1 & \longrightarrow & \mathbb{G}_{m,\mathbb{C}} & \longrightarrow & \mathbb{G}_{m,\Bbbk'} & \longrightarrow & \mathbb{G}_{m,\mathbb{C}} & \longrightarrow & 1
\end{array}$$

where the first morphism maps z to (z, z^{-1}), and the second one maps (z, w) to zw, and where the $\mathbb{R}$-group structures are defined by $\tau_1 : z \mapsto \bar{z}^{-1}$, $\tau_2 : (z, w) \mapsto (\bar{w}, \bar{z})$, and $\tau_0 : z \mapsto \bar{z}$ (see Examples 1.2.9 and 2.2.17).

Example 2.2.27 (See Example 2.3.26)**.** Let $\Bbbk := \mathbb{C}(t)$ and $\Bbbk' := \Bbbk[u]/(u^2 - t)$. The extension $\Bbbk'/\Bbbk$ is Galois with Galois group $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$. We obtain:

$$\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}) = \mathrm{Spec}\left(\frac{\Bbbk[x_1, y_1, x_2, y_2]}{(x_1x_2 + ty_1y_2 - 1, x_1y_2 + x_2y_1)}\right).$$

The corresponding norm one torus of dimension 1 is

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}) = \mathrm{Spec}(\Bbbk[x, y]/(x^2 - ty^2 - 1)).$$

Example 2.2.28 (See Example 2.3.26)**.** Let $\Bbbk := \mathbb{C}(t)$ and $\Bbbk' := \Bbbk[u]/(u^3 - t)$. The extension $\Bbbk'/\Bbbk$ is Galois with Galois group $\Gamma \cong \mathbb{Z}/3\mathbb{Z}$. We obtain:

$$\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}) = \mathrm{Spec}\left(\frac{\Bbbk[x_1, y_1, z_1, x_2, y_2, z_2]}{(x_1x_2 + ty_1z_2 + tz_1y_2 - 1, x_1y_2 + x_2y_1 + tz_1z_2, x_1z_2 + x_1z_2 + x_2z_1)}\right).$$

The corresponding norm one torus of dimension 2 is

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}) = \mathrm{Spec}(\Bbbk[x, y, z]/(x^3 + ty^3 + t^2z^3 - 3txyz - 1)).$$

Example 2.2.29 (See Example 2.3.22)**.** Let $P := X^3 + X^2 - 2X - 1 \in \mathbb{Q}[X]$, let $w := \exp(2i\pi/7)$, let $a = w + w^{-1} = 2\cos(2\pi/7)$, let $b = w^2 + w^{-2}$ and let $c := w^3 + w^{-3}$. Then, P is the minimal polynomial of a, $b = a^2 - 2$ and $c = a^3 - 3a$. The field extension $\mathbb{Q}(a)/\mathbb{Q}$ is a Galois extension of degree 3 and $\{1, a, a^2\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}(a)$. Let $\gamma : a \mapsto b$, it follows that $\mathrm{Gal}(\mathbb{Q}(a)/\mathbb{Q}) = \{id, \gamma, \gamma^2\}$. The Weil restriction $\mathbb{Q}$-torus of dimension three is defined by the equations

$$\begin{aligned} x_1x_2 + y_1z_2 + z_1y_2 - z_1z_2 &= 1, \\ x_1y_2 + x_2y_1 + 2y_1z_2 + 2y_2z_1 - z_1z_2 &= 0, \\ x_1z_2 + y_1y_2 - y_1z_2 - z_1y_2 + z_1x_2 + 3z_1z_2 &= 0. \end{aligned}$$

Then,

$$N_{\Bbbk'/\Bbbk}(\Xi) = \prod_{i\in\{0,1,2\}} \left(x + \gamma^i\left(a\right)y + \gamma^i\left(a^2\right)z\right)$$

and we obtain the 2-dimensional norm one $\mathbb{Q}$-torus

$$\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right) = \mathrm{Spec}\left(\frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3-2xy^2+6xz^2-x^2y+5x^2z-2yz^2-y^2z-xyz-1)}\right).$$

Example 2.2.30 (Biquadratic field extension [CT14, §3])**.** Recall that $\Bbbk'/\Bbbk$ is a Galois extension with Galois group isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if $\Bbbk' = \Bbbk\left(\sqrt{a}, \sqrt{b}\right)$ for some $a, b \in \Bbbk$ such that none of a, b, or ab is a square in $\Bbbk$. So, let a, $b \in \Bbbk$ satisfying this property. Then, $\left\{1, \sqrt{a}, \sqrt{b}, \sqrt{ab}\right\}$ is a $\Bbbk$-basis of $\Bbbk'$ and the Galois group is $\{id, \gamma_1, \gamma_2, \gamma_1\gamma_2\}$, where

$$\gamma_1 : \sqrt{a} \mapsto -\sqrt{a}, \quad \sqrt{b} \mapsto \sqrt{b},$$

$$\gamma_2 : \sqrt{a} \mapsto \sqrt{a}, \quad \sqrt{b} \mapsto -\sqrt{b}.$$

Then,

$$N_{\Bbbk'/\Bbbk}(\Xi) = \prod_{\gamma\in\Gamma} \left(x + \gamma\left(\sqrt{a}\right)y + \gamma\left(\sqrt{b}\right)z + \gamma\left(\sqrt{ab}\right)w\right).$$

Therefore, the corresponding norm one $\Bbbk$-torus $\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)$ of dimension 3 is defined by the equation:

$$x^4+a^2y^4+b^2z^4+a^2b^2w^4-2ax^2y^2-2bx^2z^2-2ab(x^2w^2+y^2z^2)-2a^2by^2w^2-2ab^2z^2w^2+8abxyzw = 1.$$

See Example 2.3.22.

2.3 Torus actions

2.3.1 Split torus actions on affine varieties

This section is inspired from [Lie10, Section 1.3.3]. Let $\Bbbk$ be a field. Let $\mathbb{T} := \mathrm{Spec}(\Bbbk[M])$ be a split $\Bbbk$-torus and let X be an affine $\mathbb{T}$-variety. Then, we recall and prove that the algebra $\Bbbk[X]$ is M-graded, that is, there exists a direct sum decomposition

$$\Bbbk[X] = \bigoplus_{m\in M} \Bbbk[X]_m,$$

such that for all $m, m' \in M$, $\Bbbk[X]_m \cdot \Bbbk[X]_{m'} \subset \Bbbk[X]_{m+m'}$.

Example 2.3.1. Let $M = \mathbb{Z}^2$. Consider the action of $\mathbb{G}_{m,\mathbb{C}}^2 = \mathrm{Spec}(\mathbb{C}[M])$ on $\mathbb{A}_{\mathbb{C}}^3 = \mathrm{Spec}(\mathbb{C}[x,y,z])$ given by

$$\mu((s,t),(x,y,z)) = (sx, ty, stz).$$

Then, $\mu^\sharp(x) = x \otimes \chi^{(1,0)}$, $\mu^\sharp(y) = y \otimes \chi^{(0,1)}$, and $\mu^\sharp(z) = z \otimes \chi^{(1,1)}$. Therefore, the coordinate ring of $\mathbb{A}_{\mathbb{C}}^3$ is $\mathbb{N}^2$-graded,

$$\mathbb{C}[x,y,z] = \bigoplus_{(k,l)\in\mathbb{N}^2} \mathbb{C}[x,y,z]_{(k,l)},$$

where

$$\begin{aligned} \mathbb{C}[x,y,z]_{(k,l)} &:= \left\{ f \in \mathbb{k}[x,y,z] \,\middle|\, \mu^\sharp(f) = f \otimes \chi^m \right\} \\ &= \bigoplus_{(k_1,k_2,k_3)\in\mathbb{N}^2, k_1+k_3=k, k_2+k_3=l} x^{k_1} y^{k_2} z^{k_3}. \end{aligned}$$

Theorem 2.3.2 ([KR82])**.** *Let $\mathbb{T} := \mathrm{Spec}(\mathbb{k}[M])$ be a split $\mathbb{k}$-torus and let X be an affine variety. There is a bijective correspondence between the $\mathbb{T}$-actions on X and the M-gradings on $\mathbb{k}[X]$.*

Proof. Let $\mathbb{T} := \mathrm{Spec}(\mathbb{k}[M])$ be a split $\mathbb{k}$-torus and let X be an affine variety equipped with an action $\mu : \mathbb{T} \times X \to X$. The comorphism of μ is $\mu^\sharp : \mathbb{k}[X] \to \mathbb{k}[X] \otimes \mathbb{k}[M]$. Let $m \in M$, and let

$$\mathbb{k}[X]_m := \left\{ f \in \mathbb{k}[X] \,\middle|\, \mu^\sharp(f) = f \otimes \chi^m \right\}.$$

Note that we have constructed an M-graded algebra $\bigoplus_{m\in M} \mathbb{k}[X]_m \subset \mathbb{k}[X]$. Let $f \in \mathbb{k}[X]$, then there exists a finite number of elements $f_m \in \mathbb{k}[X]$ such that $\mu^\sharp(f) = \sum_m f_m \otimes \chi^m$. Since μ is an action (see Section A.3), the following diagrams commute

$$\begin{array}{ccc} \mathbb{k}[X] & \xrightarrow{\mu^\sharp} & \mathbb{k}[X]\otimes\mathbb{k}[M] \\ {\scriptstyle \mu^\sharp}\downarrow & & \downarrow{\scriptstyle \mu^\sharp\otimes id} \\ \mathbb{k}[X]\otimes\mathbb{k}[M] & \xrightarrow[id\otimes\alpha^\sharp]{} & \mathbb{k}[X]\otimes\mathbb{k}[M]\otimes\mathbb{k}[M] \end{array} \qquad \begin{array}{ccc} \mathbb{k}[X]\otimes\mathbb{k}[M] & \xrightarrow{id\times e^\sharp} & \mathbb{k}[X]\otimes\mathbb{k} \\ & {\scriptstyle \mu^\sharp}\nwarrow & \downarrow\cong \\ & & \mathbb{k}[X] \end{array}$$

where $\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is the multiplication (action by translation on $\mathbb{T}$), and where $e : \mathrm{Spec}(\mathbb{k}) \to \mathbb{T}$ is the unity. Therefore, from the left hand side diagram, $\mu^\sharp(f_m) = f_m \otimes \chi^m$, hence $f_m \in \mathbb{k}[X]_m$. From the right hand side digram, $f = \sum_m f_m$. Finally $\bigoplus_{m\in M} \mathbb{k}[X]_m = \mathbb{k}[X]$ □

Definition 2.3.3. Let $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$ be a torus acting on an affine variety X. The $\mathbb{T}$-action on X is called *effective* if the set $\{m \in M \mid \mathbb{k}[X]_m \neq 0\}$ is not contained in a proper sublattice of M.

Example 2.3.4. The complex torus $\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[\mathbb{Z}])$ acts on $\mathbb{A}_{\mathbb{C}}$ by $t \cdot x = t^2 x$. Note that this action is not effective. Then, $\mathbb{k}[\mathbb{A}_{\mathbb{C}}] := \mathbb{k}[x] = \bigoplus_{m\in\mathbb{Z}} \mathbb{k}x^m$, where $\mathbb{k}x^m = \mathbb{k}[x]_{2m}$. The set $\{m \in M \mid \mathbb{k}[X]_m \neq 0\}$ is contained in $2\mathbb{Z}$, which is a proper sublattice of M.

Let $\mathbb{T} := \mathrm{Spec}(\mathbb{k}[M])$ be a split $\mathbb{k}$-torus and let X be an affine $\mathbb{T}$-variety. Then, we have $\mathbb{k}[X] = \bigoplus_{m\in M} \mathbb{k}[X]_m$. The *weight cone* of the $\mathbb{T}$-variety X is the cone $\omega_M \subset M_{\mathbb{Q}}$ spanned by the *weight monoid* $\{m \in M | \mathbb{k}[X]_m \neq 0\}$. An element $f_m \in \mathbb{k}[X]_m$ is called a *semi-invariant* of weight m. Therefore, we can write

$$\mathbb{k}[X] = \bigoplus_{m\in\omega_M\cap M} \mathbb{k}[X]_m.$$

For all $m \in \omega_M \cap M$, we denote

$$\Bbbk(X)_m := \left\{ \frac{f}{g} \in \Bbbk(X) \;\middle|\; \exists m' \in M,\ f \in \Bbbk[X]_{m+m'},\ \ g \in \Bbbk[X]_{m'} \right\} \subset \Bbbk(X),$$

where $\Bbbk(X) = \mathrm{Frac}(\Bbbk[X])$. Obviously, $\Bbbk(X)_0$ is a field and $\Bbbk(X)_0 = \Bbbk(X)^{\mathbb{T}}$. We get a tower of field extensions $\Bbbk \subset \Bbbk(X)^{\mathbb{T}} \subset \Bbbk(X)$.

For an algebraic torus $\mathbb{T}$ acting on an algebraic variety X, the complexity of this action is defined as the transcendence degree of $\Bbbk(X)^{\mathbb{T}}$ over $\Bbbk$ (see [LV83, Vin86]). If the $\mathbb{T}$-action is effective, then the complexity is $\dim(X) - \dim(\mathbb{T})$. Moreover, the complexity of the $\mathbb{T}$-action is equal to the codimension of a general orbit.

Assume that $\mathbb{T}$ acts effectively on X. Then, the set $\{m \in M \mid \Bbbk[X]_m \neq 0\}$ is not contained in a proper sublattice of M, and the weight cone ω_M of the $\mathbb{T}$-action is full dimensional. Therefore, for every $m \in M$, $\Bbbk(X)_m \neq 0$. Let $\{e_1, \dots, e_n\}$ be a $\mathbb{Z}$-basis of M. There exists elements $u_1 \in \Bbbk(X)_{e_1}, \dots, u_n \in \Bbbk(X)_{e_n}$. Consider the group morphism

$$\begin{aligned} u : M &\to \Bbbk(X)^* \\ m = m_1e_1 + \cdots + m_ne_n &\mapsto u(m) := u_1^{m_1} \cdots u_n^{m_n} \in \Bbbk(X)_m. \end{aligned}$$

Note that, for all $m \in M$, $\Bbbk(X)_m = \Bbbk(X)^{\mathbb{T}} u(m)$. Then, since $\Bbbk[X]_m \subset \Bbbk(X)_m$, we can write

$$\Bbbk[X]_m = \widetilde{\Bbbk[X]}_m u(m),$$

with $\widetilde{\Bbbk[X]}_m \subset \Bbbk(X)^{\mathbb{T}}$. Then , we get

$$\Bbbk[X] = \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk[X]}_m u(m).$$

Consider the $\Bbbk(X)^{\mathbb{T}}$-algebra

$$\Bbbk(X)^{\mathbb{T}}[M] := \left\{ \sum a_m \chi^m \;\middle|\; m \in M, a_m \in \Bbbk(X)^{\mathbb{T}} \right\},$$

and identify χ^m with $u(m)$. So, we can write

$$\Bbbk[X] \cong \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk[X]}_m \chi^m \subset \Bbbk(X)^{\mathbb{T}}[M].$$

In other words, for all $m \in M$, $u(m)$ can be identified with a character of the $\Bbbk(X)^{\mathbb{T}}$-torus $\mathbb{T}_{\Bbbk(X)^{\mathbb{T}}} = \mathrm{Spec}(\Bbbk(X)^{\mathbb{T}}[M])$.

Remark 2.3.5. If X is a $\Bbbk$-variety endowed with an effective $\mathbb{T}$-action of complexity zero, then $\Bbbk(X)^{\mathbb{T}} = \Bbbk$. Furthermore, if X is normal, then $\widetilde{\Bbbk[X]}_m = \Bbbk$ for all $m \in \omega_M \cap M$, and $\Bbbk[X] \cong \Bbbk[\omega_M \cap M] \subset \Bbbk[M]$ (see Theorem 2.4.2, or [CLS11, Theorem 1.3.5]).

Let X be a normal variety endowed with a $\mathbb{T}$-action. Then, by Sumihiro's Theorem (see [Sum74]), X is covered by a finite number of $\mathbb{T}$-invariant affine open subsets. Therefore, the description of normal $\mathbb{T}$-varieties can be down in two steps. First, the description of normal affine $\mathbb{T}$-varieties, and then the study of how to glue them together.

2.3.2 Torus actions over arbitrary fields

In this section, $\Bbbk'/\Bbbk$ is a non-necessarily finite Galois extension in $\overline{\Bbbk}$.

Let $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$ be a split torus acting on an affine $\Bbbk'$-variety X. Then, recall that the coordinate ring $\Bbbk'[X]$ of X is M-graded

$$\Bbbk'[X] := \bigoplus_{m \in M} \Bbbk'[X]_m.$$

In the next lemma, we describe the graded ring of the $\mathbb{T}$-variety X when a $\Bbbk$-torus $(\mathbb{T}, \tau)$ acts on (X, σ).

Lemma 2.3.6. *Let $(\mathbb{T} = \operatorname{Spec}(\mathbb{k}'[M]), \tau)$ be a $\mathbb{k}$-torus acting on the affine $\mathbb{k}$-variety (X,σ). Let ω_M be the weight cone of the $\mathbb{T}$-action on X. Then $\tilde{\tau}(\omega_M) = \omega_M$ and for all $m \in M$ and $\gamma \in \Gamma$*

$$\sigma_\gamma^\sharp\left(\mathbb{k}'[X]_m\right) = \mathbb{k}'[X]_{\tilde{\tau}_\gamma(m)}.$$

Proof. Let $m \in M$, $\gamma \in \Gamma$, and let $f \in \mathbb{k}'[X]_m$. We obtain the diagram of Definition 1.2.11

$$\begin{array}{ccc} \mathbb{T}\times X & \xrightarrow{\mu} & X \\ {\scriptstyle \tau_\gamma\times\sigma_\gamma}\downarrow & & \downarrow{\scriptstyle \sigma_\gamma} \\ \mathbb{T}\times X & \xrightarrow[\mu]{} & X. \end{array}$$

Therefore,

$$\left(\mu^\sharp \circ \sigma_\gamma^\sharp\right)(f) = \left(\left(\tau_\gamma^\sharp \times \sigma_\gamma^\sharp\right) \circ \mu^\sharp\right)(f) = \tau_\gamma^\sharp\left(\chi^m\right) \otimes \sigma_\gamma^\sharp(f) = \chi^{\tilde{\tau}_\gamma(m)} \otimes \sigma_\gamma^\sharp(f).$$

Hence $\sigma_\gamma^\sharp\left(\mathbb{k}'[X]_m\right) \subset \mathbb{k}'[X]_{\tilde{\tau}_\gamma(m)}$. Moreover, if $g \in \mathbb{k}'[X]_{\tilde{\tau}_\gamma(m)}$, then $g = \sigma_\gamma^\sharp\left(\sigma_{\gamma^{-1}}^\sharp(g)\right)$. Hence, $\sigma_\gamma^\sharp\left(\mathbb{k}'[X]_m\right) = \mathbb{k}'[X]_{\tilde{\tau}_\gamma(m)}$. □

Example 2.3.7 (See Examples 1.2.12 and 2.3.1)**.** Consider the action of $\mathbb{G}_{m,\mathbb{C}}^2$ on $\mathbb{A}_\mathbb{C}^3$ given by $(s,t)\cdot(x,y,z) = (sx, ty, stz)$. The Weil restriction $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ acts effectively on $(\mathbb{A}_\mathbb{C}^3, \sigma')$, where $\sigma'(x,y,z) = (\overline{y}, \overline{x}, \overline{z})$. Let $(k,l) \in \mathbb{N}^2$, and let $f \in \mathbb{C}[x,y,z]_{(k,l)}$. Then, $\sigma'^\sharp(f) \in \mathbb{C}[x,y,z]_{(l,k)}$.

A $\mathbb{k}$-torus T acts effectively on X if the $T_{\mathbb{k}'}$-action on $X_{\mathbb{k}'}$ is effective (see Definition 2.3.3), where $\mathbb{k}'/\mathbb{k}$ is a field extension that splits the $\mathbb{k}$-torus T.

Example 2.3.8 (See Example 2.3.7)**.** Consider the action of $\mathbb{G}_{m,\mathbb{C}}^2$ on $\mathbb{A}_\mathbb{C}^3$ given by $(s,t)\cdot(x,y,z) = (sx, ty, stz)$. The Weil restriction $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ acts effectively on $(\mathbb{A}_\mathbb{C}^3, \sigma')$, where $\sigma'(x,y,z) = (\overline{y}, \overline{x}, \overline{z})$.

2.3.3 Torsors

In this Section, $\mathbb{k}'/\mathbb{k}$ is a non-necessarily finite Galois extension in $\overline{\mathbb{k}}$. Split toric $\mathbb{k}$-varieties, that is $\mathbb{k}$-varieties endowed with an action of a split $\mathbb{k}$-torus $\mathbb{T} = \operatorname{Spec}(\mathbb{k}[M])$ having a dense open orbit isomorphic to $\mathbb{T}$ itself, are determined by a fan in $N_\mathbb{Q}$; the $\mathbb{Q}$-vector space associated to the cocharacter lattice of $\mathbb{T}$ (see for instance [ELFST14, §2.1]). If the acting torus is a non-necessarily split $\mathbb{k}$-torus, its character lattice is not n-dimensional, and the definition of toric $\mathbb{k}$-varieties we use in this paper involves the notion of torsor.

Definition and first properties

Definition 2.3.9 (See Section A.3)**.** Let T be a $\mathbb{k}$-torus. A T-torsor is a $\mathbb{k}$-variety V together with a left action $\mu : T\times V \to V$ such that the map $(\mu, pr_2) : T\times V \to V\times V$ is an isomorphism. Two T-torsors V and V' are isomorphic if there exists a T-equivariant isomorphism $V \to V'$ of $\mathbb{k}$-varieties. A T-torsor is trivial if it is isomorphic to T acting on itself by translation.

A T-torsor V is trivial if and only if the set $V(\mathbb{k})$ of $\mathbb{k}$-points is not empty. In particular, any torsor over an algebraically closed field is trivial. Furthermore, by Hilbert's 90 Theorem (see Theorem 2.3.18), any $\mathbb{G}_{m,\mathbb{k}}^n$-torsor is trivial, and by [CT76, III.3], a quasi-trivial torus has no non-trivial torsors (see [CTS77, §2, §5]).

Remark 2.3.10. In our setting, there is an equivalent definition of a T-torsor. A T-torsor is a $\Bbbk$-variety V such that $T_{\overline{\Bbbk}} \cong V_{\overline{\Bbbk}}$ and such that the action $\mu_{\overline{\Bbbk}} : T_{\overline{\Bbbk}} \times V_{\overline{\Bbbk}} \to V_{\overline{\Bbbk}}$ corresponds to the action by translation on $T_{\overline{\Bbbk}}$. Furthermore, any $\mathbb{G}^n_{m,\Bbbk'}$-torsor is trivial. Therefore, a $(\mathbb{G}^n_{m,\Bbbk'}, \tau)$-torsor is a $\Bbbk$-variety $(\mathbb{G}^n_{m,\Bbbk'}, \sigma)$ endowed with a $(\mathbb{G}^n_{m,\Bbbk'}, \tau)$-action, where σ is a $\Bbbk$-structure on $\mathbb{G}^n_{m,\Bbbk'}$.

Since a $(\mathbb{G}^n_{m,\Bbbk'}, \tau)$-torsor is a $\Bbbk$-form of $\mathbb{G}^n_{m,\Bbbk'}$ viewed as a $\mathbb{G}^n_{m,\Bbbk'}$-variety for the usual action by translation, from Proposition 1.4.18 and Remark 2.3.10 we obtain the following result.

Proposition 2.3.11. *Let* $(\mathbb{T} = \operatorname{Spec}(\Bbbk'[M]), \tau)$ *be a* $\Bbbk$*-torus. Consider the (continuous)* Γ*-action on the abstract group* $\operatorname{Aut}^{\mathbb{T}}(\mathbb{T})$ *of* $\mathbb{T}$*-equivariant* $\Bbbk'$*-automorphisms of* $\mathbb{T}$ *given by* $\gamma \cdot \varphi = \tau_\gamma^{-1} \circ \varphi \circ \tau_\gamma$. *There is a bijection*

$$\begin{aligned} \mathrm{H}^1_{cont}\left(\Gamma, \operatorname{Aut}^{\mathbb{T}}(\mathbb{T})\right) &\simeq \{\textit{isomorphism classes of } (\mathbb{T}, \tau)\textit{-torsors}\}; \\ (\gamma \mapsto c_\gamma) &\mapsto (\mathbb{T},\ (\gamma \mapsto c_\gamma \circ \tau_\gamma)) \end{aligned}$$

that sends the trivial cocycle to a trivial $(\mathbb{T}, \tau)$*-torsor.*

Proof. Let V be a $(\mathbb{T}, \tau)$-torsor over $\Bbbk$. Then, $V_{\Bbbk'}$ is a $\mathbb{T}$-torsor over $\Bbbk'$. By Hilbert's 90 Theorem (see Theorem 2.3.18) $V_{\Bbbk'}$ is a trivial torsor, that is $V_{\Bbbk'}$ is $\mathbb{T}$-isomorphic to $\mathbb{T}$. Hence V is a $\Bbbk$-form of $\mathbb{T}$ endowed with a $(\mathbb{T}, \tau)$-action, and $\mathrm{H}^1\left(\Gamma, \operatorname{Aut}^{\mathbb{T}}(\mathbb{T})\right)$ classifies such varieties (see Proposition 1.4.18). □

Remark 2.3.12 (Details on the construction of $(\mathbb{T}, \tau)$-torsors)**.** Let $(\mathbb{T}, \tau)$ be a $\Bbbk$-torus. Let $G := \operatorname{Hom}_{gr}(M, (\Bbbk')^*)$ be endowed with the continuous Γ-action $\gamma \cdot f := \gamma \circ f \circ \tilde{\tau}_\gamma^{-1}$. Since there is a Γ-equivariant group isomorphism

$$\begin{aligned} \operatorname{Aut}^{\mathbb{T}}(\mathbb{T}) &\to G = \operatorname{Hom}_{gr}(M, (\Bbbk')^*) \cong \mathbb{T}(\Bbbk') \\ f &\mapsto h, \end{aligned}$$

where $f^\sharp : a_m\chi^m \mapsto a_m h(m)\chi^m$ (see the proof of Lemma 2.2.5), we obtain isomorphisms of pointed sets

$$\mathrm{H}^1_{cont}\left(\Gamma, \operatorname{Aut}^{\mathbb{T}}(\mathbb{T})\right) \cong \mathrm{H}^1_{cont}(\Gamma, G) \cong \mathrm{H}^1_{cont}(\Gamma, \mathbb{T}(\Bbbk')).$$

- Let $h \in G$ be a cocycle. That is,

$$\forall m \in \omega_M^\vee \cap M,\ \forall \gamma_1, \gamma_2 \in \Gamma, \qquad h_{\gamma_1}(m)\gamma_1\left(h_{\gamma_2}\left(\tilde{\tau}_{\gamma_1}^{-1}(m)\right)\right) = h_{\gamma_1\gamma_2}(m).$$

Then $(\mathbb{T}, \sigma_h)$, where the $\Bbbk$-structure σ_h is such that

$$\begin{aligned} \sigma_h^\sharp : \Gamma &\to \operatorname{Aut}_{\Bbbk}(\Bbbk'[M]) \\ \gamma &\mapsto \left(\sigma_{h\,\gamma}^{\,\sharp} : a_m\chi^m \mapsto \gamma(a_m)h_\gamma(\tilde{\tau}_\gamma(m))\chi^{\tilde{\tau}_\gamma(m)}\right), \end{aligned}$$

is a $(\mathbb{T}, \tau)$-torsor.

- Conversely, let $(\mathbb{T}, \sigma)$ be a $(\mathbb{T}, \tau)$-torsor. The map

$$\begin{aligned} h : \Gamma &\to G \\ \gamma &\mapsto \left(h_\gamma : m \mapsto \frac{\sigma_\gamma^\sharp\left(\chi^{\tilde{\tau}_\gamma^{-1}(m)}\right)}{\chi^m}\right) \end{aligned}$$

is a cocycle.

Example 2.3.13. Let $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$ be the norm one $\mathbb{R}$-torus $\mathbb{S}^1$. Let $G := \mathrm{Hom}_{gr}(\mathbb{Z}, \mathbb{C}^*) \cong \mathbb{C}^*$. The Γ-action on $G \cong \mathbb{C}^*$ is given by $\gamma \cdot z := \bar{z}^{\ -1}$. A cocycle is thus a complex number $z \in \mathbb{C}^*$ such that $z\bar{z}^{\ -1} = 1$, that is a real number. This cocycle is equivalent to $1 \in \mathbb{C}^*$ if there exists $w \in G \cong \mathbb{C}^*$ such that $z = w^{-1}\bar{w}^{\ -1} = |w|^{-2} > 0$. Then, $\mathrm{H}^1(\Gamma, G) \cong \{\pm 1\}$.

Remark 2.3.14 (Product of tori and torsors)**.** Let T_1, T_2 be two $\mathbb{k}$-tori. From the Remark 2.3.12, it is a straightforward computation to prove that $T_1 \times T_2$-torsors are product of T_1-torsors by T_2-torsors.

Example 2.3.15. For any quadratic field extension $\mathbb{k}'/\mathbb{k}$, there is no non-trivial torsors for $\mathbb{G}_{m,\mathbb{k}}^{n_0} \times \mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})^{n_2}$ (see Theorem 2.3.19, see also [Gil22a]), but $\mathbb{S}^1$ has a non-trivial torsor, which is $X := \mathrm{Spec}(\mathbb{R}[x,y]/(x^2 + y^2 + 1))$. The $\mathbb{R}$-group structure on $\mathbb{G}_{m,\mathbb{C}}$ corresponding to $\mathbb{S}^1$ is $\tau_1 : z \mapsto \bar{z}^{-1}$, and the $\mathbb{R}$-structure on $\mathbb{G}_{m,\mathbb{C}}$ corresponding to X is $\sigma_1 : z \mapsto -\bar{z}^{-1}$.

If σ is a $\mathbb{k}$-structure on $\mathbb{T} := \mathrm{Spec}(\mathbb{k}'[M])$, then there exists a cocycle $h : \Gamma \to \mathrm{Hom}_{gr}(M, (\mathbb{k}')^*)$, and a group morphism $\tilde{\sigma} : \Gamma \to \mathrm{GL}(M)$ such that

$$\forall \gamma \in \Gamma, \quad \sigma_\gamma^\sharp : \chi^m \mapsto h_\gamma(\tilde{\sigma}_\gamma(m))\chi^{\tilde{\sigma}_\gamma(m)}.$$

We get a $\mathbb{k}$-group structure τ^σ on $\mathbb{T}$ defined by $\chi^m \mapsto \chi^{\tilde{\sigma}_\gamma(m)}$ for all $\gamma \in \Gamma$. Then, $(\mathbb{T}, \sigma)$ is a $(\mathbb{T}, \tau^\sigma)$-torsor. The next result is also a corollary of Proposition 2.2.6.

Corollary 2.3.16. *Let $\mathbb{T} := \mathrm{Spec}(\mathbb{k}'[M])$ be a split $\mathbb{k}'$-torus and let σ be a $\mathbb{k}$-structure on $\mathbb{T}$. There exists a $\mathbb{k}$-group structure τ^σ on $\mathbb{T}$ such that $(\mathbb{T}, \sigma)$ is a $(\mathbb{T}, \tau^\sigma)$-torsor.*

Let $(\mathbb{T}, \tau)$ be a $\mathbb{k}$-torus and let $\mathcal{G}$ be the image of $\hat{\tau}$ in $\mathrm{GL}(N)$. By Remark 2.2.10, there exists a finite Galois extension $\mathbb{k}_1/\mathbb{k}$ in $\mathbb{k}'$ that splits the $\mathbb{k}$-torus $(\mathbb{T}, \tau)$ and such that $\mathrm{Gal}(\mathbb{k}_1/\mathbb{k}) \cong \mathcal{G}$. In this context, we describe in the next proposition how are related finite and infinite Galois cohomology sets that classify $(\mathbb{T}, \tau)$-torsors.

Proposition 2.3.17 ([ELFST14, Proposition 3.7], inspired from [Ser97, I.2.6])**.** *Let $(\mathbb{T}, \tau)$ be a $\mathbb{k}$-torus, where $\mathbb{T} = \mathrm{Spec}(\mathbb{k}'[M])$. Let $\mathbb{k}_1 := \mathbb{k}'^{ker(\hat{\tau})}$, where $\hat{\tau} : \Gamma \to \mathrm{GL}(N)$, and let τ_1 be the induced $\mathbb{k}$-structure on the split $\mathbb{k}_1$-form $\mathbb{T}_1$ of $\mathbb{T}$ (see Proposition 1.4.9). Then*

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right) \cong \mathrm{H}^1\left(\mathrm{Gal}(\mathbb{k}_1/\mathbb{k}), \mathrm{Aut}^{\mathbb{T}_1}(\mathbb{T}_1)\right).$$

Examples of torsors

Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension. In this paragraph, we study torsors of some $\mathbb{k}$-tori.

Recall that the torus $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$ corresponds to $(\mathbb{T}_{\mathbb{k}'}, \tau_0)$, where τ_0 is the $\mathbb{k}$-group structure of Example 2.2.9, and the Γ-action on M is the trivial one.

Theorem 2.3.18 (Hilbert's 90 Theorem)**.** *Let $\mathbb{T}$ be a split $\mathbb{k}$-torus, and let $\mathbb{k}'/\mathbb{k}$ be any Galois extension of Galois group Γ. Then*

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}_{\mathbb{k}'}}(\mathbb{T}_{\mathbb{k}'})\right) \cong \{1\},$$

where the Γ-action on $\mathrm{Aut}^{\mathbb{T}_{\mathbb{k}'}}(\mathbb{T}_{\mathbb{k}'})$ is given by $\gamma \cdot \varphi = \tau_{0\gamma}^{-1} \circ \varphi \circ \tau_{0\gamma}$.

Proof. It is a classical result, particularly for finite Galois extensions. We give a proof in the case of a non-necessarily finite Galois extension. The proof is divided in two steps.

Step 1. Let $n \in \mathbb{N}^*$, let $\mathbb{k}'/\mathbb{k}$ be any finite Galois extension of Galois group Γ, and let $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$ be a split $\mathbb{k}$-torus. We show that

$$\mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}_{\mathbb{k}'}}(\mathbb{T}_{\mathbb{k}'})\right) \cong \{1\}.$$

We will prove that $\mathrm{H}^1(\Gamma, G_n) \cong \{1\}$, where $G_n = \mathrm{Hom}(\mathbb{Z}^n, (\Bbbk')^*)$, and where the Γ-action on G_n is given, for all $\gamma \in \Gamma$ and $g \in G_n$, by $\gamma \cdot g := \gamma \circ g$.

We prove this result by induction on n. Let $n = 1$, then $G_1 \cong (\Bbbk')^*$. By Hilbert's 90 Theorem, $\mathrm{H}^1(\Gamma, G_1) = \{1\}$. Let $n \geq 1$, and assume that the assertion is true for this fixed n. We have a Γ-equivariant short exact sequence of Γ-groups (see [Ber10, Definition II.3.1])

$$1 \longrightarrow G_n \xrightarrow{\Psi} G_{n+1} \xrightarrow{\Psi'} G_1 \longrightarrow 1,$$

with $\Psi(f) : \mathbb{Z}^n \oplus \mathbb{Z} \to (\Bbbk')^*, (k_1, \ldots, k_n, k) \mapsto f(k_1, \ldots, k_n)$ and $\Psi'(g) : \mathbb{Z} \to (\Bbbk')^*, k \mapsto g(0, \ldots, 0, k)$, where $f \in G_n$ and $g \in G_{n+1}$. This induces an exact sequence of Galois cohomology groups (or [Ber10, Proposition II.4.7])

$$1 \longrightarrow G_n^\Gamma \longrightarrow G_{n+1}^\Gamma \longrightarrow G_1^\Gamma \longrightarrow \mathrm{H}^1(\Gamma, G_n) \longrightarrow \mathrm{H}^1(\Gamma, G_{n+1}) \longrightarrow \mathrm{H}^1(\Gamma, G_1).$$

By induction, $\mathrm{H}^1(\Gamma, G_n) = \{1\}$ and $\mathrm{H}^1(\Gamma, G_1) = \{1\}$. Therefore, $\mathrm{H}^1(\Gamma, G_{n+1}) = \{1\}$.

Finally, by Remark 2.3.12,

$$\mathrm{H}^1(\Gamma, G) \cong \mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}_{\Bbbk'}}(\mathbb{T}_{\Bbbk'})\right),$$

where $G = \mathrm{Hom}(M, (\Bbbk')^*)$.

Step 2. Let $\Bbbk'/\Bbbk$ be a non-necessarily finite Galois extension. We show that

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}_{\Bbbk'}}(\mathbb{T}_{\Bbbk'})\right) \cong \{1\}.$$

We will prove that $\mathrm{H}^1_{cont}(\Gamma, G) \cong \{1\}$, where $G = \mathrm{Hom}(M, (\Bbbk')^*)$, and where the Γ-action on G is given, for all $\gamma \in \Gamma$ and $g \in G$, by $\gamma \cdot g := \gamma \circ g$.

Let $\Bbbk_1/\Bbbk$ be a finite Galois extension in $\Bbbk'$. Note that $G_1 := G^{\mathrm{Gal}(\Bbbk'/\Bbbk_1)} = \mathrm{Hom}_{gr}(M, \Bbbk_1^*)$. Then, $G = \mathrm{colim}\ G_i$, where the colimit is over all finite Galois extension $\Bbbk_i/\Bbbk$ in $\Bbbk'$. Finally, by [Ser97, §2.2, Proposition 8 or Corollary 1],

$$\mathrm{H}^1_{cont}(\Gamma, G) = \lim\ \mathrm{H}^1(\mathrm{Gal}(\Bbbk_i/\Bbbk), G_i),$$

where the limit is taken over all finite Galois extension $\Bbbk_i/\Bbbk$ in $\Bbbk'$. By Step 1, for any finite Galois extension $\Bbbk_1/\Bbbk$, $\mathrm{H}^1(\mathrm{Gal}(\Bbbk_1/\Bbbk), G_1) = \{1\}$, hence $\mathrm{H}^1_{cont}(\Gamma, G) = \{1\}$. □

The following proposition, based on Shapiro's Lemma [GS06, Corollary 3.3.2], provides a class of tori having no non-trivial torsors.

Proposition 2.3.19 ([CT76, Lemma III.4], [CTS77, §2, §5])**.** *Let T be a quasi-trivial $\Bbbk$-torus. Then:*

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{T_{\bar{\Bbbk}}}(T_{\bar{\Bbbk}})\right) = \{1\}.$$

Remark 2.3.20. We can prove this proposition for Weil restriction $\Bbbk$-tori $(\mathbb{G}^d_{m,\Bbbk'}, \tau^R)$ by showing that any cocycle is equivalent to the trivial one.

We saw that quasi-trivial tori have no non-trivial torsors. Torsors of norm one tori are described by the next proposition. The Brauer group of a field extension is used in the description; so we give in Proposition 2.3.25 some cases where this Brauer group is trivial.

Proposition 2.3.21 ([CT76, III.3], [CTS77, §2])**.** *Let $\Bbbk'/\Bbbk$ be a finite Galois extension, and let $T \cong \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})$. Then*

$$\mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{T_{\Bbbk'}}(T_{\Bbbk'})\right) \cong \Bbbk^*/\mathrm{Im}\left(N_{\Bbbk'/\Bbbk}(\Bbbk)\right),$$

where $N_{\Bbbk'/\Bbbk}(\Bbbk) : \mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'})(\Bbbk) = (\Bbbk')^* \to \mathbb{G}_{m,\Bbbk}(\Bbbk) = \Bbbk^*$ *is the norm map obtained from the functor* $N_{\Bbbk'/\Bbbk}$ *mentioned in Proposition 2.2.20. Furthermore (see [GS06, Corollary 4.4.10], and Lemma 2.2.25), if the extension* $\Bbbk'/\Bbbk$ *is cyclic, there is a canonical isomorphism*

$$\Bbbk^*/\mathrm{Im}\left(N_{\Bbbk'/\Bbbk}(\Bbbk)\right) \cong \mathrm{Br}(\Bbbk'/\Bbbk),$$

where $\mathrm{Br}(\Bbbk'/\Bbbk)$ *is the kernel of the morphism* $\mathrm{Br}(\Bbbk) \to \mathrm{Br}(\Bbbk')$*, and* $\mathrm{Br}(\Bbbk)$ *is the Brauer group of* $\Bbbk$*.*

Proof. Let $\Bbbk'/\Bbbk$ be a finite Galois extension of degree d. Let $M_0 := \mathbb{Z}$, let $M_R := \mathbb{Z}^d$, and let $M := \mathbb{Z}^{d-1}$. By definition, $(\mathbb{G}_{m,\Bbbk})_{\Bbbk'} = \mathrm{Spec}(\Bbbk'[M_0])$, $(\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}))_{\Bbbk'} = \mathrm{Spec}(\Bbbk'[M_R])$, and $(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}(\mathbb{G}_{m,\Bbbk'}))_{\Bbbk'} = \mathrm{Spec}(\Bbbk'[M])$. From the exact sequence of Proposition 2.2.20, we obtain a Galois-equivariant short exact sequence of lattices

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \xrightarrow{P^*} & M_R & \xrightarrow{F^*} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{\tau}_{0\gamma}} & & \downarrow{\scriptstyle \tilde{\tau}^R_\gamma} & & \downarrow{\scriptstyle \tilde{\tau}^{(1)}_\gamma} & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & M_R & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Let $G := \mathrm{Hom}(M, (\Bbbk')^*)$, let $G_R := \mathrm{Hom}(M_R, (\Bbbk')^*)$, and let $G_0 := \mathrm{Hom}(M_0, (\Bbbk')^*)$. The above sequence induces in turns a short exact sequence

$$0 \longrightarrow G \xrightarrow{\Phi} G_R \xrightarrow{\Psi} G_0 \longrightarrow 0,$$

where

$$\Phi : f \mapsto f \circ F^*, \quad \text{and} \quad \Psi : f \mapsto f \circ P^*.$$

By [Ber10, Proposition II.4.4], we obtain an exact sequence of Galois-cohomology groups

$$0 \longrightarrow G^\Gamma \xrightarrow{\Phi} G_R^\Gamma \xrightarrow{\Psi} G_0^\Gamma \xrightarrow{\delta^0} \mathrm{H}^1(\Gamma, G) \longrightarrow \mathrm{H}^1(\Gamma, G_R).$$

We describe the map δ^0. Let $f \in G_0^\Gamma$, then $\delta_0(f)$ is the equivalence class of a cocycle $h : \Gamma \to G$, defined by the relations

$$\Phi(h_\gamma) = g^{-1}\gamma \cdot g = \frac{\gamma \circ g \circ \tilde{\tau}^R_{\gamma^{-1}}}{g},$$

for an arbitrary preimage g of f under Ψ. Let $g : (m_1, \ldots, m_d) \mapsto f(m_d)$. Then, we have

$$\Phi(h_\gamma)(m_1, \ldots, m_d) = h_\gamma(m_1 - m_d, \ldots, m_{d-1} - m_d) = f(1)^{m_{i_\gamma} - m_d},$$

where $\tilde{\tau}_{R\gamma^{-1}}(m_1, \ldots, m_d) = (\ldots, m_{i_\gamma})$. Using Proposition 2.3.19, we get

$$0 \longrightarrow G^\Gamma \xrightarrow{\Phi} G_R^\Gamma \xrightarrow{\Psi} G_0^\Gamma \xrightarrow{\delta^0} \mathrm{H}^1(\Gamma, G) \longrightarrow 0.$$

Note that $G_0^\Gamma \cong \Bbbk^*$, and $\mathrm{Ker}(\delta^0) = \mathrm{Im}(\Psi) \cong \mathrm{Im}(N_{\Bbbk'/\Bbbk}(\Bbbk))$. Hence

$$\begin{aligned} \mathrm{H}^1(\Gamma, G) &\cong \Bbbk^*/\mathrm{Im}\left(N_{\Bbbk'/\Bbbk}(\Bbbk)\right) \\ [h] = \delta^0(f) &\mapsto f(1). \end{aligned}$$

∎

Example 2.3.22. Assume that $\mathbb{k}'/\mathbb{k}$ is a finite Galois-extension and consider the norm one $\mathbb{k}$-torus $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$ defined by the equation $N_{\mathbb{k}'/\mathbb{k}}(\Xi) = 1$. Let $\alpha \in \mathbb{k}^*$. Then, the $\mathbb{k}$-variety defined by the equation $N_{\mathbb{k}'/\mathbb{k}}(\Xi) = \alpha$ is a $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$-torsor, and any $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$-torsor has this form. By Proposition 2.3.21, this torsor is trivial if and only if $\alpha \in \mathrm{Im}\left(N_{\mathbb{k}'/\mathbb{k}}(\mathbb{k})\right)$. For instance:

1. If $\mathbb{k}' = \mathbb{C}$ and $\mathbb{k} = \mathbb{R}$, then $-1 \notin \mathrm{Im}\left(N_{\mathbb{k}'/\mathbb{k}}(\mathbb{k})\right)$ and $X_1 := \mathrm{Spec}\left(\mathbb{R}[x,y]/(x^2+y^2+1)\right)$ is, up to isomorphism, the only non-trivial $\mathbb{S}^1$-torsor (see Examples 2.2.22, 2.3.23 and [DL18, Proposition 3.1]);
2. If $\mathbb{k}' = \mathbb{k}(\cos(2\pi/7))$ and $\mathbb{k} = \mathbb{Q}$, then $2; 4 \notin \mathrm{Im}\left(N_{\mathbb{k}'/\mathbb{k}}(\mathbb{k})\right)$ (See Examples 2.2.29, 2.3.24 and [Lam91, 14 p 234-240]);
3. If $\mathbb{k}' = \mathbb{k}\left(\sqrt{13}, \sqrt{17}\right)$ and $\mathbb{k} = \mathbb{Q}$, then for instance $-1; 25; 49 \notin \mathrm{Im}\left(N_{\mathbb{k}'/\mathbb{k}}(\mathbb{k})\right)$ (see Example 2.2.30 and [CF67, Example 5.3]);

Example 2.3.23. Consider the field extension $\mathbb{C}/\mathbb{R}$ and $T = \mathbb{S}^1$. Then, $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{T_\mathbb{C}}(T_\mathbb{C})) = \{\pm 1\}$ (see [DL18, Proposition 3.1] or [Gil22a, §4.3]). The non-trivial $\mathbb{S}^1$-torsor is (see Example 2.3.22)

$$X_1 = \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1)).$$

The corresponding $\mathbb{R}$-structure σ on $\mathbb{G}_{m,\mathbb{C}}$ is defined by

$$\sigma^\sharp : a_m\chi^m \mapsto \bar{a}_m(-1)^m\chi^{-m}.$$

Example 2.3.24. We pursue Example 2.2.29. Let $\mathbb{k}' = \mathbb{k}(a)$, $\mathbb{k} = \mathbb{Q}$ and

$$X_\alpha = \mathrm{Spec}\left(\frac{\mathbb{Q}[x,y,z]}{(x^3+y^3+z^3-2xy^2+6xz^2-x^2y+5x^2z-2yz^2-y^2z-xyz-\alpha)}\right),$$

where $\alpha \in \mathbb{k}^*$. Then, X_α is a $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$-torsor (see Example 2.3.22). Using Example 2.2.29, Remark 2.3.12, and the isomorphism of Proposition 2.3.21, we can show that X_α corresponds to the $\mathbb{k}$-structure σ on $\mathbb{G}^2_{m,\mathbb{k}'}$ defined by

$$\sigma^\sharp_\gamma\left(a_{(k,l)}\chi^{(k,l)}\right) = \gamma(a_{(k,l)})h_\gamma\left(\tilde{\tau}_\gamma(k,l)\right)\chi^{\tilde{\tau}_\gamma(k,l)} = \gamma(a_{(k,l)})\alpha^{-l}\chi^{(-l,k-l)};$$

and

$$\sigma^\sharp_{\gamma^2}\left(a_{(k,l)}\chi^{(k,l)}\right) = \gamma(a_{(k,l)})h_{\gamma^2}\left(\tilde{\tau}_{\gamma^2}(k,l)\right)\chi^{\tilde{\tau}_{\gamma^2}(k,l)} = \gamma^2(a_{(k,l)})\alpha^{-k}\chi^{(-k+l,-k)}.$$

Recall that if $\alpha \in \{2;4\} \not\subseteq \mathrm{Im}(N_{\mathbb{k}'/\mathbb{k}}(\mathbb{k}))$ (see Example 2.3.22), then X_α is a non-trivial $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$-torsor.

We recall in Proposition 2.3.25 some results on the Brauer groups (see [GS06, Sections 2.4 and 4.4] or [Ser79, Sections 4 to 7], or [Sta, Chapter 073W]). These results are used in Examples 2.3.26 and 4.2.9.

Proposition 2.3.25 ([Ser79, §7 Examples of Brauer Groups])**.**

(i) If $\mathbb{k}$ *is algebraically (resp. separably) closed, then* $\mathrm{Br}(\mathbb{k}) = \{1\}$*;*

(ii) If $\mathbb{k}$ *is an extension of transcendence degree 1 over an algebraically closed field, then* $\mathrm{Br}(\mathbb{k}) = \{1\}$ *(Tsen's Theorem), or more generally, if* $\mathbb{k}$ *is* C_1*, then* $\mathrm{Br}(\mathbb{k}) = \{1\}$ *(see [GS06, §6.2]);*

(iii) If $\mathbb{k}$ *is an algebraic extension of* $\mathbb{Q}$ *containing all the roots of 1, then* $\mathrm{Br}(\mathbb{k}) = \{1\}$.

Example 2.3.26. We pursue Examples 2.2.27 and 2.2.28.

(i) Let $\mathbb{k} := \mathbb{C}(t)$ and $\mathbb{k}' := \mathbb{k}[u]/(u^2-t)$. By Proposition 2.3.25 and by Proposition 2.3.21, the one-dimensional $\mathbb{k}$-torus $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$ has no non-trivial torsors.

(ii) Let $\mathbb{k} := \mathbb{C}(t)$ and $\mathbb{k}' := \mathbb{k}[u]/(u^3-t)$. By Proposition 2.3.25 and by Proposition 2.3.21, the two-dimensional $\mathbb{k}$-torus $\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$ has no non-trivial torsors.

2.4 Toric varieties

2.4.1 Split toric varieties

Most of the literature on split toric varieties is over an algebraically closed field of characteristic zero (see [Ful93], [KKMSD73], [CLS11], ...), but in fact these constructions work over any field. See for instance Huruguen's thesis for a point on this theory [Hur, §1]. Let $\Bbbk$ be an arbitrary field, we briefly recall the theory of split toric $\Bbbk$-varieties.

Definition 2.4.1. A normal $\Bbbk$-variety X such that there exists a split $\Bbbk$-torus $\mathbb{T}$ acting effectively on X with a dense open orbit U is called a *split* $\mathbb{T}$*-toric* $\Bbbk$*-variety.*

Note that the dense open orbit U is a $\mathbb{T}$-torsor. In this case, since there are no non-trivial torsors by Hilbert's 90 Theorem (see Theorem 2.3.18), $\mathbb{T}$ acts on X with a dense open orbit $U \cong \mathbb{T}$.

Let N and M be dual lattices of rank n (see Section 2.1). Let ω_N in $N_{\mathbb{Q}}$ be a pointed cone. Then, by Gordan's Lemma (see [CLS11, Proposition 1.2.17]), $\omega_N^\vee \cap M$ is a finitely generated semi-group. Therefore, the integral domain $\Bbbk[\omega_N^\vee \cap M]$ is a finite type $\Bbbk$-algebra.

To a pointed cone ω_N in $N_{\mathbb{Q}}$, we associate the affine $\Bbbk$-variety $X_{\omega_N} := \mathrm{Spec}(\Bbbk[\omega_N^\vee \cap M])$. The inclusion $\Bbbk[\omega_N^\vee \cap M] \hookrightarrow \Bbbk[M]$ induces an effective action of $\mathbb{T} := \mathrm{Spec}(\Bbbk[M])$ on X_{ω_N}. Indeed, by Theorem 2.3.2, the $\Bbbk$-algebra morphism

$$\begin{aligned} \Bbbk[\omega_N^\vee \cap M] &\to \Bbbk[\omega_N^\vee \cap M] \otimes_{\Bbbk} \Bbbk[M] \\ \chi^m &\mapsto \chi^m \otimes \chi^m \end{aligned}$$

corresponds to an effective $\mathbb{T}$-action

$$\mathbb{T} \times_{\mathrm{Spec}(\Bbbk)} X_{\omega_N} \to X_{\omega_N}$$

having a dense open orbit isomorphic to $\mathbb{T}$. (indeed, since $\Bbbk(\omega_N^\vee \cap M)^{\mathbb{T}} = \Bbbk$, the complexity of the $\mathbb{T}$-action is 0).

Finally, since $\omega_N^\vee \cap M$ is saturated (see [CLS11, Proposition 1.3.5]), X_{ω_N} is a normal $\Bbbk$-variety. Therefore, X_{ω_N} is a split $\mathbb{T}$-toric $\Bbbk$-variety.

Proposition 2.4.2 ([CLS11, Theorem 1.1.17 and Theorem 1.3.5])**.** *Let X be an affine $\Bbbk$-variety. Then, X admits a structure of a split toric variety if and only if there exists a lattice M and a pointed cone $\omega_N \subset N_{\mathbb{Q}}$ such that $X \cong X_{\omega_N}$.*

Proof. If $X \cong \mathrm{Spec}(\Bbbk[\omega_N^\vee \cap M])$, then we saw that X is a split $\mathbb{T}$-toric $\Bbbk$-variety, where $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$. Conversely, let X be an affine split $\mathbb{T}$-toric $\Bbbk$-variety. Then, by Theorem 2.3.2, we get

$$\Bbbk[X] = \bigoplus_{m \in \omega_M \cap M} \Bbbk[X]_m,$$

where ω_M is the weight cone of the $\mathbb{T}$-action on X. Since the action is effective, ω_M is full dimensional, and there exists a group morphism (see Section 2.3.1)

$$\begin{aligned} u : M &\to \Bbbk(X)^* = \Bbbk^* \\ m &\mapsto u(m) \in \Bbbk(X)_m. \end{aligned}$$

We get

$$\Bbbk[X] = \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk[X]}_m u(m),$$

where $\widetilde{\Bbbk[X]}_m \in \{\Bbbk, 0\}$. Since $\Bbbk[X]$ is normal, then $\widetilde{\Bbbk[X]}_m = \Bbbk$ for all $m \in \omega_M \cap M$. Identifying for all $m \in M$, $u(m)$ with χ^m, we get

$$\Bbbk[X] \cong \bigoplus_{m \in \omega_M \cap M} \Bbbk\chi^m = \Bbbk[\omega_M \cap M] \subset \Bbbk[M].$$

Therefore, X is a split $\mathbb{T}$-toric $\Bbbk$-variety associated to the pointed cone $\omega_M^\vee \subset N_\mathbb{Q}$. □

Remark 2.4.3 (Normality assumption and graduation)**.** The cuspidal curve

$$C = \operatorname{Spec}(\mathbb{C}[x,y]/(x^3 - y^2)) \subset \mathbb{A}^2_\mathbb{C}$$

is a non-normal variety endowed with an effective torus action having a dense open orbit. The $\mathbb{G}_{m,\mathbb{C}}$-action on C comes from the $\mathbb{G}_{m,\mathbb{C}}$-action on $\mathbb{A}^2_\mathbb{C}$ defined by $t \cdot (x,y) = (t^2x, t^3y)$. The weight monoid of the $\mathbb{G}_{m,\mathbb{C}}$-action on C is $\{0, 2, 3, 4, 5, \ldots\} \subset M = \mathbb{Z}$; it is not saturated. The weight cone is $\omega_M = \mathbb{Q}_{\geq 0}$.

Given a fan Λ in $N_\mathbb{Q}$, we construct the $\Bbbk$-variety X_Λ by gluing the affine varieties X_{ω_N}, for ω_N a cone of Λ. First, note that if ω'_N is a face of ω_N, then $\omega_N^\vee \subset {\omega'_N}^\vee$, and the induced map $X_{\omega'_N} \to X_{\omega_N}$ is an open immersion. If ω_N and ω'_N are two cones of Λ, then we glue these varieties along their common subvarieties $X_{\omega_N \cap \omega'_N}$. Since every pointed cone ω_N contains 0 as a face, $X_0 = \mathbb{T} = \operatorname{Spec}(\Bbbk[M])$ is contained in X_{ω_N} for every cone $\omega_N \in \Lambda$. The $\mathbb{T}$-actions on the X_{ω_N} are compatible with the open immersions $X_{\omega'_N} \hookrightarrow X_{\omega_N}$ induced by by the inclusions $\omega'_N \subset \omega_N$. Thus $\mathbb{T}$ acts effectively on X_Λ with a dense open orbit, and we get a split $\mathbb{T}$-toric $\Bbbk$-variety (see [CLS11, Theorem 3.1.5]).

Conversely, let X be a $\mathbb{T}$-toric $\Bbbk$-variety. Then, by Sumihiro's Theorem (see [Sum74]), we can cover X by a finite number of $\mathbb{T}$-invariant affine open subsets. We obtain the next result.

Theorem 2.4.4 ([CLS11, Corollary 3.1.8])**.** *Let X be a $\Bbbk$-variety. Then, X admits a structure of a split toric variety if and only if there exists a lattice M and a fan Λ in $N_\mathbb{Q}$ such that $X \cong X_\Lambda$.*

2.4.2 General case

Since a $\Bbbk$-torus is not necessarily split, we now focus on toric $\Bbbk$-varieties equipped with an arbitrary $\Bbbk$-torus action (see for instance [Hur11], [ELFST14] and [Dun16]).

Definition 2.4.5 (Toric $\Bbbk$-varieties)**.** A normal $\Bbbk$-variety X such that there exits a $\Bbbk$-torus T acting effectively on X with a dense open orbit U is called a *T-toric $\Bbbk$-variety* .

Note that the dense open orbit U is a T-torsor.

The next result, which is well known for specialists, can be seen as a warm-up to the generalized Altmann-Hausen presentation that will arrive in the next chapter (see Theorem 3.2.3).

Proposition 2.4.6. *Let $\Bbbk'/\Bbbk$ be a Galois extension of Galois group Γ. Let $\mathbb{T} := \operatorname{Spec}(\Bbbk'[M])$ be a $\Bbbk'$-torus, and let τ be a $\Bbbk$-group structure on $\mathbb{T}$. Let $G := \operatorname{Hom}_{gr}(M, (\Bbbk')^*)$.*

(i) Let ω_N be a pointed cone in $N_\mathbb{Q}$ that is stable under the Γ-action induced by $\hat{\tau}$, and let $h : \Gamma \to G$ be a map such that

$$\forall m \in \omega_M^\vee \cap M,\ \forall \gamma_1, \gamma_2 \in \Gamma, \qquad h_{\gamma_1}(m)\gamma_1\left(h_{\gamma_2}\left(\tilde{\tau}_{\gamma_1}^{-1}(m)\right)\right) = h_{\gamma_1\gamma_2}(m). \tag{2.1}$$

Then, the affine $\mathbb{T}$-toric $\Bbbk'$-variety X_{ω_N} admits a $\Bbbk$-structure $\sigma_{X_{\omega_N,h}}$ such that $(X_{\omega_N}, \sigma_{X_{\omega_N,h}})$ is a $(\mathbb{T},\tau)$-toric $\Bbbk$-variety that contains the torsor encoded by h (see Remark 2.3.12) as a $(\mathbb{T},\tau)$-stable dense open subset.

(ii) Let (X,σ) be an affine $(\mathbb{T},\tau)$-toric $\Bbbk$-variety. There exists a pointed cone ω_N in $N_\mathbb{Q}$ that is stable under $\hat{\tau}$, and a map h satisfying (2.1) such that the $(\mathbb{T},\tau)$-toric $\Bbbk$-varieties (X,σ) and $(X_{\omega_N},\sigma_{X_{\omega_N},h})$ are isomorphic.

Proof. • Let ω_N be a pointed cone in $N_\mathbb{Q}$, and let $h:\Gamma\to G$ be a map satisfying (2.1). We denote by μ the natural $\mathbb{T}$-action on $X_{\omega_N} := \mathrm{Spec}(\Bbbk'[\omega_M\cap M])$ defined by

$$\begin{aligned}\mu^\sharp : \Bbbk'[\omega_N^\vee\cap M] &\to \Bbbk'[M]\otimes\Bbbk'[\omega_N^\vee\cap M]\\ a_m\chi^m &\mapsto \chi^m\otimes a_m\chi^m.\end{aligned}$$

Since for all $\gamma\in\Gamma$ $\hat{\tau}_\gamma(\omega_N)=\omega_N$, there exists a $\Bbbk$-structure $\sigma_{X_{\omega_N},h}$ on X_{ω_N} defined by

$$\begin{aligned}\sigma^\sharp_{X_{\omega_N},h} : \Gamma &\to \mathrm{Aut}_\Bbbk(\Bbbk'[\omega_N^\vee\cap M])\\ \gamma &\mapsto \left(a_m\chi^m\mapsto \gamma(a_m)h_\gamma(\tilde{\tau}_\gamma(m))\chi^{\tilde{\tau}_\gamma(m)}\right).\end{aligned}$$

Then, $(X_{\omega_N},\sigma_{X_{\omega_N},h})$ is a $(\mathbb{T},\tau)$-toric $\Bbbk$-variety since the following diagram commutes for all $\gamma\in\Gamma$

$$\begin{array}{ccc}\Bbbk'[\omega_N^\vee\cap M] & \xrightarrow{\mu^\sharp} & \Bbbk[M]\otimes\Bbbk'[\omega_N^\vee\cap M]\\ \left(\sigma^\sharp_{X_{\omega_N},h}\right)_\gamma\Big\downarrow & & \Big\downarrow \tau^\sharp_\gamma\otimes\left(\sigma^\sharp_{X_{\omega_N},h}\right)_\gamma\\ \Bbbk'[\omega_N^\vee\cap M] & \xrightarrow[\mu^\sharp]{} & \Bbbk[M]\otimes\Bbbk'[\omega_N^\vee\cap M].\end{array}$$

Finally, from the inclusion $\Bbbk'[\omega_N^\vee\cap M]\hookrightarrow\Bbbk'[M]$, the $(\mathbb{T},\tau)$-torsor $(\mathbb{T},\sigma_h)$ (see Remark 2.3.12) is contained in $(X_{\omega_N},\sigma_{X_{\omega_N},h})$ as a $(\mathbb{T},\tau)$-stable dense open subset.

• Let (X,σ) be an affine $(\mathbb{T},\tau)$-toric $\Bbbk$-variety. Then, X is a $\mathbb{T}$-toric $\Bbbk'$-variety (see Corollary 1.4.13); we denote by ω_M its weight cone. By Lemma 2.3.6, $\tilde{\tau}(\omega_M)=\omega_M$. From the proof of Proposition 2.4.2, we get

$$\Bbbk'[X] = \bigoplus_{m\in\omega_M\cap M}\Bbbk' u(m)\cong\bigoplus_{m\in\omega_M\cap M}\Bbbk'\chi^m,$$

where $u: M\to\Bbbk'(X)^*$ is a group morphism such that for all $m\in\omega_M\cap M$, $u(m)\in\Bbbk'(X)_m$. Let $m\in\omega_M\cap M$, let $\gamma\in\Gamma$, and let

$$h_\gamma(m) := \frac{\sigma^\sharp_\gamma(u(\tilde{\tau}_{\gamma^{-1}}(m)))}{u(m)}.$$

From Lemma 2.3.6, it follows that $h_\gamma(m)\in(\Bbbk')^*$, hence we get a map $h:\Gamma\to G$. Let $\omega_N := \omega_M^\vee$. Note that h satisfies (2.1), so we can construct a $\Bbbk$-structure $\sigma_{X_{\omega_N},h}$ on $X_{\omega_N} := \mathrm{Spec}(\Bbbk'[\omega_M\cap M])$. Finally, the next diagram commutes

$$\begin{array}{ccc}\Bbbk'[X] & \xrightarrow{\sigma^\sharp_\gamma} & \Bbbk'[X]\\ \cong\Big\uparrow & & \Big\uparrow\cong\\ \Bbbk'[\omega_M\cap M] & \xrightarrow{\sigma^\sharp_{X_{\omega_N},h}} & \Bbbk'[\omega_M\cap M]\end{array}\qquad\qquad \begin{array}{ccc} a_m u(m) & \mapsto & \gamma(a_m)h_\gamma(\tilde{\tau}_\gamma(m))u(\tilde{\tau}_\gamma(m))\\ \updownarrow & & \updownarrow\\ a_m\chi^m & \longmapsto & \gamma(a_m)h_\gamma(\tilde{\tau}_\gamma(m))\chi^{\tilde{\tau}_\gamma(m)}.\end{array}$$

Therefore, the $(\mathbb{T},\tau)$-toric $\Bbbk$-varieties (X,σ) and $(X_{\omega_N},\sigma_{X_{\omega_N}})$ are isomorphic. □

Example 2.4.7. In Example 4.1.1, we give an example of a $\mathbb{G}^2_{m,\mathbb{C}}$-toric $\mathbb{C}$-variety that does not admit an $\mathbb{R}$-form in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties.

Let $\mathbb{T} := \mathrm{Spec}(\Bbbk'[M])$ be a split $\Bbbk'$-torus, let τ be a $\Bbbk$-group structure on $\mathbb{T}$, and let $G := \mathrm{Hom}_{gr}(M, (\Bbbk')^*)$ be endowed with the (continuous) Γ-action $\gamma \cdot f := \gamma \circ f \circ \tilde{\tau}_\gamma^{-1}$. Recall that there is an isomorphism of pointed sets (see Remark 2.3.12)

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right) \cong \mathrm{H}^1_{cont}(\Gamma, G).$$

Furthermore, the map h of Proposition 2.4.6 is a cocycle, that is, $h \in \mathrm{H}^1_{cont}(\Gamma, G)$. Therefore, $\Bbbk$-forms of an affine $\mathbb{T}$-toric variety X in the category of $(\mathbb{T}, \tau)$-toric varieties are parametrized by

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right).$$

The latter parametrizes $(\mathbb{T}, \tau)$-torsors. We will extend this result to non-necessarily affine varieties in the next proposition.

Let (X, σ) be a $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety, where $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$. Note that X is a split $\mathbb{T}$-toric $\Bbbk'$-variety (see Corollary 1.4.13). Therefore, there exists a fan $\Lambda \subset N_{\mathbb{Q}}$ such that $X \cong X_\Lambda$. In the next remark, and in the next proposition, we focus on the $\Bbbk$-forms of X in the category of $(\mathbb{T}, \tau)$-toric varieties.

Remark 2.4.8. Let (X, σ) be a $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety, where $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$. Note that $\sigma_\gamma(X_{\mathbb{T}}) = X_{\mathbb{T}}$ for all $\gamma \in \Gamma$, where $X_{\mathbb{T}} \cong \mathbb{T}$ is the dense open orbit of the toric $\Bbbk'$-variety X. Hence, σ induces a $\Bbbk$-structure $\sigma_{\mathbb{T}}$ on $X_{\mathbb{T}}$ and $(X_{\mathbb{T}}, \sigma_{\mathbb{T}})$ is a $(\mathbb{T}, \tau)$-torsor. Furthermore, $\mathrm{Aut}^{\mathbb{T}}(X) \cong \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$ as abstract groups, hence

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X)\right) \cong \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right).$$

By Proposition 1.4.18, $\Bbbk$-forms of X in the category of $(\mathbb{T}, \tau)$-toric $\Bbbk$-varieties are parametrized by $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T}))$, which classifies $(\mathbb{T}, \tau)$-torsors (see Proposition 2.3.11).

Let (X, σ) be a $(\mathbb{T}, \tau)$-toric $\Bbbk'$-variety, where $\mathbb{T} := \mathrm{Spec}\,(\Bbbk'[M])$. Then, by Remark 2.4.8, the dense open orbit $X_{\mathbb{T}} \cong \mathbb{T}$ of X is Γ-invariant and we have an induced $\Bbbk$-structure $\sigma_{\mathbb{T}}$ on $X_{\mathbb{T}}$. Conversely, if σ is a $\Bbbk$-structure on $X_{\mathbb{T}} \cong \mathbb{T}$, then the next proposition gives a condition to extend σ on X.

Proposition 2.4.9 (Based on [Hur11, Proposition 1.19])**.** *Let $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$ be a split $\Bbbk'$-torus and let Λ be a fan in $N_{\mathbb{Q}}$. Let X_Λ be the associated split toric $\Bbbk'$-variety and let σ be a $\Bbbk$-structure on the dense open orbit $(X_\Lambda)_{\mathbb{T}} \cong \mathbb{T}$. Let τ^σ be the $\Bbbk$-group structure on $\mathbb{T}$ constructed in Corollary 2.3.16.*

- *The $\Bbbk$-structure σ extends to a $\Bbbk$-structure σ_Λ on X_Λ if and only if the fan Λ is stable under the Γ-action $\hat{\tau}^\sigma$ on $N_{\mathbb{Q}}$ (i.e., for any cone $\lambda \in \Lambda$ and for any $\gamma \in \Gamma$, $\hat{\tau}^\sigma_\gamma(\lambda) \in \Lambda$).*
- *Furthermore, if σ_Λ is such a $\Bbbk$-structure, then*

$$\mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X_\Lambda)\right) \cong \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right),$$

where the Γ-action on $\mathrm{Aut}(\mathbb{T})$ is given by $\gamma \cdot \varphi = \sigma_\gamma^{-1} \circ \varphi \circ \sigma_\gamma$ for all $\gamma \in \Gamma$ and $\varphi \in \mathrm{Aut}(\mathbb{T})$.

In [Hur11, §1.3], there is an example of a three-dimensional toric $\Bbbk'$-variety X_Λ endowed with a $\Bbbk$-structure σ such that the fan is Γ-stable, but which does not admit a $\Bbbk$-form in the category of $(\mathbb{G}^2_{m,\Bbbk'}, \tau^\sigma)$-toric $\Bbbk$-varieties.

Remark 2.4.10. For the effectiveness of the descent datum defined by σ_Λ on X_Λ, see [Hur11, Theorem 1.22]. However, by [Hur11, Theorem 1.25], this descent datum is always effective if the torus is two-dimensional.

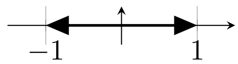
Proof. Let $\Bbbk_1/\Bbbk$ be a finite Galois extension that splits $\mathbb{T}/\Gamma$, i.e. $(\mathbb{T}/\Gamma)_{\Bbbk_1} \cong \mathbb{T}_1 := \operatorname{Spec}(\Bbbk_1[M])$. By Proposition 1.4.8, there exists a $\Bbbk$-structure $\sigma_1 : \operatorname{Gal}(\Bbbk_1/\Bbbk) \cong \Gamma/\operatorname{Gal}(\Bbbk'/\Bbbk_1) \to \operatorname{Aut}(\mathbb{T}_{1/\Bbbk})$ such that the diagram of Proposition 1.4.8 commutes for all $\gamma \in \Gamma$. Moreover, by Remark 2.2.10, we can factorize the Γ-action on $N_{\mathbb{Q}}$:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\hat{\tau}^{\sigma}} & \operatorname{GL}(N) \\ \downarrow & \nearrow_{\hat{\tau}^{\sigma_1}} & \\ \Gamma/\operatorname{Gal}\left(\overline{\Bbbk}/\Bbbk_1\right) & & \end{array}$$

Since the fan Λ is $\operatorname{Gal}(\Bbbk_1/\Bbbk)$-stable, by [Hur11, Proposition 1.19], we can extend the $\Bbbk$-structure σ_1 on $\mathbb{T}_1$ to a $\Bbbk$-structure $\sigma_{\Lambda,1}$ on $X_{\Lambda,1}$ (the $\Bbbk_1$-toric variety constructed from Λ) and $(X_{\Lambda,1}, \sigma_{\Lambda,1})$ is a $(\mathbb{T}_1, \tau^{\sigma_1})$-toric $\Bbbk_1$-variety. Then, we obtain a $\Bbbk$-structure σ_Λ on $X_\Lambda \cong X_{\Lambda,1} \times_{\operatorname{Spec}(\Bbbk_1)} \operatorname{Spec}(\Bbbk')$, defined by $\gamma \in \Gamma \mapsto \sigma_{\Lambda,1_\gamma} \times \operatorname{Spec}(\gamma)$, that extends σ and such that $(X_\Lambda, \sigma_\Lambda)$ is a $(\mathbb{T}, \tau^\sigma)$-toric $\Bbbk$-variety. □

Remark 2.4.11. Note that the condition on the fan mentioned in Proposition 2.4.9 only depends on the $\Bbbk$-group structure τ^σ. In other words, if $(\mathbb{T}, \sigma')$ is another $(\mathbb{T}, \tau^\sigma)$-torsor, then the $\Bbbk$-structure σ' extends to a $\Bbbk$-structure σ'_Λ on X_Λ.

Example 2.4.12. Let $\Bbbk'/\Bbbk$ be a Galois extension. Consider the following fan in $N_{\mathbb{Q}} = \mathbb{Q}$ corresponding to $\mathbb{P}^1_{\Bbbk'}$.



- Consider the split $\Bbbk$-torus $\mathbb{G}_{m,\Bbbk}$ corresponding to $(\mathbb{G}_{m,\Bbbk'}, \tau_0)$, where $\hat{\tau}_0 : \Gamma \to \operatorname{GL}(N)$ is the trivial Γ-representation. Note that the fan of $\mathbb{P}^1_{\Bbbk'}$ is Γ-stable. By Hilbert's 90 Theorem (see Theorem 2.3.18), there is no non-trivial torsor, we obtain

$$\mathrm{H}^1\left(\Gamma, \operatorname{Aut}^{\mathbb{G}_{m,\Bbbk'}}(\mathbb{P}^1_{\Bbbk'})\right) \cong \mathrm{H}^1\left(\Gamma, \operatorname{Aut}^{\mathbb{G}_{m,\Bbbk'}}(\mathbb{G}_{m,\Bbbk'})\right) = \{1\}.$$

Hence, up to isomorphism, there is a single $\Bbbk$-form of $\mathbb{P}^1_{\Bbbk'}$ in the category of $\mathbb{G}_{m,\Bbbk}$-toric $\Bbbk$-varieties. Furthermore, the $\Bbbk$-structure τ_0 extends to a $\Bbbk$-structure σ_0 on $\mathbb{P}^1_{\Bbbk'}$ such that for all $\gamma \in \Gamma$,

$$\sigma_{0_\gamma} : [u, v] \mapsto [\gamma^{-1}(u) : \gamma^{-1}(v)].$$

Therefore, the only $\Bbbk$-form of $\mathbb{P}^1_{\Bbbk'}$ in the category of $\mathbb{G}_{m,\Bbbk}$-toric $\Bbbk$-varieties is $\mathbb{P}^1_{\Bbbk}$.

- Assume that $\Bbbk = \mathbb{R}$ and $\Bbbk' = \mathbb{C}$. Consider the non-split $\mathbb{R}$-torus $\mathbb{S}^1$ corresponding to $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$, where $\hat{\tau}_1 = -id_N$. Note that the fan of $\mathbb{P}^1_{\Bbbk'}$ is Γ-stable. Up to isomorphism, there are two $\mathbb{S}^1$-torsors, which are $\mathbb{S}^1$ and $X_1 := \operatorname{Spec}(\mathbb{R}[x, y]/(x^2 + u^2 + 1))$. We get

$$\mathrm{H}^1\left(\Gamma, \operatorname{Aut}^{\mathbb{G}_{m,\Bbbk'}}(\mathbb{P}^1_{\Bbbk'})\right) \cong \mathrm{H}^1\left(\Gamma, \operatorname{Aut}^{\mathbb{G}_{m,\Bbbk'}}(\mathbb{G}_{m,\Bbbk'})\right) = \{\pm 1\}.$$

Hence, up to isomorphism, there are two $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{S}^1$-toric $\mathbb{R}$-varieties. First, the $\mathbb{R}$-structure

$$\tau_1 : t \mapsto t^{-1}$$

on $\mathbb{G}_{m,\mathbb{C}}$ corresponding to the trivial $\mathbb{S}^1$-torsor extends to the $\mathbb{R}$-structure

$$[u, v] \mapsto [\bar{v} : \bar{u}].$$

This $\mathbb{R}$ structure is equivalent to the following one

$$[u, v] \mapsto [\bar{u} : \bar{v}].$$

Therefore, $\mathbb{P}^1_{\mathbb{R}}$ is an $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{S}^1$-toric $\mathbb{R}$-varieties. Then, the $\mathbb{R}$-structure

$$\sigma_1 : t \mapsto -t^{-1}$$

on $\mathbb{G}_{m,\mathbb{C}}$ corresponding to the non-trivial $\mathbb{S}^1$-torsor extends to the $\mathbb{R}$-structure

$$[u, v] \mapsto [\bar{v} : -\bar{u}].$$

Therefore, $C = \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2))$ is the other $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{S}^1$-toric $\mathbb{R}$-varieties.

This section ends with the next result.

Proposition 2.4.13. *Let $(\mathbb{T}, \tau)$ be a $\Bbbk$-torus. Assume there exists a $(\mathbb{T}, \tau)$-toric variety (X, σ). Then, the set*

$$\mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X)\right)$$

is finite.

Proof. Since the complexity of the $(\mathbb{T}, \tau)$-action is zero, $\mathrm{Aut}^{\mathbb{T}}(X) \cong \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$. Then, since $\mathrm{Aut}^{\mathbb{T}}(\mathbb{T})$ is a linear group, $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X))$ is finite (see [BS64, §6]). □

2.5 Smooth toric Del Pezzo surfaces

In this section, we will use Proposition 2.4.9 to classify smooth toric Del Pezzo $\mathbb{R}$-surfaces. Del Pezzo surfaces are used in Section 4.2.2 to study the torsors emerging in the Altmann-Hausen presentation over arbitrary fields of characteristic zero.

2.5.1 Definition and properties

Del Pezzo surfaces over algebraically closed fields are studied in [Man86, Chapter IV §24]. See also [Man86, Appendix, §3], [Isk79], [Isk67], and [KM98] for a classification of minimal rational surfaces over non-algebraically closed fields using Del Pezzo surfaces. In Appendix D, we recall some tools of birational geometry that we use in this section.

Definition 2.5.1. Let $\Bbbk$ be a field. A smooth projective $\Bbbk$-surface V is called a Del Pezzo surface if the dual of its canonical sheaf Ω_V^{-1} is ample. The degree of V is the degree of $V_{\bar{\Bbbk}}$, that is the self-intersection number of any anti-canonical Weil divisor $-K_{V_{\bar{\Bbbk}}}$ on $V_{\bar{\Bbbk}}$.

The property of being ample by definition means that there exists an integer $n \in \mathbb{N}^*$ and a closed embedding $\iota : V \hookrightarrow \mathbb{P}^n_{\Bbbk}$, such that $\Omega_V^{-n} \cong \iota^*(\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(1))$. If one can takes $n = 1$, then the sheaf Ω_V^{-1} is very ample.

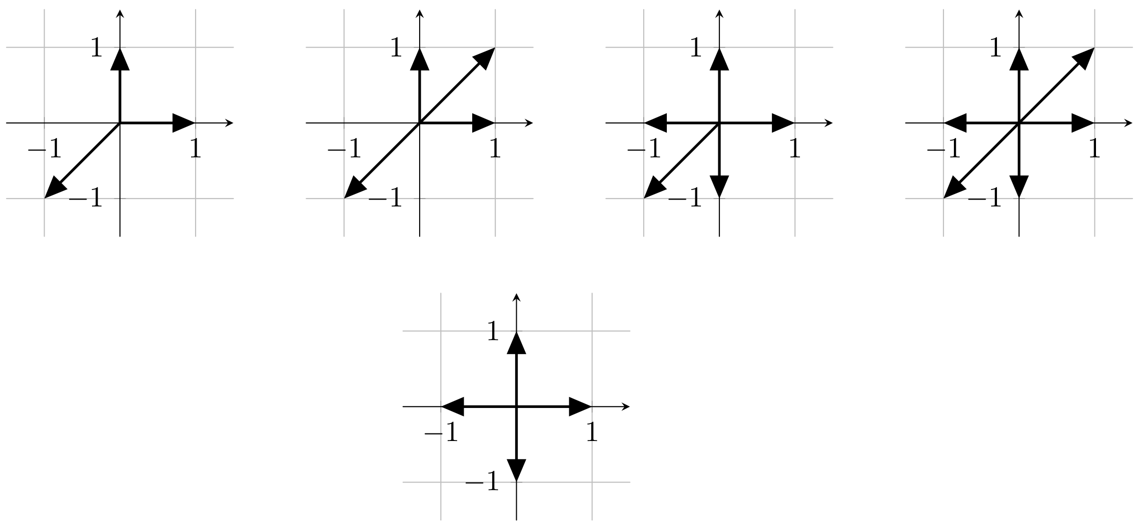
Del Pezzo surfaces over an algebraically (or separably) closed field are classified, up to deformation, by their degree.

Proposition 2.5.2 ([Man86, Theorem 24.4])**.** *Let $\Bbbk$ be an algebraically closed field, and let V be a Del Pezzo $\Bbbk$-surface of degree d. Then,*

(i) $1 \leq d \leq 9$, and the Picard group of V is a free abelian group of rank $10 - d$;

(ii) Either V is isomorphic to the blow-up of $\mathbb{P}^2_{\Bbbk}$ at $r = 9 - d$ $\Bbbk$-points in general position, or $d = 8$ and $V \cong \mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$. Conversely, any smooth projective $\Bbbk$-surface satisfying this condition is a Del Pezzo $\Bbbk$-surface.

Furthermore, V is a $\mathbb{G}^2_{m,\Bbbk}$-toric Del Pezzo $\Bbbk$-surface if and only if either V is isomorphic to the blow-up of $\mathbb{P}^2_{\Bbbk}$ at $r = 9 - d$ $\Bbbk$-toric points for $d \in \{9, 8, 7, 6\}$, or $d = 8$ and $V \cong \mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$. The fan in $N_{\mathbb{Q}} = \mathbb{Q}^2$ are respectively



Remark 2.5.3 ([Man86, Remark 24.4.1]). Let $\mathbb{k}$ be an algebraically closed field. For $5 \leq d \leq 7$, all Del Pezzo surfaces of the same degree are isomorphic.

Let $\mathbb{k}$ be a perfect field. By Galois descent (see [Sta, Lemma 0D2P]), V is a Del Pezzo $\mathbb{k}$-surface if and only if $V_{\overline{\mathbb{k}}}$ is a Del Pezzo $\overline{\mathbb{k}}$-surface. Therefore, either $V_{\overline{\mathbb{k}}}$ is isomorphic to the blow-up of $\mathbb{P}^2_{\overline{\mathbb{k}}}$ at $r = 9 - d$ $\overline{\mathbb{k}}$-points in general position, or $d = 8$ and $V_{\overline{\mathbb{k}}} \cong \mathbb{P}^1_{\overline{\mathbb{k}}} \times \mathbb{P}^1_{\overline{\mathbb{k}}}$. If $d = 9$, then V is a Severi-Brauer $\mathbb{k}$-surface; i.e. a $\mathbb{k}$-form of $\mathbb{P}^2_{\overline{\mathbb{k}}}$.

2.5.2 Classification of real smooth toric Del Pezzo surfaces

In this section, we give a classification of real smooth toric Del Pezzo surfaces based on Proposition 2.4.9 (compare with [Rus02, Proposition 1.2 and Corollary 2.4] and [Ben16, Theorem 4.2, Propositions 4.7, 4.8 and 4.10]. See also [Wal87]). In [SZ21], there is a classification of rational Del Pezzo surfaces of degree 8 and 6 over a perfect field.

Up to isomorphism, the $\mathbb{R}$-forms of $\mathbb{G}^2_{m,\mathbb{C}}$ in the category of $\mathbb{R}$-groups are (see Proposition 4.1.2)

$$\mathbb{G}^2_{m,\mathbb{R}}, \quad \mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1, \quad \mathbb{S}^1 \times \mathbb{S}^1, \quad \text{and} \quad \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}).$$

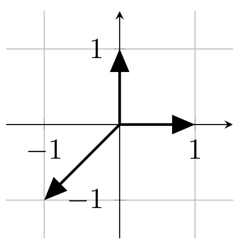
Up to equivalence, the corresponding $\mathbb{R}$-group structures are respectively

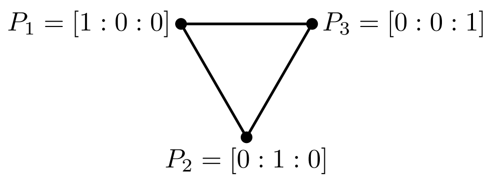
$$\tau_0 \times \tau_0, \quad \tau_0 \times \tau_1, \quad \tau_1 \times \tau_1, \quad \text{and} \quad \tau_2,$$

where $\hat{\tau}_0 = id_N$, $\hat{\tau}_1 = -id_N$ and $\hat{\tau} : (k, l) \mapsto (l, k)$.

Degree 9

• Consider the $\mathbb{R}$-torus $\mathbb{G}^2_{m,\mathbb{R}}$. Then by Hilbert's 90 Theorem (see Theorem 2.3.18), there is no non-trivial torsor. The following fan is Γ-stable (for $\hat{\tau}_0 \times \hat{\tau}_0$). The right hand side picture represent the toric divisors associated to the rays and their intersections.





Therefore, by Proposition 2.4.9, there is up to isomorphism only one $\mathbb{R}$-form of $\mathbb{P}^2_{\mathbb{C}}$ in the category of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})^{\Gamma} = \mathbb{Z}L,$$

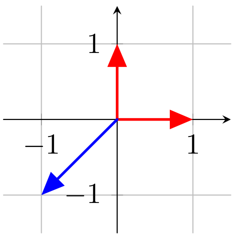
where L is a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i. Moreover, the $\mathbb{R}$-form is a minimal surface since it does not contain (-1)-curves.

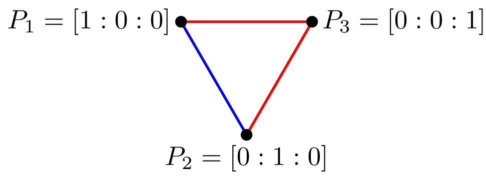
The $\mathbb{R}$-structure $\tau_0 \times \tau_0$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

$$\sigma_0 : [u : v : w] \mapsto [\bar{u} : \bar{v} : \bar{w}],$$

and $(\mathbb{P}^2_{\mathbb{C}}, \sigma_0)$ corresponds to $\mathbb{P}^2_{\mathbb{R}}$. Therefore, $\mathbb{P}^2_{\mathbb{R}}$ is a $\mathbb{G}^2_{m,\mathbb{C}}$-toric $\mathbb{R}$-variety. By Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})^{\Gamma} = \mathrm{Pic}(\mathbb{P}^2_{\mathbb{R}})$.

• Consider the $\mathbb{R}$-torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Then by Theorem 2.3.19, there is no non-trivial torsor. The following fan is Γ-stable (for $\hat{\tau}_2$).





Therefore, by Proposition 2.4.9, there is up to isomorphism only one $\mathbb{R}$-form of $\mathbb{P}^2_{\mathbb{C}}$ in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})^{\Gamma} = \mathbb{Z}(L + \sigma(L)),$$

where L is a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, and where σ is the induced $\mathbb{R}$-structure on $\mathbb{P}^2_{\mathbb{C}}$. Moreover, the $\mathbb{R}$-form is a minimal surface.

The $\mathbb{R}$-structure τ_2 on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

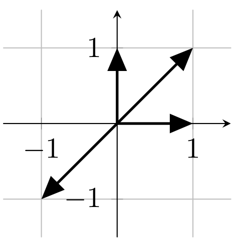
$$\sigma : [u : v : w] \mapsto [\bar{v} : \bar{u} : \bar{w}],$$

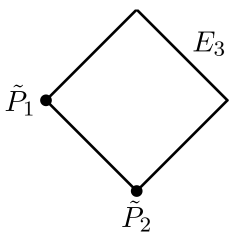
which is equivalent to σ_0, and $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$ corresponds to $\mathbb{P}^2_{\mathbb{R}}$. Another way to see that $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$ corresponds to $\mathbb{P}^2_{\mathbb{R}}$ is to note that P_3 is fixed by σ. Indeed, a Severi-Brauer $\mathbb{k}$-surface with a $\mathbb{k}$-point is $\mathbb{P}^2_{\mathbb{k}}$. Therefore, $\mathbb{P}^2_{\mathbb{R}}$ is a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-variety. By Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})^{\Gamma} = \mathrm{Pic}(\mathbb{P}^2_{\mathbb{R}})$.

• One can show that we cannot find a Γ-representation equivalent to $\hat{\tau}_0 \times \hat{\tau}_1$ (resp. $\hat{\tau}_1 \times \hat{\tau}_1$) such that the above fan of $\mathbb{P}^2_{\mathbb{C}}$ is Γ-stable. Therefore, we cannot endow $\mathbb{P}^2_{\mathbb{C}}$ with an $\mathbb{R}$-structure σ such that $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$ is a $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$-toric $\mathbb{R}$-variety (resp. $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-variety).

Degree 8 (a)

• Consider the $\mathbb{R}$-torus $\mathbb{G}^2_{m,\mathbb{R}}$. There is no non-trivial torsor. The following fan is Γ-stable.





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}L \oplus \mathbb{Z}E_3,$$

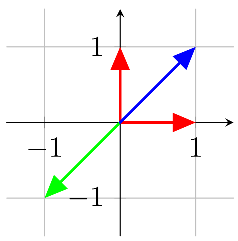
where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, and where E_3 is the exceptional divisor of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_3. The $\mathbb{R}$-form is not minimal since it contains a Γ-invariant (-1)-curve: we can contract E_3, and we obtain $\mathbb{P}^2_{\mathbb{R}}$.

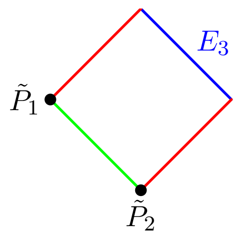
Recall that $\mathbb{P}^2_{\mathbb{R}}$ is a $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-surface. It corresponds to $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$, where

$$\sigma : [u : v : w] \mapsto [\bar{u} : \bar{v} : \bar{w}].$$

Let $P_3 := [0 : 0 : 1]$. It is a toric $\mathbb{C}$-point (i.e. it is the unique fixed point for the $\mathbb{G}^2_{m,\mathbb{C}}$-action on the affine open subset $\{[u : v : 1] \mid u, v \in \mathbb{C}\}$). Since $\sigma(P_3) = P_3$, it corresponds to a toric $\mathbb{R}$-point. Therefore, $\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}})$ is a $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-surface. Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}}))$.

• Consider the $\mathbb{R}$-torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. There is no non-trivial torsor. The following fan is Γ-stable (for $\hat{\tau}_2$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}(L + \sigma(L)) \oplus \mathbb{Z}E_3,$$

where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, E_3 is the exceptional divisor of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_3, and where σ is the induced $\mathbb{R}$-structure on $\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}})$. The $\mathbb{R}$-form is not minimal since it contains a Γ-invariant (-1)-curve: we can contract E_3 and we obtain $\mathbb{P}^2_{\mathbb{R}}$.

Recall that $\mathbb{P}^2_{\mathbb{R}}$ is a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-surface. It corresponds to $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$, where

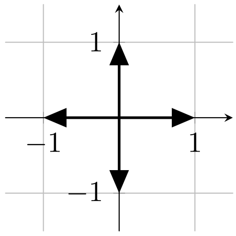
$$\sigma : [u : v : w] \mapsto [\bar{v} : \bar{u} : \bar{w}].$$

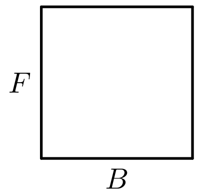
Let $P_3 := [0 : 0 : 1]$ be a toric $\mathbb{C}$-point. Since $\sigma(P_3) = P_3$, it corresponds to a toric $\mathbb{R}$-point. Therefore, $\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}})$ is a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-surface. Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}}))$.

Degree 8 (b)

Compare with [Ben16, Theorem 4.2].

• Consider the $\mathbb{R}$-torus $\mathbb{G}^2_{m,\mathbb{R}}$. There is no non-trivial torsor. The following fan is Γ-stable (for $\hat{\tau}_0 \times \hat{\tau}_0$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties, having Γ-invariant Picard group

$$\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}B \oplus \mathbb{Z}F.$$

The $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is minimal since it contains no (-1)-curves (by Theorem D.0.8).

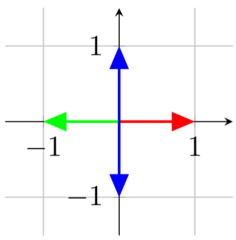
The $\mathbb{R}$-structure $\tau_0 \times \tau_0$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

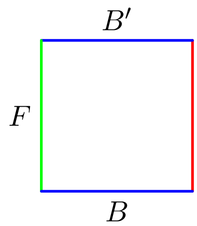
$$\sigma_0 : ([u:v],[u':v']) \mapsto ([\bar{u}:\bar{v}],[\bar{u}':\bar{v}'])$$

on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. Hence, $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ is a $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-variety. Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$. Note that

$$\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \cong \mathrm{Proj}\left(\frac{\mathbb{R}[x,y,z,w]}{(xw-yz)}\right) \cong \mathrm{Proj}\left(\frac{\mathbb{R}[x,y,z,w]}{(x^2+y^2-z^2-w^2)}\right).$$

• Consider the $\mathbb{R}$-torus $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$. There are two $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$-torsors: $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$ and $\mathbb{G}_{m,\mathbb{R}} \times X_1$, where X_1 is the non trivial $\mathbb{S}^1$-torsor defined in Example 2.3.22 (see Proposition 2.5.4). Note that the following fan is Γ-stable (for $\hat{\tau}_0 \times \hat{\tau}_1$).





Therefore, up to isomorphism, there are two $\mathbb{R}$-forms of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$-toric $\mathbb{R}$-varieties, having Γ-invariant Picard group

$$\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}(B+B') \oplus \mathbb{Z}F.$$

The $\mathbb{R}$-structure $\tau_0 \times \tau_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

$$([u:v],[u':v']) \mapsto ([\bar{u}:\bar{v}],[\bar{v}':\bar{u}'])$$

on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ (see Example 2.4.12). We get a $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$-toric $\mathbb{R}$-variety. By Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$.

The $\mathbb{R}$-structure $\tau_0 \times \sigma_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$, where $\sigma_1 : z \mapsto -\bar{z}^{-1}$, extends to an $\mathbb{R}$-structure

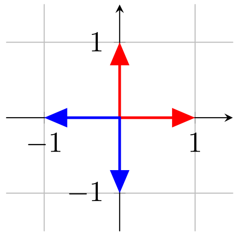
$$([u:v],[u':v']) \mapsto ([\bar{u}:\bar{v}],[\bar{v}':-\bar{u}'])$$

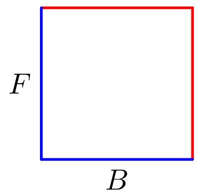
on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathbb{P}^1_{\mathbb{R}} \times C$, where

$$C := \mathrm{Proj}\left(\frac{\mathbb{R}[x,y,z]}{(x^2+y^2+z^2)}\right).$$

We get a $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$-toric $\mathbb{R}$-variety with no $\mathbb{R}$-points.

• Consider the $\mathbb{R}$-torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. There is no non-trivial torsor. Note that the following fan is Γ-stable (for $\hat{\tau}_2$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties, having Γ-invariant Picard group $\mathbb{Z}(B+F)$.

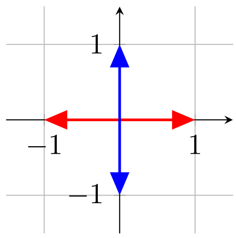
The $\mathbb{R}$-structure $\sigma_1 \times \sigma_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

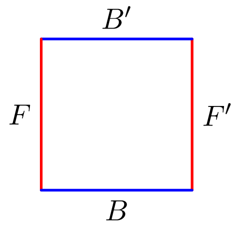
$$([u:v],[u':v']) \mapsto ([\bar{u}':\bar{v}'],[\bar{u}:\bar{v}])$$

on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$. We get a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-variety. By Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}))$. Note that

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathrm{Proj}\left(\frac{\mathbb{R}[x,y,z,w]}{(x^2-y^2-z^2-w^2)}\right).$$

• Consider the $\mathbb{R}$-torus $\mathbb{S}^1 \times \mathbb{S}^1$. There are three $\mathbb{S}^1 \times \mathbb{S}^1$-torsors: $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \times X_1$ and $X_1 \times X_1$ (see Proposition 2.5.4). Note that the following fan is Γ stable (for $\hat{\tau}_1 \times \hat{\tau}_1$).





Therefore, up to isomorphism, there are three $\mathbb{R}$-forms of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-varieties, having Γ-invariant Picard group $\mathbb{Z}(B+B') \oplus \mathbb{Z}(F+F')$.

The $\mathbb{R}$-structure $\tau_1 \times \tau_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

$$([u:v],[u':v']) \mapsto ([\bar{v}:\bar{u}],[\bar{v}':\bar{u}'])$$

on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. We get a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-variety containing the trivial $\mathbb{S}^1 \times \mathbb{S}^1$-torsor as an $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset. By Lemma D.0.9, $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$.

The $\mathbb{R}$-structure $\tau_1 \times \sigma_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

$$([u:v],[u':v']) \mapsto ([\bar{v}:\bar{u}],[\bar{v}':-\bar{u}'])$$

on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathbb{P}^1_{\mathbb{R}} \times C$. We get a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-variety with no $\mathbb{R}$-points containing the $\mathbb{S}^1 \times \mathbb{S}^1$-torsor $\mathbb{S}^1 \times X_1$ as an $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset.

The $\mathbb{R}$-structure $\sigma_1 \times \sigma_1$ on $\mathbb{G}^2_{m,\mathbb{C}}$ extends to an $\mathbb{R}$-structure

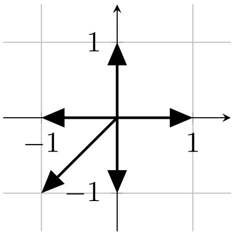
$$([u:v],[u':v']) \mapsto ([\bar{v}:-\bar{u}],[\bar{v}':-\bar{u}'])$$

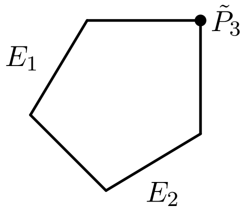
on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $C \times C$. We get a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-variety with no $\mathbb{R}$-points containing the $\mathbb{S}^1 \times \mathbb{S}^1$-torsor $X_1 \times X_1$ as an $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset. Note that

$$C \times C \cong \mathrm{Proj}\left(\frac{\mathbb{R}[x,y,z,w]}{(x^2+y^2+z^2+w^2)}\right).$$

Degree 7

• Consider the $\mathbb{R}$-torus $\mathbb{G}^2_{m,\mathbb{R}}$. Then there is no non-trivial torsor. Note that the fan is Γ-stable (for $\hat{\tau}_0 \times \hat{\tau}_0$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2,$$

where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, and where E_1 and E_2 are the exceptional divisors of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_1 and P_2. This $\mathbb{R}$-surface is not minimal, since it contains a Γ-invariant (-1)-curve. Indeed, if we contract it, we get $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$.

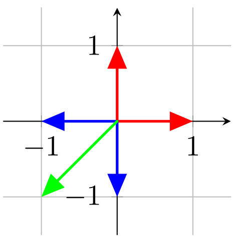
Therefore, $\mathrm{Bl}_Q(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$, where Q is a toric $\mathbb{R}$-point on $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, is a $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-surface.

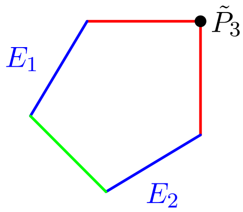
We can also contract the two disjoint (-1)-curves and we get $\mathbb{P}^2_{\mathbb{R}}$. Then, $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{R}})$ is a $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-surface. Hence, we get an isomorphism of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties

$$\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_Q(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}).$$

Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}}))^\Gamma = \mathrm{Pic}(\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{R}}))$.

• Consider the $\mathbb{R}$-torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Then there is no non-trivial torsor. Note that the fan is Γ-stable (for $\hat{\tau}_2$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathbb{Z}(L + \sigma(L)) \oplus \mathbb{Z}(E_1 + E_2),$$

where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, where E_1 and E_2 are the exceptional divisors of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_1 and P_2, and where σ is the induced $\mathbb{R}$-structure on $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}})$. This $\mathbb{R}$-surface is not minimal, since it contains a Γ-invariant (-1)-curve; we can contract it we get an $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. We can also contract the pair of Γ-invariant (-1)-curves and we get $\mathbb{P}^2_{\mathbb{R}}$.

Let $P_1 := [1:0:0]$ and let $P_2 := [0:1:0]$ be two toric $\mathbb{C}$-points in $\mathbb{P}^2_{\mathbb{C}}$. Consider the $\mathbb{R}$-structure on $\mathbb{P}^2_{\mathbb{C}}$

$$\sigma : [u : v : w] \mapsto [\bar{v} : \bar{u} : \bar{w}].$$

Recall that $(\mathbb{P}^2_{\mathbb{C}}, \sigma)$ corresponds to $\mathbb{P}^2_{\mathbb{R}}$, which is a T-toric $\mathbb{R}$-variety. Then, $P_1 \sqcup P_2$ forms a Γ-orbit, hence it defines a closed point (corresponding to a maximal ideal) in $\mathbb{P}^2_{\mathbb{R}}$, which is not an $\mathbb{R}$-point. Hence $\mathrm{Bl}_{P_1 \sqcup P_2}(\mathbb{P}^2_{\mathbb{R}})$ is a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric Del Pezzo $\mathbb{R}$-surface. Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{Bl}_{P_1 \sqcup P_2}(\mathbb{P}^2_{\mathbb{R}}))$.

Let $Q := ([0:1],[0:1]) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ be a toric $\mathbb{C}$-point, and let

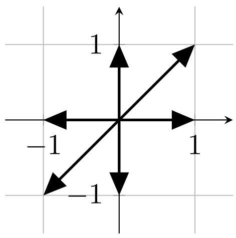
$$\sigma : ([u:v],[u':v']) \mapsto ([\bar{u}':\bar{v}'],[\bar{u}:\bar{v}])$$

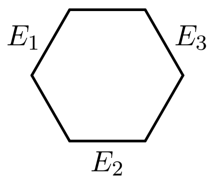
be an $\mathbb{R}$-structure on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$. Then, Q defines an $\mathbb{R}$-point on $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$. Then $\mathrm{Bl}_Q(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}))$ is also a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. Therefore, we get an isomorphism of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties

$$\mathrm{Bl}_Q(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})) \cong \mathrm{Bl}_{P_1 \sqcup P_2}(\mathbb{P}^2_{\mathbb{R}}).$$

Degree 6

• Consider the $\mathbb{R}$-torus $\mathbb{G}^2_{m,\mathbb{R}}$. There is no non-trivial torsor. Note that the fan is Γ-stable (for $\hat{\tau}_0 \times \hat{\tau}_0$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3,$$

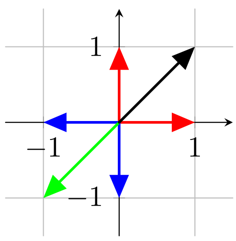
where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, where E_1, E_2, and E_3 are the exceptional divisors of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_1, P_2, and P_3, and where σ is the induced $\mathbb{R}$-structure on $\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}})$. This $\mathbb{R}$-surface is not minimal, we can contract three disjoint (-1)-curves and we get $\mathbb{P}^2_{\mathbb{R}}$. We can also contract two disjoint 'opposite' (-1)-curves and we get $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$.

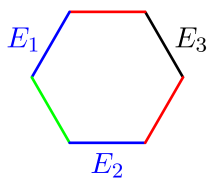
Therefore, we get an isomorphism of $\mathbb{G}^2_{m,\mathbb{R}}$-toric $\mathbb{R}$-varieties

$$\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_{Q_1,Q_2}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}),$$

where Q_1 and Q_2 are two toric $\mathbb{R}$-points on $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. Moreover, by Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}))$.

• Consider the $\mathbb{R}$-torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. There is no non-trivial torsor. Note that the fan is Γ-stable (for $\hat{\tau}_2$).





Therefore, up to isomorphism, there is only one $\mathbb{R}$-form of $\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. The Γ-invariant Picard group is

$$\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathbb{Z}(L+\sigma(L)) \oplus \mathbb{Z}(E_1+E_2) \oplus \mathbb{Z}E_3,$$

where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, where E_1, E_2, and E_3 are the exceptional divisors of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_1, P_2, and P_3, and where σ is the induced $\mathbb{R}$-structure on $\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$. This $\mathbb{R}$-surface is not minimal, we can contract an orbit of two (-1)-curves and an invariant (-1)-curve that does not intersect the first two one, and we get $\mathbb{P}^2_{\mathbb{R}}$. Note that if we contract the two Γ-invariant (-1)-curves, we obtain $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$.

Therefore, $\mathrm{Bl}_{P_1\sqcup P_2,P_3}(\mathbb{P}^2_{\mathbb{R}})$, where $P_1 \sqcup P_2$ is a closed point that is not an $\mathbb{R}$-point, is a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-surface. By Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}}))^{\Gamma} = \mathrm{Pic}(\mathrm{Bl}_{P_1\sqcup P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}))$.

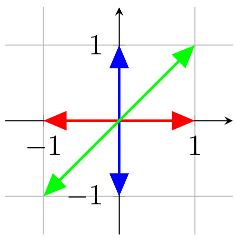
Let $Q_1 := ([0:1],[0:1])$, and let $Q_2 := ([1:0],[1:0])$ be two toric $\mathbb{C}$-points in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, and let

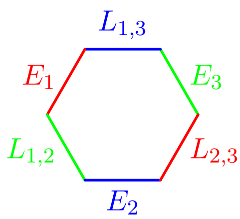
$$\sigma : ([u:v],[u':v']) \mapsto ([\bar{u}':\bar{v}'],[\bar{u}:\bar{v}])$$

be an $\mathbb{R}$-structure on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ corresponding to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$. Then, Q_1 and Q_2 define toric $\mathbb{R}$-points on $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$. Note that $\mathrm{Bl}_{Q_1,Q_2}(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}))$ is also a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties. Therefore, we get an isomorphism of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-toric $\mathbb{R}$-varieties

$$\mathrm{Bl}_{Q_1,Q_2}(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})) \cong \mathrm{Bl}_{P_1\sqcup P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}).$$

- Consider the $\mathbb{R}$-torus $\mathbb{S}^1 \times \mathbb{S}^1$. There are three $\mathbb{S}^1 \times \mathbb{S}^1$-torsors: $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \times X_1$ and $X_1 \times X_1$ (see Proposition 2.5.4). Note that the fan is Γ stable (for $\hat{\tau}_1 \times \hat{\tau}_1$).





Therefore, up to isomorphism, there are three $\mathbb{R}$-forms of $\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$ in the category of $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-varieties, having Γ-invariant Picard group

$$\mathbb{Z}(L+\sigma(L))+\mathbb{Z}(E_1+L_{2,3})+\mathbb{Z}(E_2+L_{1,3})+\mathbb{Z}(E_3+L_{1,2}) = \mathbb{Z}(-E_2-L)\oplus\mathbb{Z}(E_2-E_1-E_3-L)\oplus\mathbb{Z}(L+\sigma(L)),$$

where L is the strict transform of a line in $\mathbb{P}^2_{\mathbb{C}}$ that does not contain any of the P_i, E_1, E_2, and E_3 are the exceptional divisors of the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in P_1, P_2, and P_3, and where σ is the induced $\mathbb{R}$-structure on $\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$. These $\mathbb{R}$-surfaces are not minimal, we can contract an orbit of two (-1)-curves, we get an $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ endowed with a $\mathbb{S}^1 \times \mathbb{S}^1$-action.

Let $Q_1 := ([0:1],[1:0])$, and let $Q_2 := ([1:0],[0:1])$ be two toric $\mathbb{C}$-points in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Then, $\mathrm{Bl}_{Q_1,Q_2}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}) \cong \mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$ is a Del Pezzo $\mathbb{C}$-surface of degree six.

Recall that $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ is a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface of degree eight corresponding to $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \sigma)$, where σ is the following $\mathbb{R}$-structure on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$$\sigma : ([u:v],[u':v']) \mapsto ([\bar{v}:\bar{u}],[\bar{v}':\bar{u}']),$$

and note that $Q_2 = \sigma(Q_1)$. Therefore, the blow-up of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ in the closed point $Q_1 \sqcup Q_2$ is a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface containing the trivial $\mathbb{S}^1 \times \mathbb{S}^1$-torsor as an $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset (see [Man86, Theorem 30.3.1]). By Lemma D.0.9, $\mathrm{Pic}(\mathrm{Bl}_{Q_1,Q_2}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}))^\Gamma = \mathrm{Pic}(\mathrm{Bl}_{Q_1 \sqcup Q_2}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}))$.

Recall that $\mathbb{P}^1_{\mathbb{R}} \times C$ is a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface of degree eight corresponding to $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \sigma)$, where σ is the following $\mathbb{R}$-structure on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$$\sigma : ([u : v], [u' : v']) \mapsto ([\bar{v} : \bar{u}], [\bar{v}' : -\bar{u}']),$$

and note that $Q_2 = \sigma(Q_1)$. Therefore, the blow-up of $\mathbb{P}^1_{\mathbb{R}} \times C$ in the closed point $Q_1 \sqcup Q_2$ is a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface containing the $\mathbb{S}^1 \times \mathbb{S}^1$-torsor $\mathbb{S}^1 \times X_1$ as an $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset.

Recall that $C \times C$ is a $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface of degree eight corresponding to $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \sigma)$, where σ is the following $\mathbb{R}$-structure on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$$\sigma : ([u : v], [u' : v']) \mapsto ([\bar{v} : -\bar{u}], [\bar{v}' : -\bar{u}']),$$

and note that $Q_2 = \sigma(Q_1)$. Therefore, the blow-up of $C \times C$ in the closed point $Q_1 \sqcup Q_2$ is an $\mathbb{S}^1 \times \mathbb{S}^1$-toric $\mathbb{R}$-surface containing the $\mathbb{S}^1 \times \mathbb{S}^1$-torsor $X_1 \times X_1$ as a $\mathbb{S}^1 \times \mathbb{S}^1$-stable dense open subset.

With the above notations, we get the next proposition (compare with [Rus02, Proposition 1.2 and Corollary 2.4] and [Ben16, Theorem 4.2, Propositions 4.7, 4.8 and 4.10]. See also [Wal87] and [SZ21].

Proposition 2.5.4 (Classification of smooth toric Del Pezzo $\mathbb{R}$-surfaces)**.**

X	T	$\Bbbk$-*form of* X *in the category of* T-*toric* $\mathbb{R}$-*varieties*
$\mathbb{P}^2_{\mathbb{C}}$	$\mathbb{G}^2_{m,\mathbb{R}}$	$\mathbb{P}^2_{\mathbb{R}}$
	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$	$\mathbb{P}^2_{\mathbb{R}}$
$\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{C}})$	$\mathbb{G}^2_{m,\mathbb{R}}$	$\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}})$
	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$	$\mathrm{Bl}_{P_3}(\mathbb{P}^2_{\mathbb{R}})$
$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$	$\mathbb{G}^2_{m,\mathbb{R}}$	$\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$
	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$
	$\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1$	$\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, $\mathbb{P}^1_{\mathbb{R}} \times C$
	$\mathbb{S}^1 \times \mathbb{S}^1$	$\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, $\mathbb{P}^1_{\mathbb{R}} \times C$, $C \times C$
$\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{C}})$	$\mathbb{G}^2_{m,\mathbb{R}}$	$\mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_Q(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$
	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$	$\mathrm{Bl}_{P_1 \sqcup P_2}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_Q(R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}))$
$\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{C}})$	$\mathbb{G}^2_{m,\mathbb{R}}$	$\mathrm{Bl}_{P_1,P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_{Q_1,Q_2}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$
	$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$	$\mathrm{Bl}_{P_1 \sqcup P_2,P_3}(\mathbb{P}^2_{\mathbb{R}}) \cong \mathrm{Bl}_{Q_1,Q_2}(R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}}))$
	$\mathbb{S}^1 \times \mathbb{S}^1$	$\mathrm{Bl}_{Q_1 \sqcup Q_2}(\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}})$, $\mathrm{Bl}_{Q_1 \sqcup Q_2}(\mathbb{P}^1_{\mathbb{R}} \times C)$, $\mathrm{Bl}_{Q_1 \sqcup Q_2}(C \times C)$

Chapter 3

Altmann-Hausen presentation

Normal affine algebraic varieties endowed with an effective torus action over an algebraically closed field of characteristic zero admit a geometrico-combinatorial presentation due to Altmann and Hausen in [AH06]. In this context, an n-dimensional torus is an algebraic group $\mathbb{T}$ isomorphic to $\mathbb{G}_{m,\mathbb{k}}^n$. Over a non closed field $\mathbb{k}$, a $\mathbb{k}$-torus is defined as an algebraic group T such that

$$T_{\overline{\mathbb{k}}} := T \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(\overline{\mathbb{k}}) \cong \mathbb{G}_{m,\overline{\mathbb{k}}}^n,$$

for some $n \in \mathbb{N}^*$. Over $\overline{\mathbb{k}}$ all tori are *split*, i.e. isomorphic to $\mathbb{G}_{m,\overline{\mathbb{k}}}^n$ for some $n \in \mathbb{N}^*$, but over $\mathbb{k}$ a torus may be non-split and may have non-trivial torsors. The existence of non split tori and non-trivial torsors imply to insert additional data in the geometrico-combinatorial presentation over non closed fields, as we have seen in the toric case in Section 2.4.2.

Over the past few years, using Galois descent methods, the Altmann-Hausen presentation was extended in different directions. Langlois in [Lan15, Lan21] extended it for complexity one torus actions over an arbitrary field in any characteristic using a specific construction. In the $\mathbb{C}/\mathbb{R}$ setting, an $\mathbb{R}$-torus is a product of copies of the split torus $\mathbb{G}_{m,\mathbb{R}}$, of the real circle $\mathbb{S}^1 = \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1))$, and of the Weil restriction $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ (see Proposition 4.1.2). Contrary to $\mathbb{G}_{m,\mathbb{R}}$ and $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, the circle $\mathbb{S}^1$ has a non-trivial torsor, namely $\mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1))$. Dubouloz, Liendo and Petitjean focused on normal affine $\mathbb{R}$-varieties endowed with $\mathbb{S}^1$-actions in [DL18, DP20], and they observed that an additional datum is needed to encode these varieties because of the existence of the non-trivial $\mathbb{S}^1$-torsor. More generally, a geometrico-combinatorial presentation of normal affine $\mathbb{R}$-varieties endowed with an arbitrary $\mathbb{R}$-torus action based on [AH06] was established by the author in [Gil22a]. These three presentations agree in the sense that they use a similar language and that we recover the first two presentations from the last one in the $\mathbb{C}/\mathbb{R}$ setting.

As noted in [Gil22a], it is natural and reasonable to expect that a general presentation of normal affine varieties endowed with torus actions over arbitrary fields of characteristic zero can be obtained by combining Altmann-Hausen theory for split torus actions together with appropriate Galois descent methods: this is precisely the goal of this chapter and this is the reason why we work over characteristic zero fields. We extend the Altmann-Hausen presentation to arbitrary fields of characteristic zero using a similar approach to the one adopted in [Gil22a] (see the preprint [Gil22b]). However, in this context, the Galois extension $\overline{\mathbb{k}}/\mathbb{k}$ may be infinite and the description of $\mathbb{k}$-tori is more involved (except in dimension 1). We give a complete description of normal affine varieties endowed with a torus action over arbitrary fields of characteristic zero. From this presentation, we recover the one given in [Gil22a] when $\mathbb{k} = \mathbb{R}$.

3.1 Altmann-Hausen presentation over algebraically closed fields

In this section, $\Bbbk$ is an algebraically closed field of characteristic zero.

3.1.1 Preliminaries in convex geometry

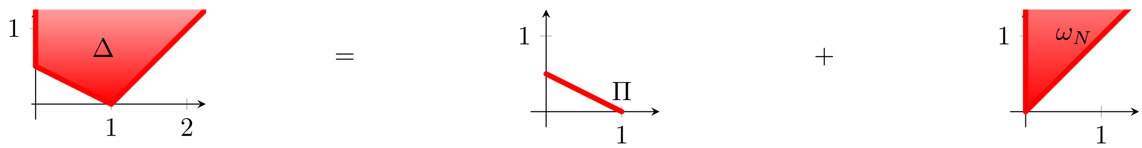
A subset $\Pi \subset N_{\mathbb{Q}}$ is called a *polytope* if there exists a finite set $S \subset N_{\mathbb{Q}}$ such that Π is the convex hull of S, and it is called a *rational polytope* if S can be taken inside the lattice N. A proper face Π' of Π is the intersection of Π with a supporting affine hyperplane.

A *convex polyhedron* is the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. For us, a *polyhedron* in $N_{\mathbb{Q}}$ is always a convex polyhedron. The *relative interior* of a polyhedron Δ, denoted by $\mathrm{Relint}(\Delta)$, is obtained by removing all proper faces from Δ. Moreover, any polyhedron Δ in $N_{\mathbb{Q}}$ admits a Minkowski sum decomposition:

$$\Delta = \Pi + \omega_N,$$

where $\Pi \subset N_{\mathbb{Q}}$ is a polytope and $\omega_N \subset N_{\mathbb{Q}}$ is a cone. In this decomposition, the cone ω_N is unique and called the *tail cone* of Δ (see [AH06, §1]).

Example 3.1.1. $\Delta = \Pi + \omega_N$



Definition 3.1.2. Let ω_N be a pointed cone in $N_{\mathbb{Q}}$. By a *ω_N-polyhedron* in $N_{\mathbb{Q}}$, we mean a polyhedron in $N_{\mathbb{Q}}$ having the cone ω_N as its tail cone. We denote the set of all ω_N-polyhedra in $N_{\mathbb{Q}}$ by $\mathrm{Pol}^+_{\omega_N}(N_{\mathbb{Q}})$.

The Minkowski sum of two ω_N-polyhedra in $N_{\mathbb{Q}}$ is again a ω_N-polyhedron in $N_{\mathbb{Q}}$. Thus, endowed with Minkowski sum, $\mathrm{Pol}^+_{\omega_N}(N_{\mathbb{Q}})$ is an abelian monoid, whose neutral element is ω_N [AH06, §1].

3.1.2 Polyhedral divisors

We now introduce the language of polyhedral divisors and proper polyhedral divisors. The idea is to replace rational coefficients by tailed polyhedra. This section is based on [Gil22a, §3.1], but for a more detailed version, see the original one in [AH06, §2].

Let Y be a normal $\Bbbk$-variety. The group of Weil divisors on Y is denoted $\mathrm{WDiv}(Y)$ and the group of Cartier divisors on Y is denoted by $\mathrm{CDiv}(Y)$. Since Y is normal, we have an inclusion $\mathrm{CDiv}(Y) \subset \mathrm{WDiv}(Y)$. A Cartier (resp. Weil) $\mathbb{Q}$-divisor is an element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{CDiv}(Y)$ (resp $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y)$). The sheaf of sections $\mathcal{O}_Y(D)$ of a Weil $\mathbb{Q}$-divisor D on Y is defined by

$$\mathrm{H}^0(V, \mathcal{O}_Y(D)) := \left\{ f \in \Bbbk(Y) \;\middle|\; \mathrm{div}(f)|_V + D|_V \geq 0 \right\} \cup \{0\},$$

where $V \subset Y$ is an open subset. Now we turn to divisors with tailed polyhedra coefficients.

Definition 3.1.3. Let ω_N be a pointed cone in $N_{\mathbb{Q}}$. An *ω_N-polyhedral divisor* on Y is a formal sum

$$\mathcal{D} = \sum_Z \Delta_Z \otimes Z \in \mathrm{Pol}^+_{\omega_N}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y)$$

over all prime divisors $Z \subset Y$, and $\Delta_Z = \omega_N$ for all but finitely prime divisors Z.

Let $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$ be a ω_N-polyhedral divisor on Y. For a prime divisor Z on Y we denote the support function of Δ_Z by

$$h_{\Delta_Z} : \omega_N^\vee \to \mathbb{Q},\ m \mapsto \min\{\langle m|v\rangle \mid v \in \Delta_Z\}.$$

This map is piece-wise linear; if Λ_{Δ_Z} is the normal quasi-fan associated to Δ_Z having $\omega_N^\vee$ as support, then h_{Δ_Z} is linear on the cones of Λ_{Δ_Z} (see Appendix C.1).

For every $m \in \omega_N^\vee$ we can evaluate $\mathcal{D}$ in m by letting $\mathcal{D}(m)$ be the Weil $\mathbb{Q}$-divisor on Y defined by

$$\mathcal{D}(m) := \sum_Z h_{\Delta_Z}(m) \otimes Z \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{WDiv}(Y).$$

The map $m \mapsto \mathcal{D}(m)$ is a convex piece-wise linear map; it is linear on the cones of the quasi-fan having $\omega_N^\vee$ as support constructed from the common refinement of the fan Λ_{Δ_Z} (see Proposition C.1.5). The convex property means that

$$\forall m,\ m' \in M \cap \omega_N^\vee,\quad \mathcal{D}(m + m') \geq \mathcal{D}(m) + \mathcal{D}(m').$$

Before introducing proper polyhedral divisors, we recall the following definitions:

Definition 3.1.4 ([CLS11, Definition 6.0.23])**.** A Cartier $\mathbb{Q}$-divisor D on a $\Bbbk$-variety Y is called *semi-ample* if it admits a *basepoint-free* multiple. That is, D is semi-ample if there exits $n \in \mathbb{N}^*$ such that for all $y \in Y$, there exits $f \in \mathrm{H}^0(Y, \mathcal{O}_Y(nD))$ with $f(y) \neq 0$ (the set of open subsets $Y_f := Y \backslash \mathrm{Supp}(\mathrm{div}(f) + D)$, with $f \in \mathrm{H}^0(Y, \mathcal{O}_Y(nD))$, cover Y).

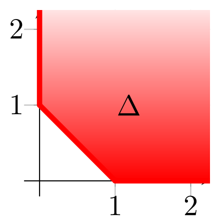
Definition 3.1.5. A Cartier $\mathbb{Q}$-divisor D on Y is called *big* if, for some $n \in \mathbb{N}^*$, there exists a section $f \in \mathrm{H}^0(Y, \mathcal{O}_Y(nD))$ with an affine non-vanishing locus $Y_f := Y \backslash \mathrm{Supp}(\mathrm{div}(f) + D)$.

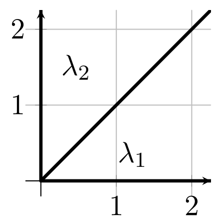
Definition 3.1.6. A *proper ω_N-polyhedral divisor* on a $\Bbbk$-variety Y, abbreviated an *ω_N-pp-divisor*, is an ω_N-polyhedral divisor $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$ on Y satisfying the following properties:
(i) for all $m \in \omega_N^\vee \cap M$, $\mathcal{D}(m)$ is a semi-ample Cartier $\mathbb{Q}$-divisor on Y; and
(ii) for all $m \in \mathrm{Relint}(\omega_N^\vee) \cap M$, $\mathcal{D}(m)$ is big.

The sum of two ω_N-pp-divisors with respect to a given cone ω_N is again an ω_N-pp-divisor. Thus, ω_N-pp-divisors form a monoid denoted by $\mathrm{PPDiv}_{\mathbb{Q}}(Y, \omega_N)$.

Example 3.1.7. Let $N = \mathbb{Z}^2$, let $\omega_N = \mathbb{Q}^2_{\geq 0}$, and let Δ be the ω_N-polyhedron defined below. The normal fan associated to Δ consists of the two cones λ_1 and λ_2 refining the cone $\omega_N^\vee = \mathbb{Q}^2_{\geq 0}$ of the dual lattice $M = \mathbb{Z}^2$ (see Section C.1). The support function $h_\Delta : \omega_N^\vee \to \mathbb{Q},\ m \mapsto \min\{\langle m|v\rangle \mid v \in \Delta\}$ is linear on each λ_i, and we obtain:

$$h_\Delta(m_1, m_2) = \begin{cases} m_2 & \text{if } (m_1, m_2) \in \lambda_1 \\ m_1 & \text{if } (m_1, m_2) \in \lambda_2 \end{cases}$$

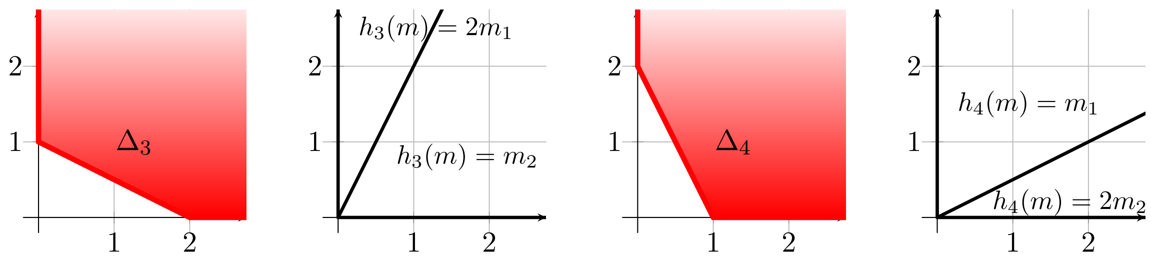




Consider a divisor $\mathcal{D} := \Delta \otimes D$ on a normal variety Y, where D is a prime divisor. Then,

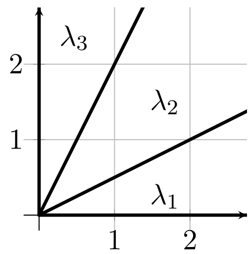
$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes D & \text{if } (m_1, m_2) \in \lambda_1 \\ m_1 \otimes D & \text{if } (m_1, m_2) \in \lambda_2 \end{cases}$$

Example 3.1.8. Let $N = \mathbb{Z}^2$, let $\omega_N = \mathbb{Q}^2_{\geq 0}$, let $\Delta_1 = \Delta_2 = \omega_N$ and let Δ_3 and Δ_4 be the ω_N-polyhedra defined in the following illustrations. The normal quasi-fan associated to Δ_3 (resp. Δ_4) consists of two cones refining the cone $\omega_N^\vee = \mathbb{Q}^2_{\geq 0}$ of the dual lattice $M = \mathbb{Z}^2$. The support function of the polyhedron Δ_i is denoted $h_i : \omega_N^\vee \to \mathbb{Q},\ m \mapsto \min\{\langle m|v\rangle \mid v \in \Delta_i\}$. Note that $h_1 = h_2 = 0$.



Consider the divisor $\mathcal{D} := \Delta_1 \otimes D_1 + \Delta_2 \otimes D_2 + \Delta_3 \otimes D_3 + \Delta_4 \otimes D_4$ on a normal variety Y, where the D_i are prime divisors. We have $\mathcal{D} = \Delta_3 \otimes D_3 + \Delta_4 \otimes D_4$. Considering the fan refining these two normal fans, we obtain:

$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes D_3 + 2m_2 \otimes D_4 & \text{if } (m_1, m_2) \in \lambda_1 \\ m_2 \otimes D_3 + m_1 \otimes D_4 & \text{if } (m_1, m_2) \in \lambda_2 \\ 2m_1 \otimes D_3 + m_1 \otimes D_4 & \text{if } (m_1, m_2) \in \lambda_3 \end{cases}$$



3.1.3 Altmann and Hausen main theorems

Let us present the main results of [AH06] about the geometrico-combinatorial presentation of normal affine $\Bbbk$-varieties endowed with an effective torus action.

Definition 3.1.9. A $\Bbbk$-variety Y is said to be *semi-projective* if its $\Bbbk$-algebra of global functions $\mathrm{H}^0(Y, \mathcal{O})$ is finitely generated and Y is projective over $Y_0 = \mathrm{Spec}(\mathrm{H}^0(Y, \mathcal{O}))$.

Remark 3.1.10. Note that affine $\Bbbk$-varieties and projective $\Bbbk$-varieties are semi-projective. A semi-projective variety is quasi-projective. Indeed, $Y \to Y_0$ is a projective morphism, moreover Y_0 is an affine $\Bbbk$-variety (so quasi-projective). Then the morphism $Y \to \mathrm{Spec}(\Bbbk)$ is quasi-projective.

Let Y be a normal semi-projective $\Bbbk$-variety, let ω_N be a pointed cone in $N_{\mathbb{Q}}$ and let $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$ be an ω_N-pp-divisor on Y. By [AH06, Proposition 2.11], for all $m,\ m' \in M \cap \omega_N^\vee$, we have $\mathcal{D}(m + m') \geq \mathcal{D}(m) + \mathcal{D}(m')$. So, for all $m,\ m' \in \omega_N^\vee \cap M$, we have a map:

$$\mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m))\right) \otimes \mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m'))\right) \to \mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m + m'))\right).$$

This ensures that the $\mathrm{H}^0\left(Y, \mathcal{O}_Y\right)$-sub-modules $\mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m))\right)$ of $\Bbbk(Y)$ can be put together into an M-graded $\Bbbk$-algebra

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \omega_N^\vee \cap M} \mathrm{H}^0\left(Y, \mathcal{O}_Y(\mathcal{D}(m))\right) \mathfrak{X}_m \subset \Bbbk(Y)[M],$$

where $\mathfrak{X}_m$ is an indeterminate of weight m such that for all $m, m' \in M$, $\mathfrak{X}_m \mathfrak{X}_{m'} = \mathfrak{X}_{m+m'}$. We denote by $X[Y, \mathcal{D}] := \mathrm{Spec}(A[Y, \mathcal{D}])$ the associated affine $\mathbb{T}$-scheme.

Definition 3.1.11 (AH-datum)**.** Let $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$ be a $\Bbbk$-torus. An *AH-datum* is a triple $(\omega_N, Y, \mathcal{D})$, where ω_N is a pointed cone in $N_{\mathbb{Q}}$, Y is a normal semi-projective $\Bbbk$-variety, and $\mathcal{D}$ is an ω_N-pp-divisor on Y.

Let X be a $\Bbbk$-variety endowed with an effective $\mathbb{T}$-action. Recall that we can write

$$\Bbbk[X] = \bigoplus_{m \in \omega_M \cap M} \Bbbk[X]_m,$$

the spaces $\Bbbk[X]_m$ consisting of semi-invariant regular functions on X, and where ω_M is the weight cone of the $\mathbb{T}$-action on X (see Section 2.3.1). The general idea of the construction of Altmann-Hausen is to identify $\Bbbk(X)^{\mathbb{T}}$ with $\Bbbk(Y)$ for some semi-projective variety Y, and use an appropriate pp-divisor on Y to construct the grading of $\Bbbk[X]$ via an identification between $\Bbbk[X]_m$ and $\mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))\,\mathfrak{X}_m$.

Theorem 3.1.12 ([AH06, Theorems 3.1 and 3.4], see also Appendix E)**.** *Fix a* $\Bbbk$*-torus* $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$*.*

(i) Let $(\omega_N, Y, \mathcal{D})$ *be an* AH-datum*. The affine scheme* $X[Y, \mathcal{D}]$ *is a normal* $\Bbbk$*-variety, of dimension* $\dim(Y) + \dim(\mathbb{T})$*, endowed with an effective* $\mathbb{T}$*-action of weight cone* $\omega_N^\vee$*.*

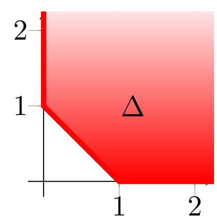
(ii) Conversely, let X *be an affine normal variety endowed with an effective* $\mathbb{T}$*-action, and let* ω_N *be the cone in* $N_{\mathbb{Q}}$ *dual to the weight cone. There exists an* AH-datum $(\omega_N, Y, \mathcal{D})$ *such that there is an isomorphism of* $\mathbb{T}$*-varieties* $X \cong X[Y, \mathcal{D}]$*, i.e. the graded* $\Bbbk$*-algebras* $\Bbbk[X]$ *and* $A[Y, \mathcal{D}]$ *are isomorphic.*

The first item of Theorem 3.1.12 is a part of the result obtained by Altmann-Hausen in [AH06]. See Appendix E for the complete result and for some examples in the context of $\mathbb{G}_{m,\mathbb{C}}$-actions on affine normal surfaces.

Remark 3.1.13 ([AH06, §4, Step 1 of *proof of Theorem 3.1*])**.** Let $(Y, \mathcal{D})$ be an AH-datum. Since $\mathcal{D}(m)$ is a semi-ample Cartier $\mathbb{Q}$-divisor for all $m \in \omega_N^\vee \cap M$, $A[Y, \mathcal{D}]$ is of finite type and integrally closed.

Example 3.1.14 (See Example 2.3.1)**.** The affine variety $\mathbb{A}^3_{\mathbb{C}}$ endowed with the action of $\mathbb{G}^2_{m,\mathbb{C}}$ given by $(s,t)\cdot(x,y,z) = (sx, ty, stz)$ is described by a semi-projective variety $Y := \mathbb{P}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \cup \{\infty\}$, and a pp-divisor on Y defined by $\mathcal{D} := \Delta \otimes \{\infty\}$, where Δ is the polyhedral defined below. Using Example 3.1.7, we have:

$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes \{\infty\} & \text{if } (m_1, m_2) \in \lambda_1 \\ m_1 \otimes \{\infty\} & \text{if } (m_1, m_2) \in \lambda_2 \end{cases}$$



3.1.4 A proof of Altmann and Hausen main theorems

Given a normal affine variety endowed with an effective torus action, we provide a proof for the construction of an AH-datum using a method that is mentioned in [AH06]. In Section 3.2, we will extend the Altmann-Hausen presentation using a Galois-equivariant version of the proof of Theorem 3.1.12.

Proposition 3.1.15 (Downgrading torus action)**.** *Let* X *be an affine* $\Bbbk$*-variety endowed with an action of the* d*-dimensional* $\Bbbk$*-torus* $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$*. Then, there exist* $n \in \mathbb{N}, n \geq d$ *such that:*

(i) $\mathbb{T}$ *is a closed subgroup of* $\mathbb{G}^n_{m,\Bbbk}$*; and*

(ii) X *is a closed subvariety of* $\mathbb{A}^n_{\Bbbk}$ *and* $X \hookrightarrow \mathbb{A}^n_{\Bbbk}$ *is* $\mathbb{T}$*-equivariant. Moreover,* X *intersects the dense open orbit of* $\mathbb{A}^n_{\Bbbk}$ *for the natural* $\mathbb{G}^n_{m,\Bbbk}$*-action, and the weight cone of* $\mathbb{A}^n_{\Bbbk}$ *is the weight cone of* X*.*

Proof. The algebra $\Bbbk[X]$ is finitely generated, so we can write $\Bbbk[X] = \Bbbk[\tilde{g}_1, \dots, \tilde{g}_k]$ with $\tilde{g}_i \in \Bbbk[X]\backslash\{0\}$. Since the split torus $\mathbb{T}$ acts on X, the $\Bbbk$-algebra $\Bbbk[X]$ is M-graded, that is $\Bbbk[X] = \bigoplus_{m \in M} \Bbbk[X]_m$. So, there exists homogeneous elements $\tilde{g}_{i,j}$ such that $\tilde{g}_i = \tilde{g}_{i,1} + \cdots +$

$\tilde{g}_{i,k_i}$. Hence we can assume that there exists $n \in \mathbb{N}$ such that $\Bbbk[X] = \Bbbk[g_1, \ldots, g_n]$, where the g_i are homogeneous of degree $m_i \in M$.

(i) The $\Bbbk$-algebra morphism

$$\psi : \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \Bbbk[M]$$
$$x_i \mapsto \chi^{m_i}$$

is surjective since the $\mathbb{T}$-action on X is effective. Hence, $\mathbb{T}$ is a closed subgroup of $\mathbb{G}_{m,\Bbbk}^n$.

(ii) The $\Bbbk$-algebra morphism

$$\varphi : \Bbbk[x_1, \ldots, x_n] \to \Bbbk[X]$$
$$x_i \mapsto g_i$$

is surjective and induces a $\Bbbk$-algebra isomorphism $\Bbbk[g_1, \ldots, g_n] \cong \Bbbk[x_1, \ldots, x_n]/\mathfrak{a}$, with $\mathfrak{a} = \operatorname{Ker}(\varphi)$. Hence, X is a closed subvariety of $\mathbb{A}_{\Bbbk}^n$. Moreover, the comorphism of the $\mathbb{T}$-action on $\mathbb{A}_{\Bbbk}^n$ is given by:

$$\tilde{\mu}^{\sharp} : \Bbbk[x_1, \ldots, x_n] \to \Bbbk[M] \otimes \Bbbk'[x_1, \ldots, x_n]$$
$$x_i \mapsto \chi^{m_i} \otimes x_i.$$

Therefore, φ is $\mathbb{T}$-equivariant, so the closed immersion $X \hookrightarrow \mathbb{A}_{\Bbbk}^n$ is $\mathbb{T}$-equivariant. Finally, note that for all $i \in \{1, \ldots, n\}$, $x_i \notin \mathfrak{a}$, hence X intersects the dense open orbit of $\mathbb{G}_{m,\Bbbk}^n$.

Let $m \in M$ such that $\Bbbk[x_1, \ldots, x_n]_m \neq 0$. Note that

$$\Bbbk[x_1, \ldots, x_n]_m = \bigoplus_{(k_1,\ldots,k_n) \in \mathbb{N}^n, m_1k_1 + \cdots + m_nk_n = m} \Bbbk x_1^{k_1} \cdots x_n^{k_n}.$$

Hence, since for all $i \in \{1, \ldots, n\}$, $x_i \notin \mathfrak{a}$, it follows that $(\Bbbk[x_1, \ldots, x_n]/\mathfrak{a})_m \neq 0$, and the weight cone of $\mathbb{A}_{\Bbbk}^n$ is the weight cone of X. □

Example 3.1.16. We pursue Example 2.3.8 (see also Example 3.1.14). Consider the action of $\mathbb{G}_{m,\mathbb{C}}^2$ on $\mathbb{A}_{\mathbb{C}}^3$ given by $(s,t) \cdot (x,y,z) = (sx, ty, stz)$. The action of $\mathbb{G}_{m,\mathbb{C}}^2$ on $\mathbb{A}_{\mathbb{C}}^3$ comes from the inclusion of $\mathbb{G}_{m,\mathbb{C}}^2$ in $\mathbb{G}_{m,\mathbb{C}}^3$ given by $(s,t) \mapsto (s,t,st)$. We denote by N and N' the cocharacter lattices of $\mathbb{G}_{m,\mathbb{C}}^2$ and $\mathbb{G}_{m,\mathbb{C}}^3$ respectively. Then, we obtain the exact sequence

$$0 \longrightarrow N \xrightarrow{F} N' \xrightarrow{P} N_Y \longrightarrow 0,$$

with

$$F := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad P := \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}.$$

Proof of the second item of Theorem 3.1.12.

• ***Step 0:*** *Preliminaries.*

Let X be a normal affine $\Bbbk$-variety endowed with an effective $\mathbb{T}$-action. Let ω_M be the weight cone of the $\mathbb{T}$-action on X, and let $\omega_N = \omega_M^{\vee}$. Using Proposition 3.1.15, there exists $n \in \mathbb{N}$ such that $\mathbb{T}$ is a closed subgroup of $\mathbb{G}_{m,\Bbbk}^n$ and X is a closed $\mathbb{T}$-equivariant subvariety of $\mathbb{A}_{\Bbbk}^n$. So, let $\mathfrak{a}$ be the ideal of $\Bbbk[\mathbb{A}_{\Bbbk}^n] = \Bbbk[x_1, \ldots, x_n]$ such that $\Bbbk[X]$ is $\mathbb{T}$-equivariantly isomorphic to $\Bbbk[\mathbb{A}_{\Bbbk}^n]/\mathfrak{a}$. We write $\Bbbk[X] = \Bbbk[\mathbb{A}_{\Bbbk}^n]/\mathfrak{a}$ and we denote by $p_X : \Bbbk[\mathbb{A}_{\Bbbk}^n] \twoheadrightarrow \Bbbk[X]$ the surjective algebra morphism induced from the inclusion $X \subset \mathbb{A}_{\Bbbk}^n$. Let M' be the character lattice of $\mathbb{G}_{m,\Bbbk}^n$, and let M_Y be the sublattice of M' constructed in Remark 2.2.11. We have the exact sequences of Remark 2.2.11

$$0 \longrightarrow N \xrightarrow{F} N' \xrightarrow{P} N_Y \longrightarrow 0.$$

$$0 \longrightarrow M_Y \xrightarrow{P^*} M' \xrightarrow{F^*} M \longrightarrow 0.$$

Recall that there always exists a section $s^* : M \to M'$ (see Lemma C.3.2), and a cosection $t^* : M' \to M_Y$. These homomorphisms satisfy $F^* \circ s^* = Id_M$, $t^* \circ P^* = Id_{M_Y}$ and $P^* \circ t^* = Id_{M'} - s^* \circ F^*$.

Since $\mathrm{Frac}(\Bbbk[M']) = \Bbbk(\mathbb{A}^n_\Bbbk)$, the section s^* induces a morphism

$$u : \omega_M \cap M \to \Bbbk\left(\mathbb{A}^n_\Bbbk\right)^* \\ m \mapsto \chi^{s^*(m)}$$

such that for all $m \in \omega_M \cap M$, $u(m) \in \Bbbk(\mathbb{A}^n_\Bbbk)_m$.

By Section 2.3.1, we can write

$$\Bbbk\left[\mathbb{A}^n_\Bbbk\right] = \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk\left[\mathbb{A}^n_\Bbbk\right]}_m u(m) \subset \Bbbk\left(\mathbb{A}^n_\Bbbk\right)_0 [M],$$

where $\Bbbk[\mathbb{A}^n_\Bbbk]_m = \widetilde{\Bbbk[\mathbb{A}^n_\Bbbk]}_m u(m)$, with $\widetilde{\Bbbk[\mathbb{A}^n_\Bbbk]}_m \subset \Bbbk(\mathbb{A}^n_\Bbbk)_0$.

Since for all i, $x_i \notin \mathfrak{a}$, for all $m \in \omega_M \cap M$ we have $u_X(m) \in \Bbbk(X)_m$, where $u_X(m)$ is obtained from the surjective morphism $p_X : \Bbbk[\mathbb{A}^n_\Bbbk] \twoheadrightarrow \Bbbk[X]$. It follows a morphism $u_X : \omega_M \cap M \to \Bbbk(X)^*$. Hence we can write

$$\Bbbk[X] = \bigoplus_{m \in \omega_M \cap M} \Bbbk[X]_m = \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk[X]}_m u_X(m) \subset \Bbbk(X)_0[M],$$

where, for all $m \in \omega_M \cap M$, $\Bbbk[X]_m = \widetilde{\Bbbk[X]}_m u_X(m)$ and $\widetilde{\Bbbk[X]}_m \subset \Bbbk(X)_0$. Furthermore, if $f/g \in \widetilde{\Bbbk\left[\mathbb{A}^n_\Bbbk\right]}_m$, observe that $p_X(g) \neq 0$. Hence, $p_X(f/g)$ is well defined, $p_X(\widetilde{\Bbbk[\mathbb{A}^n_\Bbbk]}_m) = \widetilde{\Bbbk[X]}_m$, and the surjective graded algebra morphism p_X can be written

$$p_X : \bigoplus_{m \in \omega_M \cap M} \widetilde{\Bbbk\left[\mathbb{A}^n_\Bbbk\right]}_m u(m) \to \bigoplus_{m \in \omega_M \cap M} \Bbbk[X]_m \\ \frac{f}{g} u(m) \mapsto \frac{p_X(f)}{p_X(g)} u_X(m).$$

The aim of this proof is to identify $\widetilde{\Bbbk[\mathbb{A}^n_\Bbbk]}_m$ with $\mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))$ for some pp-divisor $\mathcal{D}$ on some normal $\Bbbk$-variety Y, and to identify $\widetilde{\Bbbk[X]}_m$ with $\mathrm{H}^0(\tilde{Y}_X, \mathcal{O}_{\tilde{Y}_X}(\mathcal{D}_X(m)))$ for some pp-divisor $\mathcal{D}_X$ on some normal $\Bbbk$-variety $\tilde{Y}_X$, in view of obtaining graded algebra isomorphisms $\Bbbk[\mathbb{A}^n_\Bbbk] \cong A[Y, \mathcal{D}]$ and $\Bbbk[X] \cong A[\tilde{Y}_X, \mathcal{D}_X]$.

• ***Step 1****: Altmann-Hausen quotient and divisors.*

Let $\{e_1, \dots, e_n\}$ be the standard basis of N'. The cone in $N'_\mathbb{Q}$ of the toric variety $\mathbb{A}^n_\Bbbk$ is $\mathbb{Q}^n_{\geq 0}$, and $F^*(\mathbb{Q}^n_{\geq 0}) = \omega_M$.

Let Λ_Y be the fan in $(N_Y)_\mathbb{Q}$ generated by $\{P(e_1), \dots, P(e_n)\}$. Let Y be the toric $\Bbbk$-variety obtained from the fan Λ_Y; it is a semi-projective variety (see [CLS11, Proposition 7.2.9]).

Let Y_X be the closure of the image of $X \cap \mathbb{G}^n_{m,\Bbbk}$ in Y by the surjective group homomorphism $\pi : \mathbb{G}^n_{m,\Bbbk} \twoheadrightarrow \mathbb{T}_Y$ composed with the inclusion $\mathbb{T}_Y \hookrightarrow Y$. The normalization $\tilde{Y}_X$ of Y_X, with morphism $\nu : \tilde{Y}_X \to Y_X$, is a semi-projective $\Bbbk$-variety (see [Sta, Lemmas 0BXR, 0B3I, 0C4P]). For each ray of the fan Λ_Y, we denote by v_i its first lattice vector, $i \in \{1, \ldots, n'\}$. To a ray spanned by v_i corresponds a toric divisor D_{v_i} on Y (see Appendix C.2). We will see in Proposition 3.1.19 that the divisor

$$\mathcal{D} = \sum_{i=1}^{n'} \Delta_{v_i} \otimes D_{v_i},$$

where $\Delta_{v_i} := s(P^{-1}(v_i) \cap \mathbb{Q}^n_{\geq 0})$, is an ω_N-pp-divisor on Y. Furthermore, we can pullback $\mathcal{D}$ on $\tilde{Y}_X$ and we obtain an ω_N-pp-divisor $\mathcal{D}_X$ on $\tilde{Y}_X$ (see Proposition 3.1.19). More precisely,

$$\mathcal{D}_X := (\iota \circ \nu)^* \mathcal{D} = \sum_{i=1}^{n'} \Delta_{v_i} \otimes (\iota \circ \nu)^* D_{v_i},$$

where $\iota : Y_X \hookrightarrow Y$.

- ***Step 2:*** *Isomorphisms* $\Bbbk(Y) \cong \Bbbk(\mathbb{A}^n_{\Bbbk})_0$ *and* $\Bbbk(Y_X) \cong \Bbbk(X)_0$.

Observe that $\pi^\sharp : \Bbbk[M_Y] \to \Bbbk[M']_0$ is an isomorphism. Hence, $\pi^\sharp$ induces an isomorphism

$$\begin{aligned} \mathrm{Frac}(\Bbbk[M_Y]) &\to \mathrm{Frac}(\Bbbk[M']_0) \\ \frac{f}{g} = \frac{\sum a_i \chi^{m_i}}{\sum b_j \chi^{m_j}} &\mapsto \frac{\pi^\sharp(f)}{\pi^\sharp(g)} = \frac{\sum a_i \chi^{P^*(m_i)}}{\sum b_j \chi^{P^*(m_j)}}. \end{aligned}$$

For all $m \in M$, recall that $(F^*)^{-1}(m) = s^*(m) + \mathrm{Ker}(F^*)$ (see Lemma C.3.2). Hence, $\mathrm{Frac}(\Bbbk[M'])_0 = \mathrm{Frac}(\Bbbk[M']_0)$. Since $\mathbb{T}_Y$ is a dense open subset of Y, we have $\Bbbk(Y) = \Bbbk(\mathbb{T}_Y)$. Since $\mathbb{G}^n_{m,\Bbbk}$ is a dense open subset of $\mathbb{A}^n_{\Bbbk}$, we have $\mathrm{Frac}(\Bbbk[M']) = \Bbbk(\mathbb{A}^n_{\Bbbk})$. Therefore, $\mathrm{Frac}(\Bbbk[M'])_0 = \Bbbk(\mathbb{A}^n_{\Bbbk})_0$. Finally, we obtain an isomorphism

$$\varphi : \Bbbk(Y) \to \Bbbk\left(\mathbb{A}^n_{\Bbbk}\right)_0 .$$

Moreover, the inclusion $X \hookrightarrow \mathbb{A}^n_{\Bbbk}$ is $\mathbb{T}$-equivariant, the variety $\mathbb{T}_Y \cap Y_X$ is affine and the following diagram commutes

$$\begin{array}{ccc} \mathbb{G}^n_{m,\Bbbk} & \xrightarrow{\quad\pi\quad} & \mathbb{T}_Y \\ \uparrow & & \uparrow \iota \\ \mathbb{G}^n_{m,\Bbbk} \cap X & \longrightarrow & \mathbb{T}_Y \cap Y_X = \pi\left(\mathbb{G}^n_{m,\Bbbk} \cap X\right). \end{array}$$

Therefore the following diagram commutes

$$\begin{array}{ccc} \Bbbk\left[\mathbb{T}_Y\right] & \xrightarrow{\quad\pi^\sharp\quad} & \Bbbk\left[\mathbb{G}^n_{m,\Bbbk}\right]_0 \\ \iota^\sharp \downarrow & & \downarrow p_X \\ \Bbbk\left[\mathbb{T}_Y \cap Y_X\right] & \longrightarrow & \Bbbk\left[\mathbb{G}^n_{m,\Bbbk} \cap X\right]_0 . \end{array}$$

It follows an isomorphism $\Bbbk(\mathbb{T}_Y \cap Y_X) \to \mathrm{Frac}(\Bbbk[\mathbb{G}^n_{m,\Bbbk} \cap X]_0)$ over $\varphi : \Bbbk(Y) = \Bbbk(\mathbb{T}_Y) \to \Bbbk(\mathbb{A}^n_{\Bbbk})_0 = \mathrm{Frac}(\Bbbk[\mathbb{G}^n_{m,\Bbbk}]_0)$. Furthermore, since,

$$\mathrm{Frac}\left(\Bbbk\left[\mathbb{G}^n_{m,\Bbbk} \cap X\right]_0\right) = \Bbbk\left(\mathbb{G}^n_{m,\Bbbk} \cap X\right)_0, \ \Bbbk\left(Y_X\right) = \Bbbk\left(\mathbb{T}_Y \cap Y_X\right), \text{ and } \Bbbk\left(X\right) = \Bbbk\left(\mathbb{G}^n_{m,\Bbbk} \cap X\right),$$

we obtain an isomorphism $\Bbbk(Y_X) \cong \Bbbk(X)_0$.

By a property of the normalization (see [Sta, Lemma 0BXR]), we have an isomorphism $\Bbbk(Y_X) \cong \Bbbk(\tilde{Y}_X)$, therefore, we have constructed an isomorphism

$$\varphi_X : \Bbbk\left(\tilde{Y}_X\right) \to \Bbbk(X)_0$$

over $\varphi : \Bbbk(Y) \to \Bbbk(\mathbb{A}^n_{\Bbbk})_0$.

• ***Step 3:*** *Isomorphisms* $A[Y, \mathcal{D}] \cong \Bbbk[\mathbb{A}^n_{\Bbbk}]$ *and* $A[\tilde{Y}_X, \mathcal{D}_X] \cong \Bbbk[X]$.

Let $m \in \omega_M \cap M$. Consider the polyhedron $\Delta(m) := (F^*)^{-1}(m) \cap \mathbb{Q}^n_{\geq 0} \subset M'_{\mathbb{Q}}$ and the polyhedron $\Delta_Y(m) = t^*(\Delta(m)) \subset (M_Y)_{\mathbb{Q}}$ (see Lemma C.3.3). Note that

$$\widetilde{\Bbbk[\mathbb{A}^n_{\Bbbk}]}_m := \bigoplus_{m' \in \Delta(m) \cap M'} \Bbbk\chi^{m' - s^*(m)} = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \Bbbk\chi^{P^*(m_Y)} = \varphi\left(\bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \Bbbk\chi^{m_Y}\right).$$

Combining Proposition C.2.2 and Lemma 3.1.20, we get the following equality:

$$\mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \Bbbk\chi^{m_Y}.$$

Therefore, we obtain a graded algebra isomorphism:

$$\begin{aligned} \Phi : A[Y, \mathcal{D}] &\to \Bbbk\left[\mathbb{A}^n_{\Bbbk}\right] \\ f\mathfrak{X}_m &\mapsto \varphi(f)u(m). \end{aligned}$$

Consider the morphism defined by

$$\begin{aligned} \Phi_X : A[\tilde{Y}_X, \mathcal{D}_X] &\to \Bbbk(X)_0[M] \\ f\mathfrak{X}_m &\mapsto \varphi_X(f)u_X(m). \end{aligned}$$

Since $\mathcal{D}_X$ is the pull back of $\mathcal{D}$ on $\tilde{Y}_X$, for all $m \in \omega_M \cap M$ there is a well defined map

$$\begin{aligned} \mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) &\to \mathrm{H}^0(\tilde{Y}_X, \mathcal{O}_{\tilde{Y}_X}(\mathcal{D}_X(m))) \\ f &\to (\nu^\sharp \circ \iota^\sharp)(f) \end{aligned}$$

Then, the induced map

$$\begin{aligned} \Psi : A[Y, \mathcal{D}] &\to A[\tilde{Y}_X, \mathcal{D}_X] \\ f\mathfrak{X}_m &\to (\nu^\sharp \circ \iota^\sharp)(f)\mathfrak{X}_m \end{aligned}$$

is a graded algebra morphism.

Using [AH06] (see also [AH03, Proposition 8.1])[1], we obtain a graded algebra isomorphism

$$\begin{aligned} \Phi_X : A\left[\tilde{Y}_X, \mathcal{D}_X\right] &\to \Bbbk\left[X\right] \\ f\mathfrak{X}_m &\mapsto \varphi_X(f)u_X(m) \end{aligned}$$

such that the following diagram commutes

$$\begin{array}{ccc} A[Y, \mathcal{D}] & \xhookrightarrow{\quad\Phi\quad} & \Bbbk[\mathbb{A}^n_{\Bbbk}] \\ \Psi \big\downarrow & & \big\downarrow p_X \\ A[\tilde{Y}_X, \mathcal{D}_X] & \xhookrightarrow{\quad\Phi_X\quad} & \Bbbk[X]. \end{array}$$

□

[1] In view of extending the Altmann-Hausen presentation over arbitrary perfect fields, it will be interesting to get a proof of the surjectivity of Ψ without using GIT tools (such tools are used in [AH06]).

Remark 3.1.17. The construction of the $\Bbbk$-variety $X[Y,\mathcal{D}]$ does not depend on the choice of the cosection. Indeed, for $j \in \{1,2\}$, let $s_j : N' \to N$ be two cosections, let $\mathcal{D}_j := \sum_i \Delta^j_{v_i} \otimes D_{v_i}$ be the two associated pp-divisors. Note that for all $m \in M$, $(s_1^* - s_2^*)(m) \in \mathrm{Ker}(F^*) = \mathrm{Im}(P^*)$, thus there exists a lattice homomorphism $s_0 : N_Y \to N$ such that $s_1 - s_2 = s_0 \circ P$. Let $g : M \to \Bbbk(Y)^*$ be the morphism defined by $g(m) := \chi^{s_0^*(m)}$. Let $m \in \omega_M \cap M$. Since $\Delta^1_{v_i} = s_0(v_i) + \Delta^2_{v_i}$, we have :

$$\mathcal{D}_1(m) = \sum_i \langle s_0^*(m)|v_i\rangle \otimes D_{v_i} + \mathcal{D}_2(m) = \mathrm{div}_Y(g(m)) + \mathcal{D}_2(m).$$

Therefore the M-graded algebras $A[Y,\mathcal{D}_1]$ and $A[Y,\mathcal{D}_2]$ are isomorphic via:

$$\begin{aligned} A\left[Y,\mathcal{D}_1\right] &\to A\left[Y,\mathcal{D}_2\right] \\ f\mathfrak{X}_m &\mapsto fg(m)\mathfrak{X}_m. \end{aligned}$$

Remark 3.1.18 (AH-datum of a $\mathbb{G}^n_{m,\Bbbk}$-toric $\mathbb{T}$-variety X_δ, where $\mathbb{T} \subset \mathbb{G}^n_{m,\Bbbk}$)**.** In the proof of Theorem 3.1.12, we construct explicitly an AH-datum for the $\mathbb{T}$-variety $\mathbb{A}^n_\Bbbk$ using the closed embedding $\mathbb{T} \hookrightarrow \mathbb{G}^n_{m,\Bbbk}$, and the fact that $\mathbb{A}^n_\Bbbk$ is a $\mathbb{G}^n_{m,\Bbbk}$-toric $\mathbb{T}$-variety of cone $\mathbb{Q}^n_{\geq 0}$. We get an AH-datum $(\omega_N, Y, \mathcal{D})$, where

$$\Delta_{v_i} := s(P^{-1}(v_i) \cap \mathbb{Q}^n_{\geq 0}), \quad Y = X_{\Lambda_Y},$$

where Λ_Y is the coarset fan in $(N_Y)_\mathbb{Q}$ refining all cones $P(\delta_0)$ such that δ_0 is a face of $\mathbb{Q}^n_{\geq 0} \subset M'_\mathbb{Q}$, and where the v_i are the first lattice vectors of the one-dimensional cones of Λ_Y. More generally, if δ is a pointed cone, we can replace $\mathbb{Q}^n_{\geq 0}$ by δ, and $\mathbb{A}^n_\Bbbk$ by X_δ (see [AH06, Section 11]). We get an AH-datum $(\omega_N, Y, \mathcal{D})$, where

$$\Delta_{v_i} := s(P^{-1}(v_i) \cap \delta), \quad Y = X_{\Lambda_Y},$$

where Λ_Y is the coarset fan in $(N_Y)_\mathbb{Q}$ refining all cones $P(\delta_0)$ such that δ_0 is a face of $\delta \subset M'_\mathbb{Q}$, and where the v_i are the first lattice vectors of the one-dimensional cones of Λ_Y. In other words, for any $\mathbb{G}^n_{m,\Bbbk}$-toric $\mathbb{T}$-variety X_δ, we can compute directly an AH-datum without using a closed embedding of X_δ in some affine space. Therefore, in this context, we do not use the normalization morphism, so this part of this construction works in any characteristic (see Section 5.1.3).

The next two results are used in the proof of Theorem 3.1.12. With the notations of the proof of Theorem 3.1.12, we have the following result.

Proposition 3.1.19 ([AH03, Proposition 8.1])**.** *The divisor $\mathcal{D} =\in \{i,\ldots,n'\}\Delta_{v_i} \otimes D_{v_i}$, where $\Delta_{v_i} := s(\mathbb{Q}^n_{\geq 0} \cap P^{-1}(v_i))$, is an ω_N-pp-divisor on Y. Furthermore, we can pullback $\mathcal{D}$ on $\tilde{Y}_X$ and we obtain an ω_N-pp-divisor $\mathcal{D}_X$ on $\tilde{Y}_X$.*

We give another proof of [AH03, Proposition 8.1] based on the situation of Proposition 3.1.15.

Proof. • We prove that $\mathcal{D}$ is an ω_N-pp-divisor on Y. First, for all $i \in \{1,\ldots,n'\}$, the tail cone of Δ_{v_i} is $\omega_M^\vee$ (see Lemma C.3.4). Let $m \in \omega_M \cap M$, then, by Lemma C.3.5, we have

$$\mathcal{D}(m) = \sum_{i=1}^{n'} \min \langle m\,|\Delta_{v_i}\rangle \otimes D_{v_i} = \sum_{i=1}^{n'} -\min \langle \Delta_Y(m)|\, v_i\rangle \otimes D_{v_i} = \sum_{i=1}^{n'} -\langle m_i|v_i\rangle \otimes D_{v_i},$$

where the m_i realize the minimum on $\Delta_Y(m)$. There exists an integer $k > 0$ such that $km_i \in \mathbb{N}$ for all $i \in \{1,\ldots,n'\}$. Therefore, $k\mathcal{D}(m) = \mathcal{D}(km)$ is a Weil $\mathbb{Z}$-divisor, and we assume that $\mathcal{D}(m)$ is a Weil $\mathbb{Z}$-divisor.

Using [CLS11, Theorem 4.2.8], we show that $\mathcal{D}(m)$ is a Cartier divisor on Y. Indeed, let $\lambda \in \Lambda$, and let $\{v_{i_1}, \dots, v_{i_p}\}$ denotes the set of lattice vectors that span the rays of λ. Then, since λ is strictly convex, it follows $\cap_{j=1}^{p}(m_{i_j} + v_{i_j}^{\perp}) \neq \emptyset$, where $m_{i_j} \in \Delta_Y(m) \cap M_Y$ for all $j \in \{1, \dots, p\}$. So, there exits $m_\lambda \in M_Y$ such that $\cap_{j=1}^{p}(m_{i_j} + v_{i_j}^{\perp}) = m_\lambda + \cap_{j=1}^{p} v_{i_j}^{\perp}$. Note that, for all $j \in \{1, \dots, p\}$, $\langle m_\lambda | v_{i_j}\rangle = \langle m_{i_j} | v_{i_j}\rangle$. Hence $\mathcal{D}(m)|_{U_\lambda} = \mathrm{div}(\chi^{-m_\lambda})$, where U_λ is the affine toric open subset of Y defined by λ. Therefore, $\mathcal{D}(m)$ is a Cartier $\mathbb{Z}$-divisor on Y

Using [CLS11, Theorem 6.1.7], one can show that $\mathcal{D}(m)$ is semi-ample (we have to show that there exists $k \in \mathbb{N}^*$ such that $k\mathcal{D}(m)$ is basepoint-free).

Let $m \in \mathrm{Relint}(\omega_M) \cap M$. Combining [CLS11, Proposition 6.3.12] together with [CLS11, §9.3 Iitaka Dimension and Big Divisors], we conclude that $\mathcal{D}(m)$ is big. Finally, $\mathcal{D}$ is an ω_N-pp-divisor on Y.

• We prove that $\mathcal{D}_X$ is an ω_N-pp-divisor on $\tilde{Y}_X$. First, $X \cap \mathbb{G}_{m,\Bbbk}^n \neq \emptyset$. Then, by construction, the variety Y_X is not contained in $\cup_i D_{v_i}$ (D_{v_i} are toric divisors). Therefore, for all i, the pullback of D_{v_i} on Y_X is well defined (see [Sta, Lemma 02OO, (3)]), i.e the closed subscheme $\iota^{-1}(D_{v_i}) = \iota^* D_{v_i}$ is an effective Cartier divisor. Then, since $\nu : \tilde{Y}_X \to Y_X$ is a dominant morphism, we obtain effective Cartier divisors $(\iota \circ \nu)^* D_{v_i}$ on $\tilde{Y}_X$ (see [Sta, Lemma 02OO, (2)]). Let

$$\mathcal{D}_X := (\iota \circ \nu)^* \mathcal{D} = \sum_i \Delta_{v_i} \otimes (\iota \circ \nu)^* D_{v_i}.$$

Let $m \in \omega_M \cap M$, then $\mathcal{D}_X(m)$ is semi-ample. Let $m \in \mathrm{Relint}(\omega_M) \cap M$, then $\mathcal{D}_X(m)$ is big. Finally, $\mathcal{D}_X$ is an ω_N-pp-divisor $\mathcal{D}_X$ on $\tilde{Y}_X$. □

Lemma 3.1.20. *Consider the ω_N-polyhedral divisor $\mathcal{D} = \sum_{v_i} \Delta_{v_i} \otimes D_{v_i}$, where $\Delta_{v_i} := s(P^{-1}(v_i) \cap \mathbb{Q}_{\geq 0}^n)$. Then, for all $m \in \omega_M \cap M$,*

$$\Delta(\mathcal{D}(m)) = \Delta_Y(m),$$

where

$$\Delta(\mathcal{D}(m)) := \{u_Y \in (M_Y)_{\mathbb{Q}} \mid \langle u_Y | v_i\rangle \geq -\min\langle m | \Delta_{v_1}\rangle \ \forall i \in \{1, \dots, n'\}\},$$

$$\Delta_Y(m) := \{u_Y \mid u_Y \in (M_Y)_{\mathbb{Q}}, \langle P^*(u_Y) | e_i\rangle \geq -\langle s^*(m) | e_i\rangle \ \forall i \in \{1, \dots, n\}\}.$$

Proof. Recall that, by Lemma C.3.3, $\Delta_Y(m) = t^*(\Delta(m)) = t^*((F^*)^{-1}(m) \cap \mathbb{Q}_{\geq 0}^n)$. By the Lemma C.3.5, we can write

$$\Delta(\mathcal{D}(m)) := \{u_Y \in (M_Y)_{\mathbb{Q}} \mid \langle u_Y | v_i\rangle \geq \min\langle \Delta_Y(m) | v_i\rangle \ \forall i \in \{1, \dots, n'\}\}.$$

Therefore, $\Delta_Y(m) \subset \Delta(\mathcal{D}(m))$. Let $u_Y \in \Delta(\mathcal{D}(m))$, we show that $u_Y \in \Delta_Y(m)$. By definition, we have for all j that $\langle u_Y | v_j\rangle \geq \min\langle \Delta_Y(m) | v_j\rangle$. Fix some e_i such that $P(e_i) \neq 0$. There exists $j \in \{1, \dots, n\}$ and $\alpha_j \in \mathbb{N}^*$ such that $P(e_i) = \alpha_j v_j$. We have $\alpha_j \langle u_Y | v_j\rangle = \langle P^*(u_Y) | e_i\rangle$. Moreover, by Lemma C.3.3,

$$\alpha_j \min\langle \Delta_Y(m) | v_j\rangle = \min\langle P^*(\Delta_Y(m)) | e_i\rangle = \min\langle \Delta(m) - s^*(m) | e_i\rangle.$$

Since $\Delta(m) \subset \mathbb{Q}_{\geq 0}^n$, we have $\langle \Delta(m) | e_i\rangle \geq 0$, hence $\alpha_j \min\langle \Delta_Y(m) | v_j\rangle \geq -\langle s^*(m) | e_i\rangle$, and $\min\langle \Delta_Y(m) | P(e_i)\rangle \geq -\langle s^*(m) | e_i\rangle$. Therefore, $\langle P^*(u_Y) | e_i\rangle \geq \min\langle \Delta_Y(m) | P(e_i)\rangle$. If $P(e_i) = 0$, then $0 = \langle P^*(u_Y) | e_i\rangle = \min\langle \Delta_Y(m) | e_i\rangle \geq -\langle s^*(m) | e_i\rangle$. We conclude that $u_Y \in \Delta_Y(m)$. □

3.1.5 Functoriality

Theorem 3.1.12 establishes correspondences between affine $\Bbbk$-varieties endowed with a $\Bbbk$-torus action and pairs $(Y, \mathcal{D})$. In this section, we focus on this correspondence. In general, there is no one-to-one correspondence between $\mathbb{T}$-varieties and pairs $(Y, \mathcal{D})$; see [AH06, Section 8] for a precise statement. However, Altmann and Hausen define the notion of minimal pp-divisor in [AH06, Section 8] that leads to the following result. Divisors constructed by the method exhibited in the proof of Theorem 3.1.12 are minimal.

First, recall that if $\mathcal{D}' = \sum_i \Delta_{v_i} \otimes D'_{v_i}$ is an ω'_N-pp-divisor on a semi-projective variety Y', then we can pullback $\mathcal{D}$ under a morphism $\psi : Y \to Y'$, where Y is a semi-projective variety, and ϕ is such that none of the supports of D_{v_i} contains $\psi(Y)$. We define the non-necessarily proper ω'_N-polyhedral divisor (see [AH06, Definition 8.3], see also [Sta, Lemma 02OO, (3)])

$$\psi^*\mathcal{D} := \sum_i \Delta_{v_i} \otimes \psi^* D_{v_i}.$$

Theorem 3.1.21 ([AH06, Theorem 8.8]). *Let $\mathbb{T} = \operatorname{Spec}(\Bbbk[M])$ be a $\Bbbk$-torus. Let $\omega_N \subset N_{\mathbb{Q}}$ (resp. $\omega'_N \subset N_{\mathbb{Q}}$) be a pointed cone, let (Y, σ_Y) (resp. (Y', σ'_Y)) be a normal semi projective $\Bbbk$-variety and $\mathcal{D}$ be a minimal proper ω_N-polyhedral divisor on Y (resp. $\mathcal{D}'$ be a minimal proper ω'_N-polyhedral divisor on Y'). The affine $\Bbbk$-varieties $X[Y, \mathcal{D}]$ and $X[Y', \mathcal{D}']$ are $\mathbb{T}$-isomorphic if and only if the following holds:*

(i) there exists a $\Bbbk$-isomorphism $\psi : Y \to Y'$;
(ii) there exists a lattice automorphism $L : N \to N$ such that $L(\omega_N) = \omega'_N$;
(iii) there exists a monoid morphism $g : \omega_N^\vee \cap M \to \Bbbk'(Y)$;
(iv) for all $m \in \omega'_M \cap M$, $\psi^*(\mathcal{D}'(m)) = \mathcal{D}(L^*(m)) + \operatorname{div}_Y(g(m))$;

Proof. The morphisms ψ and L induces a $\mathbb{T}$-equivariant isomorphism of graded algebras:

$$\Psi : A[Y', \mathcal{D}'] \to A[Y, D], \ f\mathfrak{X}_m \mapsto \psi^\sharp(f) g(m) \mathfrak{X}_{L^*(m)}.$$

□

3.2 Altmann-Hausen presentation over arbitrary fields

In this section, $\Bbbk$ is a characteristic zero field. Let $\overline{\Bbbk}$ be a fixed algebraic closure of $\Bbbk$ and let $\Bbbk'$ be a non necessarily finite Galois extension of $\Bbbk$ in $\overline{\Bbbk}$ of Galois group Γ.

We extend the Altmann-Hausen presentation of torus actions on affine varieties over algebraically closed fields to arbitrary fields of characteristic zero (see Theorem 3.2.3). We show that the cocycle appearing in the presentation over non closed fields encodes a torsor, and that a simpler presentation is possible when this torsor is trivial.

3.2.1 Altmann-Hausen presentation over an arbitrary field

Let $\Bbbk$ be an arbitrary field, let $\Bbbk' = \overline{\Bbbk}$, and let $\Gamma := \operatorname{Gal}(\overline{\Bbbk}/\Bbbk)$. Let $\mathbb{T} = \operatorname{Spec}(\overline{\Bbbk}[M])$ be a $\overline{\Bbbk}$-torus, let N be the dual lattice of M.

Let ω_N be a pointed cone in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, let Y be a normal semi-projective $\overline{\Bbbk}$-variety (see Definition 3.1.9), and let $\mathcal{D} := \sum \Delta_i \otimes D_i$ be a proper ω_N-polyhedral divisor (see Definition 3.1.6). This means that the D_i are prime divisors on Y and the coefficients Δ_i are convex polyhedra in $N_{\mathbb{Q}}$ having ω_N as tail cone. Then, for every $m \in \omega_N^\vee \cap M$, we can evaluate $\mathcal{D}$ in m to obtain a Weil $\mathbb{Q}$-divisor $\mathcal{D}(m) := \sum \min\{\langle m|\Delta_i\rangle\} \otimes D_i$. From the triple $(\omega_N, Y, \mathcal{D})$ (called an AH-datum), Altmann and Hausen construct an M-graded $\overline{\Bbbk}$-algebra:

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \omega_N^\vee \cap M} \mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))\mathfrak{X}_m \subset \overline{\Bbbk}(Y)[M].$$

By [AH06, Theorems 3.1], the affine scheme $X[Y,\mathcal{D}] := \mathrm{Spec}(A[Y,\mathcal{D}])$ is a normal $\overline{\mathbb{k}}$-variety endowed with a $\mathbb{T}$-action of weight cone $\omega_N^\vee$ (see Theorem 3.1.12).

We will state one of the main results of this article. We extend the Altmann-Hausen presentation over an arbitrary field. In this setting, the acting torus is non-necessarily split. In the next definition, we adapt the notion of AH-datum to our setting.

Definition 3.2.1 (Generalized AH-datum)**.** Let $\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$ be a $\overline{\mathbb{k}}$-torus and let τ be a $\mathbb{k}$-group structure on $\mathbb{T}$. A *generalized AH-datum* $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ over $\overline{\mathbb{k}}$ is an AH-datum $(\omega_N, Y, \mathcal{D})$ over $\overline{\mathbb{k}}$ such that ω_N is stable under the Γ-action induced by $\hat{\tau}$, and together with a $\mathbb{k}$-structure σ_Y on Y and with a map

$$h : \Gamma \to \mathrm{Hom}(\omega_N^\vee \cap M, \overline{\mathbb{k}}(Y)^*)$$

such that

$$\forall m \in \omega_M^\vee \cap M,\ \forall \gamma \in \Gamma, \qquad {\sigma_Y}^*_\gamma(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}_\gamma(m)) + \mathrm{div}_Y\left(h_\gamma\left(\tilde{\tau}_\gamma(m)\right)\right), \quad \text{and} \tag{3.1}$$

$$\forall m \in \omega_M^\vee \cap M,\ \forall \gamma_1, \gamma_2 \in \Gamma, \qquad h_{\gamma_1}(m){\sigma_Y}^\sharp_{\gamma_1}\left(h_{\gamma_2}\left(\tilde{\tau}_{\gamma_1}^{-1}(m)\right)\right) = h_{\gamma_1\gamma_2}(m). \tag{3.2}$$

Remark 3.2.2 (The map h of Theorem 3.2.3 is a cocycle (see Remark 2.3.12))**.** Let $(\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M]), \tau)$ be a $\mathbb{k}$-torus, let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum, let $\mathbb{L} := \overline{\mathbb{k}}(Y)$, and let $\mathbb{K} = \mathbb{L}^\Gamma$. By Lemma 3.2.8, the extension $\mathbb{L}/\mathbb{K}$ is Galois with Galois group Γ. Let $G := \mathrm{Hom}_{gr}(M, \mathbb{L}^*)$ be endowed with the continuous Γ-action $\gamma \cdot f := {\sigma_Y}^\sharp_\gamma \circ f \circ \tilde{\tau}_\gamma^{-1}$. Note that the map h mentioned in Theorem 3.2.3 is a cocycle, that is $h \in \mathrm{H}^1_{cont}(\Gamma, G)$. Let $\mathbb{T}_\mathbb{L} := \mathrm{Spec}(\mathbb{L}[M])$ be the $\mathbb{L}$-torus associated to M, and let $\tau_\mathbb{L}$ be the $\mathbb{K}$-group structure on $\mathbb{T}_\mathbb{L}$ induced by $\tilde{\tau}$. Since there is a Γ-equivariant group isomorphism

$$\begin{aligned} \mathrm{Aut}^{\mathbb{T}_\mathbb{L}}(\mathbb{T}_\mathbb{L}) &\to \mathrm{Hom}_{gr}(M, \mathbb{L}^*) \\ f &\mapsto h, \end{aligned}$$

where $f^\sharp : a_m\chi^m \mapsto a_m h(m)\chi^m$, we obtain

$$\mathrm{H}^1_{cont}(\Gamma, G) \cong \mathrm{H}^1_{cont}\left(\Gamma, \mathrm{Aut}^{\mathbb{T}_\mathbb{L}}\left(\mathbb{T}_\mathbb{L}\right)\right).$$

Therefore, the cocycle h encoded a $(\mathbb{T}_\mathbb{L}, \tau_\mathbb{L})$-torsor. This torsor is over the generic point of (Y, σ_Y) (see Lemma 3.2.8).

From a generalized AH-datum over $\overline{\mathbb{k}}$, we can easily construct an affine $\mathbb{k}$-variety endowed with a $(\mathbb{T}, \tau)$-action (see the proof of the next theorem in Section 3.2.2). We get the next result.

Theorem 3.2.3 (Torus actions on normal affine varieties over arbitrary fields)**.** *Let $\mathbb{k}$ be a field, and let $\Gamma := \mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. Let $\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$ be a $\overline{\mathbb{k}}$-torus, and let τ be a $\mathbb{k}$-group structure on $\mathbb{T}$.*

(i) Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum over $\overline{\mathbb{k}}$. The affine $\mathbb{T}$-variety $X[Y,\mathcal{D}]$ admits a $\mathbb{k}$-structure $\sigma_{X[Y,\mathcal{D}]}$ such that $(\mathbb{T}, \tau)$ acts effectively on $(X[Y,\mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

(ii) Let (X, σ) be a normal affine $\mathbb{k}$-variety endowed with an effective $(\mathbb{T}, \tau)$-action. There exists a generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ such that $(X, \sigma) \cong (X[Y,\mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$ as $(\mathbb{T}, \tau)$-varieties.

Example 3.2.4 (The case of affine toric varieties)**.** Let (X, σ) be an affine $(\mathbb{T}, \tau)$-toric variety, where $\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$ (for instance a $(\mathbb{T}, \tau)$-torsor). Let ω_M be the weight cone of the $\mathbb{T}$-action on X. As we have seen in the proof Proposition 2.4.6 (or as we will see in the proof of Theorem 3.2.3), there exists a map $h : \Gamma \to \mathrm{Hom}(\omega_M \cap M, \overline{\mathbb{k}}^*)$ such that for all $\gamma \in \Gamma$

$$\begin{aligned} \sigma_\gamma^\sharp : \overline{\mathbb{k}}[\omega_M \cap M] &\to \overline{\mathbb{k}}[\omega_M \cap M] \\ a_m\chi^m &\mapsto \gamma(a_m)h_\gamma(\tilde{\tau}_\gamma(m))\chi^{\tilde{\tau}_\gamma(m)}. \end{aligned}$$

In this case, the generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ is as follows: $\omega_N = \omega_M^\vee$, $Y = \mathrm{Spec}(\overline{\Bbbk})$, σ_Y is the $\Bbbk$-structure defined by $\gamma \mapsto \mathrm{Spec}(\gamma)$, $\mathcal{D}$ is trivial, and h is the map defined above. Therefore, the Altman-Hausen presentation is given by ω_M (that encodes the toric $\mathbb{T}$-variety X since $\overline{\Bbbk}[X] \cong A[Y, \mathcal{D}] = \overline{\Bbbk}[\omega_M \cap M]$), and by h (that encodes the $\Bbbk$-structure σ on X compatible with τ).
Moreover, note that $\sigma_\gamma(X_\mathbb{T}) = X_\mathbb{T}$ for all $\gamma \in \Gamma$, where $X_\mathbb{T} \cong \mathbb{T}$ is the dense open orbit of the toric $\overline{\Bbbk}$-variety X. Hence, σ induces a $\Bbbk$-structure $\sigma_\mathbb{T}$ on $X_\mathbb{T}$ and $(X_\mathbb{T}, \sigma_\mathbb{T})$ is a $(\mathbb{T}, \tau)$-torsor (see Remark 2.4.8 and Proposition 2.4.9). We get

$$\begin{aligned} {\sigma_\mathbb{T}}^\sharp_\gamma : \overline{\Bbbk}[M] &\to \overline{\Bbbk}[M] \\ a_m \chi^m &\mapsto \gamma(a_m) h_\gamma(\tilde{\tau}_\gamma(m)) \chi^{\tilde{\tau}_\gamma(m)}. \end{aligned}$$

Remark 3.2.5. If the complexity of the $\mathbb{T}$-action is greater than zero, then

$$\overline{\Bbbk}(X)^\mathbb{T} \cong \overline{\Bbbk}(Y),$$

and the map $h : \Gamma \to \mathrm{Hom}(\omega_M \cap M, \overline{\Bbbk}(Y)^*)$ encodes a torsor defined by

$$\begin{aligned} {\sigma_\mathbb{T}}^\sharp_\gamma : \overline{\Bbbk}(Y)[M] &\to \overline{\Bbbk}(Y)[M] \\ a_m \chi^m &\mapsto \gamma(a_m) h_\gamma(\tilde{\tau}_\gamma(m)) \chi^{\tilde{\tau}_\gamma(m)}. \end{aligned}$$

3.2.2 Proof of Theorem 3.2.3

Toric downgrading

Given an effective action of a $\overline{\Bbbk}$-torus $\mathbb{T}$ on a normal affine $\overline{\Bbbk}$-variety X, Altmann and Hausen indicate in [AH06, §11] a recipe on how to determine a semi-projective variety Y_X and a pp-divisor $\mathcal{D}_X$ mentioned in Theorem 3.1.12. The idea is to embed $\mathbb{T}$-equivariantly X into a toric variety $\mathbb{A}^n_{\overline{\Bbbk}}$ such that X intersects the dense open orbit of $\mathbb{A}^n_{\overline{\Bbbk}}$ for the natural $\mathbb{G}^n_{m,\overline{\Bbbk}}$-action. They construct a normal semi-projective $\overline{\Bbbk}$-variety Y and a pp-divisor $\mathcal{D}$ describing the $\mathbb{T}$-action on $\mathbb{A}^n_{\overline{\Bbbk}}$. From these data, they obtain Y_X and $\mathcal{D}_X$ describing the $\mathbb{T}$-action on X.

In this section, we describe a $\mathbb{T}$-equivariant embedding $X \hookrightarrow \mathbb{A}^n_{\overline{\Bbbk}}$ that is also Γ-equivariant (Proposition 3.2.6), and we use this embedding to extend the Altmann-Hausen presentation to the case of $\Bbbk$-torus actions on affine $\Bbbk$-varieties (Theorem 3.2.3).

Proposition 3.2.6 (Downgrading torus action)**.** *Let X be an affine $\Bbbk'$-variety endowed with an action of the d-dimensional $\Bbbk'$-torus $\mathbb{T} = \mathrm{Spec}(\Bbbk'[M])$. Let σ be a $\Bbbk$-structure on X, and let τ be a $\Bbbk$-group structure on $\mathbb{T}$. If the $\Bbbk$-torus $(\mathbb{T}, \tau)$ acts effectively on (X, σ), then there exist $n \in \mathbb{N}, n \geq d$ such that:*

(i) *There is a $\Bbbk$-group structure τ' on $\mathbb{G}^n_{m,\Bbbk'}$ that extends to a $\Bbbk$-structure σ' on $\mathbb{A}^n_{\Bbbk'}$;*
(ii) *$(\mathbb{T}, \tau)$ is a closed subgroup of $(\mathbb{G}^n_{m,\Bbbk'}, \tau')$; and*
(iii) *(X, σ) is a closed subvariety of $(\mathbb{A}^n_{\Bbbk'}, \sigma')$ and $(X, \sigma) \hookrightarrow (\mathbb{A}^n_{\Bbbk'}, \sigma')$ is $(\mathbb{T}, \tau)$-equivariant. Moreover, X intersects the dense open orbit of $\mathbb{A}^n_{\Bbbk'}$ for the natural $\mathbb{G}^n_{m,\Bbbk'}$-action, and the weight cone of $\mathbb{A}^n_{\Bbbk'}$ is the weight cone of X.*

Proof. Let $\Bbbk_1/\Bbbk$ be a finite Galois extension in $\Bbbk'$ that splits the $\Bbbk$-torus $(\mathbb{T}, \tau)$. We have a tower of Galois extensions $\Bbbk \subset \Bbbk_1 \subset \Bbbk'$. Let $H := \mathrm{Gal}(\Bbbk'/\Bbbk_1)$. By the Galois correspondence (see Theorem B.3.1), H is a normal subgroup of Γ and $\Gamma/H \cong \mathrm{Gal}(\Bbbk_1/\Bbbk)$. The $\Bbbk$-structure σ on X restricts to a $\Bbbk_1$-structure $\sigma_H := \sigma|_H$ on X, and the $\Bbbk$-group structure τ on $\mathbb{T}$ restricts to a $\Bbbk_1$-group structure $\tau_H := \tau|_H$ on $\mathbb{T}$. Let $X_1 := X/H$ and $\mathbb{T}_1 := \mathbb{T}/H$ be the associated $\Bbbk_1$-varieties (see Theorem 1.4.11). We obtain an induced $\Bbbk$-structure σ_1 on X_1 and a $\Bbbk$-group structure τ_1 on $\mathbb{T}_1$ such that $X_1/\mathrm{Gal}(\Bbbk_1/\Bbbk) \cong X/\Gamma$ and $\mathbb{T}_1/\mathrm{Gal}(\Bbbk_1/\Bbbk) \cong \mathbb{T}/\Gamma$ (see

Propositions 1.4.8 and 1.4.9). Moreover $(\mathbb{T}_1, \tau_1)$ acts on (X_1, σ_1). Since $\Bbbk_1$ is a splitting field of $\mathbb{T}/\Gamma$, $\mathbb{T}_1 \cong \mathbb{G}_{m,\Bbbk_1}^n$, the H-action on M is trivial and $\Bbbk'[M]^H = \Bbbk_1[M]$ (see Remark 2.2.10).

(i) The algebra $\Bbbk_1[X_1] = \Bbbk'[X]^H$ is finitely generated, so we can write $\Bbbk_1[X_1] = \Bbbk_1[\tilde{g}_1, \dots, \tilde{g}_k]$ with $\tilde{g}_i \in \Bbbk_1[X_1] \backslash \{0\}$. Since the split torus $\mathbb{T}_1 \cong \operatorname{Spec}(\Bbbk_1[M])$ acts on X_1, the $\Bbbk_1$-algebra $\Bbbk_1[X_1]$ is M-graded, that is $\Bbbk_1[X_1] = \bigoplus_{m \in M} \Bbbk_1[X_1]_m$. So, there exists homogeneous elements $\tilde{g}_{i,j}$ such that $\tilde{g}_i = \tilde{g}_{i,1} + \dots + \tilde{g}_{i,k_i}$. Note that $\Bbbk_1[X_1]$ is generated as a $\Bbbk_1$-algebra by $\{\tilde{g}_{i,j}, {\sigma_1}^\sharp_\gamma(\tilde{g}_{i,j}) \mid \gamma \in \operatorname{Gal}(\Bbbk_1/\Bbbk)\}$. Moreover, by Lemma 2.3.6, an homogeneous element is sent to an homogeneous element by ${\sigma_1}^\sharp_\gamma$ for all $\gamma \in \operatorname{Gal}(\Bbbk_1/\Bbbk)$. Hence, we can assume there exists $n \in \mathbb{N}$ and homogeneous elements g_i of degree $m_i \in M$, such that $\Bbbk_1[X_1] = \Bbbk_1[g_1, \dots, g_n]$ and such that the set $\{g_i \mid 1 \leq i \leq n\}$ is stable under the $\operatorname{Gal}(\Bbbk_1/\Bbbk)$-action ${\sigma_1}^\sharp$ on $\Bbbk_1[X_1]$.

We obtain a Γ-equivariant isomorphism $\Bbbk'[X]^H \otimes_{\Bbbk_1} \Bbbk' \cong \Bbbk'[X]$, where the Γ-action on the left hand side is given by $\gamma \mapsto {\sigma_1}^\sharp_\gamma \otimes \gamma$ for all $\gamma \in \Gamma$, and where the Γ-action on $\Bbbk'[X]$ is given by $\sigma^\sharp$ (see Proposition 1.4.8). Therefore, we can write $\Bbbk'[X] = \Bbbk'[g_1, \dots, g_n]$, where the set $\{g_i \mid 1 \leq i \leq n\}$ is stable under the Γ-action $\sigma^\sharp$.

Let $\gamma \in \Gamma$, let ${\tau'}_\gamma$ and let ${\sigma'}_\gamma$ be the maps induced by the semilinear maps ${\tau'}^\sharp_\gamma(x_i) = x_j$, and ${\sigma'}^\sharp_\gamma(x_i) = x_j$, where $\sigma^\sharp_\gamma(g_i) = g_j$. This induces a $\Bbbk$-group structure τ' on $\mathbb{G}_{m,\Bbbk'}^n = \operatorname{Spec}(\Bbbk'[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ and a $\Bbbk$-structure σ' on $\mathbb{A}_{\Bbbk'}^n = \operatorname{Spec}(\Bbbk'[x_1, \dots, x_n])$.

(ii) The $\Bbbk'$-algebra morphism

$$\begin{aligned} \psi : \Bbbk'[x_1^{\pm 1}, \dots, x_n^{\pm 1}] &\to \Bbbk'[M] \\ x_i &\mapsto \chi^{m_i} \end{aligned}$$

is surjective since the $\mathbb{T}$-action on X is effective. Since $(\mathbb{T}, \tau)$ acts on (X, σ), ψ is Γ-equivariant. So, the $\Bbbk$-algebra morphism $\psi^\Gamma : \Bbbk'[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\Gamma \to \Bbbk'[M]^\Gamma$ is well defined and surjective. Hence, $(\mathbb{T}, \tau)$ is a closed subgroup of $(\mathbb{G}_{m,\Bbbk'}^n, \tau')$.

(iii) The $\Bbbk'$-algebra morphism

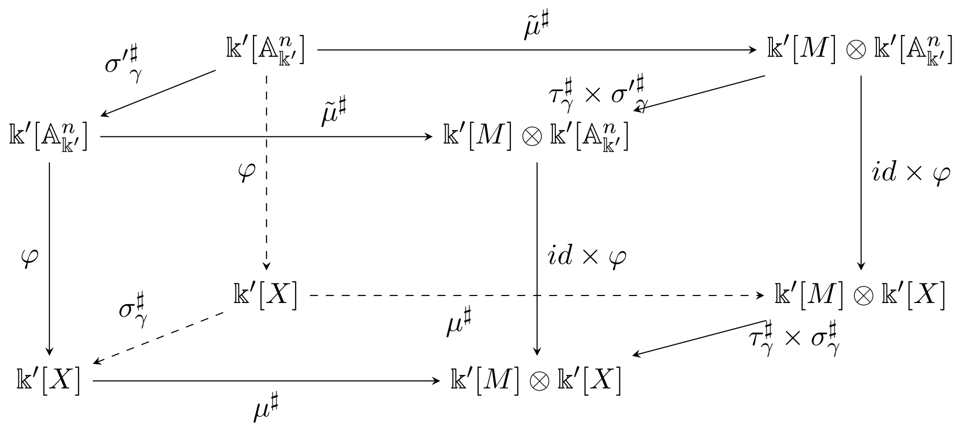
$$\begin{aligned} \varphi : \Bbbk'[x_1, \dots, x_n] &\to \Bbbk'[X_1] \\ x_i &\mapsto g_i \end{aligned}$$

is surjective and induces a $\Bbbk'$-algebra isomorphism $\Bbbk'[g_1, \dots, g_n] \cong \Bbbk'[x_1, \dots, x_n]/\mathfrak{a}$, with $\mathfrak{a} = \operatorname{Ker}(\varphi)$. Moreover, the morphism φ is Γ-equivariant. So, the $\Bbbk$-algebra morphism $\varphi^\Gamma : \Bbbk'[x_1, \dots, x_n]^\Gamma \to \Bbbk'[X_1]^\Gamma$ is well defined and surjective. Hence, (X, σ) is a closed subvariety of $(\mathbb{A}_{\Bbbk'}^n, \sigma')$.

Note that φ is $\mathbb{T}$-equivariant, so the closed immersion $X \hookrightarrow \mathbb{A}_{\Bbbk'}^n$ is $\mathbb{T}$-equivariant. Moreover, the comorphism of the $\mathbb{T}$-action on $\mathbb{A}_{\Bbbk'}^n$ is given by:

$$\begin{aligned} \tilde{\mu}^\sharp : \Bbbk'[x_1, \dots, x_n] &\to \Bbbk'[M] \otimes \Bbbk'[x_1, \dots, x_n] \\ x_i &\mapsto \chi^{m_i} \otimes x_i. \end{aligned}$$

Then, the following diagram commutes for all $\gamma \in \Gamma$:



Hence, the morphism φ is $(\mathbb{T}, \tau)$-equivariant, so (X, σ) is a closed subvariety of $(\mathbb{A}^n_{\Bbbk'}, \sigma')$, and $(X, \sigma) \hookrightarrow (\mathbb{A}^n_{\Bbbk'}, \sigma')$ is $(\mathbb{T}, \tau)$-equivariant.

Finally, note that for all $i \in \{1, \dots, n\}$, $x_i \notin \mathfrak{a}$, hence X_1 intersects the dense open orbit of $\mathbb{G}^n_{m,\Bbbk'}$.

Let $m \in M$ such that $\Bbbk'[x_1, \dots, x_n]_m \neq 0$. Note that

$$\Bbbk'[x_1, \dots, x_n]_m = \bigoplus_{(k_1,\dots,k_n) \in \mathbb{N}^n, m_1k_1 + \cdots + m_nk_n = m} \Bbbk' x_1^{k_1} \cdots x_n^{k_n}.$$

Hence, since for all $i \in \{1, \dots, n\}$, $x_i \notin \mathfrak{a}$, it follows that $(\Bbbk'[x_1, \dots, x_n]/\mathfrak{a})_m \neq 0$, and the weight cone of $\mathbb{A}^n_{\Bbbk'}$ is the weight cone of X. $\square$

Example 3.2.7. We pursue Example 3.1.16. Consider the action of $\mathbb{G}^2_{m,\mathbb{C}}$ on $\mathbb{A}^3_{\mathbb{C}}$ given by $(s,t) \cdot (x,y,z) = (sx, ty, stz)$. The Weil restriction $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$ acts on $(\mathbb{A}^3_{\mathbb{C}}, \sigma')$, where $\sigma'(x,y,z) = (\overline{y}, \overline{x}, \overline{z})$. The action of $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$ on $(\mathbb{A}^3_{\mathbb{C}}, \sigma')$ comes from the Γ-equivariant inclusion of $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$ in $(\mathbb{G}^3_{m,\mathbb{C}}, \tau')$ given by $(s,t) \mapsto (s,t,st)$, where τ' is the $\mathbb{R}$-group structure defined by $\tau' = \tau_2 \times \tau_0$. We denote by M and M' the character lattices of $\mathbb{G}^2_{m,\mathbb{C}}$ and $\mathbb{G}^3_{m,\mathbb{C}}$ respectively. Then, we obtain the diagrams of Remark 2.2.11 with:

$$F := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad P := \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \qquad \hat{\tau}_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\tau}' := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat{\tau}_Y := \begin{bmatrix} 1 \end{bmatrix}$$

A result from Artin

The next lemma is another key ingredient used in the generalization of the Altmann-Hausen construction over non closed fields.

Lemma 3.2.8. *Let Y be a quasi-projective $\Bbbk'$-variety, let σ be a $\Bbbk$-strucure on Y, and let $\mathbb{L} := \Bbbk'(Y)$. Then, σ induces a faithful Γ-action on $\mathbb{L}$ by field automorphisms, and $\mathbb{L}/\mathbb{L}^\Gamma$ is a Galois extension of absolute Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{L}^\Gamma) \cong \Gamma$. Moreover, $\mathbb{L}^\Gamma = \Bbbk(Y/\Gamma)$.*

Proof. (compare with [FJ05, Lemma 1.3.2]) Since Y is an integral scheme, for any Γ-invariant affine open subset $U \subset Y$, the ring $\mathcal{O}_Y(U)$ is integral and $\mathbb{L} := \Bbbk'(Y) = \mathrm{Frac}(\mathcal{O}_Y(U))$. Hence, σ induces a group homomorphism $\varphi : \Gamma \to \mathrm{Aut}_{\Bbbk}(\mathbb{L})$, $\gamma \mapsto \varphi_\gamma := \sigma^\sharp_\gamma$.

Let $\Bbbk_1/\Bbbk$ and let $\Bbbk_2/\Bbbk$ be finite Galois extensions in $\Bbbk'$. Since there exists a finite Galois extension $\Bbbk_3/\Bbbk$ that contains $\Bbbk_1$ and $\Bbbk_2$ [Sta, Lemmas 0EXM and 09DT], we can assume that $\Bbbk_2/\Bbbk$ is a finite Galois extension such that $\Bbbk \subset \Bbbk_1 \subset \Bbbk_2 \subset \Bbbk'$. By the Galois correspondence (see Theorem B.3.1), $H_2 := \mathrm{Gal}(\Bbbk'/\Bbbk_2)$ is a normal subgroup of $H_1 := \mathrm{Gal}(\Bbbk'/\Bbbk_1)$. Consider the $\Bbbk_1$-variety $Y_1 := Y/H_1$ and the $\Bbbk_2$-variety $Y_2 := Y/H_2$.

We have $Y/\Gamma \cong Y_1/\mathrm{Gal}(\Bbbk_1/\Bbbk) \cong Y_2/\mathrm{Gal}(\Bbbk_2/\Bbbk)$ (see Propositions 1.4.8 and 1.4.9). Moreover, since $\mathrm{Gal}(\Bbbk_2/\Bbbk_1)$ is a finite group[2], then $\Bbbk_1(Y_1)^{\mathrm{Gal}(\Bbbk_1/\Bbbk)} = \Bbbk_2(Y_2)^{\mathrm{Gal}(\Bbbk_2/\Bbbk)}$. Denote $\mathbb{K} := \Bbbk_1(Y_1)^{\mathrm{Gal}(\Bbbk_1/\Bbbk)}$, this field does not depend on the finite Galois extension $\Bbbk_1/\Bbbk$. By [Lan02, Theorem 1.8 (Artin)], the field extension $\Bbbk_1(Y_1)/\mathbb{K}$ is a finite Galois extension of Galois group $\mathrm{Gal}(\Bbbk_1(Y_1)/\mathbb{K}) \cong \mathrm{Gal}(\Bbbk_1/\Bbbk)$. Furthermore, the field $\mathbb{L} := \Bbbk'(Y)$ is the union of all above $\Bbbk_1(Y_1)$. Then (see [Sta, Lemma 0BU2]),

$$\mathrm{Gal}(\Bbbk'/\Bbbk) = \lim \ \mathrm{Gal}\left(\Bbbk_i/\Bbbk\right) \cong \lim \ \mathrm{Gal}\left(\Bbbk_i(Y_i)/\mathbb{K}\right) = \mathrm{Gal}(\mathbb{L}/\mathbb{K}),$$

where the limits are over all finite Galois extension $\Bbbk_i/\Bbbk$ in $\Bbbk'$. □

Proof of Theorem 3.2.3

Proof of the first item. Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum over $\overline{\Bbbk}$. By Theorem 3.1.12 (1), $X[Y,\mathcal{D}] := \mathrm{Spec}(A[Y,\mathcal{D}])$ is a normal affine $\overline{\Bbbk}$-variety endowed with a $\mathbb{T}$-action, of weight cone $\omega_N^\vee$. This action is obtained from the following comorphism:

$$\begin{aligned} \mu^\sharp : A[Y,\mathcal{D}] &\to \overline{\Bbbk}[M] \otimes A[Y,\mathcal{D}] \\ f\mathfrak{X}_m &\mapsto \chi^m \otimes f\mathfrak{X}_m. \end{aligned}$$

We now construct a $\Bbbk$-structure on $X[Y,\mathcal{D}]$ such that $(\mathbb{T},\tau)$ acts on $(X[Y,\mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$. Let $\gamma \in \Gamma$. By Condition (3.1), we obtain isomorphisms of $A[Y,\mathcal{D}]_0$-modules:

$$\begin{aligned} \alpha_{\gamma_m} : \mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))\mathfrak{X}_m &\to \mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(\tilde{\tau}_\gamma(m))))\mathfrak{X}_{\tilde{\tau}_\gamma(m)} \\ f\mathfrak{X}_m &\mapsto {\sigma_Y}^\sharp_\gamma(f) h_\gamma(\tilde{\tau}_\gamma(m))\mathfrak{X}_{\tilde{\tau}_\gamma(m)}. \end{aligned}$$

These isomorphisms collect into an isomorphism of $A[Y,\mathcal{D}]_0$-modules $\oplus_{m\in\omega_N^\vee\cap M}\alpha_m$ on the direct sum $A[Y,\mathcal{D}]$. By Condition (3.2), the latter isomorphism corresponds to a $\Bbbk$-structure $\sigma_{X[Y,\mathcal{D}]}$ on $X[Y,\mathcal{D}]$. Finally, $(\mathbb{T},\tau)$ acts on $(X[Y,\mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$ since the following diagram commutes for all $\gamma \in \Gamma$:

$$\begin{array}{ccc} A[Y,\mathcal{D}] & \xrightarrow{\mu^\sharp} & \Bbbk[M] \otimes A[Y,\mathcal{D}] \\ {\sigma^\sharp}_{X[Y,\mathcal{D}]_\gamma}\downarrow & & \downarrow \tau^\sharp_\gamma \otimes \sigma^\sharp_{X[Y,\mathcal{D}]_\gamma} \\ A[Y,\mathcal{D}] & \xrightarrow[\mu^\sharp]{} & \Bbbk[M] \otimes A[Y,\mathcal{D}] \end{array}$$

□

Proof of the second item.

• ***Step 0**: Preliminaries.*

Let (X, σ_X) be a normal affine $\Bbbk$-variety endowed with an effective $(\mathbb{T},\tau)$-action. Let ω_M be the weight cone of the $\mathbb{T}$-action on X, and let $\omega_N = \omega_M^\vee$. From Lemma 2.3.6, for all $\gamma \in \Gamma$, $\hat{\tau}_\gamma(\omega_N) = \omega_N$.

Using Proposition 3.2.6, there exists $n \in \mathbb{N}$ such that $(\mathbb{T},\tau)$ is a closed subgroup of $(\mathbb{G}^n_{m,\overline{\Bbbk}}, \tau')$ and (X, σ_X) is a closed $(\mathbb{T},\tau)$-equivariant subvariety of $(\mathbb{A}^n_{\overline{\Bbbk}}, \sigma)$. So, let $\mathfrak{a}$ be the ideal of $\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}] = \overline{\Bbbk}[x_1,\ldots,x_n]$ such that $\overline{\Bbbk}[X]$ is $(\Gamma\times\mathbb{T})$-equivariantly isomorphic to $\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}]/\mathfrak{a}$. We write $\overline{\Bbbk}[X] = \overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}]/\mathfrak{a}$. Let M' be the character lattice of $\mathbb{G}^n_{m,\Bbbk'}$, and let M_Y be the sublattice of M' constructed in Remark 2.2.11. We have the commutative diagrams of Remark 2.2.11.

---

[2]If G is a finite group acting on an integral ring A by ring morphisms, then $\mathrm{Frac}(A^G) = \mathrm{Frac}(A)^G$. Indeed, $\mathrm{Frac}(A^G) \subset \mathrm{Frac}(A)^G$ and since $\dfrac{a}{b} = \dfrac{a\prod_{g\in G, g\neq e_G} g\cdot b}{\prod_{g\in G} g\cdot b}$, then $\mathrm{Frac}(A)^G \subset \mathrm{Frac}(A^G)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y & \longrightarrow & 0 \\
 & & \big\downarrow \hat{\tau}_\gamma & & \big\downarrow \hat{\tau}'_\gamma & & \big\downarrow \hat{\tau_Y}_\gamma & & \\
0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M & \longrightarrow & 0 \\
 & & \big\downarrow \tilde{\tau_Y}_\gamma & & \big\downarrow \tilde{\tau}'_\gamma & & \big\downarrow \tilde{\tau}_\gamma & & \\
0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M & \longrightarrow & 0
\end{array}$$

Recall that there always exists a section $s^* : M \to M'$ (see Lemma C.3.2), but not necessarily a Γ-equivariant one, and a cosection $t^* : M' \to M_Y$. These homomorphisms satisfy $F^* \circ s^* = Id_M$, $t^* \circ P^* = Id_{M_Y}$ and $P^* \circ t^* = Id_{M'} - s^* \circ F^*$.

Since $\mathrm{Frac}(\overline{\Bbbk}[M']) = \overline{\Bbbk}(\mathbb{A}^n_{\overline{\Bbbk}})$, the section s^* induces a morphism

$$\begin{aligned} u : \omega_M \cap M &\to \overline{\Bbbk}\left(\mathbb{A}^n_{\overline{\Bbbk}}\right)^* \\ m &\mapsto \chi^{s^*(m)} \end{aligned}$$

such that for all $m \in \omega_M \cap M$, $u(m) \in \overline{\Bbbk}(\mathbb{A}^n_{\overline{\Bbbk}})_m$.

By Section 2.3.1, we can write

$$\overline{\Bbbk}\left[\mathbb{A}^n_{\overline{\Bbbk}}\right] = \bigoplus_{m \in \omega_M \cap M} \widetilde{\overline{\Bbbk}\left[\mathbb{A}^n_{\overline{\Bbbk}}\right]}_m u(m) \subset \overline{\Bbbk}\left(\mathbb{A}^n_{\overline{\Bbbk}}\right)_0 [M],$$

where $\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}]_m = \widetilde{\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}]}_m u(m)$, with $\widetilde{\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}]}_m \subset \overline{\Bbbk}(\mathbb{A}^n_{\overline{\Bbbk}})_0$.

Since for all i, $x_i \notin \mathfrak{a}$, for all $m \in \omega_M \cap M$ we have $u_X(m) \in \overline{\Bbbk}(X)_m$, where $u_X(m)$ is obtained from the surjective morphism $\overline{\Bbbk}[\mathbb{A}^n_{\overline{\Bbbk}}] \twoheadrightarrow \overline{\Bbbk}[X]$. It follows a morphism $u_X : \omega_M \cap M \to \overline{\Bbbk}(X)^*$. Hence we can write

$$\overline{\Bbbk}[X] = \bigoplus_{m \in \omega_M \cap M} \overline{\Bbbk}[X]_m = \bigoplus_{m \in \omega_M \cap M} \widetilde{\overline{\Bbbk}[X]}_m u_X(m) \subset \overline{\Bbbk}(X)_0[M],$$

where, for all $m \in \omega_M \cap M$, $\overline{\Bbbk}[X]_m = \widetilde{\overline{\Bbbk}[X]}_m u_X(m)$ and $\widetilde{\overline{\Bbbk}[X]}_m \subset \overline{\Bbbk}(X)_0$.

- ***Step 1**: Altmann-Hausen quotient and divisors.*

Let $\{e_1, \dots, e_n\}$ be the standard basis of N'. The cone in $N'_{\mathbb{Q}}$ of the toric variety $\mathbb{A}^n_{\overline{\Bbbk}}$ is $\mathbb{Q}^n_{\geq 0}$, and $F^*(\mathbb{Q}^n_{\geq 0}) = \omega_M$. Since the fan in $N'_{\mathbb{Q}}$ generated by $\{e_1, \dots, e_n\}$ is Γ-stable (for $\hat{\tau}'$) and P is Γ-equivariant, the fan Λ_Y in $(N_Y)_{\mathbb{Q}}$ generated by $\{P(e_1), \dots, P(e_n)\}$ is Γ-stable (for $\hat{\tau}_Y$).

Let Y be the toric $\overline{\Bbbk}$-variety obtained from the fan Λ_Y; it is a semi-projective variety (see [CLS11, Proposition 7.2.9]). Since Λ_Y is Γ-stable, the $\Bbbk$-group structure τ_Y on $\mathbb{T}_Y := \mathrm{Spec}(\overline{\Bbbk}[M_Y])$ extends to an $\Bbbk$-structure σ_Y on Y by Proposition 2.4.9.

Let Y_X be the closure of the image of $X \cap \mathbb{G}^n_{m,\overline{\Bbbk}}$ in Y by the surjective group homomorphism $\pi : \mathbb{G}^n_{m,\overline{\Bbbk}} \twoheadrightarrow \mathbb{T}_Y$ composed with the inclusion $\mathbb{T}_Y \hookrightarrow Y$. Since these morphisms are Γ-equivariant, the $\Bbbk$-structure on Y restricts to an $\Bbbk$-structure σ_{Y_X} on Y_X. The normalization $\tilde{Y}_X$ of Y_X, with morphism $\nu : \tilde{Y}_X \to Y_X$, is a semi-projective $\overline{\Bbbk}$-variety. Using universal property of normalization (see [Har77, Exercise 3.8]) and the fact that ν is an isomorphism on a dense open subset of Y_X (see [Sta, Lemma 0BXR]), there exists an $\Bbbk$-structure $\sigma_{\tilde{Y}_X}$ on $\tilde{Y}_X$ that makes the following diagram commute for all $\gamma \in \Gamma$

$$\begin{array}{ccc} \tilde{Y}_X & \xrightarrow{\sigma_{\tilde{Y}_X\gamma}} & \tilde{Y}_X \\ \nu \downarrow & & \downarrow \nu \\ Y_X & \xrightarrow{\sigma_{Y_X\gamma}} & Y_X . \end{array}$$

For each ray of the fan Λ_Y, we denote by v_i its first lattice vector. To a ray spanned by v_i corresponds a toric divisor D_{v_i} on Y (see Appendix C.2).

By Proposition 3.1.19, the divisor $\mathcal{D} = \sum_{v_i} \Delta_{v_i} \otimes D_{v_i}$, where $\Delta_{v_i} := s(\mathbb{Q}^n_{\geq 0} \cap P^{-1}(v_i))$, is an ω_N-pp-divisor on Y. Furthermore, we can pull back $\mathcal{D}$ on $\tilde{Y}_X$ and we obtain an ω_N-pp-divisor $\mathcal{D}_X$ on $\tilde{Y}_X$.

- ***Step 2:*** *Isomorphisms* $\overline{\mathbb{k}}(Y) \cong \overline{\mathbb{k}}(\mathbb{A}^n_{\overline{\mathbb{k}}})_0$ *and* $\overline{\mathbb{k}}(Y_X) \cong \overline{\mathbb{k}}(X)_0$.

Observe that $\pi^\sharp : \overline{\mathbb{k}}[M_Y] \to \overline{\mathbb{k}}[M']_0$ is an isomorphism. Hence, $\pi^\sharp$ induces an isomorphism

$$\begin{aligned} \mathrm{Frac}(\overline{\mathbb{k}}[M_Y]) &\to \mathrm{Frac}(\overline{\mathbb{k}}[M']_0) \\ \frac{f}{g} = \frac{\sum a_i \chi^{m_i}}{\sum b_j \chi^{m_j}} &\mapsto \frac{\pi^\sharp(f)}{\pi^\sharp(g)} = \frac{\sum a_i \chi^{P^*(m_i)}}{\sum b_j \chi^{P^*(m_j)}}. \end{aligned}$$

For all $m \in M$, note that $(F^*)^{-1}(m) = s^*(m) + \mathrm{Ker}(F^*)$. Hence, $\mathrm{Frac}(\overline{\mathbb{k}}[M'])_0 = \mathrm{Frac}(\overline{\mathbb{k}}[M']_0)$. Since $\mathbb{T}_Y$ is a dense open subset of Y, we have $\overline{\mathbb{k}}(Y) = \overline{\mathbb{k}}(\mathbb{T}_Y)$. Since $\mathbb{G}^n_{m,\overline{\mathbb{k}}}$ is a dense open subset of $\mathbb{A}^n_{\overline{\mathbb{k}}}$, we have $\mathrm{Frac}(\overline{\mathbb{k}}[M']) = \overline{\mathbb{k}}(\mathbb{A}^n_{\overline{\mathbb{k}}})$. Therefore, $\mathrm{Frac}(\overline{\mathbb{k}}[M'])_0 = \overline{\mathbb{k}}(\mathbb{A}^n_{\overline{\mathbb{k}}})_0$. Finally, we obtain a Γ-equivariant isomorphism

$$\varphi : \overline{\mathbb{k}}(Y) \to \overline{\mathbb{k}}\left(\mathbb{A}^n_{\overline{\mathbb{k}}}\right)_0 .$$

Moreover, the inclusion $X \hookrightarrow \mathbb{A}^n_{\overline{\mathbb{k}}}$ is $\mathbb{T}$-equivariant, the variety $\mathbb{T}_Y \cap Y_X$ is affine and the following diagram commutes

$$\begin{array}{ccc} \mathbb{G}^n_{m,\overline{\mathbb{k}}} & \xrightarrow{\pi} & \mathbb{T}_Y \\ \uparrow & & \uparrow \\ \mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X & \twoheadrightarrow & \mathbb{T}_Y \cap Y_X = \pi\left(\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X\right). \end{array}$$

Therefore the following diagram commutes

$$\begin{array}{ccc} \overline{\mathbb{k}}\left[\mathbb{T}_Y\right] & \xrightarrow{\pi^\sharp} & \overline{\mathbb{k}}\left[\mathbb{G}^n_{m,\overline{\mathbb{k}}}\right]_0 \\ \downarrow & & \downarrow \\ \overline{\mathbb{k}}\left[\mathbb{T}_Y \cap Y_X\right] & \hookrightarrow & \overline{\mathbb{k}}\left[\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X\right]_0 . \end{array}$$

It follows an isomorphism $\overline{\mathbb{k}}(\mathbb{T}_Y \cap Y_X) \to \mathrm{Frac}(\overline{\mathbb{k}}[\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X]_0)$. Since,

$$\mathrm{Frac}\left(\overline{\mathbb{k}}\left[\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X\right]_0\right) = \overline{\mathbb{k}}\left(\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X\right)_0, \ \overline{\mathbb{k}}\left(Y_X\right) = \overline{\mathbb{k}}\left(\mathbb{T}_Y \cap Y_X\right), \text{ and } \overline{\mathbb{k}}\left(X\right) = \overline{\mathbb{k}}\left(\mathbb{G}^n_{m,\overline{\mathbb{k}}} \cap X\right),$$

we obtain a Γ-equivariant isomorphism $\overline{\mathbb{k}}(Y_X) \cong \overline{\mathbb{k}}(X)_0$. Note that we have a Γ-equivariant isomorphism $\overline{\mathbb{k}}(Y_X) \cong \overline{\mathbb{k}}(\tilde{Y}_X)$, therefore, the isomorphism

$$\varphi_X : \overline{\mathbb{k}}\left(\tilde{Y}_X\right) \to \overline{\mathbb{k}}(X)_0$$

is Γ-equivariant.

- ***Step 3:*** *Isomorphisms* $A[Y, \mathcal{D}] \cong \overline{\mathbb{k}}[\mathbb{A}^n_{\overline{\mathbb{k}}}]$ *and* $A[\tilde{Y}_X, \mathcal{D}_X] \cong \overline{\mathbb{k}}[X]$.

Let $m \in \omega_M \cap M$. Consider the polyhedron $\Delta(m) := (F^*)^{-1}(m) \cap \mathbb{Q}^n_{\geq 0} \subset M'_{\mathbb{Q}}$ and the polyhedron $\Delta_Y(m) := t^*(\Delta(m)) \subset (M_Y)_{\mathbb{Q}}$. Note that

$$\widetilde{\overline{\Bbbk}[\mathbb{A}^n_{\Bbbk}]}_m := \bigoplus_{m' \in \Delta(m) \cap M'} \overline{\Bbbk}\chi^{m' - s^*(m)} = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \overline{\Bbbk}\chi^{P^*(m_Y)} = \varphi\left(\bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \overline{\Bbbk}\chi^{m_Y}\right).$$

It follows from Proposition C.2.2 and Lemma 3.1.20 that

$$\mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \overline{\Bbbk}\chi^{m_Y}.$$

Therefore, we obtain a graded algebra isomorphism

$$\begin{aligned} \Phi : A[Y, \mathcal{D}] &\to \overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right] \\ f\mathfrak{X}_m &\mapsto \varphi(f)u(m). \end{aligned}$$

Consider the morphism

$$\begin{aligned} \Phi_X : A[\tilde{Y}_X, \mathcal{D}_X] &\to \overline{\Bbbk}(X)_0[M] \\ f\mathfrak{X}_m &\mapsto \varphi_X(f)u_X(m). \end{aligned}$$

As in the proof of Theorem 3.1.12, the natural map $\Psi : A[Y, \mathcal{D}] \to A[\tilde{Y}_X, \mathcal{D}_X]$ is a surjective graded algebra morphism. Therefore, the image of Φ_X is contained in $\overline{\Bbbk}[X]$, and we obtain a graded algebra isomorphism

$$\begin{aligned} \Phi_X : A\left[\tilde{Y}_X, \mathcal{D}_X\right] &\to \overline{\Bbbk}\left[X\right] \\ f\mathfrak{X}_m &\mapsto \varphi_X(f)u_X(m) \end{aligned}$$

such that the following diagram commutes

$$\begin{array}{ccc} A\left[Y, \mathcal{D}\right] & \xrightarrow{\ \Phi\ } & \overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right] \\ \Psi \downarrow & & \downarrow \\ A\left[\tilde{Y}_X, \mathcal{D}_X\right] & \xrightarrow{\ \Phi_X\ } & \overline{\Bbbk}\left[X\right]. \end{array}$$

- ***Step 4***: *Equality* $\sigma^*_{\tilde{Y}_X}(\mathcal{D}_X(m)) = \mathcal{D}_X(\tilde{\tau}(m)) + \operatorname{div}(h(\tilde{\tau}(m)))$, *for all* $m \in \omega_M \cap M$.

Let $m \in \omega_M \cap M$ and let $\gamma \in \Gamma$. Let

$$h'_\gamma(m) := \frac{\sigma^\sharp_\gamma\left(u\left(\tilde{\tau}_{\gamma^{-1}}(m)\right)\right)}{u(m)}.$$

By Lemma 2.3.6,

$$\sigma^\sharp_\gamma\left(\overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right]_{\tilde{\tau}_\gamma(m)}\right) = \overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right]_m.$$

It follows that $h'_\gamma(m) \in \overline{\Bbbk}(\mathbb{A}^n_{\Bbbk})^*_0$, and we get a monoid morphism

$$h'_\gamma : \omega_M \cap M \to \overline{\Bbbk}(\mathbb{A}^n_{\Bbbk})^*_0.$$

Moreover, note that $h'_{\gamma_1\gamma_2}(m) = h'_{\gamma_1}(m)\sigma^\sharp_{\gamma_1}(h'_{\gamma_2}(\tilde{\tau}_{\gamma_1^{-1}}(m)))$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Let $\gamma \in \Gamma$ and consider the monoid morphism

$$h_\gamma := \varphi^{-1} \circ h'_\gamma : \omega_M \cap M \to \overline{\Bbbk}(Y)^*.$$

We construct a $\Bbbk$-structure on $X[Y, \mathcal{D}]$ from the $\Bbbk$-structure σ on $\mathbb{A}^n_{\Bbbk}$ using the following commutative diagram

$$\begin{array}{ccc} \overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right] & \xrightarrow{\sigma^\sharp_\gamma} & \overline{\Bbbk}\left[\mathbb{A}^n_{\Bbbk}\right] \\ \Phi\uparrow & & \uparrow\Phi \\ A\left[Y,\mathcal{D}\right] & \longrightarrow & A\left[Y,\mathcal{D}\right] \end{array} \qquad \begin{array}{ccc} \varphi(f)u(m) & \longmapsto & \sigma^\sharp_\gamma\left(\varphi(f)\right)h'_\gamma(\tilde{\tau}_\gamma(m))u(\tilde{\tau}_\gamma(m)) \\ \updownarrow & & \updownarrow \\ f\mathfrak{X}_m & \longmapsto & \varphi^{-1}\left(\sigma^\sharp_\gamma\left(\varphi(f)\right)h_\gamma(\tilde{\tau}_\gamma(m))\right)\mathfrak{X}_{\tilde{\tau}_\gamma(m)}. \end{array}$$

Since φ is Γ-equivariant, we have

$$\varphi^{-1}\left(\sigma^\sharp\left(\varphi(f)\right)h'_\gamma(\tilde{\tau}_\gamma(m))\right) = {\sigma_Y}^\sharp_\gamma(f)h_\gamma(\tilde{\tau}_\gamma(m)).$$

Hence, the morphism

$$\begin{aligned} A[Y,\mathcal{D}] &\to A[Y,\mathcal{D}] \\ f\mathfrak{X}_m &\mapsto {\sigma_Y}^\sharp_\gamma(f)h_\gamma(\tilde{\tau}_\gamma(m))\mathfrak{X}_{\tilde{\tau}_\gamma(m)} \end{aligned}$$

induces a $\Bbbk$-structure σ' on $X[Y,\mathcal{D}]$. From this we deduce that, for all $\gamma \in \Gamma$,

$${\sigma_Y}^*_\gamma(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}_\gamma(m)) + \mathrm{div}_Y(h(\tilde{\tau}_\gamma(m))). \tag{3.3}$$

Moreover, $h_{\gamma_1\gamma_2}(m) = h_{\gamma_1}(m){\sigma_Y}^\sharp_{\gamma_1}(h_{\gamma_2}(\tilde{\tau}_{\gamma_1^{-1}}(m)))$ for all $\gamma_1,\gamma_2 \in \Gamma$. By the same reasoning, we construct a $\Bbbk$-structure σ'' on $X[\tilde{Y}_X,\mathcal{D}_X]$ from the following morphism

$$\begin{aligned} A\left[\tilde{Y}_X,\mathcal{D}_X\right] &\to A\left[\tilde{Y}_X,\mathcal{D}_X\right] \\ f\mathfrak{X}_m &\mapsto {\sigma_{\tilde{Y}_X}}^\sharp_\gamma(f)h_X(\tilde{\tau}_\gamma(m))\mathfrak{X}_{\tilde{\tau}_\gamma(m)}, \end{aligned}$$

where h_X is obtained from h by the projection $\overline{\Bbbk}[\mathbb{A}^n_{\Bbbk}] \twoheadrightarrow \overline{\Bbbk}[X]$. Thus we obtain, for all $\gamma \in \Gamma$,

$${\sigma_{\tilde{Y}_X}}^*_\gamma\left(\mathcal{D}_X(m)\right) = \mathcal{D}_X\left(\tilde{\tau}_\gamma(m)\right) + \mathrm{div}_{\tilde{Y}_X}\left(h_{X\gamma}\left(\tilde{\tau}_\gamma(m)\right)\right)$$

and, for all $\gamma_1,\gamma_2 \in \Gamma$, $h_{X\gamma_1\gamma_2}(m) = h_{X\gamma_1}(m){\sigma_{\tilde{Y}_X}}^\sharp_{\gamma_1}(h_{X\gamma_2}(\tilde{\tau}_{\gamma_1^{-1}}(m)))$.

Therefore, we get a generalized AH-datum $(\omega_N, \tilde{Y}_X, \mathcal{D}_X, \sigma_{\tilde{Y}_X}, h_X)$, and the $\Bbbk$-structure $\sigma_{X[\tilde{Y}_X,\mathcal{D}_X]}$ constructed from this datum (see the proof of the first item) is exactly σ''. Therefore, $(X,\sigma_X) \cong (X[Y,\mathcal{D}],\sigma_{X[Y,\mathcal{D}]})$ as $(\mathbb{T},\tau)$-varieties. □

Remark 3.2.9. The construction of the $\Bbbk$-variety $(X[Y,\mathcal{D}],\sigma_{X[Y,\mathcal{D}]})$ does not depend on the choice of the cosection. Indeed, for $j \in \{1,2\}$, let $s_j : N' \to N$ be two cosections, let $\mathcal{D}_j := \sum_i \Delta^j_{v_i} \otimes D_{v_i}$ be the two associated pp-divisors, and let $h_j : \Gamma \to \mathrm{Hom}\left(\omega_N^\vee \cap M, \overline{\Bbbk}(Y)^*\right)$ be the map constructed in Step 4. Note that for all $m \in M$, $(s_1^* - s_2^*)(m) \in \mathrm{Ker}(F^*) = \mathrm{Im}(P^*)$, thus there exists a lattice homomorphism $s_0 : N_Y \to N$ such that $s_1 - s_2 = s_0 \circ P$. Let $g : M \to \overline{\Bbbk}(Y)^*$ be the morphism defined by $g(m) := \chi^{s_0^*(m)}$. Let $m \in \omega_M \cap M$. Since $\Delta^1_{v_i} = s_0(v_i) + \Delta^2_{v_i}$, we have :

$$\mathcal{D}_1(m) = \sum_i \langle s_0^*(m)|v_i\rangle \otimes D_{v_i} + \mathcal{D}_2(m) = \mathrm{div}_Y(g(m)) + \mathcal{D}_2(m).$$

Therefore the M-graded algebras $A[Y,\mathcal{D}_1]$ and $A[Y,\mathcal{D}_2]$ are isomorphic via:

$$\begin{aligned} A\left[Y,\mathcal{D}_1\right] &\to A\left[Y,\mathcal{D}_2\right] \\ f\mathfrak{X}_m &\mapsto fg(m)\mathfrak{X}_m. \end{aligned}$$

Moreover, since for all $\gamma \in \Gamma$,

$$\frac{\sigma_{Y}{}^{\sharp}_{\gamma}\left(g\left(\tilde{\tau}_{\gamma}^{-1}(m)\right)\right)}{g(m)} = \frac{\sigma_{Y}{}^{\sharp}_{\gamma}\left(\chi^{s_0^*\left(\tilde{\tau}_{\gamma}^{-1}(m)\right)}\right)}{\chi^{s_0^*(m)}} = \varphi_X^{-1}\left(\frac{\sigma_{\gamma}^{\sharp}\left(\chi^{P^*\circ s_0^*\left(\tilde{\tau}_{\gamma}^{-1}(m)\right)}\right)}{\chi^{P^*\circ s_0^*(m)}}\right)$$

$$= \varphi_X^{-1}\left(\frac{\sigma_{\gamma}^{\sharp}\left(u_1(\tilde{\tau}_{\gamma}^{-1}(m))\right)u_2(m)}{u_1(m)\sigma_{\gamma}^{\sharp}\left(u_2(\tilde{\tau}_{\gamma}^{-1}(m))\right)}\right) = \frac{h_{1\gamma}(m)}{h_{2\gamma}(m)},$$

the following diagram commutes:

$$\begin{array}{ccc} A\left[Y,\mathcal{D}_1\right] & \xrightarrow{\sigma_{X[Y,\mathcal{D}_1]}{}^{\sharp}_{\gamma}} & A\left[Y,\mathcal{D}_1\right] \\ \cong\downarrow & & \downarrow\cong \\ A\left[Y,\mathcal{D}_2\right] & \xrightarrow{\sigma_{X[Y,\mathcal{D}_2]}{}^{\sharp}_{\gamma}} & A\left[Y,\mathcal{D}_2\right] \end{array} \qquad \begin{array}{ccc} f\mathfrak{X}_m & \longmapsto & \sigma_{Y}{}^{\sharp}_{\gamma}(f)h_{1\gamma}(\tilde{\tau}_{\gamma}(m))\mathfrak{X}_{\tilde{\tau}_{\gamma}(m)} \\ \downarrow & & \downarrow \\ fg(m)\mathfrak{X}_m & \mapsto & \sigma_{Y}{}^{\sharp}_{\gamma}(fg(m))h_{2\gamma}(\tilde{\tau}_{\gamma}(m))\mathfrak{X}_{\tilde{\tau}_{\gamma}(m)} \end{array}$$

Hence, the $\Bbbk$-varieties $(X[Y,\mathcal{D}_1],\sigma_{X[Y,\mathcal{D}_1]})$ and $(X[Y,\mathcal{D}_2],\sigma_{X[Y,\mathcal{D}_2]})$ are $(\mathbb{T},\tau)$-equivariantly isomorphic.

3.2.3 Galois cohomology, torsors and Altmann-Hausen presentation

In the non-necessarily split version of the Altmann-Hausen construction (Theorem 3.2.3), a cocycle h appears in the combinatorial presentation. We will see that Theorem 3.2.3 simplifies (i.e we can take $h = 1$) if $\mathrm{H}^1_{cont}(\Gamma, G) = \{1\}$ (see Remark 3.2.2).

Proposition 3.2.10. *Let $\Bbbk$ be a field, let $\mathbb{T} = \operatorname{Spec}(\overline{\Bbbk}[M])$ be a $\overline{\Bbbk}$-torus, and let τ be a $\Bbbk$-group structure on $\mathbb{T}$. Let (X,σ) be a normal affine variety endowed with an effective action of $(\mathbb{T},\tau)$. Let (Y,σ_Y) be the $\Bbbk$-variety of Theorem 3.2.3. If $\mathrm{H}^1_{cont}(\Gamma,G) = \{1\}$ (see Remark 3.2.2), then there exists an ω_N-pp divisor $\mathcal{D}$ on Y such that*

$$\forall m \in \omega_M \cap M, \forall \gamma \in \Gamma,\ \sigma_{Y}{}^{*}_{\gamma}(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}_{\gamma}(m)),$$

and such that the varieties (X,σ) and $(X[Y,\mathcal{D}],\sigma_{X[Y,\mathcal{D}]})$ are $(\mathbb{T},\tau)$-equivariantly isomorphic.

Proof. By Theorem 3.2.3, there exists a generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ such that the varieties (X,σ_X) and $(X[Y,\mathcal{D}'],\sigma_{X[Y,\mathcal{D}']})$ are $(\mathbb{T},\tau)$-equivariantly isomorphic.
Since $\mathrm{H}^1_{cont}(\Gamma,G) = \{1\}$, the trivial cocycle is equivalent to h, so there exists $g \in G$ such that

$$h_{\gamma}(m) = g^{-1}(m)\sigma_{Y}{}^{\sharp}_{\gamma}(g(\tilde{\tau}_{\gamma}(m))).$$

Let $m \in \omega_M \cap M$ and $\gamma \in \Gamma$, then

$$\sigma_{Y}{}^{*}_{\gamma}(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}_{\gamma}(m)) + \operatorname{div}_Y(h_{\gamma}(\tilde{\tau}_{\gamma}(m))) = \mathcal{D}(\tilde{\tau}_{\gamma}(m)) + \sigma_{Y}{}^{*}_{\gamma}\operatorname{div}_Y(g(m)) - \operatorname{div}_Y(g(\tilde{\tau}_{\gamma}(m)))$$

$$\iff \sigma_{Y}{}^{*}_{\gamma}\left(\mathcal{D}(m) - \operatorname{div}_Y(g(m))\right) = \mathcal{D}(\tilde{\tau}_{\gamma}(m)) - \operatorname{div}_Y(g(\tilde{\tau}_{\gamma}(m))).$$

So, if $\mathcal{D}'$ is the pp-divisor defined by

$$\mathcal{D}'(m) := \mathcal{D}(m) - \operatorname{div}_Y(g(m)),$$

then $\sigma_{Y}{}^{*}_{\gamma}(\mathcal{D}'(m)) = \mathcal{D}'(\tilde{\tau}_{\gamma}(m))$, and the M-graded algebras $A[Y,\mathcal{D}]$ and $A[Y,\mathcal{D}']$ are isomorphic with respect to $\sigma^{\sharp}_{X[Y,\mathcal{D}]}$ and $\sigma^{\sharp}_{X[Y,\mathcal{D}']}$. Hence the varieties (X,σ_X) and $(X[Y,\mathcal{D}'],\sigma_{X[Y,\mathcal{D}']})$ are $(\mathbb{T},\tau)$-equivariantly isomorphic. □

Quasi-trivial $\mathbb{k}$-tori have no non-trivial torsors (see Proposition 2.3.19). Therefore, the Altmann-Hausen presentation simplifies for quasi-trivial $\mathbb{k}$-torus actions on normal affine varieties.

Proposition 3.2.11 (Quasi-trivial torus actions on normal affine varieties over arbitrary fields)**.** *Let* $\mathbb{k}$ *be a field, and denote* $\Gamma := \mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. *Let* $(\mathbb{T} = \mathrm{Spec}(\overline{\mathbb{k}}[M]), \tau)$ *be a quasi-trivial* $\mathbb{k}$*-torus.*

(i) Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h = 1)$ *be a generalized AH-datum over* $\overline{\mathbb{k}}$. *Then the affine* $\mathbb{T}$*-variety* $X[Y, \mathcal{D}]$ *admits a* $\mathbb{k}$*-structure* $\sigma_{X[Y,\mathcal{D}]}$ *such that* $(\mathbb{T}, \tau)$ *acts effectively on* $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

(ii) Let (X, σ) *be a normal affine* $\mathbb{k}$*-variety endowed with an effective* $(\mathbb{T}, \tau)$*-action. Then there exists a generalized AH-datum* $(\omega_N, Y, \mathcal{D}, \sigma_Y, h = 1)$ *such that there is an isomorphism of* $(\mathbb{T}, \tau)$*-varieties* $(X, \sigma) \cong (X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

Proof. If $(\mathbb{T}, \tau)$ is a quasi-trivial $\mathbb{k}$-torus acting on a normal affine variety, then the $\mathbb{K}$-torus $(\mathbb{T}_{\mathbb{L}}, \tau_{\mathbb{L}})$ is still quasi-trivial (see Remark 3.2.2). Therefore, $\mathrm{H}^1_{cont}(\Gamma, G) = \{1\}$. Conversely, let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h = 1)$ be a generalized AH-datum over $\overline{\mathbb{k}}$. Then, one can easily shows that the affine $\mathbb{T}$-variety $X[Y, \mathcal{D}]$ admits a $\mathbb{k}$-structure $\sigma_{X[Y,\mathcal{D}]}$ such that $(\mathbb{T}, \tau)$ acts on $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$ (see the proof of the first item of Theorem 3.2.3). □

Remark 3.2.12. In the $\mathbb{C}/\mathbb{R}$ case, this simplification is always possible when the acting torus is quasi-trivial, that is it has no $\mathbb{S}^1$-factors ([Gil22a, Lemma 4.12]). Recall that any real torus is isomorphic to a torus of the form $\mathbb{G}_{m,\mathbb{R}}^{n_0} \times (\mathbb{S}^1)^{n_1} \times \mathrm{R}_{\mathbb{C}/\mathbb{R}}^{n_2}(\mathbb{G}_{m,\mathbb{C}})$, with $n_0, n_1, n_2 \in \mathbb{N}$ (see Proposition 4.1.2).

3.2.4 Split $\mathbb{k}$-torus actions

A consequence of Hilbert's 90 Theorem (see Theorem 2.3.18) on Theorem 3.2.3, and on Proposition 3.2.11, is that the Altmann-Hausen presentation of [AH06] extends *mutatis mutandis* to split torus action on normal affine varieties over an arbitrary field.

Proposition 3.2.13 (Split torus actions on normal affine varieties over arbitrary fields)**.** *Let* $\mathbb{k}$ *be a field, and let* $\mathbb{T} = \mathrm{Spec}(\mathbb{k}[M])$ *be a split* $\mathbb{k}$*-torus.*

(i) Let $(\omega_N, Y, \mathcal{D})$ *be an AH-datum over* $\mathbb{k}$. *The affine scheme* $X[Y, \mathcal{D}] := \mathrm{Spec}(A[Y, \mathcal{D}])$ *is a normal* $\mathbb{k}$*-variety endowed with an effective* $\mathbb{T}$*-action.*

(ii) Let X *be an affine normal* $\mathbb{k}$*-variety endowed with an effective* $\mathbb{T}$*-action. There exists an AH-datum* $(\omega_N, Y, \mathcal{D})$ *over* $\mathbb{k}$ *such that there is an isomorphism of* $\mathbb{T}$*-varieties* $X \cong X[Y, \mathcal{D}]$.

Proof. Let $\tau := \tau_0$ be the natural $\mathbb{k}$-structure on $\mathbb{T}_{\overline{\mathbb{k}}} = \mathrm{Spec}(\overline{\mathbb{k}}[M])$ (see Example 2.2.9). Recall that $\tilde{\tau}_\gamma = id_M$ for all $\gamma \in \Gamma := \mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. The idea of the proof is to show that, in our setting, a generalized AH-datum $(\omega_N, Y', \mathcal{D}', \sigma_Y, h = 1)$ over $\overline{\mathbb{k}}$ corresponds to an AH-datum $(\omega_N, Y, \mathcal{D})$ over $\mathbb{k}$.

(1) Let σ_Y be the natural $\mathbb{k}$-structure on $Y_{\overline{\mathbb{k}}}$. Then $(\omega_N, Y_{\overline{\mathbb{k}}}, \mathcal{D}_{\overline{\mathbb{k}}})$ is an AH-datum such that ${\sigma_Y}^*_\gamma(\mathcal{D}_{\overline{\mathbb{k}}}(m)) = \mathcal{D}_{\overline{\mathbb{k}}}(m)$ for all $\gamma \in \Gamma$ and for all $m \in \omega_N^\vee \cap M$. By Theorem 3.2.3, there exists a $\mathbb{k}$-structure $\sigma_{X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}},\mathcal{D}_{\overline{\mathbb{k}}}]}$ on $X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}}, \mathcal{D}_{\overline{\mathbb{k}}}]$ such that $(\mathbb{T}_{\overline{\mathbb{k}}}, \tau)$ acts on $(X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}}, \mathcal{D}_{\overline{\mathbb{k}}}], \sigma_{X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}},\mathcal{D}_{\overline{\mathbb{k}}}]})$. By construction of $\sigma_{X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}},\mathcal{D}_{\overline{\mathbb{k}}}]}$, observe that $X[Y, \mathcal{D}]$ corresponds to $(X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}}, \mathcal{D}_{\overline{\mathbb{k}}}], \sigma_{X_{\overline{\mathbb{k}}}[Y_{\overline{\mathbb{k}}},\mathcal{D}_{\overline{\mathbb{k}}}]})$ (see Remark 3.2.2).

(2) Let σ be the natural $\mathbb{k}$-structure on $X_{\overline{\mathbb{k}}}$. Since $(\mathbb{T}_{\overline{\mathbb{k}}}, \tau)$ acts on $(X_{\overline{\mathbb{k}}}, \sigma)$, by Theorem 3.2.3, there exists a generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ such that there is an isomorphism of $(\mathbb{T}_{\overline{\mathbb{k}}}, \tau)$-varieties $(X_{\overline{\mathbb{k}}}, \sigma) \cong (X_{\overline{\mathbb{k}}}[Y, \mathcal{D}], \sigma_{X_{\overline{\mathbb{k}}}[Y,\mathcal{D}]})$. Let $Y_0 := Y/\Gamma$ and let $\mathbb{L} := \overline{\mathbb{k}}(Y)$. By Remark 3.2.2 and by Theorem 2.3.18, $H^1_{cont}(\Gamma, G) \cong \{1\}$, where $G := \mathrm{Hom}_{gr}(M, \mathbb{L}^*)$. Then, by Corollary 3.2.10, there exists an ω_N-pp divisor $\mathcal{D}'$ on Y such that, ${\sigma_Y}^*_\gamma(\mathcal{D}'(m)) = \mathcal{D}'(m)$

for all $\gamma \in \Gamma$ and for all $m \in \omega_N^\vee \cap M$. Therefore we obtain a divisor $\mathcal{D}_0$ on the $\Bbbk$-variety Y_0 (see [Man86, Lemma 21.8.1]). Moreover, $X[Y_0, \mathcal{D}_0]$ corresponds to $(X_{\overline{\Bbbk}}[Y, \mathcal{D}], \sigma_{X_{\overline{\Bbbk}}[Y,\mathcal{D}]})$. □

Proposition 3.2.13 has a consequence on Theorem 3.2.3: we can replace the infinite Galois extension $\overline{\Bbbk}/\Bbbk$ by any finite Galois extension that splits the acting torus.

Corollary 3.2.14 (Torus actions on normal affine varieties over arbitrary fields)**.** *Let $\Bbbk'/\Bbbk$ be a finite Galois extension of Galois group Γ. Let $(\mathbb{T} = \operatorname{Spec}(\Bbbk'[M]), \tau)$ be a $\Bbbk$-torus that splits over $\Bbbk'$.*

(i) Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum over $\Bbbk'$. Then the affine $\mathbb{T}$-variety $X[Y, \mathcal{D}]$ admits a $\Bbbk$-structure $\sigma_{X[Y,\mathcal{D}]}$ such that $(\mathbb{T}, \tau)$ acts effectively on $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

(ii) Let (X, σ) be a normal affine $\Bbbk$-variety endowed with an effective $(\mathbb{T}, \tau)$-action. Then there exists a generalized AH-datum $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ such that there is an isomorphism of $(\mathbb{T}, \tau)$-varieties $(X, \sigma) \cong (X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$.

Proof. In the proof of Theorem 3.2.3, we use the Altmann-Hausen presentation over an algebraically closed field given in [AH06] combined with a toric downgrading (see Proposition 3.2.6). By Proposition 3.2.13, this presentation extends over any field. Hence, combining Proposition 3.2.13 together with Proposition 3.2.6, we obtain the desired result. □

By Corollary 3.2.14, we obtain an effective method to compute an AH-datum of a T-action on a $\Bbbk$-variety X. We consider a finite Galois extension $\Bbbk'/\Bbbk$ that splits T. Then, we determine an AH-datum for the $T_{\Bbbk'}$-action on $X_{\Bbbk'}$ using a $\operatorname{Gal}(\Bbbk'/\Bbbk)$-equivariant embedding $X_{\Bbbk'}$ in some $\mathbb{A}^n_{\Bbbk'}$ (Proposition 3.2.6) and then we deduce an AH-datum for the T action on a X.

Remark 3.2.15. Corollary 3.2.14 generalizes a result of Langlois (see [Lan15, Theorem 5.10] and [Lan21]) that focuses on $\Bbbk$-torus actions of complexity one, where $\Bbbk$ is an arbitrary field. Indeed, the geometrico-combinatorial presentation described in [Lan15] is over a splitting field $\Bbbk'$ of the $\Bbbk$-torus (so $\Bbbk'/\Bbbk$ is a finite Galois extension).

3.2.5 Functoriality

Theorem 3.2.3 establishes correspondences between affine $\Bbbk$-varieties endowed with a $\Bbbk$-torus action and triples $(Y, \mathcal{D}, h)$. In this section, we focus on this correspondence. In general, there is no one-to-one correspondence between $(\mathbb{T}, \tau)$-varieties and triples $(Y, \mathcal{D}, h)$; see [AH06, Section 8] for a precise statement. However, Altmann and Hausen define the notion of minimal pp-divisor in [AH06, Section 8] that leads us to the following result. Recall that divisors constructed as in the proof of Theorem 3.2.3 are minimal.

Theorem 3.2.16. *Let $(\mathbb{T} = \operatorname{Spec}(\Bbbk'[M]), \tau)$ be a $\Bbbk$-torus. Let $\omega_N \subset N_\mathbb{Q}$ (resp. $\omega'_N \subset N_\mathbb{Q}$) be a pointed cone, let (Y, σ_Y) (resp. (Y', σ'_Y)) be a normal semi projective $\Bbbk$-variety, $\mathcal{D}$ be a minimal proper ω_N-polyhedral divisor on Y (resp. $\mathcal{D}'$ be a minimal proper ω'_N-polyhedral divisor on Y') and $h : \Gamma \to \operatorname{Hom}(\omega_N^\vee \cap M, \Bbbk'(Y)^*)$ satisfying conditions (3.1) and (3.2) of Theorem 3.2.3. (resp. $h' : \Gamma \to \operatorname{Hom}\left({\omega'_N}^\vee \cap M, \Bbbk'(Y')^*\right)$). The affine $\Bbbk$-varieties $\left(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]}\right)$ and $\left(X[Y', \mathcal{D}'], \sigma_{X[Y',\mathcal{D}']}\right)$ are $(\mathbb{T}, \tau)$-isomorphic if and only if the following holds:*

(i) there exists a $\Bbbk$-isomorphism $\psi : (Y, \sigma_Y) \to (Y', \sigma'_Y)$;

(ii) there exists a lattice automorphism $L : N \to N$ such that $L(\omega_N) = \omega'_N$, and such that, for all $\gamma \in \Gamma$, $L \circ \hat{\tau}_\gamma = \hat{\tau}_\gamma \circ L$;

(iii) there exists a monoid morphism $g : \omega_N^\vee \cap M \to \Bbbk'(Y)$;

(iv) for all $m \in \omega'_M \cap M$, $\psi^*(\mathcal{D}'(m)) = \mathcal{D}(L^*(m)) + \operatorname{div}_Y(g(m))$;

(v) *for all* $m \in \omega'_M \cap M$ *and* $\forall \gamma \in \Gamma$, $\dfrac{\sigma_{Y}{}^{\sharp}_{\gamma}(g(\tilde{\tau}_{\gamma^{-1}}(m)))}{g(m)} = \dfrac{\psi^{\sharp}(h'_{\gamma}(m))}{h_{\gamma}(L^{*}(m))}$ *(i.e the cocycles defined by* $h_\gamma \circ L$ *and* $\psi^{\sharp} \circ h'_\gamma$ *are equivalent).*

Proof. By Theorem 3.1.21, the affine $\mathbb{k}$-varieties $X[Y, \mathcal{D}]$ and $X[Y', \mathcal{D}']$ are $\mathbb{T}$-isomorphic if and only if the following holds:
(i) there exists an isomorphism $\psi : Y \to Y'$;
(ii) there exists a lattice automorphism $L : N \to N$ such that $L(\omega_N) = \omega'_N$;
(iii) there exists a monoid morphism $g : M \to \mathbb{k}'(Y)$;
(iv) for all $m \in \omega'_M \cap M$, $\psi^{*}(\mathcal{D}'(m)) = \mathcal{D}(L^{*}(m)) + \mathrm{div}_Y(g(m))$;
The morphisms ψ and L induce a $\mathbb{T}$-equivariant isomorphism of graded algebras:

$$\Psi : A[Y', \mathcal{D}'] \to A[Y, D],\ \ f\mathfrak{X}_m \mapsto \psi^{\sharp}(f)g(m)\mathfrak{X}_{L^{*}(m)}.$$

Therefore the following diagram commutes for all $\gamma \in \Gamma$ if and only if the conditions of Theorem 3.2.16 are fulfilled.

$$\begin{array}{ccc} A[Y', \mathcal{D}'] & \xrightarrow{\ \Psi\ } & A[Y, \mathcal{D}] \\ \sigma_{Y}{}^{\sharp\,\prime}_{\gamma} \downarrow & & \downarrow \sigma_{Y}{}^{\sharp}_{\gamma} \\ A[Y', \mathcal{D}'] & \xrightarrow{\ \Psi\ } & A[Y, \mathcal{D}] \end{array}$$

□

Chapter 4

Altmann-Hausen presentation: some specific cases

In this chapter, which is based on the above with simplifying assumptions, we focus on the Altmann-Hausen presentation over $\mathbb{R}$, and then on affine varieties endowed with a two-dimensional torus action.

To extend the Altmann-Hausen presentation to the real setting, we use the language of $\mathbb{R}$-structures on algebraic $\mathbb{C}$-varieties. It is a particular case of Definition 1.2.1 where $\Bbbk' = \mathbb{C}$, $\Bbbk = \mathbb{R}$, and $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ (see Remark 1.2.3). Recall that an $\mathbb{R}$-structure on an algebraic $\mathbb{C}$-variety X is an involution of $\mathbb{R}$-schemes σ on X such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \bar{z})} & \mathrm{Spec}(\mathbb{C}). \end{array}$$

An $\mathbb{R}$-morphism between two $\mathbb{C}$-varieties X and X' endowed with $\mathbb{R}$-structures σ and σ' is a morphism of $\mathbb{C}$-varieties $f : X \to X'$ such that $\sigma' \circ f = f \circ \sigma$. An $\mathbb{R}$-group structure τ on a complex algebraic group G is an $\mathbb{R}$-structure on G such that the multiplication $G \times G \to G$, the inverse $G \to G$ and the unity $\mathrm{Spec}(\mathbb{C}) \to G$ are $\mathbb{R}$-morphisms. Let us note that an $\mathbb{R}$-group structure τ on a complex torus $\mathbb{T}$ corresponds to a lattice involution $\tilde{\tau}$ on its character lattice $\mathrm{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}}) \cong M$.

There is an equivalence of categories between the category of quasi-projective algebraic $\mathbb{R}$-varieties (resp. real algebraic groups) and the category of quasi-projective algebraic $\mathbb{C}$-varieties endowed with an $\mathbb{R}$-structure (resp. complex algebraic groups endowed with an $\mathbb{R}$-group structure); see Theorem 1.4.1 with $\Bbbk' = \mathbb{C}$ and $\Bbbk = \mathbb{R}$. Therefore we will often write (X, σ) to refer to an algebraic $\mathbb{R}$-variety and (G, τ) to refer to a real algebraic group.

The present chapter focuses on real torus actions on normal affine $\mathbb{R}$-varieties and on two-dimensional torus actions. The main result of Section 4.1 (see Corollary 4.1.6) gives a presentation of real torus actions in the language of [AH06] extended to affine $\mathbb{C}$-varieties with $\mathbb{R}$-structures.

Section 4.2 focuses on two-dimensional torus action over arbitrary fields. Tori of dimension n over an arbitrary field $\Bbbk$ are related to the conjugacy classes of finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$. Therefore, a one-dimensional $\Bbbk$-torus, up to isomorphism, is either a split $\Bbbk$-torus or a norm one $\Bbbk$-torus (see Definition 2.2.21). For instance, if $\Bbbk = \mathbb{R}$, the norm one $\mathbb{R}$-torus is $\mathbb{S}^1$. The classification of the conjugacy classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ is well known and there is an explicit description of two-dimensional $\Bbbk$-tori given by Voskresenski in [Vos65] (see

Proposition 4.2.4). Based on this classification, there is a complete description of the Galois cohomology set that classifies torsors appearing in Theorem 3.2.3 in [ELFST14, Theorems 5.3 & 5.5] (see Theorem 4.2.7). Furthermore, using birational geometry tools for surfaces, we give in Section 4.2.3 a method that can help to determine if a torsor is trivial or not. More exactly, we relate the torsor encoded by the cocycle h appearing in the Altmann-Hausen presentation to a certain Del Pezzo surface.

4.1 Altmann-Hausen presentation over $\mathbb{R}$ and examples

In this section, we focus on the case where $\Bbbk' = \mathbb{C}$ and $\Bbbk = \mathbb{R}$. The Galois group Γ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

4.1.1 Real tori and torsors

Results

The torus $\mathbb{G}_{m,\mathbb{C}}$ has two non-isomorphic $\mathbb{R}$-forms: the real split torus $\mathbb{G}_{m,\mathbb{R}}$ and the real circle $\mathbb{S}^1 := \operatorname{Spec}\left(\mathbb{R}[x,y]/(x^2+y^2-1)\right)$. Since $\operatorname{Aut}_{gr}(\mathbb{G}_{m,\mathbb{C}}) = \{id, -id\}$, the equivalence class of an $\mathbb{R}$-group structure on $\mathbb{G}_{m,\mathbb{C}}$ has only one element. So, the $\mathbb{R}$-group structures on $\mathbb{G}_{m,\mathbb{C}}$ associated to $\mathbb{G}_{m,\mathbb{R}}$ and $\mathbb{S}^1$ are respectively

$$\tau_0 : z \mapsto \overline{z} \quad \text{and} \quad \tau_1 : z \mapsto \overline{z}^{\,-1}.$$

The group structure on $\mathbb{S}^1$ is given by

$$(x,y)\cdot(x',y') = (xx'-yy', xy'+yx').$$

Recall that $\mathbb{S}^1$ is isomorphic to $\mathrm{SO}_2(\mathbb{R})$ (see [DL18, §1.2]). The *Weil restriction* of $\mathbb{G}_{m,\mathbb{C}}$ is

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) := \operatorname{Spec}\left(\mathbb{R}[x_1,y_1,x_2,y_2]/(x_1y_1-x_2y_2-1, x_2y_1+x_1y_2)\right).$$

It is an $\mathbb{R}$-form of $\mathbb{G}^2_{m,\mathbb{C}}$. An $\mathbb{R}$-group structure on $\mathbb{G}^2_{m,\mathbb{C}}$ associated to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is

$$\tau_2 : (z,w) \mapsto (\overline{w}, \overline{z}).$$

The group structure on $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is given by

$$(x_1,y_1,x_2,y_2)\cdot(x_1',y_1',x_2',y_2') = (x_1x_1'-x_2x_2', y_1y_1'-y_2y_2', x_1x_2'+x_2x_1', y_1y_2'+y_2y_1').$$

By abuse, we call Weil restriction any $\mathbb{R}$-torus isomorphic to $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Here, $\operatorname{Aut}_{gr}(\mathbb{G}^2_{m,\mathbb{C}}) \cong \mathrm{GL}_2(\mathbb{Z})$, so the equivalence class of an $\mathbb{R}$-group structure on $\mathbb{G}^2_{m,\mathbb{C}}$ has infinitely many elements. For instance

$$\tau_2' : (z,w) \mapsto (\overline{w}^{\,-1}, \overline{z}^{\,-1}) \quad \text{and} \quad \tau_2'' : (z,w) \mapsto (\overline{z}^{\,-1}\overline{w}, \overline{w})$$

are $\mathbb{R}$-group structures equivalent to τ_2.

Example 4.1.1. Consider the cone

$$\omega_N = \operatorname{Cone}\left\{\begin{bmatrix}1\\0\end{bmatrix};\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

in $N_{\mathbb{Q}} = \mathbb{Q}^2$. There are no $\mathbb{R}$-group structure τ equivalent to τ_2 such that $\hat{\tau}(\omega_N) = \omega_N$, so we cannot endow X_{ω_N} with an $\mathbb{R}$-structure compatible with a $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-action.

The following result, which describes $\mathbb{R}$-forms of $\mathbb{G}_{m,\mathbb{C}}^n$ in the category of groups, is proved in the next paragraph.

Proposition 4.1.2 (Classification of $\mathbb{R}$-group structures on $\mathbb{G}_{m,\mathbb{C}}^n$, [MJT21, Proposition 1.5])**.** *Let $n \in \mathbb{N}^*$. Then, an $\mathbb{R}$-group structure on $\mathbb{G}_{m,\mathbb{C}}^n$ is equivalent to exactly one $\mathbb{R}$-group structure of the form*

$$\tau_0^{\times n_0} \times \tau_1^{\times n_1} \times \tau_2^{\times n_2},$$

with $n_0 + n_1 + 2n_2 = n$. That is, any n-dimensional $\mathbb{R}$-torus is isomorphic to a torus of the form

$$\mathbb{G}_{m,\mathbb{R}}^{n_0} \times \left(\mathbb{S}^1\right)^{n_1} \times \mathrm{R}_{\mathbb{C}/\mathbb{R}}^{n_2}(\mathbb{G}_{m,\mathbb{C}}).$$

We focus know on $\mathbb{R}$-forms of $\mathbb{G}_{m,\mathbb{C}}^n$ in the category of varieties. If V is an $\mathbb{R}$-form of $\mathbb{G}_{m,\mathbb{C}}^n$, then by Corollary 2.3.16 there exists an n-dimensional $\mathbb{R}$-torus T such that V is a T-torsor. For instance, $\mathbb{S}^1$ has a non-trivial torsor, namely

$$\mathrm{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1)).$$

The corresponding $\mathbb{R}$-structure is

$$\sigma_1 : z \mapsto -\bar{z}^{-1}.$$

Proposition 4.1.3 (Classification of $\mathbb{R}$-structures on $\mathbb{G}_{m,\mathbb{C}}^n$, [MJT21, Remark 1.6])**.** *Let $n \in \mathbb{N}^*$. Then, an $\mathbb{R}$-structure on $\mathbb{G}_{m,\mathbb{C}}^n$ is equivalent to exactly one real structure of the form*

$$\tau_0^{\times n_0} \times \tau_1^{\times n_1} \times \sigma_1^{\times n_{-1}} \times \tau_2^{\times n_2},$$

with $n_0 + n_1 + n_{-1} + 2n_2 = n$

Proof. Let $(\mathbb{G}_{m,\mathbb{C}}^n, \sigma)$ be an $\mathbb{R}$-form of $\mathbb{G}_{m,\mathbb{C}}^n$ in the category of varieties. Then, by Corollary 2.3.16, there exits an n-dimensional $\mathbb{R}$-torus $(\mathbb{G}_{m,\mathbb{C}}^n, \tau)$ such that $(\mathbb{G}_{m,\mathbb{C}}^n, \sigma)$ is a $(\mathbb{G}_{m,\mathbb{C}}^n, \tau)$-torsor. By Proposition 4.1.2, we can assume that

$$\tau = \tau_0^{\times n_0} \times \tau_2^{\times n_2} \times \tau_1^{\times n_1'},$$

where $n_0 + 2n_2 + n_1' = n$. Note that $(\mathbb{G}_{m,\mathbb{C}}^{n_0+n_2}, \tau_0^{\times n_0} \times \tau_2^{\times n_2})$ is a quasi-trivial $\mathbb{R}$-torus. Hence, it has no non-trivial torsor (see Proposition 2.3.19). Then, by Proposition 2.3.21 and by Remark 2.3.14, $(\mathbb{G}_{m,\mathbb{C}}^{n_1'}, \tau_1^{\times n_1'})$-torsors are, up to isomorphism, $(\mathbb{G}_{m,\mathbb{C}}^{n_1'}, \tau_1^{\times n_1} \times \sigma_1^{\times n_{-1}})$, where $n_1 + n_{-1} = n_1'$ (see also Example 4.1.14). Therefore, we obtain the desired result. □

Proof of Proposition 4.1.2

The proof of Proposition 4.1.2 is based on the following Lemma.

Lemma 4.1.4. *The conjugacy classes of order 2 elements in $\mathrm{GL}_n(\mathbb{Z})$ are exactly represented by diagonal block matrices*

$$\mathrm{diag}\left(1, \ldots, 1, -1, \ldots, -1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

Proof. This proof is due to Lucy Moser-Jauslin. Let $\mathcal{C}$ be one of the conjugacy classes of $\mathrm{GL}_n(\mathbb{Q})$ of elements of order 2 and let $\mathcal{C}_0 := \mathrm{GL}_n(\mathbb{Z}) \cap \mathcal{C}$. We consider the action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathcal{C}_0$ and we want to study the $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes. Note that any element of $\mathcal{C}$ is diagonalizable by matrices in $\mathrm{GL}_n(\mathbb{Q})$, and the only eigenvalues are -1 and 1 since the polynomial $X^2 - 1 = (X-1)(X+1)$ canceled any matrices in $\mathcal{C}$. Moreover, if $M \in \mathcal{C}$, there exists integers r and s such that the characteristic polynomial of M is $\chi_M = (X-1)^r(X+1)^s$, with $r+s=n$, and if N is another matrix in $\mathcal{C}$, $\chi_N = \chi_M$ since M and N are conjugate. Hence $\mathcal{C} = \mathcal{C}(r)$ is the set of diagonalizable matrices with characteristic polynomial $(X-1)^r(X+1)^s$ with $r+s=n$. Let $d := \min(r,s)$. We will prove that $\mathcal{C}_0(r)$ has $d+1$ orbits under the action of $\mathrm{GL}_n(\mathbb{Z})$ and each one is represented by an element of the form

$$\begin{bmatrix} I_r & A_k \\ 0 & -I_s \end{bmatrix},$$

where A_k is a matrix of the form

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

and k is an integer satisfying $0 \leq k \leq d$. Then, note that by simply exchanging lines and columns, this matrix is similar over $\mathbb{Z}$ to a diagonal block matrix, where the blocks consist of k 2×2 blocks of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

and $r-k$ blocks (1), and $s-k$ blocks (-1). Note that one can also replace the 2×2 blocks by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

since they are conjugate in $\mathrm{GL}_2(\mathbb{Z})$.

We prove this result in several steps. First, we show that any matrix $M \in \mathcal{C}_0(r)$ can be triangulized over $\mathbb{Z}$ where the diagonal is of the form $\mathrm{diag}(1,\dots,1,-1,\dots,-1)$. We prove this by induction on r:
$\mathcal{P}(r') :=$ " *for all* $n' \geq r'$, *for all* $M \in \mathrm{GL}_{n'}(\mathbb{Z})$ *diagonalizable in* $\mathrm{GL}_{n'}(\mathbb{Q})$ *such that* $\chi_M = (X-1)^{r'}(X+1)^{s'}$ *and* $r'+s'=n'$, M *is triangulizable over* $\mathrm{GL}_{n'}(\mathbb{Z})$".
For $r'=0$, it's true because in this case $\mathcal{C}_0(0)$ has only one element : $-I_n$. Let $r'>0$ and assume $\mathcal{P}(r')$ is true. Let $n' \geq r'+1$ and $M \in \mathrm{GL}_{n'}(\mathbb{Z})$ diagonalizable in $\mathrm{GL}_{n'}(\mathbb{Q})$ such that $\chi_M = (X-1)^{r'+1}(X+1)^{s'}$ and $r'+1+s'=n'$. There exists an eigenvector $v \in \mathbb{Z}^{n'}$ such that $Mv=v$, and the coefficients of v in $\mathbb{Z}$ are globally relatively prime. We can complete $\{v\}$ to a basis of the lattice $\mathbb{Z}^{n'}$. Let P be the transition matrix from this new basis to the initial one ($P \in \mathrm{GL}_{n'}(\mathbb{Z})$), and let $M' = P^{-1}MP$. We have

$$M' = \begin{bmatrix} 1 & * \\ 0 & N \end{bmatrix},$$

with $N \in \mathrm{GL}_{n'-1}(\mathbb{Z})$, $n'-1 \geq r'$, $\chi_N = (X-1)^{r'}(X+1)^{s'}$ and $r'+s'=n'-1$. By induction, N is triangulizable over $\mathbb{Z}$, hence M is also triangulizable over $\mathbb{Z}$ and the property is true. Now, Let $M \in \mathcal{C}_0(r)$. Since M is triangulizable and satisfies $M^2 = I_n$, then M is conjugate in $\mathrm{GL}_n(\mathbb{Z})$ to

$$M_1 := \begin{bmatrix} I_r & B \\ 0 & -I_s \end{bmatrix},$$

for some matrix $B \in \mathrm{M}_{r,s}(\mathbb{Z})$. Using Smith's normal form theorem, there exist two matrices $N_r \in \mathrm{GL}_r(\mathbb{Z})$ and $N_s \in \mathrm{GL}_s(\mathbb{Z})$, an integer k satisfying $0 \leq k \leq d$, and integers $\{\delta_1,\dots,\delta_k\}$ satisfying $\delta_i | \delta_{i+1}$ for all $1 \leq i \leq k-1$ such that

$$N_r B N_s^{-1} = \begin{bmatrix} \delta_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \delta_k & \\ 0 & & & 0 \end{bmatrix}.$$

Note that by conjugating M_1 with
$$\begin{bmatrix} N_r & 0 \\ 0 & N_s \end{bmatrix},$$
the matrix M is conjugate in $\mathrm{GL}_n(\mathbb{Z})$ to
$$M_2 := \begin{bmatrix} I_r & N_r B N_s^{-1} \\ 0 & -I_s \end{bmatrix}.$$
Now, note that if δ_j is even, then all δ_i are even for $i \geq j$. By conjugating with appropriate matrices, we will cancel the δ_i for $i \geq j$ and transform the others into 1. To do this, note that for an arbitrary matrix $R \in \mathrm{M}_{r,s}(\mathbb{Z})$, after conjugating the matrix M_2 by
$$\begin{bmatrix} I_r & R \\ 0 & I_s \end{bmatrix},$$
we obtain
$$\begin{bmatrix} I_r & N_r B N_s^{-1} + 2R \\ 0 & I_s \end{bmatrix}.$$
Hence, the matrix M is conjugate in $\mathrm{GL}_n(\mathbb{Z})$ to
$$\begin{bmatrix} I_r & A_k \\ 0 & -I_s \end{bmatrix}.$$
Finally, we have to show that two of the $d+1$ matrices that we constructed are not conjugate in $\mathrm{GL}_n(\mathbb{Z})$. One way to do this is to consider the matrices in $\mathbb{Z}/2\mathbb{Z}$. If they were conjugate over $\mathbb{Z}$, they would be conjugate over $\mathbb{Z}/2\mathbb{Z}$. However, they have different Jordan normal form. □

Proof of Proposition 4.1.2. All $\mathbb{R}$-group structures given in the proposition are inequivalent because their real loci are
$$(\mathbb{R}^*)^{n_0} \times (\mathbb{S}^1)^{n_1} \times (\mathbb{R}^2 \backslash \{(0,0)\})^{n_2},$$
which are pairwise *non-diffeomorphic.*

Conversely, let τ be an $\mathbb{R}$-group structure, there exists $\varphi \in \mathrm{Aut}_{gr}(\mathbb{G}_{m,\mathbb{C}}^n)$ such that
$$\tau = \varphi \circ (\tau_0)^{\times n}$$
Note that all group automorphisms commute with $(\tau_0)^{\times n}$, so τ is an involution if and only if φ is an involution. Let φ_0 represent the conjugacy class of φ. Then $\varphi = \psi \circ \varphi_0 \circ \psi^{-1}$, with $\psi \in \mathrm{Aut}_{gr}(\mathbb{G}_{m,\mathbb{C}}^n)$. Hence
$$\psi^{-1} \circ \tau \circ \psi = \varphi_0 \circ (\tau_0)^{\times n}$$
Moreover, the real group structure $\tau' = \psi^{-1} \circ \tau \circ \psi$ is equivalent to τ. Finally, recall that conjugacy classes of order 2 elements in $\mathrm{GL}_n(\mathbb{Z})$ are exactly represented by block diagonal matrices of the form
$$\mathrm{diag}\left(1, \ldots, 1, -1, \ldots, -1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$
Each block corresponds respectively to τ_0, τ_1, and τ_2. □

4.1.2 Altmann-Hausen presentation over $\mathbb{R}$

We present the main theorical results of [Gil22a] concerning the presentation of affine $\mathbb{R}$-varieties endowed with real torus actions. It is a particular case of Theorem 3.2.3. This section is based on [Gil22a, §5].

The next definition is a translation of Definition 3.2.1 to the context of the $\mathbb{C}/\mathbb{R}$ descent.

Definition 4.1.5 (Generalized AH-datum)**.** Let $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M])$ be a $\mathbb{C}$-torus and let τ be an $\mathbb{R}$-group structure on $\mathbb{T}$. A *generalized AH-datum* $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ over $\mathbb{C}$ is an AH-datum $(\omega_N, Y, \mathcal{D})$ over $\mathbb{C}$ such that ω_N is stable under the Γ-action induced by $\hat{\tau}$, and together with an $\mathbb{R}$-structure σ_Y on Y and with a monoid morphism $h : \omega_N^\vee \cap M \to \mathbb{C}(Y)^*$ such that

$$\forall m \in \omega_M^\vee \cap M \qquad {\sigma_Y}^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \operatorname{div}_Y\left(h\left(\tilde{\tau}(m)\right)\right), \quad \text{and} \tag{4.1}$$

$$\forall m \in \omega_M^\vee \cap M, \qquad h(m){\sigma_Y}^\sharp\left(h\left(\tilde{\tau}^{-1}(m)\right)\right) = 1. \tag{4.2}$$

The next result is a direct corollary of Theorem 3.2.3. See also [Gil22a] for a self-contained proof in the context of the $\mathbb{C}/\mathbb{R}$-descent.

Corollary 4.1.6. *Let* $(\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M], \tau)$ *be a real torus.*

(i) Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ *be a generalized AH-datum over the field* $\mathbb{C}$*, then there exists an* $\mathbb{R}$*-structure* $\sigma_{X[Y,\mathcal{D}]}$ *on the normal affine variety* $X[Y, \mathcal{D}]$ *such that* $(\mathbb{T}, \tau)$ *acts effectively on* $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$*.*

(ii) Let (X, σ_X) *be a normal affine* $\mathbb{R}$*-variety endowed with an effective* $(\mathbb{T}, \tau)$*-action. There exists a generalized AH-datum* $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ *over* $\mathbb{C}$ *such that the affine varieties* (X, σ_X) *and* $(X[Y, \mathcal{D}], \sigma_{X[Y,\mathcal{D}]})$ *are* $(\mathbb{T}, \tau)$*-equivariantly isomorphic.*

In the case where the $\mathbb{R}$-torus T is quasi-split (i.e T has no $\mathbb{S}^1$-factors), our presentation simplifies. Indeed, if (X, σ) is endowed with a T-action and if (Y, σ_Y) is the variety mentioned in Corollary 4.1.6, we see in Proposition 3.2.11 that there exists a proper polyhedral divisor $\mathcal{D}$ on Y such that $\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m))$ for all $m \in \omega \cap M$; i.e we can *take* $h = 1$. On the other hand, this simplification is not always possible for $\mathbb{S}^1$-actions: see Section 4.1.5 for details and examples. In this case we recover the presentation for $\mathbb{S}^1$-actions given by Dubouloz and Liendo in [DL18].

Recall that we gave an explicit method to compute an AH-datum of an affine variety endowed with a torus action. This method is based on an equivariant embedding, see Proposition 3.2.6. In the case where $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is a subtorus of a real torus $(\mathbb{G}_{m,\mathbb{C}}^n, \tau')$, we have the following result.

Lemma 4.1.7. *Let* $(\mathbb{G}_{m,\mathbb{C}}^{2q} = \operatorname{Spec}(\mathbb{C}[M]), \tau_2^{\times q})$ *be a subtorus of* $(\mathbb{G}_{m,\mathbb{C}}^n = \operatorname{Spec}(\mathbb{C}[M']), \tau')$*. Then, there exists a* Γ*-equivariant section* $s^* : M \to M'$ *(i.e.* $F^* \circ s^* = id$ *and* $\tau' \circ s^* = s^* \circ \tau_2^{\times q}$*).*

Proof. The Galois group $\Gamma = \{id, \gamma\}$ acts on M via $\tilde{\tau}_2^{\times q}$, so M is a Γ-module. We have the following short exact sequences of $\mathbb{Z}[\Gamma]$-modules:

$$0 \longrightarrow M_Y \xrightarrow{P^*} M' \xrightarrow{F^*} M \longrightarrow 0.$$

Note that we have an isomorphism of $\mathbb{Z}[\Gamma]$-module:

$$\begin{aligned} M &\to \mathbb{Z}[\Gamma]^q \\ (k_1, l_1, \ldots, k_q, l_q) &\mapsto (k_1\chi^{id} + l_1\chi^\gamma, \ldots, k_q\chi^{id} + l_q\chi^\gamma). \end{aligned}$$

Hence M is free $\mathbb{Z}[\Gamma]$-module of rank q, so it is a projective $\mathbb{Z}[\Gamma]$-module. By [Eis95, Proposition A3.1], there exists a morphism $s^* : M \to M'$ of $\mathbb{Z}[\Gamma]$-module such that $F^* \circ s^* = id_M$. □

Remark 4.1.8.

(i) The interpretation of Lemma 4.1.7 is : $(\mathbb{G}_{m,\mathbb{C}}^n, \tau') \cong \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})^q \times T'$, where T' is a real torus of dimension $n-2q$.

(ii) For $\mathbb{G}_{m,\mathbb{R}}$-actions, the $\mathbb{Z}[\Gamma]$-module $M=\mathbb{Z}$, with Γ-action given by $\tilde{\tau}_0 = id$, is not a projective $\mathbb{Z}[\Gamma]$-module. Indeed, we have a Γ-equivariant isomorphism

$$M \to \mathbb{Z}[\Gamma]/(\chi^\gamma), \quad m \mapsto [m\chi^{id} + m\chi^\gamma].$$

(iii) For $\mathbb{S}^1$-actions, the $\mathbb{Z}[\Gamma]$-module $M=\mathbb{Z}$, with Γ-action given by $\tilde{\tau}_1 = -id$, is not a projective $\mathbb{Z}[\Gamma]$-module. Indeed, we have a Γ-equivariant isomorphism

$$M \to \mathbb{Z}[\Gamma]/(\chi^\gamma), \quad m \mapsto [m\chi^{id} + (-m)\chi^\gamma].$$

Example 4.1.9 (See Lemma 2.2.25)**.** The real tori $\mathbb{G}_{m,\mathbb{R}}$ and $\mathbb{S}^1$ are real subtori of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. The inclusion are given by:

$$(\mathbb{G}_{m,\mathbb{C}}, \tau_0) \to (\mathbb{G}_{m,\mathbb{C}}^2, \tau_2), \ t \mapsto (t,t) \ \text{ and } \ (\mathbb{G}_{m,\mathbb{C}}, \tau_1) \to (\mathbb{G}_{m,\mathbb{C}}^2, \tau_2), \ t \mapsto (t, t^{-1}).$$

We obtain the diagrams of Remark 2.2.11 with $M' = \mathbb{Z}^2$, $M=\mathbb{Z}$, $M_Y = \mathbb{Z}$, $F^* = [1,1]$ and $P^* = [1,-1]$ for $\mathbb{G}_{m,\mathbb{R}}$, and $F^* = [1,-1]$ and $P^* = [1,1]$ for $\mathbb{S}^1$. In these two cases, there does not exist a Γ-equivariant section since τ_2 is not equivalent to $\tau_1 \times \tau_0$.

Example 4.1.10. Consider a $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})^q$-variety Z, and an equivariant embedding as in Proposition 3.2.6. Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum describing the $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})^q$-variety Z. By Lemma 4.1.7, there exists a Γ-equivariant section, then, by construction of h, we can take $h=1$.

4.1.3 Split $\mathbb{R}$-torus actions on normal affine $\mathbb{R}$-varieties

We give examples of actions of the split $\mathbb{R}$-torus $\mathbb{G}_{m,\mathbb{R}}^n$ on affine $\mathbb{R}$-varieties (see Proposition 3.2.13). By definition, a $\mathbb{G}_{m,\mathbb{R}}^n$-action on an $\mathbb{R}$-variety (X, σ_X) is an action of the $\mathbb{R}$-torus $(\mathbb{G}_{m,\mathbb{C}}^n, \tau_0^{\times n})$ on (X, σ_X).

Example 4.1.11 (See Example E.2.1 for more details on the construction of an AH-datum)**.** In the case of a $\mathbb{G}_{m,\mathbb{R}}$-action, the sequence obtained from the inclusion $(X,\sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$ of Proposition 3.2.6 does not always have a Γ-equivariant section. Indeed, consider the affine variety $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$, where $\sigma(x,y) = (\bar{y}, \bar{x})$. This variety is $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}_{\mathbb{C}}) \cong \mathbb{A}_{\mathbb{R}}^2$ (see Example A.7.4; one can also show that σ is equivalent to $\sigma_0 : (u,v) \mapsto (\bar{u}, \bar{v})$, via the change of variables $\varphi : (u,v) \mapsto (u+iv, u-iv)$).
Note that the torus $(\mathbb{G}_{m,\mathbb{C}}, \tau_0)$ acts on $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$ by $t \cdot (x,y) = (tx, ty)$. Then, we have the following split short exact sequence (see Example 4.1.9)

$$1 \longrightarrow \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{C}}^2 \longrightarrow \mathbb{G}_{m,\mathbb{C}} \longrightarrow 1,$$

with $\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}^2, t \mapsto (t,t)$ and $\mathbb{G}_{m,\mathbb{C}}^2 \to \mathbb{G}_{m,\mathbb{C}}, (s,t) \mapsto s/t$. We obtain the diagrams of Remark 2.2.11 with $M' = \mathbb{Z}^2$, $M = \mathbb{Z}$, $M_Y = \mathbb{Z}$, and with the following lattice homomorphisms:

$$F^* := \begin{bmatrix} 1 & 1 \end{bmatrix}; \qquad P^* := \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \qquad \tilde{\tau}' := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad \tilde{\tau}_Y := \begin{bmatrix} -1 \end{bmatrix}; \qquad \tilde{\tau} = \begin{bmatrix} 1 \end{bmatrix}.$$

We can show that there is no Γ-equivariant section $s^* : M \to \mathbb{Z}^2$. Indeed, note that if such a section exists, we obtain $\mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1 \cong \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, which is false (see Proposition 4.1.2).
Let

$$s^* := \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

it is a section. Then, an Altmann-Hausen presentation of the $\mathbb{G}_{m,\mathbb{R}}$-action on $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$ is (we refer to the proof of Theorem 3.2.3 for details, particularly for the relation between s and h)

- $\omega_N := \mathbb{Q}_{\geq 0}$;
- $Y := \mathbb{P}^1 = \mathbb{A}^1_{\mathbb{C}} \cup \{\infty\}$;
- $\sigma_Y : v \mapsto \overline{v}^{\,-1}$;
- $\mathcal{D} := [1, +\infty[\otimes\{\infty\}$;
- $h := w = x/y$.

Now we give a $(\Gamma \times \mathbb{G}_{m,\mathbb{C}})$-equivariant inclusion $(X, \sigma) \hookrightarrow (\mathbb{A}^n_{\mathbb{C}}, \sigma')$ that admits a Γ-equivariant section. First, note that $\mathbb{A}^2_{\mathbb{C}} \cong \mathrm{Spec}(\mathbb{C}[x, y, z]/(x+y-z)) \subset \mathbb{A}^3_{\mathbb{C}}$, where the closed embedding is given by

$$\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}},\ (x, y) \mapsto (x, y, x+y).$$

Consider the action of $\mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^3_{\mathbb{C}}$ defined by $t\cdot(x, y, z) = (tx, ty, tz)$. This action is obtained from the inclusion

$$\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}^3_{m,\mathbb{C}},\ t \mapsto (t, t, t).$$

Consider the $\mathbb{R}$-group structure on $\mathbb{G}^3_{m,\mathbb{C}}$ defined by $\sigma'(t_1, t_2, t_3) = (\overline{t_2}, \overline{t_1}, \overline{t_3})$. The closed immersion $\mathbb{A}^2_{\mathbb{C}} \cong \mathrm{Spec}(\mathbb{C}[x, y, z]/(x+y-z)) \subset \mathbb{A}^3_{\mathbb{C}}$ is $(\Gamma \times \mathbb{G}_{m,\mathbb{C}})$-equivariant. We obtain the diagrams of Remark 2.2.11 with $M' = \mathbb{Z}^3$, $M = \mathbb{Z}$, $M_Y = \mathbb{Z}^2$, and with the following lattice homomorphisms:

$$F^* := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}; \qquad P^* := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}; \qquad s^* := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\tilde{\tau}' := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \tilde{\tau}_Y := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The section s^* is Γ-equivariant. An Altmann-Hausen presentation of the $\mathbb{G}_{m,\mathbb{R}}$-action on $(\mathbb{A}^3_{\mathbb{C}}, \sigma')$ is given by:

- $\omega_N := \mathbb{Q}_{\geq 0}$;
- $Y := \mathbb{P}^2 = U_1 \cup U_2 \cup U_3$, where

$$U_1 = \mathrm{Spec}(\mathbb{C}[v_1, w_1]),\ U_2 = \mathrm{Spec}(\mathbb{C}[v_2, w_2]),\ \text{and } U_3 = \mathrm{Spec}(\mathbb{C}[v_3, w_3]),$$

  with gluing morphism obtained from $(v_1 = x/z, w_1 = y/z)$, $(v_2 = y/x, w_2 = z/x)$ and $(v_3 = z/y, w_3 = x/y)$;
- σ_Y is the $\mathbb{R}$-structure exchanging x and y and fixing z;
- $\mathcal{D} := [1, +\infty[\otimes D$, where $D\big|_{U_1} = 0$, $D\big|_{U_2} = \{w_2 = 0\}$ and $D\big|_{U_3} = \{v_3 = 0\}$; and
- $h := 1$.

We deduce from this an Altmann-Hausen presentation of the $\mathbb{G}_{m,\mathbb{R}}$-action on $(\mathrm{Spec}(\mathbb{C}[x, y, z]/(x+y-z)), \sigma')$ (we refer to the proof of Theorem 3.2.3 for details):

- $\omega_{NX} = \omega_N = \mathbb{Q}_{\geq 0}$;
- $Y_X := U_{1X} \cup U_{2X} \cup U_{3X} \cong \mathbb{P}^1$, where

$$U_{1X} = \mathrm{Spec}(\mathbb{C}[v_1, w_1]/(v_1 + w_1 - 1)),$$

$$U_{2X} = \mathrm{Spec}(\mathbb{C}[v_2, w_2])/(v_2 - w_2 + 1),$$

$$U_{3X} = \mathrm{Spec}(\mathbb{C}[v_3, w_3]/(v_3 - w_3 - 1));$$

- $\sigma_{Y_X} = \sigma_Y\big|_X$;
- $\mathcal{D}_X := [1, +\infty[\otimes D$, where $D\big|_{U_1} = 0$, $D\big|_{U_2} = \{(1, 0)\}$ and $D\big|_{U_3} = \{(0, 1)\}$; and
- $h_X := 1$.

4.1.4 Weil restriction actions on normal affine $\mathbb{R}$-varieties

By definition, an $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-action on an $\mathbb{R}$-algebraic variety (X, σ_X) is an action of the $\mathbb{R}$-torus $(\mathbb{G}_{m,\mathbb{C}}, \tau_2)$ on (X, σ_X). We give examples of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-actions on an $\mathbb{R}$-algebraic variety (see Proposition 3.2.11).

Example 4.1.12. The sequences of Example 3.2.7 admit a Γ-equivariant section $s : N' \to N$ defined by

$$s := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

An Altmann-Hausen presentation of the $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$-action on $(\mathbb{A}^3_{\mathbb{C}}, \sigma')$ is given by

$$(Y, \sigma_Y, \mathcal{D}, h),$$

where (see Example 3.1.14):

- $\omega_N := \mathbb{Q}^2_{\geq 0}$;
- $Y := \mathbb{P}^1 = \mathbb{A}^1_{\mathbb{C}} \cup \{\infty\}$;
- σ_Y is the complex conjugation on the coordinates;
- $\mathcal{D} := \Delta \otimes \{\infty\}$; and
- $h := 1$.

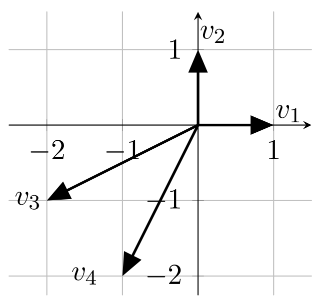
Example 4.1.13. Consider the hypersurface X of $\mathbb{A}^4_{\mathbb{C}} := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4])$ defined by $x_1x_3 = x_2x_4$. The torus $\mathbb{G}^2_{m,\mathbb{C}}$ acts on $\mathbb{A}^4_{\mathbb{C}}$ by $(s,t)\cdot(x_1,x_2,x_3,x_4) := (sx_1, tx_2, st^2x_3, s^2tx_4)$. Since the polynomial $x_1x_3 - x_2x_4$ is homogeneous, $\mathbb{G}^2_{m,\mathbb{C}}$ acts on X. Let σ' be the $\mathbb{R}$-structure on $\mathbb{A}^4_{\mathbb{C}}$ defined by $\sigma'(x_1,x_2,x_3,x_4) = (\overline{x_2}, \overline{x_1}, \overline{x_4}, \overline{x_3})$ and let σ be the induced $\mathbb{R}$-structure on X. Then, the real torus $(\mathbb{G}^2_{m,\mathbb{C}}, \tau_2)$ acts on $(\mathbb{A}^4_{\mathbb{C}}, \sigma')$ and on (X, σ).
Consider the immersion $\mathbb{T} := \mathbb{G}^2_{m,\mathbb{C}} \hookrightarrow \mathbb{G}^4_{m,\mathbb{C}}$, $(s,t) \mapsto (s, t, st^2, s^2t)$. We denote $\mathbb{T}_Y := \mathbb{G}^4_{m,\mathbb{C}}/\mathbb{T}$. We denote $\mathbb{T} = \mathrm{Spec}(\mathbb{C}[M])$, $\mathbb{G}^4_{m,\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[M'])$, and $\mathbb{T}_Y = \mathrm{Spec}(\mathbb{C}[M_Y])$. Then, we have the split short exact sequences of Remark 2.2.11 with

$$F^* := \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}; \quad P^* := \begin{bmatrix} -1 & -2 \\ -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \tilde{\tau}' := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \tilde{\tau}_Y = \tilde{\tau} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that the section $s : N' \to N$, defined by

$$s := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

is Γ-equivariant. Let Y be the toric variety defined by the following fan obtained from P.



Since this fan is stable under the lattice involution $\hat{\tau}_Y$, the $\mathbb{R}$-group structure τ_Y extends to an $\mathbb{R}$-structure σ_Y on Y (see Proposition 2.4.9). Let $\mathcal{D}$ be the divisor defined in Example 3.1.8, where $D_1, \ldots, D_4$ are the toric divisors obtained from the rays $v_1, \ldots, v_4$ respectively.

An Altmann-Hausen presentation of the $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$-action on $(\mathbb{A}_{\mathbb{C}}^4, \sigma')$ is given by

$$(\omega_N := \mathbb{Q}_{\geq 0}^2, Y, \mathcal{D}, \sigma_Y, h = 1).$$

An Altmann-Hausen presentation of the $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$-action on (X, σ) is given by

$$(\omega_N, Y_X, \mathcal{D}_X, \sigma_{Y_X}, h_X),$$

where (see Example 3.1.8)

- Y_X is the closure of the image of $\mathbb{G}_{m,\mathbb{C}}^4 \cap X$ in Y;
- $\mathcal{D}_X := \Delta_3 \otimes D_3 \cap Y_X + \Delta_4 \otimes D_4 \cap Y_X$;
- $\sigma_{Y_X} = \sigma_Y\big|_{Y_X}$; and
- $h_X := 1$.

4.1.5 Circle actions on normal affine $\mathbb{R}$-varieties

By definition, an $\mathbb{S}^1$-action on an algebraic $\mathbb{R}$-variety (X, σ_X) is an action of the torus $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$ on (X, σ_X). Note that $\mathbb{G}_{m,\mathbb{C}}$ acts on X and the algebra $\mathbb{C}[X]$ is graded by $M \cong \mathbb{Z}$. By [DL18, Lemma 1.7] (see also [FZ03]), $\mathbb{C}[X]_m \neq 0$ for all $m \in M$, so the weight cone of this action is $\omega_M := M_{\mathbb{Q}} = \mathbb{Q}$.

In this case, the generalized AH-datum $(\omega_N = \omega_M^\vee, Y, \mathcal{D}, \sigma_Y, h)$ mentioned in Corollary 4.1.6 consists of a proper polyhedral divisor $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ and a Γ-invariant rational function h on Y such that $\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(-m) + \operatorname{div}_Y(h^{-m})$ for all $m \in M$ (we recover [DL18, Theorem 2.7]).

In the case of $\mathbb{S}^1$-actions, we do not necessarily have $\mathrm{H}^1(\Gamma, G) = \{1\}$ (see Proposition 2.3.21). Indeed, in contrast to the split case, we cannot apply Hilbert's theorem 90 because the action defined on $\mathbb{C}(Y)^*$ does not extend to an action on the field $\mathbb{C}(Y)$.

Example 4.1.14. (See [DL18, Proposition 3.1]). There are only two $\mathbb{R}$-forms of $\mathbb{G}_{m,\mathbb{C}}$ compatible with an $\mathbb{S}^1$-action: the real circle $X_1 = \mathbb{S}^1$ and $X_{-1} = \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1))$. An $\mathbb{R}$-structure associated to X_1 is $\sigma_1 := \tau_1 : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}, z \mapsto \overline{z}^{\,-1}$ and an $\mathbb{R}$-structure associated to X_{-1} is $\sigma_{-1} : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}, z \mapsto -\overline{z}^{\,-1}$. Consider the action by translation $\mu : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}, (t,x) \mapsto tx$. The varieties X_1 and X_{-1} are endowed with an $\mathbb{S}^1$-action since the following diagram commutes for $i \in \{-1, 1\}$.

$$\begin{array}{ccc}
\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{\mu} & \mathbb{G}_{m,\mathbb{C}} \\
{\scriptstyle \tau_1 \times \sigma_i}\downarrow & & \downarrow{\scriptstyle \sigma_i} \\
\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \xrightarrow[\mu]{} & \mathbb{G}_{m,\mathbb{C}}
\end{array}$$

The $\mathbb{R}$-variety $(Y = \operatorname{Spec}(\mathbb{C}), \sigma_Y)$, where σ_Y is the complex conjugation, is a 'real Altmann-Hausen quotient' of both X_1 and X_{-1}.

A generalized AH-datum $(\omega_N = 0, Y, \mathcal{D}_1, \sigma_Y, h_1)$ for $\mathbb{S}^1$ acting on X_1 consists of the trivial divisor and the real number $h_1 = 1 \in \mathbb{C}(Y)^* = \mathbb{C}^*$. Note however that $\mathrm{H}^1(\Gamma, \mathbb{C}^*) \neq \{1\}$.

A generalized AH-datum $(\omega_N = 0, Y, \mathcal{D}_{-1}, \sigma_Y,_{-1})$ for $\mathbb{S}^1$ acting on X_{-1} consists of the trivial divisor and the real number $h_{-1} = -1 \in \mathbb{C}(Y)^* = \mathbb{C}^*$. Note that we cannot find a complex number $g \in \mathbb{C}(Y)^*$ such that the cocycle h_{-1} satisfies $h_{-1} = g\bar{g}$, so we cannot find a presentation where the Γ-invariant rational function h mentioned in Corollary 4.1.6 is equal to 1.

Example 4.1.15 (See [DL18, Proposition 3.4])**.** From Example E.2.3, consider the $\mathbb{S}^1$-action on $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}_\mathbb{C}) \cong \mathbb{A}^2_\mathbb{R}$ defined by

$$\begin{aligned} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{A}^2_\mathbb{C} &\to \mathbb{A}^2_\mathbb{C} \\ (t,(x,y)) &\mapsto (tx, t^{-1}y), \end{aligned}$$

where $\mathbb{G}_{m,\mathbb{C}}$ is endowed with the $\mathbb{R}$-group structure τ_1, and $\mathbb{A}^2_\mathbb{C}$ is endowed with the $\mathbb{R}$-structure $\sigma : (x,y) \mapsto (\bar{y}, \bar{x})$. The generalized AH-datum is $(\omega_N, Y, \sigma_Y, \mathcal{D}, h)$, where $(\omega_N = \{0\}, Y = \mathbb{A}^1_\mathbb{C} = \mathrm{Spec}(\mathbb{C}[z]), \mathcal{D} = [0,1] \otimes \{0\})$ is the AH-datum describing the $\mathbb{G}_{m,\mathbb{C}}$-action on $\mathbb{A}^2_\mathbb{C}$, where σ_Y is the $\mathbb{R}$-structure on Y defined by $z \mapsto \bar{z}$, and where $h : m \mapsto z^{-m}$. Furthermore, one can show that the cocycle h is not equivalent two the trivial cocycle.
Using the change of variables $\varphi : (u,v) \mapsto (u+iv, u-iv)$, the action on $\mathbb{S}^1$ on $\mathbb{A}^2_\mathbb{R}$ is

$$\begin{aligned} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{A}^2_\mathbb{C} &\to \mathbb{A}^2_\mathbb{C} \\ (t,(u,v)) &\mapsto \left(\frac{(t+t^{-1})u + i(t-t^{-1})v}{2}, \frac{(t+t^{-1})v - i(t-t^{-1})u}{2} \right), \end{aligned}$$

where $\mathbb{G}_{m,\mathbb{C}}$ is endowed with the $\mathbb{R}$-group structure τ_1, and $\mathbb{A}^2_\mathbb{C}$ is endowed with the $\mathbb{R}$-structure $\sigma_0 : (x,y) \mapsto (\bar{x}, \bar{y})$. Therefore, using the Γ-equivariant isomorphisms $\mathbb{C}[t^{\pm 1}] \cong \mathbb{C}[u_0, v_0]/(u_0v_0 - 1) \cong \mathbb{C}[x_0, y_0]/(x_0^2 + y_0^2 - 1)$, the action of $\mathbb{S}^1 = \mathrm{Spec}(\mathbb{R}[x_0,y_0]/(x_0^2+y_0^2-1))$ on $\mathbb{A}^2_\mathbb{R}$ is

$$\begin{aligned} \mathbb{S}^1 \times \mathbb{A}^2_\mathbb{R} &\to \mathbb{A}^2_\mathbb{R} \\ ((x_0,y_0),(u,v)) &\mapsto (x_0u - y_0v, x_0v + y_0u)\,. \end{aligned}$$

Note that the action of $\mathbb{S}^1$ on $\mathbb{A}^2_\mathbb{R} \cap \mathbb{S}^1$ is the action of $\mathbb{S}^1$ on itself by multiplication (see Section 4.1.1), and that the $\mathbb{S}^1$-action on $\mathbb{A}^2_\mathbb{R}$ is the canonical action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathbb{A}^2_\mathbb{R}$.

4.2 Two-dimensional torus actions

We focus on two-dimensional torus actions on normal affine varieties. In Sections 4.2.1 and 4.2.2, we have compiled results from the literature. We describe the set of two-dimensional tori in Proposition 4.2.4 and we give a description of the Galois-cohomology set classifying T-torsors in Theorem 4.2.7: it is exactly the result we need to determine if the Altmann-Hausen presentation simplifies in the case of two-dimensional torus actions on normal affine varieties. In addition, using birational geometry tools, we give in Section 4.2.3 a method that can help to determine whether a torsor is trivial or not (see Proposition 4.2.12).

4.2.1 Two-dimensional tori and finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$

Let $\mathbb{k}'/\mathbb{k}$ be a non-necessarily finite Galois extension and denote by Γ its Galois group. Let $(\mathbb{G}^n_{m,\mathbb{k}'}, \tau)$ be a $\mathbb{k}$-torus. Recall that τ induces a Γ-action $\tilde{\tau}$ on the character lattice $M \cong \mathbb{Z}^n$ satisfying $\tilde{\tau}_{\gamma_1\gamma_2} = \tilde{\tau}_{\gamma_1} \circ \tilde{\tau}_{\gamma_2}$, and a dual Γ-action $\hat{\tau}$ on the cocharacter lattice $N \cong \mathbb{Z}^n$ satisfying $\hat{\tau}_{\gamma_1\gamma_2} = \hat{\tau}_{\gamma_2} \circ \hat{\tau}_{\gamma_1}$ (see Remark 2.2.7). Two Γ-representations ρ and ρ' on $\mathrm{GL}_n(\mathbb{Z})$ are equivalent if there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $\rho'(\gamma) = P \circ \rho(\gamma) \circ P^{-1}$ for all $\gamma \in \Gamma$.

Let $n \in \mathbb{N}^*$. Recall that there is a one-to-one correspondence between $\mathbb{k}$-group structures on $\mathbb{G}^n_{m,\mathbb{k}'}$ and Γ-representations in $\mathrm{GL}_n(\mathbb{Z})$ (see Lemma 2.2.8). Furthermore, equivalent classes of $\mathbb{k}$-group structures on $\mathbb{G}^n_{m,\mathbb{k}'}$ correspond to equivalent classes of Γ-representations in $\mathrm{GL}_n(\mathbb{Z})$.

The image of a Γ-representation is a finite subgroup $\mathcal{G}$ of $\mathrm{GL}_n(\mathbb{Z})$ (see Remark 2.2.10). Moreover, if τ' is a $\mathbb{k}$-group structure on $\mathbb{G}^n_{m,\mathbb{k}'}$ equivalent to τ, then the associated finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ are conjugate. The converse is false (see Example 4.2.2), except in dimension 1. Therefore, since $\mathrm{GL}_1(\mathbb{Z}) = \{1, -1\}$, we get the next result.

Proposition 4.2.1. *A one-dimensional* $\Bbbk$*-torus is, up to isomorphism, either a split* $\Bbbk$*-torus or a norm one* $\Bbbk$*-torus.*

Example 4.2.2 (Inequivalent Γ-representation on $\mathrm{GL}_2(\mathbb{Z})$ having same image $\mathcal{G}_6$ in $\mathrm{GL}_2(\mathbb{Z})$)**.** Let $\Bbbk'/\Bbbk$ be a finite Galois extension of degree 4 such that

$$\mathrm{Gal}(\Bbbk'/\Bbbk) \cong \mathscr{C}_2 \times \mathscr{C}_2 = \{(1;1), (1;-1), (-1;1), (-1;-1)\}.$$

Let

$$s := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider $\rho, \rho', , \rho''$ be three representations of Γ in $\mathrm{GL}_2(\mathbb{Z})$ defined by

$$\begin{aligned} \rho &: \Gamma \to \mathrm{GL}_2(\mathbb{Z});\ (1;1) \mapsto I;\ (1;-1) \mapsto s;\ (-1;1) \mapsto -s;\ (-1;-1) \mapsto -I; \\ \rho' &: \Gamma \to \mathrm{GL}_2(\mathbb{Z});\ (1;1) \mapsto I;\ (1;-1) \mapsto s;\ (-1;1) \mapsto -I;\ (-1;-1) \mapsto -s; \\ \rho'' &: \Gamma \to \mathrm{GL}_2(\mathbb{Z});\ (1;1) \mapsto I;\ (1;-1) \mapsto -I;\ (-1;1) \mapsto -s;\ (-1;-1) \mapsto s. \end{aligned}$$

One can show that these representations are inequivalent.

We denote by $\mathscr{D}_n$ the dihedral group of order n, by $\mathscr{C}_n$ the cyclic group of order n, and by $\mathscr{S}_n$ the group of permutations of a set with n elements.

Proposition 4.2.3 ([Lor05, §1.10.1], see Appendix F)**.** *Let*

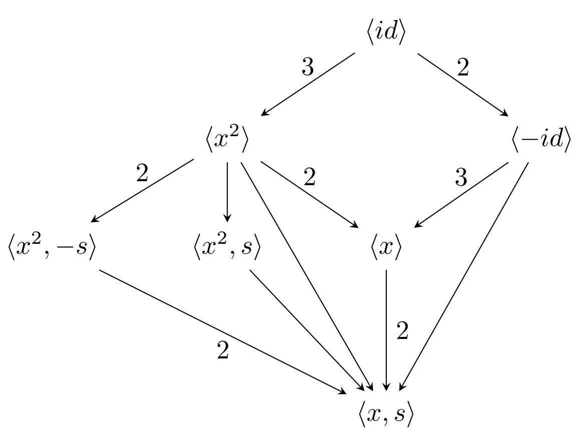
$$d := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix};\ s := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\ \textit{and } x := \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

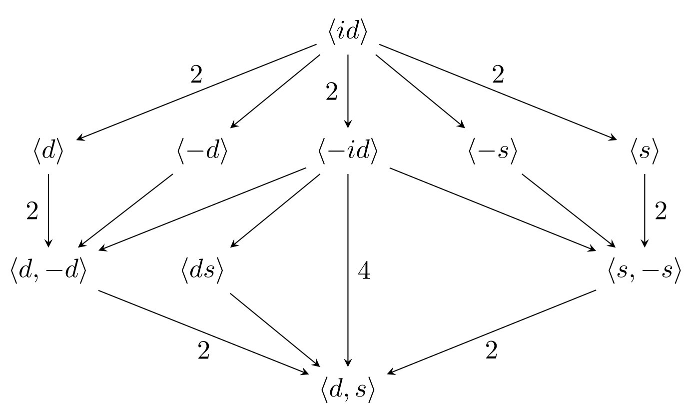
Then, the non-trivial conjugacy classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ *are:*

Label	*Order*	*Generators*	*Isomorphism type*
$\mathcal{G}_1$	*12*	x, s	$\mathscr{D}_{12} \cong \mathscr{S}_3 \times \mathscr{C}_2$
$\mathcal{G}_2$	*8*	d, s	$\mathscr{D}_8$
$\mathcal{G}_3$	*6*	x^2, s	$\mathscr{D}_6 \cong \mathscr{S}_3$
$\mathcal{G}_4$	*6*	$x^2, -s$	$\mathscr{D}_6 \cong \mathscr{S}_3$
$\mathcal{G}_5$	*4*	$d, -d$	$\mathscr{C}_2 \times \mathscr{C}_2$
$\mathcal{G}_6$	*4*	$s, -s$	$\mathscr{C}_2 \times \mathscr{C}_2$

Label	*Order*	*Generators*	*Isomorphism type*
$\mathcal{G}_7$	*6*	x	$\mathscr{C}_6$
$\mathcal{G}_8$	*4*	ds	$\mathscr{C}_4$
$\mathcal{G}_9$	*3*	x^2	$\mathscr{C}_3$
$\mathcal{G}_{10}$	*2*	$x^3 = -id$	$\mathscr{C}_2$
$\mathcal{G}_{11}$	*2*	d	$\mathscr{C}_2$
$\mathcal{G}_{12}$	*2*	s	$\mathscr{C}_2$

The following diagrams represent the inclusions of normal subgroups (the number is the index):





Based on Proposition 4.2.3, a classification of two-dimensional $\mathbb{k}$-tori is available in [Vos65]. Let $n \in \mathbb{N}^*$. If $\mathbb{k}'/\mathbb{k}$ is a Galois extension such that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathscr{D}_{2n}$, we denote

$$\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle \varrho, \varsigma | \varrho^n = \varsigma^2 = e \text{ and } \varsigma\varrho = \varrho^{-1}\varsigma \rangle.$$

If $\mathbb{k}'/\mathbb{k}$ is a Galois extension such that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathscr{C}_n$, we denote

$$\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle \zeta | \zeta^n = e \rangle.$$

If $\mathbb{k}'/\mathbb{k}$ is a Galois extension such that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathscr{C}_2 \times \mathscr{C}_2$, we denote

$$\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle \xi, \xi' | \xi^2 = \xi'^2 = 1 \rangle.$$

Proposition 4.2.4 ([Vos65])**.** *Let T be a two-dimensional $\mathbb{k}$-torus, and let τ be the corresponding $\mathbb{k}$-group structure on $T_{\overline{\mathbb{k}}}$. Let $\mathbb{k}'$ be a splitting field of T such that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathcal{G}$,* ***where $\mathcal{G}$ is the image of $\tilde{\tau}$****. Then, T is isomorphic to one of the following tori, where all intersections are taken inside $\mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$, with the canonical inclusions.*

	$\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathcal{G}$	*Corresponding $\mathbb{k}$-tori*
$\mathcal{G}_1$	$\langle \varrho, \varsigma \rangle \cong \langle x, s \rangle$	$\mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right) \cap \mathrm{R}_{\mathbb{k}_2/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}_3/\mathbb{k}_2}(\mathbb{G}_{m,\mathbb{k}_3})\right)$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \varrho^2 \rangle}$ is a Galois extension of $\mathbb{k}$ of degree 4, where $\mathbb{k}_2 = \mathbb{k}'^{\langle \varrho^i\varsigma, \varrho^3 \rangle}$, and where $\mathbb{k}_3 = \mathbb{k}'^{\langle \varrho^i\varsigma \rangle}$ for $i \in \{2, 3\}$. The extension $\mathbb{k}_3/\mathbb{k}_2$ is a Galois extension of degree 2.*
$\mathcal{G}_2$	$\langle \varrho, \varsigma \rangle \cong \langle ds, s \rangle$	$\mathrm{R}_{\mathbb{k}_2/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}_1/\mathbb{k}_2}(\mathbb{G}_{m,\mathbb{k}_1})\right)$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \varrho\varsigma \rangle}$ and $\mathbb{k}_2 = \mathbb{k}'^{\langle \varrho\varsigma, \varrho^2 \rangle}$, or $\mathbb{k}_1 = \mathbb{k}'^{\langle \varsigma \rangle}$ and $\mathbb{k}_2 = \mathbb{k}'^{\langle \varsigma, \varrho^2 \rangle}$.*
$\mathcal{G}_3$	$\langle \varrho, \varsigma \rangle \cong \langle x^2, s \rangle$	$\mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right) \cap \mathrm{R}_{\mathbb{k}_2/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}_2})$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \varrho \rangle}$ is a Galois extension of degree 2 of $\mathbb{k}$, and $\mathbb{k}_2 = \mathbb{k}'^{\langle \varsigma \rangle}$ is a cubic subfield.*
$\mathcal{G}_4$	$\langle \varrho, \varsigma \rangle \cong \langle x^2, -s \rangle$	$\mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right) \cap \mathrm{R}_{\mathbb{k}_2/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_2}(\mathbb{G}_{m,\mathbb{k}'})\right)$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \varrho \rangle}$ is a Galois extension of degree 2 of $\mathbb{k}$, and where $\mathbb{k}_2 = \mathbb{k}'^{\langle \varsigma \rangle}$ is a cubic subfield.*
$\mathcal{G}_5$	$\langle \xi, \xi' \rangle \cong \langle d, -d \rangle$	$\mathrm{R}^{(1)}_{\mathbb{k}_1/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}_1}) \times \mathrm{R}^{(1)}_{\mathbb{k}_2/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}_2})$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \xi' \rangle}$ and $\mathbb{k}_2 = \mathbb{k}'^{\langle \xi \rangle}$ are Galois extension of degree 2 of $\mathbb{k}$, or $\mathbb{k}_1 = \mathbb{k}'^{\langle \xi\xi' \rangle}$ and $\mathbb{k}_2 = \mathbb{k}'^{\langle \xi \rangle}$, or $\mathbb{k}_1 = \mathbb{k}'^{\langle \xi\xi' \rangle}$ and $\mathbb{k}_2 = \mathbb{k}'^{\langle \xi' \rangle}$.*
$\mathcal{G}_6$	$\langle \xi, \xi' \rangle \cong \langle s, -s \rangle$	$\mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right)$, *where $\mathbb{k}_1$ goes through the three quadratic Galois extensions $\mathbb{k}'^{\langle \xi \rangle}$, $\mathbb{k}'^{\langle \xi' \rangle}$ and $\mathbb{k}'^{\langle \xi\xi' \rangle}$.*
$\mathcal{G}_7$	$\langle \zeta \rangle \cong \langle x \rangle$	$\mathrm{R}_{\mathbb{k}_2/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_2}(\mathbb{G}_{m,\mathbb{k}'})\right) \cap \mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right)$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \zeta^2 \rangle}$ is a Galois extension of degree 2 of $\mathbb{k}$, and $\mathbb{k}_2 = \mathbb{k}'^{\langle \zeta^3 \rangle}$ is the unique Galois extension of degree 3 of $\mathbb{k}$.*
$\mathcal{G}_8$	$\langle \zeta \rangle \cong \langle ds \rangle$	$\mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}(\mathbb{G}_{m,\mathbb{k}'})\right)$, *where $\mathbb{k}_1 = \mathbb{k}'^{\langle \zeta^2 \rangle}$ is the unique Galois extension of degree 2 of $\mathbb{k}$.*
$\mathcal{G}_9$	$\langle \zeta \rangle \cong \langle x^2 \rangle$	$\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$.
$\mathcal{G}_{10}$	$\langle \zeta \rangle \cong \langle -id \rangle$	$\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \times \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$.
$\mathcal{G}_{11}$	$\langle \zeta \rangle \cong \langle d \rangle$	$\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'}) \times \mathbb{G}_{m,\mathbb{k}}$.
$\mathcal{G}_{12}$	$\langle \zeta \rangle \cong \langle s \rangle$	$\mathrm{R}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_{m,\mathbb{k}'})$.
$\mathcal{G}_{13}$	$\langle \zeta \rangle \cong \langle id \rangle$	$\mathbb{G}_{m,\mathbb{k}'}$.

Remark 4.2.5 (On the notations of Proposition 4.2.4)**.** Let $(\mathbb{T} = \mathrm{Spec}(\mathbb{k}'[M]), \tau)$ be a two-dimensional $\mathbb{k}$-torus. Assume for instance that the image of $\tilde{\tau}$ is $< x >$. Then, if $\gamma \in \Gamma$ is such that $\tilde{\tau}_\gamma = x^2$, then $\mathbb{k}_1 = \mathbb{k}'^{<\gamma>}$. In Proposition 4.2.4, we have $\tilde{\tau}_{\zeta^2} = x^2$.

Proof. Let T be a two-dimensional $\Bbbk$-torus. It corresponds to a pair $(\mathbb{T} = \operatorname{Spec}(\Bbbk'[M]), \tau)$, for some Galois extension $\Bbbk'/\Bbbk$. By Remark 2.2.10, we can assume that $\Bbbk'/\Bbbk$ is a finite Galois extension such that the Γ-representation

$$\tilde{\tau} : \Gamma \to \operatorname{Aut}(M) \cong \operatorname{GL}_2(\mathbb{Z})$$

is injective. Therefore, the image of $\tilde{\tau}$ is isomorphic to $\Gamma = \operatorname{Gal}(\Bbbk'/\Bbbk)$, and $\Bbbk'$ is a splitting field of the $\Bbbk$-torus. Let R be a $\Bbbk$-algebra. Then, the set of R-points of T is (see [Sta, Section 01J5])

$$T(R) := \operatorname{Hom}(\operatorname{Spec}(R), T) \cong \operatorname{Hom}(\Bbbk[T], R).$$

Moreover, from Theorem 1.4.11,

$$\operatorname{Hom}(\Bbbk[T], R) \cong \operatorname{Hom}_\Gamma(\Bbbk'[M], R_{\Bbbk'}),$$

where $\operatorname{Hom}_\Gamma(\Bbbk[M], R)$ denotes the Γ-equivariant $\Bbbk'$-algebra morphism of $\operatorname{Hom}(\Bbbk'[M], R_{\Bbbk'})$, and where $R_{\Bbbk'} := R \otimes_\Bbbk \Bbbk'$. The Γ-action on $\Bbbk'[M]$ is given by $\tau^\sharp$ and the Γ-action on $R_{\Bbbk'}$ is defined by $id \otimes \gamma$. Finally, observe that

$$T(R) \cong \operatorname{Hom}_\Gamma(\Bbbk'[M], R_{\Bbbk'}) \cong \operatorname{Hom}_\Gamma(M, R^*_{\Bbbk'}),$$

where $\operatorname{Hom}_\Gamma(M, R^*_{\Bbbk'})$ denotes the Γ-equivariant $\Bbbk'$-group morphisms from M to $R^*_{\Bbbk'}$. The Γ-action on M is given by $\tilde{\tau}$. Fix a $\mathbb{Z}$-basis $\{e_1, e_2\}$ of $M \cong \mathbb{Z}^2$. An element $\phi \in \operatorname{Hom}_\Gamma(M, R^*_{\Bbbk'})$ is determined by the relations

$$\gamma \cdot v_i = \phi(\tilde{\tau}_\gamma(e_i)), \forall \gamma \in \Gamma,$$

where $v_i = \phi(e_i)$. Note that it is sufficient to compute these relations on the generators of $\Gamma \cong \mathcal{G}_i$. Recall that, in Appendix F, we describe the conjugacy classes of finite subgroups of $\operatorname{GL}_2(\mathbb{Z})$.

- Assume that $\operatorname{Gal}(\Bbbk'/\Bbbk) = \langle \zeta \rangle \cong \langle s \rangle$.

Up to equivalence, there is a single Γ-representation in $\operatorname{GL}(M)$ defined by $\tilde{\tau}_\zeta = s$. An element $\phi \in \operatorname{Hom}_\Gamma(M, R^*_{\Bbbk'})$ is determined by the relations

$$\zeta \cdot v_1 = \phi(\tilde{\tau}_\zeta(e_1)) = v_2, \;\; \zeta \cdot v_2 = \phi(\tilde{\tau}_\zeta(e_2)) = v_1.$$

Then, we obtain (see Proposition 2.2.20):

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \zeta \cdot v_1 = v_2 \right\} \\ &= \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R). \end{aligned}$$

- Assume that $\operatorname{Gal}(\Bbbk'/\Bbbk) = \langle \zeta \rangle \cong \langle d \rangle$.

Up to equivalence, there is a single Γ-representation in $\operatorname{GL}(M)$ defined by $\tilde{\tau}_\zeta = d$. Then, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \zeta \cdot v_1 = v_1^{-1}, \; \zeta \cdot v_2 = v_2 \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_1(\zeta \cdot v_1) = 1 \right\} \times \left\{ v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \zeta \cdot v_2 = v_2 \right\} \\ &= \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \times \mathbb{G}_{m,\Bbbk}(R). \end{aligned}$$

- Assume that $\operatorname{Gal}(\Bbbk'/\Bbbk) = \langle \zeta \rangle \cong \langle -id \rangle$.

Up to equivalence, there is a single Γ-representation in $\operatorname{GL}(M)$ defined by $\tilde{\tau}_\zeta = -id$. Then, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \zeta \cdot v_1 = v_1^{-1}, \; \zeta \cdot v_2 = v_2^{-1} \right\} \\ &= \{ v_1 \in R^*_{\Bbbk'} | \, v_1(\zeta \cdot v_1) = 1 \} \times \{ v_2 \in R^*_{\Bbbk'} | \, v_2(\zeta \cdot v_2) = 1 \} \\ &= \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \times \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R). \end{aligned}$$

• Assume that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle\zeta\rangle \cong \langle x^2\rangle$.
Up to equivalence, there is a single Γ-representation in $\mathrm{GL}(M)$ defined by $\tilde{\tau}_\zeta = x^2$ (indeed, x^2 and x^4 are conjugate in $\mathrm{GL}_2(\mathbb{Z})$, therefore the Γ-representation defined by $\tilde{\tau}'_\zeta = x^4$ is equivalent to $\tilde{\tau}$). Then, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, \zeta\cdot v_1 = v_2,\ \zeta\cdot v_2 = v_1^{-1}v_2^{-1} \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, v_1(\zeta\cdot v_1)(\zeta^2\cdot v_1) = 1 \right\} \\ &= \ \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R). \end{aligned}$$

• Assume that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle\zeta\rangle \cong \langle ds\rangle$.
Up to equivalence, there is a single Γ-representation defined by $\tilde{\tau}_\zeta = ds$ (indeed, ds is conjugate to $(ds)^3$). Let $\mathbb{k}_1 := \mathbb{k}'^{\langle\zeta^2\rangle}$. Then, $\mathbb{k}_1/\mathbb{k}$ is a finite Galois extension of degree 2 in $\mathbb{k}'$. We obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, \zeta\cdot v_1 = v_2,\ \zeta\cdot v_2 = v_1^{-1} \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, v_1(\zeta^2\cdot v_1) = 1 \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R_{\mathbb{k}_1}) \middle| \, v_1(\zeta^2\cdot v_1) = 1 \right\} \\ &= \ \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}_1}\right)(R_{\mathbb{k}_1}) \\ &= \ \mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}_1}\right)\right)(R). \end{aligned}$$

• Assume that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) = \langle\zeta\rangle \cong \langle x\rangle$.
Up to equivalence, there is a single Γ-representation in $\mathrm{GL}(M)$ defined by $\tilde{\tau}_\zeta = x$ (indeed, x is conjugate to x^5). Let $\mathbb{k}_1 := \mathbb{k}'^{\langle\zeta^2\rangle}$. The subgroup $\langle\zeta^2\rangle$ of $\langle\zeta\rangle$ is a normal one, thus $\mathbb{k}_1/\mathbb{k}$ is a Galois extension of degree 2. Let $\mathbb{k}_2 := \mathbb{k}'^{\langle\zeta^3\rangle}$. The subgroup $\langle\zeta^3\rangle$ of $\langle\zeta\rangle$ is a normal one, thus $\mathbb{k}_2/\mathbb{k}$ is a Galois extension of degree 3.

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, \zeta\cdot v_1 = v_1v_2,\ \zeta\cdot v_2 = v_1^{-1} \right\} \\ &= \left\{ v_1, v_2 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, \zeta\cdot v_1 = v_1v_2,\ (\zeta^3\cdot v_2)(\zeta^2\cdot v_1) = 1,\ \zeta^2\cdot v_1 = v_2 \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, v_2(\zeta^2\cdot v_2)(\zeta^4\cdot v_2) = 1,\ v_2(\zeta^3\cdot v_2) = 1 \right\} \\ &= T_1(R) \cap T_2(R), \end{aligned}$$

where

$$\begin{aligned} T_1(R) &:= \left\{ v \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, v(\zeta^2\cdot v)(\zeta^4\cdot v) = 1 \right\} \\ &= \left\{ v \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R_{\mathbb{k}_1}) \middle| \, v(\zeta^2\cdot v)(\zeta^4\cdot v) = 1 \right\} \\ &= \mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R_{\mathbb{k}_1}) \\ &= \mathrm{R}_{\mathbb{k}_1/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_1}\left(\mathbb{G}_{m,\mathbb{k}'}\right)\right)(R), \end{aligned}$$

and where

$$\begin{aligned} T_2(R) &:= \left\{ v \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R) \middle| \, v(\zeta^3\cdot v) = 1 \right\} \\ &= \left\{ v \in \mathrm{R}_{\mathbb{k}'/\mathbb{k}}\left(\mathbb{G}_{m,\mathbb{k}'}\right)(R_{\mathbb{k}_2}) \middle| \, v(\zeta^3\cdot v) = 1 \right\} \\ &= \mathrm{R}_{\mathbb{k}_2/\mathbb{k}}\left(\mathrm{R}^{(1)}_{\mathbb{k}'/\mathbb{k}_2}\left(\mathbb{G}_{m,\mathbb{k}'}\right)\right)(R). \end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle \xi, \xi' \rangle \cong \langle s, -s \rangle$.

Up to equivalence, there are three non-equivalent Γ-representations defined by:

$$\tilde{\tau}_{\xi} = s, \quad \tilde{\tau}_{\xi'} = -s; \qquad \tilde{\tau}^{(2)}_{\xi} = s, \quad \tilde{\tau}^{(2)}_{\xi'} = -id; \qquad \tilde{\tau}^{(3)}_{\xi} = -id, \quad \tilde{\tau}^{(3)}_{\xi'} = s.$$

Let $\Bbbk_1 = \Bbbk'^{\langle \xi\xi' \rangle}$. Then, $\Bbbk_1/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$. Then, for $\tilde{\tau}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_2,\ \xi \cdot v_2 = v_1,\ \xi' \cdot v_1 = v_2^{-1},\ \xi' \cdot v_2 = v_1^{-1} \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_1((\xi\xi') \cdot v_1) = 1 \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_1}) \middle| \, v_1((\xi\xi') \cdot v_1) = 1 \right\} \\ &= \ \mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_1}) \\ &= \ \mathrm{R}_{\Bbbk_1/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R). \end{aligned}$$

Let $\Bbbk_2 = \Bbbk'^{\langle \xi' \rangle}$. Then, $\Bbbk_2'/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$. Then, for $\tilde{\tau}^{(2)}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_2,\ \xi \cdot v_2 = v_1,\ \xi' \cdot v_1 = v_1^{-1},\ \xi' \cdot v_2 = v_2^{-1} \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_2}) \middle| \, v_1(\xi' \cdot v_1) = 1 \right\} \\ &= \ \mathrm{R}_{\Bbbk_2/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R). \end{aligned}$$

Let $\Bbbk_3 = \Bbbk'^{\langle \xi \rangle}$. Then, $\Bbbk_3'/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$. Then, for $\tilde{\tau}^{(3)}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_1^{-1},\ \xi \cdot v_2 = v_2^{-1},\ \xi' \cdot v_1 = v_2,\ \xi' \cdot v_2 = v_1 \right\} \\ &= \left\{ v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk_3}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_3}) \middle| \, v_1(\xi \cdot v_1) = 1 \right\} \\ &= \ \mathrm{R}_{\Bbbk_3/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_3}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R). \end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle \xi, \xi' \rangle \cong \langle d, -d \rangle$.

Up to equivalence, there are three non-equivalent Γ-representations defined by:

$$\tilde{\tau}_{\xi} = d, \quad \tilde{\tau}_{\xi'} = -d; \qquad \tilde{\tau}^{(2)}_{\xi} = d, \quad \tilde{\tau}^{(2)}_{\xi'} = -id; \qquad \tilde{\tau}^{(3)}_{\xi} = -id, \quad \tilde{\tau}^{(3)}_{\xi'} = d.$$

Let $\Bbbk_1 = \Bbbk'^{\langle \xi' \rangle}$, and let $\Bbbk_2 = \Bbbk'^{\langle \xi \rangle}$. Then, $\Bbbk_1/\Bbbk$ and $\Bbbk_2/\Bbbk$ are Galois extension of degree 2 in $\Bbbk'$, having Galois groups isomorphic to $\langle \zeta \rangle = \{e, \zeta\}$. Then, for $\tilde{\tau}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_1^{-1},\ \xi \cdot v_2 = v_2,\ \xi' \cdot v_1 = v_1,\ \xi' \cdot v_2 = v_2^{-1} \right\} \\ &= \left\{ v_1 \in R^*_{\Bbbk'} | \, \xi \cdot v_1 = v_1^{-1},\ \xi' \cdot v_1 = v_1 \right\} \times \left\{ v_2 \in R^*_{\Bbbk'} | \, \xi \cdot v_2 = v_2,\ \xi' \cdot v_2 = v_2^{-1} \right\} \\ &= \{ v_1 \in R^*_{\Bbbk_1} | \, v_1(\zeta \cdot v_1) = 1 \} \times \{ v_2 \in R^*_{\Bbbk_2} | \, v_2(\zeta \cdot v_2) = 1 \} \\ &= \ \mathrm{R}^{(1)}_{\Bbbk_1/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_1}\right)(R) \times \mathrm{R}^{(1)}_{\Bbbk_2/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_2}\right)(R). \end{aligned}$$

Let $\Bbbk_3 = \Bbbk'^{<\xi\xi'>}$. Then, $\Bbbk_3/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$, having Galois groups isomorphic to $\langle \zeta \rangle = \{e, \zeta\}$. Then, for $\tilde{\tau}^{(2)}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_1^{-1},\ \xi \cdot v_2 = v_2,\ \xi' \cdot v_1 = v_1^{-1},\ \xi' \cdot v_2 = v_2^{-1} \right\} \\ &= \{ v_1 \in R^*_{\Bbbk'} | \, (\xi\xi') \cdot v_1 = v_1,\ v_1(\xi \cdot v_1) = 1 \} \times \{ v_2 \in R^*_{\Bbbk'} | \, \xi \cdot v_2 = v_2,\ v_2(\xi' \cdot v_2) = 1 \} \\ &= \{ v_1 \in R^*_{\Bbbk_3} | \, v_1(\zeta \cdot v_1) = 1 \} \times \{ v_2 \in R^*_{\Bbbk_2} | \, v_2(\zeta \cdot v_2) = 1 \} \\ &= \ \mathrm{R}^{(1)}_{\Bbbk_3/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_3}\right)(R) \times \mathrm{R}^{(1)}_{\Bbbk_2/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_2}\right)(R). \end{aligned}$$

Then, for $\tilde{\tau}^{(3)}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \xi \cdot v_1 = v_1^{-1},\ \xi \cdot v_2 = v_2^{-1},\ \xi' \cdot v_1 = v_1^{-1},\ \xi' \cdot v_2 = v_2 \right\} \\ &= \left\{ v_1 \in R_{\Bbbk'}^* \,|\, (\xi\xi') \cdot v_1 = v_1,\ v_1(\xi \cdot v_1) = 1 \right\} \times \left\{ v_2 \in R_{\Bbbk'}^* \,|\, \xi' \cdot v_2 = v_2,\ v_2(\xi \cdot v_2) = 1 \right\} \\ &= \left\{ v_1 \in R_{\Bbbk_3}^* \,|\, v_1(\zeta \cdot v_1) = 1 \right\} \times \left\{ v_2 \in R_{\Bbbk_1}^* \,|\, v_2(\zeta \cdot v_2) = 1 \right\} \\ &= \ \mathrm{R}^{(1)}_{\Bbbk_3/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_3}\right)(R) \times \mathrm{R}^{(1)}_{\Bbbk_1/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_1}\right)(R). \end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle \varrho, \varsigma \rangle \cong \langle x^2, s \rangle$.

Up to equivalence, there is a single Γ-representation defined by:

$$\tilde{\tau}_\varrho = x^2, \quad \tilde{\tau}_\varsigma = s.$$

Let $\Bbbk_1 = \Bbbk'^{\langle \varrho \rangle}$. Then, $\Bbbk_1/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$. Let $\Bbbk_2 = \Bbbk'^{\langle \varrho\varsigma \rangle}$. Then, for $\tilde{\tau}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho \cdot v_1 = v_2,\ \varrho \cdot v_2 = v_1^{-1}v_2^{-1},\ \varsigma \cdot v_1 = v_2 \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho \cdot v_2)(\varrho^2 \cdot v_2) = 1,\ (\varrho\varsigma) \cdot v_2 = v_2 \right\} \\ &= \ \mathrm{R}_{\Bbbk_1/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R) \cap \mathrm{R}_{\Bbbk_2/\Bbbk}\left(\mathbb{G}_{m,\Bbbk_2}\right)(R). \end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle \varrho, \varsigma \rangle \cong \langle x^2, -s \rangle$.

Up to equivalence, there is a single Γ-representation defined by:

$$\tilde{\tau}_\varrho = x^2, \quad \tilde{\tau}_\varsigma = -s.$$

Let $\Bbbk_1 = \Bbbk'^{\langle \varrho \rangle}$. Then, $\Bbbk_1/\Bbbk$ is a Galois extension of degree 2 in $\Bbbk'$. Let $\Bbbk_2 = \Bbbk'^{\langle \varrho\varsigma \rangle}$. Note that $\langle \varrho\varsigma \rangle$ is not a normal subgroup of $\langle \varrho, \varsigma \rangle$. Then, for $\tilde{\tau}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho \cdot v_1 = v_2,\ \varrho \cdot v_2 = v_1^{-1}v_2^{-1},\ \varsigma \cdot v_1 = v_2^{-1} \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho \cdot v_2)(\varrho^2 \cdot v_2) = 1,\ v_2((\varrho\varsigma) \cdot v_2) = 1 \right\} \\ &= \ \mathrm{R}_{\Bbbk_1/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R) \cap \mathrm{R}_{\Bbbk_2/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R). \end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle \varrho, \varsigma \rangle \cong \langle ds, s \rangle$.

Up to equivalence, there are two non equivalent Γ-representations defined by:

$$\tilde{\tau}_\varrho = ds, \quad \tilde{\tau}_\varsigma = s, \quad \text{and} \quad \tilde{\tau}^{(2)}_\varrho = ds,\ \tilde{\tau}^{(2)}_{\varrho\varsigma} = s.$$

Let $\Bbbk_1 = \Bbbk'^{\langle \varrho\varsigma \rangle}$. Then, $\Bbbk_1/\Bbbk$ is a field extension of degree 4 in $\Bbbk'$. Let $\Bbbk_2 = \Bbbk'^{\langle \varrho\varsigma, \varrho^2 \rangle}$. Then $\Bbbk_1/\Bbbk_2$ is a Galois extension of degree 2 of Galois group $\langle \zeta \rangle = \{e, \zeta\}$. Therefore, for $\tilde{\tau}$, we obtain:

$$\begin{aligned} T(R) &= \left\{ v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho \cdot v_1 = v_2,\ \varrho \cdot v_2 = v_1^{-1},\ \varsigma \cdot v_1 = v_2 \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, (\varrho\varsigma) \cdot v_2 = v_2,\ v_2(\varrho^2 \cdot v_2) = 1 \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_2}) \middle| \, (\varrho\varsigma) \cdot v_2 = v_2,\ v_2(\varrho^2 \cdot v_2) = 1 \right\} \\ &= \left\{ v_2 \in \mathrm{R}_{\Bbbk_1/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk_1}\right)(R_{\Bbbk_2}) \middle| \, v_2(\zeta \cdot v_2) = 1 \right\} \\ &= \ \mathrm{R}_{\Bbbk_2/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk_1/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk_1}\right)\right)(R). \end{aligned}$$

Note that $\Bbbk_1/\Bbbk$ is not a Galois extension, therefore $\Bbbk_1$ is not a splitting field of T.
Let $\Bbbk_3 = \Bbbk'^{\langle\varsigma\rangle}$. Then, $\Bbbk_3/\Bbbk$ is a field extension of degree 4 in $\Bbbk'$. Let $\Bbbk_4 = \Bbbk'^{\langle\varsigma,\varrho^2\rangle}$. Then $\Bbbk_3/\Bbbk_4$ is a Galois extension of degree 2 of Galois group $\langle\zeta\rangle = \{e,\zeta\}$. Therefore, for $\tilde{\tau}^{(2)}$, we obtain:

$$\begin{aligned}
T(R) &= \left\{v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho\cdot v_1 = v_2,\ \varrho\cdot v_2 = v_1^{-1},\ (\varrho\varsigma)\cdot v_1 = v_2\right\}\\
&= \left\{v_1 \in \mathrm{R}_{\Bbbk'/\Bbbk_4}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_4}) \middle| \, \varsigma\cdot v_1 = v_1,\ v_1(\varrho^2\cdot v_1) = 1\right\}\\
&= \left\{v_1 \in \mathrm{R}_{\Bbbk_3/\Bbbk_4}\left(\mathbb{G}_{m,\Bbbk_3}\right)(R_{\Bbbk_4}) \middle| \, v_1(\zeta\cdot v_1) = 1\right\}\\
&= \ \mathrm{R}_{\Bbbk_4/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk_3/\Bbbk_4}\left(\mathbb{G}_{m,\Bbbk_3}\right)\right)(R)
\end{aligned}$$

- Assume that $\mathrm{Gal}(\Bbbk'/\Bbbk) = \langle\varrho,\varsigma\rangle \cong \langle x, s\rangle$.

Up to equivalence, there are two non equivalent Γ-representations defined by:

$$\tilde{\tau}_\varrho = x,\quad \tilde{\tau}_\varsigma = s; \qquad \tilde{\tau}^{(2)}_\varrho = x,\quad \tilde{\tau}^{(2)}_{\varrho\varsigma} = s.$$

Let $\Bbbk_1 := \Bbbk'^{\langle\varrho^2\rangle}$ and let $\Bbbk_2 := \Bbbk'^{\langle\varrho^2\varsigma,\varrho^3\rangle}$. Then, $\Bbbk_1/\Bbbk$ is a Galois extension of degree 4 in $\Bbbk'$, and $\Bbbk_2/\Bbbk$ is a field extension of degree 3 in $\Bbbk'$. Let $\Bbbk_3 := \Bbbk'^{\langle\varrho^2\varsigma\rangle}$. Then, $\langle\varrho^2\varsigma\rangle$ is a normal subgroup of $\langle\varrho^2\varsigma,\varrho^3\rangle$. Hence, $\Bbbk_3/\Bbbk_2$ is a Galois extension of degree 2 of Galois group $\langle\zeta\rangle = \{e,\zeta\}$. Then, for $\tilde{\tau}$, we obtain:

$$\begin{aligned}
T(R) &= \left\{v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho\cdot v_1 = v_1v_2,\ \varrho\cdot v_2 = v_1^{-1},\ \varsigma\cdot v_1 = v_2\right\}\\
&= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho^2\cdot v_2)(\varrho^4\cdot v_2) = 1,\ v_2(\varrho^3\cdot v_2) = 1,\ (\varrho^2\varsigma)\cdot v_2 = v_2\right\}\\
&= \ T_1(R)\cap T_2(R),
\end{aligned}$$

where

$$\begin{aligned}
T_1(R) &= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho^2\cdot v_2)(\varrho^4\cdot v_2) = 1\right\}\\
&= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_1}) \middle| \, v_2(\varrho^2\cdot v_2)(\varrho^4\cdot v_2) = 1\right\}\\
&= \ \mathrm{R}_{\Bbbk_1/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R),
\end{aligned}$$

and where

$$\begin{aligned}
T_2(R) &= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho^3\cdot v_2) = 1,\ (\varrho^2\varsigma)\cdot v_2 = v_2\right\}\\
&= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk'}\right)(R_{\Bbbk_2}) \middle| \, v_2(\varrho^3\cdot v_2) = 1,\ (\varrho^2\varsigma)\cdot v_2 = v_2\right\}\\
&= \left\{v_2 \in \mathrm{R}_{\Bbbk_3/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk_3}\right)(R_{\Bbbk_2}) \middle| \, v_2(\zeta\cdot v_2) = 1\right\}\\
&= \ \mathrm{R}_{\Bbbk_2/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk_3/\Bbbk_2}\left(\mathbb{G}_{m,\Bbbk_3}\right)\right)(R).
\end{aligned}$$

Let $\Bbbk_2^{(2)} := \Bbbk'^{\langle\varrho^3\varsigma,\varrho^3\rangle}$, and let $\Bbbk_3^{(2)} := \Bbbk'^{\langle\varrho^3\varsigma\rangle}$. Then, $\langle\varrho^3\varsigma\rangle$ is a normal subgroup of $\langle\varrho^3\varsigma,\varrho^3\rangle$. Hence, for $\tilde{\tau}^{(2)}$, we obtain:

$$\begin{aligned}
T(R) &= \left\{v_1, v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, \varrho\cdot v_1 = v_1v_2,\ \varrho\cdot v_2 = v_1^{-1},\ (\varrho\varsigma)\cdot v_1 = v_2\right\}\\
&= \left\{v_2 \in \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}\right)(R) \middle| \, v_2(\varrho^2\cdot v_2)(\varrho^4\cdot v_2) = 1,\ v_2(\varrho^3\cdot v_2) = 1,\ (\varrho^3\varsigma)\cdot v_2 = v_2\right\}\\
&= \ \mathrm{R}_{\Bbbk_1/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk'/\Bbbk_1}\left(\mathbb{G}_{m,\Bbbk'}\right)\right)(R)\cap \mathrm{R}_{\Bbbk_2^{(2)}/\Bbbk}\left(\mathrm{R}^{(1)}_{\Bbbk_3^{(2)}/\Bbbk_2^{(2)}}\left(\mathbb{G}_{m,\Bbbk_3^{(2)}}\right)\right)(R).
\end{aligned}$$

□

This section ends on the following example.

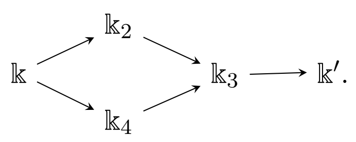
Example 4.2.6. Using inverse Galois theory, we can illustrate each of the conjugacy classes $\mathcal{G}_i$ by a $\mathbb{Q}$-torus that splits over a finite Galois extension $\mathbb{k}'/\mathbb{k}$, such that $\mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \cong \mathcal{G}_i$.

Label	Galois extension of $\mathbb{Q}$ of Galois group $\mathcal{G}$
$\mathcal{G}_1$	$\mathbb{Q}\left(\sqrt[3]{1+\sqrt{2}}\right)$ [Esc01, Exercise 8.9], [Wei09, Example 2.9.6]
$\mathcal{G}_2$	$\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ [Esc01, §8.7.3]
$\mathcal{G}_3$	$\mathbb{Q}\left(\sqrt[3]{2}, j\right)$ [Esc01, §8.3]
$\mathcal{G}_4$	$\mathbb{Q}\left(\sqrt[3]{2}, j\right)$
$\mathcal{G}_5$	$\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (see Example 2.2.30)
$\mathcal{G}_6$	$\mathbb{Q}(\sqrt{2}, \sqrt{3})$
$\mathcal{G}_7$	$\mathbb{Q}\left(e^{\frac{2i\pi}{7}}\right)$ [Esc01, §9.5]
$\mathcal{G}_8$	$\mathbb{Q}\left(e^{\frac{2i\pi}{5}}\right)$ [Esc01, §9.5]
$\mathcal{G}_9$	$\mathbb{Q}\left(\cos\left(\frac{2\pi}{7}\right)\right)$ (see Example 2.2.29)
$\mathcal{G}_{10}$	$\mathbb{Q}(i)$
$\mathcal{G}_{11}$	$\mathbb{Q}(i)$
$\mathcal{G}_{12}$	$\mathbb{Q}(i)$

4.2.2 Two-dimensional tori and their torsors

In [ELFST14, §5], there is a complete description of the Galois-cohomology set used to classify T-torsors, where T is a two-dimensional $\mathbb{k}$-torus. Let $\mathbb{k}'/\mathbb{k}$ be a finite Galois extension that splits T, and let τ be the associated $\mathbb{k}$-group structure on $T_{\mathbb{k}'} \cong \mathrm{Spec}(\mathbb{k}'[M])$. We can assume that $\tilde{\tau} : \mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \to \mathrm{GL}_2(\mathbb{Z})$ is injective, therefore $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$ is isomorphic to is a finite subgroup $\mathcal{G}$ of $\mathrm{GL}_2(\mathbb{Z})$ (see Remark 2.2.10 and Proposition 2.3.17). An expression of $H^1(\mathrm{Gal}(\mathbb{k}'/\mathbb{k}), \mathrm{Aut}^{T_{\mathbb{k}'}}(T_{\mathbb{k}'}))$ depending on a Brauer group is obtained in [ELFST14, §5]. In order to state their results, we briefly introduce some notations.

Let $\mathbb{k}_2$, $\mathbb{k}_4$ and $\mathbb{k}_3$ be fields extensions such that the following diagram commutes



Consider the homomorphism $\beta : \mathrm{Br}(\mathbb{k}_2/\mathbb{k}) \to \mathrm{Br}(\mathbb{k}_3/\mathbb{k}_4)$ obtained by base extension from $\mathbb{k}$ to $\mathbb{k}_4$. Consider the homomorphism $\eta : \mathrm{Br}(\mathbb{k}_3/\mathbb{k}_2) \to \mathrm{Br}(\mathbb{k}_4/\mathbb{k})$ induced by the norm map $N_{\mathbb{k}_2/\mathbb{k}}(\mathbb{k}) : \mathbb{k}_2^* \to \mathbb{k}^*$ (see Proposition 2.2.20). We denote the kernel of η by $\mathrm{Br}_\eta(\mathbb{k}_2/\mathbb{k}|\mathbb{k}_3/\mathbb{k}_4)$.

In Theorem 4.2.7, we summarize main results of [ELFST14]. Compare with Proposition 4.2.4 combined with Proposition 2.3.19 and with Proposition 2.3.21.

Theorem 4.2.7 ([ELFST14, Theorems 5.3 & 5.5])**.** *Let* $(\mathbb{T} = \mathrm{Spec}(\mathbb{k}'[M]), \tau)$ *be a two-dimensional* $\mathbb{k}$*-torus. By Proposition 2.3.17, we can assume that* $\mathbb{k}'/\mathbb{k}$ *is a finite Galois extension such that the* Γ*-representation* $\tilde{\tau} : \mathrm{Gal}(\mathbb{k}'/\mathbb{k}) \to \mathrm{GL}(M) \cong \mathrm{GL}_2(\mathbb{Z})$ *is injective. We denote its image by* $\mathcal{G}$*. Then, with the notation of Proposition 4.2.4, we get the following Galois cohomology set.*

	$\mathrm{Gal}(\Bbbk'/\Bbbk) \cong \mathcal{G}$	$\mathrm{H}^1\left(\mathrm{Gal}(\Bbbk'/\Bbbk), \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})\right)$
$\mathcal{G}_1$	$\langle \varrho, \varsigma\rangle \cong \langle x, s\rangle$	$\dfrac{\mathrm{Br}\,(\Bbbk_3/\Bbbk_4)}{\beta\,(\mathrm{Br}\,(\Bbbk_2/\Bbbk))} \oplus \mathrm{Br}_\eta\,(\Bbbk_2/\Bbbk\vert\Bbbk_3/\Bbbk_4)$, *where* $\Bbbk_4 = \Bbbk'^{\langle \varrho^2, \varrho^i\varsigma\rangle}$, *where* $\Bbbk_2 = \Bbbk'^{\langle \varrho^i\varsigma, \varrho^3\rangle}$, *and where* $\Bbbk_3 = \Bbbk'^{\langle \varrho^i\varsigma\rangle}$ *for* $i \in \{2,3\}$.
$\mathcal{G}_2$	$\langle \varrho, \varsigma\rangle \cong \langle ds, s\rangle$	$\mathrm{Br}\,(\Bbbk_1/\Bbbk_2)$, *where* $\Bbbk_1 = \Bbbk'^{\langle \varrho\varsigma\rangle}$ *and* $\Bbbk_2 = \Bbbk'^{\langle \varrho\varsigma, \varrho^2\rangle}$, *or* $\Bbbk_1 = \Bbbk'^{\langle \varsigma\rangle}$ *and* $\Bbbk_2 = \Bbbk'^{\langle \varsigma, \varrho^2\rangle}$.
$\mathcal{G}_3$	$\langle \varrho, \varsigma\rangle \cong \langle x^2, s\rangle$	$\mathrm{Br}\,(\Bbbk_2/\Bbbk)$, *where* $\Bbbk_2 = \Bbbk'^{\langle \varsigma\rangle}$.
$\mathcal{G}_4$	$\langle \varrho, \varsigma\rangle \cong \langle x^2, -s\rangle$	$\dfrac{\mathrm{Br}\,(\Bbbk'/\Bbbk_1)}{\beta\,(\mathrm{Br}(\Bbbk_2/\Bbbk))}$, *where* $\Bbbk_1 = \Bbbk'^{\langle \varrho\rangle}$, *and where* $\Bbbk_2 = \Bbbk'^{\langle \varsigma\rangle}$.
$\mathcal{G}_5$	$\langle \xi, \xi'\rangle \cong \langle d, -d\rangle$	$\mathrm{Br}\,(\Bbbk_1/\Bbbk) \oplus \mathrm{Br}\,(\Bbbk_2/\Bbbk)$, *where* $\Bbbk_1 = \Bbbk'^{\langle \xi'\rangle}$ *and* $\Bbbk_2 = \Bbbk'^{\langle \xi\rangle}$, *or* $\Bbbk_1 = \Bbbk'^{\langle \xi\xi'\rangle}$ *and* $\Bbbk_2 = \Bbbk'^{\langle \xi\rangle}$, *or* $\Bbbk_1 = \Bbbk'^{\langle \xi\xi'\rangle}$ *and* $\Bbbk_2 = \Bbbk'^{\langle \xi'\rangle}$.
$\mathcal{G}_6$	$\langle \xi, \xi'\rangle \cong \langle s, -s\rangle$	$\mathrm{Br}\,(\Bbbk'/\Bbbk_1)$, *where* $\Bbbk_1 = \Bbbk'^{\langle \xi'\rangle}$, *or* $\Bbbk_1 = \Bbbk'^{\langle \xi'\rangle}$, *or* $\Bbbk_1 = \Bbbk'^{\langle \xi\xi'\rangle}$.
$\mathcal{G}_7$	$\langle \zeta\rangle \cong \langle x\rangle$	$\dfrac{\mathrm{Br}\left(\Bbbk'/\Bbbk'^{\langle \zeta^2\rangle}\right)}{\beta\,(\mathrm{Br}(\Bbbk'/\Bbbk))} \oplus \mathrm{Br}_\eta\left(\Bbbk'/\Bbbk\vert\Bbbk'/\Bbbk'^{\langle \zeta^2\rangle}\right)$
$\mathcal{G}_8$	$\langle \zeta\rangle \cong \langle ds\rangle$	$\mathrm{Br}\left(\Bbbk'/\Bbbk'^{\langle \zeta^2\rangle}\right)$
$\mathcal{G}_9$	$\langle \zeta\rangle \cong \langle x^2\rangle$	$\mathrm{Br}(\Bbbk'/\Bbbk)$
$\mathcal{G}_{10}$	$\langle \zeta\rangle \cong \langle -id\rangle$	$\mathrm{Br}(\Bbbk'/\Bbbk) \oplus \mathrm{Br}(\Bbbk'/\Bbbk)$
$\mathcal{G}_{11}$	$\langle \zeta\rangle \cong \langle d\rangle$	$\mathrm{Br}(\Bbbk'/\Bbbk)$
$\mathcal{G}_{12}$	$\langle \zeta\rangle \cong \langle s\rangle$	$\{0\}$
$\mathcal{G}_{13}$	$\langle \zeta\rangle \cong \langle id\rangle$	$\{0\}$

Example 4.2.8. Let $\Bbbk'/\Bbbk$ be any Galois extension and $(\mathbb{T} = \mathrm{Spec}(\Bbbk'[M]), \tau)$ be a two-dimensional $\Bbbk$-torus such that the image of $\tilde{\tau}$ is conjugate to $\mathcal{G}_{12}$; for instance $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Then, $H^1(\mathrm{Gal}(\Bbbk'/\Bbbk), \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})) = \{1\}$. Therefore, if $(\mathbb{T}, \tau)$ acts on an affine normal variety (X, σ), then the Altmann-Hausen presentation simplifies (see Corollary 3.2.10).

Example 4.2.9. Let $(\mathbb{T} = \mathrm{Spec}(\Bbbk'[M]), \tau)$ be a two-dimensional $\Bbbk$-torus such that the image $\mathcal{G}$ of $\tilde{\tau}$ is isomorphic to Γ. Let $(\omega_N, Y, \mathcal{D}, \sigma_Y, h)$ be a generalized AH-datum. Let $\mathbb{L} := \Bbbk'(Y)$ and let $\mathbb{K} := \mathbb{L}^\Gamma$. By Lemma 3.2.8, $\mathbb{L}/\mathbb{K}$ is a Galois extension of Galois group Γ. If $\mathbb{K}$ satisfies one of the conditions of Proposition 2.3.25 (therefore $\mathbb{K}$ has cohomological dimension one (see [Ser97, Chapter 2, §3, in particular Corollary of Proposition 8]), then the Altmann-Hausen presentation simplifies (see Corollary 3.2.10) since all the Brauer group mentioned in Theorem 4.2.7 are trivial (see [Ser97, Chapter 2, Proposition 6]).

4.2.3 Γ-equivariant MMP

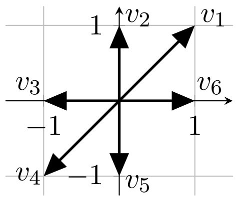
In this section, $\Bbbk' = \overline{\Bbbk}$ and $\Gamma = \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$.

For a point on the (toric) MMP, see for instance [CLS11, §15.4 & §15.5]. See also [KM98, §1.4 & Chapter 2]. See Appendic D for some basic tools of birational geometry. Compactification of torsors is studied in [Vos82, Section 1, Theorem 1 and its Corollary], and also in [ELFST14, Theorem 3.9]. In this section we need the following lemma, and more exactly its proof.

Lemma 4.2.10. *Let* $(\mathbb{T}, \sigma)$ *be a* $(\mathbb{T}, \tau)$*-torsor, where* $\mathbb{T} = \mathrm{Spec}(\overline{\Bbbk}[M])$ *and* $M \cong \mathbb{Z}^2$. *Then* $(\mathbb{T}, \sigma)$ *admits a smooth projective compactification that is a* $(\mathbb{T}, \tau)$*-toric Del Pezzo* $\Bbbk$*-surface containing* $(\mathbb{T}, \sigma)$ *as a* $(\mathbb{T}, \tau)$*-stable dense open subset.*

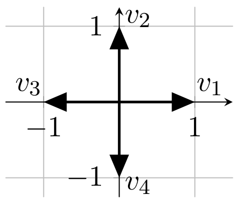
Proof. The $\Bbbk$-group structure τ induces a Γ-representation $\hat{\tau}$ on $\mathrm{GL}(N)$. The image of $\hat{\tau}$ is a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

• *Case 1:* The image of $\hat{\tau}$ is conjugate to a normal subgroup of $\mathcal{G}_1 = \langle x, s\rangle$ (see Proposition 4.2.3). Let Λ be the fan of the projective toric variety $\mathrm{Bl}_3(\mathbb{P}^2_{\overline{\Bbbk}})$ in $N_{\mathbb{Q}}$:



This fan is Γ-stable (for $\hat{\tau}$), therefore, by Proposition 2.4.9, the $\Bbbk$-structure σ on $\mathbb{T}$ extends to a $\Bbbk$-structure σ_Λ on $X_\Lambda = \mathrm{Bl}_3(\mathbb{P}^2_{\overline{\Bbbk}})$, the variety $(X_\Lambda, \sigma_\Lambda)$ is a (compact) $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety (a toric Del Pezzo $\Bbbk$-surface of degree 6), and we obtain a $(\mathbb{T}, \tau)$-equivariant open immersion $(\mathbb{T}, \sigma) \hookrightarrow (X_\Lambda, \sigma_\Lambda)$.

• *Case 2:* The image of $\hat{\tau}$ is conjugate to a normal subgroup of $\mathcal{G}_2 = \langle d, s\rangle$ (see Proposition 4.2.3). Let Λ be the fan of the projective toric variety $\mathbb{P}^1_{\overline{\Bbbk}} \times \mathbb{P}^1_{\overline{\Bbbk}}$ in $N_{\mathbb{Q}}$:



This fan is Γ-stable (for $\hat{\tau}$), therefore, by Proposition 2.4.9, the $\Bbbk$-structure σ on $\mathbb{T}$ extends to a $\Bbbk$-structure σ_Λ on $X_\Lambda = \mathbb{P}^1_{\overline{\Bbbk}} \times \mathbb{P}^1_{\overline{\Bbbk}}$, the variety $(X_\Lambda, \sigma_\Lambda)$ is a (compact) $(\mathbb{T}, \tau)$-toric $\Bbbk$-variety (a toric Del Pezzo $\Bbbk$-surface of degree 8), and we obtain a $(\mathbb{T}, \tau)$-equivariant open immersion $(\mathbb{T}, \sigma) \hookrightarrow (X_\Lambda, \sigma_\Lambda)$. □

Note that Lemma 4.2.10 is still true for any field extension $\Bbbk'/\Bbbk$. We assume $\Bbbk' = \overline{\Bbbk}$ in view of applying an MMP (see Proposition 4.2.12).

The next proposition illustrates the reason why we are interested in embedding a $(\mathbb{T}, \tau)$-torsor in some $(\mathbb{T}, \tau)$-toric Del Pezzo surface.

Proposition 4.2.11. *Let $(\mathbb{T}, \tau)$ be a two-dimensional $\Bbbk$-torus and let $(\mathbb{T}, \sigma)$ be a $(\mathbb{T}, \tau)$-torsor. Let $(X_\Lambda, \sigma_\Lambda)$ be a $(\mathbb{T}, \tau)$-toric Del Pezzo surface that contains $(\mathbb{T}, \sigma)$ as a $(\mathbb{T}, \tau)$-stable dense open subset. The torsor is trivial if and only if $(X_\Lambda, \sigma_\Lambda)$ has a $\Bbbk$-point.*

Proof. If the torsor is trivial, then $(X_\Lambda, \sigma_\Lambda)$ has a $\Bbbk$-point. Conversely, if the toric Del Pezzo surface $(X_\Lambda, \sigma_\Lambda)$ contains a $\Bbbk$-point (this hypothesis is fulfilled if the degree is 7), then $(X_\Lambda, \sigma_\Lambda)$ is $\Bbbk$-birational to $\mathbb{P}^2_{\Bbbk}$, and the set of $\Bbbk$-points is dense (see [VA13, Theorem 2.1], see also [Pie12, §2.2, §5.2]). Therefore, the torsor $(\mathbb{T}, \sigma)$ has a $\Bbbk$-point, hence it is trivial. □

Combining Lemma 4.2.10 with an MMP, we get the next result.

Proposition 4.2.12. *Let $(\mathbb{T}, \tau)$ be a two-dimensional $\Bbbk$-torus and let $(\mathbb{T}, \sigma)$ be a $(\mathbb{T}, \tau)$-torsor. Taking a smooth projective compactification, and then applying a $(\mathbb{T}, \sigma)$-equivariant MMP, we obtain a smooth toric Del Pezzo $\Bbbk$-surface X, birational to $(\mathbb{T}, \sigma)$, that contains $(\mathbb{T}, \sigma)$ as a $(\mathbb{T}, \tau)$-stable dense open subset. More precisely, we have the following possibilities for X.*

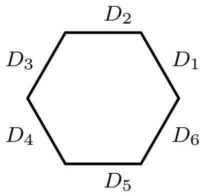
(i) X is a $\Bbbk$-form (of Picard rank one) of $\mathbb{P}^2_{\overline{\Bbbk}}$. In this case, the image of $\hat{\tau}$ is conjugate to $\mathcal{G}_3$, $\mathcal{G}_9$, $\mathcal{G}_{12}$, or to $\langle id\rangle$.

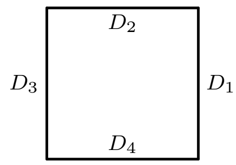
(ii) X is a $\Bbbk$-form of Picard rank one of the Del Pezzo $\overline{\Bbbk}$-surface of degree 6. In this case, the image of $\hat{\tau}$ is conjugate to $\mathcal{G}_1$, $\mathcal{G}_4$, or to $\mathcal{G}_7$.

(iii) X is a $\Bbbk$-form of Picard rank one of $\mathbb{P}^1_{\overline{\Bbbk}} \times \mathbb{P}^1_{\overline{\Bbbk}}$. In this case, the image of $\hat{\tau}$ is conjugate to $\mathcal{G}_6$, $\mathcal{G}_{12}$, $\mathcal{G}_2$, or to $\mathcal{G}_8$; or

(iv) X is a $\Bbbk$-form of Picard rank two of $\mathbb{P}^1_{\overline{\Bbbk}} \times \mathbb{P}^1_{\overline{\Bbbk}}$ (if $(\mathbb{T}, \tau)$ is reducible). In this case, the image of $\hat{\tau}$ is conjugate to $\mathcal{G}_{10}$, $\mathcal{G}_5$, $\mathcal{G}_{11}$, or to $\langle id\rangle$.

Proof. By Lemma 4.2.10, a compactification of $(\mathbb{T},\sigma)$ is a $(\mathbb{T},\tau)$-toric Del Pezzo surface $(X_\Lambda,\sigma_\Lambda)$. We can assume that $X_\Lambda = \mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$ or $X_\Lambda = \mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ (see the Proof of Lemma 4.2.10). Recall that to a ray v_i of a fan of a toric variety corresponds a toric divisor D_i. The toric divisors associated to $X_\Lambda = \mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$ and to $X_\Lambda = \mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ are respectively:





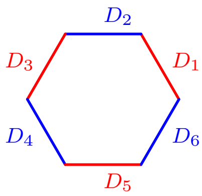
By construction, $\mathbb{T} = X_\Lambda\backslash\cup D_i$ and $(\mathbb{T},\sigma) = (X_\Lambda\backslash\cup D_i,\sigma_\Lambda)$ [Man86, Theorem 30.3.1 p166] [Man66, Theorem 3.10 p108].

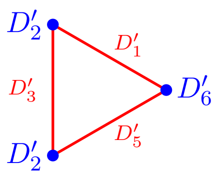
Recall that $\mathbb{P}^2_{\Bbbk}$ has Picard rank one. Therefore, a $\Bbbk$-form of $\mathbb{P}^2_{\Bbbk}$ has also Picard rank one (see Example D.0.7) Then, recall that $\mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$ has Picard rank four and that the Picard group is generated by three disjoint toric divisors and a line that does not intersect any of the six toric divisors. Therefore, a $\Bbbk$-form of $\mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$ has Picard rank between one and four. Finally, recall that $\mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ has Picard rank two and that the Picard group is generated by two intersecting toric divisors. Therefore, a $\Bbbk$-form of $\mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ has Picard rank one or two.

We apply a Galois-equivariant MMP to the $\Bbbk$-surface $(X_\Lambda,\sigma_\Lambda)$. This algorithm consists of contracting (-1)-curves in a way compatible with the Galois-action.

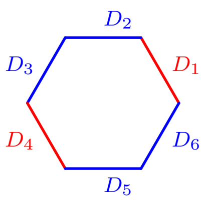
Case 1: $X_\Lambda = \mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$, it has six (-1)-curves.

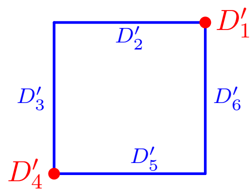
- If the image of $\hat{\tau}$ is conjugate to $\langle x,s\rangle$, $\langle x^2,-s\rangle$, or to $\langle x\rangle$ (the acting torus is irreducible), the six (-1)-curves form a unique Galois-orbit, hence we obtain a minimal surface that is a $\Bbbk$-form of $\mathrm{Bl}_3(\mathbb{P}^2_{\Bbbk})$ of Picard rank one.
- If the image of $\hat{\tau}$ is conjugate to $\langle x^2,s\rangle$ or to $\langle x^2\rangle$ (the acting torus is irreducible), the six (-1)-curves form two Galois-orbits. We can contract one of these two orbits and we obtain a minimal surface that is a $\Bbbk$-form of $\mathbb{P}^2_{\Bbbk}$ (of Picard rank one) where the (1)-curves form a single Γ-orbits.



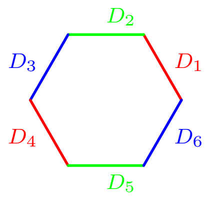


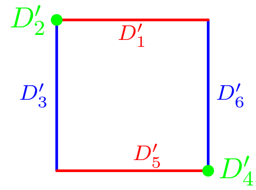
- If the image of $\hat{\tau}$ is conjugate to $\langle s,-s\rangle$ (the acting torus is irreducible), the six (-1)-curves form two Galois-orbits. We can contract the orbit consisting of two (-1)-curves and we obtain a minimal surface that is a $\Bbbk$-form of $\mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ where the (0)-curves form a single Γ-orbit, hence it is of Picard rank one (the Picard group is generated by $D'_2+D'_6+D'_5+D'_3$).



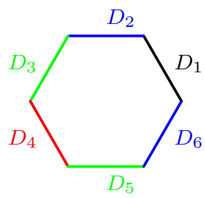


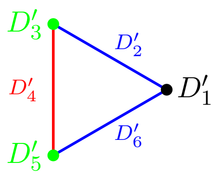
- If the image of $\hat{\tau}$ is conjugate to $\langle x^3\rangle$ (the acting torus is reducible), the six (-1)-curves form three Galois-orbits. We can contract the orbit consisting of two (-1)-curves and we obtain a minimal surface that is a $\Bbbk$-form of $\mathbb{P}^1_{\Bbbk}\times\mathbb{P}^1_{\Bbbk}$ of Picard rank two where the (0)-curves have two Γ-orbits (the Picard group is generated by $D'_1 + D'_5$ and by $D'_6 + D'_3$).

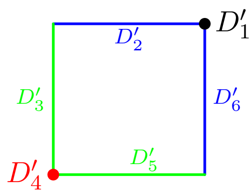




- If the image of $\hat{\tau}$ is conjugate to $\langle s \rangle$ (the acting torus is irreducible), the six (-1)-curves form four Galois-orbits. We can contract an orbit consisting of two (-1)-curves, then we contract the invariant (-1)-curve and we obtain the minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^2_{\mathbb{k}}$ (of Picard rank one) where the (1)-curves have two Γ-orbits. The other possibility is to contract a Γ-invariant (-1)-curve. Then we can contract the pair of (-1)-curves or the invariant (-1)-curve. We obtain respectively a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank one (the Picard group is generated by $D'_3 + D'_5$) where the (0)-curves have two Γ-orbits, or a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^2_{\mathbb{k}}$ where the (1)-curves have two Γ-orbits.



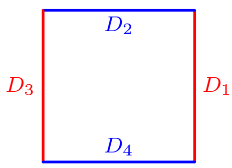




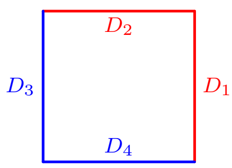
- If the image of $\hat{\tau}$ is conjugate to $\langle id \rangle$ (the acting torus is reducible), the six (-1)-curves form six Galois-orbit, hence we can contract and we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^2_{\mathbb{k}}$ (of Picard rank one) where the (1)-curves have three Γ-orbits.

Case 2: $X_\Lambda = \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$, it contains no (-1)-curves. Hence we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$.

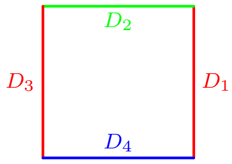
- If the image of $\hat{\tau}$ is conjugate to $\langle d, s \rangle$, $\langle s, -s \rangle$, or to $\langle ds \rangle$ (the acting torus is irreducible), the four (0)-curves form a unique Galois-orbit. Therefore, we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank one.
- If the image of $\hat{\tau}$ is conjugate to $\langle d, -d \rangle$ or to $\langle x^3 \rangle$ (the acting torus is reducible), the four (0)-curves form two Galois-orbits. We obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank two (the Picard group is generated by $D_2 + D_4$ and by $D_1 + D_3$).



- If the image of $\hat{\tau}$ is conjugate to $\langle s \rangle$ (the acting torus is irreducible), the four (0)-curves form two Galois-orbits. Therefore, we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank one (the Picard group is generated by $D_2 + D_1$).



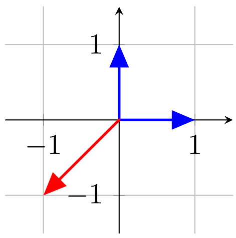
- If the image of $\hat{\tau}$ is conjugate to $\langle d \rangle$ (the acting torus is reducible), the four (0)-curves form three Galois-orbits. Therefore, we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank two (the Picard group is generated by D_2 and by $D_1 + D_3$).

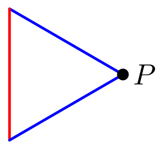


- If the image of $\hat{\tau}$ is conjugate to $\langle id \rangle$ (the acting torus is reducible), the four (0)-curves form four Galois-orbit. Therefore, we obtain a minimal surface that is a $\mathbb{k}$-form of $\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}}$ of Picard rank two.

□

Example 4.2.13 (Another proof of $\mathrm{H}^1(\mathcal{G}_{12}, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})) = \{1\}$)**.** Let $(\mathbb{T}, \tau)$ be a $\mathbb{k}$-torus such that the image of the Γ-representation $\hat{\tau}$ on N is $\mathcal{G}_{12} = \langle s \rangle$. For instance $\overline{\mathbb{k}} = \mathbb{C}$ and $\mathbb{k} = \mathbb{R}$ and $(\mathbb{T}, \tau)$ is the Weil Restriction $\mathbb{R}$-torus. Note that, by Proposition 2.3.17, we can assume that $\mathbb{T} = \mathrm{Spec}(\mathbb{k}'[M])$, where $\mathbb{k}'/\mathbb{k}$ is a finite Galois extension of Galois group isomorphic to $\mathcal{G}_{12}$. Let $(\mathbb{T}, \sigma)$ be a $(\mathbb{T}, \tau)$-torsor. We will prove that this torsor is trivial. The fan of the toric Del Pezzo surface $\mathbb{P}^2_{\mathbb{k}'}$ is Galois-stable, therefore, by Proposition 2.4.9, $(\mathbb{T}, \sigma)$ is a dense $(\mathbb{T}, \tau)$-stable open subset of a Severi Brauer $\mathbb{k}$-surface $(\mathbb{P}^2_{\mathbb{k}'}, \sigma')$. The Galois action on the fan exchanges two rays and fixes the other one; therefore the Galois action on the toric divisors exchanges two toric divisors and fixes a point P. Thus, $(\mathbb{P}^2_{\mathbb{k}'}, \sigma')$ contains a $\mathbb{k}$-point P. A Severi Brauer suface with a $\mathbb{k}$-point is isomorphic to $\mathbb{P}^2_{\mathbb{k}}$. Therefore, $(\mathbb{T}, \sigma)$ is a trivial $(\mathbb{T}, \tau)$-torsor since it contains $\mathbb{k}$-points, and $\mathrm{H}^1(\mathcal{G}_{12}, \mathrm{Aut}^{\mathbb{T}}(\mathbb{T})) = \{1\}$.





Example 4.2.14 (See [Har77, Chapter V, Example 4.2.3])**.** Let $\mathscr{D}_{12} := \langle x, s \rangle$. Recall that we have an isomorphism $\mathscr{D}_{12} \cong \mathscr{S}_3 \times \mathscr{C}_2$, that identifies for instance x with $((1,2,3), -id)$, and s with $((2,3), id)$. Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension with Galois group $\Gamma = \langle \gamma, \gamma' \rangle \cong \mathscr{D}_{12}$. Let $(\mathbb{G}^2_{m,\mathbb{k}'}, \tau)$ be the $\mathbb{k}$-torus such that

$$\hat{\tau}_\gamma = x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\tau}_{\gamma'} = s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider the surface X of $\mathbb{P}^2_{\mathbb{k}'} \times \mathbb{P}^2_{\mathbb{k}'}$ defined by $x_i y_i = x_j y_j$, $i \in \{1,2,3\}$, where

$$\mathbb{P}^2_{\mathbb{k}'} \times \mathbb{P}^2_{\mathbb{k}'} = \{([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]) \mid x_i, y_j \in \mathbb{k}'\}.$$

It is the graph of the standard quadratic birational Cremona transformation. This defines a Del Pezzo $\mathbb{k}'$-surface of degree six. The six (-1)-curves are the elements of the orbit of the line $([0:0:1], [y_1 : y_2 : 0])$ by the action of $\mathscr{D}_{12}$. The $\mathbb{k}$-torus $(\mathbb{G}^2_{m,\mathbb{k}'}, \tau)$ acts on (X, σ), where σ is the $\mathbb{k}$-structure on X defined in the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \\ \tau_\gamma \times \sigma_\gamma \downarrow & & \downarrow \sigma_\gamma \\ \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \\ \tau_{\gamma'} \times \sigma_{\gamma'} \downarrow & & \downarrow \sigma_{\gamma'} \\ \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \end{array}$$

Indeed, we have

$$\begin{array}{ccc} ((s,t),([x_1:x_2:x_3],[y_1:y_2:y_3])) & \longmapsto & ([x_1:sx_2:tx_3],[y_1:s^{-1}y_2:t^{-1}y_3]) \\ \downarrow & & \downarrow \\ ((\gamma(t^{-1}s),\gamma(s)),([\gamma(y_2):\gamma(y_3):\gamma(y_1)],[\gamma(x_2):\gamma(x_3):\gamma(x_1)]) & \longmapsto & ([\gamma(s^{-1}y_2):\gamma(t^{-1}y_3):\gamma(y_1)],[\gamma(sx_2):\gamma(tx_3):\gamma(x_1)]) \end{array}$$

$$\begin{array}{ccc} ((s,t),([x_1:x_2:x_3],[y_1:y_2:y_3])) & \longmapsto & ([x_1:sx_2:tx_3],[y_1:s^{-1}y_2:t^{-1}y_3]) \\ \downarrow & & \downarrow \\ ((\gamma'(t),\gamma'(s)),([\gamma'(x_1):\gamma'(x_3):\gamma'(x_2)],[\gamma'(y_1):\gamma'(y_3):\gamma'(y_2)]) & \longmapsto & ([\gamma'(x_1):\gamma'(tx_3):\gamma'(sx_2)],[\gamma'(y_1):\gamma'(t^{-1}y_3):\gamma'(s^{-1}y_2)]) \end{array}$$

Then, (X,σ) is a toric Del Pezzo $\Bbbk$-surface. Note that the open orbit is a trivial $(\mathbb{G}_{m,\Bbbk'}^2,\tau)$-torsor since it contains the invariant point $([1:1:1],[1:1:1])$.

Example 4.2.15 (See [Har77, Chapter V, Example 4.2.3])**.** Let $\mathscr{D}_6 := \langle x^2, s\rangle$. Recall that we have an isomorphism $\mathscr{D}_6 \cong \mathscr{S}_3$, that identifies for instance x^2 with $(1,2,3)$, and s with $(2,3)$. Let $\Bbbk'/\Bbbk$ be a Galois extension with Galois group $\Gamma = \langle \gamma, \gamma'\rangle \cong \mathscr{D}_6$. Let $(\mathbb{G}_{m,\Bbbk'}^2,\tau)$ be the $\Bbbk$-torus such that

$$\tilde{\tau}_\gamma = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\tau}_{\gamma'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider the surface X of $\mathbb{P}_{\Bbbk'}^2 \times \mathbb{P}_{\Bbbk'}^2$ defined by $x_iy_i = x_jy_j$, $i \in \{1,2,3\}$, where

$$\mathbb{P}_{\Bbbk'}^2 = \{([x_1:x_2:x_3],[y_1:y_2:y_3]) \mid x_i, y_j \in \Bbbk'\}.$$

This defines a Del Pezzo $\Bbbk'$-surface of degree six. The $\Bbbk$-torus $(\mathbb{G}_{m,\Bbbk'}^2,\tau)$ acts on (X,σ), where σ is the $\Bbbk$-structure on X defined in the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \\ \tau_\gamma \times \sigma_\gamma \downarrow & & \downarrow \sigma_\gamma \\ \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \\ \tau_{\gamma'} \times \sigma_{\gamma'} \downarrow & & \downarrow \sigma_{\gamma'} \\ \mathbb{G}_{m,\Bbbk'}^2 \times X & \xrightarrow{\mu} & X \end{array}$$

Indeed, we have

$$\begin{array}{ccc} ((s,t),([x_1:x_2:x_3],[y_1:y_2:y_3])) & \longmapsto & ([x_1:sx_2:tx_3],[y_1:s^{-1}y_2:t^{-1}y_3]) \\ \downarrow & & \downarrow \\ ((\gamma(ts^{-1}),\gamma(s)^{-1}),([\gamma(x_2):\gamma(x_3):\gamma(x_1)],[\gamma(y_2):\gamma(y_3):\gamma(y_1)]) & \longmapsto & ([\gamma(sx_2):\gamma(tx_3):\gamma(x_1)],[\gamma(s^{-1}y_2):\gamma(t^{-1}y_3):\gamma(y_1)]) \end{array}$$

$$\begin{array}{ccc} ((s,t),([x_1:x_2:x_3],[y_1:y_2:y_3])) & \longmapsto & ([x_1:sx_2:tx_3],[y_1:s^{-1}y_2:t^{-1}y_3]) \\ \downarrow & & \downarrow \\ ((\gamma'(t),\gamma'(s)),([\gamma'(x_1):\gamma'(x_3):\gamma'(x_2)],[\gamma'(y_1):\gamma'(y_3):\gamma'(y_2)]) & \longmapsto & ([\gamma'(x_1):\gamma'(tx_3):\gamma'(sx_2)],[\gamma'(y_1):\gamma'(t^{-1}y_3):\gamma'(s^{-1}y_2)]) \end{array}$$

Then, (X,σ) is a toric Del Pezzo $\Bbbk$-surface. Note that the open orbit is a trivial $(\mathbb{G}_{m,\Bbbk'}^2,\tau)$-torsor since it contains the invariant point $([1:1:1],[1:1:1])$.

Chapter 5

Some open questions

In this short chapter, we list some lines of study that should be explored further.

5.1 Altman-Hausen description of normal varieties endowed with an effective torus action

In Chapter 3, we gave a complete description of normal affine $\Bbbk$-varieties endowed with an effective $\Bbbk$-torus action, where $\Bbbk$ is a field of characteristic zero. In this section, we highlight some cases to explore.

5.1.1 Toric downgrading using Weil restriction

To determine an AH-datum of a normal affine $\Bbbk$-variety X endowed with an effective $\Bbbk$-torus action of T, we used a method called *toric downgrading* (see Proposition 3.2.6). We also used this method to extend the Altman-Hausen description to arbitrary fields of characteristic zero in Section 3.2.2. Another possibility is based on the properties of Weil restriction (see in Proposition A.7.5). Let X be a normal affine $\Bbbk$-variety endowed with an effective $\Bbbk$-torus action μ of T. Let $\Bbbk'/\Bbbk$ be a finite Galois extension that splits T, i.e. $T_{\Bbbk'} \cong \mathbb{G}_{m,\Bbbk'}^n$ for some $n \in \mathbb{N}^*$. Then, there are closed immersions

$$X \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right) \quad \text{and} \quad T \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(T_{\Bbbk'}\right),$$

and the following diagram is commutative

$$\begin{array}{ccc} T \times_{\mathrm{Spec}(\Bbbk)} X & \xrightarrow{\quad \mu \quad} & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{R}_{\Bbbk'/\Bbbk}(T_{\Bbbk'}) \times_{\mathrm{Spec}(\Bbbk)} \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'}) & \xrightarrow{\mathrm{R}_{\Bbbk'/\Bbbk}(\mu)} & \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'}). \end{array}$$

Question 1. From the Altmann-Hausen presentation for quasi-split tori combined with toric downgrading via $X \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right)$ and $T \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(\mathbb{G}_{m,\Bbbk'}^n\right)$, is it possible to obtain the Altmann-Hausen presentation of the T-action on X ?

Note that, in the AH-datum of the $\mathrm{R}_{\Bbbk'/\Bbbk}\left(T_{\Bbbk'}\right)$-action on $\mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right)$, we can take $h = 1$ since there are no non-trivial torsors. However, this is not necessarily the case in the AH-datum for the T-action on X. So, how to recover this datum from this embedding ?

5.1.2 Altman-Hausen description of normal varieties

In [AHS08], the Altmann-Hausen presentation of normal affine varieties endowed with a torus action over an algebraically closed field of characteristic zero was extended to arbitrary normal varieties by gluing affine torus actions via divisorial fans. We expect that this presentation will extend to arbitrary torus actions over characteristic zero fields by gluing the presentation constructed in this thesis in Chapter 3.

Question 2. Is it possible to have an Altmann-Hausen presentation of arbitrary varieties over characteristic zero fields endowed with an effective torus action ? In particular, Is it possible to have an Altmann-Hausen presentation of projective varieties over characteristic zero fields endowed with an effective torus action ?

Application. Determine a classification of Fano varieties over characteristic zero fields. Indeed, in [HHS11, §4], the AH-presentation is used to study some three-dimensional Fano varieties over an algebraically closed field of characteristic zero.

5.1.3 Altman-Hausen description in positive characteristic

The main results of this thesis are based on the Altmann-Hausen presentation of affine varieties endowed with an effective torus action over an algebraically closed field of characteristic zero [AH06]. Indeed, we extend this presentation over arbitrary fields of characteristic zero using Galois descent tools (see Section 3.2.2). If the Altmann-Hausen presentation is valid for affine varieties endowed with an effective split torus action over an arbitrary field of any characteristic, we expect that we can extend it using Galois descent tools to affine varieties endowed with any effective torus action (by Remark 3.1.18, in the context of toric varieties endowed with a subtorus action, this extension is possible). Therefore, we may investigate in this way. A key observation for this generalization is the following one. In the proof of Theorem 3.1.12, the normalization of a variety is used to construct the Altmann-Hausen quotient. In the context of this thesis, the Altmann-Hausen quotient is a variety since the normalization morphism is finite. However, it is not necessary the case in this general setting.

Question 3. Is there an Altmann-Hausen presentation of normal affine varieties endowed with an effective torus action in positive characteristic ?

Question 4. More generally, does there exist an Altmann-Hausen presentation of normal affine varieties endowed with an effective torus action over rings ? In this context, how to define an AH-datum, and more precisely, what are the properties of the *AH-quotient* Y ?

5.2 Study of the infiniteness of $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X))$

Let $\Bbbk$ be a characteristic zero field, and let $\Bbbk'/\Bbbk$ be a finite Galois extension (see Proposition 2.3.17).

Let $\mathbb{T}$ be a $\Bbbk'$-torus and let τ be an $\Bbbk$-group structure on $\mathbb{T}$. Assume there exists a $(\mathbb{T}, \tau)$-variety denoted (X, σ). This means that the following diagram commutes for all $\gamma \in \Gamma$ (see Definition 1.2.11)

$$\begin{array}{ccc} \mathbb{T} \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \tau_\gamma \times \sigma_\gamma}\downarrow & & \downarrow{\scriptstyle \sigma_\gamma} \\ \mathbb{T} \times X & \xrightarrow{\mu} & X. \end{array}$$

If the complexity of the $(\mathbb{T}, \tau)$-action on (X, σ) is zero, then there exists a finite number of $\Bbbk$-forms of X in the category of $(\mathbb{T}, \tau)$-varieties (see Proposition 2.4.13).

Let $c \in \mathbb{N}^*$.

Question 5. Does there exist a $(\mathbb{T}, \tau)$-variety (X, σ) of complexity c such that the set $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X))$ is infinite ?

If $c \geq 2$ and if $\Bbbk' = \mathbb{C}$ and $\Bbbk = \mathbb{R}$, we have the next result.

Proposition 5.2.1. *Let $(\mathbb{T}, \tau)$ be an $\mathbb{R}$-torus and let $c \geq 2, c \in \mathbb{N}$. There exists a $(\mathbb{T}, \tau)$-variety (X, σ) such that the complexity of the $(\mathbb{T}, \tau)$-action on (X, σ) is c and such that the set*

$$\mathrm{H}^1\left(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X)\right)$$

is infinite.

Proof. Let $X = \mathbb{T} \times Z$, where Z is a complex variety of dimension ≥ 2 having infinitely many $\mathbb{R}$-forms pairwise non-isomorphic (see [Bot21] and [DOY20]). Note that if $\varphi \in \mathrm{Aut}^{\mathbb{T}}(X)$, then there exists $\varphi_{\mathbb{T}} : X \to \mathbb{T}$ and $\varphi_Z \in \mathrm{Aut}(Z)$ such that $\varphi = \varphi_{\mathbb{T}} \times \varphi_Z$. Therefore, non-equivalent $\mathbb{R}$-structures σ and σ' on Z gives non-equivalent $\mathbb{R}$-structures $\tau \times \sigma$ and $\tau \times \sigma'$ on X. □

Therefore, Question 5 in the $\mathbb{C}/\mathbb{R}$ setting becomes:

Question 6. Does there exist a $(\mathbb{T}, \tau)$-variety (X, σ) over $\mathbb{R}$ of complexity 1 such that the set $\mathrm{H}^1(\Gamma, \mathrm{Aut}^{\mathbb{T}}(X))$ is infinite ?

5.3 Three-dimensional tori

In Section 4.2, we gave a classification of two-dimensional $\Bbbk$-tori based on the classification of conjugacy classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$. There are 13 conjugacy classes of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$. From this classification, Voskresenskiĭ classified two-dimensional tori in [Vos65], and he showed that all two-dimensional tori are rational in [Vos67].

There are 73 conjugacy classes of finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$ (see [Tah71]). Algebraic $\Bbbk$-tori of dimension 3 are classified up to birational equivalence by [Kun87], and he found non-rational $\Bbbk$-tori. In [Lem15], the case of four-dimensional tori is studied, and more exactly some rationality properties. There are 710 conjugacy classes of finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$.

Question 7. How does the classification of three-dimensional tori may simplify the Altmann-Hausen presentation ?

In particular, we may look for a result similar to the one of [ELFST14] (see Theorem 4.2.7).

5.4 Presentation of G-varieties of complexity one

Another direction to explore, which extends the case of torus actions, is the following one: we consider reductive algebraic groups actions instead of torus actions.

Let G be a reductive algebraic group, and let X be a variety endowed with a G-action. The Altmann-Hausen presentation is no longer valid, but there is a well-established combinatorial description for algebraic varieties of complexity ≤ 1 over an algebraically closed field (see [LV83, Kno91, Tim11]). So, it is natural to ask if such a description can be generalized over an arbitrary perfect base field.

The complexity-zero case, corresponding to spherical varieties, was extended recently to arbitrary perfect base fields (see [Hur11, Wed18, MJT21, BG21]).

Question 8. How to extend this combinatorial description of complexity-one varieties to an arbitrary perfect base field ?

5.5 Forms of affine space

It is well known that, up to isomorphism, the only $\mathbb{R}$-form of $\mathbb{A}_{\mathbb{C}}$ is $\mathbb{A}_{\mathbb{R}}$, (see for instance [Ben16, Théorème 4.5]). The case of the affine plane was studied in [Kam75].

Theorem 5.5.1 ([Kam75]). *Let $\mathbb{k}'/\mathbb{k}$ be a finite separable field extension. Then, up to isomorphism, $\mathbb{A}^2_{\mathbb{k}}$ is the only $\mathbb{k}$-form of $\mathbb{A}^2_{\mathbb{k}'}$. In particular, any $\mathbb{R}$-structure on $\mathbb{A}^2_{\mathbb{C}}$ is equivalent to $\sigma_0 : (x, y) \mapsto (\bar{x}, \bar{y})$).*

For the affine space, there is the next conjecture.

Conjecture 5.5.2. *Up to isomorphism, $\mathbb{A}^3_{\mathbb{R}}$ is the only $\mathbb{R}$-form of $\mathbb{A}^3_{\mathbb{C}}$*

To test this conjecture, we can study a more rigid class of varieties, namely we can study varieties $(\mathbb{A}^3_{\mathbb{C}}, \sigma)$ endowed with an affective $\mathbb{R}$-torus action using the generalized Altmann-Hausen presentation. More generally, we get the following question.

Question 9. Let $\mathbb{k}'/\mathbb{k}$ be a finite separable field extension. How to use the generalized Altmann-Hausen presentation to study the $\mathbb{k}$-forms of $\mathbb{A}^n_{\mathbb{k}'}$?

Appendices

Appendix A

Scheme theory

A.1 Functor of points

Given a scheme X, we can define a contravariant functor from the category of schemes to the category of sets, called the functor of points of X (See [Sta, Section 001L]):

$$\begin{aligned} h_X : \mathrm{Sch}^{op} &\to \mathrm{Sets} \\ T &\mapsto X(T) := \mathrm{Hom}(T, X) \\ (\phi : T' \to T) &\mapsto \left(X(\phi) : X(T) \to X(T'); f \mapsto f \circ \phi\right). \end{aligned}$$

By Yoneda lemma, the functor $h : X \mapsto h_X$ is fully faithful. Therefore, to give a scheme morphism $f : X \to Y$ is equivalent to give for every scheme T a map of sets $g_T : X(T) \to Y(T)$ such that for any scheme morphism $\phi : T' \to T$, the following diagram commutes

$$\begin{array}{ccc} X(T) & \xrightarrow{g_T} & Y(T) \\ {\scriptstyle X(\phi)}\downarrow & & \downarrow{\scriptstyle Y(\phi)} \\ X(T') & \xrightarrow{g_{T'}} & Y(T'). \end{array}$$

Definition A.1.1 (See [Sta, Section 01JF])**.** Let $F : \mathrm{Sch}^{op} \to \mathrm{Sets}$ be a contravariant functor. We say that F is *representable* if there exists a scheme X such that $h_X \cong F$.

If $X = \mathrm{Spec}(A)$ is an affine $\Bbbk$-scheme, then we consider the following functor instead of h_X

$$\begin{aligned} \mathrm{Alg}_{\Bbbk} &\to \mathrm{Sets} \\ R &\mapsto \mathrm{Hom}(R, A). \end{aligned}$$

Example A.1.2. Let $\Bbbk$ be a field. The functor from the category of $\Bbbk$-algebras to the category of sets

$$\begin{aligned} \mathrm{Alg}_{\Bbbk} &\to \mathrm{Sets} \\ R &\mapsto R \end{aligned}$$

is representable by the affine scheme $\mathbb{A}_{\Bbbk} := \mathrm{Spec}(\Bbbk[\mathbb{N}])$.

A.2 Group schemes

A reference for the group scheme theory is [GP11]. In this section, S is a scheme. In this thesis, we often have $S = \mathrm{Spec}(\Bbbk)$ for some field $\Bbbk$.

Definition A.2.1 (See [Sta, §022R])**.** An S-group scheme is a $\Bbbk$-scheme G together with three $\Bbbk$-morphisms $m : G \times_{\mathrm{Spec}(\Bbbk)} G \to G$, $e : \mathrm{Spec}(\Bbbk) \to G$, $s : G \to G$ such that the following diagrams are commutative:

(i) Associativity

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m \times id} & G \times_S G \\ {\scriptstyle id \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{m} & G \end{array}$$

(ii) Left and right inverse (where Δ is the diagonal morphism)

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G \times_S G & \xrightarrow{s \times id} & G \times_S G \\ \downarrow & & & & \downarrow{\scriptstyle m} \\ S & & \xrightarrow{e} & & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G \times_S G & \xrightarrow{id \times s} & G \times_S G \\ \downarrow & & & & \downarrow{\scriptstyle m} \\ S & & \xrightarrow{e} & & G \end{array}$$

(iii) Left and right neutral element

$$\begin{array}{ccc} S \times_S G & \xrightarrow{e \times id} & G \times_S G \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{id} & G \end{array} \qquad \begin{array}{ccc} G \times_S S & \xrightarrow{id \times e} & G \times_S G \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{id} & G. \end{array}$$

Let G be an S-group scheme, and let T be an S-scheme. Then, the set $G(T) := \mathrm{Hom}_S(T, G)$ is an abstract group such that for any S-morphism $\phi : T' \to T$, the induced map of sets $G(\phi) : G(T) \to G(T')$ is a group homomorphism. Therefore, the functor of points of an S-group scheme is the contravariant functor from the category of schemes to the category of abstract groups

$$\begin{aligned} h_G : \mathrm{Sch}^{op} &\to \mathrm{Grp} \\ T &\mapsto X(T) := \mathrm{Hom}(T, X). \end{aligned}$$

Conversely, if G is an S-scheme such that for every S-scheme T, $G(T)$ is an abstract group, and for every S-morphism $\phi : T' \to T$, the induced map of sets $G(\phi) : G(T) \to G(T')$ is a group homomorphism, then G is an S-group scheme.

Let $\Bbbk$ be a field, let $S = \mathrm{Spec}(\Bbbk)$, and let $G = \mathrm{Spec}(A)$ be an affine $\Bbbk$-scheme. Then G is a $\Bbbk$-group scheme if and only if A is a Hopf algebra over $\Bbbk$. The multiplication $G \times G \to G$ corresponds to the coproduct $A \otimes_{\Bbbk} A \to A$, the unity $e : \mathrm{Spec}(\Bbbk) \to G$ corresponds to the counity $A \to \Bbbk$, and the inverse $s : G \to G$ corresponds to the antipode $A \to A$.

Example A.2.2. Let $\Bbbk$ be a field. The functor

$$\begin{aligned} \mathrm{Alg}_{\Bbbk} &\to \mathrm{Grp} \\ R &\mapsto R^*. \end{aligned}$$

is representable by $\mathbb{G}_{m,\Bbbk} := \mathrm{Spec}(\Bbbk[\mathbb{Z}])$.

Example A.2.3. Consider the multiplicative group $\mathbb{G}_{m,\Bbbk} := \mathrm{Spec}(\Bbbk[t^{\pm 1}])$. The coproduct is defined by $t \mapsto t \otimes t$, the counity is defined by $t \mapsto 1$, and the antipode is defined by $t \mapsto t^{-1}$.

Example A.2.4. Consider the additive group $\mathbb{G}_{a,\Bbbk} := \mathrm{Spec}(\Bbbk[t])$. The coproduct is defined by $t \mapsto t \otimes 1 + 1 \otimes t$, the counity is defined by $t \mapsto 0$, and the antipode is defined by $t \mapsto -t$.

Definition A.2.5. Let G and G' be two S-group schemes. An S-morphism $\varphi : G \to G'$ is an *S-group morphism* if the following diagram commutes

$$\begin{array}{ccc} G \times_S G & \xrightarrow{\mu} & G \\ {\scriptstyle \varphi \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G' \times_S G' & \xrightarrow{\mu'} & G' \end{array}$$

In other words, for any S-scheme T, the induced map of sets $\varphi(T) : G(T) \to G'(T)$ is a group morphism such that for any S-scheme morphism $\phi : T' \to T$ the following diagram commutes

$$\begin{array}{ccc} G(T) & \xrightarrow{\varphi(T)} & G'(T) \\ {\scriptstyle G(\phi)}\downarrow & & \downarrow{\scriptstyle G'(\phi)} \\ G(T') & \xrightarrow{\varphi(T')} & G'(T') \end{array}$$

A.3 Group schemes action

A reference for the group scheme theory is [GP11]. This section is inspired from [Dub04, §A.2].

Definition A.3.1. An S-group scheme action on an S-scheme X is an S-scheme morphism $\mu : G \times_S X \to X$ such that the following diagrams commute

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{id \times \mu} & G \times_S X \\ {\scriptstyle m \times id}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times_S X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} S \times_S X & \xrightarrow{e \times id} & G \times_S X \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle id} \\ X & \xrightarrow{\mu} & X. \end{array}$$

An equivalent definition is to give, for any S-scheme T, an action of the abstract group $G(T)$ on the set $X(T)$ such that for any S-morphism $\phi : T' \to T$ the induced map of sets $X(\phi) : X(T) \to X(T')$, and the induced group morphism $G(\phi) : G(T) \to G(T')$, satisfy the following commutative diagram

$$\begin{array}{ccc} G(T) \times_{S(T)} X(T) & \xrightarrow{\mu(T)} & X(T) \\ {\scriptstyle G(\phi) \times X(\phi)}\downarrow & & \downarrow{\scriptstyle X(\phi)} \\ G(T') \times_{S(T')} X(T') & \xrightarrow{\mu(T')} & X(T') \end{array}$$

Definition A.3.2. Let G and G' be two S-group schemes, and let $\varphi : G \to G'$ be an S-group scheme morphism. Let X and X' be two S-schemes endowed respectively with an action of G and G'

$$\mu : G \times_S X \to X, \quad \text{and} \quad \mu' : G' \times_S X' \to X'.$$

An S-scheme morphism $\phi : X \to X'$ is φ-equivariant if the following diagram commutes

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\mu} & X \\ {\scriptstyle \varphi \times \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ G' \times_S X' & \xrightarrow{\mu'} & X' \end{array}$$

Furthermore, if $G = G'$, the morphism $\phi : X \to X'$ is G-equivariant if it is id_G-equivariant.

In this thesis, we are led to study torsors for $\Bbbk$-tori. In the next definition, we see that the notion of torsor is more general.

Definition A.3.3. (see [Sta, Section 0497]) Let G be an S-group scheme. An S-scheme X endowed with a G-action $\mu : G \times_S X \to X$ is called a *pseudo G-torsor* if the induced morphism $\mu \times pr_2 : G \times_S X \to X \times_S X$ is an isomorphism of S-schemes (pr_2 is the canonical projection on the second factor). A pseudo G-torsor X is *trivial* if there exists a G-equivariant isomorphism $G \to X$ over S, where G acts on G by left multiplication. A pseudo G-torsor is a *G-torsor* if there exists an fpqc covering $\{S_i \to S\}$ of S such that $X \times_S S_i$ is a trivial pseudo $G \times_S S_i$-torsor.

A G-torsor over a field is trivial if and only if it contains a $\Bbbk$-point. Indeed, the condition is necessary since $G(\Bbbk) \neq 0$. Conversely, if $y \in Y(\Bbbk)$, then G is isomorphic to Y via $g \mapsto \mu(g, y)$.

Example A.3.4. Let $\Bbbk$ be an algebraically closed field. Then, any pseudo G-torsor is trivial. Let $\Bbbk$ be a perfect field. Then any pseudo G-torsor is a G-torsor since $\mathrm{Spec}(\overline{\Bbbk}) \to \mathrm{Spec}(\Bbbk)$ is an fpqc covering.

A.4 Finite group actions on schemes

Let $\Bbbk$ be a field, and let $\Bbbk'$ be a non-necessarily finite Galois extension of $\Bbbk$. Recall that quasi-projective $\Bbbk$-varieties correspond to quasi-projective $\Bbbk'$-varieties endowed with a $\Bbbk$-structure (see Theorem 1.4.11). A $\Bbbk$-structure on a $\Bbbk'$-variety X induces a right action of $\mathrm{Gal}(\Bbbk'/\Bbbk)$ by $\Bbbk$-automorphisms on X. If X is a quasi-projective $\Bbbk'$-variety endowed with a $\Bbbk$-structure σ, then the corresponding $\Bbbk$-variety is the categorical quotient $X/\mathrm{Gal}(\Bbbk'/\Bbbk)$. In this section, we give the definition and some properties on the quotient of a $\Bbbk$-scheme by a finite group action. This section is based on [GW20, §12.7] and on [SGA03, Exposé V, §1].

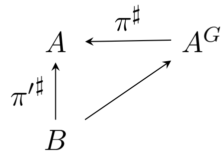
Definition A.4.1. Let X be a $\Bbbk$-scheme, and let G be a group of $\Bbbk$-automorphisms of X. A *categorical quotient* of X by G is a pair (Y, π) such that

- Y is a $\Bbbk$-scheme;
- $\pi : X \to Y$ is a G-invariant $\Bbbk$-morphism (i.e for all $g \in G$, $\pi \circ g = \pi$), and;
- for every G-invariant $\Bbbk$-scheme morphism $\pi' : X \to Z$, there exists a unique $\Bbbk$-morphism $f : Y \to Z$ such that $f \circ \pi = \pi'$.

Clearly, if such a quotient exists, it is unique up to a unique $\Bbbk$-isomorphism. In this case, we write X/G instead of Y. For affine scheme, we can easily obtain this quotient.

Proposition A.4.2. *Let $X = \mathrm{Spec}(A)$ be an affine $\Bbbk$-scheme, and let G be a group of $\Bbbk$-automorphisms of X. Let $A^G := \{a \in A \mid g^\sharp(a) = a \ \forall g \in G\}$ be the ring of invariants. Let $Y := \mathrm{Spec}(A^G)$, and let $\pi : X \to Y$ be the morphism induced by the inclusion $A^G \hookrightarrow A$. Then the pair (Y, π) is a categorical quotient of X by G.*

Proof. Let $Z = \operatorname{Spec}(B)$ be a $\Bbbk$-scheme, and let $\pi' : X \to Z$ be a G-invariant $\Bbbk$-morphism. The corresponding $\Bbbk$-algebra morphism $\pi'^{\sharp} : B \to A$ satisfies $g^{\sharp} \circ \pi'^{\sharp} = \pi'^{\sharp}$. Therefore, for all $b \in B$, $\pi'^{\sharp}(b) \in A^G$, and we get a unique $\Bbbk$-morphism $\pi'^{\sharp} : B \to A^G$ that makes the following diagram commutative



□

We give the existence of such a quotient and its properties in the case where G is a finite group. Note that, if $X = \operatorname{Spec}(A)$ is an affine $\Bbbk$-scheme, then a $\Bbbk$-scheme automorphism of X corresponds to a $\Bbbk$-algebra automorphism of A. To obtain a left action of G on A, we assume that we have a right action of G on X.

Proposition A.4.3 ([SGA03, Exposé V, §1], or [GW20, §12.7])**.** *Let $X = \operatorname{Spec}(A)$ be an affine $\Bbbk$-scheme, and let G be a finite group of $\Bbbk$-automorphisms of X. Let*

$$A^G := \{a \in A \mid g^{\sharp}(a) = a \ \forall g \in G\}$$

be the ring of invariants. Let $Y := \operatorname{Spec}(A^G)$, and let $\pi : X \to Y$ be the S-morphism corresponding to the inclusion $A^G \hookrightarrow A$.

(i) The pair (Y, π) is a categorical quotient of X by G.
(ii) Fibers of π are the orbits of the G-action on X.
(iii) The homomorphism $p^{\sharp} : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \cong (\pi_* \mathcal{O}_X)^G$.
(iv) For every open subset U of Y, the pair $(U, \pi|_{\pi^{-1}(U)})$ is the categorical quotient of $\pi^{-1}(U)$ by G.
(v) If X is integral, then so is Y.
(vi) The morphism π is integral, surjective, and has finite fibers.
(vii) Let X be of finite type over $\Bbbk$. Then, π is finite and $Y = X/G$ is of finite type over $\Bbbk$.

Example A.4.4. Let $\Bbbk'/\Bbbk$ be a finite Galois extension, and let $X = \operatorname{Spec}(A))$, where $A := \Bbbk'[x_1, \ldots, x_n]$. Consider the following left $\Gamma := \operatorname{Gal}(\Bbbk'/\Bbbk)$-action on $\Bbbk'[x_1, \ldots, x_n]$.

$$\Gamma \times A \to A, \quad \left(\gamma, \sum a_{k_1,\ldots,k_n} x_1^{k_1} \ldots x_n^{k_n}\right) \mapsto \sum \gamma(a_{k_1,\ldots,k_n}) x_1^{f_1} \ldots x_n^{k_n}.$$

This action corresponds to a right action of Γ on X. Note that X is affine and of finite type over $\Bbbk$, and that Γ is a finite group acting on X by $\Bbbk$-automorphisms. Then, $X/\Gamma = \operatorname{Spec}(\Bbbk[x_1, \ldots, x_n])$.

The notion of quasi-projective variety often appears in the Galois descent theory because of the following result. A scheme X is called an FA-scheme if every finite subset of X is contained in an affine open subset of X (see [GLL13]). In particular, an FA-scheme is separated. A quasi-projective scheme over an affine scheme is FA. More generally, a scheme admitting an ample invertible sheaf is FA (See [Gro61, II. Corollary 4.5.4]). In particular, we have the next result.

Lemma A.4.5 (See [Liu06, Proposition 3.3.36])**.** *A quasi-projective $\Bbbk$-variety endowed with a finite group action G can be covered by G-invariants affine open subsets.*

Proposition A.4.6 ([SGA03, Corollaire 1.5, Proposition 1.8])**.** *Let X be a $\Bbbk$-scheme and let G be a finite group of $\Bbbk$-automorphisms of X. The categorical quotient X/G exists if and only if X is covered by G-invariants affine open subsets. The quotient X/G is obtained by gluing the quotient of the G-invariants affine open subsets of X. In particular, if X is a quasi-projective $\Bbbk$-scheme, then the quotient X/G exists. Furthermore, X is affine (resp. separated) if and only if Y is affine (resp. separated).*

Proposition A.4.7 (See [Ben16, Proposition 3.9])**.** *Let $\Bbbk'/\Bbbk$ be a finite Galois extension of Galois group Γ, let X be a $\Bbbk$-scheme, and let $X_{\Bbbk'} := X \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk')$. Consider the Γ-action on $X_{\Bbbk'}$ defined by $\gamma \mapsto id \times \mathrm{Spec}(\gamma)$. Then, (X, pr_1) is the categorical quotient of $X_{\Bbbk'}$ by Γ, where $pr_1 : X_{\Bbbk'} \to X$ is the canonical projection.*

Proof. Let $\Bbbk'/\Bbbk$ be a finite Galois extension of Galois group Γ, and let X be a $\Bbbk$-scheme. Since pr_1 is an affine morphism, we can cover $X_{\Bbbk'}$ by Γ-invariant affine open subsets. Indeed, let $U = \mathrm{Spec}(A)$ be an affine open subset of X. Then, $pr_1^{-1}(U) = U \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk') = \mathrm{Spec}(A \otimes_{\Bbbk} \Bbbk')$ is a Γ-invariant affine open subset of $X_{\Bbbk'}$. Therefore, the inverse image of an affine cover of X is a Γ-invariant affine cover of $X_{\Bbbk'}$. By Proposition A.4.6, the quotient of X by Γ exists.

Let $U = \mathrm{Spec}(A)$ be an affine open subset of X. Then, the projection pr_1 corresponds to the $\Bbbk$-morphism

$$A \to A \otimes_{\Bbbk} \Bbbk', \quad a \mapsto a \otimes 1.$$

By Proposition A.4.2; $(U \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk'))/\Gamma = \mathrm{Spec}((A \otimes_{\Bbbk} \Bbbk')^{\Gamma})$. By a classical result (see [Ben16, Lemme 3.8]),

$$(A \otimes_{\Bbbk} \Bbbk')^{\Gamma} = A^{\Gamma} \otimes_{\Bbbk} \Bbbk = A.$$

Therefore, $(U, pr_1|_{pr_1^{-1}(U)})$ is the quotient of $U \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk')$ by Γ. Finally, by Proposition A.4.6, (X, pr_1) is the quotient of $X_{\Bbbk'}$ by Γ. □

Remark A.4.8. Using fpqc descent, one can shows that Proposition A.4.7 is still true if $\Bbbk'/\Bbbk$ is an non-necessarily finite Galois extension (see the proof of Proposition 1.4.8).

A.5 Limits and colimits

The Galois group of an infinite Galois extension is the *limit* of the Galois group of all intermediate finite Galois extension. Since there is a lot of confusions on the vocabulary around the notion of *limits* and *colimits*, we fix here the notations.

Definition A.5.1 ([Sta, Definition 0030])**.** Let $(I, \geq)$ be a preordered set. Let $\mathcal{C}$ be a category.

(i) An *inductive system* over I in $\mathcal{C}$ is given by objects C_i of $\mathcal{C}$, and for every $i \leq i'$ a morphism $f_{i,i'} : C_i \to C_{i'}$ such that $f_{i,i} = id$ and such that $f_{i,i''} = f_{i',i''} \circ f_{i,i'}$ whenever $i \leq i' \leq i''$.

(ii) A *projective system* over I in $\mathcal{C}$ is given by objects C_i of $\mathcal{C}$, and for every $i \leq i'$ a morphism $f_{i',i} : C_{i'} \to C_i$ such that $f_{i,i} = id$ and such that $f_{i'',i} = f_{i',i} \circ f_{i'',i'}$ whenever $i \leq i' \leq i''$.

We denote a system by $(C_i, f_{i,i'})$. The maps $f_{i,i'}$ are called the *transition maps.*

We could take (co)limits of any system over I. However, it is customary to take only colimits of inductive systems over I and only limits of projective system over I. See [Sta, §002Z] for more details.

Definition A.5.2. Let $(I, \leq)$ be a preordered set. We say that an inductive system (resp. a projective system) is a directed inductive system (resp. a directed projective system) if I is a directed set ([Sta, Definition 00D3]): I is non-empty and for all $i_1, i_2 \in I$, there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

Definition A.5.3 ([Sta, Definition 002E])**.** Let $(I, \leq)$ be a preordered set and let $(C_i, f_{i,i'})$ be a projective system. A *limit* of this projective system in $\mathcal{C}$ is an object $\lim_{i\in I} C_i$ in $\mathcal{C}$ together with morphisms $p_i : \lim_{i\in I} C_i \to C_i$ such that

(i) for $i \leq i'$, we have $p_i = f_{i',i} \circ p_{i'}$

(ii) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to C_i$ such that for all $i \leq i'$ we have $q_i = f_{i',i} \circ q_{i'}$, there exists a unique morphism $q : W \to \lim_{i\in I} C_i$ such that $q_i = p_i \circ q$ for all $i \in I$.

Definition A.5.4 ([Sta, Definition 002E])**.** Let $(I, \leq)$ be a preordered set and let $(C_i, f_{i,i'})$ be an inductive system. A *colimit* of this inductive system in $\mathcal{C}$ is an object $\mathrm{colim}_{i\in I} C_i$ in $\mathcal{C}$ together with morphisms $s_i : C_i \to \mathrm{colim}_{i\in I} C_i$ such that

(i) for $i \leq i'$, we have $s_i = s_{i'} \circ f_{i,i'}$

(ii) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : C_i \to W$ such that for all $i \leq i'$ we have $t_i = t_{i'} \circ f_{i,i'}$, there exists a unique morphism $t : \mathrm{colim}_{i\in I} C_i \to W$ such that $t_i = t \circ s_i$ for all $i \in I$.

Limits (resp. colimits), if they exist, are unique up to a unique isomorphism by the uniqueness requirement in these definitions.

Example A.5.5. Let $\Bbbk'/\Bbbk$ be a Galois extension. Then, the set of finite Galois extension $\Bbbk_i/\Bbbk$ in $\Bbbk'$ equipped with the inclusion is a directed inductive system. One can show that $\mathrm{colim}\ \Bbbk_i = \Bbbk'$ (see [Sta, §002U]). If $\Bbbk \subset \Bbbk_i \subset \Bbbk_j$ is a tower of finite Galois extension in $\Bbbk'$, we get a morphism $\mathrm{Spec}(\Bbbk_j) \to \mathrm{Spec}(\Bbbk_i)$, and therefore we obtain a directed projective system. One can show that $\lim\ \mathrm{Spec}(\Bbbk_i) = \mathrm{Spec}(\Bbbk')$.

We end this section with the notion of profinite groups. We will see that the Galois group of a non-necessarily finite Galois extension is profinite (see Proposition B.1.8).

Definition A.5.6. A topological group is *profinite* if it is isomorphic, as topological group, to a limit of finite groups (each of them being endowed with the discrete topology).

A.6 Varieties

In this thesis, we consider $\Bbbk$-varieties defined as follow.

Definition A.6.1. Let $\Bbbk$ be a field. A $\Bbbk$-scheme X is a $\Bbbk$-*variety* if X is a separated and geometrically integral scheme of finite type over $\Bbbk$.

Below, we give some details about the definition of a $\Bbbk$-variety. We will see for instance that a $\Bbbk$-variety is also quasi-compact, Noetherian, reduced, irreducible, ...

Definition A.6.2 (See [Sta, Section 01KH])**.** A scheme X is *separated* if the diagonal morphism $\Delta : X \to X \times_{\mathrm{Spec}(\mathbb{Z})} X$ is a closed immersion.

Definition A.6.3 (See [Sta, Section 0364])**.** A scheme X is *irreducible* if it is non-empty and it can not be a union of two proper closed subsets. Let $\Bbbk$ be a field. A $\Bbbk$-scheme X is *geometrically irreducible* if the scheme $X_{\Bbbk'} := X \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk')$ is irreducible for every field extension $\Bbbk'$ of $\Bbbk$.

A reduced ring is a ring with no non-zero nilpotent elements.

Definition A.6.4 (See [Sta, Sections 01IZ, 035U])**.** A scheme X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced. Let $\Bbbk$ be a field, and let $x \in X$. We say X is *geometrically reduced* at x if for any field extension $\Bbbk'/\Bbbk$ and any point $x' \in X_{\Bbbk'}$ lying over x, the local ring $\mathcal{O}_{X_{\Bbbk'},x'}$ is reduced. A $\Bbbk$-scheme X is *geometrically reduced over* $\Bbbk$ if X is geometrically reduced at every point of X.

A scheme X is reduced if and only if for every open $U \subset R$, the ring $\mathcal{O}_X(U)$ is reduced. A $\Bbbk$-scheme X is geometrically reduced at $x \in X$ if and only if the ring $\mathcal{O}_{X,x}$ is geometrically reduced over $\Bbbk$ (that is $\mathcal{O}_{X,x} \otimes_{\Bbbk} \Bbbk'$ is reduced for every field extension $\Bbbk'/\Bbbk$).

A domain is a non-zero ring with no zero divisors. An integral domain is a commutative domain.

Definition A.6.5 (See [Sta, Sections 01OJ, 0366])**.** A scheme X is *integral* if it is non-empty and for every non-empty affine open $\mathrm{Spec}(R) = U \subset X$, the ring R is an integral domain. Let $\Bbbk$ be a field. A $\Bbbk$-scheme X is *geometrically integral* if the scheme $X_{\Bbbk'}$ is integral for every field extension $\Bbbk'$ of $\Bbbk$.

A scheme X is geometrically integral if and only if it is both geometrically reduced and geometrically irreducible over $\Bbbk$

A topological space is quasi-compact if every open covering of X has a finite subcover. A continuous map of topological spaces is quasi-compact if the inverse image of every quasi-compact open subset is quasi compact.

Definition A.6.6 (See [Sta, Section 01K2])**.** A scheme X is *quasi-compact* if its underlying topological space is quasi-compact. A morphism of schemes is called *quasi-compact* if the underlying map of topological space is quasi-compact. An S-scheme X is *quasi-compact* if its structural morphism $X \to S$ is quasi-compact.

A ring map $R \to A$ is of finite type, or A is a finite type R-algebra, if there exists $n \in \mathbb{N}$ and a surjection of R-algebras $R[x_1, \ldots, x_n] \to A$.

Definition A.6.7 (See [Sta, Section 01K2])**.** A morphism of scheme $f : X \to S$ is of finite type at $x \in X$ if there exists an affine open neighborhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type. A morphism of scheme $f : X \to S$ is of *finite type* if it is of finite type at every point of X and quasi-compact. An S-scheme X is of *finite type* if its structural morphism $X \to S$ is of finite type.

Recall that a ring R is Noetherian if it satisfies the ascending chain condition of ideals. Equivalently, every ideal of R is finitely generated. Every finite type commutative R-algebra A is Noetherian.

Definition A.6.8 (See [Sta, Section 01OU])**.** A scheme X is *locally Noetherian* if every $x \in X$ has an affine open neighborhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is Noetherian. A scheme X is *Noetherian* if it is locally Noetherian and quasi-compact.

A domain is called normal if it is integrally closed in its field of fraction. A ring R is called *normal* if for every prime $\mathfrak{p} \subset R$, the localization $R_{\mathfrak{p}}$ is a normal domain (see details in [Sta, Section 037B]).

Definition A.6.9 (See [Sta, Section 033H])**.** A scheme X is normal if and only if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

A scheme X is normal if and only if for every affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is normal.

A.7 Weil restriction

This section is based on [BLR90, §7.6] and on the notes [JMS]. See also [CGP10, §A.5]. Let $p : S' \to S$ be a scheme morphism. We give some conditions that ensure the existence of the Weil restriction of an S'-scheme X'.

Definition A.7.1. Let $p : S' \to S$ be a scheme morphism. Given an S'-scheme X', consider the contravariant functor

$$\begin{aligned}\mathrm{R}_{S'/S}(X') : \mathrm{Sch}_S^{op} &\to \mathrm{Sets}\\ T &\mapsto X'(T \times_S S') := \mathrm{Hom}_{S'}(T_{S'}, X').\end{aligned}$$

If the functor $\mathrm{R}_{S'/S}(X')$ is representable by an S-scheme X, then we say that X is the Weil restriction of X' along p, and we denote it $\mathrm{R}_{S'/S}(X')$.

Theorem A.7.2 ([BLR90, Theorem 4], [CGP10, Proposition A.5.8])**.** *Let $p : S' \to S$ be a finite and locally free morphism (see [Sta, Definition 02KA and Lemma 02KB]. Let X' be an S'-scheme such that for any $s \in S$ and any finite set $P \subset X' \times_S \mathrm{Spec}(\kappa(s))$, there exists an affine open subscheme $U' \subset X'$ containing P. Then, the functor $\mathrm{R}_{S'/S}(X')$ is representable by an S-scheme. In particular, if X' is a quasi-projective S'-scheme, then the Weil restriction of X' exists. Furthermore, if p is a finite flat morphism of Noetherian schemes, if X' is a quasi-projective S'-scheme, then $\mathrm{R}_{S'/S}(X')$ is a quasi-projective S-scheme.*

The role of the quasi-projective hypothesis on X' is to ensure that any finite set of points in the underlying topological space of X' is contained in an affine open subscheme of X'. Under this hypothesis, $\mathrm{R}_{S'/S}(X')$ is constructed as a gluing of S-schemes $\mathrm{R}_{S'/S}(U_i')$ for a suitable finite affine open covering $\{U_i'\}$ of X'.

We list some properties of the Weil restriction in the next proposition.

Proposition A.7.3 ([CGP10, Propositions A.5.2, A.5.4, and A.5.5])**.** *Let $p : S' \to S$ be a finite and flat morphism of affine Noetherian schemes, and let X' be an S'-scheme such that the Weil restriction $\mathrm{R}_{S'/S}(X')$ exists as an S-scheme.*

(i) If X' is affine of finite type over S', then $\mathrm{R}_{S'/S}(X')$ is affine of finite type over S;

(ii) If T is an S-scheme and $T' := T \times_S S'$, then there is an isomorphism

$$\mathrm{R}_{T'/T}(X' \times_{S'} T') \cong \mathrm{R}_{S'/S}(X') \times_S T;$$

(iii) There is a natural isomorphism

$$\mathrm{R}_{S'/S}(X' \times_{Z'} Y') \cong \mathrm{R}_{S'/S}(X') \times_{\mathrm{R}_{S'/S}(Z')} \mathrm{R}_{S'/S}(Y'),$$

where Y' and Z' are quasi-projective S' scheme such that there are morphisms $X' \to Z'$ and $Y' \to Z'$. In particular, if X' is an S'-group scheme, then $\mathrm{R}_{S'/S}(X')$ will be an S-group scheme;

(iv) If $f : X' \to Y'$ is a smooth (resp. étale, open immersion), then so is $\mathrm{R}_{S'/S}(f)$.

(v) Consider the S-scheme $S' := \sqcup S_i \to S$, where $\{S_i\}$ is a finite collection of S-schemes, and the S'-scheme $X' := \sqcup X_i$, where X_i is an S_i-scheme for all i. Then,

$$\mathrm{R}_{S'/S}(X') \cong \prod \mathrm{R}_{S_i/S}(X_i);$$

(vi) If X' is a G'-torsor for a smooth quasi-projective S'-group scheme G', then $\mathrm{R}_{S'/S}(X')$ equipped with the natural $\mathrm{R}_{S'/S}(G')$-action is a $\mathrm{R}_{S'/S}(G')$-torsor;

(vii) If X' is quasi-compact, then so is $\mathrm{R}_{S'/S}(X')$;

(viii) $\mathrm{R}_{S'/S}(X')$ is separated.

Example A.7.4. Let $\Bbbk'/\Bbbk$ be a finite extension of degree d. Then $\mathrm{R}_{\Bbbk'/\Bbbk}(\mathrm{Spec}(\Bbbk')) = \mathrm{Spec}(\Bbbk)$, $\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{A}^1_{\Bbbk'}) = \mathbb{A}^d_{\Bbbk}$, and $\mathrm{R}_{\Bbbk'/\Bbbk}(\mathbb{G}^1_{m,\Bbbk'}) = \mathbb{G}^d_{m,\Bbbk}$. Let $X := \mathrm{Spec}(\Bbbk'[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$. Then

$$\mathrm{R}_{\Bbbk'/\Bbbk}(X) = \mathrm{Spec}(\Bbbk[y_{i,j}]/(g_{k,l})),$$

where $1 \leq i \leq n$, $1 \leq j \leq d$, $1 \leq k \leq m$, $1 \leq l \leq d$, and where the $g_{k,l}$ are polynomials in the indeterminate $y_{i,j}$ defined as follows. Take a $\Bbbk$-basis $(e_1, \ldots, e_d)$ of $\Bbbk'$, and write $x_i = y_{i,1}e_1 + \cdots + y_{i,d}e_d$. Then, there exists polynomials $g_{k,l}$ such that $f_j = g_{j,1}e_1 + \cdots + g_{j,d}e_d$. See for instance Example 1.2.10.

This section ends on a result similar to the embedding described in Proposition 3.2.6.

Proposition A.7.5. *Let X be a normal affine $\Bbbk$-variety endowed with an effective $\Bbbk$-torus action μ of T. Let $\Bbbk'/\Bbbk$ be a finite Galois extension. Then, there are closed immersions*

$$X \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right) \quad \text{and} \quad T \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}\left(T_{\Bbbk'}\right).$$

Furthermore, the following diagram is commutative

$$\begin{array}{ccc} T \times_{\mathrm{Spec}(\Bbbk)} X & \xrightarrow{\quad\mu\quad} & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{R}_{\Bbbk'/\Bbbk}(T_{\Bbbk'}) \times_{\mathrm{Spec}(\Bbbk)} \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'}) & \xrightarrow{\mathrm{R}_{\Bbbk'/\Bbbk}(\mu)} & \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'}). \end{array}$$

Proof. Indeed, from the definition of the Weil restriction, we have

$$\mathrm{Hom}_{\Bbbk}\Big(X, \mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right)\Big) \cong \mathrm{Hom}_{\Bbbk'}\left(X_{\Bbbk'}, X_{\Bbbk'}\right).$$

Therefore, the $\Bbbk'$-morphism $id_{X_{\Bbbk'}}$ corresponds to a $\Bbbk$-morphism $X \to \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'})$. Then, by Proposition A.7.3, we get a natural isomorphism

$$\mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right) \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk') \cong \mathrm{R}_{\Bbbk' \otimes_{\Bbbk} \Bbbk'/\Bbbk'}\Big(X_{\Bbbk'} \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk')\Big).$$

Therefore (see also Lemma 1.3.1),

$$\mathrm{R}_{\Bbbk'/\Bbbk}\left(X_{\Bbbk'}\right) \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk') \cong \prod_{\Gamma} X_{\Bbbk'} = X_{\Bbbk'}^{d}.$$

Then, the base change

$$X_{\Bbbk'} \to \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'})_{\Bbbk'} \cong X_{\Bbbk'}^{d}$$

corresponds to the diagonal morphism, which is a closed immersion (since X is separated). Finally, by Theorem 1.1.4, we get a closed immersion $X \hookrightarrow \mathrm{R}_{\Bbbk'/\Bbbk}(X_{\Bbbk'})$. □

Appendix B

Galois theory

In this section, we state with no proof the main results of the Galois theory. For a point on field extension and on Galois theory, see for instance [Mor96] or [Esc01].

B.1 Generalities

Definition B.1.1. Let $\mathbb{k}$ be a field and let $\mathbb{k}'$ be an extension of $\mathbb{k}$. The set of $\mathbb{k}$-automorphisms of $\mathbb{k}'$ is equipped with a group structure whose group law is the composition of $\mathbb{k}$-automorphisms. We will denote this group by $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$, and call it the *Galois group of the extension* $\mathbb{k}'$ *over* $\mathbb{k}$.

In this thesis, we often consider Galois groups of normal extensions; non-normal one do not possess enough $\mathbb{k}$-automorphisms to make the fundamental theorem of the Galois correspondence (see Theorems B.2.1 and B.3.1).

Definition B.1.2 ([Esc01, §7.2])**.** A *normal extension* of a field $\mathbb{k}$ is an algebraic extension $\mathbb{k}'$ of $\mathbb{k}$ such that every irreducible polynomial in $\mathbb{k}[x]$ having a root in $\mathbb{k}'$ has all its roots in $\mathbb{k}'$. In other words, all conjugates of elements of $\mathbb{k}'$ must lie in $\mathbb{k}'$.

Definition B.1.3 ([Esc01, §15.1])**.** Let $\mathbb{k}'/\mathbb{k}$ be a field extension. An element $a \in \mathbb{k}'$ that is algebraic over $\mathbb{k}$ is said to be *separable over* $\mathbb{k}$ if it is a simple root of its minimal polynomial. An algebraic field extension $\mathbb{k}'/\mathbb{k}$ is *separable* if every element of $\mathbb{k}'$ is separable over $\mathbb{k}$.

Definition B.1.4. A *Galois extension* of a field $\mathbb{k}$ is normal and *separable*.

Remark B.1.5. In the case of infinite fields of non-zero characteristic, it is necessary to distinguish between *normal extensions* and *Galois extension.* In characteristic zero, the two notions are equivalent. If $\mathbb{k}$ is a perfect field, the two notions are equivalent since all extension of such a field are separable (see [Esc01, Chapter 15], see also [Sta, Lemma 030P]).

Definition B.1.6. Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension. The *Krull topology* on $\mathbb{k}'/\mathbb{k}$ is the unique topology such that for all $\gamma \in \mathrm{Gal}(\mathbb{k}'/\mathbb{k})$, the family of subsets

$$\{\gamma \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_0) \mid \mathbb{k}_0/\mathbb{k} \text{ is a finite Galois extension, } \mathbb{k}_0 \subset \mathbb{k}'\}$$

is a basis of open neighborhoods of γ. Hence, a subset S of $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ is open if $S = \emptyset$ or if $S = \bigcup_i \gamma_i \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_i)$ for some $\gamma_i \in \mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ and for some finite Galois extensions $\mathbb{k}_i/\mathbb{k}$ in $\mathbb{k}'$.

Remark B.1.7 ([Tho])**.** Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension. The Krull topology is discrete if and only if the extension $\mathbb{k}'/\mathbb{k}$ is finite.

From the next proposition, we get that the Galois group of a non-necessarily finite Galois extension is a profinite group (see Definition A.5.6).

Proposition B.1.8 ([Ber10, Theorem I.2.18], see also [Sta, §0BMI])**.** *Let $\Bbbk'/\Bbbk$ be a Galois extension. We have an isomorphism of topological groups*

$$\mathrm{Gal}(\Bbbk'/\Bbbk) \cong \lim \ \mathrm{Gal}(\Bbbk_i/\Bbbk),$$

where the limit is over all finite Galois extension $\Bbbk_i/\Bbbk$ in $\Bbbk'$.

Let H be a normal subgroup of Γ, and let ${\Bbbk'}^H$ be the field of fixed elements (see [Mor96, Chater I, §2]). Then, since H is normal in Γ, for every $\gamma \in \Gamma$ we can restrict γ on ${\Bbbk'}^H$, and we get an automorphism $\gamma|_{{\Bbbk'}^H} \in \mathrm{Gal}({\Bbbk'}^H/\Bbbk)$. Therefore, we obtain a continuous surjective homomorphism

$$\Gamma \to \mathrm{Gal}\left(\Bbbk_1/\Bbbk\right), \quad \gamma \mapsto \gamma|_{{\Bbbk'}^H},$$

with kernel $\mathrm{Gal}\left(\Bbbk'/{\Bbbk'}^H\right)$.

Conversely, let $\Bbbk \subset \Bbbk_1 \subset \Bbbk'$ be a Galois extensions of $\Bbbk$, and let $\gamma \in \Gamma$. Since $\Bbbk_1$ is a normal subextension in $\Bbbk'$, $\gamma(\Bbbk_1) = \Bbbk_1$. We get an automorphism $\gamma|_{\Bbbk_1} : \Bbbk_1 \to \Bbbk_1$ ([Sta, Lemmas 09HQ and 0BME]). Therefore, we obtain a continuous surjective homomorphism

$$\Gamma \to \mathrm{Gal}\left(\Bbbk_1/\Bbbk\right), \quad \gamma \mapsto \gamma|_{\Bbbk_1},$$

with kernel $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$ ([Sta, Lemmas 0BMK, 0BMM and Theorem 0BML]). In other words, we obtain a short exact sequence

$$1 \longrightarrow \mathrm{Gal}\left(\Bbbk'/\Bbbk_1\right) \longrightarrow \Gamma := \mathrm{Gal}\left(\Bbbk'/\Bbbk\right) \longrightarrow \mathrm{Gal}\left(\Bbbk_1/\Bbbk\right) \longrightarrow 1$$

of profinite topological groups. Hence, $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$ is a normal subgroup of Γ and we have an isomorphism

$$\begin{aligned} \Gamma/\mathrm{Gal}(\Bbbk'/\Bbbk_1) &\to \mathrm{Gal}(\Bbbk_1/\Bbbk) \\ \gamma\mathrm{Gal}(\Bbbk'/\Bbbk_1) &\mapsto \gamma|_{\Bbbk_1}. \end{aligned}$$

B.2 Galois theory for finite field extensions

This section is based on [Esc01, §8.4 & 8.5]. See also [Mor96, Chapter I, §5].

Theorem B.2.1 (Fundamental Theorem of Galois Theory over finite field extensions)**.** *Let $\Bbbk'/\Bbbk$ be a normal extension of finite degree. Let $\mathscr{E}$ be the set of intermediate extension between $\Bbbk$ and $\Bbbk'$, and let $\mathscr{G}$ be the set of subgroups of $\mathrm{Gal}(\Bbbk'/\Bbbk)$. Let $I : \mathscr{G} \to \mathscr{E}$ be the map that associates to a subgroup H of $\mathrm{Gal}(\Bbbk'/\Bbbk)$ the field of invariants ${\Bbbk'}^H$, and let $G : \mathscr{E} \to \mathscr{G}$ be the map which associates to an extension $\Bbbk_0$ of $\mathscr{E}$ the group $\mathrm{Gal}(\Bbbk'/\Bbbk_0)$.*

(i) Let $\Bbbk_0$ be an intermediate normal extension of $\Bbbk$. The map $\mathrm{Gal}(\Bbbk'/\Bbbk) \to \mathrm{Gal}(\Bbbk_0/\Bbbk)$ obtained by taking the restriction to $\Bbbk_0$ of a $\Bbbk$-automorphism of $\Bbbk'$ is a surjective group homomorphism with kernel equal to $\mathrm{Gal}(\Bbbk'/\Bbbk_0)$. Furthermore, $\mathrm{Gal}(\Bbbk'/\Bbbk_0)$ is a normal subgroup of $\mathrm{Gal}(\Bbbk'/\Bbbk)$ and

$$\mathrm{Gal}(\Bbbk_0/\Bbbk) \cong \mathrm{Gal}(\Bbbk'/\Bbbk)/\mathrm{Gal}(\Bbbk'/\Bbbk_0).$$

(ii) I and G define inverse bijections that are decreasing for the inclusion relation.

(iii) By restriction, I and G define inverse bijections of the set $\mathscr{E}'$ of normal extensions of $\Bbbk$ contained in $\Bbbk'$ to the set $\mathscr{G}'$ of normal subgroups of $\mathrm{Gal}(\Bbbk'/\Bbbk)$.

(iv) If $\Bbbk_0$ and $\Bbbk_1$ are intermediate extension between $\Bbbk$ and $\Bbbk'$, then $\Bbbk_1$ is a normal extension of $\Bbbk_0$ if and only if $\mathrm{Gal}(\Bbbk'/\Bbbk_1)$ is a normal subgroup of $\mathrm{Gal}(\Bbbk'/\Bbbk_0)$, in which case

$$\mathrm{Gal}(\Bbbk_1/\Bbbk_0) \cong \mathrm{Gal}(\Bbbk'/\Bbbk_0)/\mathrm{Gal}(\Bbbk'/\Bbbk_1).$$

(v) If $\Bbbk_0$ and $\Bbbk_1$ are intermediate extension between $\Bbbk$ and $\Bbbk'$ with $\Bbbk_0 \subset \Bbbk_1$, then

$$[\Bbbk_1 : \Bbbk_0] = \left|\mathrm{Gal}(\Bbbk'/\Bbbk_0)\right| / \left|\mathrm{Gal}(\Bbbk'/\Bbbk_1)\right|.$$

B.3 Galois theory for infinite field extensions

This section is based on [Ber10, Chapter 1]. The proof may be found in [Mor96, Chapter IV].

Theorem B.3.1 (Fundamental Theorem of Galois Theory over finite field extensions)**.** *Let $\mathbb{k}'/\mathbb{k}$ be a Galois extension. There is a one-to-one correspondence between the following sets:*

(i) The set of subfields $\mathbb{k}_0$ of $\mathbb{k}'$ containing $\mathbb{k}$ and the set of closed subgroups of $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$.

(ii) The set of subfields $\mathbb{k}_0$ of $\mathbb{k}'$ containing $\mathbb{k}$ such that $\mathbb{k}_0/\mathbb{k}$ is a finite field extension and the set of open subgroups of $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$.

(iii) The set of subfields $\mathbb{k}_0$ of $\mathbb{k}'$ containing $\mathbb{k}$ such that $\mathbb{k}_0/\mathbb{k}$ is a finite Galois extension and the set of open normal subgroups of $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$.

In all cases, the correspondence is given by

$$\mathbb{k}_0 \mapsto \mathrm{Gal}(\mathbb{k}'/\mathbb{k}_0)$$
$$\mathbb{k}'^H \leftarrow\!\shortmid H$$

Moreover, if H is an open normal subgroup of $\mathrm{Gal}(\mathbb{k}'/\mathbb{k})$, then we have

$$\mathrm{Gal}(\mathbb{k}'^H/\mathbb{k}) \cong \mathrm{Gal}(\mathbb{k}'/\mathbb{k})/H.$$

In particular, for any finite Galois extension $\mathbb{k}_0/\mathbb{k}$ in $\mathbb{k}'$, we have

$$\mathrm{Gal}(\mathbb{k}_0/\mathbb{k}) \cong \mathrm{Gal}(\mathbb{k}'/\mathbb{k})/\mathrm{Gal}(\mathbb{k}'/\mathbb{k}_0).$$

Appendix C

Convex geometry

C.1 Normal fan, polytope, and polyhedron

This section is based on [CLS11, §2.2 and §7.1].

Let M and N be dual lattices. Let $\Delta \subset M_{\mathbb{Q}}$ be a polyhedron; it is the intersection of finitely many closed half-spaces:

$$\Delta = \bigcap_i \{u \in M_{\mathbb{Q}} \mid \langle u|v_i\rangle \geq -a_i\},$$

where $a_i \in \mathbb{Q}$ and $v_i \in N$. Furthermore, if Δ is full dimensional, each facet (i.e a face of codimension one) has a unique supporting affine hyperplane. Therefore,

$$\Delta = \{u \in M_{\mathbb{Q}} \mid \langle u|v_i\rangle \geq -a_i \text{ for all facets } \delta_i \text{ of } \Delta\},$$

where v_i is the generator of the *inward-pointing facet normals* of the facet δ_i.

Recall that there is a Minkowski sum decomposition $\Delta = \omega_M + \Pi$, where ω_M is a cone and Π is a polytope. In this decomposition, the cone is unique and called the tail cone of Δ. Here, the the tail cone of Δ is

$$\omega_M := \{u \in M_{\mathbb{Q}} \mid \langle u|v_i\rangle \geq 0 \text{ for all facets } \delta_i \text{ of } \Delta\}.$$

The *normal fan* $\Lambda_\Delta \subset N_{\mathbb{Q}}$ of a full dimensional polyhedron Δ consists of cones λ_δ indexed by faces δ of Δ, where

$$\lambda_\delta := \operatorname{Cone}(v_i \mid \delta_i \text{ facet containing } \delta).$$

In practice, this normal fan is constructed as follows. A vertex $u \in \Delta$ gives the cone

$$C_u := \operatorname{Cone}(\Delta \cap M - u) \subset M_{\mathbb{Q}} \text{ and } C_u^\vee \in \Lambda_\Delta.$$

The vertices of Δ correspond to the maximal cone in Λ_Δ, and the facets of Δ correspond to the rays in Λ_Δ. The ray generators of the normal fan Λ_Δ are generators of the inward-pointing facet normal v_i.

If the polyhedron Δ is not full dimensional, then by the same recipe, we construct a normal quasi-fan Λ_Δ, having support the dual of the tail cone of Δ.

Example C.1.1. A full dimensional polyhedron Δ, with tail cone $\omega = \mathbb{Q}^2_{\geq 0}$, and its normal fan.

Example C.1.2 ([CLS11, Example 2.3.4])**.** A full dimensional polytope and its normal fan.

Example C.1.3 ([CLS11, Example 2.3.4])**.** A polytope and its normal quasi-fan.

A polyhedron leads to a support function on the normal quasi-fan Λ_Δ. The precise definition of this function is the following.

Definition C.1.4. Let Λ be a quasi-fan in $N_\mathbb{Q}$ having support the cone ω_N. A *support function* is a function $h : \omega_N \to \mathbb{Q}$ that is linear on each cone of Λ.

Proposition C.1.5 ([AH06, Lemma 1.4])**.** *Let ω_N be a pointed cone in $N_\mathbb{Q}$, let Δ be a ω_N-polyhedron, and let $\Lambda_\Delta \subset M_\mathbb{Q}$ be the normal quasi-fan of Δ. Consider the map*

$$h_\Delta : \omega_N^\vee \to \mathbb{Q}; \quad u \mapsto \min(\langle u|v\rangle \mid u \in \Delta).$$

(i) The map h_Δ is well defined and is a support function for Λ_Δ;
(ii) The map h_Δ is convex. That means that for any two vectors u, $u' \in \omega_N^\vee$, we have

$$h_\Delta(u) + h_\Delta(u') \leq h_\Delta(u + u').$$

Proof. The statements are standard in the case where Δ is a full dimensional lattice polytope, see [CLS11, Proposition 4.2.14], or [Oda88, Appendix A]). According to [AH06, Lemma 1.4], the proof can be easily adapted to our setting. □

C.2 Convex geometry and toric divisors

This appendix is based on [Ful93] and [CLS11].

On a variety X, a Weil divisor is a finite formal sum $\sum a_i \otimes D_i$ of irreducible closed subvarieties D_i of codimension one in X and such that $a_i \in \mathbb{Z}$. The sum of two Weil divisors is again a Weil divisor, therefore we get a group denoted $\mathrm{WDiv}(X)$. A Weil $\mathbb{Q}$-divisor is an element of $\mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. A Cartier divisor is given by the data of a covering of X by affine open subsets U_i, and non-zero rational functions f_i on U_i, called *local equations*, such that f_i/f_j are nowhere zero regular functions on $U_i \cap U_j$. We get a group denoted $\mathrm{CDiv}(X)$. A Cartier $\mathbb{Q}$-divisor is an element of $\mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. When X is a normal variety, we have an embedding $\mathrm{CDiv}(X) \subset \mathrm{WDiv}(X)$. Furthermore, if X is smooth, then $\mathrm{CDiv}(X) = \mathrm{WDiv}(X)$, and we denote $\mathrm{Div}(X)$ the group of all divisors on X.

Let X_Λ be a split $\mathbb{T}$-toric (normal) $\Bbbk$-variety of fan $\Lambda \subset N_{\mathbb{Q}}$. We are interested in divisors that are mapped to themselves by the torus $\mathbb{T} = \mathrm{Spec}(\Bbbk[M])$. The irreducible subvarieties of codimension one that are $\mathbb{T}$-invariant correspond to rays of the fan. Number the rays $\langle v_1 \rangle$, $\dots$, $\langle v_q \rangle$, where v_i is the first lattice vector of the ray $\langle v_i \rangle$. To a ray $\langle v_i \rangle$, the associated prime divisor D_{v_i} is the orbit closure of the distinguished point of $\mathrm{Spec}(\Bbbk[\langle v_i \rangle^\vee \cap M])$. The $\mathbb{T}$-invariant Weil divisors, denoted $\mathrm{WDiv}_{\mathbb{T}}(X_\Lambda)$, are of the form

$$D = \sum a_i \otimes D_{v_i},$$

where a_i are integer. Note that $D|_{\mathbb{T}}$ is the trivial divisor.

For $m \in M$ (see [CLS11, Proposition 4.1.2], or [Ful93, Lemma in §3.3]), we have

$$\mathrm{div}(\chi^m) = \sum \langle m | v_i \rangle \otimes D_{v_i}.$$

A $\mathbb{T}$-invariant Weil divisor is Cartier if and only if for each $\lambda \in \Lambda$, there exists $m_\lambda \in M$ such that for all ray $\langle v_i \rangle$ of λ, $\langle m_\lambda | v_i \rangle = -a_i$ (see [CLS11, Theorem 4.2.6]). Let $D = \sum a_i \otimes D_{v_i}$ be a Cartier divisor on X_Λ, let $\lambda \in \Lambda$, and let U_λ be the affine open subset associated to λ. Then, $D|_{U_\lambda} = \mathrm{div}(\chi^{-m_\lambda})$, and $\{(U_\lambda, \chi^{-m_\lambda})\}_{\lambda \in \Lambda}$ is a local data for D (see [CLS11, Proposition 4.2.8, Definition 4.0.12]).

Example C.2.1. Let $\Delta \subset M_{\mathbb{Q}}$ be a full dimensional lattice polyhedron

$$\Delta = \{u \in M_{\mathbb{Q}} \mid \langle u | v_i \rangle \geq -a_i \text{ for all facets } \delta_i \text{ of } \Delta\},$$

where $a_i \in \mathbb{Z}$. Let Λ_Δ the corresponding normal fan, and let X_{Λ_Δ} be the associated toric variety. Then, to the polyhedron Δ, is associated a Cartier divisor on X_{Λ_Δ} (see [CLS11, Proposition 4.2.10])

$$D_\Delta := \sum_{\delta_i \text{ facet of } \Delta} a_i \otimes D_{v_i}.$$

Recall that if D is a Weil $\mathbb{Q}$-divisor on a $\Bbbk$-variety X, then we define the sheaf of sections $\mathcal{O}_X(D)$ by:

$$\mathrm{H}^0(V, \mathcal{O}_X(D)) := \left\{ f \in \Bbbk(X) \;\middle|\; \mathrm{div}(f)|_V + D|_V \geq 0 \right\} \cup \{0\},$$

where $V \subset X$ is an open subset.

Let X_Λ be a normal toric variety, and let $D := \sum a_i \otimes D_{v_i}$ be a $\mathbb{T}$-invariant Weil divisor. Let $m \in M$. Note that, $\mathrm{div}(\chi^m) + D \geq 0$ is equivalent to

$$\langle m | v_i \rangle + a_i \geq 0 \text{ for all ray } \langle v_i \rangle \in \Lambda,$$

where v_i is the first lattice vector of the ray $\langle v_i \rangle$. Therefore, we define a polyhedron from the divisor D by

$$\Delta(D) := \{u \in M_{\mathbb{Q}} \mid \langle u | v_i \rangle \geq -a_i, \ \forall i\}.$$

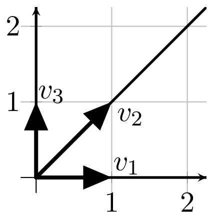
Proposition C.2.2 ([CLS11, Proposition 4.3.3], see also Lemma in [Ful93, §3.4])**.** *Let X be a (normal) toric variety of fan $\Lambda \subset N_{\mathbb{Q}}$. Let $\{v_1, \ldots, v_q\}$ be the lattice generators of the rays of Λ, let $\{a_1, \ldots, a_q\}$ be rational numbers, and let $D := \sum a_i \otimes D_{v_i}$ be a Weil $\mathbb{Q}$-divisor. Then,*

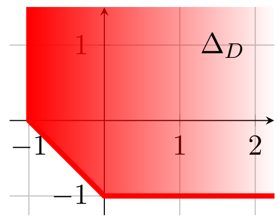
$$\mathrm{H}^0\left(X, \mathcal{O}_X(D)\right) = \sum_{m \in \Delta(D) \cap M} \Bbbk \chi^m,$$

where $\Delta(D)$ is the polyhedron $\Delta(D) := \{u \in M_{\mathbb{Q}} \mid \langle u | v_i \rangle \geq -a_i, \ \forall i \in \{1, \ldots, q\}\}$.

Proof. Observe that $\mathrm{H}^0(X, \mathcal{O}_X(D)) = \mathrm{H}^0(X, \mathcal{O}_X(\lfloor D \rfloor))$ and that $\Delta(D) \cap M = \Delta(\lfloor D \rfloor) \cap M$. Therefore, we conclude by [CLS11, Proposition 4.3.3] (or Lemma in [Ful93, §3.4]). □

Example C.2.3 ([CLS11, Example 4.3.4])**.** Consider the fan of the blow-up of $\mathbb{A}^2_{\mathbb{C}}$ at the origin, and consider the torus invariant Weil divisor $D := D_{v_1} + D_{v_2} + D_{v_3}$. Then, Δ_D is defined by:





C.3 Convex geometry and short exact sequences of lattices

We give technical details used in the proof of the Altmann-Hausen presentation via *toric downgrading*.

Notation C.3.1. We consider the following dual short exact sequences of lattices:

$$0 \longrightarrow N \xrightarrow{F} N' \xrightarrow{P} N_Y \longrightarrow 0$$

$$0 \longrightarrow M_Y \xrightarrow{P^*} M' \xrightarrow{F^*} M \longrightarrow 0$$

In the next lemma, we list some classical properties of a split short exact sequence of lattices.

Lemma C.3.2. *Consider the short exact sequences of Notation C.3.1. There exists a section $t : N_Y \to N'$ and a cosection $s : N' \to N$ such that*

1. $F^* \circ s^* = id_M$
2. $t^* \circ P^* = id_{M_Y}$
3. $P^* \circ t^* = id_{M'} - s^* \circ F^*$
4. $s \circ F = id_N$
5. $P \circ t = id_{N_Y}$
6. $t \circ P = id_{N'} - F \circ s$
7. $M' = \mathrm{Im}(P^*) \oplus \mathrm{Ker}(t^*)$
8. $M' = \mathrm{Ker}(F^*) \oplus \mathrm{Im}(s^*)$
9. $\mathrm{Ker}(t^*) = \mathrm{Im}(s^*)$
10. $N' = \mathrm{Im}(F) \oplus \mathrm{Ker}(s)$
11. $N' = \mathrm{Ker}(P) \oplus \mathrm{Im}(t)$
12. $\mathrm{Ker}(s) = \mathrm{Im}(t)$

Proof. For the existence of the section, see [Eis95, Proposition A.3.1], and for the list of properties, see [Lan02, Proposition 3.2]. □

Denote by $\{e_1, \dots, e_n\}$ the canonical basis of N', this set generates the pointed cone $\mathbb{Q}^n_{\geq 0} \subset N'_{\mathbb{Q}}$. Observe that $P(\mathbb{Q}^n_{\geq 0}) \subset N_{\mathbb{Q}}$ is the cone generated by the set $\{P(e_1), \dots, P(e_n)\}$. Let v_i be the first lattice vector of the ray v_i and consider the following polyhedron

$$\Delta_{v_i} := s\left(P^{-1}(v_i) \cap \mathbb{Q}^n_{\geq 0}\right) \subset N_{\mathbb{Q}}.$$

Observe that the polyhedron $s(P^{-1}(0) \cap \mathbb{Q}^n_{\geq 0}) \subset N_{\mathbb{Q}}$ is by definition a pointed cone, and Δ_{v_i} are $s(P^{-1}(0) \cap \mathbb{Q}^n_{\geq 0})$-polyhedra.

Consider the cone $\omega_M := F^*(\mathbb{Q}^n_{\geq 0}) \subset M_{\mathbb{Q}}$. Then $\Delta(0) := (F^*)^{-1}(0) \cap \mathbb{Q}^n_{\geq 0} \subset M'_{\mathbb{Q}}$ is a pointed cone, and for all $m \in \omega_M \cap M$, observe that the polyhedron

$$\Delta(m) := (F^*)^{-1}(m) \cap \mathbb{Q}^n_{\geq 0} \subset M'_{\mathbb{Q}}$$

has tail cone $\Delta(0)$. We define the polyhedron

$$\Delta_Y(m) := \{u_Y \in (M_Y)_{\mathbb{Q}} \mid \langle P^*(u_Y)|e_i\rangle \geq -\langle s^*(m)|e_i\rangle \ \forall i \in \{1, \dots, n\}\} \subset (M_Y)_{\mathbb{Q}}.$$

Lemma C.3.3. *Let $m \in \omega_M \cap M$, then*
(i) $\Delta(m) = P^*(\Delta_Y(m)) + s^*(m)$*;*
(ii) $\Delta_Y(m) = t^*(\Delta(m)) = t^*\left((F^*)^{-1}(m) \cap \mathbb{Q}^n_{\geq 0}\right)$*;*
(iii) $\Delta_Y(m)$ *has tail cone* $t^*(\Delta(0)) = P(\mathbb{Q}^n_{\geq 0})^\vee \subset (M_Y)_{\mathbb{Q}}$*.*

Proof. *(i)* Since $(F^*)^{-1}(m) = s^*(m) + \mathrm{Ker}(F^*) = s^*(m) + \mathrm{Im}(P^*)$ (see Lemma C.3.2), we have

$$\begin{aligned}\Delta(m) &= \{u' = P^*(u_Y) + s^*(m) \mid u_Y \in (M_Y)_{\mathbb{Q}}, \langle u'|e_i\rangle \geq 0 \ \forall i \in \{1, \dots, n\}\} \\ &= \{P^*(u_Y) \mid u_Y \in (M_Y)_{\mathbb{Q}}, \langle P^*(u_Y)|e_i\rangle \geq -\langle s^*(m)|e_i\rangle \ \forall i \in \{1, \dots, n\}\} + s^*(m) \\ &= P^*\left(\{u_Y \mid u_Y \in (M_Y)_{\mathbb{Q}}, \langle P^*(u_Y)|e_i\rangle \geq -\langle s^*(m)|e_i\rangle \ \forall i \in \{1, \dots, n\}\}\right) + s^*(m) \\ &= P^*(\Delta_Y(m)) + s^*(m)\end{aligned}$$

(ii) We have $t^* \circ s^*(m) = 0$. Thus $t^*(\Delta(m)) = \Delta_Y(m)$.

(iii) Let $u_Y \in t^*(\Delta(0))$, there exists $u' \in \Delta(0)$ such that $u_Y = t^*(u')$. We have for all $v' \in \mathbb{Q}^n_{\geq 0}$:

$$\langle t^*(u')|P(v')\rangle = \langle P^*(t^*(u'))|v'\rangle = \langle u' - s^*(F^*(u'))|v'\rangle = \langle u'|v'\rangle \geq 0$$

Hence $u_Y \in P(\mathbb{Q}^n_{\geq 0})^\vee$. Conversely, let $u_Y \in P(\mathbb{Q}^n_{\geq 0})^\vee$. We have $u_Y = t^*(P^*(u_Y))$, we show that $P^*(u_Y) \in \Delta(0)$. First, $F^*(P^*(u_Y)) = 0$. Moreover, since $\langle P^*(u_Y)|v'\rangle = \langle u_Y|P(v')\rangle \geq 0$ for all $v' \in \mathbb{Q}^n_{\geq 0}$, we have $P^*(u_Y) \in \mathbb{Q}^n_{\geq 0}$. Hence, $P^*(u_Y) \in \Delta(0)$ and $u_Y \in t^*(\Delta(0))$. □

Lemma C.3.4. *The following equality holds:*

$$s\left(F(N_{\mathbb{Q}}) \cap \mathbb{Q}^n_{\geq 0}\right) = s\left(P^{-1}(0) \cap \mathbb{Q}^n_{\geq 0}\right) = F^*\left(\mathbb{Q}^n_{\geq 0}\right)^\vee \subset N_{\mathbb{Q}}.$$

Thus, the tail cone of Δ_{v_i} *is* $\omega_N := s\left(P^{-1}(0) \cap \mathbb{Q}^n_{\geq 0}\right) = F^*\left(\mathbb{Q}^n_{\geq 0}\right)^\vee = \omega_M^\vee$*.*

Proof. It is a consequence of Lemma C.3.3. □

Lemma C.3.5. *for all i, we have*

$$\min\langle \Delta_Y(m)|v_i\rangle + \min\langle m|\Delta_{v_i}\rangle = 0.$$

It is a corollary of the next lemma, with $\omega = \mathbb{Q}^n_{\geq 0}$, $u = s^*(m)$, and $v = t(v_i)$.

Lemma C.3.6 (from the proof of [AH03, Proposition 8.5])**.** *Consider the short exact sequences of Notation C.3.1. Let ω be a pointed cone in $N'_{\mathbb{Q}}$, let $v \in \omega$ and let $u \in \omega^\vee$. Then,*

$$\min\left\langle u \,\middle|\, P^{-1}(P(v)) \cap \omega\right\rangle + \min\left\langle (F^*)^{-1}\left(F^*(u)\right) \cap \omega^\vee \,\middle|\, v\right\rangle = \langle u|v\rangle\,.$$

Appendix D

Birational geometry

In this Appendix, based on [Har77, Chapter V] and [Man86, Chapter III], we recall basic tools of birational geometry. See also [Pie12].

Let X be a smooth projective surface over an algebraically closed field. We denote $\mathrm{Div}(X)$ the group of all divisors on X, and by $\mathrm{Pic}(X)$ the group of invertible sheaf up to isomorphism, which is isomorphic to the group of divisors modulo linear equivalence (see [Har77, Chapter II, §6]). There exists a unique pairing

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z},$$

called the *intersection pairing* (see [Har77, Chapter V, Theorem 1.1]). For instance, if C and D are smooth curves which meet transversely at r points, then the intersection number is r.

Definition D.0.1. Let $\Bbbk$ be an algebraically closed field, and let X be a smooth projective $\Bbbk$-surface. A (-1)*-curve* C is a curve on X such that it self intersection number is $C^2 = -1$, and $C \cong \mathbb{P}^1_{\Bbbk}$.

Now we come to Castelnuovo's criterion for contracting a curve on a surface over an algebraically closed field. We refer to [Har77, Chapter I, §4] for the construction of the blow-up at a point of a surface. Recall that if E is the exceptional curve of a blow-up at a single point of a surface, then E is a (-1)-curve (see [Har77, Proposition 3.1]). The following theorem tell us that any (-1)-curve is the exceptional curve for some blow-up.

Theorem D.0.2 (Castelnuovo, see [Har77, Theorem 5.7])**.** *Let $\Bbbk$ be an algebraically closed field. Let X be a smooth projective $\Bbbk$-surface, and let C be a (-1)-curve. There exists a morphism $f : X \to X'$, where X' is a smooth projective surface, and a point $x \in X'$ such that X is isomorphic via f to the blow-up of X' at x, and C is the exceptional curve.*

Remark D.0.3. Over non-algebraically closed field, an exceptional curve may have self intersection number lower than -1, see [Man86, Theorem 21.5] for more details.

Definition D.0.4 ([Man86, Definition 21.7])**.** Let $\Bbbk$ be a field. A smooth projective $\Bbbk$-surface X is called *minimal* if any birational $\Bbbk$-morphism $X \to X'$ ([Man86, Definition 12.3]), where X' is a smooth projective $\Bbbk$-surface, is an isomorphism.

Lemma D.0.5. *Let $\Bbbk$ be an algebraically closed field. A smooth projective $\Bbbk$-surface X is minimal if and only if there are no (-1)-curves on X.*

Remark D.0.6. Over non-algebraically closed field, a $\Bbbk$-surface X is minimal if and only if there are no exceptional curves on X.

Let $\Bbbk'/\Bbbk$ be a Galois extension of Galois group Γ. If X is a $\Bbbk$-variety, then the natural map $D \mapsto D_{\Bbbk'}$ identifies $\mathrm{Div}(X)$ with the subgroup of Γ-invariant divisors $(\mathrm{Div}(X_{\Bbbk'}))^{\Gamma}$ (see [Man86, Lemma 21.8.1]).

Example D.0.7 (Severi-Brauer varieties)**.** Let $\mathbb{k}$ be a characteristic zero field, and let $n \in \mathbb{N}^*$. A Severi-Brauer $\mathbb{k}$-variety of dimension n is a $\mathbb{k}$-form of $\mathbb{P}^n_{\overline{\mathbb{k}}}$ (see [GS06, Chapter 5]), in particular it is a smooth $\mathbb{k}$-variety. We say that a Severi-Brauer $\mathbb{k}$-variety X of dimension n splits over an algebraic extension $\mathbb{k}'/\mathbb{k}$ in $\overline{\mathbb{k}}$ if $X_{\mathbb{k}'} \cong \mathbb{P}^n_{\mathbb{k}'}$. By [Poo17, Proposition 4.5.10 (Châtelet)], the variety X splits over $\mathbb{k}'$ if and only if $X_{\mathbb{k}'}(\mathbb{k}') \neq \emptyset$. Furthermore, there exists a finite Galois extension $\mathbb{k}'/\mathbb{k}$ that splits X (see also [Pie12, Theorem 3.29, Corollary 3.33]).

Recall that $\mathbb{P}^n_{\overline{\mathbb{k}}}$ has Picard rank one with Picard group generated by any hyperplane of $\mathbb{P}^n_{\overline{\mathbb{k}}}$, and that a principal divisor has degree zero (see [Har77, Chapter II, Proposition 6.4]).

We show that the Picard rank of a Severi-Brauer $\mathbb{k}$-variety is one (see [GS06, Theorem 5.4.10]). Let σ be a $\mathbb{k}$-structure on $\mathbb{P}^n_{\overline{\mathbb{k}}}$ and consider the Severi-Brauer $\mathbb{k}$-variety $(\mathbb{P}^n_{\overline{\mathbb{k}}}, \sigma)$. Let H be an hyperplane of $\mathbb{P}^n_{\overline{\mathbb{k}}}$, then

$$D = \sum_{\gamma \in \Gamma} \sigma_\gamma(H)$$

is an effective Γ-invariant divisor (the sum is finite, see for instance Proposition 1.4.8). Therefore, D corresponds to a divisor on $(\mathbb{P}^n_{\overline{\mathbb{k}}}, \sigma)$ (see [Man86, Lemma 21.8.1]). Since the Picard rank of $\mathbb{P}^n_{\overline{\mathbb{k}}}$ is one, there exists a rational function $g \in \overline{\mathbb{k}}(\mathbb{P}^n_{\overline{\mathbb{k}}})^*$ such that $D = \deg(D)H + \operatorname{div}(g)$. If the Picard rank of $(\mathbb{P}^n_{\overline{\mathbb{k}}}, \sigma)$ is zero, then there exists an invertible rational function $f \in \mathbb{k}(\mathbb{P}^n_{\overline{\mathbb{k}}}/\Gamma) = \overline{\mathbb{k}}(\mathbb{P}^n_{\overline{\mathbb{k}}})^\Gamma$ (see Lemma 3.2.8) such that $D = \operatorname{div}(f)$, hence $\deg(D)H$ is a principal divisor on $\mathbb{P}^n_{\overline{\mathbb{k}}}$, which is a contradiction. Finally, any two hyperplanes of $(\mathbb{P}^n_{\overline{\mathbb{k}}}, \sigma)$ are linearly equivalents since $\mathbb{P}^n_{\overline{\mathbb{k}}}$ has Picard rank one. Therefore, the Picard rank of a Severi-Brauer $\mathbb{k}$-variety is one.

Let $\mathbb{k}$ be a perfect field, let (X, σ) be a smooth projective $\mathbb{k}$-surface, and let $\{E_1, \ldots, E_r\}$ be a Γ-invariant collection of pairwise disjoint (-1)-curves on X, where X is over $\overline{\mathbb{k}}$. There exists a $\overline{\mathbb{k}}$-surface X', r $\overline{\mathbb{k}}$-points $\{P_1, \ldots, P_r\} \subset X'$, and a birational $\overline{\mathbb{k}}$-morphism $X \to X'$ that is the blow-up of X' at $P_1, \ldots, P_r$ such that $E_1, \ldots, E_r$ are the exceptional curves respectively. The $\mathbb{k}$-structure on X extends to a $\mathbb{k}$-structure on X' such that $\{P_1, \ldots, P_r\}$ is Γ-invariant. One can show that $\{E_1, \ldots, E_r\}$ corresponds to an exceptional curve E of X/Γ, and that $\{P_1, \ldots, P_r\}$ corresponds to a closed point P of X'/Γ. So we get a birational $\mathbb{k}$-morphism $X/\Gamma \to X'/\Gamma$ that is the blow-up of X'/Γ at P with E as exceptional curve.

Then, we get the next result.

Theorem D.0.8 ([Man86, Theorem 21.8])**.** *Let (X, σ) be a smooth projective $\mathbb{k}$-surface over a prefect field $\mathbb{k}$, where X is over $\overline{\mathbb{k}}$. The surface (X, σ) is minimal if and only if for every (-1)-curve E on X, there exists an element $\gamma \in \Gamma$ such that $\sigma_\gamma(E) \neq E$ and the intersection $E \cap \sigma_\gamma(E)$ is non-empty.*

The Γ-action on $\operatorname{Div}(X)$ induces a Γ-action on $\operatorname{Pic}(X)$.

Lemma D.0.9 ([Sta, Lemma 0CDT], see also [CTS21, §2.5])**.** *Let $\mathbb{k}$ be a field. Let X be a quasi-compact and quasi-separated scheme over $\mathbb{k}$ with $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{k}$. If X has a $\mathbb{k}$-rational point, then for any Galois extension $\mathbb{k}'/\mathbb{k}$ we have*

$$\operatorname{Pic}(X) = \operatorname{Pic}(X_{\mathbb{k}'})^\Gamma.$$

Moreover, the action of the Galois group on $\operatorname{Pic}(X_{\mathbb{k}'})$ is continuous.

Remark D.0.10 ([Sta, Lemma 0BUG])**.** Let $\mathbb{k}$ be a field. If X is a proper geometrically integral $\mathbb{k}$-scheme (in particular, if X is a Del Pezzo $\mathbb{k}$-surface), then $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{k}$.

The next proposition is a compilation of classical results on the blow-up of $\mathbb{P}^2_{\overline{\mathbb{k}}}$ in r closed points in general position.

Proposition D.0.11 ([Pie12]). *Let $\mathbb{k}$ be an algebraically closed field. Let $f : X \to \mathbb{P}^2_{\mathbb{k}}$ be a blow-up with center r closed points in general position $P_1, \dots, P_r \in \mathbb{P}^2_{\mathbb{k}}$, $1 \leq r \leq 8$. For $i = 1, \dots, r$, let E_i be the inverse image of P_i under f, and let L be the inverse image of a line in $\mathbb{P}^2_{\mathbb{k}}$ that does not contain any of the P_i. Then*

(i) The Picard group is $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$*;*

(ii) $L^2 = 1$, $E_i^2 = -1$, $L \cdot E_i = 0$, $E_i \cdot E_j = 0 \ \forall i, j \in \{i, \dots, r\}, i \neq j$, *and;*

(iii) A canonical divisor is $K_X = -3L + E_1 + \cdots + E_r$, *and* $K_X^2 = 9 - r$.

Furthermore, if $r \geq 2$, let $L_{i,j}$, where $i < j$, be the strict transform under f of the line containing P_i and P_j.

(iv) $L_{i,j} \cdot E_k = 0$ *if and only if* $i \neq k \neq j$, *and* $L_{i,j} \cdot E_k = 1$ *otherwise;*

(v) $L_{i,j} \cdot L_{k,l} = 0$ *if and only if* $\{i, j\} \cap \{k, l\} \neq \emptyset$, *and* $L_{i,j} \cdot L_{k,l} = 1$ *otherwise.*

Appendix E

Altmann-Hausen presentation

In this appendix, $\Bbbk$ is an algebraically closed field of characteristic zero.

E.1 Altmann-Hausen presentation

Given a normal semi-projective variety Y together with a pp-divisor $\mathcal{D}$ on Y, we state the complete result obtained by Altmann and Hausen in [AH06].

Definition E.1.1 ([AH06], see also [BH06] and [BH06, Definition 3.1])**.** Let $\mathbb{T}$ be a $\Bbbk$-torus acting on a variety X.

(i) A *categorical quotient* is a variety Y together with a $\mathbb{T}$-invariant morphism $\pi : X \to Y$ satisfying the following universal property: any other $\mathbb{T}$-invariant morphism $X \to Y$ admits a unique factorization through π.

(ii) A morphism $\pi : X \to Y$ is called a *good quotient* for the $\mathbb{T}$-action on A if it is affine, $\mathbb{T}$-invariant, surjective, and the pullback map $\pi^\sharp : \mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^{\mathbb{T}}$ is an isomorphism.

(iii) A *geometric quotient* is a good quotient $\pi : X \to Y$ that is also an orbit space, i.e. $\pi^{-1}(y)$ is a single $\mathbb{T}$-orbit for each closed point $y \in Y$.

If a good quotient exists, then it is categorical. In particular, a good quotient is unique up to a unique isomorphism, and the quotient space is denoted $X//\mathbb{T}$. Furthermore, a geometric quotient is a good quotient.

Theorem E.1.2 ([AH06, Theorem 3.1])**.** *Let Y be a normal semi-projective, let N be a lattice, let $\omega_N \subset N_{\mathbb{Q}}$ be a pointed cone, and let M be the dual lattice of N. Given an ω_N-pp-divisor on Y, consider the $\mathcal{O}_Y$-algebra*

$$\mathcal{A} := \bigoplus_{m \in \sigma_N^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(m))\mathfrak{X}_m,$$

and the relative spectrum $\tilde{X} := \mathrm{Spec}_Y(\mathcal{A})$ (see [Sta, §01LL and §01LQ] for the construction of the relative spectrum). Let $\mathbb{T} := \mathrm{Spec}(\Bbbk[M])$.

(i) The scheme $\tilde{X}$ is a normal variety of dimension $dim(Y) + dim(\mathbb{T})$, and the grading of $\mathcal{A}$ defines an effective torus action $\mathbb{T} \times \tilde{X} \to \tilde{X}$ having the canonical map $\pi : \tilde{X} \to Y$ as a good quotient.

(ii) The ring of global sections $A[Y, \mathcal{D}] := \mathrm{H}^0(Y, \mathcal{A}) = \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a finitely generated M-graded normal $\Bbbk$-algebra, and we have a proper, birational $\mathbb{T}$-invariant contraction morphism $\tilde{X} \to X[Y, \mathcal{D}]$, where $X[Y, \mathcal{D}] = \mathrm{Spec}(A[Y, \mathcal{D}])$.

(iii) Let $m \in \omega_N^\vee \cap M$, and let $f \in \mathrm{H}^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))$. Then, $\pi(\tilde{X}_{f\mathfrak{X}_m}) = Y_f$. In particular, if Y_f is affine, then so is $\tilde{X}_{f\mathfrak{X}_m}$, and the canonical map $\tilde{X}_{f\mathfrak{X}_m} \to X_f$ is an isomorphism. Moreover, even for non-affine Y_f, we have

$$\mathrm{H}^0(Y_f, \mathcal{A}) = \bigoplus_{m \in \omega_N^\vee \cap M} (A[Y, \mathcal{D}])_{f\mathfrak{X}_m})_m.$$

Remark E.1.3. Let $(\omega_N, Y, \mathcal{D})$ be an AH-datum.

1. There is a $\mathbb{T}$-invariant rational map $X \dashrightarrow Y$.
2. If Y is affine, then $\tilde{X} = X[Y, \mathcal{D}]$ (see [Sta, Lemmas 01LP, 01LT01LV]), and we get a good quotient $\pi : X[Y, \mathcal{D}] \to Y$.

E.2 Illustration: $\mathbb{G}_{m,\mathbb{C}}$-actions on normal affine surfaces

Actions of the one-dimensional torus $\mathbb{G}_{m,\mathbb{C}}$ on normal affine $\mathbb{C}$-surfaces were described by Flenner and Zaidenberg in [FZ03]; they described such surfaces using some divisor on a smooth curve. The Altmann-Hausen presentation extends, in some way, the one obtained by Flenner-Zaidenberg: from an AH-datum associated to a $\mathbb{G}_{m,\mathbb{C}}$-surface, one can recover the datum exhibited in [FZ03] (see explanations in [AH06]). Following [FZ03], one distinguish three cases of $\mathbb{G}_{m,\mathbb{C}}$-actions on normal affine $\mathbb{C}$-surfaces. In this section, we illustrate each case by a standard example.

E.2.1 Elliptic action

A $\mathbb{G}_{m,\mathbb{C}}$-action on a normal affine surface X is called elliptic if it has a unique fixed point which belongs to the closure of every one-dimensional orbit. The complement of the unique fixed point is fibered by the one-dimensional orbits over a projective curve. The coordinate ring is $\mathbb{N}$-graded, the weight cone is $\mathbb{Q}_{\geq 0}$, and $\mathbb{C}[x]_0 = \mathbb{C}$. More precisely, there exists $m_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq m_0, \mathbb{C}[X]_m \neq \{0\}$. A typical example is the following one.

Example E.2.1. Consider the action of $\mathbb{T} := \mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x, y])$ defined by

$$\mu(t, (x, y)) = (tx, ty).$$

The coordinate ring of $\mathbb{A}^2_{\mathbb{C}}$ is $\mathbb{N}$-graded:

$$\mathbb{C}[x, y] = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[x, y]_m,$$

where $\mathbb{C}[x, y]_m$ denotes the set of homogeneous polynomials of degree m. Note that $\mathbb{C}[x, y]_0 = \mathbb{C}$. Therefore, the weight cone of this action is $\omega_M = \mathbb{Q}_{\geq 0}$. Its dual cone is $\omega_N = \mathbb{Q}_{\geq 0}$.

We compute an AH-datum $(\omega_N, Y, \mathcal{D})$ for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$. Note that $\mathbb{A}^2_{\mathbb{C}}$ is a $\mathbb{G}^2_{m,\mathbb{C}}$-toric variety, and that the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ comes from the closed immersion

$$\mathbb{T} \hookrightarrow \mathbb{G}^2_{m,\mathbb{C}}, \quad t \mapsto (t, t).$$

Let $M = \mathbb{Z}$, let $M' = \mathbb{Z}^2$, and let (e_1, e_2) be the canonical basis of $(M')_{\mathbb{Q}}$. We have $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M])$ and $\mathbb{G}^2_{m,\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[M'])$. Then, we get the following short exact sequences of lattices

$$0 \longrightarrow N \xrightarrow{F} N' \xrightarrow{P} N_Y \longrightarrow 0$$

$$0 \longrightarrow M_Y \xrightarrow{P^*} M' \xrightarrow{F^*} M \longrightarrow 0,$$

admitting a cosection $s : N' \to N$, where $M_Y \cong \mathbb{Z}$,

$$P = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad s = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let $\mathbb{T}_Y := \mathrm{Spec}(\mathbb{C}[M_Y])$. Then, an AH-quotient for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ is the $\mathbb{T}_Y$-toric variety obtained from the fan in $(N_Y)_{\mathbb{Q}}$ generated by $v_1 := P(e_1)$ and $v_2 := P(e_2)$. Hence, $Y = \mathbb{P}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \cup \{\infty\}$, where $\mathbb{A}^1_{\mathbb{C}}$ is the affine open subset of $\mathbb{P}^1_{\mathbb{C}}$ obtained from the cone generated by v_1. The toric divisors associated to the rays $< v_1 >$ and $< v_2 >$ are respectively $D_{v_1} = \{0\}$ and $D_{v_2} = \{\infty\}$. The coefficients of the ω_N-pp divisor

$$\mathcal{D} = \Delta_{v_1} \otimes \{0\} + \Delta_{v_2} \otimes \{\infty\}$$

are

$$\Delta_{v_1} = s(P^{-1}(v_1) \cap \mathbb{Q}^2_{\geq 0}) = s(e_1 + \ker(P)) = [0, +\infty[= \omega_N,$$

$$\Delta_{v_2} = s(P^{-1}(v_2) \cap \mathbb{Q}^2_{\geq 0}) = s(e_2 + \ker(P)) = [1, +\infty[.$$

Therefore,

$$\mathcal{D} = [1, +\infty[\otimes\{\infty\}.$$

Note that we get a $\mathbb{T}$-invariant rational map $\mathbb{A}^2_{\mathbb{C}} \dashrightarrow Y$ defined by the geometric quotient

$$\pi : \mathbb{A}^2_{\mathbb{C}}\backslash\{(0,0)\} \to Y, \quad (x,y) \mapsto \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases}.$$

We know describe the graded algebra isomorphism $\mathbb{C}[x,y] \cong A[Y, \mathcal{D}]$. Let $\mathbb{C}(Y) = \mathbb{C}(z)$. Let $m \in \mathbb{Z}$.

$$\mathcal{D}(m) = m \otimes \{\infty\}.$$

We get

$$A[Y, \mathcal{D}] = \bigoplus_{m \in \mathbb{N}} (\mathbb{C} \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^m)\, \mathfrak{X}_m \subset \mathbb{C}(Y)[M].$$

On the other hand, we get

$$\mathbb{C}[x,y] = \bigoplus_{m \in \mathbb{N}} \left(\mathbb{C} \oplus \mathbb{C}\frac{x}{y} \oplus \cdots \oplus \mathbb{C}\frac{x^m}{y^m}\right) y^m \subset \mathrm{Frac}(\mathbb{C}[x,y])_0[x].$$

Finally, we identify z with $\frac{x}{y}$, and $\mathfrak{X}_m$ with y^m.

E.2.2 Parabolic action

A $\mathbb{G}_{m,\mathbb{C}}$-action on a normal affine surface X is called parabolic if the set of its fixed point is one-dimensional. The $\mathbb{G}_{m,\mathbb{C}}$-surface is fibered over an affine curve, and this fibration is invariant under the $\mathbb{G}_{m,\mathbb{C}}$-action. The coordinate ring is $\mathbb{N}$-graded, the weight cone is $\mathbb{Q}_{\geq 0}$, and $\mathbb{C}[x]_0 \neq \mathbb{C}$. More precisely, for all $m \in \mathbb{N}, \mathbb{C}[X]_m \neq \{0\}$. A simple example is the cylinder $Y \times \mathbb{A}^1_{\mathbb{C}}$ over an affine curve Y, where $\mathbb{G}_{m,\mathbb{C}}$ acts on the second factors.

Example E.2.2. Consider the action of $\mathbb{T} := \mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y])$ defined by

$$\mu(t, (x,y)) = (tx, y).$$

The coordinate ring of $\mathbb{A}^2_{\mathbb{C}}$ is $\mathbb{N}$-graded:

$$\mathbb{C}[x,y] = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[x,y]_m = \mathbb{C}[y]x^m.$$

Note that $\mathbb{C}[x,y]_0 = \mathbb{C}[y]$. Therefore, the weight cone of this action is $\omega_M = \mathbb{Q}_{\geq 0}$. Its dual cone is $\omega_N = \mathbb{Q}_{\geq 0}$.

We compute an AH-datum $(\omega_N, Y, \mathcal{D})$ for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$. Note that $\mathbb{A}^2_{\mathbb{C}}$ is a $\mathbb{G}^2_{m,\mathbb{C}}$-toric variety, and that the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ comes from the closed immersion

$$\mathbb{T} \hookrightarrow \mathbb{G}^2_{m,\mathbb{C}}, \quad t \mapsto (t, 1).$$

Let $M = \mathbb{Z}$, let $M' = \mathbb{Z}^2$, and let (e_1, e_2) be the canonical basis of $(M')_{\mathbb{Q}}$. We have $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M])$ and $\mathbb{G}^2_{m,\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[M'])$. Then, we get the short exact sequences of lattices of Example E.2.1 with

$$P = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let $\mathbb{T}_Y := \operatorname{Spec}(\mathbb{C}[M_Y])$. Then, an AH-quotient for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ is the $\mathbb{T}_Y$-toric variety obtained from the fan in $(N_Y)_{\mathbb{Q}}$ generated by $v_2 := P(e_2)$. Hence, $Y = \mathbb{A}^1_{\mathbb{C}}$. The toric divisor associated to the ray $< v_2 >$ is $D_{v_2} = \{0\}$. The coefficient of the ω_N-pp divisor

$$\mathcal{D} = \Delta_{v_2} \otimes \{0\}$$

is

$$\Delta_{v_2} = s(P^{-1}(v_2) \cap \mathbb{Q}^2_{\geq 0}) = s(e_2 + \ker(P)) = [0, +\infty[= \omega_N.$$

Therefore, $\mathcal{D}$ is the trivial ω_N-pp divisor on Y.

Note that we get a good quotient

$$\pi : \mathbb{A}^2_{\mathbb{C}} \to Y \cong \operatorname{Spec}(\mathbb{C}[x, y]^{\mathbb{T}}), \quad (x, y) \mapsto y.$$

This quotient is not a geometric one since there are non-closed orbits.

We know describe the graded algebra isomorphism $\mathbb{C}[x, y] \cong A[Y, \mathcal{D}]$. Let $Y = \operatorname{Spec}(\mathbb{C}[z])$. Let $m \in \mathbb{Z}$.

$$\mathcal{D}(m) = 0.$$

We get

$$A[Y, \mathcal{D}] = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[z]\mathfrak{X}_m \subset \mathbb{C}(Y)[M].$$

On the other hand, we get

$$\mathbb{C}[x, y] = \bigoplus_{m \in \mathbb{N}} \mathbb{C}[y]x^m \subset \operatorname{Frac}(\mathbb{C}[x, y])_0[x].$$

Finally, we identify z with y, and $\mathfrak{X}_m$ with x^m.

E.2.3 Hyperbolic action

A $\mathbb{G}_{m,\mathbb{C}}$-action on a normal affine surface X is called hyperbolic if there is only a finite number of fixed point, and each one of them belongs to the closure of exactly two one-dimensional orbits. The $\mathbb{G}_{m,\mathbb{C}}$-surface is fibered over an affine curve, and this fibration is invariant under the $\mathbb{G}_{m,\mathbb{C}}$-action. The coordinate ring is $\mathbb{Z}$-graded, the weight cone is $\mathbb{Q}$, and $\mathbb{C}[x]_0 \neq \mathbb{C}$. More precisely, for all $m \in \mathbb{Z}, \mathbb{C}[X]_m \neq \{0\}$.

Example E.2.3. Consider the action of $\mathbb{T} := \mathbb{G}_{m,\mathbb{C}}$ on $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x, y])$ defined by

$$\mu(t, (x, y)) = (tx, t^{-1}y).$$

The coordinate ring of $\mathbb{A}^2_{\mathbb{C}}$ is $\mathbb{Z}$-graded:

$$\mathbb{C}[x, y] = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}[x, y]_m,$$

where

$$\mathbb{C}[x, y]_m = \begin{cases} \mathbb{C}[xy]x^m & \text{if } m > 0 \\ \mathbb{C}[xy]y^{-m} & \text{if } m < 0 \end{cases}.$$

Note that $\mathbb{C}[x,y]_0 = \mathbb{C}[xy]$. Therefore, the weight cone of this action is $\omega_M = \mathbb{Q}$. Its dual cone is $\omega_N = \{0\}$.

We compute an AH-datum $(\omega_N, Y, \mathcal{D})$ for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$. Note that $\mathbb{A}^2_{\mathbb{C}}$ is a $\mathbb{G}^2_{m,\mathbb{C}}$-toric variety, and that the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ comes from the closed immersion

$$\mathbb{T} \hookrightarrow \mathbb{G}^2_{m,\mathbb{C}}, \quad t \mapsto (t, t^{-1}).$$

Let $M = \mathbb{Z}$, let $M' = \mathbb{Z}^2$, and let (e_1, e_2) be the canonical basis of $(M')_{\mathbb{Q}}$. We have $\mathbb{T} = \mathrm{Spec}(\mathbb{C}[M])$ and $\mathbb{G}^2_{m,\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[M'])$. Then, we get the short exact sequences of lattices of Example E.2.1 with

$$P = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let $\mathbb{T}_Y := \mathrm{Spec}(\mathbb{C}[M_Y])$. Then, an AH-quotient for the $\mathbb{T}$-action on $\mathbb{A}^2_{\mathbb{C}}$ is the $\mathbb{T}_Y$-toric variety obtained from the fan in $(N_Y)_{\mathbb{Q}}$ generated by $v_1 := P(e_1)$. Hence, $Y = \mathbb{A}^1_{\mathbb{C}}$. The toric divisor associated to the ray $< v_1 >$ is $D_{v_1} = \{0\}$. The coefficient of the ω_N-pp divisor

$$\mathcal{D} = \Delta_{v_1} \otimes \{0\}$$

is

$$\Delta_{v_1} = s(P^{-1}(v_1) \cap \mathbb{Q}^2_{\geq 0}) = s(e_1 + \ker(P)) = [0, 1].$$

Therefore,

$$\mathcal{D} = [0, 1] \otimes \{0\}.$$

Note that we get a good quotient

$$\pi : \mathbb{A}^2_{\mathbb{C}} \to Y \cong \mathrm{Spec}(\mathbb{C}[x,y]^{\mathbb{T}}), \quad (x, y) \mapsto xy.$$

This quotient is not a geometric one since there are non-closed orbits.

We know describe the graded algebra isomorphism $\mathbb{C}[x,y] \cong A[Y, \mathcal{D}]$. Let $Y = \mathrm{Spec}(\mathbb{C}[z])$. Let $m \in \mathbb{Z}$.

$$\mathcal{D}(m) = \begin{cases} 0 \otimes \{0\} & \text{if } m \geq 0 \\ m \otimes \{0\} & \text{if } m < 0 \end{cases}.$$

We get

$$A[Y, \mathcal{D}] = \bigoplus_{m \in \mathbb{Z}_{<0}} \mathbb{C}[z] z^m \mathfrak{X}_m \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}[z] \mathfrak{X}_m \subset \mathbb{C}(Y)[M].$$

On the other hand, we get

$$\mathbb{C}[x,y] = \bigoplus_{m \in \mathbb{Z}_{<0}} \mathbb{C}[xy](xy)^{-m} x^m \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}[xy] x^m \subset \mathrm{Frac}(\mathbb{C}[x,y])_0[x].$$

Finally, we identify z with xy, and $\mathfrak{X}_m$ with x^m.

Appendix F

Finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$

In this appendix, we describe the conjugacy classes of the finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ exhibited in Proposition 4.2.3.

$$\begin{aligned}\mathcal{G}_1 = \langle x, s\rangle &= \left\{id, x, x^2, x^3, x^4, x^5, s, xs, x^2s, x^3s, x^4s, x^5s\right\}\\ &= \left\{ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}; \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}; \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}; \right.\\ &\qquad \left. \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}; \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}; \begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix}; \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix} \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix} \right\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_2 = \langle ds, s\rangle &= \left\{id, ds, (ds)^2, (ds)^3, s, d, (ds)^2s, (ds)^3s\right\}\\ &= \left\{ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}; \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}; \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}; \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}; \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}; \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}; \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \right\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_3 = \langle x^2, s\rangle &= \left\{id, x^2, x^4, s, x^2s, x^4s\right\}\\ &= \left\{ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}; \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}; \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}; \begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix}; \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix} \right\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_4 = \langle x^2, -s\rangle &= \left\{id, x^2, x^4, -s, -x^2s, -x^4s\right\}\\ &= \left\{ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}; \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}; \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}; \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}; \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} \right\}\end{aligned}$$

$$\mathcal{G}_5 = \langle d, -d \rangle = \{id, d, -id, -d\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$\mathcal{G}_6 = \langle s, -s \rangle = \{id, s, -id, -s\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{G}_7 = \langle x \rangle = \left\{ id, x, x^2, x^3, x^4, x^5 \right\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$$

$$\mathcal{G}_8 = \langle ds \rangle = \left\{ id, ds, (ds)^2, (ds)^3 \right\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{G}_9 = \langle x^2 \rangle = \left\{ id, x^2, x^4 \right\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{G}_{10} = \langle -id \rangle = \{id, -id\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$\mathcal{G}_{11} = \langle d \rangle = \{id, d\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{G}_{12} = \langle s \rangle = \{id, s\}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{G}_{13} = \langle id \rangle = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

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Abstract

Over an algebraically closed field of characteristic zero, normal affine varieties endowed with an effective torus action were described by Altmann and Hausen in 2006 by a geometrico-combinatorial presentation. Using Galois descent tools, we extend this presentation to the case where the ground field is an arbitrary field of characteristic zero. In this context, the acting torus may be non split and may have non-trivial torsors, thus we need additional data to encode such varieties. We provide some situations where the generalized Altmann-Hausen presentation simplifies. For instance, if the acting torus is split, we recover *mutatis mutandis* the original Altmann-Hausen presentation. Finally, we focus on the real setting and on affine varieties endowed with a two-dimensional torus action. In the latter case we relate the torsor appearing in the generalized Altmann-Hausen presentation to some smooth toric Del Pezzo surface.

Résumé

Les variétés affines normales munies d'une action effective d'un tore ont été décrites en 2006 par Altmann et Hausen à l'aide de données géométrico-combinatoires dans le cas où le corps de base est algébriquement clos et de caractéristique zéro. En utilisant des outils de descente galoisienne, nous étendons la présentation Altmann-Hausen des variétés affines normales munies d'une action effective d'un tore au cas où le corps de base est un corps quelconque de caractéristique zéro. Dans ce cadre, un tore n'est pas nécessairement déployé et peut de plus avoir des torseurs non triviaux. Nous avons donc besoin de données supplémentaires pour décrire de telles variétés. Nous fournissons ensuite des situations dans lesquelles la présentation Altmann-Hausen généralisée se simplifie. Ainsi, si le tore considéré est déployé, nous retrouvons *mutatis mutandis* la présentation Altmann-Hausen originelle. Enfin, nous nous intéressons au cas où le corps de base est $\mathbb{R}$, ainsi qu'au cas d'actions de tores de dimension 2. Dans ce dernier cas, nous faisons un lien entre le torseur apparaissant dans la présentation Altmann-Hausen généralisée et certaines surfaces toriques lisses de Del Pezzo.