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Local monomialization of generalized real analytic functions

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Résumé

Les fonctions analytiques généralisées sont définies par des séries convergentes de monômes à coefficients réels et exposants réels positifs. Nous étudions l'extension de la géométrie analytique réelle associée à ces algèbres de fonctions. Nous introduisons pour cela la notion de variété analytique réelle généralisée. Il s'agit de variétés topologiques à bord munies de la structure du faisceau des fonctions analytiques réelles généralisées. Notre résultat principal est un théorème de monomialisation locale de ces fonctions.

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Introduction

Resolution of Singularities is an important subject in many fields of mathematics, being a fundamental tool in the resolution of lots of important problems as well as a set of elaborated techniques resulting very useful in full of different contexts. This explain that, even if it is a classical discipline, it is gaining in importance and constantly progressing.

The general setting is well known: given an object that we want to study (manifolds, varieties, functions, foliations, vector fields, diffeomorphisms, families) it may present singularities making it non trivial. The strategy to understand the richness behind these singularities consists on modify the ambient space by terms of compositions of a particular kind of well known transformations (blowing-ups) given rise to an object with "simpler singularities" easier to study. The problem is then translated to the understanding of the combination of the blowing-up transformations and the relation between the geometry of the object obtained and the initial one.

This method was applied to the classical case of algebraic varieties by using algebro-geometric techniques by Zariski and the Italian School which constituted the foundations of modern Algebraic Geometry. The Hironaka's work of 1964 suppose an inflection point in the resolution of singularities theory. It shows resolution of singularities on algebraic varieties of characteristic zero. Since then, many of the important progress in resolution of singularities has been based on this work: resolution of singularities on real and complex analytic manifolds, the effective resolution, embedded resolution of singularities, local uniformization, monomialization, resolution of singularities on complex foliations of codimension one, resolution of singularities on vector fields, rectiliniarization of subanalytic sets,...

The framework of this doctoral thesis is resolution of singularities on real analytic sets. One of the vicissitudes of this resolution of singularities is that of monomialization of germs of real analytic functions, consisting on the process to transform such a germ f on a function $\tilde{f}$ which can be locally expressed as

$$\tilde{f}(x_1, \ldots, x_m) = x_1^{a_1} \cdots x_m^{a_m} g, \quad \text{with } g(0) \neq 0,$$

that is, the product of a monomial times a function which does not vanish (we say that $\tilde{f}$ **is a locally monomial function**). In this way, the set of zeroes of $\tilde{f}$ is locally very simple: it consists on the coordinate hyperplanes with respect to a given coordinate system. For instance, the process of monomialization of germs of real analytic functions is well known. As we can see in [2], which serve us as a model for our work, there are two crucial arguments to show this result: first one is the Noetherianity of the ring of analytic germs and the other one is Weierstrass Preparation Theorem.

Monomialization of wider classes of germs is proved in more general contexts where neither Noetherianity nor Weierstrass Preparation Theorem holds. For instance, [3] about quasi-analytic Denjoy-Carleman classes or [4] about quasi-analytic classes appearing as formal solutions of a certain kind of differential equations. They are quite important classes on the framework of o-minimal structures and model theory. In these cases, the germs considered admit usual formal series (with natural exponents) as asymptotic expansions with the property of quasi-analyticity, or uniqueness of such an expansion, which is essential for the adaptation of the proof in the analytic setting to this more general case.

In this work we remain inside the class of *real generalized power series* $\mathbb{R}[[X^*]]$. These are series of the form

$$s = \sum_{\alpha \in [0,\infty)^m} s_\alpha X_1^{\alpha_1} \cdots X_m^{\alpha_m}$$

with $s_\alpha \in \mathbb{R}$ such that the support $\mathrm{Supp}(s) = \{\alpha / s_\alpha \neq 0\}$ is contained in a cartesian product $S_1 \times \cdots \times S_m$ where $S_j \subset [0, \infty)$ is a well ordered subset for the usual order in $\mathbb{R}$. We focus on the subclass $\mathbb{R}\{X^*\}$ of *real convergent generalized power series* for the usual notion of convergence of infinite sums of functions (see definition 1.2.3) which are, so to speak, the smallest quasi-analytic subclass of $\mathbb{R}[[X^*]]$. It follows, from the definition, that we do not have Noetherianity on these classes. For instance, if $m = 1$, the ideal generated by $\{X^\alpha : \alpha > 0\}$ is not finitely generated.

Formal generalized power series as well as convergent, appear associated to natural problems on differential or functional equations. By example, the function

$$x \mapsto \zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n} : [0, e^{-2}] \longrightarrow \mathbb{R}$$

where ζ is the Riemann zeta function,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Elements in $\mathbb{R}\big\{X^*\big\}$ give rise to real functions by passing to the limit, which, being the exponents of the variables not necessary integers, are not defined in a whole neighborhood of the origin in $\mathbb{R}^m$. Those functions are then defined on the hyper-cube $[0, \varepsilon]^m$ where they are continuous, and as we will see, in the interior of $[0, \varepsilon]^m$ they are analytic. We will call them **real generalized analytic functions** or, for short, **$\mathcal{G}$-functions**.

This kind of functions has been deeply studied by Van den Dries and Speissegger in [1] from the point of view of o-minimal properties: roughly speaking, sets defined by equalities and inequalities using these functions and the linear projections of these sets have the same geometrical behavior as real (global) subanalytic sets: finitude of the number of connected components, finite analytic stratifications, triangulations, etc. The condition on the well ordered support replace, in some way, Noetherianity in the proof of those finitude results. An other crucial ingredient, proved also in [1], is the version of the Weierstrass Preparation Theorem with respect to regular "analytic variables" (appearing only with integer exponents).

Using as a thread the mentioned work [1] and the techniques on resolution of singularities appearing in [2] and [3] we present in this work the **local monomialization of real generalized analytic functions**.

In order to present it in a general geometrical context we construct the category of **real generalized analytic manifolds**. We use the generalized power series analogously to the power series in the classical case of analytic manifolds. One of the main peculiarities is that generalized analytic manifolds will be manifolds with boundary and corners. This is a geometrical consequence of the existence of non analytic variables in the generalized case: a function like x^λ for a non integer λ is only defined for positive values of the variable x.

For a better comprehension of the differences with the classical analytic case, we will use analytic manifolds with boundary and corners. We present at the beginning of chapter two a brief recall of these objects and their properties in the language of subsheaves on $\mathbb{R}$-algebras of continuous functions (called locally ringed spaces).

The Appendix is devoted to a brief exposition of the general concepts and basic properties in this theory. In a few words, we consider the category $\mathfrak{C}$ where an object of $\mathfrak{C}$ is a pair $X = (|X|, \mathfrak{C}_X)$ where $|X|$ is a topological space and $\mathfrak{C}_X$ is a sheaf of $\mathbb{R}$-algebras of continuous functions over $|X|$ such that, for each $p \in |X|$, the stalk $\mathfrak{C}_{X,p}$ is a local $\mathbb{R}$-algebra. The morphisms between two objects $X = (|X|, \mathfrak{C}_X)$ and $Y = (|Y|, \mathfrak{C}_Y)$ are pairs $(\varphi, \varphi^\sharp)$ where $\varphi : |X| \to |Y|$ is a continuous map and $\varphi^\sharp : \mathfrak{C}_Y \to \varphi_*\mathfrak{C}_X$ is the associated morphism of sheaves determined by φ by composition; that is, if $f \in \mathfrak{C}_Y(V)$ is a section over the open set V of $|Y|$, then $\varphi^\sharp(f) = f \circ \varphi \in \varphi_*\mathfrak{C}_X(V) = \mathfrak{C}_X(\varphi^{-1}(V))$.

We will define $\mathcal{G}$ the category of real generalized analytic manifolds and $\mathcal{O}$ the category of real analytic manifolds with boundary and corners as subcategories of $\mathfrak{C}$. In both cases $\mathcal{O}$ and $\mathcal{G}$, an object will be a locally ringed space on $\mathbb{R}$-algebras of continuous functions whose underlying topological space is a topological manifold with boundary of pure dimension, all of them locally homeomorphic to a local model $\mathbb{R}^k_{\geq 0}$ for some k. By a convenient choice of the second component of the object (that is the sheaf of continuous functions), objects in the subcategory $\mathcal{O}$ will be the (standard) real analytic manifolds with boundary and corners, when the chosen sheaf is such that it is locally isomorphic to the sheaf of analytic functions in the local model (those which are sums of standard real convergent power series). Objects of the subcategory $\mathcal{G}$, on the contrary, are defined with the property that the sheaf is locally isomorphic to the sheaf of generalized analytic functions on the local model. They will be called generalized real analytic manifolds.

Once the geometrical context is given, we concentrate on the statement and the proof of the main result, Theorem 3.4.2.

Local Monomialisation of $\mathcal{G}$-analytic functions.- Let M be a generalized analytic manifold and $f \in \mathcal{G}(M)$ a $\mathcal{G}$-analytic function. Given $p \in |M|$ there exists a finite family

$$\Sigma = \{\pi_j : W_j \to M, L_j\}_{j \in J}$$

where

1. each π_j is the composition of a sequence of finitely many local blowing-ups (with admissible centers)
$$\pi_j : W_j = W_{j,n_j} \overset{\pi_{j,n_j}}{\to} W_{j,n_j-1} \overset{\pi_{j,n_j-1}}{\to} W_{j,n_j-2} \cdots \overset{\pi_{j,1}}{\to} W_{j,0} = M$$
2. each L_j is a compact subset of $|W_j|$ such that $\cup_{j \in J} \pi_j(L_j)$ is a compact neighborhood of p in $|M|$.

such that for all $j \in J$, $f \circ \pi_j : W_j \to \mathbb{R}$ is locally monomial at any point of L_j (i.e. it writes in certain coordinates as a monomial times a nowhere vanishing function). We can furthermore take such a family Σ such that any of the local blowing-ups involved in it is with an admissible center of codimension ≤ 2.

Let us explain the terminology involved in the statement of the main theorem. First, an admissible center of a generalized or standard manifold is a *submanifold* of the ambient space (a similar notion to that of a smooth analytic submanifold of an analytic manifold without boundary) which is locally given by the zeros of some local coordinates. Geometrically, it has "normal crossings" with the boundary of the ambient manifold.

Let us now get into the definition of blowing up morphism with closed admissible center in the category of generalized analytic manifolds. We can proceed as follows.

First, we recall what a blowing-up morphism is in the category of (standard) real analytic manifolds with boundary and corners. This is a quite well known notion in the category of analytic manifolds without boundary. Essentially, it is a proper analytic morphism that replaces the center of blowing-up by an hypersurface taking account of the set of lines in a normal bundle of the center, inducing an isomorphism outside this hypersurface, called the exceptional divisor of the blowing-up. In our point of view, since the analytic manifolds that we consider have boundary and corners, we follow the suitable approach of considering the so called *oriented real blowing-up*, in contrast with the (relatively more usual) *projective real blowing-up*. The main difference is that, in the former case, points of the center of blowing-ups are replaced by the set of half-lines, normal to the center, defined by means of a system of coordinates; while for the projective blowing-up, points are replaced by the set of normal lines through them. At boundary points, we have no entire but half-lines, thus showing the convenience of the use of oriented blowing-up.

As a consequence, the exceptional divisor (the inverse image of the center) always becomes a new boundary component to the blown-up space even if the center of blowing-up is contained in the interior of the standard analytic manifold (where normal entire lines are defined). The choice for this kind of blowing-up also at interior points is based only on consideration of coherency.

In compensation, we do not alter the properties of orientability of the manifold, although in these pages, where we only use local blowing-ups (that is, whose center is just a closed "subvariety" on some open domain), this point does not give us an advantage.

In order to introduce the concept of blowing-up morphism in the category of generalized analytic manifolds, we notice first a (a priori unexpected) peculiarity that does not occur in the standard case: if we proceed defining directly the blowing-up for the local model (as we may do in the standard case) by "gluing" the local charts, we could obtain different (non-isomorphic) blowing-up morphisms for different choices of local coordinates. Thus, our concept of blowing-up morphism is not only attached to an admissible center of blowing-up, but relative also to the choice of coordinates.

A convenient procedure to define blowing-ups in the category of generalized manifolds uses the concept of *standardization*. In few words, a generalized manifold is said to be *standardizable* if it is isomorphic (the isomorphism will be called a standardization) to a generalized analytic manifold obtained from a standard analytic manifold (with boundary and corners) by ”enriching” its structure of analytic functions by the procedure of adding to the sheaf of analytic functions in a coordinate atlas those generalized analytic functions in the same coordinate atlas, just in a similar way as we consider an algebraic variety as having an analytic structure by adding analytic functions to the algebraic ones. The theory of enrichments and standardizable manifolds is developed in section 2.4.

Once we have a standardizable generalized manifold M and a fixed standardization ϕ to the enrichment of some standard manifold A, we can translate blowing-ups with admissible centers in A (in the standard setting) to corresponding admissible centers in the generalized manifold M via the standardization. The details of this definition are presented in section 3.3.

As we can expect, the peculiarity noticed above on the dependence on the coordinates is reflected in the fact that the blowing-up so defined depends on the considered standardization ϕ of M.

The term ”local blowing-up” in the statement of the main theorem stands, as usual, for blowing-up with an admissible center which is locally closed, that is, closed in some open subset of the ambient space considered. The existence of such local blowing-ups is guaranteed by the Proposition 3.1.10 below where we prove that any point in a generalized manifold has a neighborhood which is standardizable (this is just given by the existence of local coordinates).

However, the global situation is not that easy. We show in 2.4.2 concrete examples of generalized analytic manifolds which are not standardizable. Such examples are interpreted as exotic examples that could complicate the theory of generalized analytic manifolds in its full generality. In fact, with this peculiarity in mind, no good notion of blowing-up is possible when the closed admissible center to be blown-up has not a neighborhood which admits a standardization. This is the case of the example in 2.3.5: it consists of a three dimensional generalized manifold whose boundary consists of a circle which has no standardizable open neighborhood. The geometric interpretation of this pathological example is that this center has not a good "global normal bundle" of half-lines.

The existence of such pathological examples of non-standardizable generalized manifolds may constitute an essential point of difficulty on the attempt to prove a *Global Resolution of Singularities* of generalized functions.

This problem is, roughly speaking, as follows. Start with M a neighborhood of a given fix point of the manifold. Can we improve our statement of Local Monomialization Theorem so that the family Σ consists of a single sequence of blowing-ups ($|J| = 1$)

$$\pi : M_n \overset{\pi_n}{\to} M_{n-1} \overset{\pi_{n-1}}{\to} M_{n-2} \cdots \overset{\pi_1}{\to} M_0 = M$$

and such that, moreover, each π_j is a *global* blowing-up; that is, a blowing-up with respect to a closed admissible center of the whole manifold M_{j-1} ?

A global resolution of singularities in the category of generalized manifolds and generalized functions is a desirable result which we have not attacked and a natural continuation of the subject that we present in this text. It remains as an open question of, in our modest opinion, great interest.

Chapter 1

Generalized power series.

In this chapter we introduce the algebra of generalized power series both in the formal and convergent setting. Most of the basic properties on these series are presented and proved in the work of Van den Dries and Speissegger [1]. We prove here a new property, proposition 1.1.20, which will be fundamental for our purposes.

1.1 Formal generalized power series.

1.1.1 Basic definitions.

Let $[0,\infty)$ denote the set of non-negative real numbers and $(0,\infty)$ the set of positive real numbers. For reasons to be clear below, once we have fixed a natural number m, elements in $[0,\infty)^m$ will be called **exponents** and they will be usually denoted by α, β, etc. On the other hand, elements of $(0,\infty)^m$ will be called **weight vectors** and they will be usually denoted by ρ, τ, etc.

For exponents $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and a weight vector $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ we put as usual $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2 \ldots, \alpha_m + \beta_m)$ and $|\alpha|_\rho = \rho_1\alpha_1 + \rho_2\alpha_2 + \ldots + \rho_m\alpha_m$. When $\rho = (1, ..., 1)$, sometimes we use the standard notation $|\alpha| = \alpha_1 + \cdots + \alpha_m$ for $|\alpha|_\rho$.

We partially order $[0,\infty)^m$ as follows. For exponents α and β,

$$\alpha = (\alpha_1, \ldots, \alpha_m) \leq \beta = (\beta_1, \ldots, \beta_m) \Leftrightarrow \alpha_i \leq \beta_i \;\; \forall i \in \{1, \ldots, m\}$$

We call the order given the **division order**.

Definition 1.1.1. A subset of $[0,\infty)^m$ will be called **good** if it is contained in a cartesian product of well ordered subsets of $[0,\infty)$.

Proposition 1.1.2. Let $m \in \mathbb{N}$ and let $S, T \subseteq [0,1)^m$ be good subsets of $[0,\infty)^m$ and ρ be a weight vector. Then

i) S is countable.

ii) The set $\{(\rho_1\alpha_1, \rho_2\alpha_2, \ldots, \rho_m\alpha_m) : \alpha \in S\}$ is good.

iii) The set $S_{\min}$ of minimal elements of S for the division order is finite, and each element $\beta \in S$ is greater or equal to some element of $S_{\min}$.

iv) The set $S \cup T$ is a good subset of $[0,\infty)^m$.

v) The set
$$\sum(S) := \{\alpha^1 + \ldots + \alpha^k : k \in \mathbb{N} \text{ and } \alpha^1, \ldots, \alpha^k \in S\}$$
is a good subset of $[0,\infty)^m$. In particular, by *iv)*, $S + T := \{s + t : s \in S, t \in T\}$ is a good subset of $[0,\infty)^m$ too, since $S + T \subseteq \sum(S \cup T)$.

vi) The set $\{|\alpha|_\rho : \alpha \in S\}$ is a well ordered subset of $[0,\infty)$ and for any $c \in [0,\infty)$ the set $S_\rho(c) := \{\alpha \in S : |\alpha|_\rho = c\}$ is finite.

Proof. For *i)*, it is enough to show the result for $m = 1$, but this is a well known result : given $x \in S$ there exists its successor, x^+ defined by

$$x^+ := \min\{y \in S : y > x\}$$

and we can find a rational number $q_x \in \mathbb{Q}$ between x and x^+.

For *ii)*, as S is good, $S \subseteq S_1 \times S_2 \times \cdots \times S_m$ with $S_i \subseteq [0,\infty)$ well ordered for all $i \in \{1,2,\ldots,m\}$. Then, $\{(\rho_1\alpha_1, \rho_2\alpha_2, \ldots, \rho_m\alpha_m) : \alpha \in S\} \subseteq \rho_1 S_1 \times \rho_2 S_2 \times \cdots \times \rho_m S_m$.

iii), *iv)* and *v)* are proved in lemma 4.2 and 4.3 in [1]. *vi)* is proved in the same paper, for the special case of $\rho = (1,...,1)$. The proof for general $\rho \in (0,\infty)^m$ goes in the same lines: If $\{|\alpha|_\rho : \alpha \in S\}$ is not well ordered we can take an infinite sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ in S such that the sequence $\{|\alpha^n|_\rho\}_{n\in\mathbb{N}}$ is strictly decreasing. This implies that at least one of the projections $\{\alpha^n_j\}_{n\in\mathbb{N}}$ must contain a strictly decreasing subsequence against the fact that S is good.

Assume now that we can take an infinite sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ such that $|\alpha^n|_\rho = c$. As it is infinite there must be one infinite projection. If there are no strictly decreasing subsequences in this projection, there must be an increasing subsequence. As the value of $|\alpha^n|_\rho$ is constant, there must exists a strictly decreasing sequence in other projection, which is impossible because S is good. □

Let $X = (X_1, X_2, \ldots, X_m)$ be variables and let X^* denote the multiplicative monoid whose elements are the monomials $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_m^{\alpha_m}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in [0,\infty)^m$ multiplied according to $X^\alpha \cdot X^\beta = X^{\alpha+\beta}$. The identity element of X^* is $X^0 = 1$, where $0 = (0,\ldots,0)$.

Definition 1.1.3. Let A be a commutative ring with $1 \neq 0$. A **formal generalized power series** in the variables X with coefficients in A is a map $s : [0,\infty)^m \longrightarrow A$, that we write as the formal series

$$s = s(X) := \sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha,$$

where $s_\alpha = s(\alpha)$, such that the set $\mathrm{Supp}\,(s) := \{\alpha \in [0,\infty)^m : s_\alpha \neq 0\}$, called the **support** of s, is a good subset of $[0,\infty)^m$.

Let $A[[X^*]]$ denote the set of generalized power series in the variables X with coefficients in A. If the support of s is finite we say that s is a **generalized polynomial** on X^*, and we write $A[X^*]$ for the set of generalized polynomials on X^* with coefficients in A. We consider the power series ring $A[[X]]$ also as subset of $A[[X^*]]$, namely as the subset of all series s as above for which $\mathrm{Supp}(s) \subseteq \mathbb{N}^m$. (Note that $\mathbb{N}^m$ is a good subset of $[0,1)^m$.)

The operations of sum and product on $A[[X^*]]$ are defined as usually : Given $a \in A$ and $s,t \in A[[X^*]]$ with $s = \sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha$ and $t = \sum_{\alpha\in[0,\infty)^m} t_\alpha X^\alpha$

$$as := \sum_{\alpha\in[0,\infty)^m} (as_\alpha) X^\alpha$$

$$s + t := \sum_{\alpha\in[0,\infty)^m} (s_\alpha + t_\alpha) X^\alpha$$

$$st := \sum_{\alpha\in[0,\infty)^m} (\sum_{\beta+\gamma=\alpha} s_\beta t_\gamma) X^\alpha$$

It is obvious that $as \in A[[X^*]]$. On the other hand, notice that $\text{Supp}(s+t) \subseteq \text{Supp}(s) \cup \text{Supp}(t)$. So, by proposition 1.1.2, $\text{Supp}(s+t)$ is good and $s+t$ is a well defined element in $A[[X^*]]$.

Notice also that for every $\alpha \in [0,\infty)^m$ we have the inclusion

$$\{\beta+\gamma \in \text{Supp}(s)+\text{Supp}(t) : \beta+\gamma=\alpha\} \subseteq \{\delta \in \text{Supp}(s)+\text{Supp}(t) : |\delta|=|\alpha|\}$$

So, by proposition 1.1.2, for each $\alpha \in [0,\infty)^m$ there are only a finite number of $\beta \in \text{Supp}(s)$ and $\gamma \in \text{Supp}(t)$ such that $\beta+\gamma=\alpha$, so $\sum_{\beta+\gamma=\alpha} s_\beta t_\gamma$ is a finite sum in the ring A and then a well defined element of A. Hence the series st as above is well defined as a map from $[0,\infty)^m$ to A. Moreover, since $\text{Supp}(st) \subseteq \text{Supp}(s)+\text{Supp}(t)$, by proposition 1.1.2 again, st is an element in $A[[X^*]]$.

The set $A[[X^*]]$ with these operations is an A-algebra. Notice also that these operations are compatible with the standard operations in the ring $A[X]$: considering a variable X_i as the series with support equal to $\{(0,\dots,0,1^{(i)},0,\dots,0)\}$, taking a natural power X_i^n is just the series with support $\{(0,...,0,n^{(i)},...,0)\}$. Moreover, the generalized polynomials $A[X^*]$ and the formal power series $A[[X]]$ with their standard operations, are subalgebras of $A[[X^*]]$.

The **constant term** of a series $s=\sum s_\alpha X^\alpha \in A[[X^*]]$ is the element $s_0=s(0)\in A$. Notice that the map

$$s=\sum s_\alpha X^\alpha \in A[[X^*]] \longmapsto s_0 \in A$$

sending a series to its constant term is an A-algebra homomorphism.

Fix a weight vector $\rho=(\rho_1,\dots,\rho_m)\in(0,\infty)^m$. Let $s=\sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha \in A[[X^*]]$. The **$\rho$-order** of s is defined as:

$$\text{ord}_\rho(s)=\begin{cases} \min\{|\alpha|_\rho : s_\alpha \neq 0\} & \text{if } s\neq 0. \\ \infty & \text{if } s=0. \end{cases}$$

In the special case of weight vector $\rho=(1,\dots,1)$, the ρ-order of a series s will be called simply order of s and denoted by $\text{ord}(s)$.

Given $s_1,s_2\in A[[X^*]]$ we have that

i) $\text{ord}_\rho(s_1+s_2) \geq \min\{\text{ord}_\rho(s_1),\text{ord}_\rho(s_2)\}$

ii) $\text{ord}_\rho(s_1 s_2) \geq \text{ord}_\rho(s_1)+\text{ord}_\rho(s_2)$, with equality if A is an integral domain.

As a consequence, we obtain that $A[[X^*]]$ is an integral domain if A is an integral domain.

Definition 1.1.4. Given a weight vector ρ and a series $s=\sum s_\alpha X^\alpha \in A[[X^*]]$ we define the **initial part of** s (relative to ρ) as

$$\text{In}_\rho(s)=\sum_{|\alpha|_\rho=\text{ord}_\rho(s)} s_\alpha X^\alpha$$

The series $s\in A[[X^*]]$ will be called **ρ-homogeneous** if it is equals to its initial part relative to ρ.

A series is **quasi-homogeneous** if it is ρ-homogeneous for a weight vector ρ.

Finally, for any series $s\in A[[X^*]]$ and any weight vector ρ we write:

$$s=\text{In}_\rho(s)+\text{res}_\rho(s)$$

where $\text{res}(s)_\rho=\sum_{|\alpha|_\rho>\text{ord}_\rho(s)} s_\alpha X^\alpha$, is called the **$\rho$-residual part of** s. It is a series whose ρ-order is strictly greater than that of the series s.

Remark 1.1.5. Notice that, by property *vi)* in 1.1.2, the initial part $\mathrm{In}_\rho(s)$ is in fact a polynomial.

Definition 1.1.6. We say that a family $\{s_j\}_{j\in J}$ in $A[[X^*]]$ is **sumable** if :

i) For each $\alpha \in [0,\infty)^m$ there are only finitely many $j \in J$ such that $\alpha \in$Supp(s_j), and

ii) $\bigcup_{j\in J}$Supp(s_j) is a good subset of $[0,\infty)^m$.

In this case, if we put $s_j = \sum_{\alpha\in[0,\infty)^m} s_{j_\alpha} X^\alpha$ for every $j \in J$, we define the sum of $\{s_j\}_{j\in J}$ denoted by $\sum_{j\in J} s_j$ to be the map from $[0,\infty)^m$ to A which we write in series notation as

$$\sum_{j\in J} s_j := \sum_{\alpha\in[0,\infty)^m} (\sum_{j\in J} s_{j_\alpha}) X^\alpha.$$

Notice that it is well defined by condition *i)*. We claim that $\sum_{j\in J} s_j \in A[[X^*]]$: The support of $\sum_{j\in J} s_j$ is the set

$$\mathrm{Supp}(\sum_{j\in J} s_j) = \{\alpha \in [0,\infty)^m : \sum_{j\in J} s_{j_\alpha} \neq 0\}.$$

If $\alpha \in$Supp$(\sum_{j\in J} s_j)$, $\sum_{j\in J} s_{j_\alpha} \neq 0$ so there exists at least some $j \in J$ such that $s_{j_\alpha} \neq 0$. Thus Supp$(\sum_{j\in J} s_j) \subseteq \bigcup_{j\in J}$Supp$(s_j)$. As $\bigcup_{j\in J}$Supp(s_j) is a good subset by condition *ii)*, then so is Supp$(\sum_{j\in J} s_j)$.

Notice that if $s = \sum_\alpha s_\alpha X^\alpha$ is a generalized power series then the family $\{s_\alpha X^\alpha\}_{\alpha\in \mathrm{Supp}(s)}$ is summable and that its sum is nothing but s.

The following lemma (cf. 4.2 of [1]) characterize the set of **units** in $A[[X^*]]$. We reproduce here its proof in order to start getting familiar with the kind of arguments that we use repeatedly in the sequel.

Lemma 1.1.7. Let $s = \sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha \in A[[X^*]]$. Then s is a unit in $A[[X^*]]$ if and only if its constant term s_0 is a unit in A.

Proof. . If $ss' = 1$ with $s' = \sum_{\beta\in[0,\infty)^m} s'_\beta X^\beta$, then $s_0 s'_0 = 1$, so s_0 is a unit in A. Conversely, if $bs_0 = 1$ with $b \in A$, then $bs = 1 - s'$ with $\mathrm{ord}(s') > 0$. Let us see first that the family $\{s'^n\}_{n\in N}$ is summable : if $\alpha \in [0,\infty)^m$ since $\mathrm{ord}(s') > 0$ and $\mathrm{ord}(s'^n) \geq n\,\mathrm{ord}(s')$ there exists $N \in \mathbb{N}$ big enough such that $N\mathrm{ord}(s') > |\alpha|$. Then for any $n \geq N$, $\alpha \notin$Supp(s'^n) and condition *i)* of 1.1.6 is satisfied. For condition *ii)* notice that Supp$(s'^n) \subseteq \sum($Supp$(s'))$ for all $n \in \mathbb{N}$, so $\bigcup_{n\in\mathbb{N}}$Supp$(s'^n) \subseteq \sum$Supp(s'), which is a good subset of $[0,\infty)^m$. So there exists the sum $\sum_{n\in\mathbb{N}} s'^n \in A[[X^*]]$. As $1 = (1-s')\sum_{n\in\mathbb{N}} s'^n$, we have that $1 = bs(\sum_{n\in\mathbb{N}} s'^n)$ so s is a unit in $A[[X^*]]$. □

Remark 1.1.8. In the proof of 1.1.7 it is proved implicitly that if $s \in A[[X^*]]$ with $\mathrm{ord}(s) > 0$ then $\{s^n\}_{n\in\mathbb{N}}$ is a summable family.

Definition 1.1.9. Given $s \in A[[X^*]]$, we define the **minimal support** of s as the set

$$\mathrm{Supp}_{\min}(s) := \{\alpha \in \mathrm{Supp}(s) : \alpha \text{ is minimal for the division order}\}$$

Notice that, differently of the classical formal setting, the algebra $A[[X^*]]$ is not Noetherian: the ideal generated by $\{X_1^{1/N} : N \in \mathbb{N}\}$ is not finitely generated. If it was the case, take the generator with smallest order in the variable X_1, say s with $\mathrm{ord}_{X_1}(s) > 0$. We can find $N \in \mathbb{N}$ such that $1/N <\mathrm{ord}_{X_1}(s)$, so s does not divide $X^{1/N}$.

Nevertheless, we have the following finiteness property, which is a consequence of property *iii)* of proposition 1.1.2.

Proposition 1.1.10. Given $s \in A[[X^*]]$ its minimal support is finite and the series s can be written as

$$s = \sum_{\alpha \in \mathrm{Supp}_{\min}(s)} X^\alpha u_\alpha \tag{1.1}$$

where $u_\alpha \in A[[X^*]]$ satisfies $u_\alpha(0) \neq 0$ for any $\alpha \in \mathrm{Supp}_{\min}(s)$.

The expression (1.1) is called a **monomial presentation** for s. It is unique up to a change on the elements u_α, for instance, taking $s(X, Y) = X + Y + XY$ we have two possible choices, $X(1 + Y) + Y$ and $X + Y(1 + X)$.

Definition 1.1.11. A series $s \in A[[X^*]]$ will be called **of monomial type** if $s = X^\alpha u$ where $u \in A[[X^*]]$ with $u(0) \neq 0$. A series is of monomial type if and only if its minimal support has only one element.

Lemma 1.1.12. If $s = s_1 s_2 \in \mathbb{R}[[X^*]]$ is the product of two series $s_1, s_2 \in \mathbb{R}[[X^*]]$ and s is of monomial type, then s_1 and s_2 are both of monomial type.

Proof. Put $s_1 s_2 = X^\alpha u$ where $u \in A[[X^*]]$ with $u(0) \neq 0$. Write a monomial presentation for s_1 and s_2:

$$s_1 = \sum_{\alpha^1 \in \mathrm{Supp}_{\min}(s_1)} X^{\alpha^1} u_{\alpha^1}; \qquad s_2 = \sum_{\alpha^2 \in \mathrm{Supp}_{\min}(s_2)} X^{\alpha^2} v_{\alpha^2}$$

Since $s_1 s_2 = X^\alpha u$ there exists $\beta^1 \in$Supp(s_1), $\beta^2 \in$Supp(s_2) such that $\alpha = \beta^1 + \beta^2 \leq \alpha^1 + \alpha^2$ for any $\alpha^1 \in$Supp(s_1) and $\alpha^2 \in$Supp(s_2). Suppose that there exists $\alpha^1 \in$Supp(s_1) such that $\beta^1 \not\leq \alpha^1$. This implies that there exists $j \in \{1, \ldots, m\}$ such that $\beta^1_j > \alpha^1_j$. Then, $\alpha_j = \beta^1_j + \beta^2_j > \alpha^1_j + \beta^2_j$, an so $\alpha \not\leq \alpha^1 + \beta^2$, contradiction. Thus $\beta^1 \leq \alpha^1$ for any $\alpha^1 \in$Supp(s_1) so s_1 is of monomial type. Similar for s_2. □

Mixed series. Let $(X, Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ be a tuple of $(m+n)$ distinct variables. Let

$$s = \sum_{(\alpha,\beta) \in [0,\infty)^{m+n}} s_{\alpha\beta} X^\alpha Y^\beta \in A[[(X, Y)^*]]$$

From now on we put

$$\mathrm{Supp}_X(s) := \{\alpha \in [0, \infty)^m : \text{exists } \beta \in [0, \infty)^n \text{ with } (\alpha, \beta) \in \mathrm{Supp}(s)\} = pr_X(\mathrm{Supp}(s))$$

$$\mathrm{Supp}_Y(s) := \{\beta \in [0, \infty)^n : \text{exists } \alpha \in [0, \infty)^m \text{ with } (\alpha, \beta) \in \mathrm{Supp}(s)\} = pr_Y(\mathrm{Supp}(s)).$$

where pr_X (respectively pr_Y) denotes the projection onto the first m coordinates (respectively last n coordinates) of $\mathbb{R}^{m+n}$.

We consider for $\beta \in$Supp$_Y(s)$, the following series in the X-variables

$$s_{\cdot,\beta}(X) := \sum_{\alpha \in \mathrm{Supp}_Y(s)} s_{\alpha\beta} X^\alpha$$

Recall that Supp$(s_{\cdot,\beta}(X)) \subseteq$Supp$_X(s)$ for each β which is good because is the projection of a good subset, so $s_{\cdot,\beta}(X) \in A[[X^*]]$.

If we define for $\beta \in$Supp$_Y(s)$, $s_\beta := s_{\cdot,\beta}(X) Y^\beta$ we can see s_β both as an element of $A[[(X, Y)^*]]$ and as an element of $(A[[X^*]])[[Y^*]]$. In both cases the family $\{s_\beta\}_{\beta \in \mathrm{Supp}_Y(s)}$ is clearly summable, so we can consider its sum

$$\sum_{\beta \in \mathrm{Supp}_Y(s)} s_\beta = \sum_{\beta \in \mathrm{Supp}_Y(s)} (\sum_{\alpha \in \mathrm{Supp}_X(s)} s_{\alpha\beta} X^\alpha) Y^\beta$$

as an element in $A[[(X,Y)^*]]$ or in $(A[[X^*]])[[Y^*]]$. Notice that in the former case, this gives nothing but s. This procedure permits to identify $A[[(X,Y)^*]]$ with a subring of $(A[[X^*]])[[Y^*]]$ via the injective ring homomorphism

$$\begin{aligned} A[[(X,Y)^*]] &\to (A[[X^*]])[[Y^*]] \\ \sum s_{\alpha\beta} X^\alpha Y^\beta &\mapsto \sum_\beta (\sum_\alpha s_{\alpha\beta} X^\alpha) Y^\beta \end{aligned}$$

Note that this homomorphism is not surjective in general: with $m, n > 0$, the series $\sum_{k=1}^\infty X_1^{1/k} Y_1^k$ is in $(A[[X^*]])[[Y^*]]$, but not in (the image of) $A[[(X,Y)^*]]$. Notice, however, that we have a natural inclusion, $(A[[X^*]])[Y^*] \subseteq A[[(X,Y)^*]]$.

We shall also work with the subring $A[[X^*, Y]]$ of $A[[(X,Y)^*]]$, consisting of those $s \in A[[(X,Y)^*]]$ in which the Y-variables have only natural numbers as exponents, that is whose support is included in $\mathbb{R}^m_{\geq 0} \times \mathbb{N}^n$, i.e., such that $\mathrm{Supp}_Y(s) \subseteq \mathbb{N}^n$. Similarly to the above, we identify $A[[X^*, Y]]$ with the corresponding subring of $A[[X^*]][[Y]]$; notice again that the example above shows that $A[[X^*, Y]] \subsetneq (A[[X^*]])[[Y]]$. On the other hand, we have the equality $A[[X^*, Y]] = (A[[Y]])[[X^*]]$.

As a matter of terminology, in the ring $A[[X^*, Y]]$, variables X will be called **generalized** (or **non-analytic**) and variables Y will be called **analytic**.

Partial derivatives. The operation $s \mapsto \frac{\partial s}{\partial X_i} \in A[[X]]$ does not extend naturally to $A[[X^*]]$, but the modified operation $s \mapsto X_i \frac{\partial s}{\partial X_i}$ on $A[[X]]$ does have a good extension ∂_i to $A[[X^*]]$: given $s = \sum s_\alpha X^\alpha \in A[[X^*]]$, we define

$$\partial_i s := \sum \alpha_i s_\alpha X^\alpha \in A[[X^*]]$$

On the other hand, considering $s \in A[[X^*, Y]]$ as an element of $A[[X^*]][[Y]]$, the partial derivatives $\partial s / \partial Y_j$ defined as usual belong to $A[[X^*, Y]]$, and in fact $Y_j \partial s / \partial Y_j = \partial_{m+j} s$.

1.1.2 Newton polyhedron of generalized series.

In this paragraph, let us use the following quite well known terminology about polyhedron that can be found in the modern book [11], for instance.

A subset Δ of a real affine space $\mathbb{E}$ is called a **(finite) convex polyhedron of** $\mathbb{E}$ if it is a finite intersection of closed half-spaces in $\mathbb{E}$ (a closed half-space is the closure of one of the two connected components of $\mathbb{E} \setminus H$ where H is an affine hyperplane in $\mathbb{E}$). The dimension of Δ is the minimum dimension of an affine subspace of $\mathbb{E}$ containing Δ. It has dimension equal to that of $\mathbb{E}$ if and only if Δ has a non-empty interior in E.

An affine hyperplane H in $\mathbb{R}^n$ is called a **supporting hyperplane** for Δ if Δ is contained in one of the two closed half-spaces determined by H. A **face** of Δ is the intersection of Δ with a supporting hyperplane. It is easy to see that there are only finitely many faces of a convex polyhedron Δ and that a face is a convex polyhedron in the supporting hyperplane. A face of a face of Δ is called a **subface** of Δ. A face which is not equal to the whole Δ is called a **proper** face. A face of dimension 0 is called a **vertex** and a face of dimension one is called an **edge** .

It is a well known result that a bounded convex polyhedron is nothing more than the convex hull of its vertices and, reciprocally, the convex hull of finitely many points in $\mathbb{E}$ is a bounded convex polyhedron in $\mathbb{E}$.

Finally, a **(finite) polyhedral complex in** $\mathbb{E}$ is a finite union of convex polyhedra in $\mathbb{E}$ such that the intersection of two of them is either empty or a common face of both. For example, if Δ is a convex polyhedron in $\mathbb{E}$ with non-empty interior, then its frontier is a polyhedral complex, equal to the union of all proper faces of Δ.

Now, given $s \in A[[X^*]]$, we can define its **Newton polyhedron** in the usual way. Consider $\mathcal{N}(s) :=$Supp$(s) + \mathbb{R}^m_{\geq 0}$ and define the Newton Polyhedron $\Delta(s)$ as

$$\Delta(s) = \text{convex hull of } \ (\mathcal{N}(s))$$

Using the property of finite monomial presentation of s (cf. Proposition 1.1), we have that $\text{Supp}_{\min}(s)$ is finite and that

$$\mathcal{N}(s) = \text{Supp}_{\min}(s) + \mathbb{R}^m_{\geq 0}.$$

In this situation, we can assure that the Newton polyhedron $\Delta(s)$ is a finite convex polyhedron as we have defined above, which justifies the given name.

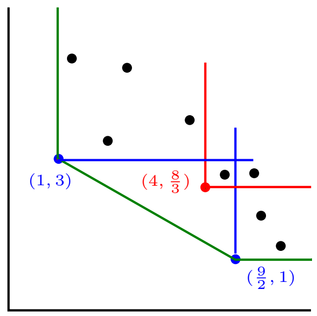


Figure 1.1: Newton polygon and minimal support.

Notice that every vertex of the polyhedron is an element of the minimal support of s but not reciprocally (see Fig. 1.1). By property *iii*) of 1.1.2 we conclude that the Newton polyhedron of a generalized power series has finitely many vertices.

Given a weight vector $\rho \in (0, \infty)^m$, the initial part $\text{In}_\rho(s)$ of a given series with respect to ρ can be determined geometrically using the Newton polyhedron of s in the usual way. For any non negative constant $c \in \mathbb{R}_{\geq 0}$, we define the hyperplane of $\mathbb{R}^m$

$$H_{\rho,c} := \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \rho_1 x_1 + \rho_2 x_2 + \ldots + \rho_m x_m = c\}$$

Lemma 1.1.13. Fix $\rho \in (0, \infty)^m$ a weight vector. Given a series $s \in A[[X^*]]$,

i) The ρ-order of s is given by $\text{ord}_\rho(s) = \sup\{c \in \mathbb{R}_{\geq 0} : H_{\rho,c} \cap \Delta(s) = \emptyset\}$

ii) We have that
$$H_{\rho,\text{ord}_\rho(s)} \cap \text{Fr}(\Delta(s)) = \text{Convex Hull}(\text{Supp}(\text{In}_\rho(s)))$$

Proof. First notice that if $c \in \mathbb{R}_{\geq 0}$ is such that $H_{\rho,c} \cap \Delta(s) \neq \emptyset$ then for all $c' > c$, $H_{\rho,c} \cap \Delta(s) \neq \emptyset$ by definition of $\Delta(s)$. On the other hand, $H_{\rho,\text{ord}_\rho(s)} \cap \mathcal{N}(s) \neq \emptyset$, by definition of $\text{ord}_\rho(s)$. Thus $\text{ord}_\rho(s)$ is an upper bound of $\{c \in \mathbb{R}_{\geq 0} : H_{\rho,c} \cap \mathcal{N}(s) = \emptyset\}$. Let $c' = \sup\{c \in \mathbb{R}_{\geq 0} : H_{\rho,c} \cap \mathcal{N}(s) = \emptyset\}$. Notice that, since $\Delta(s)$ is connected, for any c for which $H_{\rho,c} \cap \Delta(s) = \emptyset$, the hyperplane $H_{\rho,c}$ is a supporting hyperplane of the polyhedron $\Delta(s)$. By continuity, we must have also that $H_{\rho,c'}$ is a supporting hyperplane and, moreover, $H_{\rho,c'} \cap \Delta(s) \neq \emptyset$. But then $H_{\rho,c'} \cap \Delta(s) = H_{\rho,c'} \cap \text{Fr}(\Delta(s))$ which is a face of the polyhedron. This face contains at least one vertex of $\Delta(s)$, that is an element $\alpha \in \Delta(s)$. Since $\alpha \in H_{\rho,c'}$, we have $|\alpha|_\rho = c'$ and thus, by definition of the ρ-order, we obtain

$\mathrm{ord}_\rho(s) \leq c'$, giving the required equality.

For the second part of the lemma, notice that we have proved that $H_{\rho,\mathrm{ord}_\rho(s)}$ is a supporting hyperplane of Δ and thus it cuts the polyhedron in a face F of it. This face contains no line parallel to a coordinate axis, so F is a bounded face and hence, F is the convex hull of its vertices. Being this set of vertices included in the hyperplane $H_{\rho,\mathrm{ord}_\rho(s)}$, it is contained in $\mathrm{Supp}(\mathrm{In}_\rho(s))$ and thus $H_{\rho,\mathrm{ord}_\rho(s)} \cap \Delta(s) \subset$ Convex Hull$(\mathrm{Supp}(\mathrm{In}_\rho(s)))$. The other inclusion is obvious since $\mathrm{Supp}(\mathrm{In}_\rho(s)) \subset F$ and F is convex. □

1.1.3 Composition morphisms.

Recall that in the classical framework of formal power series, the composition of series makes sense: we can change variables by series with no constant term. Formally, if $s \in A[[Y]]$, and $t = (t_1, t_2, \ldots, t_n) \in A[[W]]^n$, where $W = (W_1, \ldots, W_n)$, with $t_1(0) = \ldots = t_n(0) = 0$ we may substitute t for Y in s and obtain an element $s(t(W)) \in A[[W]]$. This operation of substitution satisfies the following natural property: for any fixed n-tuple of series $t(W) \in A[[W]]^n$, the map $s \mapsto s(t(W))$ from $A[[Y]]$ to $A[[W]]$ is an A-algebra homomorphism.

We can proceed similarly in the situation of mixed power series, already studied in [1], page 4393, when we just substitute analytic variables by formal series. More precisely, let $s \in A[[X^*, Y]]$, where X is m-dimensional and Y is n-dimensional, and let $t = (t_1, t_2, \ldots, t_n) \in A[[W]]^n$ with $t_1(0) = \ldots = t_n(0) = 0$. Since $A[[X^*, Y]] \subseteq A[[X^*]][[Y]]$, we may substitute t for Y in s and obtain an element $s(X, t(W)) \in A[[X^*]][[W]]$. One easily checks that in fact $s(X, t(W)) \in A[[X^*, W]]$ (see part *i*) of Proposition 1.1.14 below). Again, once $t(W)$ is fixed, the map $s \in A[[X^*, Y]] \mapsto s(X, t(W)) \in A[[X^*, W]]$ is an algebra homomorphism.

However, the general problem of composition of generalized power series is much more delicate. Take for instance just the simple example $s = Y^{1/2} \in \mathbb{R}[[Y^*]]$ with Y a single variable. If we want that substitution gives rise to an algebra homomorphism (or if we want any reasonable definition of substitution), to substitute Y by a generalized power series $t(W)$ must be interpreted as a "square root" of $t(W)$. But then, choosing for instance $t(W) = W_1 + W_2$ in two variables, there is no reasonable candidate in $\mathbb{R}[[W^*]]$ whose square is equal to $W_1 + W_2$. However, in the special case of $A = \mathbb{R}$, there is a subset of real generalized power series which is characterized precisely by this condition, the series of monomial type. Let us prove that they are exactly those series that can be plugged into variables in every generalized power series (propositions 1.1.14 and 1.1.20).

Proposition 1.1.14. Let $X = (X_1, X_2, \ldots, X_m)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, $Z = (Z_1, Z_2, \ldots, Z_r)$ and $W = (W_1, W_2, \ldots, W_l)$ denote multi-variables.

i) Let $s = \sum_{(\alpha,I)\in[0,\infty)^m\times\mathbb{N}^n} s_{(\alpha,I)} X^\alpha Y^I \in A[[X^*, Y]]$ and let $t = (t_1, t_2, \ldots, t_n) \in A[[W]]^n$ with $t_1(0) = \ldots = t_n(0) = 0$. The family

$$\{s_{(\alpha,I)} X^\alpha t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}\}_{\substack{\alpha\in\mathrm{Supp}_X(s)\\ I=(i_1,i_2,\ldots,i_n)\in\mathbb{N}^n}}$$

is summable and its sum, denoted by $s(X, t_1, t_2, \ldots, t_n)$, or for short, $s(X, t(W))$, is in $A[[X^*, W]]$. Moreover, the map $s \mapsto s(X, t(W))$ is an A-algebra homomorphism from $A[[X^*, Y]]$ to $A[[X^*, W]]$.

ii) Let $s = \sum_{(\alpha,I)\in[0,\infty)^m\times\mathbb{N}^n} s_{(\alpha,I)} X^\alpha Y^I \in A[[X^*, Y]]$ and let $t = (t_1, t_2, \ldots, t_n) \in A[[Z^*]]^n$ with $t_1(0) = \ldots = t_n(0) = 0$. The family

$$\{s_{(\alpha,I)} X^\alpha t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}\}_{\substack{\alpha\in\mathrm{Supp}_X(s)\\ I=(i_1,i_2,\ldots,i_n)\in\mathbb{N}^n}}$$

is summable and its sum, denoted by $s(X, t_1, t_2, \dots, t_n)$, or for short, $s(X, t(Z))$, is in $A[[X^*, Z^*]]$. Moreover, the map $s \mapsto s(X, t(Z))$ is an A-algebra homomorphism from $A[[X^*, Y]]$ to $A[[X^*, Z^*]]$.

iii) If $u = \sum_{\alpha \in [0,\infty)^m} u_\alpha X^\alpha \in \mathbb{R}[[X^*]]$ is such that $u_0 > 0$, the family$\{(u - u_0)^k\}_{k\in\mathbb{N}}$ is summable and then we can define for every $a > 0$

$$u^a := \sum_{k\in\mathbb{N}} \binom{a}{k} u_0^{a-k} (u - u_0)^k \in \mathbb{R}[[X^*]]$$

iv) Let $s = \sum s_\alpha X^\alpha \in \mathbb{R}[[X^*]]$ and $t = (t_1, t_2, \dots, t_m) \in \mathbb{R}[[Z^*]]^m$. If $t_i = Z^{\beta^i} u_i$, with $\beta^i \neq (0, ..., 0)$, $u_i \in \mathbb{R}[[Z^*]]$ and $u_i(0) > 0$ for all $i \in \{1, 2, \dots, m\}$ (that is, t_i is of monomial type), the family $\{s_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_m^{\alpha_m}\}_{\alpha \in \mathrm{Supp}(s)}$ is summable and its sum, denoted by $s(t_1, t_2, \dots, t_n)$ is in $\mathbb{R}[[Z^*]]$. Moreover, the map $s \mapsto s(t_1, \dots, t_n)$ is an $\mathbb{R}$-algebra homomorphism from $\mathbb{R}[[X^*]]$ to $\mathbb{R}[[Z^*]]$.

Proof. For *i*), let us call for any $\alpha \in \mathrm{Supp}_X(s)$ and $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$

$$q_{(\alpha,I)} := s_{(\alpha,I)} X^\alpha t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

Notice that if $(\gamma, J) \in [0, \infty)^m \times \mathbb{N}^l$, $(\gamma, J) \in$Supp$(q_{(\alpha,I)})$ if $\gamma = \alpha$. Since $t_1(0) = \ldots = t_n(0) = 0$, for any $1 \leq i \leq n$ there exists $\widetilde{J}_i \in \mathbb{N}^l$ with $\widetilde{J}_i \neq \underline{0}$ such that $W^{\widetilde{J}_i}$ divides t_i. Then, for $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$, $W^{i_1\widetilde{J}_1 + i_2\widetilde{J}_2 + \cdots + i_n\widetilde{J}_n}$ divides $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$. As there are only finitely many $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$ such that $i_1\widetilde{J}_1 + i_2\widetilde{J}_2 + \cdots + i_n\widetilde{J}_n \leq J$ we have condition *i*) of 1.1.6 summable family. On the other hand,

$$\bigcup_{\substack{\alpha \in \mathrm{Supp}_X(s) \\ I=(i_1,i_2,\dots,i_n)\in\mathbb{N}^n}} \mathrm{Supp}(q_{(\alpha,I)}) \subseteq \mathrm{Supp}_X(s) \times \mathbb{N}^l$$

which is a good set.

We can reason analogously for *ii*), but in this case, using the analogous notation,

$$\bigcup_{\substack{\alpha \in \mathrm{Supp}_X(s) \\ I=(i_1,i_2,\dots,i_n)\in\mathbb{N}^n}} \mathrm{Supp}(q_{(\alpha,I)}) \subseteq \mathrm{Supp}_X(s) \times \sum (\cup_{i=1}^n \mathrm{Supp}(t_i))$$

which is a good set by properties 1.1.2.

Part *iii*) is an immediate consequence of remark 1.1.8.

For part *iv*), we write

$$t_i = Z^{\beta^i}(u_i(0) + \varepsilon_i)$$

where $\varepsilon_i(0) = 0$ and $\beta^i = (\beta_1^i, \beta_2^i, \dots, \beta_r^i) \neq 0$ for $i = 1, 2, \dots, m$. We define for any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathrm{Supp}(s)$,

$$q_\alpha := s_\alpha Z^{\alpha_1\beta^1 + \alpha_2\beta^2 + \cdots + \alpha_m\beta^m} (u_1(0) + \varepsilon_1)^{\alpha_1} (u_2(0) + \varepsilon_2)^{\alpha_2} \cdots (u_m(0) + \varepsilon_m)^{\alpha_m} \tag{1.2}$$

By part *iii*), $q_\alpha \in \mathbb{R}[[Z^*]]$ for any $\alpha \in \mathrm{Supp}(s)$. We have to prove that the family $\{q_\alpha\}_{\alpha\in\mathrm{Supp}(s)}$ is summable. For $i \in \{1, 2, \dots, m\}$,

$$(u_i(0) + \varepsilon_i)^{\alpha_i} = \sum_{k\in\mathbb{N}} \binom{\alpha_i}{k} u_i(0)^{\alpha_i - k} \varepsilon_i^k$$

Let $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_r)\in[0,\infty)^r$. If $\gamma\in$Supp(q_α),

$$\gamma=\alpha_1\beta^1+\alpha_2\beta^2+\cdots+\alpha_m\beta^m+\delta(\alpha) \tag{1.3}$$

where $\delta(\alpha)=(\delta(\alpha)_1,\delta(\alpha)_2,\dots,\delta(\alpha)_r)\in\sum(\cup_{i=1}^m\text{Supp}(\varepsilon_i))$. Suppose that there are infinitely many $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)\in$ Supp(s) such that $\gamma\in$Supp(q_α). Take a sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ of different elements in Supp(s) such that $\gamma\in$Supp(q_{α^n}) for $n\in\mathbb{N}$. As Supp(s) is good, there exists $j\in\{1,2,\dots,m\}$ such that $\{\alpha_j^n\}_{n\in\mathbb{N}}$ is strictly increasing. Take $k\in\{1,2,\dots,m\}$ such that $\beta_k^j\neq 0$. Since

$$\gamma_k=\alpha_1^n\beta_k^1+\alpha_2^n\beta_k^2+\cdots+\alpha_m^n\beta_k^m+\delta(\alpha^n)_k$$

and all the terms involved are non-negative, either $\{\delta(\alpha^n)_k\}_{n\in\mathbb{N}}$ or $\{\alpha_i^n\}_{n\in\mathbb{N}}$ for at least one $i\neq j$ should be strictly decreasing which is impossible because $\sum(\cup_{i=1}^m\text{Supp}(\varepsilon_i))$ and Supp(s) are good.

On the other hand, by (1.3), $\bigcup_{\alpha\in\text{Supp}(s)}\text{Supp}(q_\alpha)\subseteq\sum(\cup_{i=1}^m\text{Supp}_{X_i}(s)\beta^i\cup\text{Supp}\varepsilon_i))$ where $\text{Supp}_{X_i}(s)$ is the projection on the i^{th}-component of Supp(s), and by proposition 1.1.2

$$\sum(\cup_{i=1}^m\text{Supp}_{X_i}(s)\beta^i\cup\text{Supp}\varepsilon_i))$$

is good. □

Remark 1.1.15. Let $s=\sum s_\alpha X^\alpha\in\mathbb{R}[[X^*]]$, $M_1,M_2,\dots,M_m\in\mathbb{R}[[Z^*]]$ be monomials ($M_i=Z^{\beta^i}$ with $\beta^i\neq\underline{0}$), $W=(W_1,W_2,\dots,W_m)$ be variables and $\lambda_1,\lambda_2,\dots,\lambda_m\in\mathbb{R}_{>0}$. If we define for any $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)\in$ Supp(s)

$$t_\alpha:=s_\alpha M_1^{\alpha_1}(\lambda_1+W_1)^{\alpha_1}M_2^{\alpha_2}(\lambda_2+W_2)^{\alpha_2}\cdots M_m^{\alpha_m}(\lambda_m+W_m)^{\alpha_m}$$

by part *iii*) of proposition 1.1.14 above, $t_\alpha\in\mathbb{R}[[Z^*,W]]$. In fact, the sum of the family

$$\{t_\alpha:=s_\alpha M_1^{\alpha_1}(\lambda_1+W_1)^{\alpha_1}M_2^{\alpha_2}(\lambda_2+W_2)^{\alpha_2}\cdots M_m^{\alpha_m}(\lambda_m+W_m)^{\alpha_m}\}_{\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)\in\text{Supp}(s)}$$

(summable in $\mathbb{R}[[Z^*,W^*]]$ by part *iv*)), belongs to $\mathbb{R}[[Z^*,W]]$. This is a consequence of the proof of part *iv*). We denote this sum by $s(M_1^{\alpha_1}(\lambda_1+W_1)^{\alpha_1},M_2^{\alpha_2}(\lambda_2+W_2)^{\alpha_2},\dots,M_m^{\alpha_m}(\lambda_m+W_m)^{\alpha_m})$.

Examples 1.1.16. *i*) Let G_{m+n} denote the group of permutations of $m+n$ elements, and $G_{m,n}$ the subgroup of G_{m+n} permuting on the one hand the first m elements between them and the n last elements on the other. Then if $\sigma\in G_{m,n}$, it induces an A-algebra automorphism of $A[[X^*,Y]]$ by putting

$$\sigma(\sum s_{\alpha,\beta}X^\alpha Y^\beta)=\sum s_{\alpha,\beta}\sigma(X^\alpha Y^\beta)$$

where $\sigma(X^\alpha Y^\beta):=X_{\sigma(1)}^{\alpha_1}\cdots X_{\sigma(m)}^{\alpha_m}Y_{\sigma(m+1)-m}^{\beta_1}\cdots Y_{\sigma(m+n)-m}^{\beta_n}$. We usually write σs for $\sigma(s)$, where $s\in A[[X^*,Y]]$. Also corresponding to σ we define a map $\sigma:\mathbb{R}^{m+n}\to\mathbb{R}^{m+n}$ by $\sigma(x,y)=(x_{\sigma(1)},\dots,x_{\sigma(m)},y_{\sigma(m+1)-m}\cdots,y_{\sigma(m+n)-m})$. (For a polyradius $r=(r_1,\dots,r_m)$ the case $n=0$ applies, so that $\sigma(r)=(r_{\sigma(1)},\dots,r_{\sigma(m)})$.)

ii) Assume $m\geq 2$. Given distinct $i,j\in\{1,2,\dots,m\}$ and $\gamma>0$, we define an injective monoid homomorphism $\varsigma_{ij}^\gamma:X^*\to X^*$ such that $\varsigma_{ij}^\gamma(X_k)=X_k$ for $k\neq i$ and $\varsigma_{ij}^\gamma(X_i)=X_iX_j^\gamma$, as follows:

$$\varsigma_{ij}^\gamma(X^\alpha):=X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_{j-1}^{\alpha_{j-1}}X_j^{\gamma\alpha_i+\alpha_j}X_{j+1}^{\alpha_{j+1}}\cdots X_m^{\alpha_m}=X^\alpha X_j^{\gamma\alpha_i}$$

It extends to an injective A-algebra endomorphism of $A[[X^*]]$ by putting $\varsigma_{ij}^\gamma(\sum s_\alpha X^\alpha):=\sum s_\alpha\varsigma_{ij}^\gamma(X^\alpha)$. To avoid too many nested parentheses, we will write $\varsigma_{ij}^\gamma s$ instead of $\varsigma_{ij}^\gamma(s)$.

Proposition 1.1.14 shows that, in the context of real generalized power series, substitution of variables X_i by other series t_i is possible if the substitute series t_i are of monomial type. In the following proposition we prove a reciprocal result, that is, if a real generalized power series t, in any given number of variables, can be the "substitute" of a variable in any generalized power series, then t must be of monomial type. A correct statement of this reciprocal property makes use of the special series $X^{1/N}$, for $N \in \mathbb{N}$, where to substitute the variable X by t; that is, existence of N^{th}-roots of t for any N. For our purposes, we state this result in the slightly more general context of mixed series.

Definition 1.1.17. Let $s \in \mathbb{R}[[X^*, Y]]$ be a formal generalized real power series where $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$ are respectively the generalized and the analytic variables. Suppose that $s \not\equiv 0$. For $N \in \mathbb{N}_{\geq 0}$ we say that s has has an N^{th}-root if there exists $s_N \in \mathbb{R}[[X^*, Y]]$ such that $(s_N)^N = s$.

Lemma 1.1.18. Let $s \in \mathbb{R}[[X^*]]$. Suppose that $s \not\equiv 0$ and that s has an N^{th}-root for any $N \in \mathbb{N}$. Then, for any weight vector $\rho \in (0, \infty)^m$, the initial part of s, $\mathrm{In}_\rho(s)$ have an N^{th}-root for any $N \in \mathbb{N}$. Moreover, any N^{th}-root of $\mathrm{In}_\rho(s)$ is ρ-homogeneous whose ρ-degree is equal to $\mathrm{ord}_\rho(s)/N$.

Proof. Put $s = \mathrm{In}_\rho(s)$+$\mathrm{res}_\rho(s)$ where $\mathrm{res}_\rho(s)$ is the residual part of s with respect to ρ. Let $s_N \in \mathbb{R}[[X^*, Y]]$ be an N^{th}-root of s and put $s_N =\mathrm{In}_\rho(s_N)$+$\mathrm{res}_\rho(s_N)$. We have

$$s = (s_N)^N = (\mathrm{In}_\rho(s_N) + \mathrm{res}_\rho(s_N))^N = (\mathrm{In}_\rho(s_N))^N + \sum_{k=1}^{N} \binom{N}{k} \mathrm{In}_\rho(s_N)^{N-k} \mathrm{res}_\rho(s_N)^k$$

Since $\mathrm{ord}_\rho(\mathrm{In}_\rho(s_N)) <\mathrm{ord}_\rho(\mathrm{res}_\rho(s_N))$, $\mathrm{ord}_\rho((\mathrm{In}_\rho(s_N))^N) <\mathrm{ord}_\rho(\mathrm{In}_\rho(s_N)^{N-k}\mathrm{res}_\rho(s_N)^k)$ for all $k \in \{1, 2, \ldots, N\}$ which implies that $\mathrm{In}_\rho(s) =(\mathrm{In}_\rho(s_N))^N$. This argument also shows that any N^{th}-root of $\mathrm{In}_\rho(s)$ is ρ-homogeneous and its ρ-degree is equal to $\mathrm{ord}_\rho(s)/N$ by property 2 of the order function ord_ρ. □

Corollary 1.1.19. Let $s \in \mathbb{R}[[X^*]]$. Suppose that $s \not\equiv 0$ and that s has an N^{th}-root for any $N \in \mathbb{N}$. Let, for all $N \in \mathbb{N}$, $s_N \in \mathbb{R}[[X^*]]$ be an N^{th}-root of s, that is $(s_N)^N = s$. Then, for any weight vector ρ, $\mathrm{In}_\rho(s) = (\mathrm{In}_\rho(s_N))^N$ and so $\mathrm{ord}_\rho(\mathrm{In}(s_N)) =\mathrm{ord}_\rho(\mathrm{In}(s))/N$.

Proposition 1.1.20. Let $s \in \mathbb{R}[[X^*, Y]]$ be a formal generalized real power series where $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$ are respectively the generalized and the analytic variables. Suppose that $s \not\equiv 0$ and that for any integer $N \in \mathbb{N}_{\geq 0}$ there exists a N^{th}-root $s_N \in \mathbb{R}[[X^*, Y]]$ of s, that is $(s_N)^N = s$. Then $s = X^\alpha u$, where $\alpha \in [0, \infty)^m$ and $u \in \mathbb{R}[[X^*, Y]]$ is a unit such that $u(0, 0) > 0$.

Proof. <u>If $m = 0$</u>, the result is well know : If $s = s(Y) \in \mathbb{R}[[Y]]$ is a usual formal power series with all N-roots then s is a unit. Otherwise, any N^{th}-root of s is not a unit. Thus $\mathrm{ord}(s_N) \geq 1$, because $s_N \in \mathbb{R}[[Y]]$, and then the order of s would be greater or equal to N for all $N \in \mathbb{N}$ and thus $s = 0$. In addition, $s(0) > 0$ because $s_2(0)^2 = s(0)$.

<u>If $m > 0$</u>. Consider s as an element of $(\mathbb{R}[[Y]])[[X^*]]$. Suppose that the Newton polyhedron of s (as an element of $(\mathbb{R}[[Y]])[[X^*]]$) has only one vertex, that is, $s = X^\alpha u(X, Y)$ with $u(0, Y) \neq 0$. If $\mathrm{ord}(u(0, Y)) = 0$, $u(0, 0) \neq 0$ and in particular $u(0, 0) > 0$. If not, $(s_N)^N = s = X^\alpha u$; if $X^{\alpha/N}$ does not divide s_N, there exists $i \in \{1, 2, \ldots, m\}$ such that $X_i^{\alpha_i/N}$ does not divides s, that is, such that $\alpha_i/N < \min(\mathrm{Supp}_{X_i}(s_N))$ which implies that $\alpha_i < \min(\mathrm{Supp}_{X_i}(s_N^N = s))$, contradiction. Thus, $X^{\alpha/N}$ divides s_N, so $s_N = X^{\alpha/N} t_N$ and $X^\alpha (t_N)^N = (s_N)^N = s = X^\alpha u$. Then u has all the N^{th}-roots which implies that $u(0, Y)$ is 0 or it is a unit by the case $m = 0, n \in \mathbb{N}$.

Now we prove that the Newton polyhedron of s (as an element of $(\mathbb{R}[[Y]])[[X^*]]$) can not have more that one vertex. Suppose that it has at least two different vertices. Then, the Newton polyhedron has at least one edge $[\alpha, \beta]$ with $\alpha \neq \beta$ which is not parallel to a coordinate axis. Then there exists a weight vector $\rho = (\rho_1, \ldots, \rho_m) \in (0, \infty)^m$ and a supporting hyperplane of $\Delta(s)$ of the form $H_{\rho,c} = \{\rho_1 x_1 + \rho_2 x_2 + \ldots + \rho_m x_m = c\}$ with $c \geq 0$ that cuts the Newton polyhedron exactly in the edge $[\alpha, \beta]$.

Write s as the sum $s = \mathrm{p}_\rho(s) + \mathrm{r}_\rho(s)$ where $\mathrm{p}_\rho(s)$ is the ρ-homogeneous part of s and $\mathrm{r}_\rho(s)$ is the residual part, whose ρ-order is strictly bigger than $\mu = \mathrm{ord}_\rho(s)$. Recall that p_ρ is a polynomial in $\mathbb{R}[[Y]][X^*]$ (see properties 1.1.2). Moreover, our choice of ρ implies that $\mathrm{Supp}(p_\rho)$ is contained in the segment $[\alpha, \beta]$ and that its extremities α and β both belong to $\mathrm{Supp}(p_\rho)$

For any $N \in \mathbb{N}$, let s_N be a N^{th}-root of s. As we have seen in Corollary 1.1.19, the ρ-initial part $p_{\rho,N} = \mathrm{In}_\rho(s_N)$ of s_N is an N^{th}-root of p_ρ. Notice also that $p_{\rho,N}$ is ρ-homogeneous of degree $\mathrm{ord}_\rho(s)/N$.

Thus, our proposition will be finished once we prove the following claim, which is a particular case of the proposition:

Claim: Suppose that $\mathrm{Supp}_X(s)$ is contained in the segment $[\alpha, \beta]$ where $\alpha \neq \beta$, non parallel to any of the coordinate axis, and that $\alpha, \beta \in \mathrm{Supp}_X(s)$. Then s can not have an N^{th}-root in $\mathbb{R}[[X^*, Y]]$ for any natural number N.

Proof of the Claim.- Assume that s has an N^{th}-root $s_N \in \mathbb{R}[[X^*, Y]]$ for any N. Consider $m-1$ independent weight vectors $\rho^1, \ldots, \rho^{m-1}$ such that the line containing the segment $[\alpha, \beta]$ is the intersection of hyperplanes of the form H_{ρ^j, c_j} with $c_j \geq 0$, for $j = 1, \ldots, m-1$. Then s is ρ^j-homogeneous for any j and, by Lemma 1.1.18, the N^{th}-root s_N is ρ^j-homegeneous too; that is, its support is contained in the hyperplane of the form H_{ρ^j, d_j} (in fact $d_j = c_j/N$). Therefore $\mathrm{Supp}_X(s_N)$ is contained in a line which is parallel to $[\alpha, \beta]$ (in fact in the line containing the segment $[\alpha/N, \beta/N]$). We can write

$$s = \sum_{\lambda \in [0,1]} s_\lambda X^{(1-\lambda)\alpha + \lambda\beta}$$

Notice that this sum is finite since s_N is a quasi-homogeneous polynomial.

Let us call $\mathrm{Supp}^*(s) := \{\lambda \in [0,1] : s_\lambda \neq 0\}$. Recall that our hypothesis that $\alpha, \beta \in \mathrm{Supp}_X(s)$ implies that $0, 1 \in \mathrm{Supp}^*(s)$. As $(s_N)^N = s$, we have that for $\lambda \in \mathbb{R}$,

$$s_\lambda = \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_N = N\lambda, \lambda_j \in \mathrm{Supp}^*(s_N)} s_{N,\lambda_1} s_{N,\lambda_2} \cdots s_{N,\lambda_N} \tag{1.4}$$

Let $\lambda_0 := \min(\mathrm{Supp}^*(s_N))$ and $\lambda_1 := \max(\mathrm{Supp}^*(s_N))$. Let us show that $\lambda_0 = 0$ and that $\lambda_1 = 1$. In fact, taking $\lambda = \lambda_0$ in the expression (1.4), we see that there is just a summand in that expression which is $(s_{N,\lambda_0})^N \neq 0$. We can also see that if $\lambda < \lambda_0$ then $s_\lambda = 0$ in the expression (1.4) by the definition of λ_0. Since $s_\lambda = 0$ for $\lambda < 0$ and $s_0 \neq 0$, this shows that $\lambda_0 = 0$. Analogously, we show that $\lambda_1 = 1$.

Let N big enough such that if $\lambda \in \mathrm{Supp}^*(s)$, then $\lambda = 0$ or $\lambda > 1/N$ (this is possible because $\mathrm{Supp}^*(s)$ is finite). For $\lambda = 1/N$, we have in the expression (1.4) the summands corresponding to the tuples of the form

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) = (0, \ldots, 1^{(j^{th})}, \ldots, 0)$$

for $j = 1, 2, \ldots, N$. Each of them gives rise to the same coefficient $(s_{N,1})(s_{N,0})^{N-1} \neq 0$ because $s_{N,1} \neq 0 \neq s_{N,0}$. On the other hand, since $\lambda = 1/N \notin \mathrm{Supp}^*(s)$ there must exist other N-tuples

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) \in [[0,1] \cap \mathrm{Supp}^*(s)]^N$$

which are different of the N-tuples $(0,\ldots,1^{(j^{th})},\ldots,0)$ and such that $\lambda_1+\lambda_2+\cdots+\lambda_N=1$. Since all $\lambda_j \geq 0$, there exists $\lambda^1 \in$Supp$^*(s_N)$ with $0<\lambda^1<1$. Now, for $\lambda=\lambda^1/N$ we have in (1.4) the summands corresponding to the tuples

$$(\lambda_1,\lambda_2,\ldots,\lambda_N)=(0,\ldots,\overbrace{\lambda^1}^{j^{th}},\ldots,0)$$

for $j=1,2,\ldots,N$. They give rise to the same summand

$$(s_{N,\lambda_1})(s_{N,0})^{N-1}\neq 0$$

As $\lambda^1/N \notin$Supp$^*(s)$, there must be N-tuples

$$(\lambda_1,\lambda_2,\ldots,\lambda_N)\in[\text{Supp}^*(s)]^N$$

different from $(0,\ldots,\overbrace{\lambda^1}^{j^{th}},\ldots,0)$ with $\lambda_1+\lambda_2+\cdots+\lambda_N=\lambda^1$. Since all $\lambda_j\geq 0$, there exist $\lambda^2 \in$Supp$^*(s_N)$ with $0<\lambda^2<\lambda^1<1$. We construct in this way a strictly decreasing sequence in Supp$^*(s_N)$ which is impossible. □

1.1.4 The b invariant

Let us introduce here a numerical invariant, that will be used in the proof of the main theorem in chapter 3, associated to a series that measures how far it is a series from being of monomial type. Below, we will see how does this invariant behaves under some specific transformations of the type of example *ii)* in 1.1.16 (those corresponding to the local expression of certain blowing-up morphism to be defined in chapter 3). Both the invariant and its behavior is already introduced and discussed in the paper [1]; we just reproduce here the same arguments since they are crucial to our purposes.

Let $\alpha,\beta\in[0,\infty)^m$ be exponents. Put $\inf(\alpha,\beta):=(\min\{\alpha_1,\beta_1\},\ldots,\min\{\alpha_m,\beta_m\})$. If $\inf(\alpha,\beta)\in\{\alpha,\beta\}$, then put $d(\alpha,\beta)=0$. If $\inf(\alpha,\beta)\notin\{\alpha,\beta\}$, there are two possibilities:

i) $\inf(\alpha,\beta)=\underline{0}$. Let $a:=|\{j\in\{1,\ldots,m\}:\alpha_j\neq 0\}|$ and $b:=|\{j\in\{1,\ldots,m\}:\beta_j\neq 0\}|$. Then, $d(\alpha,\beta)=a+b$.

ii) $\inf(\alpha,\beta)\neq\underline{0}$. Then, $d(\alpha,\beta):=d(\alpha-\inf(\alpha,\beta),\beta-\inf(\alpha,\beta))$.

Finally, write $X^\alpha|X^\beta$ or "X^α divides X^β" iff $\alpha\leq\beta$, $\gcd(X^\alpha,X^\beta):=X^{\inf(\alpha,\beta)}$ and $d(X^\alpha,X^\beta):=d(\alpha,\beta)$.

The mapping $d:[0,\infty)^m\to\mathbb{N}$ measures how far is $\{\alpha,\beta\}$ to be totally ordered by the division order.

For $m\geq 2$, different $i,j\in\{1,\ldots,m\}$ and $\gamma>0$ let ς_{ij}^{γ} denote the morphism given in example *ii)* of 1.1.16.

Lemma 1.1.21. *i)* $d(X^\alpha,X^\beta)=0$ if and only $\{\alpha,\beta\}$ is totally ordered by the division order, or equivalently, either $X^\alpha\mid X^\beta$ or $X^\beta\mid X^\alpha$.

ii) If $m=1$, $d(X^\alpha,X^\beta)=0$.

iii) If $m\geq 2$ and $d(X^\alpha,X^\beta)=0$, then $d(\varsigma_{ij}^{\gamma}(X^\alpha),\varsigma_{ij}^{\gamma}(X^\beta))=0$ for any different $i,j\in\{1,\ldots,m\}$ and $\gamma>0$.

iv) $d(X^\alpha,X^\beta)=d(X^\beta,X^\alpha)$.

v) If $d(X^\alpha, X^\beta) \neq 0$, then there exists different $i, j \in \{1, \dots, m\}$ and $\gamma > 0$ such that

$$d(\varsigma_{ij}^{\gamma}(X^\alpha), \varsigma_{ij}^{\gamma}(X^\beta)) < d(X^\alpha, X^\beta)$$

and

$$d(\varsigma_{ji}^{1/\gamma}(X^\alpha), \varsigma_{ji}^{1/\gamma}(X^\beta)) < d(X^\alpha, X^\beta)$$

Proof. We reproduce the proof given in [1] of point *v)* because of the relevance of that point on the proof of the main result of this work. Suppose $d(X^\alpha, X^\beta) \neq 0$. Suppose first that $\gcd(X^\alpha, X^\beta) = 1$. Then we can choose different $i, j \in \{1, \dots, m\}$ such that $\alpha_i \neq 0$ and $\beta_j \neq 0$. Let $\gamma := \beta_j/\alpha_i$. Then $\varsigma_{ij}^{\gamma}(X^\alpha) = X^\alpha X_j^{\beta_j}$ and $\varsigma_{ij}^{\gamma}(X^\beta) = X^\beta$. Dividing X^β and $X^\alpha X_j^{\beta_j}$ by its gcd, $X_j^{\beta_j}$, we obtain $d(\varsigma_{ij}^{\gamma}(X^\alpha), \varsigma_{ij}^{\gamma}(X^\beta)) < d(X^\alpha, X^\beta)$. Analogously, $d(\varsigma_{ji}^{1/\gamma}(X^\alpha), \varsigma_{ji}^{1/\gamma}(X^\beta)) < d(X^\alpha, X^\beta)$.

For the general case, take $i, j \in \{1, \dots, m\}$ and $\gamma > 0$ such that

$$d(\varsigma_{ij}^{\gamma}(X^{\alpha-\omega}), s_{ij}^{\gamma}(X^{\beta-\omega})) < d(X^\alpha, X^\beta)$$

and

$$d(\varsigma_{ji}^{1/\gamma}(X^{\alpha-\omega}), \varsigma_{ji}^{1/\gamma}(X^{\beta-\omega})) < d(X^\alpha, X^\beta)$$

where $\omega = \inf(\alpha, \beta)$. The identity $\varsigma_{ij}^{\gamma}(X^\alpha) = \varsigma_{ij}^{\gamma}(X^{\alpha-\omega})\varsigma_{ij}^{\gamma}(X^\omega)$ then implies

$$d(\varsigma_{ij}^{\gamma}(X^\alpha), \varsigma_{ij}^{\gamma}(X^\beta)) = d(\varsigma_{ij}^{\gamma}(X^{\alpha-\omega}), \varsigma_{ij}^{\gamma}(X^{\beta-\omega}));$$

hence, $d(\varsigma_{ij}^{\gamma}(X^\alpha), \varsigma_{ij}^{\gamma}(X^\beta)) < d(X^\alpha, X^\beta)$. The case of $s_{ji}^{1/\gamma}$ is again similar. □

Definition 1.1.22. Given $s \in A[[X^*]]$, we define

$$b(s) = (b_1(s), b_2(s)) := (\#\mathrm{Supp}_{\min}(s) - 1, b_2(s)) \in \mathbb{N}^2 \tag{1.5}$$

where

$$b_2(s) = \begin{cases} 0 & \text{if } b_1(s) = 0 \\ \min\{d(\alpha, \beta) : \alpha, \beta \in \mathrm{Supp}_{\min}(s), \alpha \neq \beta\} & \text{if } b_1(s) \neq 0. \end{cases}$$

Notice that $b(s) = (0, 0)$, if and only if s is of monomial type. Consequently, if $m = 1$, $b(s) = (0, 0)$ for any $s \in A[[X^*]]$.

We order $\mathbb{N}^2$ lexicographically in what follows.

Proposition 1.1.23. Let $s \in A[[X^*]]$.

i) If $b(s) = (0, 0)$ and $m \geq 2$, then for any different $i, j \in \{1, \dots, m\}$ and $\gamma > 0$

$$b(\varsigma_{ij}^{\gamma}(s)) = (0, 0)$$

ii) If $b(s) \neq (0, 0)$, then there exists different $i, j \in \{1, \dots, m\}$ and $\gamma > 0$ such that

$$b(\varsigma_{ij}^{\gamma}(s)) < b(s) \;\text{ and }\; b(\varsigma_{ji}^{1/\gamma}(s) < b(s)$$

Proof. It follows from the definition of b and lemma 1.1.21. For a detailed proof see [1], proposition 4.14. □

1.1.5 Weierstrass preparation theorem.

Now we state a **Weierstrass Division and Preparation Theorem** for formal generalized series as it appears in [1].

Definition 1.1.24. Let $n > 0$. A power series $s \in A[[X^*, Y]]$ is called **regular in** Y_n of order d if

$$s(0, 0, Y_n) = uY_n^d + \text{terms of higher degree in } Y_n$$

with u a unit in A. Put $Y' := (Y_1, \dots, Y_{n-1})$.

Theorem 1.1.25. Let $n > 0$ and let $s \in A[[X^*, Y]]$ be regular in Y_n of order d.

1. There is for each $s' \in A[[X^*, Y]]$ a unique pair (Q, R) with $Q \in A[[X^*, Y]]$ and $R \in A[[X^*, Y']][Y_n]$, such that

$$s' = Qs + R \text{ and } deg_{Y_n}(R) < d.$$

2. s factors uniquely as $s = uP$, where u is a unit in $A[[X^*, Y]]$ and $P \in A[[X^*, Y']][Y_n]$ is a monic polynomial of degree d in Y_n.

Note that the polynomial P has the form

$$P = Y_n^d + a_1(X, Y')Y_n^{d-1} + \dots + a_d(X, Y')$$

with $a_i(0, 0)$ non units in A for $1 \leq i \leq d$ because it is monic and s is regular in Y_n of order d (if there exists i such that $a_i(0, 0)$ is a unit, s would be regular of order smaller or equal than $i < d$).

Implicit functions. We obtain as a corollary an Implicit Functions Theorem:

Corollary 1.1.26. Let $s = (s_1, s_2, \dots, s_k) \in A[[X^*, Y, W]]^k$ where $X = (X_1, X_2, \dots, X_m)$, $Y = (Y_1, Y_2, \dots, Y_n)$ and $W = (W_1, W_2, \dots, W_k)$. Suppose that $s_j(\underline{0}) = 0$ for $j = 1, 2 \dots, k$ and that the matrix $\left(\frac{\partial s_j}{\partial W_i}(\underline{0})\right)_{1 \leq i,j \leq k}$ is not singular. Then there exists $t_1, t_2, \dots, t_k \in A[[X^*, Y]]$ with $t_i(\underline{0}) = 0$ such that $s_j(X, Y, t_1(X, Y), t_2(X, Y), \dots, t_k(X, Y)) \equiv 0$ for $j = 1, 2 \dots, k$.

Proof. By induction on k. If $k = 1$, since $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$, s_1 is regular of order 1 in W_1. By Weierstrass preparation,

$$s_1 = (W_1 - a(X, Y, W_2, W_3, \dots, W_k))u_1$$

We take $t_1 = a$, which solves the problem.

Let $k \geq 2$ and suppose the result true for $k-1$. We can suppose $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$ and $\frac{\partial s_j}{\partial W_1}(\underline{0}) = 0$ for $j = 2, 3 \dots, k$ (if this is not the case, change the order of the s_i to have $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$ and then pick $\widetilde{s_1} := s_1$ and $\widetilde{s_j} := s_j - \frac{\frac{\partial s_j}{\partial W_1}(\underline{0})}{\frac{\partial s_1}{\partial W_1}(\underline{0})} s_1$ for $j = 2, 3 \dots, k$. If the result is proved for the $\widetilde{s_j}$ we obtain $t_1, t_2, \dots, t_k \in A[[X^*, Y]]$ with $t_i(\underline{0}) = 0$ such that $\widetilde{s_j}(X, Y, t_1(X, Y), t_2(X, Y), \dots, t_k(X, Y)) \equiv 0$. Notice that the same t_i solve the initial problem.)

Since $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$, s_1 is regular of order 1 in W_1. By Weierstrass preparation,

$$s_1 = (W_1 - a(X, Y, W_2, W_3, \dots, W_k))u_1$$

We define for $j = 2, 3 \dots, k$,

$$\overline{s_j} := s_j(X, Y, a(X, Y, W_2, W_3, \dots, W_k), W_2, W_3, \dots, W_k)$$

which are in $A[[X^*, Y, W_2, W_3, \dots, W_k]]$ because $a(\underline{0}) = 0$ (see proposition 1.1.14). On the other hand,

$$\frac{\partial \overline{s_j}}{\partial W_i}(\underline{0}) = \frac{\partial s_j}{\partial W_1}(a(\underline{0}))\frac{\partial a}{\partial W_i}(\underline{0}) + \frac{\partial s_j}{\partial W_i}(\underline{0}) = \frac{\partial s_j}{\partial W_i}(\underline{0})$$

for $1 \leq i, j \leq k$. Then, the matrix $\left(\frac{\partial \overline{s_j}}{\partial W_i}(\underline{0})\right)_{2\leq i,j\leq k} = \left(\frac{\partial s_j}{\partial W_i}(\underline{0})\right)_{2\leq i,j\leq k}$ which is not singular because $\frac{\partial s_j}{\partial W_1}(\underline{0}) = 0$ for $j \in \{2 \dots, k\}$, so by the induction assumption, there exits $t_2, \dots, t_k \in A[[X^*, Y]]$ with $t_i(\underline{0}) = 0$ such that

$$s_j(X, Y, a(t_2(X, Y), \dots, t_k(X, Y)), t_2(X, Y), \dots, t_k(X, Y)) = \overline{s_j}(X, Y, t_2(X, Y), \dots, t_k(X, Y)) \equiv 0$$

for $j = 2, 3 \dots, k$. Take $t_1 = a(t_2(X, Y), \dots, t_k(X, Y))$. □

1.2 Convergent generalized power series

In this section, we consider the subring of convergent series in the ring of formal generalized power series, where convergence is defined in a very natural way. The most part of the concepts and results are already given in the paper [1] but we reproduce here some of them when the arguments are useful for our purposes.

Convergent generalized power series give rise, passing to the limit in the partial sums, to functions in their domains of convergence, as much as the convergent standard power series give rise to the analytic functions. We will call those functions "generalized analytic functions". They will be our objects of study during the rest of this text.

1.2.1 Basic definitions.

Given any family $\{c_j\}_{j\in J}$ of positive real numbers, we can consider its sum

$$\sum_{j\in J} c_j \in [0, \infty]$$

With this notation we mean, as usual, that $\sum_{j\in J} c_j$ is equal to $c \in [0, \infty)$ if for any $\epsilon > 0$ there exists a finite set $J(\epsilon) \subset J$ such that for any finite subset $\tilde{J}$ of J containing $J(\epsilon)$ we have,

$$|\sum_{j\in J} c_j - c| < \epsilon.$$

If $\sum_{j\in J} c_j$ is not equal to c for any $c \in [0, \infty)$ we say that $\sum_{j\in J} c_j$ is equal to ∞. The reader familiarized with this concept can go directly to Definition 1.2.5.

We recall a property about interchanging index of summation in these kind of infinite sums.

Lemma 1.2.1. Let $\{c_{i,j}\}_{(i,j)\in I\times J}$ be a family of positive real numbers, $c_{i,j} > 0$ for any $(i, j) \in I \times J$. It is equivalent:

i) $\sum_{(i,j)\in I\times J} c_{i,j} = C < \infty$.

ii) For each $i \in I$, $\sum_{j\in J} c_{i,j} = C_i < \infty$ and $\sum_{i\in I} C_i = C$.

iii) For each $j \in J$, $\sum_{i\in I} c_{i,j} = C^j < \infty$ and $\sum_{j\in J} C^j = C$.

Proof. Let us show the equivalence between $i)$ and $ii)$, being the equivalence between $i)$ and $iii)$ analogous.

$i) \Rightarrow ii)$. Suppose that there exists $C \in \mathbb{R}_{>0}$ such that $\sum_{(i,j)\in I\times J} c_{i,j} = C$. Let $i \in I$, and J_0 a finite subset of J. Since the $c_{i,j}$ are positif and $\sum_{(i,j)\in I\times J} c_{i,j} = C$, $\sum_{j\in J_0} c_{i,j} \leq C$. Let

$$C_i := \sup_{\substack{J_0\subseteq J\\ J_0 \text{finite}}} \{\sum_{j\in J_0} c_{i,j}\}$$

Notice that $C_i \leq C$, for any $i \in I$. We claim that $C_i = \sum_{j\in J} c_{i,j}$: first, if $J_1 \subseteq J$ is finite, $\sum_{j\in J_1} c_{i,j} \leq C_i$ by definition of C_i. Let $\epsilon > 0$. Since $C_i - \epsilon$ is not an upper bound of de family $\{\sum_{j\in J_0} c_{i,j}\}_{\substack{J_0\subseteq J\\ J_0\text{finite}}}$, there exists $J_0 \subseteq J$ finite such that $C_i - \epsilon < \sum_{j\in J_0} c_{i,j}$, which implies that $C_i - \sum_{j\in J_0} c_{i,j} < \epsilon$. Thus, $C_i = \sum_{j\in J} c_{i,j}$.

Let us prove now that $\sum_{i\in I} C_i = C$. Notice that if $I_0 \subseteq I$ is finite, $\sum_{i\in I_0} C_i \leq C$ (if not, there should exist a finite $I_0 \subseteq I$ such that $\sum_{i\in I_0} C_i = C + \epsilon$ with $\epsilon > 0$. Let, for $i \in I_0$, $J_0(i) \subseteq J$ finite such that $C_i - \sum_{j\in J_0(i)} c_{i,j} < \frac{\epsilon}{\sharp I_0}$, where $\sharp I_0$ denotes the number of elements in I_0. Let $J_0 = \cup_{i\in I_0} J_0(i)$. Then,

$$C + \epsilon - \sum_{(i,j)\in I_0\times J_0} c_{i,j} = \sum_{i\in I_0} C_i - \sum_{(i,j)\in I_0\times J_0} c_{i,j} < \epsilon$$

which implies that $C < \sum_{(i,j)\in I_0\times J_0} c_{i,j}$. Contradiction.) If $\epsilon > 0$, there exists $I_0 \times J_0 \subseteq I \times J$ finite such that $C - \sum_{(i,j)\in I_0\times J_0} c_{i,j} < \epsilon$. But,

$$\sum_{(i,j)\in I_0\times J_0} c_{i,j} = \sum_{i\in I_0}(\sum_{j\in J_0} c_{i,j}) \leq \sum_{i\in I_0} C_i$$

Then,

$$C - \sum_{i\in I_0} C_i \leq C - \sum_{(i,j)\in I_0\times J_0} c_{i,j} < \epsilon$$

So, $C = \sum_{i\in I} C_i$.

$ii) \Rightarrow i)$. Let us show first that if $\Lambda \subseteq I \times J$ is finite, then $\sum_{(i,j)\in\Lambda} c_{i,j} \leq C$. Suppose $\Lambda = I_0 \times J_0$ with I_0 and J_0 finite, then $\sum_{(i,j)\in\Lambda} c_{i,j} = \sum_{i\in I_0}(\sum_{j\in J_0} c_{i,j}) \leq \sum_{i\in I_0} C_i \leq C$.

Let $\epsilon > 0$. As $\sum_{i\in I} C_i = C$, there exists $I_0 \subseteq I$ finite such that $C - \sum_{i\in I_0} C_i < \frac{\epsilon}{2}$. For each $i \in I_0$ let $J_0(i) \subseteq J$ finite such that $C_i - \sum_{j\in J_0(i)} c_{i,j} < \frac{\epsilon}{2\sharp I_0}$ where $\sharp I_0$ denotes the number of elements in I_0. Let $J_0 := \cup_{i\in I_0} J_0(i)$. It is finite and for any $i \in I_0$, $C_i - \sum_{j\in J_0} c_{i,j} < \frac{\epsilon}{2\sharp I_0}$. Then, $\sum_{i\in I_0} C_i - \sum_{(i,j)\in I_0\times J_0} c_{i,j} < \frac{\epsilon}{2}$. Thus, $C - \sum_{(i,j)\in I_0\times J_0} c_{i,j} < \epsilon$. □

Now, if A is a normed ring with norm $|\cdot|$, we can generalize the concept of the sum of a family of elements in A.

Lemma 1.2.2. Given any family $\{a_j\}_{j\in J}$ of elements of A, there is at most one element $a \in A$ such that

$$\begin{aligned} &\text{for each } \epsilon > 0 \text{ there is a finite subset } J(\epsilon) \subseteq J \text{ with} \\ &|\sum_{j\in\tilde{J}} a_j - a| < \epsilon \text{ for any finite set } \tilde{J} \subseteq J \text{ that contains } J(\epsilon). \end{aligned} \tag{1.6}$$

Proof. Suppose $a, b \in A$ satisfying (1.6). For $\epsilon > 0$, there exists $J_a(\epsilon), J_b(\epsilon) \subseteq J$ finite such that $|\sum_{j\in J_a(\epsilon)} a_j - a| < \frac{\epsilon}{2}$ and $|\sum_{j\in J_b(\epsilon)} a_j - b| < \frac{\epsilon}{2}$. Then, with $J(\epsilon) = J_a(\epsilon) \cup J_b(\epsilon)$, $|a - b| \leq |a - \sum_{j\in J(\epsilon)} a_j| + |\sum_{j\in J(\epsilon)} a_j - b| < \epsilon$ for any $\epsilon > 0$. □

Definition 1.2.3. With the notation of Lemma 1.2.2 above, if $a \in A$ has property (1.6), we say that $\sum_{j\in I} a_j$ **exists in** A and define $\sum_{j\in I} a_j := a$.

We show here some properties of these kind of sums which will be useful for the rest of the chapter.

Lemma 1.2.4. Let $\{a_j\}_{j\in J}$ be a family of elements of A.

i) If $\sum_{j\in J} a_j$ exists in A, then, for any finite subset $\tilde{J} \subseteq J$, $\sum_{j\notin \tilde{J}} a_j$ exists in A and $\sum_{j\notin \tilde{J}} a_j = \sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j$.

ii) If $\sum_{j\in J} a_j$ exists in A, then, for any $\epsilon > 0$ there exists a finite subset $J(\epsilon) \subseteq J$, such that $|\sum_{j\notin \tilde{J}} a_j| < \epsilon$ for any $\tilde{J} \subseteq J$ finite containing $J(\epsilon)$.

iii) If A is complete and $\sum_{j\in I} |a_j| < \infty$, $\sum_{j\in I} a_j$ exists in A.

iv) If $\sum_{j\in J} a_j$ exists in A and $\sum_{j\in J} |a_j| < \infty$, then, $|\sum_{j\in J} a_j| \leq \sum_{j\in J} |a_j|$.

Proof. For *i*), let $\tilde{J} \subseteq J$ be a finite subset of J. Let $\epsilon > 0$ and $J(\epsilon) \subseteq J$ be finite such that $|\sum_{j\in J} a_j - \sum_{j\in \bar{J}} a_j| < \epsilon$ for any $\bar{J} \subseteq J$ finite containing $J(\epsilon)$. Let $J^*(\epsilon) := J(\epsilon) \cap (J \setminus \tilde{J})$. If J^* is a finite subset of $J \setminus \tilde{J}$ with $J^*(\epsilon) \subseteq J^*$, $J(\epsilon) \subseteq \tilde{J} \cup J^*$, so

$$|(\sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j) - \sum_{j\in J^*} a_j| = |\sum_{j\in J} a_j - \sum_{j\in \tilde{J}\cup J^*} a_j| < \epsilon$$

For *ii*), let $\epsilon > 0$ and $J(\epsilon) \subseteq J$ be finite such that $|\sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j| < \epsilon$ for $\tilde{J} \subseteq J$ finite containing $J(\epsilon)$. By part *i*), $|\sum_{j\notin \tilde{J}} a_j| = |\sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j|$.

For *iii*), we claim that under these hypothesis, $a_j \neq 0$ for only countably many $j \in J$. For that, it suffices to prove that if X is a subset of strictly positive real numbers and $C > 0$ a constant such that for any finite subset $Y \subseteq X$, $\sum_{x\in Y} x \leq C$, then X is countable. Suppose that there exists $\{x_n\}_{n\in\mathbb{N}}$ a strictly increasing sequence of elements of X. Then, for any $N \in \mathbb{N}$, $Nx_1 < \sum_{i=1}^{N} x_i \leq C$, which is impossible. So given $x \in X$ there exists its antecessor, x^- defined by

$$x^- := \max\{y \in X : y < x\}$$

and we can find a rational number $q_x \in \mathbb{Q}$ between x^- and x. So we can suppose $J = \mathbb{N}$. The sequence $\{S_n := \sum_{j=1}^{n} a_j\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A, because for $m < n$, $|S_n - S_m| \leq \sum_{k=m}^{n} |a_k| \xrightarrow[n,m\to\infty]{} 0$. Since A is complete, $\sum_{n\in\mathbb{N}} a_n$ exists in A.

For *iv*), let $\epsilon > 0$. Let $J(\epsilon) \subseteq J$ be finite such that $|\sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j| < \epsilon$ for $\tilde{J} \subseteq J$ finite containing $J(\epsilon)$. Thus

$$|\sum_{j\in J} a_j - \sum_{j\in \tilde{J}} a_j| \geq \Big||\sum_{j\in J} a_j| - |\sum_{j\in \tilde{J}} a_j|\Big| \geq |\sum_{j\in J} a_j| - |\sum_{j\in \tilde{J}} a_j| \geq |\sum_{j\in J} a_j| - \sum_{j\in \tilde{J}} |a_j| \geq |\sum_{j\in J} a_j| - \sum_{j\in J} |a_j|$$

which implies $|\sum_{j\in J} a_j| - \sum_{j\in J} |a_j| < \epsilon$ for any $\epsilon > 0$. □

From now on, unless indicated otherwise, we let A denote a normed ring with norm $|\cdot|$. We let $r, l \in (0,\infty)^m$ denote polyradii, and we write $r \leq l$ if $r_i \leq l_i$ for all i, and $r < l$ if $r_i < l_i$ for all i (notice that $r < l$ does not mean $r \leq l$ and $r \neq l$). Also if $\alpha \in [0,\infty)^m$, we put $r^\alpha = r_1^{\alpha_1} \cdots r_m^{\alpha_m}$.

Definition 1.2.5. For $s = \sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha \in A[[X^*]]$ and a polyradius r we define

$$\|s\|_r := \sum_{\alpha\in[0,\infty)^m} |s_\alpha| r^\alpha \in [0,\infty]$$

We have, for $s, t \in A[[X^*]]$ and polyradii $r, l \in (0,\infty)^m$ (see [1], page 4391):

1. $\|s\|_r = 0$ if and only if $s = 0$;

2. $\|s + t\|_r \leq \|s\|_r + \|t\|_r$;

3. $\|st\|_r \leq \|s\|_r \|t\|_r$;

4. if $r \leq l$, then $\|s\|_r \leq \|s\|_l$.

We now define

$$A\{X^*\}_r := \{s \in A[[X^*]] : \|s\|_r < \infty\}$$

Note that $A\{X^*\}_r$ is a normed ring with norm $\|\cdot\|_r$. It is clearly a subring of $A[[X^*]]$ containing $A[X^*]$. We put

$$A\{X^*\} := \bigcup_r A\{X^*\}_r$$

Since $A\{X^*\}_r \supseteq A\{X^*\}_l$ if $r \leq l$, $A\{X^*\}$ is also a subring of $A[[X^*]]$. Put also, for mixed variables $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$,

$$A\{X^*, Y\} := A[[X^*, Y]] \cap A\{(X, Y)^*\},$$

and

$$A\{X^*, Y\}_{(r,l)} := A[[X^*, Y]] \cap A\{(X, Y)^*\}_{(r,l)}$$

for polyradii $r = (r_1, \ldots, r_m)$, $l = (l_1, \ldots, l_n)$.

Now, always for a normed ring A, we generalize the concept of summable family (cf. 1.1.6) of formal generalized power series in the following way (see 5.7 of [1]):

Definition 1.2.6. Let J be any index set and assume that $\{s_j = \sum_\alpha s_{j,\alpha} X^\alpha\}_{j \in J}$ is a family in $A[[X^*]]$ such that

i) for each $\alpha \in [0, 1)^m$ we have $\sum_{j \in J} |s_{j,\alpha}| < \infty$ and $\sum_{j \in J} s_{j,\alpha}$ exists in A

ii) $\cup_{j \in J} \mathrm{Supp}(s_j)$ is a good subset of $[0, 1)^m$.

Then, if we define $\sum_{j \in J} s_j := \sum_\alpha (\sum_{j \in J} s_{j,\alpha}) X^\alpha$, $\sum_{j \in J} s_j \in A[[X^*]]$.

Proposition 1.2.7. Let $\{s_j = \sum_\alpha s_{j,\alpha} X^\alpha\}_{j \in J}$ be a family in $A[[X^*]]$ satisfying *i)* and *ii)* of Definition 1.2.6. Suppose that $\sum_{j \in J} \|s_j\|_r < \infty$. Then $\|\sum_{j \in J} s_j\|_r \leq \sum_{j \in J} \|s_j\|_r$ and we obtain that

i) $\sum_{j \in J} s_j$ actually belongs to $A\{X^*\}_r$ and

ii) $\sum_{j \in J} s_j$ is also the sum of the family $\{s_j\}_{j \in J}$ in the normed ring $(A\{X^*\}_r, \|\cdot\|_r)$.

Proof. By lemma 1.2.1,

$$\sum_{j \in J} \|s_j\|_r = \sum_{j \in J} (\sum_\alpha |s_{j,\alpha}| r^\alpha) = \sum_\alpha (\sum_{j \in J} |s_{j,\alpha}|) r^\alpha$$

On the other hand,

$$\|\sum_{j \in J} s_j\|_r = \sum_\alpha |\sum_{j \in J} s_{j,\alpha}| r^\alpha$$

Thus, by part *iv)* of Lemma 1.2.4, $\|\sum_{j \in J} s_j\|_r \leq \sum_{j \in J} \|s_j\|_r < \infty$ which implies consequence *i)*. For *ii)*, let $\epsilon > 0$. Since $\sum_{j \in J} \|s_j\|_r < \infty$, by *i)* and *ii)* of Lemma 1.2.4 there exists $J(\epsilon) \subseteq J$ finite such that $\sum_{j \notin J(\epsilon)} \|s_j\|_r < \epsilon$. Then,

$$\|\sum_{j \in J} s_j - \sum_{j \in J(\epsilon)} s_j\|_r = \|\sum_{j \notin J(\epsilon)} s_j\|_r \leq \sum_{j \notin J(\epsilon)} \|s_j\|_r < \epsilon$$

□

Remark 1.2.8. Let $s = \sum_\alpha s_\alpha X^\alpha \in A\{X^*\}_r$. Let for any $\Lambda \in \mathcal{P}_\mathcal{F}([0,\infty)^m)$, $s_\Lambda := \sum_{\alpha\in\Lambda} s_\alpha X^\alpha$, where $\mathcal{P}_\mathcal{F}([0,\infty)^m)$ denotes the set of finite subsets of $[0,\infty)^m$. Then, by Lemma 1.2.4, for any $\epsilon > 0$ there exists a $\Lambda(\epsilon) \in \mathcal{P}_\mathcal{F}([0,\infty)^m)$ such that

$$\|s - s_\Lambda\|_r = \|\sum_{\alpha\notin\Lambda} s_\alpha X^\alpha\|_r < \epsilon$$

for any $\Lambda \in \mathcal{P}_\mathcal{F}([0,\infty)^m)$ with $\Lambda(\epsilon) \subseteq \Lambda$.

1.2.2 Properties of convergent series

We show here those properties of formal series with an analogous statement in the convergent setting: composition morphisms, Weierstrass preparation and implicit functions. We need the following lemma

Lemma 1.2.9. If $s = \sum s_\alpha X^\alpha \in A\{X^*\}$, then $\lim_{r\to 0} \|s\|_r = |s(0)|$.

Proof. (see [1], 5.5) It suffices to show that $\lim_{r\to 0} \|s - s(0)\|_r = 0$, so replacing s by $s - s(0)$ we may as well assume that $s(0) = 0$. Take l such that $\|s\|_l < \infty$, and fix $\epsilon > 0$. Let $J \subseteq$Supp(s) be finite such that $\sum_{\alpha\notin J} |s_\alpha| l^\alpha < \epsilon/2$ (Lemma 1.2.4), and let $\tilde{l} \le l$ be a polyradius such that $\sum_{\alpha\in J} |s_\alpha|\tilde{l}^\alpha < \epsilon/2$. Then for every $r \le \tilde{l}$ (Lemma 1.2.4),

$$\|s\|_r = \|\sum_{\alpha\notin J} s_\alpha X^\alpha + \sum_{\alpha\in J} s_\alpha X^\alpha\|_r \le \|\sum_{\alpha\notin J} s_\alpha X^\alpha\|_r + \|\sum_{\alpha\in J} s_\alpha X^\alpha\|_r \le \sum_{\alpha\notin J} |s_\alpha| r^\alpha + \sum_{\alpha\in J} |s_\alpha|\tilde{l}^\alpha < \epsilon$$

Since ϵ was arbitrary, this proves the lemma. □

Using the same notation as in Proposition 1.1.14 the following "convergent version" of the properties of composition of series holds:

Proposition 1.2.10. Let $X = (X_1, X_2, \ldots, X_m)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, $Z = (Z_1, Z_2, \ldots, Z_\mu)$ and $W = (W_1, W_2, \ldots, W_l)$ denote multi-variables.

i) Let $s = \sum_{(\alpha,I)\in[0,\infty)^m\times\mathbb{N}^n} s_{(\alpha,I)} X^\alpha Y^I \in A\{X^*, Y\}$ and let $t = (t_1, t_2, \ldots, t_n) \in A\{W\}^n$ with $t_1(0) = \ldots = t_n(0) = 0$. Then $s(X, t(W))$ is in $A\{X^*, W\}$. Moreover, the map $s \mapsto s(X, t(W))$ is an A-algebra homomorphism from $A\{X^*, Y\}$ to $A\{X^*, W\}$.

ii) Let $s = \sum_{(\alpha,I)\in[0,\infty)^m\times\mathbb{N}^n} s_{(\alpha,I)} X^\alpha Y^I \in A\{X^*, Y\}$ and let $t = (t_1, t_2, \ldots, t_n) \in A\{Z^*\}^n$ with $t_1(0) = \ldots = t_n(0) = 0$. Then, $s(X, t(Z))$, is in $A\{X^*, Z^*\}$. Moreover, the map $s \mapsto s(X, t(Z))$ is an A-algebra homomorphism from $A\{X^*, Y\}$ to $A\{X^*, Z^*\}$.

iii) Let $a > 0$. If $u = \sum_{\alpha\in[0,\infty)^m} u_\alpha X^\alpha \in \mathbb{R}\{X^*\}$ is such that $u_0 > 0$,

$$u^a := \sum_{k\in\mathbb{N}} \binom{a}{k} u_0^{a-k}(u - u_0)^k \in \mathbb{R}\{X^*\}$$

iv) Let $s = \sum s_\alpha X^\alpha \in \mathbb{R}\{X^*\}$ and $t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}\{Z^*\}^m$. If $t_i = Z^{\beta^i} u_i$, with $\beta^i \neq (0, ..., 0)$, $u_i \in \mathbb{R}\{Z^*\}$ and $u_i(0) > 0$ for all $i \in \{1, 2, \ldots, m\}$ (that is, t_i is of monomial type), $s(t_1, t_2, \ldots, t_m)$ is in $\mathbb{R}\{Z^*\}$. Moreover, the map $s \mapsto s(t_1, \ldots, t_m)$ is an $\mathbb{R}$-algebra homomorphism from $\mathbb{R}\{X^*\}$ to $\mathbb{R}\{Z^*\}$.

Proof. We use Proposition 1.2.7 to prove the convergence of the formal series obtained in Proposition 1.1.14

For *i)*, let us call for any $\alpha \in \mathrm{Supp}_X(s)$ and $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$

$$q_{(\alpha,I)} := s_{(\alpha,I)} X^\alpha t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

Then, it suffices to prove that there exists polyradius $\tilde{r} \in (0,\infty)^m$ and $\tilde{l} \in (0,\infty)^n$ such that $\|\sum_{(\alpha,I)} q_{(\alpha,I)}\|_{(\tilde{r},\tilde{l})} < \infty$. Take polyradius $(r,l) \in (0,\infty)^{m+n}$ such that $s \in \mathbb{R}\{X^*,Y\}_{(r,l)}$ and $t_1,\dots,t_n \in \mathbb{R}\{W\}_l$. Now, take $\tilde{l} < l$ such that $\|t_i\|_{\tilde{l}} < l_i$ (which is possible by Lemma 1.2.9 because $t_i(0) = 0$ for $i = 1,2,\dots,n$.) Thus,

$$\sum_{(\alpha,I)} \|q_{(\alpha,I)}\|_{(r,\tilde{l})} = \sum_{(\alpha,I)} |s_{(\alpha,I)}| r^\alpha \|t_1\|_{\tilde{l}_1}^{i_1} \|t_2\|_{\tilde{l}_2}^{i_2} \cdots \|t_n\|_{\tilde{l}_n}^{i_n} < \sum_{(\alpha,I)} |s_{(\alpha,I)}| r^\alpha l^I = \|s\|_{(r,l)} < \infty$$

The same argument is valid for *ii*). For *iii*), if we put $\varepsilon := u - u(0) \in \mathbb{R}\{X^*\}$, $\varepsilon(0) = 0$. So it is enough to prove that there exists a polyradius $r \in (0,\infty)^m$ such that $\sum_k \|\varepsilon\|_r^k < \infty$. Notice that this is a particular case of *ii*).

For *iv*), we define for any $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_m) \in \mathrm{Supp}(s)$,

$$q_\alpha := s_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_m^{\alpha_m}$$

By part *iii*), $q_\alpha \in \mathbb{R}\{Z^*\}$ for any $\alpha \in \mathrm{Supp}(s)$. Let $\delta = (\delta_1,\delta_2,\dots,\delta_m) \in (0,\infty)^m$ be a polyradius such that $s \in \mathbb{R}\{X^*\}_\delta$. Since $t_1(0) = t_2(0) = \dots = t_m(0) = 0$, by Proposition 1.2.9 there exists a polyradius $r \in (0,\infty)^\mu$ such that $\|t_i\|_r < \delta_i$ for $i = 1,2,\dots,m$. Then, $\|q_\alpha\|_r \le |s_\alpha| \|t_1\|_r^{\alpha_1} \|t_2\|_r^{\alpha_2} \cdots \|t_m\|_r^{\alpha_m} < |s_\alpha| \delta^\alpha$ for any $\alpha \in$Supp(s) which implies that

$$\sum_\alpha \|q_\alpha\|_r < \sum_\alpha |s_\alpha| \delta^\alpha = \|s\|_\delta < \infty$$

□

Remark 1.2.11. Let $s = \sum s_\alpha X^\alpha \in \mathbb{R}\{X^*\}$, $M_1, M_2, \dots, M_m \in \mathbb{R}\{Z^*\}$ be monomials ($M_i = Z^{\beta^i}$ with $\beta^i \neq \underline{0}$), $W = (W_1,W_2,\dots,W_m)$ be variables and $\lambda_1,\lambda_2,\dots,\lambda_m \in \mathbb{R}_{>0}$. If we define for $i = 1,2,\dots,m$, $t_i := M_i(\lambda_i + W_i)$, by remark 1.1.15, $s(t_1,t_2,\dots,t_m) \in \mathbb{R}[[Z^*,W]]$. Analogously to the proof of part *iv*) we obtain that in fact $s(t_1,t_2,\dots,t_m) \in \mathbb{R}\{Z^*,W\}$.

Example 1.2.12. Notice that in particular, with the notation of the example 1.1.16, if $s \in A\{X^*,Y\}$ and $\sigma \in G_{m,n}$, $\sigma s \in A\{X^*,Y\}$.

Corollary 1.2.13. *Let $s \in A\{X^*\}$. Then s is a unit in $A\{X^*\}$ if and only if $s(0)$ is a unit in A.*

Proof. The necessity is clear. Suppose then $s(0) \neq 0$ and write $s = s(0)(1-t)$ for some $t \in A\{X^*\}$ with $t(0) = 0$. Then $1-t$ has inverse $1+t+t^2+\ldots \in A[[X^*]]$. The series $\bar{s} := \sum_{k\in\mathbb{N}} W^k \in A\{W\}$. By part *ii*) of Proposition 1.2.10, $1+t+t^2+\ldots = \bar{s}(t) \in A\{X^*\}$. □

The Weierstrass Preparation Theorem is also true in the convergent case (see 5.10 of [1]).

Theorem 1.2.14. *Let $n > 0$ and let $s \in A\{X^*,Y\}$ be regular in Y_n of order d.*

1. *There is for each $s' \in A\{X^*,Y\}$ a unique pair (Q,R) with $Q \in A\{X^*,Y\}$ and $R \in A\{X^*,Y'\}[Y_n]$, such that*

$$s' = Qs + R \text{ and } deg_{Y_n}(R) < d.$$

2. *s factors uniquely as $s = UP$, where $U \in A\{X^*,Y\}$ is a unit and $P \in A\{X^*,Y'\}[Y_n]$ is monic of degree d in Y_n.*

Corollary 1.2.15. *Let $s = (s_1,s_2,\dots,s_k) \in A\{X^*,Y,W\}^k$ where $X = (X_1,X_2,\dots,X_m)$, $Y = (Y_1,Y_2,\dots,Y_n)$ and $W = (W_1,W_2,\dots,W_k)$. Suppose that $s_j(\underline{0}) = 0$ for $j = 1,2\dots,k$ and that the matrix $\left(\frac{\partial s_j}{\partial W_i}(\underline{0})\right)_{1\le i,j\le k}$ is not singular. Then there exists $t_1,t_2,\dots,t_k \in A\{X^*,Y\}$ with $t_i(\underline{0}) = 0$ such that $s_j(X,Y,t_1(X,Y),t_2(X,Y),\dots,t_k(X,Y)) \equiv 0$ for $j = 1,2\dots,k$.*

Proof. By induction on k. If $k=1$, since $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$, s_1 is regular of order 1 in W_1. By Weierstrass preparation,

$$s_1 = (W_1 - a(X,Y,W_2,W_3,\dots,W_k))u_1$$

We take $t_1 = a$, which solves the problem.

Let $k \geq 2$ and suppose the result true for $k-1$. We can suppose $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$ and $\frac{\partial s_j}{\partial W_1}(\underline{0}) = 0$ for $j = 2,3\dots,k$ (if this is not the case, change the order of the s_i to have $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$ and then pick $\widetilde{s_1} := s_1$ and $\widetilde{s_j} := s_j - \frac{\frac{\partial s_j}{\partial W_1}(\underline{0})}{\frac{\partial s_1}{\partial W_1}(\underline{0})} s_1$ for $j = 2,3\dots,k$. If the result is proved for the $\widetilde{s_j}$ we obtain $t_1,t_2,\dots,t_k \in A\{X^*,Y\}$ with $t_i(\underline{0}) = 0$ such that $\widetilde{s_j}(X,Y,t_1(X,Y),t_2(X,Y),\dots,t_k(X,Y)) \equiv 0$. Notice that the same t_i solve the initial problem.)

Since $\frac{\partial s_1}{\partial W_1}(\underline{0}) \neq 0$, s_1 is regular of order 1 in W_1. By Weierstrass preparation,

$$s_1 = (W_1 - a(X,Y,W_2,W_3,\dots,W_k))u_1$$

We define for $j = 2,3\dots,k$,

$$\overline{s_j} := s_j(X,Y,a(X,Y,W_2,W_3,\dots,W_k),W_2,W_3,\dots,W_k)$$

which are in $A\{X^*,Y,W_2,W_3,\dots,W_k\}$ because $a(\underline{0}) = 0$ (see proposition 1.1.14). On the other hand,

$$\frac{\partial \overline{s_j}}{\partial W_i}(\underline{0}) = \frac{\partial s_j}{\partial W_1}(a(\underline{0}))\frac{\partial a}{\partial W_i}(\underline{0}) + \frac{\partial s_j}{\partial W_i}(\underline{0}) = \frac{\partial s_j}{\partial W_i}(\underline{0})$$

for $1 \leq i,j \leq k$. Then, the matrix $\left(\frac{\partial \overline{s_j}}{\partial W_i}(\underline{0})\right)_{2\leq i,j\leq k} = \left(\frac{\partial s_j}{\partial W_i}(\underline{0})\right)_{2\leq i,j\leq k}$ which is not singular because $\frac{\partial s_j}{\partial W_1}(\underline{0}) = 0$ for $j \in \{2\dots,k\}$, so by the induction assumption, there exits $t_2,\dots,t_k \in A\{X^*,Y\}$ with $t_i(\underline{0}) = 0$ such that

$$s_j(X,Y,a(t_2(X,Y),\dots,t_k(X,Y)),t_2(X,Y),\dots,t_k(X,Y)) = \overline{s_j}(X,Y,t_2(X,Y),\dots,t_k(X,Y)) \equiv 0$$

for $j = 2,3\dots,k$. Take $t_1 = a(t_2(X,Y),\dots,t_k(X,Y))$. □

Notice that in Corollary 1.2.15 we do not ask the partial derivatives of the s_j to be convergent. However, one can ask if the formal partial derivative (defined in 1.1.1) of a convergent series is convergent too. Paragraph 5.9 of [1] answer affirmatively this question:

Lemma 1.2.16. (cf. 5.9 [1]) Let $s \in \mathbb{R}\{X^*,Y\}$. If $i \in \{1,\dots,m\}$, then the partial derivative $(\partial s/\partial X_i) \in \mathbb{R}\{X^*,Y\}$, and if $j \in \{1,\dots,n\}$, $(\partial s/\partial Y_j) \in \mathbb{R}\{X^*,Y\}$.

1.2.3 Functions defined by convergent series.

From now on we are only interested in the case $A = \mathbb{R}$, with the norm on $\mathbb{R}$ given by the usual absolute value. Note that Corollary 1.2.13 implies that $\mathbb{R}\{X^*\}$ is a local ring with maximal ideal $\{s \in \mathbb{R}\{X^*\} : s(0) = 0\}$, and if $m = 1$, then $\mathbb{R}\{X^*\}$ is a valuation ring.

Given a polyradius $\xi = (\xi_1,\dots,\xi_{m+n}) \in (0,\infty)^{m+n}$, we put

$$I_{m,n,\xi} := [0,\xi_1) \times \cdots \times [0,\xi_m) \times (-\xi_{m+1},\xi_{m+1}) \times \cdots \times (-\xi_{m+n},\xi_{m+n});$$

and

$$\mathrm{Clos}(I_{m,n,\xi}) := [0,\xi_1] \times \cdots \times [0,\xi_m] \times [-\xi_{m+1},\xi_{m+1}] \times \cdots \times [-\xi_{m+n},\xi_{m+n}]$$

we will denote $[0,\infty)^m \times \mathbb{R}^n$ by $I_{m,n,\infty}$. We also write $\mathbb{R}\{X^*,Y\}_\xi$ instead of $\mathbb{R}\{X^*,Y\}_{(r,l)}$ where $r = (\xi_1,\dots,\xi_m)$ and $l = (\xi_{m+1},\dots,\xi_{m+n})$. If $n = 0$ we write $I_{m,\xi}$ instead of $I_{m,0,\xi}$.

Most of the time we will consider polyradius whose components have all the same value, usually $\epsilon > 0$ or $\delta > 0$. In that case ϵ (respectively δ, etc.) will denote a positif constant ϵ or polyradius $\epsilon = (\epsilon, \dots, \epsilon)$ with different length, and its significant will be deduced by the context.

Finally, to emphasize the length of the multi-variables involved $X = (X_1, X_2, \dots, X_m), Y = (Y_1, Y_2, \dots, Y_n)$, etc. we put $\mathbb{R}\{X^*, Y\}_{m,n}$. Then, for instance, if $\epsilon > 0$, to denote the $\mathbb{R}$-algebra of convergent series in the variables $Z = (Z_1, Z_2), W = (W_1, W_2, W_3)$, where the variables Z are generalized and the variables W are analytic and the polyradius of converge is ϵ, we put $\mathbb{R}\{Z^*, W\}_{2,3,\epsilon}$.

Definition 1.2.17. Let $m, n \in \mathbb{N}$ and $\xi \in (0, \infty)^{m+n}$ a polyradius. To an element $s = \sum s_{\alpha,\beta} X^\alpha Y^\beta \in \mathbb{R}\{X^*, Y\}_{m,n,\xi}$ we associate a function on $I_{m,n,\xi}$ as follows. Given $(x, y) \in$Clos$(I_{m,n,\xi})$, the series $\sum s_{\alpha,\beta} x^\alpha y^\beta$ converges absolutely to a real number. Thus we can define the function

$$\begin{aligned} S_\xi(s) : \text{Clos}(I_{m,n,\xi}) &\longrightarrow \mathbb{R} \\ S_\xi(s)(x, y) &:= \sum s_{\alpha,\beta} x^\alpha y^\beta \end{aligned}$$

Lemma 1.2.18. Let $s \in \mathbb{R}\{X^*, Y\}_\xi$. If $\tilde{\xi} < \xi$, $S_{\tilde{\xi}}(s)$ is equal to the restriction of $S_\xi(s)$ to the polyinterval $I_{m,n,\tilde{\xi}}$, that is $S_{\tilde{\xi}}(s) = S_\xi(s)|_{I_{m,n,\tilde{\xi}}}$.

Proof. Immediate by definition of the sum morphism. □

For a real valued function $f : X \to \mathbb{R}$ we let $\|f\|_\infty$ denote its uniform norm, that is

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\} \in [0, \infty]$$

Lemma 1.2.19. Let $s = \sum_{(\alpha,\beta)} s_{\alpha,\beta} X^\alpha Y^\beta \in \mathbb{R}\{X^*, Y\}_{m,n,\xi}$. Then, $\|S_\xi(s)\|_\infty \leq \|s\|_\xi$. In particular, with the notation of Remark 1.2.8, for any $\Lambda \in \mathcal{P}_{\mathcal{F}}(\text{Supp}(s))$, since $S_\xi(s) - S_\xi(s_\Lambda) = S_\xi(s - s_\Lambda)$, $\|S_\xi(s) - S_\xi(s_\Lambda)\|_\infty \leq \|s - s_\Lambda\|_\xi$.

Proof. Let $(x, y) \in$Clos$(I_{m,n,\xi})$. Then,

$$|S_\xi(s)(x, y)| = |\sum_{(\alpha,\beta)} s_{\alpha,\beta} x^\alpha y^\beta| \leq \sum_{(\alpha,\beta)} |s_{\alpha,\beta}||x^\alpha||y^\beta| \leq \|s\|_\xi$$

□

Let $\mathcal{C}^0(\text{Clos}(I_{m,n,\xi}); \mathbb{R})$ denote the ring of all real valued continuous functions on Clos$(I_{m,n,\xi})$.

Proposition 1.2.20. The function $S_\xi(s)$ is continuous on Clos$(I_{m,n,\xi})$. Moreover, the map

$$\begin{aligned} S_\xi : \mathbb{R}\{X^*, Y\}_{m,n,\xi} &\longrightarrow \mathcal{C}^0(\text{Clos}(I_{m,n,\xi}); \mathbb{R}) \\ s &\mapsto S_\xi(s) \end{aligned}$$

is an $\mathbb{R}$-algebra homomorphism.

Proof. For each $\Lambda \in \mathcal{P}_{\mathcal{F}}([0, \infty)^m \times \mathbb{N}^n)$, the series $s_\Lambda = \sum_{(\alpha,\beta)\in\Lambda} s_{\alpha,\beta} X^\alpha Y^\beta \in \mathbb{R}\{X^*, Y\}_{m,n,\xi}$. Since Λ is finite the corresponding associated function $S_\xi(s_\Lambda) : (x, y) \in$Clos$(I_{m,n,\xi}) \mapsto s_\Lambda(x, y) \in \mathbb{R}$ is continuous. Let $\epsilon > 0$. By Remark 1.2.8, there exists $\Lambda \subseteq [0, \infty)^m \times \mathbb{N}^n$ finite such that $\|s - s_\Lambda\|_\xi < \frac{\epsilon}{3}$. Let $(x, y) \in$Clos$(I_{m,n,\xi})$. Then,

$$|S_\xi(s)(x, y) - S_\xi(s_\Lambda)(x, y)| = |\sum_{(\alpha,\beta)\notin\Lambda} s_{\alpha,\beta} x^\alpha y^\beta| \leq \|s - s_\Lambda\|_\xi < \frac{\epsilon}{3}$$

Since $S_\xi(s_\Lambda)$ is continuous on Clos$(I_{m,n,\xi})$, there exists $\delta > 0$ such that if $|(x,y)-(z,w)| < \delta$, then $|S_\xi(s_\Lambda)(x,y) - S_\xi(s_\Lambda)(z,w)| < \frac{\epsilon}{3}$. Thus, if $(z,w) \in$Clos$(I_{m,n,\xi})$ with $|(x,y)-(z,w)| < \delta$,

$$|S_\xi(s)(x,y) - S_\xi(s)(z,w)| \le |S_\xi(s)(x,y) - S_\xi(s_\Lambda)(x,y)| +$$
$$|S_\xi(s_\Lambda)(x,y) - S_\xi(s_\Lambda)(z,w)| + |S_\xi(s_\Lambda)(z,w) - S_\xi(s)(z,w)| < \epsilon$$

Now let us prove that S_ξ is an $\mathbb{R}$-algebra homomorphism. Let $s,t \in \mathbb{R}\{X^*,Y\}_{m,n,\xi}$ and $c \in \mathbb{R}$. First notice that if s or t have finite support, $S_\xi(cs) = cS_\xi(s)$, $S_\xi(s+t) = S_\xi(s) + S_\xi(t)$ and $S_\xi(st) = S_\xi(s)S_\xi(t)$. The result then follows from Remark 1.2.8 and Lemma 1.2.19: Let $\epsilon > 0$. By Remark 1.2.8 there exists $\Lambda = \Lambda(\epsilon) \in \mathcal{P}_{\mathcal{F}}([0,\infty)^m \times \mathbb{N}^n)$ such that $\|cs - (cs)_\Lambda\|_\xi < \frac{\epsilon}{2}$ and $|c|\|s - s_\Lambda\|_\xi < \frac{\epsilon}{2}$. By Lemma 1.2.19,

$$\begin{aligned}\|S_\xi(cs) - cS_\xi(s)\|_\infty &= \|S_\xi(cs) - cS_\xi(s) \pm cS_\xi(s_\Lambda)\|_\infty \le \\ &\le \|S_\xi(cs) - S_\xi((cs)_\Lambda)\|_\infty + \|cS_\xi(s) - cS_\xi(s_\Lambda)\|_\infty \le \\ &\le \|cs - (cs)_\Lambda\|_\xi + |c|\|s - s_\Lambda\|_\xi =< \epsilon\end{aligned}$$

Since ϵ was arbitrary, $S_\xi(cs) = cS_\xi(s)$.

Analogously for the sum, if $\epsilon > 0$, let $\Lambda = \Lambda(\epsilon) \in \mathcal{P}_{\mathcal{F}}([0,\infty)^m \times \mathbb{N}^n)$ such that $\|(s+t)-(s+t)_\Lambda\|_\xi < \frac{\epsilon}{3}$, $\|s - s_\Lambda\|_\xi < \frac{\epsilon}{3}$ and $\|t - t_\Lambda\|_\xi < \frac{\epsilon}{3}$. By Lemma 1.2.19,

$$\begin{aligned}\|S_\xi(s+t) - (S_\xi(s) + S_\xi(t))\|_\infty &= \|S_\xi(s+t) - (S_\xi(s) + S_\xi(t)) \pm S_\xi((s+t)_\Lambda)\|_\infty \le \\ &\le \|S_\xi(s+t) - S_\xi((s+t)_\Lambda)\|_\infty + \|S_\xi(s_\Lambda) - S_\xi(s)\|_\infty + \|S_\xi(t_\Lambda) - S_\xi(t)\|_\infty \le \\ &\le \|(s+t) - ((s+t)_\Lambda)\|_\xi + \|s_\Lambda - s\|_\xi + \|t_\Lambda - t\|_\xi < \epsilon\end{aligned}$$

Since ϵ was arbitrary, $S_\xi(s+t) = S_\xi(s) + S_\xi(t)$.

Analogously for the product, if $\epsilon > 0$, let $\Lambda = \Lambda(\epsilon) \in \mathcal{P}_{\mathcal{F}}([0,\infty)^m \times \mathbb{N}^n)$ such that $\|(st)-(st)_\Lambda\|_\xi < \frac{\epsilon}{3}$, $\|s\|_\xi\|t - t_\Lambda\|_\xi < \frac{\epsilon}{3}$ and $\|t\|_\xi\|s - s_\Lambda\|_\xi < \frac{\epsilon}{3}$. By Lemma 1.2.19,

$$\begin{aligned}\|S_\xi(st) - S_\xi(s)S_\xi(t)\|_\infty &= \|S_\xi(st) - S_\xi(s)S_\xi(t) \pm S_\xi((st)_\Lambda)\|_\infty = \\ &= \|S_\xi(st) - S_\xi((st)_\Lambda) + S_\xi(s_\Lambda)S_\xi(t_\Lambda) - S_\xi(s)S_\xi(t)\|_\infty \le \\ &\le \|S_\xi(st) - S_\xi((st)_\Lambda)\|_\infty + \|S_\xi(s)S_\xi(t) - S_\xi(s_\Lambda)S_\xi(t_\Lambda) \pm S_\xi(s)S_\xi(t_\Lambda)\|_\infty = \\ &\le \|S_\xi(st) - S_\xi((st)_\Lambda)\|_\infty + \|S_\xi(s)(S_\xi(t) - S_\xi(t_\Lambda)) + S_\xi(t_\Lambda)(S_\xi(s_\Lambda) - S_\xi(s))\|_\infty \le \\ &\le \|S_\xi(st) - S_\xi((st)_\Lambda)\|_\infty + \|S_\xi(s)\|_\infty\|S_\xi(t) - S_\xi(t_\Lambda)\|_\infty + \|S_\xi(t_\Lambda)\|_\infty\|S_\xi(s_\Lambda) - S_\xi(s)\|_\infty \le \\ &\le \|st - (st)_\Lambda\|_\xi + \|s\|_\xi\|t - t_\Lambda\|_\xi + \|t_\Lambda\|_\xi\|s_\Lambda - s\|_\xi < \epsilon\end{aligned}$$

Since ϵ was arbitrary, $S_\xi(st) = S_\xi(s)S_\xi(t)$. □

We call S_ξ the **sum morphism**. Using the same notation as in Proposition 1.1.14:

Proposition 1.2.21. Let $X = (X_1, X_2, \ldots, X_m)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, $Z = (Z_1, Z_2, \ldots, Z_\mu)$ and $W = (W_1, W_2, \ldots, W_k)$ denote multi-variables.

i) Let $s \in \mathbb{R}\{X^*,Y\}$ and let $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}\{W\}^n$ with $t_1(0) = \ldots = t_n(0) = 0$. Then, for convenient strictly positif ϵ,

$$S_\epsilon(s(X, t(W))(x,w) = S_\epsilon(s)(x, S_\epsilon(t_1)(w), S_\epsilon(t_2)(w), \ldots, S_\epsilon(t_n)(w))$$

for any $(x,w) \in$Clos$(I_{m,k,\epsilon})$.

ii) Let $s \in \mathbb{R}\{X^*,Y\}$ and let $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}\{Z^*\}^n$ with $t_1(0) = \ldots = t_n(0) = 0$. Then, for convenient strictly positif ϵ,

$$S_\epsilon(s(X, t(Z))(x,w) = S_\epsilon(s)(x, S_\epsilon(t_1)(z), S_\epsilon(t_2)(z), \ldots, S_\epsilon(t_n)(z))$$

for any $(x,z) \in$Clos$(I_{m+\mu,\epsilon})$.

iii) Let $s = \sum s_\alpha X^\alpha \in \mathbb{R}\{X^*\}$ and $t = (t_1, t_2, \dots, t_m) \in \mathbb{R}\{Z^*\}^m$. If $t_i = Z^{\beta^i} u_i$, with $\beta^i \neq (0, ..., 0)$, $u_i \in \mathbb{R}\{Z^*\}$ and $u_i(0) > 0$ for all $i \in \{1, 2, \dots, m\}$ (that is, t_i is of monomial type), there exists $\epsilon > 0$ such that

$$S_\epsilon(s(t_1, t_2, \dots, t_m)) = S_\epsilon(S_\epsilon(t_1)(z), S_\epsilon(t_2)(z), \dots, S_\epsilon(t_m)(z))$$

for any $z \in$Clos$(I_{\mu,\epsilon})$.

iv) If $s \in \mathbb{R}\{X^*, Y\}_{m,n}$ and $j \in \{1, 2, \dots, n\}$ there exists $\epsilon > 0$ such that for each $(x, y) \in I_{m,n,\epsilon}$ the partial derivative $(\partial(S_\epsilon(s))/\partial y_j)(x, y)$ exists and

$$S_\epsilon(\partial s/\partial Y_j)(x, y) = (\partial(S_\epsilon(s))/\partial y_j)(x, y)$$

v) If $s \in \mathbb{R}\{X^*, Y\}_{m,n}$ and $i \in \{1, 2, \dots, m\}$ there exists $\epsilon > 0$ such that for each interior point (x, y) of $I_{m,n,\epsilon}$, the partial derivative $(\partial(S_\epsilon(s))/\partial x_i)(x, y)$ exists and

$$x_i(\partial(S_\epsilon(s))/\partial x_i)(x, y) = S_\epsilon(\partial s_i)(x, y)$$

vi) If $s \in \mathbb{R}\{X^*, Y\}_{m,n}$ and $\sigma \in G_{m,n}$ (see 1.1.16) then there exists $\epsilon > 0$ such that

$$S_\epsilon(\sigma s)(x, y) = S_\epsilon(s)(\sigma(x, y))$$

for all $(x, y) \in I_{m,n,\epsilon}$.

Proof. The result is immediate if all the series involved have finite support. For general series, we apply Remark 1.2.8 and Lemma 1.2.19 as in the proof of 1.2.20. □

Proposition 1.2.22. Given $\nu \in (0, \infty)^m$ and $\xi \in (0, \infty)^{m+n}$, the sum morphisms

$$S_\xi : \mathbb{R}\{X^*\}_{m,\nu} \longrightarrow \mathcal{C}^0(\mathrm{Clos}(I_{m,\nu}); \mathbb{R})$$

and

$$S_\xi : \mathbb{R}\{X^*, Y\}_{m,n,\xi} \longrightarrow \mathcal{C}^0(\mathrm{Clos}(I_{m,n,\xi}); \mathbb{R})$$

are injective.

Proof. We reproduce the proof given in [1] for the first morphism, being analogous the proof for the mixed case.

Let $s = \sum s_\alpha X^\alpha \in \mathbb{R}\{X^*\}_\xi$ and assume $s \neq 0$; we will show that $S_\xi(s)$ cannot vanish identically on any $I_{m,\tilde{\xi}}$ with $\tilde{\xi} < \xi$ small enough (which is more than what we need). By induction on m: if $m = 1$ then $X = X_1$ and, assuming s has order δ, we can write $s = X^\delta(s_\delta + \sum_{\alpha > \delta} s_\alpha X^{\alpha - \delta})$ with $s_\delta \neq 0$. Put $t := s_\delta + \sum_{\alpha > \delta} s_\alpha X^{\alpha - \delta}$. It follows from Lemma 1.2.9 that $S_\xi(t)(x) \neq 0$ for all $x \in (0, \tilde{\xi}]$, where $\tilde{\xi} > 0$ is small enough.

Let $m > 1$; assume our claim holds for $\mathbb{R}\{(X')^*\}_{\xi'}(X' = (X_1, X_2, \dots, X_{m-1}), \xi = (\xi', \xi_m) \in (0, \infty)^m)$. Write a nonzero $s \in \mathbb{R}\{X^*\}_\xi$ as $s = \sum_{\alpha_m \geq 0} s_{\alpha_m} X_m^{\alpha_m} \in (\mathbb{R}\{(X')^*\}_{\xi'})\{X_m^*\}_{\xi_m}$, and note that $\{\alpha_m : s_{\alpha_m} \neq 0\}$ is a well ordered subset of $[0, \infty)$. Hence $\|s\|_\xi = \sum \|s_{\alpha_m}\|_{\xi'} \xi_m^{\alpha_m}$ and $S_\xi(s)(x) = \sum S_{\xi'}(s_{\alpha_m})(x') x_m^{\alpha_m}$ for all $x = (x', x_m) \in I_{m,\xi}$. Fix some $\alpha_m \in [0, \infty)$ with $s_{\alpha_m} \neq 0$; by the inductive assumption there are $x' \in I_{m-1,\xi'}$ arbitrarily close to the origin such that $S_{\xi'}(s_{\alpha_m})(x') \neq 0$. For such x' we have shown above (case $m = 1$) that $S_\xi(s)(x', x_m) = \sum S_{\xi'}(s_{\alpha_m})(x_m)^{\alpha_m}$ is nonzero for all sufficiently small $x_m \in (0, \xi_m]$. □

Taylor expansion. Let $s = \sum_{(\alpha,J)} s_{\alpha,J} X^{\alpha} Y^{I} \in \mathbb{R}\big\{X^*, Y\big\}_{m,n,\xi}$. Let $(a,b) = (a_1, \ldots, a_m, b_1, \ldots, b_n) \in I_{m,n,\xi}$. Let $s((a,b) + (Z,W))$ denote the sum of the family (summable by 1.2.10)

$$\{s_{\alpha,J}(a_1 + Z_1)^{\alpha_1} \cdots (a_m + Z_m)^{\alpha_m} (b_1 + W_1)^{j_1} \cdots (b_n + W_n)^{j_n}\}_{(\alpha,J)\in[0,\infty)^m \times \mathbb{N}^n}$$

Notice that $s((a,b)+(Z,W))(0) = \sum_{(\alpha,J)} s_{\alpha,J} a^{\alpha} b^{I}$ which is a real number because $(a,b) \in I_{m,n,\xi}$. Recall that $s((a,b) + (Z,W)) \in \mathbb{R}\{Z^*, W\}$ and that for any $i \in \{1, \ldots, m\}$ such that $a_i \neq 0$, the variable Z_i is analytic on $s((a,b) + (Z,W))$. Put $m' := |\{i \in \{1, \ldots, m\} : a_i \neq 0\}|$. Then, if σ is a permutation of $\{1, \ldots, m\}$ such that $\sigma(\{i \in \{1, \ldots, m\} : a_i \neq 0\}) = \{1, \ldots, m'\}$, $T_{(a,b)}(s) := \sigma s((a,b) + (Z,W)) \in \mathbb{R}\{(Z_1, \ldots, Z_{m'})^*, Z_{m'+1}, \ldots, Z_m, W\}$.

On the other hand, suppose that $f \in \mathcal{C}^0(I_{m,n,\xi}; \mathbb{R})$ is in the image of the sum morphism S_ξ, that is, there exists $s \in \mathbb{R}\big\{X^*, Y\big\}_{m,n,\xi}$ such that $f = S_\xi(s)$. Let $(a,b) \in I_{m,n,\xi}$. Put $m' := |\{i \in \{1, \ldots, m\} : a_i \neq 0\}|$, and let σ be a permutation of $\{1, \ldots, m\}$ such that $\sigma(\{i \in \{1, \ldots, m\} : a_i \neq 0\}) = \{1, \ldots, m'\}$. We consider the map $(z,w) = \theta_{(a,b),\sigma}(x,y) := (\sigma(x), y) - (\sigma(a), b)$. If $\epsilon > 0$ there exists $\delta > 0$ such that if (x,y) close enough to (a,b) ($\|(x,y) - (a,b)\| < \delta$), $(z,w) = \theta_{(a,b),\sigma}(x,y) \in I_{m',(m-m')+n,\epsilon}$. Let us call $f_{(a,b)} := f \circ \theta_{(a,b),\sigma}$. The next proposition assures that $f_{(a,b)}$ is the sum of a convergent series, in fact of the series $T_{(a,b)}(s)$:

Proposition 1.2.23. Given $s \in \mathbb{R}\big\{X^*, Y\big\}_{m,n,\xi}$ and $(a,b) \in I_{m,n,\xi}$ there exists $0 < \epsilon < \xi$ such that $S_\epsilon(T_{(a,b)}(s)) = f_{(a,b)}$

$$\begin{array}{ccc} \mathbb{R}\big\{X^*, Y\big\}_{m,n,\delta} & \xrightarrow{\quad S_\delta \quad} & \mathcal{C}^0(I_{m,n,\xi} \to \mathbb{R}) \\ \Big\downarrow T_{(a,b)} & & \Big\downarrow \theta_{(a,b),\sigma} \\ \mathbb{R}\{Z^*, W\}_{m',(m-m')+n,\epsilon} & \xrightarrow{\quad S_\epsilon \quad} & \mathcal{C}^0(I_{m',(m-m')+n,\epsilon} \to \mathbb{R}) \end{array}$$

We obtain as a consequence that the sum of a convergent series is analytic on the interior of its domain of definition.

Proof. See 6.7 of [1]. □

Chapter 2

Generalized analytic manifolds.

In this chapter we introduce the concept of Generalized Analytic Manifold. We use the generalized power series analogously to the power series in the classical case of analytic manifolds. One of the main peculiarities is that Generalized Analytic Manifolds will be manifolds with boundary and corners. This is a geometrical consequence of the existence of non analytic variables in the generalized case: a function like x^λ for a non integer λ is only defined for positive values of the variable x.

For a better comprehension of the differences with the classical analytic case, we will use analytic manifolds with boundary and corners. We present in the first section a brief recall of these objects and their properties in the language of subsheaves on $\mathbb{R}$-algebras of continuous functions (called locally ringed spaces).

The Appendix is devoted to a brief exposition of the general concepts and basic properties in this theory. In a few words, we consider the category $\mathfrak{C}$ where an object of $\mathfrak{C}$ is a pair $X = (|X|, \mathfrak{C}_X)$ where $|X|$ is a topological space and $\mathfrak{C}_X$ is a sheaf of $\mathbb{R}$-algebras of continuous functions over $|X|$ such that, for each $p \in |X|$, the stalk $\mathfrak{C}_{X,p}$ is a local $\mathbb{R}$-algebra. The morphisms between two objects $X = (|X|, \mathfrak{C}_X)$ and $Y = (|Y|, \mathfrak{C}_Y)$ are pairs $(\varphi, \varphi^\sharp)$ where $\varphi : |X| \to |Y|$ is a continuous map and $\varphi^\sharp : \mathfrak{C}_Y \to \varphi_* \mathfrak{C}_X$ is the associated morphism of sheaves determined by φ by composition; that is, if $f \in \mathfrak{C}_Y(V)$ is a section over the open set V of $|Y|$, then $\varphi^\sharp(f) = f \circ \varphi \in \varphi_* \mathfrak{C}_X(V) = \mathfrak{C}_X(\varphi^{-1}(V))$. In what follows, we are interested in two specific subcategories, $\mathcal{O}$ and $\mathcal{G}$ of $\mathfrak{C}$. Their objects are objects in $\mathfrak{C}$ and the morphisms between two objects are exactly those morphisms when considered as objects in $\mathfrak{C}$ (briefly, they are *full subcategories of* $\mathfrak{C}$, see definition B.0.22 of the appendix).

In both cases $\mathcal{O}$ and $\mathcal{G}$, an object will be a locally ringed space on $\mathbb{R}$-algebras of continuous functions whose underlying topological space is a topological manifold with boundary of pure dimension, all of them locally homeomorphic to a local model $\mathbb{R}^k_{\geq 0}$ for some k. By a convenient choice of the second component of the object (that is the sheaf of continuous functions), objects in the subcategory $\mathcal{O}$ will be the (standard) real analytic manifolds with boundary and corners, when the chosen sheaf is such that it is locally isomorphic to the sheaf of analytic functions in the local model (those which are sums of standard real convergent power series). Objects of the subcategory $\mathcal{G}$, on the contrary, are defined with the property that the sheaf is locally isomorphic to the sheaf of generalized analytic functions on the local model (to be defined below by means of convergent generalized power series). They will be called generalized real analytic manifolds.

At the end of this chapter, we introduce the concept of *standardizable* generalized analytic manifold which will permit to consider some generalized analytic manifolds as a standard real analytic manifolds with an enrichment of the structure. Certain well known operations in standard analytic manifolds such as blowing-ups with smooth centers could be translated to standardizable generalized analytic manifolds (and this will be the purpose of the next chapter).

However, we show in 2.4.2 that there exist examples of generalized analytic manifolds which are not standardizable. Such examples are interpreted as exotic examples that could complicate the theory of generalized analytic manifolds in its full generality.

2.1 Analytic manifolds with boundary and corners.

For $k \in \mathbb{N}$, $\mathbb{R}^k_{\geq 0}$ denotes the topological subspace of $\mathbb{R}^k$ consisting on those points $p = (p_1, p_2, \ldots, p_k)$ in $\mathbb{R}^k$ such that $p_i \geq 0$ for $i = 1, 2, \ldots, k$.

Definition 2.1.1. The **local model of (real) analytic manifold with boundary and corners of dimension** k is the locally ringed space $\mathbb{A}^k_+ := (\mathbb{R}^k_{\geq 0}, \mathcal{O}_{\mathbb{A}^k_+})$ whose underlying topological space is $\mathbb{R}^k_{\geq 0}$ and the sheaf $\mathcal{O}_{\mathbb{A}^k_+}$ is defined by the assignment, for any open subset $V \subset \mathbb{R}^k_{\geq 0}$:

$$V \mapsto \mathcal{O}_{\mathbb{A}^k_+}(V)$$

where $\mathcal{O}_{\mathbb{A}^k_+}(V)$ consists on the set of real functions $f : V \to \mathbb{R}$ for which there exists an open neighborhood of V in $\mathbb{R}^k$, $W \supseteq V$, and $\widetilde{f} : W \to \mathbb{R}$ an analytic function on W whose restriction to V is equal to f. We will simply say that f is **analytic on** V for such a function.

Notice that $\mathcal{O}_{\mathbb{A}^k_+}$ together with the restriction of functions as restrictions morphisms, certainly define a sheaf on $\mathbb{R}^k_{\geq 0}$. Moreover, it is clear that for every open set $V \subset \mathbb{R}^k_{\geq 0}$, $\mathcal{O}_{\mathbb{A}^k_+}(V)$ is a sub-$\mathbb{R}$-algebra of the $\mathbb{R}$-algebra of real continuous functions on V and that the stalk $\mathcal{O}_{\mathbb{A}^k_+,p}$ is a local $\mathbb{R}$-algebra for any p. Thus $\mathbb{A}^k_+$ is a locally ringed space on local $\mathbb{R}$-algebras of continuous functions, that is, an element of the category $\mathfrak{C}$ (see the Appendix for the details).

Definition 2.1.2. A **(real) analytic manifold with boundary and corners**, or for short, a **standard analytic manifold** of dimension k is a locally ringed space on $\mathbb{R}$-algebras of continuous functions $A = (|A|, \mathcal{O}_A) \in \text{Objects}(\mathfrak{C})$, where $|A|$ is a Hausdorff topological space with a countable open basis, such that any point of $|A|$ has an open neighborhood isomorphic in $\mathfrak{C}$ to $\mathbb{A}^k_+|_V := (V, \mathcal{O}_{\mathbb{A}^k_+}|_V)$ for some V open subset of $\mathbb{R}^k_{\geq 0}$.

In other words, a locally ringed space $A = (|A|, \mathcal{O}_A) \in \text{Objets}(\mathfrak{C})$ is a k-dimensional analytic manifold with boundary and corners if for any $p \in |A|$ there exits an open neighborhood U of p, an open $V \subseteq \mathbb{R}^k_{\geq 0}$ and an isomorphism $(\varphi : A|_U \to \mathbb{A}^k_+|_V) \in \text{Morphisms}_{\mathfrak{C}}(A|_U, \mathbb{A}^k_+|_V)$. In particular, if U is an open subset of A, the sections of $\mathcal{O}_A$ over U are exactly those continuous functions $f : U \to \mathbb{R}$ such that for any $p \in U$ there exists W an open neighborhood of p and an homeomorphism $\varphi : U \cap W \to \varphi(U \cap W) \subseteq \mathbb{R}^k_{\geq 0}$ such that $f \circ \varphi^{-1}$ is analytic (that is, it admits an analytic extension to a neighborhood of $\varphi(p)$ in $\mathbb{R}^k$).

Remark 2.1.3. If $\alpha > 0$ is not integer, then the map $x \in \mathbb{R}_{\geq 0} \mapsto x^\alpha \in \mathbb{R}_{\geq 0}$ is not a section of $\mathbb{A}^1_+$, because it has not an analytical extension to an open neighborhood of 0 in $\mathbb{R}$.

Definition 2.1.4. If $A = (|A|, \mathcal{O}_A)$ is a standard analytic manifold, an **open submanifold of** A is the locally ringed space $A|_U = (U, \mathcal{O}_A|_U)$ where U is an open subset of $|A|$ (see the appendix for the notation). It is clear that an open submanifold is also a standard analytic manifold with boundary and corners.

Given two analytic manifolds A, B with boundary and corners a **morphism** between them is by definition a morphism $\varphi : A \to B$ of the category $\mathfrak{C}$ (we will usually call it an **analytic morphism**). In this way we define the category of analytic manifolds with boundary and corners, denoted by $\mathcal{O}$ by

$$\text{objects}(\mathcal{O}) := \{A \in \text{objects}(\mathfrak{C}) : A \text{ is an analytic manifold with border and corners}\}$$
$$\text{morphisms}(\mathcal{O}) := \{(\varphi : A \to B) \in \text{morphisms}(\mathfrak{C}) : A, B \in \text{ objects}(\mathcal{O})\}$$

Thus, by definition, $\mathcal{O}$ is a full subcategory of $\mathfrak{C}$. Recall that a morphism $(\varphi, \varphi^\sharp)$ between two analytic manifolds with border and corners $A = (|A|, \mathcal{O}_A)$ and $B = (|B|, \mathcal{O}_B)$ is determined by a continuous map between the topological spaces $\varphi : |A| \to |B|$ (but not any continuous map!) because the associated morphism of sheaves $\varphi^\sharp$ is given by composition with φ: if V is an open subset of $|B|$ and $f \in \mathcal{O}_B(V)$, $\varphi^\sharp(f) = f \circ \varphi \in \mathcal{O}_A(\varphi^{-1}(V))$ (see proposition B.0.21 in the appendix). Such a morphism is an isomorphism if and only if $\varphi : |A| \to |B|$ is an homeomorphism and for all $p \in |A|$ the $\mathbb{R}$-algebras homomorphism induced in the stalk

$$\begin{aligned} \varphi^\sharp_p : \mathcal{O}_{B,\varphi(p)} &\longrightarrow \mathcal{O}_{A,p} \\ \varphi^\sharp_p(\mathbf{f}_{\varphi(p)}) &= (\mathbf{f} \circ \varphi)_p \end{aligned}$$

is an isomorphism. We will denote frequently a morphism $(\varphi, \varphi^\sharp)$ simply by the underlying continuous map φ, the associated sheaf morphism $\varphi^\sharp$ being completely determined by φ.

Remark 2.1.5. Notice that if V_1 and V_2 are respectively open subsets of $\mathbb{R}^{k_1}_{\geq 0}$ and $\mathbb{R}^{k_2}_{\geq 0}$, and $\varphi \in$Morph$_{\mathcal{O}}(\mathbb{A}^{k_1}_+|_{V_1}, \mathbb{A}^{k_2}_+|_{V_2})$, there exists an open neighborhood W of V_1 in $\mathbb{R}^{k_1}$ and an analytic mapping $\tilde{\varphi} : W \to V_2$ (in the sense that each component of $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_2})$ is an analytic function on W) such that $\tilde{\varphi}|_{V_1} = \varphi$. This is a consequence of the definition of $\mathcal{O}$ and the fact that the projection functions $\pi_j : p = (p_1, p_2, \ldots, p_{k_2}) \in V_2 \mapsto p_j \in \mathbb{R}$ are sections of $\mathcal{O}_{\mathbb{A}^{k_2}_+}|_{V_2}$ for any $j = 1, 2, \ldots, k_2$. Hence $\pi_j \circ \varphi = \varphi_j \in \mathcal{O}_{\mathbb{A}^{k_1}_+}|_{V_1}$ which implies that for any $j = 1, 2, \ldots, k_2$ there exists an open neighborhood W_j of V_1 and an analytic function $\tilde{\varphi}_j : W_j \to \mathbb{R}$ such that $\tilde{\varphi}_j|_{V_1} = \varphi_j$. We take $W = \cap_{j=1}^{k_2} W_j$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{k_2})$. Notice that by the identity principle for analytic functions $\tilde{\varphi}$ is the unique analytic function satisfying $\tilde{\varphi}|_{V_1} = \varphi$.

In particular, if $p \in V_1$ we can define the differential of φ at p, $d_p\varphi := d_p\tilde{\varphi}$, a linear map from $\mathbb{R}^{k_1}$ to $\mathbb{R}^{k_2}$. As a consequence, if $(\varphi, \varphi^\sharp)$ is an isomorphism, first of all $\varphi : V_1 \to V_2$ is an homeomorphism so $k_1 = k_2 = k$, and the inverse of φ, $\varphi^{-1} : V_2 \to V_1$ induces a morphism too. So there exists $\tilde{\psi}$ analytic on U an open neighborhood of V_2 in $\mathbb{R}^k$ with $\tilde{\psi}|_{V_2} = \varphi^{-1}$. As $\tilde{\psi}|_{V_2} = \tilde{\varphi}^{-1}|_{V_2}$ by the identity principle for analytic functions $\tilde{\psi} = \tilde{\varphi}^{-1}$ so for any $p \in V_1$, if we put $q = \varphi(p)$, $d_q(\varphi^{-1}) = (d_p\varphi)^{-1}$, that is, $d_p\varphi$ is a linear isomorphism.

We have seen that if $\varphi \in$Morph$_{\mathcal{O}}(\mathbb{A}^{k_1}_+|_{V_1}, \mathbb{A}^{k_2}_+|_{V_2})$, the components of the continuous map $(\varphi_1, \ldots, \varphi_{k_2})$ are analytic functions. Conversely, if we have k_2 analytic functions on a neighborhood of V_1 in $\mathbb{R}^{k_2}$, $\varphi_1, \ldots, \varphi_{k_2}$ such that $\varphi_j(p) \geq 0$ for any $p \in V_1$, the continuous map $\varphi = (\varphi_1, \ldots, \varphi_{k_2}) : V_1 \to \varphi(V_1) \subseteq \mathbb{R}^{k_2}_{\geq 0}$ induces a morphism $(\varphi, \varphi^\sharp) : \mathbb{A}^{k_1}_+|_{V_1} \to \mathbb{A}^{k_2}_+|_{\varphi(V_1)}$.

Examples 2.1.6. Some examples of standard analytic manifolds are

i) Let $\mathcal{O}_{\mathbb{R}^k}$ denote the sheaf of analytic functions over $\mathbb{R}^k$. Then $(\mathbb{R}^k, \mathcal{O}_{\mathbb{R}^k})$ is a standard analytic manifold. To see that, remark that the homeomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^k_{>0} \subseteq \mathbb{R}^k_{\geq 0}$ defined by $\varphi(y_1, \ldots, y_k) = (e^{y_1}, \ldots, e^{y_k})$ induces an isomorphism (of locally ringed spaces) from $(\mathbb{R}^k, \mathcal{O}_{\mathbb{R}^k})$ to $\mathbb{A}^k_+|_{\mathbb{R}^k_{>0}} = (\mathbb{R}^k_{>0}, \mathcal{O}_{\mathbb{A}^k_+}|_{\mathbb{R}^k_{>0}})$. Then, in particular, for V open subset of $\mathbb{R}^k$, if we let $\mathcal{O}_V$ denote the sheaf of analytic functions on V, $(V, \mathcal{O}_V)$ is a standard analytic manifold.

ii) More generally, if $M = (|M|, \mathcal{O}_M)$ is a real analytic manifold (with the sheaf-theoretic interpretation; that is, that $\mathcal{O}_M$ is the sheaf of real analytic function on the underlying variety $|M|$), then M is a standard analytic manifold. This is an immediate consequence of example above.

iii) For any k, an example of k dimensional standard analytic manifold is the local model $\mathbb{A}^k_+ = (\mathbb{R}^k_{\geq 0}, \mathcal{O}_{\mathbb{A}^k_+})$.

iv) Consider $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ with the product topology. Let $\Phi : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{>0} \subset \mathbb{R}^{m+n}_{\geq 0}$ be the map defined by

$$(x, y) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \xrightarrow{\Phi} (x, \varphi(y)) = (x, e^{y_1}, \ldots, e^{y_k}) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{>0}$$

It is a homeomorphism. We can endow a structure of standard analytic manifold to $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ via this homeomorphism: just consider the sheaf $\mathcal{O}_{m,n}$ defined by assigning to each open set $V \subset \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ the $\mathbb{R}$-algebra of functions $f : V \to \mathbb{R}$ such that $f \circ \Phi^{-1}|_{\Phi(V)} \in \mathbb{A}^{m+n}_+(\Phi(V))$, that is there exists W an open neighborhood of $\Phi(V)$ in $\mathbb{R}^{m+n}_{\geq 0}$ and an analytic function g on W such that $g|_{\Phi(V)} = f \circ \Phi^{-1}|_{\Phi(V)}$. For reasons that will be clear below, we call the standard analytic manifold

$$\mathbb{A}^m_+ \times \mathbb{R}^n := (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \mathcal{O}_{m,n})$$

the (m, n) mixed local model. Notice that by the moment $\mathbb{A}^m_+ \times \mathbb{R}^n$ is just a notation. We show in proposition 2.1.16 below that the category $\mathcal{O}$ has product. In particular the product of the standard analytic manifolds $\mathbb{A}^m_+$ and $\mathbb{R}^n$ has sense and it agrees with the given here.

Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold and $p \in |A|$. A **local chart at** p will be a pair (U, w) where U is an open neighborhood of p in $|A|$ and

$$\begin{aligned} w : U &\longrightarrow V \\ w(q) =&(w_1(q), \ldots, w_k(q)) \end{aligned}$$

is a homeomorphism which induces an isomorphism of standard analytic manifolds $A|_U = (U, \mathcal{O}_A|_U)$ and $\mathbb{A}^k_+|_V = (V, \mathcal{O}_{\mathbb{A}^k_+}|_V)$. The components $w_1, \ldots, w_k$ will be called **local coordinates at** p. We say that a local chart is **centered at** p if it sends p to the origin.

If $p = (p_1, \ldots, p_k) \in \mathbb{R}^k_{\geq 0}$, we put $A(p) := \{i \in \{1, \ldots, k\} : p_i = 0\}$, m_p :=number of elements in $A(p)$, $n_p := k - m_p$ and for $\epsilon > 0$, $I_{A(p),\epsilon} := B_1 \times B_2 \times \cdots B_k \subseteq \mathbb{R}^k$, where the B_i is either the interval $[0, \epsilon) \subset \mathbb{R}$ if $i \in A(p)$ or the interval $(-\epsilon, \epsilon)$ if $i \notin A(p)$. Notice that for any $p \in \mathbb{R}^k_{\geq 0}$, the set $\{(p + I_{A(p),\epsilon}) \cap \mathbb{R}^k_{\geq 0} : \epsilon > 0\}$ is a fundamental system of neighborhoods of p in $\mathbb{R}^k_{\geq 0}$.

Proposition 2.1.7. The map

$$p \in \mathbb{R}^k_{\geq 0} \longmapsto m_p \in \mathbb{N}$$

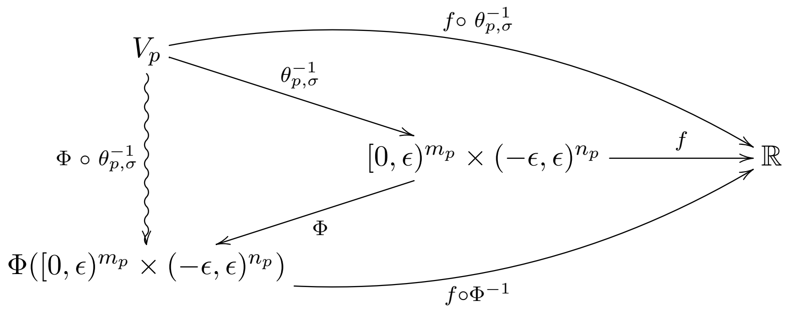
is upper semi-continuous.

Proof. Let $p = (p_1, \ldots, p_k) \in \mathbb{R}^k_{\geq 0}$ and $0 < \epsilon < \min_{i \notin A(p)}\{p_i\}$. Hence $p + I_{A(p),\epsilon}$ is an open neighborhood of p in $\mathbb{R}^k_{\geq 0}$ and if $q \in p + I_{A(p),\epsilon}$ with $q_i = 0$, then $p_i = 0$ (since $|q_i| \geq p_i - \epsilon > 0$ for any $i \notin A(p)$) which implies that $A(q) \subseteq A(p)$ and so $m_q \leq m_p$. □

Let $p \in \mathbb{R}^k_{\geq 0}$, and σ a permutation of $\{1, \ldots, k\}$ such that $\sigma(A(p)) = \{1, \ldots, m_p\}$. We denote by $\theta_{p,\sigma}$ the affine map

$$\theta_{p,\sigma}(q_1, \ldots, q_k) = p + (q_{\sigma(1)}, \ldots, q_{\sigma(k)})$$

Let $\epsilon > 0$ be such that for any $q \in (p+I_{A(p),\epsilon})$, $m_q \leq m_p$. Then $\theta_{p,\sigma}$ restricts to a homeomorphism from $[0, \epsilon)^{m_p} \times (-\epsilon, \epsilon)^{n_p}$ to $V_p := p+I_{A(p),\epsilon}$. We claim that its inverse $\theta^{-1}_{p,\sigma}$ induces an isomorphism from $\mathbb{A}_+^k|_{V_p}$ to $(\mathbb{A}_+^{m_p} \times \mathbb{R}^{n_p})|_{[0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p}}$: consider the diagram



Then it suffices to prove that $\Phi\circ\theta_{p,\sigma}^{-1}$ induces an isomorphism between $\mathbb{A}_+^k|_{V_p}$ and $\mathbb{A}_+^k|_{\Phi([0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p})}$. This follows from the fact that

$$\Phi\circ\theta_{p,\sigma}^{-1}(p+(x_1,\dots,x_{m_p},y_1,\dots y_{n_p})) = (x_{\sigma(1)},\dots,x_{\sigma(m_p)},e^{y_{\sigma(m_p+1)-m_p}},\dots,e^{y_{\sigma(k)-m_p}})$$

Hence we have proved

Proposition 2.1.8. For any point $p\in\mathbb{R}_{\geq 0}^k$ there exists $\epsilon>0$ small enough and an open neighborhood (depending on ϵ) isomorphic to $(\mathbb{A}_+^{m_p}\times\mathbb{R}^{n_p})|_{[0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p}}$.

Corollary 2.1.9. Let $\mathbb{A}_+^k|_V$ be an open submanifold of $\mathbb{A}_+^k$. Then any point $p\in V$ has an open neighborhood isomorphic to $\mathbb{A}_+^{m_p}\times\mathbb{R}^{n_p}$.

Proof. By proposition above it suffices to notice that the map

$$\varphi : [0,\epsilon)^m\times(-\epsilon,\epsilon)^n \to \mathbb{R}_{\geq 0}^m\times\mathbb{R}^n$$
$$\varphi(x_1,\dots,x_m,y_1,\dots,y_n) = (\frac{x_1}{\epsilon-x_1},\dots,\frac{x_m}{\epsilon-x_m},\frac{y_1}{\epsilon-(y_1)^2},\dots,\frac{y_n}{\epsilon-(y_n)^2})$$

induces an isomorphism between $(\mathbb{A}_+^{m_p}\times\mathbb{R}^{n_p})|_{[0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p}}$ and $(\mathbb{A}_+^{m_p}\times\mathbb{R}^{n_p})$. □

2.1.1 Stratification by the number of boundary components.

Let, for the rest of the section, $A=(|A|,\mathcal{O}_A)$ denote a k dimensional standard analytic manifold. A direct consequence of the definition is that the underlying space $|A|$ is a topological manifold of dimension k with boundary, because each point in $|A|$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}_{\geq 0}^k$, a topological manifold of dimension k with boundary (see the annex for details). Another consequence is that if $int(|A|)$ denotes the interior of this manifold then the open submanifold

$$A|_{int(A)} = (int(|A|),\mathcal{O}_A|_{int(|A|)})$$

is a real analytic manifold because any section of $\mathcal{O}_A|_{int(|A|)}$ is an analytic function. This implies that at points in the interior of the manifold, the dimension is the only local invariant by isomorphisms. As we show below, this is not the case for points at the boundary $\partial|A|$: looking at the standard local model $\mathbb{A}_+^k$, although any two points in the boundary have topologically equivalent neighborhoods, they would not have necessarily isomorphic neighborhoods in the category $\mathcal{O}$. In fact, the number of coordinate hyperplanes ("boundary components"), passing through the point will be invariant for local isomorphisms.

Let $p\in|A|$ and (U,y) be a local chart at p and define $m_p := |\{i\in\{1,\dots,k\} : y_i(p)=0\}|$. We are going to prove that m_p does not depend on the local chart chosen but only on the point p. We need the following proposition

Proposition 2.1.10. Let V_1 and V_2 two open subsets of $\mathbb{R}^k_{\geq 0}$ and suppose that the standard analytic manifolds $\mathbb{A}^k_+|_{V_1} = (V_1, \mathcal{O}_{\mathbb{A}^k_+}|_{V_1})$ and $\mathbb{A}^k_+|_{V_2} = (V_2, \mathcal{O}_{\mathbb{A}^k_+}|_{V_2})$ are isomorphic via φ. Then for each $p \in V_1$, $m_p = m_{\varphi(p)}$.

Proof. Suppose $\mathbb{A}^k_+|_{V_1} = (V_1, \mathcal{O}_{\mathbb{A}^k_+}|_{V_1})$ and $\mathbb{A}^k_+|_{V_2} = (V_2, \mathcal{O}_{\mathbb{A}^k_+}|_{V_2})$ isomorphic via φ. In particular, by Remark 2.1.5, the differential $d_p\varphi$ is a linear isomorphism for any $p \in V_1$.

Claim.- If $i \notin A(p)$ and $e_i = (0, ..., 0, 1^{(i^{th})}, 0, ..., 0)$ is the i^{th} vector of the canonical basis of $\mathbb{R}^k$ then, for any $j \in A(\varphi(p))$, the j^{th}-coordinate of $d_p\varphi(e_i)$ is equal to zero.

Once the claim proved, we obtain that $m_{\varphi(p)} \leq m_p$ because in M_p, the jacobian matrix of $d_p\varphi$, there are $k - m_p$ columns ($d_p\varphi(e_l)$ for any $l \notin A(p)$) $c_i = (c_{1,i}, \dots c_{k,i})$, $1 \leq i \leq k - m_p$, with $c_{j,i} = 0$ for at least $m_{\varphi(p)}$ positions j. Since M_p is invertible, the columns $c_1, \dots, c_{k-m_p}$ as vectors in $\mathbb{R}^k$ are linearly independent but all of them lie in the $k - m_{\varphi(p)}$ dimensional subspace $\bigcap_{j \in A(\varphi(p))} \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_j = 0\}$, so necessary $k - m_p \leq k - m_{\varphi(p)}$, i.e. $m_{\varphi(p)} \leq m_p$.

Proof of the claim.- Denote by $\tilde{\varphi}$ the extension of φ to an analytic mapping from a neighborhood of p in $\mathbb{R}^k$. Write Taylor's formula of order one:

$$\tilde{\varphi}(p + te_i) = \tilde{\varphi}(p) + t d_p\varphi(e_i) + o(t). \tag{2.1}$$

Since $i \notin A(p)$, we have that $p + te_i \in V_1$ for every $t \in \mathbb{R}$ sufficiently small and thus $\tilde{\varphi}(p + te_i) = \varphi(p + te_i) \in V_2$. Suppose that the j^{th}-coordinate of $d_p\varphi(e_i)$ is equal to $\lambda_j \neq 0$, for instance $\lambda_j > 0$. Then, for every $t < 0$ with $|t|$ sufficiently small, taking into account that the j^{th}-coordinate of $\varphi(p)$ is equal to zero, the formula (2.1) above gives that the j^{th}-coordinate of $\varphi(p + te_i)$ has the sign of $t\lambda_j$, i.e., negative which is impossible since $V_2 \subset \mathbb{R}^k_{\geq 0}$.

Since $(\varphi, \varphi^\sharp)$ is an isomorphism, we can prove symmetrically that if $i \notin A(\varphi(p))$ and $e_i = (0, ..., 0, 1^{(i^{th})}, 0, ..., 0)$ is the i^{th} vector of the canonical basis of $\mathbb{R}^k$ then, for any $j \in A(p)$, the j^{th}-coordinate of $d_{\varphi(p)}\varphi^{-1}(e_i)$ is equal to zero and hence $m_p \leq m_{\varphi(p)}$. □

Remark 2.1.11. We obtain as a corollary of proposition 2.1.10 and corollary 2.1.9 that given a standard analytic manifold $A = (|A|, \mathcal{O}_A)$, any point $p \in |A|$ has an open neighborhood U in $|A|$ such that the open submanifold $A|_U$ is isomorphic to $\mathbb{A}^{m_p}_+ \times \mathbb{R}^{n_p}$. By definition, each $p \in |A|$ is in the domain of a chart (U, ϕ) where the range of ϕ is an open subset of $\mathbb{R}^k_{\geq 0}$. By Proposition 2.1.10, we can choose such a chart *centered* at p, i.e., such that $\phi(p) = (0, ..., 0)$ if and only if $m_p = k$. If we want to have always a chart centered at any given point, we can think that there is not a single local model for standard analytic manifold of a given dimension k, namely $\mathbb{A}^k_+$, but several ones, $\mathbb{A}^m_+ \times \mathbb{R}^n$ with $m + n = k$.

Definition 2.1.12. Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold. The function $m : |A| \to \mathbb{N}$ defined by

$$m(p) := m_p = |\{i \in \{1, \dots, k\} : w_i(p) = 0\}|,$$

where $w = (w_1, \dots, w_k)$ is a local chart on p is well defined. Moreover, m is an upper semi-continuous function. Given a point p, we will say also that m_p is the **number of boundary components** of the point p.

For $j \in \{0, 1, \dots, k\}$ let

$$D(j) := \{p \in |A| : m_p = j\}$$

Let $j_0 := \max\{j \in \{0, 1, \dots, k\} : D(j) \neq \emptyset\}$. We call $D(j_0)$ the **lime** of A.

Let $\{D(j)_i\}_{i \in I_j}$ be the connected components of $D(j)$. We consider the partition of the underlying space $|A|$ by these sets

$$|A| = \bigcup_{j=0}^{k} (\cup_{i_j \in I_j} D(j)_{i_j})$$

Proposition 2.1.13. For each $j \in \{0, ..., k\}$ and each $i \in I_j$, $D(j)_i$ is a locally closed set and the restricted sheaf $(D(j)_i, \mathcal{O}_A|_{D(j)_i})$ gives rise to a real analytic manifold of dimension $k - j$. The family $\mathcal{D}_M = \{D(j)_{i_j}\}_{\substack{i_j \in I_j \\ j\in\{0,...,k\}}}$ is a locally finite stratification of $|A|$; that is, for every stratum $D(j)_i$, its boundary

$$\partial D(j)_i = \overline{D(j)_i} \setminus D(j)_i$$

is a locally finite union of strata of dimension not greater than the dimension of $D(j)_i$. Moreover, the boundary $\partial|A|$ of $|A|$ is the union of all strata of dimension strictly smaller than k.

Proof. All stated properties are true if they are true locally at each point of the manifold. Thus the proof follows from the definition of A as being locally isomorphic to open submanifolds of the local standard model $\mathbb{A}^k_+$ after checking that proposition is true for the (finite) stratification $\mathcal{D}_{\mathbb{A}^k_+}$ of $\mathbb{R}^k_{\geq 0}$. □

Definition 2.1.14. Let $|X|$ be a k dimensional Hausdorff topological space with a countable open basis. We say that a family $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ is an **$\mathcal{O}$-atlas** of $|X|$ if

i) For any $\lambda \in \Lambda$, U_λ is an open subset of $|X|$ and $\varphi_\lambda : U_\lambda \to V_\lambda := \varphi_\lambda(U_\lambda) \subseteq \mathbb{R}^k_{\geq 0}$ is an homeomorphism.

ii) $X = \bigcup_{\lambda\in\Lambda} U_\lambda$

iii) For any $\lambda, \mu \in \Lambda$, $\varphi_\lambda \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu \cap U_\lambda) \to \varphi_\lambda(U_\mu \cap U_\lambda)$ is an isomorphism in $\mathcal{O}$.

Let U an open subset of $|X|$. We denote by $\mathcal{O}_X(U)$ the set of continuous functions $f : U \to \mathbb{R}$ such that for any $p \in U$, there exists an open $V \subseteq U$ such that $f \circ \varphi_\lambda^{-1} : \varphi_\lambda(V \cap U_\lambda) \to \mathbb{R}$ has an analytic extension to an open neighborhood of $\varphi(p)$ in $\mathbb{R}^k$ for any $\lambda \in \Lambda$ such that $p \in U_\lambda$.

Proposition 2.1.15. The pair $X = (|X|, \mathcal{O}_X)$ is a standard analytic manifold.

Proof. By definition, $X \in$Obj$(\mathfrak{C})$. Let $p \in |X|$. Let $\lambda \in \Lambda$ such that $p \in U_\lambda$. Then, φ_λ^{-1} induces a morphism from $\mathbb{A}^k_+|_{\varphi_\lambda(U_\lambda)}$ to $X|_{U_\lambda}$ by definition of X. Moreover, $\varphi_\lambda : U_\lambda \to \varphi_\lambda(U_\lambda)$ induces a morphism from $X|_{U_\lambda}$ to $\mathbb{A}^k_+|_{\varphi_\lambda(U_\lambda)}$: let V be an open subset of $\varphi_\lambda(U_\lambda)$ and $g : V \to \mathbb{R}$ a section of $\mathcal{O}_{\mathbb{A}^k_+}$ over V. Then, $g \circ \varphi_\lambda \in \mathcal{O}_X(\varphi_\lambda^{-1}(V))$ because if $q \in \varphi_\lambda^{-1}(V)$, and $\mu \in \Lambda$ is such that $q \in U_\mu$, $f \circ \varphi_\lambda \circ \varphi_\mu^{-1} \in \mathcal{O}_{\mathbb{A}^k_+}(\varphi_\mu \circ \varphi_\lambda^{-1}(V))$ since by condition *iii)* of 2.1.14 $g \in \mathcal{O}_{\mathbb{A}^k_+}(V) \mapsto g \circ \varphi_\lambda \circ \varphi_\mu^{-1} \in \mathcal{O}_{\mathbb{A}^k_+}(\varphi_\mu \circ \varphi_\lambda^{-1}(V))$ is an isomorphism. □

Proposition 2.1.16. $\mathcal{O}$ is a category with product.

Proof. We show first that given $V_1 \subset \mathbb{R}^{k_1}_{\geq 0}$ and $V_2 \subset \mathbb{R}^{k_2}_{\geq 0}$ open sets, there exists a product of the open submanifolds of the local model $\mathbb{A}^{k_1}_+|_{V_1}$ and $\mathbb{A}^{k_2}_+|_{V_2}$. Let $V = V_1 \times V_2 \subset \mathbb{R}^k_{\geq 0}$ the topological product of V_1 and V_2, where $k = k_1 + k_2$. Considering V as open submanifold of $\mathbb{A}^k_+$ we claim that V, together with the usual projections $p_i : V \to V_i$, $i = 1, 2$, is a product of $\mathbb{A}^{k_1}_+|_{V_1}$ and $\mathbb{A}^{k_2}_+|_{V_2}$.

Let A be a standard manifold and $\alpha_i : A \to V_i$ morphisms of $\mathcal{O}$. Since V is the topological product of V_1 and V_2, there exists a unique continuous map $\Phi : A \to V$ such that $p_i \circ \Phi = \alpha_i$. Let us see that Φ is a morphism in the category $\mathcal{O}$. Let U be an open subset of V. It suffices to see that for every $f \in \mathcal{O}_{\mathbb{A}^k_+}(U)$, $f \circ \Phi \in \mathcal{O}_A(\Phi^{-1}(U))$. Let $a \in U$ and φ a local homeomorphism at a from W, an open neighborhood of a, to $\varphi(W) \subseteq \mathbb{R}^m_{\geq 0}$ (m =dimension of A) inducing an isomorphism between the open submanifolds $A|_W$ and $\mathbb{A}^m_+|_{\varphi(W)}$. Thus, for any $q \in \varphi(W)$

$$f \circ \Phi \circ \varphi^{-1}(q) = f(p_1 \circ \Phi \circ \varphi^{-1}(q), p_2 \circ \Phi \circ \varphi^{-1}(q)) = f(\alpha_1 \circ \varphi^{-1}(q), \alpha_2 \circ \varphi^{-1}(q))$$

Since $\alpha_i : A \to V_i$, $i = 1, 2$, are morphisms and $p_i(U)$ is an open subset of V_i, $\alpha_i \circ \varphi^{-1} \in \mathcal{O}_{\mathbb{A}^m_+}(\varphi(\alpha_i^{-1}(p_i(U))))$. Thus $\alpha_i \circ \varphi^{-1}$ have an analytic expansion on a neighborhood of $\varphi(a)$. As $f \in \mathcal{O}_{\mathbb{A}^k_+}(U)$, f has an analytic expansion on a neighborhood of $(\alpha_1(a), \alpha_2(a)) \in U$, which implies that $f(\alpha_1 \circ \varphi^{-1}, \alpha_2 \circ \varphi^{-1})$ has an analytic expansion on a neighborhood of $\varphi(a)$ as was to be proved.

Finally, just notice that if W_i is an open subset of V_i and $g_i \in \mathcal{O}_{\mathbb{A}^{k_i}_+}(W_i)$, $g_i \circ p_i \in \mathcal{O}_{\mathbb{A}^k_+}(p_i^{-1}(W_i))$ because for any $(v_{i,1}, v_{i,2}) \in p_i^{-1}(W_i)$, $g_i \circ p_i(v_{i,1}, v_{i,2}) = g_i(v_{i,i})$ which has an analytic expansion on a neighborhood of $(v_{i,1}, v_{i,2})$ since g_i has an analytic expansion on a neighborhood of $v_{i,i}$.

Now, let A_1 and A_2 be two standard analytic manifolds of dimension k_1 and k_2 respectively. We start by constructing a triplet $P = (|P|, p_1 : P \to A_1, p_2 : P \to A_2)$ as a candidate to be the product of A_1 and A_2.

It is logical to pick as underlying topological space for P the cartesian product $|P| = |A_1| \times |A_2|$ with the product topology and as morphisms $p_1 : P \to A_1, p_2 : P \to A_2$ the morphisms induced by the projections maps

$$p_1 = pr_1 : |P| = |A_1| \times |A_2| \to |A_1| \qquad p_2 = pr_2 : |P| = |A_1| \times |A_2| \to |A_2|$$

We construct now the sheaf $\mathcal{O}_P$ that will determine the structure of standard manifold for P. In order to define the sheaf $\mathcal{O}_P$ as a subsheaf of the sheaf of continuous functions, it is enough to associate to any element of a basis of open sets of the topology of $|P|$ a $\mathbb{R}$-subalgebra of continuous functions with. After that, we need to show that with this structure, $|P|$ is locally isomorphic to $\mathbb{A}^{k_1+k_2}_+$.

As a basis of open sets of the topological product $|A_1| \times |A_2|$, we can consider the set

$$\mathcal{B} = \{U_1 \times U_2 \subseteq |A_1| \times |A_2| : U_i \subset |A_i| \text{ is the domain of a coordinate chart }, i = 1, 2\}$$

Let $U_1 \times U_2 \in \mathcal{B}$. Then $A_i|_{U_i}$ is isomorphic to $\mathbb{A}^{k_i}_+|_{V_i}$ via φ_i for $i = 1, 2$. Let Φ be the map

$$\Phi = (\varphi_1, \varphi_2) : U_1 \times U_2 \to V_1 \times V_2 \subseteq \mathbb{R}^{k_1}_{\geq 0} \times \mathbb{R}^{k_2}_{\geq 0} = \mathbb{R}^{k_1+k_2}_{\geq 0}$$

Put $k = k_1 + k_2$. Then Φ is an homeomorphism and $V_1 \times V_2$ is an open subset of $\mathbb{R}^k_{\geq 0}$. Let us define

$$\Gamma_\Phi(U_1 \times U_2, \mathcal{O}_P) = \{f : U_1 \times U_2 \to \mathbb{R} / f \circ \Phi^{-1} \in \Gamma(V_1 \times V_2, \mathcal{O}_{\mathbb{A}^k_+})\}$$

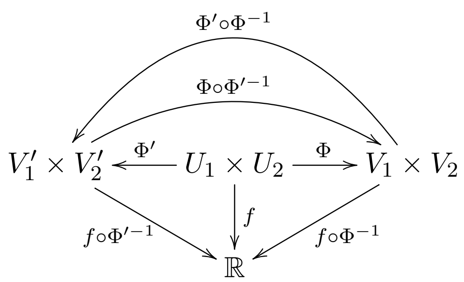
First of all let us prove that this definition does not depend on the morphisms φ_1, φ_2 such that (U_1, φ_1), (U_2, φ_2) are local charts which will endow the topological product with a well defined structure of standard analytic manifold. Let

$$\varphi'_i : U_i \to V'_i$$

be isomorphisms between $A_i|_{U_i}$ and $\mathbb{A}^{k_i}_+|_{V_i}$ and we define

$$\Phi' = (\varphi'_1, \varphi'_2) : U_1 \times U_2 \to V'_1 \times V'_2$$

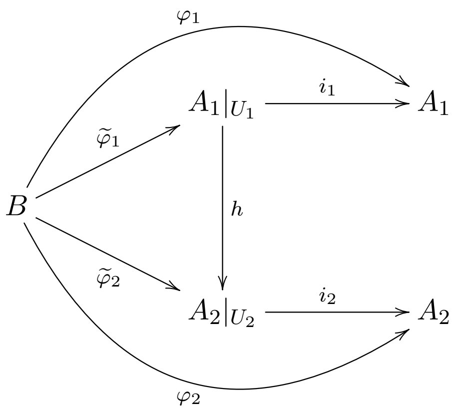
then $\Gamma_\Phi = \Gamma_{\Phi'}$. We can illustrate the situation with the diagram



The result is clear once we notice that $\Phi \circ \Phi'^{-1}$ and $\Phi' \circ \Phi^{-1}$ are morphisms of standard analytic manifolds (thus both isomorphisms), which can be seen using the definition of product and that there exists the product of open submanifolds of the local model. So $A_1 \times A_2 = (|A_1| \times |A_2|, \mathcal{O}_{A_1 \times A_2}) \in$Obj$(\mathcal{O})$. Remark that the natural projections $p_i : |A_1| \times |A_2| \to |A_i|$ are morphisms from $A_1 \times A_2$ to A_i. To finish, we have to prove that $(A_1 \times A_2, p_1, p_2)$ is a solution of the universal problem. But this is easy: if B is a standard analytic manifold and $\beta_i : B \to A_i$ are morphisms for $i = 1, 2$, the map $\Phi : B \to A_1 \times A_2$ defined by $\Phi = (\beta_1, \beta_2)$ is continuous and induce a morphism of standard analytic manifolds since this property is a local one and locally $A_1 \times A_2$ has the structure of product, by definition. ◻

Proposition 2.1.17. $\mathcal{O}$ is a category with gluing.

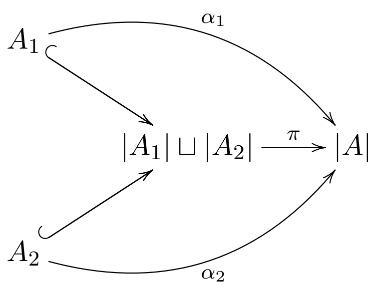
Proof. Let B, A_1, A_2 be standard analytic manifolds and $\varphi_i : B \to A_i$ be open immersions (see appendix B) decomposing in



Notice that $h = \widetilde{\varphi_2} \circ \widetilde{\varphi_1}^{-1} : A_1|_{U_1} \to A_2|_{U_2}$ is an isomorphism. Let $|A|$ be the topological space obtained by the quotient of the topological disjoint union $|A_1| \sqcup |A_2|$ by the equivalence relation

$$a_1 \sim a_2 \;\text{ if }\; a_1 = a_2 \;\text{ or }\; a_1 \in U_1,\;\; a_2 \in U_2 \;\text{ and }\; a_2 = h(a_1)$$

Denote by $\pi : |A_1| \sqcup |A_2| \to |A|$ the quotient map. For $i = 1, 2$ define $\alpha_i : |A_i| \to |A|$ as the composition of the inclusion $|A_i| \subset |A_1| \sqcup |A_2|$ with the quotient map.



Then we have that α_i is continuous, that its image $W_i = \alpha_i(|A_i|)$ is an open set of $|A|$, that $\alpha_i : |A_i| \to W_i$ is a homeomorphism and that $|A| = W_1 \cup W_2$. Now we want to define a sheaf of continuous functions (on local algebras) $\mathcal{O}_A$ on $|A|$ such that $A = (|A|, \mathcal{O}_A)$ is a standard analytic manifold and α_i is a morphism of standard analytic manifolds. Using a general construction of gluing ringed spaces (see Appendix for details), it suffices to define such a sheaf $\mathcal{O}_{W_i}$ on W_i for $i = 1, 2$ such that, for any open set $V \subset W_1 \cap W_2$, we have $\mathcal{O}_{W_1}(V) = \mathcal{O}_{W_2}(V)$: explicitly, $\mathcal{O}_A$ will be given by

$$\mathcal{O}_A(U) = \{f : U \to \mathbb{R} : f \circ \alpha_i \in \mathcal{O}_{A_i}(\alpha_i^{-1}(U)),\;\; i = 1, 2\}$$

Define

if $V \subseteq W_i$ is open, $\mathcal{O}_{W_i}(V) = \{f : V \to \mathbb{R} : f \circ \alpha_i \in \mathcal{O}_{A_i}(\alpha_i^{-1}(V))\}$.

With this definition, A_i is isomorphic (in $\mathfrak{C}$) to W_i via α_i. Now, let $V \subset W_1 \cap W_2$ be an open set. The homeomorphism $\alpha_1^{-1} \circ \alpha_2$ induces an isomorphism (of standard analytic manifolds) between the open submanifold $\alpha_2^{-1}(V)$ of A_2 and $\alpha_1^{-1}(V)$ of A_1. Thus, if $f : V \to \mathbb{R}$ is continuous, we have

$$f \circ \alpha_1 \in \mathcal{O}_{A_1}(\alpha_1^{-1}(V)) \Leftrightarrow f \circ \alpha_2 \in \mathcal{O}_{A_1}(\alpha_2^{-1}(V))$$

which shows $\mathcal{O}_{W_1}(V) = \mathcal{O}_{W_2}(V)$, as required. We claim that $A = (|A|, \mathcal{O}_A)$ is the gluing of A_1, A_2 with respect to the open immersions φ_1, φ_2. To see this, let (β_1, β_2, T) be a triplet where $T = (|T|, \mathcal{O}_T)$ is a standard analytic manifold and $\beta_i : A_i \to T$ are open immersions such that $\beta_1 \circ \varphi_1 = \beta_2 \circ \varphi_2$. We have to show that there exists an unique morphism $f : A \to T$ such that $\beta_i = f \circ \alpha_i$ for $i = 1, 2$. Uniqueness of f comes from the fact that $|A|$ is the solution of the same universal problem in the category of topological spaces: the map $f : |A| \to |T|$ must be defined by

$$f(p) = \alpha_1^{-1}(p) \text{ for } p \in W_1 \text{ and } f(p) = \alpha_2^{-1}(p) \text{ for } p \in W_2$$

We just have to prove that f is a morphism of standard analytic manifolds. This is a property that we can check locally. But f is locally defined either by $\beta_1 \circ \alpha_1^{-1}$ on W_1 or by $\beta_2 \circ \alpha_2^{-1}$ on W_2, both morphisms in the category of standard analytic manifolds. □

2.1.2 Local expression of morphisms.

Let $A = (|A|, \mathcal{O}_A)$ and $B = (|B|, \mathcal{O}_B)$ be standard analytic manifolds and $\varphi : |A| \to |B|$ a continuous map which induces a morphism from A to B. Let $p \in |A|$ and $\varphi(p) \in |B|$. We want to investigate how is the local expression of the morphism φ when we take local coordinates centered at p and at $\varphi(p)$.

More precisely, consider a local chart at p, i.e. an isomorphism $\phi : A|_{U_p} \to \mathbb{A}_+^{m_p} \times \mathbb{R}^{n_p}|_{U_0}$ where U_p is a neighborhood of p in $|A|$ and U_0 is a neighborhood of 0 in $\mathbb{R}_{\geq 0}^{m_p} \times \mathbb{R}^{n_p}$, and consider, correspondingly, a local chart $\psi : B|_{V_{\varphi(p)}} \to V_0$ at $\varphi(p)$ (one can chose U_0 and V_0 to be the whole space, according to corollary 2.1.9).

$$\begin{array}{ccc} A|_{U_p} & \xrightarrow{(\varphi,\varphi^\sharp)} & B|_{V_{\varphi(p)}} \\ \downarrow{\scriptstyle (\phi,\phi^\sharp)} & & \downarrow{\scriptstyle (\psi,\psi^\sharp)} \\ \mathbb{A}_+^{m_p} \times \mathbb{R}^{n_p} & & \mathbb{A}_+^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}} \end{array}$$

Then, the map $h := \psi \circ \varphi \circ \phi^{-1} : \mathbb{R}_{\geq 0}^{m_p} \times \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}}$ has an analytic extension to a neighborhood of $\underline{0} \in \mathbb{R}^{m_p+n_p}$.

Reciprocally, any such continuous map $h : U_0 \to V_0$ that induces a morphism (resp. isomorphism) $h : \mathbb{A}_+^{m_p} \times \mathbb{R}^{n_p}|_{U_0} \to \mathbb{A}_+^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}}|_{V_0}$ gives rise, by reversing the charts ϕ and ψ to a morphism (resp. isomorphism) from an open submanifold of A containing p to an open submanifold of B containing $\varphi(p)$.

In the following proposition, we just describe the conditions for a continuous map h to give rise to a morphism or an isomorphism between the corresponding open submanifolds of the local models $\mathbb{A}_+^m \times \mathbb{R}^n = (\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n, \mathcal{O}_{m,n})$.

Proposition 2.1.18. Let m, n, m', n' be natural numbers, $k = m+n$ and $k' = m'+n'$. Let U, V be open neighborhoods of the origin in $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ and in $\mathbb{R}_{\geq 0}^{m'} \times \mathbb{R}^{n'}$ respectively. Let $h : U \to V$ be a continuous map with $h(\underline{0}) = \underline{0}$, and $h = (h_1, ..., h_{k'})$ be the components of h as a map ranging in $\mathbb{R}^{k'}$. Denote by $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and $(z, w) = (z_1, \ldots, z_{m'}, w_1, \ldots, w_{n'})$ the coordinates in $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ and $\mathbb{R}_{\geq 0}^{m'} \times \mathbb{R}^{n'}$. Then *i)* h induces a morphism $(h, h^\sharp) : \mathbb{A}_+^m \times \mathbb{R}^n|_{U_0} \to \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}|_{V_0}$ where U_0 and V_0 are open neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively, if and only if each h_j has an analytic extension on a neighborhood of the origin in $\mathbb{R}^k$.

ii) Assume that $k = k'$ and that h induces a morphism $(h, h^\sharp) : \mathbb{A}_+^m \times \mathbb{R}^n|_U \to \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}|_V$. Then $(h, h^\sharp)$ is an isomorphism in the category $\mathcal{O}$ if and only if $m = m'$, $n = n'$, h is an homeomorphism and for any $j = 1, 2, \ldots, m$,

$$z_j = h_j(x, y) = x_{i(j)} g_j(x, y)$$

where g_j an analytic function at $\underline{0}$ such that $g_j(x, y) \neq 0$ for any $(x, y) \in W$ for W a desirable neighborhood of $\underline{0}$ in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $j \to i(j)$ a permutation of $\{1, \ldots, m\}$.

Proof. Necessity of part *i)* follows from the fact that the projections functions $pr_j : (p_1, \ldots, p_k) \in \mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'} \to p_j \in \mathbb{R}$ are sections of $\mathcal{O}_{m',n'}$ over any open neighborhood of the origin in $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$, so if h induces a morphism, each $pr_j \circ h = h_j$ is a section of $\mathcal{O}_{m,n}$ over an open neighborhood of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ which implies that they admit an analytic extension to a neighborhood of the origin in $\mathbb{R}^k$. Conversely, suppose that each h_j admits an analytic extension to $U_0 \subseteq U$ an open neighborhood of the origin in $\mathbb{R}^k$. In particular, by the open mapping theorem for analytic functions, h is an open map, so $V_0 := h(U_0)$ is an open subset of V. Let W be an open subset of V_0 and f a section of $\mathcal{O}_{m',n'}$ over W. Then $f \circ h \in \mathcal{O}_{m,n}(h^{-1}(W))$ because it admits an analytic extension for any $p \in h^{-1}(W)$.

For *ii)*, suppose that the continuous map $h : U_0 \to V_0$ induces an isomorphism of standard manifolds

$$(h, h^\sharp) : \mathbb{A}_+^m \times \mathbb{R}^n|_{U_0} \to \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}|_{V_0}$$

Since $h(\underline{0}) = \underline{0}$, by proposition 2.1.10 $m = m'$ and hence $n = n'$. Notice that h is an homeomorphism, so

$$h(\partial(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n)) = \partial(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n) = \bigcup_{i=1}^{m} \{(z, w) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n : z_i = 0\} \tag{2.2}$$

Condition (2.2) implies that for any $j = 1, 2, \ldots, m$ there exists $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_m) \in \mathbb{N}^m$ with $\alpha^j \neq \underline{0}$ such that

$$h_j(x, y) = x^{\alpha^j} g_j(x, y) = x_1^{\alpha^j_1} \cdots x_m^{\alpha^j_m} g_j(x, y) \tag{2.3}$$

with g_j analytic at $\underline{0}$, and $g_j(0, y) \neq 0$ for any $y \neq 0$ close enough to $\underline{0} \in \mathbb{R}^n$. Suppose $g_j(0, 0) = 0$. Then, there exists $i_j \in \{1, \ldots, n\}$ such that y_{i_j} divides g_j and then y_{i_j} divides h_j. This is not possible, because then we could take (x_0, y_0) an interior point of $\{(x, y) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n : y_{i_j} = 0\}$ such that $h(x_0, y_0) \in \{(z, w) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n : z_{i_j} = 0\}$, against (2.2).

Each of the first m lines of the jacobian matrix of the differential at $\underline{0} \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, $d_{\underline{0}} h$, is given by $\nabla(h_j)(0, 0)$. By (2.3)

$$x_1^{\alpha^j_1 - 1} x_2^{\alpha^j_2 - 1} \cdots x_m^{\alpha^j_m - 1} \text{ divides } \nabla(h_j)(x, y) \tag{2.4}$$

As h induces an isomorphism, $d_{\underline{0}} \varphi$ is a linear isomorphism of $\mathbb{R}^k$. Then, there are not lines of zeroes on its jacobian matrix. Since $\alpha^j \neq \underline{0}$ and (2.4), for any $j \in \{1, \ldots, m\}$ there exists a unique $i(j) \in \{1, \ldots, m\}$ such that $\alpha^j_{i(j)} = 1$ being the other components of α^j equal to zero. We have then for any $j = 1, 2, \ldots m$,

$$h_j(x, y) = x_{i(j)} g_j(x, y)$$

Now, we prove that the map $j \to i(j)$ is a permutation of $\{1, \ldots, m\}$. This follows from the fact that if we make the same construction for h^{-1},

$$h^{-1}(z, w) = (z_{l(1)} f_1(z, w), \ldots, z_{l(m)} f_m(z, w), f_{m+1}(z, w), \ldots, f_k(z, w)),$$

since $h \circ h^{-1}(z,w) = (z,w)$ and $h^{-1} \circ h(x,y) = (x,y)$, for $1 \leq j \leq m$,

$$z_j = z_{l(i(j))} f_{i(j)}(z,w) g_j(h^{-1}(z,w))$$
$$x_j = x_{i(l(j))} g_{l(j)}(x,y) f_j(h(x,y))$$

hence $i : j \in \{1,\dots,m\} \to i(j) \in \{1,\dots,m\}$ is a permutation of $\{1,\dots,m\}$ (with inverse l). $\square$

Definition 2.1.19. Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold and $p \in |A|$. Given a local chart $(U, \varphi = (x_1,\dots,x_k))$ of A at p and $f \in \mathcal{O}(U)$ we say that f **is monomial at** p **with respect to the local chart** (U,φ) **(or with respect to the coordinates** x) if the Taylor expansion of the germ $\mathbf{f}_p$ with respect to the coordinates x is of monomial type. In other words, that we can write locally f in the coordinates x as

$$f(x) = x_1^{\alpha_1} \cdots x_m^{\alpha_m} g(x), \quad x \in U,$$

where $g \in \mathcal{O}(U)$, vanishes nowhere in U, and each $\alpha_i \in \mathbb{N}$. We say that f **is monomial at the point** p if it is monomial with respect to some local chart at p. Finally, we say that f **is (locally) monomial** if it is monomial at every point of A.

Definition 2.1.20. Let $\varphi : A \to B$ be a morphism of standard analytic manifolds. We say that φ is **locally monomial** if for any $p \in |A|$ there exists local coordinates $(U, \phi = (x_1,\dots,x_k))$ centered at p such that all the components of φ are monomial at p with respect to these coordinates.

Examples 2.1.21. *i)* The morphism $(x,y) \in \mathbb{L} \times \mathbb{R} \to (x, x+y) \in \mathbb{A}_+ \times \mathbb{R}$ is locally monomial because with respect to the new coordinates $(x',y') = (x, x+y)$ its components are monomial.

ii) As a consequence of proposition 2.1.18 the morphism $(x,y) \in \mathbb{A}_+ \times \mathbb{R} \to (x, x^2(x^2+y^2)) \in \mathbb{L} \times \mathbb{R}$ is not locally monomial.

2.2 $\mathcal{G}$-analytic functions.

In this section we define the concept of generalized analytic function. These are the functions on open subsets of quadrants $\mathbb{R}^k_{\geq 0}$ which can be represented locally by real convergent generalized power series, in the same way as the classical real analytic functions are those locally described by convergent power series. The principal difference is that depending on the position of the point with respect to the boundary of the quadrant we are considering, the series will have a number of analytic or generalized variables. We need some notation.

Let $k, m, n \in \mathbb{N}$, $A \subseteq \{1,\dots,k\}$ and $\xi = (\xi_1,\dots,\xi_k) \in (0,\infty)^k$ be a polyradius. We put

$$I_{A,\xi} := B_1 \times B_2 \times \cdots B_k \subseteq \mathbb{R}^k,$$

where the B_i is either the interval $[0,\xi_i) \subset \mathbb{R}$ if $i \in A$ or the interval $(-\xi_i, \xi_i)$ if $i \notin A$. For a positive real number ϵ, we also write $I_{A,\epsilon}$ for $I_{A,(\epsilon,\dots,\epsilon)}$. Notice that, if $m+n = k$ and $A = \{1,\dots,m\}$, then we have, according to the first chapter, a second notation $I_{A,\xi} = I_{m,n,\xi}$ which will be also used here.

Let G_k denote the group of permutations of $\{1,\dots,k\}$ and $G_{m,n}$ the subgroup of G_{m+n} consisting on those permutations of $\{1,\dots,m+n\}$ such that they induce separately permutations of $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$. Given $\sigma \in G_k$,

$$\sigma : \mathbb{R}^k \to \mathbb{R}^k$$
$$\sigma(w_1,\dots,w_k) = (w_{\sigma(1)},\dots,w_{\sigma(k)})$$

With this notation, σ denotes a permutation of $\{1,...,k\}$ or a map from $\mathbb{R}^k$ to $\mathbb{R}^k$. We will deduce the meaning of σ from the context.

From now on, consider $A \subseteq \{1,\ldots,k\}$ and put $m = m(A) = |A|$ and $n = n(A) = k - m$. Let G_A denote the subset of permutations of $\{1,...,k\}$ sending A to $\{1,\ldots,m\}$.

Remark 2.2.1. Given $\sigma, \tau \in G_A$, $\sigma \circ \tau^{-1} \in G_{m(A),n(A)}$.

Notice that if $\delta > 0$ is sufficiently small, then, σ restricts to an homeomorphism $\sigma : I_{A,\delta} \to I_{m,n,\delta}$ (notice the abuse, again, of notation) whose inverse is also the restriction of a linear automorphism of $\mathbb{R}^k$ induced by a permutation of $\{1,...,k\}$, the inverse σ^{-1}, of σ.

If $p = (p_1,\ldots,p_k) \in \mathbb{R}^k_{\geq 0}$, we put

$$A(p) := \{i \in \{1,\ldots,k\} : p_i = 0\}$$

$$m_p = m(A(p)) := |A(p)|$$

$$n_p = n(A(p)) := k - m_p$$

$$G_p := G_{A(p)}$$

Notice that the family of sets $\{p + I_{A(p),\epsilon}\}$, where $\epsilon > 0$ is sufficiently small, is a fundamental system of neighborhoods of p in $\mathbb{R}^k_{\geq 0}$. By 2.1.7 the map

$$p \in \mathbb{R}^k_{\geq 0} \longmapsto m_p \in \mathbb{N}$$

is upper semi-continuous so for $\epsilon > 0$ small enough, if $q \in p + I_{A(p),\epsilon}$, then $A(q) \subseteq A(p)$, and therefore $m_q \leq m_p$.

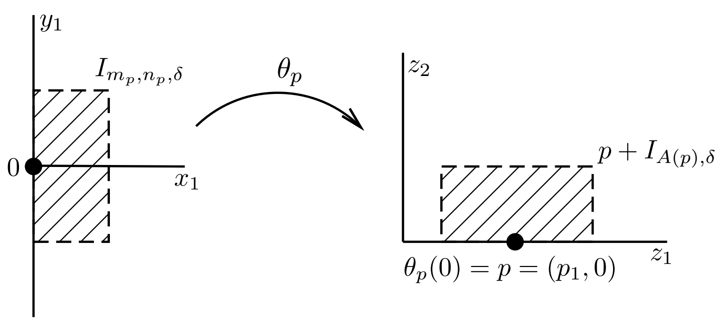
Given $p \in \mathbb{R}^k_{\geq 0}$ and $\sigma \in G_p$ we define $\theta_{p,\sigma}$ as the restriction to $\mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^{n_p}$ of the affine map given by

$$(w_1,\ldots,w_k) \mapsto p + \sigma(w_1,\ldots,w_k) = (p_1 + w_{\sigma(1)},...,p_k + w_{\sigma(k)}). \tag{2.5}$$

Coordinates in $\mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^{n_p}$ will be denoted, more conveniently, by $(x_1,\ldots,x_{m_p},y_1,\ldots,y_{n_p})$, reflecting the number and position of factors which are half real lines and those which are real lines.

Notice that for any small $\delta > 0$, $\theta_{p,\sigma}$ restricts to an homeomorphism from $I_{m_p,n_p,\delta}$ to $p + I_{A(p),\delta}$ sending $0 \in I_{m_p,n_p,\delta}$ to $p \in p + I_{A(p),\delta}$.

Graphically,



$$(x_1, y_1) \longmapsto (p_1 + y_1, x_1)$$

Definition 2.2.2. Let V be an open set in $\mathbb{R}^k_{\geq 0}$ and let $p \in V$. A function f on V is said to be **generalized analytic** or, shortening, **$\mathcal{G}$-analytic at** p if there exists $\delta > 0$, a convergent series $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p}$ and $\sigma \in G_p$ such that

i) $(p + I_{A(p),\delta}) \subseteq V$

ii) $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$

iii) $S_\delta(s)|_{I_{m_p,n_p,\delta}} = f|_{(p+I_{A(p),\delta})} \circ \theta_{p,\sigma}$

We say that f is $\mathcal{G}$-analytic on V if it is $\mathcal{G}$-analytic at every point p of V.

Remark 2.2.3. The definition above does not depend on the choice of σ in the following sense: if δ, s and σ are as in that definition satisfying $i), ii)$ and $iii)$, then for any $\tau \in G_p$ there exists $t \in \mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$ (which depends on τ) such that $S_\delta(t)|_{I_{m_p,n_p,\delta}} = f|_{(p+I_{A(p),\delta})} \circ \theta_{p,\tau}$. To prove this claim, take $\tau \in G_p$ and let $\eta = \sigma^{-1} \circ \tau$, a permutation of $\{1, ..., k\}$. Denote by ηs the series in $\mathbb{R}\{(X,Y)^*\}$ obtained by the morphism of substitution (see Proposition 1.2.10) of the variable X_i by $X_{\eta(i)}$ and of the variable Y_i by $Y_{\eta(i+m)-m}$. Notice that this series belongs actually to $\mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$ because η induces a permutation of the generalized variables X_j and a permutation of the analytic ones Y_j. The remark follows from the observation that

$$\theta_{p,\sigma} = \theta_{p,\tau} \circ \eta$$

and the fact that $S_\delta(\eta s) = S_\delta(s) \circ \eta$, from Proposition 1.2.21.

Definition 2.2.4. Let V be an open subset of $\mathbb{R}^k_{\geq 0}$. We let $\mathcal{G}_{\mathbb{L}^k}(V)$ denote the set of $\mathcal{G}$-analytic functions on V :

$$\mathcal{G}_{\mathbb{L}^k}(V) := \{f : V \to \mathbb{R} : f \text{ is } \mathcal{G}\text{-analytic on } V\}$$

Then $\mathcal{G}_{\mathbb{L}^k}(V)$ is a $\mathbb{R}$-subalgebra of the algebra of continuous functions on V with respect to the natural inclusion $\mathbb{R} \hookrightarrow \mathcal{G}_{\mathbb{L}^k}(V)$ that identifies a real number with the corresponding constant function. It is a straightforward computation, as a consequence of the fact that the sum of convergent series is an algebra homomorphism (see Proposition 1.2.20), to check that $\mathcal{G}_{\mathbb{L}^k}(V)$ is a sub $\mathbb{R}$-algebra of the algebra of real functions on V.

Theorem 2.2.5. *i)* A $\mathcal{G}$-analytic function f at a point p is continuous at that point.

ii) A $\mathcal{G}$-analytic function at a point in the interior of $\mathbb{R}^k_{\geq 0}$ in $\mathbb{R}^k$ (that is, a point $p = (p_1, \ldots, p_k) \in \mathbb{R}^k_{\geq 0}$ such that $p_i \neq 0$ for all $1 \leq i \leq k$) is analytic at this point.

iii) Let V be an open subset of $\mathbb{R}^k_{\geq 0}$, p a point in V and $f : V \to \mathbb{R}$ a function which is $\mathcal{G}$-analytic at p. Then there exists a neighborhood of p, $W \subseteq V$, such that f is $\mathcal{G}$-analytic on W.

As a consequence, if V is an open subset of $\mathbb{R}^k_{\geq 0}$ and $f : V \to \mathbb{R}$ is $\mathcal{G}$-analytic on V, f is continuous on V and analytic on the interior in $\mathbb{R}^k$ of V.

Proof. Part $i)$ follows from the fact that f coincides with the sum of a convergent generalized power series in a neighborhood of p and such a sum is continuous by 1.2.20. Part $ii)$ follows from definition of $\mathcal{G}$-analytic function.

For $iii)$, if f is analytic at p then, by definition there exists $\delta > 0$, $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p}$ and $\sigma \in G_p$ such that

1. $(p + I_{A(p),\delta}) \subseteq V$ and $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$

2. $S_\delta(s)|_{I_{m_p,n_p,\delta}} = f|_{(p+I_{A(p),\delta})} \circ \theta_{p,\sigma}$

Take $\delta > 0$ such that $\delta \leq \min_{i \notin A(p)}\{p_i\}$. Then, for all $q \in p + I_{A(p),\delta}$, $A(q) \subseteq A(p)$ and therefore $m_q \leq m_p$ (see 2.1.7). We claim that f is $\mathcal{G}$-analytic on $p + I_{A(p),\delta}$. Let $q = (q_1, \dots, q_k)$ be a point in $p + I_{A(p),\delta}$ and consider $a = (a_1, \dots, a_k) := \theta_{p,\sigma}^{-1}(q) \in I_{m_p,n_p,\delta}$. Put

$$m' = |\{i \in \{1, \dots, m_p\} : a_i = 0\}| \text{ and } n' = k - m'.$$

By Proposition 1.2.23 given a permutation τ of $\{1, \dots, k\}$ such that $\tau(A(a)) = \{1, \dots, m'\}$ and $\epsilon > 0$ such that

$$a + \tau(w) \in I_{m_p,n_p,\delta}$$

whenever $w \in I_{m',n',\epsilon}$, there exists a unique $T_a s \in \mathbb{R}\{U^*, V\}_{m',n',\epsilon}$ such that

$$S_\epsilon(T_a s)(w) = S_\delta(s)(a + \tau(w))$$

for all $w \in I_{m',n',\epsilon}$.

Consider the composition of permutations $\eta = \tau\sigma$. We have, in one hand, that $\eta \in G_{A(q)}$: If $j \in A(q)$ then the j^{th}-coordinate q_j of q is equal to zero. But $q_j = p_j + a_{\sigma(j)}$ and, since $A(q) \subseteq A(p)$, $p_j = 0$ and thus $a_{\sigma(j)} = 0$. This implies that $\sigma(j) \in A(a)$ and then that $\eta(j) = \tau(\sigma(j)) \in \{1, \dots, m'\}$.

On the other hand, we have that

$$\theta_{a,\tau}(I_{m',n',\epsilon}) \subset a + \tau(I_{m',n',\epsilon}) \subset I_{m,n,\delta},$$

so that the composition $\theta_{p,\sigma} \circ \theta_{a,\tau}$ is well defined in $I_{m',n',\epsilon}$. But this composition is nothing more than the map $\theta_{q,\tau\sigma}$, obtaining finally

$$S_\epsilon(T_a s) = S_\delta(s) \circ \theta_{a,\tau} = f \circ \theta_{p,\sigma} \circ \theta_{a,\tau} = f \circ \theta_{q,\tau\sigma}$$

which shows that f is $\mathcal{G}$-analytic at the point q, as was to be proved. □

For $p \in \mathbb{R}^k_{\geq 0}$ we consider the $\mathbb{R}$-algebra of germs of $\mathcal{G}$-analytic functions at p in the usual way: it is the quotient of the set $\{(V, f) : p \in V, V \text{ open and } f : V \to \mathbb{R}\ \mathcal{G}\text{-analytic at } p\}$ by the equivalence relation $(V, f) \sim (U, g)$ if and only if there exists an open neighborhood of p, $W \subset U \cap V$ such that $f|_W = g|_W$. Let $\mathcal{G}_{\mathbb{L}^k,p}$ denote the $\mathbb{R}$-algebra of germs of $\mathcal{G}$-analytic functions at p.

Proposition 2.2.6. For any $p \in \mathbb{R}^k_{\geq 0}$ the $\mathbb{R}$-algebra $\mathcal{G}_{\mathbb{L}^k,p}$ is isomorphic to $\mathbb{R}\{X^*, Y\}_{m_p,n_p}$. As a corollary, the $\mathbb{R}$-algebra of germs of generalized analytic functions $\mathcal{G}_{\mathbb{L}^k,p}$ is a local $\mathbb{R}$-algebra whose maximal ideal consists of those germs of functions which take the value zero at p.

Proof. Let $p \in \mathbb{R}^k_{\geq 0}$. As in the beginning of this section, let $A(p) = \{i \in \{1, ..., k\} : p_i = 0\}$ and let G_p be the set of permutations of $\{1, \dots, k\}$ that send $A(p)$ into $\{1, \dots, m_p\}$. Fix $\sigma \in G_p$ and denote by $\theta_{p,\sigma}$ the map defined in $\mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^n_p$ by $\theta(w) = p + \sigma(w)$.

Let $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$ be a given convergent generalized series and denote, as in 1.2.20 by $S_\delta(s)$ its sum, a $\mathcal{G}$-analytic function on $I_{m_p,n_p,\delta} \subset \mathbb{R}^k_{\geq 0}$. By its very definition, the composition $S_\delta(s) \circ \theta_{p,\sigma}^{-1}$ is a $\mathcal{G}$-analytic function in some neighborhood of p. We can then consider the map

$$F_\sigma : \mathbb{R}\{X^*, Y\}_{m_p,n_p} \to \mathcal{G}_{\mathbb{L}^k,p} \tag{2.6}$$

assigning to an element $s \in \mathbb{R}\{X^*, Y\}_{m_p,n_p,\delta}$ the germ at p of $S_\delta(s) \circ \theta_{p,\sigma}^{-1}$, which is well defined (the germ of such a composition does not depend on the polyradius δ as long as the series has radius of convergence greater or equal to δ). The fact that the sum operator S_δ is an algebra homomorphism between $\mathbb{R}\big\{X^*, Y\big\}_{m_p,n_p}$ and $\mathcal{C}^0(I_{A(p),\delta};\mathbb{R})$, gives directly the result that F_σ is an $\mathbb{R}$-algebra homomorphism.

Let us see finally that F is a bijection. It is injective thanks to the fact that the sum morphism S_δ is injective. Surjectivity of F_σ comes from definition: if $\mathbf{f}_p \in \mathcal{G}_{\mathbb{L}^k,p}$ is the germ of some $\mathcal{G}$-analytic function $f \in \mathcal{G}_{\mathbb{L}^k}(U)$, by definition of $\mathcal{G}_{\mathbb{L}^k}(U)$ (and Remark 2.2.3) there exists $\delta > 0$ and $s \in \mathbb{R}\big\{X^*, Y\big\}_{m_p,n_p,\delta}$ such that $S_\delta(s) = f \circ \theta_{p,\sigma}$. Then, $F_\sigma(s) = \mathbf{f}_p$. □

Notice that we have proved that the morphism F_σ in (2.6) is an isomorphism for any $\sigma \in G_p$. Following Remark 2.2.1, if $\sigma, \tau \in G_p$, we obtain that given a germ $\mathbf{f}_p$, the two series $F_\sigma^{-1}(\mathbf{f}_p)$, $F_\tau^{-1}(\mathbf{f}_p)$, are obtained one from the other by the permutation $\sigma \circ \tau^{-1}$ (or its inverse) of the variables X, Y. This permutation belongs to the subgroup G_{m_p,n_p} of permutations of the k variables which induce separate permutations, one on the generalized variables X and another permutation on the analytic ones Y. Thus we can define

Definition 2.2.7. Given $p \in \mathbb{R}^k_{\geq 0}$ and $\mathbf{f}_p \in \mathcal{G}_{\mathbb{L}^k,p}$, the series $F_\sigma^{-1}(\mathbf{f}_p) \in \mathbb{R}\big\{X^*, Y\big\}_{m_p,n_p}$ is called the **Taylor expansion of the germ $\mathbf{f}_p$** . It is well defined modulo the action of G_{m_p,n_p} on the series.

Examples 2.2.8. Let us give some examples of $\mathcal{G}$-analytic functions.

i) If V is an open subset of $\mathbb{R}^k_{\geq 0}$ and $f : V \to \mathbb{R}$ is a function which is the restriction to V of a real analytic function on an open set of $\mathbb{R}^k$ containing V, then f is $\mathcal{G}$-analytic on V. This is an easy consequence of the fact that, given variables X and Y, we have naturally the inclusion $\mathbb{R}\{X, Y\} \subset \mathbb{R}\{X^*, Y\}$.

ii) If $s \in \mathbb{R}\big\{X^*, Y\big\}_{m,n,\delta}$, then its sum is $\mathcal{G}$-analytic on $I_{m,n,\delta}$.

iii) $\sin(x^\lambda y^\mu)$ in $\mathbb{R}^2_{\geq 0}$

iv) $\log(1 + x^\lambda)$ in $\mathbb{R}_{\geq 0}$

v) Let ζ denote the Riemann zeta function. Then,

$$\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log(n)} : [0, e^{-2}) \to \mathbb{R}$$

is $\mathcal{G}$-analytic.

2.3 Generalized analytic manifolds.

We are going to define a subcategory $\mathcal{G}$ of $\mathfrak{C}$ that will be called the category of **generalized analytic manifolds**.

In order to define $\mathcal{G}$ we proceed as follows. First of all we construct a particular object $\mathbb{L}^k$ in $\mathcal{G}$ for each $k \in \mathbb{N}$ called the Standard Local Model of dimension k. Then objects of $\mathcal{G}$ are those objects in the category $\mathfrak{C}$ which are locally isomorphic to some $\mathbb{L}^k$ as ringed spaces. Morphisms in $\mathcal{G}$ will be the morphism in $\mathfrak{C}$ when consider the objects of $\mathcal{G}$ as objects in $\mathfrak{C}$ so that $\mathcal{G}$ will be a full subcategory of $\mathfrak{C}$.

We consider $\mathbb{R}^k_{\geq 0}$ as a topological space with the topology of subspace of $\mathbb{R}^k$. If U is an open set of $\mathbb{R}^k_{\geq 0}$ the assignment $U \mapsto \mathcal{G}_{\mathbb{L}^k}(U)$ of $\mathcal{G}$-analytic functions on U (see definition 2.2.4), together with the restriction morphism

$$\mathcal{G}_{\mathbb{L}^k}(U) \to \mathcal{G}_{\mathbb{L}^k}(V), f \mapsto f\,|_V$$

each time that $V \subset U$, is a sheaf on $\mathbb{R}^k_{\geq 0}$. Moreover, by 2.2.4 and theorem 2.2.5, it is a subsheaf of local $\mathbb{R}$-algebras of the sheaf of continuous functions. We define the **standard local model of generalized analytic manifold of dimension** k as

$$\mathbb{L}^k = (\mathbb{R}^k_{\geq 0}, \mathcal{G}_{\mathbb{L}^k}).$$

Definition 2.3.1. A **generalized analytic manifold** or, for short, **$\mathcal{G}$-manifold of dimension** k is a locally ringed space $M = (|M|, \mathcal{G}_M) \in \text{Objects}(\mathfrak{C})$, where $|M|$ is a Hausdorff topological space with a countable open basis, such that for every $p \in |M|$ there exists an open neighborhood U of p and an open set $V \subset \mathbb{R}^k_{\geq 0}$ such that the restrictions $M|_U = (U, \mathcal{G}_M|_U)$ and $\mathbb{L}^k|_V = (V, \mathcal{G}_{\mathbb{L}^k}|_V)$ are isomorphic in the category $\mathfrak{C}$.

Definition 2.3.2. If $M = (|M|, \mathcal{G}_M)$ is a $\mathcal{G}$-manifold, an **open submanifold of** M is the locally ringed space $M|_U = (U, \mathcal{O}_M|_U)$ where U is an open subset of $|M|$ (see the appendix for the notation). It is clear that an open submanifold is also a G-analytic manifold.

Given two generalized analytic manifolds $M = (|M|, \mathcal{G}_M)$ and $N = (|N|, \mathcal{G}_N)$ a morphism between them is, by definition, a morphism of the category $\mathfrak{C}$. The category $\mathcal{G}$ of generalized analytic manifolds is then defined by setting

$$\begin{aligned} \text{objects}(\mathcal{G}) &:= \{M \in \text{objects}(\mathfrak{C}) : M \text{ is a generalized analytic manifold}\} \\ \text{morphisms}(\mathcal{G}) &:= \{(\varphi : M \to N) \in \text{morphisms}(\mathfrak{C}) : M, N \in \text{ objects}(\mathcal{G})\} \end{aligned}$$

Recall that a morphism φ between two generalized analytic manifolds $M = (|M|, \mathcal{G}_M)$ and $N = (|N|, \mathcal{G}_N)$ is determined by a continuous map between the topological spaces $\varphi : |M| \to |N|$ (but not every continuous map between the underlying topological spaces induces a morphism between the ringed spaces !), and that such a morphism is an isomorphism if and only if $\varphi : |M| \to |N|$ is an homeomorphism and for all $p \in |M|$ the induced homomorphism in the stalk

$$\begin{aligned} \varphi : \mathcal{G}_{N,\varphi(p)} &\longrightarrow \mathcal{G}_{M,p} \\ \varphi(\mathbf{f}|_{\varphi(p)}) &= (\mathbf{f} \circ \varphi)|_p \end{aligned}$$

is an isomorphism of $\mathbb{R}$-algebras.

Examples 2.3.3. We give some examples of generalized analytic manifolds to illustrate the definition. Most of them will be used through this work.

i) Let $\mathcal{O}_{\mathbb{R}^k}$ denote the sheaf of analytic functions over $\mathbb{R}^k$. Then $(\mathbb{R}^k, \mathcal{O}_{\mathbb{R}^k})$ is a generalized analytic manifold. To see that, remark that the homeomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^k_{>0} \subseteq \mathbb{R}^k_{\geq 0}$ defined by $\varphi(y_1, \dots, y_k) = (e^{y_1}, \dots, e^{y_k})$ induces an isomorphism (of locally ringed spaces) from $(\mathbb{R}^k, \mathcal{O}_{\mathbb{R}^k})$ to $\mathbb{L}^k|_{\mathbb{R}^k_{>0}} = (\mathbb{R}^k_{>0}, \mathcal{G}_{\mathbb{L}^k}|_{\mathbb{R}^k_{>0}})$. Then, in particular, for V open subset of $\mathbb{R}^k$, if we let $\mathcal{O}_V$ denote the sheaf of analytic functions on V, $(V, \mathcal{O}_V)$ is a generalized analytic manifold.

ii) More generally, if $M = (|M|, \mathcal{O}_M)$ is a real analytic manifold (with the sheaf-theoretic interpretation; that is, that $\mathcal{O}_M$ is the sheaf of real analytic function on the underlying variety $|M|$), then M is a generalized analytic manifold. This is an immediate consequence of example above.

iii) The local model $\mathbb{L}^k = (\mathbb{R}^k_{\geq 0}, \mathcal{G}_{\mathbb{L}^k})$ is a generalized analytic manifold of dimension k.

iv) Consider $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ with the product topology. Let $\Phi : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{>0} \subset \mathbb{R}^{m+n}_{\geq 0}$ be the map defined by

$$(x, y) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \xrightarrow{\Phi} (x, \varphi(y)) = (x, e^{y_1}, \dots, e^{y_k}) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{>0}$$

It is a homeomorphism. We can endow an structure of generalized analytic manifold to $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ via this homeomorphism: just consider the sheaf $\mathcal{G}_{m,n}$ defined by assigning to each open set $V \subset \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ the $\mathbb{R}$-algebra of functions $f : V \to \mathbb{R}$ such that $f \circ \Phi^{-1}$ is a $\mathcal{G}$-function on the open set $\Phi(V)$ of $\mathbb{R}^{m+n}_{\geq 0}$. For reasons that will be clear below, we call the generalized analytic manifold

$$\mathbb{L}^m \times \mathbb{R}^n := (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \mathcal{O}_{m,n})$$

the (m, n) mixed (generalized) local model. Notice that by the moment $\mathbb{L}^m \times \mathbb{R}^n$ is just a notation. We show in proposition 2.3.21 below that the category $\mathcal{G}$ has product. In particular the product of the standard analytic manifolds $\mathbb{L}^m$ and $\mathbb{R}^n$ has sense and it agrees with the given here.

Definition 2.3.4. Let $p \in |M|$. A **local (generalized) chart at** p will be a pair (U, z) where U is an open neighborhood of p in $|M|$ and

$$\begin{aligned} z : U &\longrightarrow V \\ z(q) =&(z_1(q), \ldots , z_k(q)) \end{aligned}$$

is a homeomorphism which induces an isomorphism of generalized analytic manifolds $M|_U = (U, \mathcal{G}_M|_U)$ and $\mathbb{L}^k|_V = (V, \mathcal{G}_{\mathbb{L}^k}|_V)$. The components $z_1, \ldots , z_k$ will be called **local coordinates at** p. We say that a local chart is **centered at** p if it sends p to the origin.

Proposition 2.3.5. For any point $p \in \mathbb{R}^k_{\geq 0}$ there exists $\epsilon > 0$ small enough and an open neighborhood (depending on ϵ) isomorphic to $(\mathbb{L}^{m_p} \times \mathbb{R}^{n_p})|_{[0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p}}$. As a consequence, given $\mathbb{L}^k|_V$, an open submanifold of $\mathbb{L}^k$, any point $p \in V$ has an open neighborhood isomorphic to $\mathbb{L}^{m_p} \times \mathbb{R}^{n_p}$.

Proof. Let $p \in \mathbb{R}^k_{\geq 0}$, $A(p) \subset \{1, ..., k\}$ and $\sigma \in G_p$ as defined in 2.2. For $\delta > 0$ sufficiently small, the map $\theta_{p,\sigma}$ as in equation (2.5) restricts to a homeomorphism from the neighborhood $I_{m_p,n_p,\delta}$ of $(0, ..., 0)$ in $\mathbb{R}^k_{\geq 0}$ to the neighborhood $I_{A(p),\delta}$ of p in $\mathbb{R}^k_{\geq 0}$. Then we have that its inverse $\theta^{-1}_{p,\sigma}$ induces an isomorphism between $\mathbb{L}^k|_{I_{A(p),\delta}}$ and $(\mathbb{L}^{m_p} \times \mathbb{R}^{n_p})|_{I_{m_p,n_p,\delta}}$. In this way we can see $\theta^{-1}_{p,\sigma}$ as a local chart at p, centered at p. Now, it suffices to notice that the map

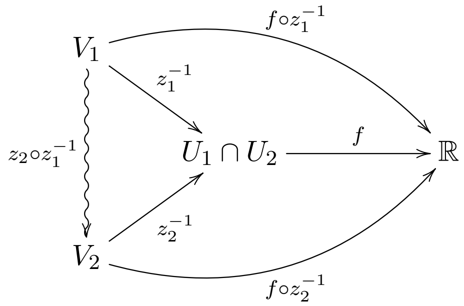
$$\begin{aligned} \varphi : [0, \epsilon)^m \times (-\epsilon, \epsilon)^n &\to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \\ \varphi(x_1, \ldots , x_m, y_1, \ldots , y_n) &= (\frac{x_1}{\epsilon - x_1}, \ldots , \frac{x_m}{\epsilon - x_m}, \frac{y_1}{\epsilon - (y_1)^2}, \ldots , \frac{y_n}{\epsilon - (y_n)^2}) \end{aligned}$$

induces an isomorphism between $(\mathbb{A}^{m_p}_+ \times \mathbb{R}^{n_p})|_{[0,\epsilon)^{m_p}\times(-\epsilon,\epsilon)^{n_p}}$ and $(\mathbb{A}^{m_p}_+ \times \mathbb{R}^{n_p})$. □

Definition 2.3.6. Let $M = (|M|, \mathcal{G}_M)$ be a k-dimensional generalized analytic manifold. Let U be an open subset of $|M|$ and $f : U \to \mathbb{R}$ a continuous function on U. Let $p \in U$. We just say that f is $\mathcal{G}$-analytic at p if the germ of f at p belongs to the local algebra $\mathcal{G}_{M,p}$. The function f will be called a $\mathcal{G}$-analytic function on U if it is $\mathcal{G}$-analytic at every point of U. Equivalently, since $\mathcal{G}_M$ is a sheaf, f is $\mathcal{G}$-analytic on U if it belongs to the algebra $\mathcal{G}_M(U)$ of sections of the structural sheaf.

By the very definition of $\mathcal{G}$-analytic manifold, we deduce that f is $\mathcal{G}$-analytic at a point $p \in U$ if and only if there exists a local chart at the point p, $z : U_p \to V \subseteq \mathbb{R}^k_{\geq 0}$ such that the function $f \circ z^{-1} : V \to \mathbb{R}$ is $\mathcal{G}$-analytic at $\varphi(p)$. If f is $\mathcal{G}$-analytic at p for all $p \in U$ we say that f is $\mathcal{G}$-analytic on U.

Remark that this property does not depend on the choice of the local chart z. If U_1, U_2 are open neighborhoods of p and $z_i : U_i \to V_i$ are isomorphisms from $M|_{U_i}$ to $\mathbb{L}^k|_{V_i}$, we have the diagram



then if $f \circ z_1^{-1}$ is $\mathcal{G}$-analytic at $z_1(p)$ it is $\mathcal{G}$-analytic on a neighborhood V' of $z_1(p)$ (see 2.2.5), that is $f \circ z_1^{-1} \in \mathcal{G}_{\mathbb{L}^k}(V')$. As $z_1 \circ z_2^{-1}$ is an isomorphism,

$$f \circ z_1^{-1} \circ z_1 \circ z_2^{-1} = f \circ z_2^{-1} \in \mathcal{G}_{\mathbb{L}^k}(W')$$

where $W' = z_2 \circ z_1^{-1}(V')$. In particular $f \circ z_2^{-1}$ is $\mathcal{G}$-analytic at $z_2(p)$.

Definition 2.3.7. Given f a $\mathcal{G}$-analytic function over M and $p \in M$, let $(U, \varphi = (x_1, \ldots, x_k))$ be a local chart of M at p. The **Taylor expansion of f at p with respect to these coordinates** is the series in $\mathbb{R}\{X^*\}$, $X = (X_1, ..., X_k)$ which is the Taylor expansion of the the germ of $f \circ \varphi^{-1}$ at $\varphi(p) \in \mathbb{R}^k_{\geq 0}$ (It is well defined up to a permutation of the generalized variables (those X_j such that $x_j(p) = 0$) and a permutation of the analytic ones (those X_j such that $x_j(p) \neq 0$).

Definition 2.3.8. Let $|M|$ be a Hausdorff topological space with a countable open basis. We say that a family $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ is an **$\mathcal{G}$-atlas** of $|M|$ if

i) For any $\lambda \in \Lambda$, U_λ is an open subset of $|M|$ and $\varphi_\lambda : U_\lambda \to \varphi_\lambda(U_\lambda) \subseteq \mathbb{R}^k_{\geq 0}$ is an homeomorphism.

ii) $X = \bigcup_{\lambda \in \Lambda} U_\lambda$

iii) For any $\lambda, \mu \in \Lambda$, $\varphi_\lambda \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu \cap U_\lambda) \to \varphi_\lambda(U_\mu \cap U_\lambda)$ is an isomorphism in $\mathcal{G}$.

Let U an open subset of $|M|$. We denote by $\mathcal{G}_M(U)$ the set of continuous functions $f : U \to \mathbb{R}$ such that for any $p \in U$, there exists an open $V \subseteq U$ such that $f \circ \varphi_\lambda^{-1} : \varphi_\lambda(V \cap U_\lambda) \to \mathbb{R}$ is $\mathcal{G}$-analytic at $\varphi(p) \in \mathbb{R}^k_{\geq 0}$ for any $\lambda \in \Lambda$ such that $p \in U_\lambda$.

Proposition 2.3.9. The pair $X = (|X|, \mathcal{O}_X)$ is a generalized analytic manifold.

Proof. By definition, $X \in$Obj$(\mathfrak{C})$. Let $p \in |X|$. Let $\lambda \in \Lambda$ such that $p \in U_\lambda$. Then, φ_λ^{-1} induces a morphism from $\mathbb{L}^k|_{\varphi_\lambda(U_\lambda)}$ to $X|_{U_\lambda}$ by definition of X. Moreover, $\varphi_\lambda : U_\lambda \to \varphi_\lambda(U_\lambda)$ induces a morphism from $X|_{U_\lambda}$ to $\mathbb{L}^k|_{\varphi_\lambda(U_\lambda)}$: let V be an open subset of $\varphi_\lambda(U_\lambda)$ and $g : V \to \mathbb{R}$ a section of $\mathcal{G}_{\mathbb{L}^k}$ over V. Then, $g \circ \varphi_\lambda \in \mathcal{G}_X(\varphi_\lambda^{-1}(V))$ because if $q \in \varphi_\lambda^{-1}(V)$, and $\mu \in \Lambda$ is such that $q \in U_\mu$, $f \circ \varphi_\lambda \circ \varphi_\mu^{-1} \in \mathcal{G}_{\mathbb{L}^k}(\varphi_\mu \circ \varphi_\lambda^{-1}(V))$ since by condition *iii)* of 2.3.8 $g \in \mathcal{G}_{\mathbb{L}^k}(V) \mapsto g \circ \varphi_\lambda \circ \varphi_\mu^{-1} \in \mathcal{G}_{\mathbb{L}^k}(\varphi_\mu \circ \varphi_\lambda^{-1}(V))$ is an isomorphism. □

2.3.1 Stratification by the number of boundary components.

Fix a k dimensional generalized analytic manifold $M = (|M|, \mathcal{G}_M)$. The first consequence of the definition is the following

Theorem 2.3.10. *i)* The underlying space $|M|$ is a topological manifold of dimension k with boundary.

ii) If $int(|M|)$ denotes the interior of this manifold then the restricted sheaf

$$int(M) = (int(|M|), \mathcal{G}_M|_{int(|M|)})$$

is a real analytic manifold.

Proof. *i)* follows from the fact that each point in $|M|$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^k_{\geq 0}$, a topological manifold of dimension k with boundary.

ii) follows from the fact that each point p in $int(|M|)$ has an open neighborhood U_p such that the restriction $M|_{U_p} = int(|M|)|_{U_p}$ is isomorphic to the restriction of $\mathbb{L}^k$ to some open set contained in the interior $\mathbb{R}^k_{>0}$ of $\mathbb{R}^k_{\geq 0}$. After this remark, use theorem 2.2.5 that asserts that a $\mathcal{G}$-analytic function at an interior point of $\mathbb{R}^k_{\geq 0}$ is analytic at that point. □

The Theorem above shows that at points in the interior of the manifold, the dimension is the only local invariant by isomorphisms. As we show below, this is not the case for points at the boundary $\partial|M|$: looking at the standard local model $\mathbb{R}^k_{\geq 0}$, although any two points in the boundary have topologically equivalent neighborhoods, they would not have necessarily isomorphic neighborhoods in the category of generalized analytic manifolds. In fact, the number of coordinate hyperplanes ("boundary components"), passing through the point will be invariant for local isomorphisms.

Let (U, z) be a local chart at p and define $m_p := |\{i \in \{1, \dots, k\} : z_i(p) = 0\}|$. We are going to prove that m_p does not depend on the local chart chosen but only on the point p. We need the following proposition

Proposition 2.3.11. Let U and V two open sets of $\mathbb{R}^k_{\geq 0}$ and suppose that the generalized analytic manifolds $\mathbb{L}^k|_U = (U, \mathcal{G}_{\mathbb{L}^k}|_U)$ and $\mathbb{L}^k|_V = (V, \mathcal{G}_{\mathbb{L}^k}|_V)$ are isomorphic via the homeomorphism $\varphi : U \to V$. Then for each $p \in U$, $m_p = m_{\varphi(p)}$ and so $n_p = n_{\varphi(p)}$.

Proof. Assume that $m_p \geq m_{\varphi(p)}$ (otherwise, take the inverse of φ). To say that $\mathbb{L}^k|_U$ and $\mathbb{L}^k|_V$ are isomorphic via φ means that, for any $q \in U$ the induced local homomorphism on the stalks

$$\begin{aligned} \varphi_q^\sharp : \mathcal{G}_{\mathbb{L}^k,\varphi(q)} &\longrightarrow \mathcal{G}_{\mathbb{L}^k,q} \\ \mathbf{f}_{\varphi(q)} &\mapsto (\mathbf{f} \circ \varphi)_q \end{aligned}$$

is an isomorphism. Let $p \in U$. By lemma 2.2.6 $\mathcal{G}_{\mathbb{L}^k,\varphi(p)}$ is isomorphic to $\mathbb{R}\big\{X^*, Y\big\}_{m_{\varphi(p)}, n_{\varphi(p)}}$ and $\mathcal{G}_{\mathbb{L}^k,p}$ is isomorphic to $\mathbb{R}\big\{Z^*, W\big\}_{m_p,n_p}$. It is important to recall what are the isomorphisms considered in that proposition: they are given by the maps

$$F := F_{\sigma,\varphi(p)} : \mathbb{R}\big\{X^*, Y\big\}_{m_{\varphi(p)}, n_{\varphi(p)}} \to \mathcal{G}_{\mathbb{L}^k,\varphi(p)}$$

$$G := F_{\tau,p} : \mathbb{R}\{Z^*, W\}_{m_p,n_p} \to \mathcal{G}_{\mathbb{L}^k,p}$$

where F (and similarly for G) sends a series $s \in \mathbb{R}\big\{X^*, Y\big\}_{m_{\varphi(p)}, n_{\varphi(p)}}$ to the germ of $S_\delta(s) \circ \theta^{-1}_{\varphi(p),\sigma}$ at $\varphi(p)$, σ being a permutation in $G_\varphi(p)$ and $\theta_{\varphi(p),\sigma}$ is defined as in (2.5) We have the diagram

$$\begin{array}{ccc} \mathcal{G}_{\mathbb{L}^k,\varphi(p)} & \xrightarrow{\quad \varphi_p \quad} & \mathcal{G}_{\mathbb{L}^k,p} \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle G} \\ \mathbb{R}\big\{X^*, Y\big\}_{m_{\varphi(p)}, n_{\varphi(p)}} & \xrightarrow{\phi := G^{-1}\circ\varphi_p\circ F} & \mathbb{R}\big\{Z^*, W\big\}_{m_p,n_p} \end{array}$$

As Z_j has all N^{th}-roots in $\mathbb{R}\big\{Z^*, W\big\}_{m_p,n_p}$, and ϕ is an algebra homomorphism, we have that $\phi(Z_j)$ also have all N^{th}-roots and then, by Proposition 1.1.20 $\phi(Z_j) = X^{\alpha_j} U_j$ for all

$j \in \{1, \ldots, m_p\}$ where the U_j are units. Notice that $\alpha_j \neq \underline{0}$ because Z_j is not a unit and an isomorphism send non units to non units. Consider at $\varphi(p)$ the local chart $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_k) = \theta^{-1}_{\varphi(p),\sigma}$ and at p the local chart $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_k) = \theta^{-1}_{p,\tau}$ (cf. proposition 2.3.5). Denote by $\tilde{\mathbf{z}}_{\mathbf{j}}$ the germ of $\tilde{z}_j$ at $\varphi(p)$, etc. By the way we have defined the isomorphism F, we have $F(Z_j) = \tilde{\mathbf{z}}_{\mathbf{j}}$ for $j = 1, ..., m_{\varphi(p)}$ and $F(W_j) = \tilde{\mathbf{z}}_{\mathbf{j}+\mathbf{m}_{\varphi(\mathbf{p})}}$ for $j = 1, ..., n$. Write then $\tilde{z} = (z, w)$ where z are the first $m_{\varphi(p)}$ components of $\tilde{z}$ and w are the last $n_{\varphi(p)}$ components. Similarly we put $\tilde{x} = (x, y)$ where x are the first m_p components of $\tilde{x}$ and y are the last n_p components.

Write the map φ in these coordinates as

$$\varphi(x, y) = (\varphi_1(x, y), \ldots, \varphi_k(x, y))$$

where $\varphi_j = \tilde{z}_j \circ \varphi$. By definition, the germ of φ_j is the image by the isomorphism $\varphi_p^\sharp$ of $\tilde{\mathbf{z}}_{\mathbf{j}}$. Using the commutative diagram above, we obtain, in a neighborhood of p, the expression

$$\varphi(x, y) = (x^{\alpha_1} u_1(x, y), \ldots, x^{\alpha_{m_{\varphi(p)}}} u_{m_{\varphi(p)}}(x, y), \varphi_{m_{\varphi(p)}+1}(x, y), \ldots, \varphi_k(x, y))$$

where u_j denotes the sum of the convergent series $U_j(X, Y) \in \mathbb{R}\{X^*, Y\}_{m_p,n_p}$. Thus we have $\varphi_j(0, y) = 0$ for $j \in \{1, \ldots, m_{\varphi(p)}\}$ and every small y. Together with the assumption $m_p \geq m_{\varphi(p)}$, this implies that φ restricts to a map from $\{(x, y)/x = 0\}$ into $\{(z, w)/z = 0\}$. This two sets being open subsets of $\mathbb{R}^{n_p}$ and $\mathbb{R}^{n_{\varphi(p)}}$ respectively, and φ being injective, the Invariance of the Domain Theorem implies that $n_p = n_{\varphi(p)}$ as was to be proved. □

Remark 2.3.12. A natural question that arises from the proof of the Proposition above is whether two algebras of convergent mixed generalized series $\mathbb{R}\{X^*, Y\}_{m,n}$ and $\mathbb{R}\{Z^*, W\}_{m',n'}$ are isomorphic if and only if $m = m'$ and $n = n'$. This is easily the case for m or m' is equal to 0 because it is the only case where such an algebra is noetherian. In our proof we have only shown that the number of analytic or non-analytic variables are the same if the isomorphism, ϕ, is given by a morphism on the sheaf structure, that is, by "composing" series under a homeomorphism.

Definition 2.3.13. Let $M = (|M|, \mathcal{G}_M)$ be a $\mathcal{G}$-manifold. The function $m : |M| \to \mathbb{N}$ defined by

$$m(p) := m_p = |\{i \in \{1, \ldots, k\} : z_i(p) = 0\}|,$$

where $z = (z_1, \ldots, z_k)$ is a chart on p is well defined. Moreover, m is an upper semi-continuous function. Given a point p, we will say also that m_p is the **number of boundary components** of the point p.

Let M be a $\mathcal{G}$-manifold of dimension k. For $j \in \{0, 1, \ldots, k\}$ let

$$D(j) := \{p \in |M| : m_p = j\}$$

Let $j_0 := \max\{j \in \{0, 1, \ldots, k\} : D(j) \neq \emptyset\}$. We call $D(j_0)$ the **lime** of M.

Let $\{D(j)_i\}_{i \in I_j}$ be the connected components of $D(j)$. We consider the partition of the underlying space $|M|$ by these sets

$$|M| = \bigcup_{j=0}^{k} (\cup_{i_j \in I_j} D(j)_{i_j})$$

Proposition 2.3.14. For each $j \in \{0, ..., k\}$ and each $i \in I_j$, $D(j)_i$ is a locally closed set and the restricted sheaf $(D(j)_i, \mathcal{G}_M|_{D(j)_i})$ gives rise to a (standard) real analytic manifold of dimension $k - j$. The family $\mathcal{D}_M = \{D(j)_{i_j}\}_{\substack{i_j \in I_j \\ j \in \{0,...,k\}}}$ is a locally finite stratification of $|M|$; that is, for every stratum $D(j)_i$, its boundary

$$\partial D(j)_i = \overline{D(j)_i} \setminus D(j)_i$$

is a locally finite union of strata of dimension not greater than the dimension of $D(j)_i$. Moreover, the boundary $\partial|M|$ of $|M|$ is the union of all strata of dimension strictly smaller than k.

Proof. All stated properties are true if they are true locally at each point of the manifold. Thus the proof follows from the definition of M as being locally isomorphic to open submanifolds of the local standard model $\mathbb{L}^k$ after checking that proposition is true for the (finite) stratification $\mathcal{D}_{\mathbb{L}^k}$ of $\mathbb{R}^k_{\geq 0}$. □

2.3.2 Local expression of morphisms

Let $M = (|M|, \mathcal{G}_M)$ be a $\mathcal{G}$-manifold. By definition, each $p \in |M|$ is in the domain of a chart (U, ϕ) where the range of ϕ is an open subset of $\mathbb{R}^k_{\geq 0}$. As a consequence of Proposition 2.3.11, we can choose such a chart *centered* at p, i.e., such that $\phi(p) = (0, ..., 0)$ if and only if $m_p = k$. However, by proposition 2.3.5, if we want to have always a chart centered at any given point, we can think that there is not a single local model for standard analytic manifold of a given dimension k, namely $\mathbb{L}^k$, but several ones, $\mathbb{L}^m \times \mathbb{R}^n$ with $m + n = k$.

Let $M = (|M|, \mathcal{O}_M)$ and $N = (|N|, \mathcal{O}_N)$ be standard analytic manifolds and $\varphi : |M| \to |N|$ a continuous map which induces a morphism from M to N. Let $p \in |M|$ and $\varphi(p) \in |N|$. We want to investigate how is the local expression of the morphism φ when we take local coordinates centered at p and at $\varphi(p)$.

More precisely, consider a local chart at p, i.e. an isomorphism $\phi : M|_{U_p} \to \mathbb{L}^{m_p} \times \mathbb{R}^{n_p}|_{U_0}$ where U_p is a neighborhood of p in $|M|$ and U_0 is a neighborhood of 0 in $\mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^{n_p}$, and consider, correspondingly, a local chart $\psi : N|_{V_{\varphi(p)}} \to V_0$ at $\varphi(p)$ (one can chose U_0 and V_0 to be the whole space, according to proposition 2.3.5).

$$\begin{array}{ccc} M|_{U_p} & \xrightarrow{(\varphi,\varphi^\sharp)} & N|_{V_{\varphi(p)}} \\ \downarrow{\scriptstyle (\phi,\phi^\sharp)} & & \downarrow{\scriptstyle (\psi,\psi^\sharp)} \\ \mathbb{L}^{m_p} \times \mathbb{R}^{n_p} & & \mathbb{L}^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}} \end{array}$$

Then, the map $h := \psi \circ \varphi \circ \phi^{-1} : \mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^{n_p} \to \mathbb{R}^{m_{\varphi(p)}}_{\geq 0} \times \mathbb{R}^{n_{\varphi(p)}}$ is $\mathcal{G}$-analytic at $\underline{0} \in \mathbb{R}^{m_p+n_p}$.

Reciprocally, any such continuous map $h : U_0 \to V_0$ that induces a morphism (resp. isomorphism) $h : \mathbb{L}^{m_p} \times \mathbb{R}^{n_p}|_{U_0} \to \mathbb{L}^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}}|_{V_0}$ gives rise, by reversing the charts ϕ and ψ to a morphism (resp. isomorphism) from an open submanifold of M containing p to an open submanifold of N containing $\varphi(p)$.

In the following proposition, we just describe the conditions for a continuous map h to give rise to a morphism or an isomorphism between the corresponding open submanifolds of the local models $\mathbb{L}^m \times \mathbb{R}^n = (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \mathcal{G}_{m,n})$.

Proposition 2.3.15. Let m, n, m', n' be natural numbers, $k = m+n$ and $k' = m'+n'$. Let U, V be open neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and in $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively. Let $h : U \to V$ be a continuous map with $h(\underline{0}) = \underline{0}$, and $h = (h_1, ..., h_{k'})$ be the components of h as a map ranging in $\mathbb{R}^{k'}$. Denote by $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and $(z, w) = (z_1, \ldots, z_{m'}, w_1, \ldots, w_{n'})$ the coordinates in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$. Then

i) h induces a morphism $(h, h^\sharp) : \mathbb{L}^m \times \mathbb{R}^n|_{U_0} \to \mathbb{L}^{m'} \times \mathbb{R}^{n'}|_{V_0}$ where U_0 and V_0 are open neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively, if and only if each h_j is $\mathcal{G}$-analytic at the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, and for $j = 1, 2, \ldots, m'$,

$$h_j(x, y) = x^{\alpha^j} g_j(x, y) = x_1^{\alpha_1^j} \cdots x_m^{\alpha_m^j} g_j(x, y)$$

for a certain $\alpha^j \in [0, \infty)^m$ and g_j a section of $\mathcal{G}_{m,n}$ with $g_j(x, y) > 0$ for any (x, y) close enough to the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, and the map $y \mapsto (h_{m+1}(0, y), \ldots, h_k(0, y))$ induces an analytic morphism from $\mathbb{R}^n$ to $\mathbb{R}^{n'}$.

ii) Assume that $k = k'$ and that h induces a morphism $(h, h^\sharp) : \mathbb{L}^m \times \mathbb{R}^n|_U \to \mathbb{L}^{m'} \times \mathbb{R}^{n'}|_V$. Then $(h, h^\sharp)$ is an isomorphism in the category $\mathcal{G}$ if and only if $m = m'$, $n = n'$, h is an homeomorphism, $y \mapsto (h_{m+1}(0, y), \ldots, h_k(0, y))$ induces an analytic automorphism of $\mathbb{R}^n$ and for any $j = 1, 2, \ldots, m$,

$$z_j = h_j(x, y) = x_{i(j)}^{a_j} g_j(x, y)$$

being $a_j > 0$, g_j an analytic function at $\underline{0}$ such that $g_j(x, y) > 0$ for any $(x, y) \in W$ for W a desirable neighborhood of $\underline{0}$ in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $j \to i(j)$ a permutation of $\{1, \ldots, m\}$.

Proof. For *i)*, suppose that h induces a morphism $(h, h^\sharp) : \mathbb{L}^m \times \mathbb{R}^n|_{U_0} \to \mathbb{L}^{m'} \times \mathbb{R}^{n'}|_{V_0}$ where U_0 and V_0 are open neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively. Since the projection maps $pr_j : (p_1, \ldots, p_k) \in \mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'} \to p_j \in \mathbb{R}$ are sections of $\mathcal{G}_{m',n'}$ over any open neighborhood of the origin, $h_j = pr_j \circ h = h^\sharp(pr_j)$ are $\mathcal{G}$-analytic at $\underline{0}$.

We have the diagram

$$\begin{array}{ccc} \mathcal{G}_{m',n',\underline{0}} & \xrightarrow{\quad h^\sharp_{\underline{0}} \quad} & \mathcal{G}_{m,n,\underline{0}} \\ F \uparrow & & \uparrow G \\ \mathbb{R}\{Z^*, W\}_{m',n'} & \xrightarrow{\phi := G^{-1} \circ h^\sharp_{\underline{0}} \circ F} & \mathbb{R}\{X^*, Y\}_{m,n} \end{array}$$

where F and G are defined as in 2.2.6. Notice that with the notations of 2.2.6 we can take θ equal to the identity for F and G. Thus, for $1 \leq j \leq m'$, $F(Z_j)$ is the germ at $\underline{0}$ of the projection map pr_j, which implies that $\phi(Z_j)$ is the Taylor expansion of h_j at $\underline{0}$. By proposition 1.1.20, since $\phi(Z_j)$ has an N^{th}-root for any $N \in \mathbb{N}$ ($\phi(Z_j^{1/N})$), there exists $\alpha^j \in [0, \infty)^m$ and a unit $U_j \in \mathbb{R}\{X^*, Y\}_{m,n}$ with $U_j(0, 0) > 0$ such that $\phi(Z_j) = X^{\alpha^j} U_j$. Let g_j denote the sum of U_j, then for any (x, y) close enough to the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ $g_j(x, y) > 0$ and by construction of G, $h_j(x, y) = x^{\alpha^j} g_j(x, y)$.

For $m'+1 \leq j \leq k'$, $h_j(0, y) = S_\epsilon(F^{-1}(h_j))(0, y) = S_\epsilon(F^{-1}(h_j)(0, Y))$ for an $\epsilon > 0$ small enough, which implies that $y \mapsto h_j(0, y)$ is analytic. Then, the map $y \mapsto (h_{m+1}(0, y), \ldots, h_k(0, y))$ induces an analytic morphism form $\mathbb{R}^n$ to $\mathbb{R}^n$.

To prove the reciprocal of part *i)*, let U_0 be an open neighborhood of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ such that $g_j(x, y) > 0$ for any $(x, y) \in U_0$ and any j with $1 \leq j \leq m'$; such that the map $y \mapsto h_{m'+j}(0, y)$ is analytic at any $y \in pr_{\mathbb{R}^n}(U_0)$ for any $j \in \{1, \ldots, n'\}$ where $pr_{\mathbb{R}^n} : (x, y) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \mapsto y \in \mathbb{R}^n$; and such that h_j is $\mathcal{G}$-analytic at any $(x, y) \in U_0$. Put $V_0 := h(U_0)$. Notice that by the hypothesis $h_j(x, y) = x^{\alpha^j} g_j(x, y)$, for $1 \leq j \leq m'$, $V_0 \subseteq \mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$. If W is an open subset of V_0 and $f \in \mathcal{G}_{m',n'}|_{V_0}(W)$, by proposition 1.2.21 we can compose the Taylor expansions of f and h at any $(x, y) \in h^{-1}(W)$, so $f \circ h \in \mathcal{G}_{m,n}|_{U_0}(h^{-1}(W))$.

To prove part *ii)*, if h induces an isomorphism, h is an homeomorphism so $k = k'$, and since $h(\underline{0}) = \underline{0}$, by proposition 2.3.11, $m = m'$ and so $n = n'$. By part *i)*, the coordinates x and z are related via h by the equations

$$\left\{ \begin{array}{l} z_1 = x^{\alpha^1} g_1(x, y) \\ z_2 = x^{\alpha^2} g_2(x, y) \\ \cdots \\ z_m = x^{\alpha_m} g_m(x, y) \end{array} \right. \qquad \left\{ \begin{array}{l} x_1 = z^{\beta^1} f_1(z, w) \\ x_2 = z^{\beta^2} f_2(z, w) \\ \cdots \\ x_m = z^{\beta_m} f_m(z, w) \end{array} \right. \tag{2.7}$$

Define the matrices $A := (\alpha_i^j)_{1 \leq i,j \leq m}$ and $B := (\beta_i^j)_{1 \leq i,j \leq m}$. Since $h \circ h^{-1} = id$, the product of the matrices AB is equal to the identity matrix Id_m. In particular the matrix A is invertible so, for all $j \in \{1, \ldots, m\}$, $(\alpha_1^j, \ldots, \alpha_m^j) \neq (0, \ldots, 0)$.

Now we claim that

$$\alpha_i^j \neq 0 \Rightarrow \alpha_i^k = 0 \tag{2.8}$$

for all $i \in \{1, \ldots, m\}$ and $j \neq k$, $j, k \in \{1, \ldots, m\}$: if there existed $i \in \{1, \ldots, m\}$ and $j \neq k$, such that $\alpha_i^j \neq 0 \neq \alpha_i^k$ we would have, by 2.7, that the homeomorphism $h|_{(\mathbb{R}^m_{\geq 0})}$ sends $\{x_i = 0\}$ to $\{z_k = 0 = z_l\}$ which is not possible because the Invariance of domain theorem.

Then, by (2.8), the columns of A have only one component not equal to zero. As A is invertible, the rows too. This implies that for any $j \in \{1, \ldots, m\}$, there exists a unique $i(j) \in \{1, \ldots, m\}$ such that $\alpha_{i(j)}^j \neq 0$ and that if $j \neq k$, $i(j) \neq i(k)$, so $j \mapsto i(j)$ is a permutation of $\{1, \ldots, m\}$. The rest of the properties and the reciprocal follow from part *i)* and the implicit functions theorem 1.2.15. $\square$

Definition 2.3.16. Let $M = (|M|, \mathcal{G}_M)$ be a $\mathcal{G}$-analytic manifold and $p \in |M|$. Given a local chart $(U, \varphi = (x_1, \ldots, x_k))$ of M at p and $f \in \mathcal{G}(U)$ we say that f **is monomial at** p **with respect to the local chart** (U, φ) **(or with respect to the coordinates** x**)** if the Taylor expansion of the germ $\mathbf{f}_p$ with respect to the coordinates x (see definition 2.3.7) is of monomial type. In other words, that we can write locally f in the coordinates x as

$$f(x) = x_1^{\alpha_1} \cdots x_m^{\alpha_m} g(x), \quad x \in U,$$

where $g \in \mathcal{G}(U)$, vanishes nowhere in U, and each $\alpha_i \in [0, \infty)$. We say that f **is monomial at the point** p if its monomial with respect to some local chart at p. Finally, we say that f **is (locally) monomial** if it is monomial at every point of M.

Remark 2.3.17. *i)* f is monomial at any point $p \in M$ such that $f(p) \neq 0$ (by definition).

ii) f is monomial at p if and only if there exists local coordinates such that the Taylor expansion of f at p with respect to these coordinates (see 2.3.7) is of monomial type. However, we can chose different local coordinates for which the Taylor expansion of f at p is not of monomial type. For instance $y_1 \in \mathbb{R}\{y_1, y_2\}$ is of monomial type but the change of coordinates $y_1 = z_1 + z_2, y_2 = z_2$ makes it not monomial.

iii) f is locally monomial if and only if $\{f = 0\}$ has normal crossing, that is, at any point p of $\{f = 0\}$ there are local coordinates such that $\{f = 0\}$ is locally at p the union of some coordinate planes.

iv) As a consequence of Lemma 1.1.12, if $f = hg$ and f is locally monomial, then so are h and g.

v) If $f_j = x^{\alpha^j} u_j(x)$ is locally monomial for $j = 1, 2, 3$, then either $\alpha^1 \leq \alpha^2$ or $\alpha^2 \leq \alpha^1$ (see, for instance Lemma 4.7 of [2]).

Proposition 2.3.18. If $f \in \mathcal{G}(M)$ is monomial at a point $p \in M$ then there exists a neighborhood U of p such that f is monomial at any point of U.

Proof. By definition there are coordinates around p, $(U, \varphi = (x_1, x_2, \ldots, x_{m_p}, y_1, y_2, \ldots, y_{n_p}))$ such that the function $f : U \to \mathbb{R}$ is given by $f(x, y) = x^\alpha y^\beta h(x, y)$ where $h \in \mathcal{G}(U)$ vanishes nowhere in U and $\alpha \in [0, \infty)^{m_p}$, $\beta \in \mathbb{N}^{n_p}$. We can moreover assume that $f \circ \varphi^{-1} \in \mathcal{G}_{\mathbb{L}^k}(\varphi(U))$ is the sum of a convergent series of monomial type

$$s(X, Y) = X^\alpha Y^\beta H(X, Y) \in \mathbb{R}\{X^*, Y\}$$

where H is a unit, and that $\varphi(U)$ is contained in the domain of convergence of s. We can see that the Taylor expansion $T_a s$ of s at any point $a \in U$ is again a series of monomial type. The proof is consequence then of Theorem 1.2.23. $\square$

Definition 2.3.19. Let $\varphi : M \to N$ be a morphism of $\mathcal{G}$-manifolds. We say that φ is **locally monomial** if for any $p \in |M|$ there exists local coordinates $(U, \phi = (x_1, \ldots, x_k))$ centered at p such that all the components of φ are monomial at p with respect to these coordinates.

Examples 2.3.20. *i)* The morphism $(x, y) \in \mathbb{L} \times \mathbb{R} \to (x, x + y) \in \mathbb{L} \times \mathbb{R}$ is locally monomial because with respect to the new coordinates $(x', y') = (x, x + y)$ its components are monomial.

ii) As a consequence of proposition 2.3.15 the morphism $(x, y) \in \mathbb{L} \times \mathbb{R} \to (x, x^2(x^2 + y^2)) \in \mathbb{L} \times \mathbb{R}$ is not locally monomial.

2.3.3 Products

Proposition 2.3.21. $\mathcal{G}$ is a category with product.

In order to prove this proposition, we state first the version for open submanifolds of the local models $\mathbb{L}^k$:

Lemma 2.3.22. Let $V_1 \subset \mathbb{R}^{k_1}_{\geq 0}$ and $V_2 \subset \mathbb{R}^{k_2}_{\geq 0}$ be open sets and let $V = V_1 \times V_2 \subset \mathbb{R}^k_{\geq 0}$, where $k = k_1 + k_2$. Considering V_1, V_2 and V as open submanifolds of $\mathbb{L}^{k_1}$, $\mathbb{L}^{k_2}$ and $\mathbb{L}^k$ respectively, we have that V, together with the usual projections $p_i : V \to V_i$, $i = 1, 2$, is a product of V_1 and V_2.

Proof. Let A be a $\mathcal{G}$-manifold and $\alpha_i : A \to V_i$ morphisms. Since V is the topological product of V_1 and V_2, there exists a unique continuous map $\Phi : A \to V$ such that $p_i \circ \Phi = \alpha_i$. Let us see that Φ is a morphism in the category of $\mathcal{G}$-manifolds. It suffices to see that for every $a \in |A|$ and for every germ $\mathbf{f}_\Phi(a) \in \mathcal{G}_{V,\Phi(a)}$ of a $\mathcal{G}$-analytic function f at $\Phi(a) \in V$, the germ of the composition $f \circ \Phi$ belongs to $\mathcal{G}_{A,a}$. Put $\Phi(a) = (b_1, b_2)$ where $b_i \in V_i$. The induced map on the stalks $\alpha_\mathbf{i} : G_{V_i,b_i} \to \mathcal{G}_{A,a}$ by the morphism α_i can be seen, taking local coordinates at $b_i \in V_i$ and $a \in A$, as a morphism between algebras of convergent generalized power series

$$\tilde{\alpha_\mathbf{i}} : \mathbb{R}\{(X^{(i)})^*, Y^{(i)}\}_{m(b_i),n(b_i)} \to \mathbb{R}\{Z^*, T\}_{m(a),n(a)},\ i = 1, 2.$$

Since $X^{(i)}_j$ has all N^{th}-roots, its image by $\tilde{\alpha_\mathbf{i}}$ has also all N^{th}-roots and, by proposition 1.1.20, it is of monomial type as a series in $\mathbb{R}\{T\}\{Z^*\}$, namely

$$\tilde{\alpha_\mathbf{i}}(X^{(i)}_k = M^{(i)}_k U^{(i)}_k \tag{2.9}$$

where $M^{(i)}_k$ is a monomial in the Z variables and $U^{(i)}_k(0) \neq 0$.

On the other hand, if we put $X = (X^{(1)}, X^{(2)})$, $Y = (Y^{(1)}, Y^{(2)})$, then $\mathcal{G}_{V,\Phi(a)}$ is isomorphic to $\mathbb{R}\{X^*, Y\}$ and under this isomorphism, the morphism induced by p_i on the corresponding stalks $\mathcal{G}_{V_i,b_i}$ and $\mathcal{G}_{V,\Phi(a)}$ is just the inclusion $\mathbb{R}\{(X^{(i)})^*, Y^{(i)}\} \subset \mathbb{R}\{X^*, Y\}$ that assigns a series in variables $(X^{(i)})^*, Y^{(i)}$ to the same series but considered in variables X^*, Y.

Now, if $\mathbf{f} \in \mathcal{G}_{V,\Phi(a)}$ is a $\mathcal{G}$-analytic germ, it is the germ of the sum of its Taylor expansion $s = T_{\Phi(a)}\mathbf{f} \in \mathbb{R}\{X^*, Y\}$ (up to a permutation of variables, see 2.2.7). Using Proposition 1.2.10 and (2.9), the series

$$t(Z, T) = s(M^{(1)}_1 U^{(1)}_1, ..., M^{(1)}_{m(b_1)} U^{(1)}_{m(b_1)}, M^{(1)}_1 U^{(1)}_1, ..., M^{(2)}_{m(b_2)} U^{(2)}_{m(b_2)}, T)$$

belongs to $\mathbb{R}\{Z^*, T\}$. By construction, the germ of its sum is the composition $\mathbf{f} \circ \Phi$ viewed in the local chart that we have considered for A at a. Thus, this composition is $\mathcal{G}$-analytic as was to be proved. □

Proof of Proposition 2.3.21.- Let M_1 and M_2 be two generalized analytic manifolds of dimension k_1 and k_2 respectively. We start by constructing a triplet $\mathcal{P} = (P, p_1 : P \to M_1, p_2 : P \to M_2)$ as a candidate to be the product of M_1 and M_2.

It is logical to pick as underlying topological space for P the cartesian product $|P| = |M_1| \times |M_2|$ with the product topology and as morphisms $p_1 : P \to M_1, p_2 : P \to M_2 \in$Morph$(\mathcal{G})$ the morphisms induced by the projections maps

$$p_1 = pr_1 : |P| = |M_1| \times |M_2| \to |M_1| \qquad p_2 = pr_2 : |P| = |M_1| \times |M_2| \to |M_2|$$

We construct now the sheave $\mathcal{G}_P$ that will determine the structure of a $\mathcal{G}$-manifold for P. In order to define the sheaf G_P as a subsheaf of the sheaf of continuous functions, it is enough to associate to any element of a basis of open sets of the topology of $|P|$ a $\mathbb{R}$-subalgebra of continuous functions with. After that, we need to show that with this structure, $|P|$ is locally isomorphic to $\mathbb{L}^{k_1+k_2}$.

As a basis of open sets of the topological product $|M_1| \times |M_2|$, we can consider the set

$$\mathcal{B} = \{U_1 \times U_2 \subseteq |M_1| \times |M_2| : U_i \subset |M_i| \text{ is the domain of a coordinate chart }, i = 1, 2\}$$

Let $U_1 \times U_2 \in \mathcal{B}$. Then $M_i|_{U_i}$ is isomorphic to $\mathbb{L}^{k_i}|_{V_i}$ via φ_i for $i = 1, 2$. Let Φ be the map

$$\Phi = (\varphi_1, \varphi_2) : U_1 \times U_2 \to V_1 \times V_2 \subseteq \mathbb{R}^{k_1}_{\geq 0} \times \mathbb{R}^{k_2}_{\geq 0} = \mathbb{R}^{k_1+k_2}_{\geq 0}$$

Put $k = k_1 + k_2$. Then Φ is an homeomorphism and $V_1 \times V_2$ is an open subset of $\mathbb{R}^k_{\geq 0}$. Let us define

$$\Gamma_\Phi(U_1 \times U_2, \mathcal{G}_P) = \{f : U_1 \times U_2 \to \mathbb{R} / f \circ \Phi^{-1} \in \Gamma(V_1 \times V_2, \mathcal{G}_{\mathbb{L}^k})\}$$

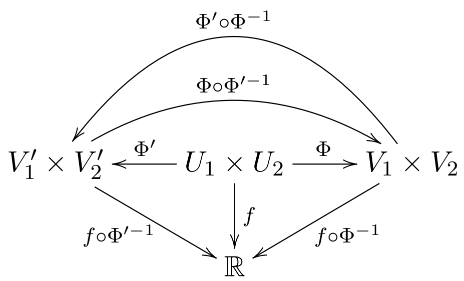
First of all let us prove that this definition does not depend on the morphisms φ_1, φ_2 such that (U_1, φ_1), (U_2, φ_2) are local charts which will endow the topological product with a well defined structure of generalized analytic manifold. Let

$$\varphi'_i : U_i \to V'_i$$

be isomorphisms between $M_i|_{U_i}$ and $\mathbb{L}^{k_i}|_{V_i}$ and we define

$$\Phi' = (\varphi'_1, \varphi'_2) : U_1 \times U_2 \to V'_1 \times V'_2$$

then $\Gamma_\Phi = \Gamma_{\Phi'}$. We can illustrate the situation with the diagram



The result is clear once we notice that $\Phi \circ \Phi'^{-1}$ and $\Phi' \circ \Phi^{-1}$ are morphisms of $\mathcal{G}$-manifolds (thus both isomorphisms), which can be seen using the definition of product and Lemma 2.3.22. So $M_1 \times M_2 = (|M_1| \times |M_2|, \mathcal{G}_{M_1 \times M_2}) \in$Obj$(\mathcal{G})$. Remark that the natural projections $p_i : |M_1| \times |M_2| \to |M_i|$ are morphisms from $M_1 \times M_2$ to M_i. To finish, we have to prove that $(M_1 \times M_2, p_1, p_2)$ is a solution of the universal problem. But this is easy: if A is a $\mathcal{G}$-manifold and $\alpha_i : A \to M_i$ are morphisms for $i = 1, 2$, the map $\Phi : A \to M_1 \times M_2$ defined by $\Phi = (\alpha_1, \alpha_2)$ is continuous and induce a morphism of $\mathcal{G}$-manifolds since this property is a local one and locally $M_1 \times M_2$ has the structure of product, by definition.

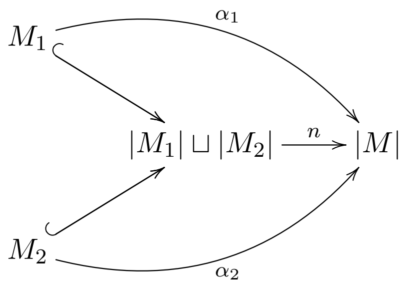
2.3.4 Gluing manifolds.

Proposition 2.3.23. $\mathcal{G}$ is a category with gluing.

Proof. Let $|M|$ be the topological space obtained by the quotient of the topological disjoint union $|M_1| \sqcup |M_2|$ by the equivalence relation

$$m_1 \sim m_2 \text{ if } m_1 = m_2 \text{ or } m_1 \in U_1, \;\; m_2 \in U_2 \text{ and } m_2 = h(m_1)$$

Denote by $\pi : |M_1| \sqcup |M_2| \to |M|$ be the quotient map. For $i = 1, 2$ define $\alpha_i : |M_i| \to |M|$ as the composition of the inclusion $|M_i| \subset |M_1| \sqcup |M_2|$ with the quotient map.



Then we have that α_i is continuous, that its image $W_i = \alpha_i(|M_i|)$ is an open set of $|M|$, that $\alpha_i : |M_i| \to W_i$ is a homeomorphism and that $|M| = W_1 \cup W_2$. Now we want to define a sheaf of continuous functions (on local algebras) $\mathcal{G}_M$ on $|M|$ such that $M = (|M|, \mathcal{G}_M)$ is a $\mathcal{G}$-manifold and α_i is a morphism of $\mathcal{G}$-manifolds. Using a general construction of gluing ringed spaces (see the Appendix for details), it suffices to define such a sheaf $\mathcal{G}_{W_i}$ on W_i for $i = 1, 2$ such that, for any open set $V \subset W_1 \cap W_2$, we have $G_{W_1}(V) = G_{W_2}(V)$: explicitly, $\mathcal{G}_M$ will be given by

$$\mathcal{G}_M(U) = \{f : U \to \mathbb{R} : f \circ \alpha_i \in \mathcal{G}_{M_i}(\alpha_i^{-1}(U)), \;\; i = 1, 2\}$$

Define

$$\text{if } V \subseteq W_i \text{ is open, } \mathcal{G}_{W_i}(V) = \{f : V \to \mathbb{R} : f \circ \alpha_i \in \mathcal{G}_{M_i}(\alpha_i^{-1}(V))\}.$$

With this definition, M_i is isomorphic (in $\mathfrak{C}$) to W_i via α_i. Now, let $V \subset W_1 \cap W_2$ be an open set. The homeomorphism $\alpha_1^{-1} \circ \alpha_2$ induces an isomorphism (of $\mathcal{G}$-manifolds) between the open submanifold $\alpha_2^{-1}(V)$ of M_2 and $\alpha_1^{-1}(V)$ of M_1. Thus, if $f : V \to R$ is continuous, we have

$$f \circ \alpha_1 \in G_{M_1}(\alpha_1^{-1}(V)) \Leftrightarrow f \circ \alpha_2 \in G_{M_1}(\alpha_2^{-1}(V))$$

which shows $G_{W_1}(V) = G_{W_2}(V)$, as required. We claim that $M = (|M|, \mathcal{G}_M)$ is the gluing of M_1, M_2 with respect to the open immersions φ_1, φ_2. To see this, let (β_1, β_2, T) be a triplet where $T = (|T|, \mathcal{G}_T)$ is a $\mathcal{G}$-manifold and $\beta_i : M_i \to T$ are open immersions such that $\beta_1 \circ \varphi_1 = \beta_2 \circ \varphi_2$. We have to show that there exists an unique morphism $f : M \to T$ such that $\beta_i = f \circ \alpha_i$ for $i = 1, 2$. Uniqueness of f comes from the fact that $|M|$ is the solution of the same universal problem in the category of topological spaces: $f : |M|$ must be defined by

$$f(p) = \alpha_1^{-1}(p) \text{ for } p \in W_1 \text{ and } f(p) = \alpha_2^{-1}(p) \text{ for } p \in W_2$$

We just have to prove that f is a morphism of $\mathcal{G}$-manifolds. This is a property that we can check locally. But f is locally defined either by $\beta_1 \circ \alpha_1^{-1}$ on W_1 or by $\beta_2 \circ \alpha_2^{-1}$ on W_2, both morphisms in the category of $\mathcal{G}$-manifolds. □

2.3.5 An example of an exotic generalized manifold.

Let $N = \mathbb{L}^1 \times \mathbb{R} \backslash \{0\}$, $\mathcal{D}_1 = \mathcal{D}_2 = \mathbb{L}^1 \times \mathbb{R}$ and $\phi_1 : N \to \mathcal{D}_1$, $\phi_{2,\alpha} : N \to \mathcal{D}_2$ be defined respectively by $\phi_1(x, y) = (x, y)$ and $\phi_{2,\alpha}(x, y) = (x, 1/y)$ if $y > 0$, $\phi_{2,\alpha}(x, y) = (x^\alpha, 1/y)$ if $y < 0$. Notice that ϕ_1 and $\phi_{2,\alpha}$ are open immersions so we can define $\mathcal{C}_\alpha$ as the gluing of $\mathcal{D}_1$ and $\mathcal{D}_2$ with respect to ϕ_1 and $\phi_{2,\alpha}$.

Remark 2.3.24. Notice that $\mathcal{C}_1$ is nothing but is the usual cylinder with the product structure $C := \mathbb{R}_{\geq 0} \times \mathbb{S}^1$ in $\mathcal{O}$. Then, the underlying topological space of $\mathcal{C}_\alpha$ is homeomorphic to the usual cylinder, the underlying topological space of $\mathcal{C}_1$.

We are going to show now that the generalized manifolds $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ are not isomorphic if $\alpha \neq \beta$, although they have homeomorphic underlying spaces. For the shake of simplicity, we just consider $\beta = 1$ and $\alpha \neq 1$.

Suppose that there exists an isomorphism

$$f : \mathcal{C}_\alpha \to \mathcal{C} = \mathcal{C}_1$$

By the very construction of the space $|\mathcal{C}_\alpha| = C_\alpha$ as the quotient space of $D_1 \sqcup D_2$ by the relation $\sim_\alpha$, if $\pi_\alpha : D_1 \sqcup D_2 \to C_\alpha$ denotes the quotient map, then $U_{\alpha,j} = \pi_\alpha(D_j)$, $j = 1, 2$, is an open set, $C_\alpha = U_{\alpha,1} \cup U_{\alpha,2}$ and we have local charts

$$\phi_{\alpha,j} = (x_{\alpha,j}, y_{\alpha,j}) : U_{\alpha,j} \to \mathbb{R}_{\geq 0} \times \mathbb{R}$$

where $x_{\alpha,j} = \phi_{\alpha,j} \circ pr_{\mathbb{R}_{\geq 0}} \in \mathcal{G}|_{U_{\alpha,j}}(U_{\alpha,j})$ and $y_{\alpha,j} = \phi_{\alpha,j} \circ pr_{\mathbb{R}} \in \mathcal{G}|_{U_{\alpha,j}}(U_{\alpha,j})$ for $j = 1, 2$. Change of coordinates is given in $U_{\alpha,1} \cap U_{\alpha,2}$ by

$$\phi_{\alpha,1} \circ \phi_{\alpha,2}^{-1}(a, b) = \begin{cases} (a, 1/b) & \text{if} \quad b > 0 \\ (a^\alpha, 1/b) & \text{if} \quad b < 0 \end{cases}$$

Remark that in $\mathcal{C}$ there exists two open subsets $U_{1,1}$, $U_{1,2}$ covering C isomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$:

$$\phi_{1,j} = (x_{1,j}, y_{1,j}) : U_{1,j} \to \mathbb{R}_{\geq 0} \times \mathbb{R}$$

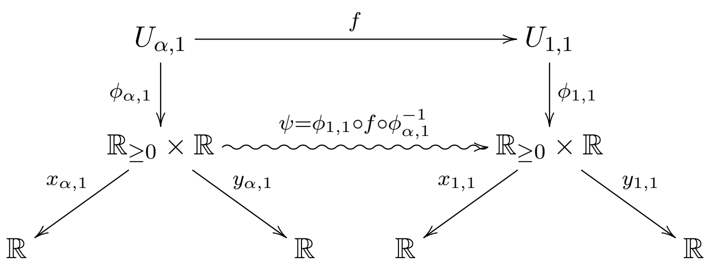
where $x_{1,j} = \phi_{1,j} \circ pr_{\mathbb{R}_{\geq 0}} \in \mathcal{O}|_{U_{1,j}}(U_{1,j})$ and $y_{1,j} = \phi_j \circ pr_{\mathbb{R}} \in \mathcal{O}|_{U_{1,j}}(U_{1,j})$ for $j = 1, 2$ such that, in $U_{1,1} \cap U_{1,2}$, the change of coordinates is given by

$$\phi_{1,1} \circ \phi_{1,2}^{-1}(a, b) = (a, 1/b)$$

Denote by the same letter the underlying homeomorphism $f : C_\alpha = |\mathcal{C}_\alpha| \to C = |\mathcal{C}_1|$. Let $p \in \partial C_\alpha$. Then $f(p) \in \partial C$. Suppose for instance that $p \in U_{\alpha,1}$ and that $f(p) \in U_{1,1}$. Using Proposition 2.3.15 on local expressions of isomorphisms between $\mathcal{G}$-manifolds, we can express f in these charts (in a neighborhood of p) as:

$$\phi_{1,1} \circ \varphi \circ \phi_{\alpha,1}^{-1}|_{\Omega_p}(x_{\alpha,1}, y_{\alpha,1}) = ((x_{\alpha,1})^\beta u(x_{\alpha,1}, y_{\alpha,1}), h(x_{\alpha,1}, y_{\alpha,1})) \tag{2.10}$$

where $\beta > 0$ and u, h are $\mathcal{G}$-functions in a neighborhood of $\phi_{\alpha,1}(p) = (0, y_{\alpha,1}(p))$ such that $u(0, y_{\alpha,1}(p)) > 0$ and $y \mapsto h(0, y)$ is an analytic isomorphism from a neighborhood of $y_{\alpha,1}(p)$ to a neighborhood of $y_{1,1}(f(p))$ in $\mathbb{R}$.



Notice that the exponent $\beta > 0$ in the expression (2.10) above depends a priori on p and on the charts $(U_{\alpha,1}, \phi_{\alpha,1})$ at p and $(U_{1,1}, \phi_{1,1})$ at $f(p)$ chosen in order to express locally the isomorphism f. We should write then (momentarily):

$$\beta = \beta(f, p, U_{\alpha,1} \to U_{1,1,}). \tag{2.11}$$

Claim. β is locally constant.

Proof of the claim.- Consider the Taylor expansion of the first coordinate $(x_{\alpha,1})^{\beta}u$ in (2.10) at the point $(0, y_{\alpha,1}(p))$; i.e. a series $s \in \mathbb{R}\{X^*, Y\}$ for some variables X, Y (notice that there is no ambiguity of the Taylor expansion here as was discussed in 2.2.7 since X and Y are 1-dimensional variables). Then s is of the form

$$s = X^{\beta}U(X, Y)$$

where U is a unit. This observation, together with Proposition 1.2.23 gives the proof of the claim.

Notice now that, if $p \in U_{\alpha,1}$ and that $f(p) \in U_{1,1} \cap U_{1,2}$ is in the domain of the two charts, then, $\beta(f, p, U_{\alpha,1} \to U_{1,1}) = \beta(f, p, U_{\alpha,1} \to U_{1,2})$ because in that domain we have $y_{1,1} = y_{1,2}$ for the second components of these chart, by construction of $\mathcal{C}_1$. So we have proved that $\beta(f, p, U_{\alpha,1} \to U_{1,i})$ **does not depend on** $i = 1, 2$ as long as $f(p) \in U_{1,1} \cap U_{1,2}$. We simply use the notation $\beta(f, p, U_{\alpha,1})$ for this number. Define analogously $\beta(f, p, U_{\alpha,2})$. Let now $p_0 := \phi_{\alpha,1}(0, 0)$. Note that $p_0 \notin U_{\alpha,2}$. Let $\beta_0 := \beta(f, p_0, U_{\alpha,1})$. By construction of $\mathcal{C}_{\alpha}$, if $p \in U_{\alpha,1} \cap U_{\alpha,2} \cap \partial C_{\alpha}$

$$\beta(f, p, U_{\alpha,1}) = \begin{cases} \beta(f, p, U_{\alpha,2}) & \text{if} \quad y_{\alpha,1}(p) > 0 \\ \alpha\beta(f, p, U_{\alpha,2}) & \text{if} \quad y_{\alpha,2}(p) < 0 \end{cases} \tag{2.12}$$

Let for $\epsilon > 0$ sufficiently small, $p_{\epsilon}^{+} := \phi_{\alpha,1}^{-1}(0, \epsilon), p_{\epsilon}^{-} := \phi_{\alpha,1}^{-1}(0, -\epsilon) \in U_{\alpha,1} \cap U_{\alpha,2} \cap \partial C_{\alpha}$. Then $\beta_0 = \beta(f, p_{\epsilon}^{+}, U_{\alpha,1}) = \beta(f, p_{\epsilon}^{-}, U_{\alpha,1})$ because β is locally constant. On the other hand, by (2.12), $\beta_0 = \beta(p_{\epsilon}^{+}, U_{\alpha,1}) = \beta(p_{\epsilon}^{+}, U_{\alpha,2})$. Also, $\beta(p_{\epsilon}^{-}, U_{\alpha,1}) = \beta(p_{\epsilon}^{+}, U_{\alpha,2}) = \beta_0$ because they are connected in U_2 but, again by the formula (2.12) above, $\beta(p_{\epsilon}^{-}, U_{\alpha,1}) = \alpha\beta(p_{\epsilon}^{-}, U_{\alpha,2})$ which implies that $\beta_0 = \alpha\beta_0$. Contradiction.

2.4 Standardizations.

Notice that $\mathcal{O}_{\mathbb{A}_{+}^{k}}$ is a subsheaf of $\mathcal{G}_{\mathbb{L}^k}$ over $\mathbb{R}_{\geq 0}^{k}$: if a function is the restriction of an analytic function to an open subset of $\mathbb{R}_{\geq 0}^{k}$ its germ at any point is the germ of the sum of a convergent power series, thus a generalized power series; this shows that this function is also $\mathcal{G}$-analytic.

In other words, the identity map $Id : \mathbb{R}_{\geq 0}^{k} \to \mathbb{R}_{\geq 0}^{k}$ induces a morphism from $\mathbb{L}^k = (\mathbb{R}_{\geq 0}^{k}, \mathcal{G}_{\mathbb{L}^k}) \to \mathbb{A}_{+}^{k} = (\mathbb{R}_{\geq 0}^{k}, \mathcal{O}_{\mathbb{A}_{+}^{k}})$ in the category $\mathfrak{C}$ of locally ringed spaces. We can also interpret this as saying that we have "enriched" the structure of the model of analytic manifold with boundary and corners $\mathbb{A}_{+}^{k}$ to an structure of $\mathcal{G}$-analytic manifold by "adding" the generalized analytic functions to the (standard) analytic ones.

In this section we describe and analyze this "enrichment" process for any analytic manifold with boundary and corners.

Proposition-definition 2.4.1. Let $A = (|A|, \mathcal{O}_A)$ be an analytic manifold with boundary and corners. Let $\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in I}$ be an $\mathcal{O}$-atlas of A. Then the subsheaf $\mathcal{G}_A$ of the sheave of continuous functions over $|A|$, whose sections over an open of $|A|$, $U \subseteq |A|$ are

$$(f : U \to \mathbb{R}) \in \mathcal{G}_A(U) \Leftrightarrow f|_{U \cap U_i} \circ \varphi_i^{-1}|_{\varphi_i(U \cap U_{ij})} \in \mathcal{G}_{\mathbb{L}^k}(\varphi_i(U \cap U_i)) \;\; \forall i \in I \text{ with } U \cap U_i \neq \emptyset.$$

$$\begin{array}{ccc} f|_{U\cap U_i} : U \cap U_i & \xrightarrow{\quad f \quad} & \mathbb{R} \\ {\scriptstyle \varphi_i}\downarrow & \nearrow {\scriptstyle \tilde{f}_i = f\circ\varphi_i^{-1} \in \mathcal{G}_{\mathbb{L}^k}(V_i)} & \\ \varphi_i(U\cap U_i) \subseteq V_i & & \end{array}$$

does not depend on the chosen atlas $\mathcal{U}$ and endows $|A|$ with a structure of $\mathcal{G}$-analytic manifold $A^e = (|A|, \mathcal{G}_A)$ such that the identity in $|A|$ induces a morphism

$$(Id_{|A|}, Id^{\sharp}_{|A|}) : A^e \to A$$

in the category $\mathfrak{C}$ of locally ringed spaces.

We will say that the $\mathcal{G}$-manifold A^e is the **enrichment** of the (standard) manifold A.

Proof. If $\{(W_j, \psi_j)\}_{j\in J}$ is another analytic atlas of A let us see that the sheaf over $|A|$, $\mathcal{G}'_A$ defined over any open of $|A|$, $U \subseteq |A|$ by

$$(f : U \to \mathbb{R}) \in \mathcal{G}'_A(U) \Leftrightarrow f|_{U\cap W_j} \circ \psi_j^{-1}|_{\psi_j(U\cap W_j)} \in \mathcal{G}_{\mathbb{L}^k}(V'_j)$$

for all $j \in J$ such that $U \cap W_j \neq \emptyset$ is exactly the sheaf $\mathcal{G}_A$: let $U \subseteq |A|$ open. Then we have the following commutative diagram

$$\begin{array}{ccccc} & & \psi_j(U\cap U_i\cap W_j) & & \\ & \overset{\psi_j}{\nearrow} & & \overset{f\circ\psi_j^{-1}}{\searrow} & \\ f|_{U\cap U_i\cap W_j} : U\cap U_i\cap W_j & & \xrightarrow{\qquad f \qquad} & & \mathbb{R} \\ & \underset{\varphi_i}{\searrow} & & \underset{f\circ\varphi_i^{-1}}{\nearrow} & \\ & & \varphi_i(U\cap U_i\cap W_j) & & \end{array}$$

Since

$$f\circ\varphi_i^{-1} = f\circ\psi_j^{-1}\circ\psi_j\circ\varphi_i^{-1}$$

and

$$f\circ\psi_j^{-1} = f\circ\varphi_i^{-1}\circ\varphi_i\circ\psi_j^{-1}$$

we only have to show that the homeomorphism $\varphi_i \circ \psi_j^{-1}$ induces an isomorphism between the open $\mathcal{G}$-submanifolds $\mathbb{L}^k|_{\psi_j(U_i\cap W_j)}$ to $\mathbb{L}^k|_{\phi_i(U_i\cap W_j)}$. But, it induces, by definition of atlas, an isomorphism between $\mathbb{A}^k_+|_{V'_j}$ and $\mathbb{A}^k_+|_{V_i}$ so by proposition 2.1.18 it is locally monomial. By Proposition 2.3.15 we deduce that it induces a morphism between $\mathbb{L}^k|_{V'_j}$ and $\mathbb{L}^k|_{V_i}$, thus an isomorphism by taking its inverse.

On the other hand, similar arguments show that we can define alternatively

Lemma 2.4.2. $f \in \Gamma(U, \mathcal{G}_A)$ if and only if for every $p \in U$ there exists some $i \in I$ with $p \in U_i$ such that $f \circ \varphi_i^{-1}$ is $\mathcal{G}$-analytic at the point $\varphi_i(p) \in \mathbb{R}^k_{\geq 0}$.

This implies that the homeomorphisms φ_i induce isomorphisms of locally ringed spaces between $\mathcal{G}_A|_{U_i}$ and $\mathbb{L}^k|_{\varphi_i(U_i)}$, which shows that A^e is a $\mathcal{G}$-manifold. □

Remark 2.4.3. For A a standard analytic manifold we could give the definition of enrichment as a generalized analytic manifold $\tilde{A}$ with the same underlying topological space and such that the identity map induces a morphism $\tilde{A} \to A$. But with this definition, we would have several different $\mathcal{G}$-manifolds as possible enrichments of the same standard analytic manifold. As an example, consider $\mathbb{L}^2 = (\mathbb{R}^2_{\geq 0}, \mathcal{G}_{\mathbb{L}^2})$ with global coordinates (y_1, y_2) on $\mathbb{R}^2_{\geq 0}$. Let

$$\phi : (\mathbb{R}^2_{\geq 0})_{(y_1,y_2)} \to (\mathbb{R}^2_{\geq 0})_{(x_1,x_2)}$$

be the map defined by

$$\phi(y_1, y_2) = (y_1^2(y_1^2 + y_2^2), y_2)$$

It is an homeomorphism with inverse

$$\phi^{-1}(x_1, x_2) = (\sqrt{\frac{\sqrt{x_2^4 + 4x_1} - x_2^2}{2}}, x_2)$$

For $V \subseteq \mathbb{R}^2_{\geq 0}$ an open subset of $\mathbb{R}^2_{\geq 0}$ we define

$$\mathcal{G}'(V) := \phi_* \mathcal{G}(\phi^{-1}(V)) = \{g : V \to \mathbb{R} : g \circ \phi|_{\phi^{-1}(V)} \in \mathcal{G}_{\mathbb{L}^2}(\phi^{-1}(V))\}$$

With this definition, $(\mathbb{L}^2)' = (\mathbb{R}^2_{\geq 0}, \mathcal{G}') \in$Obj$(\mathfrak{C})$, that is, it is a locally ringed space on local algebras of continuous functions, and the homeomorphism $\phi : \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ induces an isomorphism of locally ringed spaces, because if $g \in \mathcal{G}'(V)$, $g \circ \phi \in \mathcal{G}_{\mathbb{L}^2}(\phi^{-1}(V))$, and if $h \in \mathcal{G}_{\mathbb{L}^2}(U)$, $h \circ \phi^{-1} \in \mathcal{G}'(\phi(U))$ because $h \circ \phi^{-1} \circ \phi = h \in \mathcal{G}_{\mathbb{L}^2}(\phi^{-1}(\phi(U)))$. This implies that $(\mathbb{L}^2)'$ is a generalized analytic manifold.

Notice that the sections on open sets of the quadrant $\mathbb{R}^2_{\geq 0}$ for the sheaf $\mathcal{G}'$ contains the analytic functions; i.e. that the identity map of the quadrant induces a morphism $(\mathbb{L}^2)' \to \mathbb{A}^2_+$. However, $\mathcal{G}' \neq \mathcal{G}_{\mathbb{L}^2}$, i.e., these sections do not consist on the generalized analytic functions on open sets (moreover, the identity map on the quadrant does not induce a morphism $\mathbb{L}^2 \to (\mathbb{L}^2)'$). In fact, if $\mathcal{G}' = \mathcal{G}_{\mathbb{L}^2}$ then the function $y_1^2(y_1^2 + y_2^2) = x_1 \circ \phi$ which is a section of $\mathcal{G}'$, would have all its N^{th}-roots which is not the case. This implies that $\mathbb{L}^2$ and $(\mathbb{L}^2)'$ are two different objects and that the identity map of $\mathbb{R}^2_{\geq 0}$ does not induce an isomorphism.

However, given an analytic series $s(X_1, X_2) \in \mathbb{R}\{X_1, X_2\}_\epsilon$ convergent on a neighborhood of the origin of $(\mathbb{R}^2_{\geq 0})_{(x_1,x_2)}$ we have that $S_\epsilon(s) \circ \phi(y_1, y_2) = S_\epsilon(s)(y_1^2(y_1^2 + y_2^2), y_2)$ and $s(Y_1^2(Y_1^2 + Y_2^2), Y_2) \in \mathbb{R}\{Y_1, Y_2\}$. Which implies that $S_\epsilon(s) \in \mathcal{G}'$. This shows that the germs at the origin on $(\mathbb{L}^2)'$ contains the germs of analytic functions at zero on the usual sense. That is, $\mathbb{R}\{X_1, X_2\} \subseteq \mathcal{G}'_{(0,0)}$.

Example 2.4.4. For any $k \in \mathbb{N}$, $(\mathbb{A}^k_+)^e = \mathbb{L}^k$.

Remark 2.4.5. The enrichment is not a functor. In other words, given $A = (|A|, \mathcal{O}_A)$ and $B = (|B|, \mathcal{O}_B)$ two analytic manifolds with border and corners and $\varphi : A \to B$ a morphism, then the underlying continuous map $\varphi : |A| \to |B|$ does not induce in general a morphism between A^e and B^e. Take for instance the morphism $\varphi : \mathbb{A}^2_+ \to \mathbb{A}^1_+$ given by the map $(x, y) \to x + y$. This map does not induce a morphism between the enrichments $\mathbb{L}^2 \to \mathbb{L}^1$.

In fact, using Proposition 2.3.15, we can state:

Proposition 2.4.6. Let $A = (|A|, \mathcal{O}_A)$ and $B = (|B|, \mathcal{O}_B)$ be standard analytic manifolds and let $\pi : B \to A$ be a morphism. Then its underlying continuous map induces a morphism $\pi^e : B^e \to A^e$ of $\mathcal{G}$-manifolds if and only if π is locally monomial.

2.4.1 Standardizable generalized manifolds.

Enrichments of standard analytic manifolds are good candidates of generalized manifold to extend those operations that we know already to be well behaved for standard manifolds. In the next chapter we will follow this line of reasoning for the operation of blowing-up.

This motivates the following definition.

Definition 2.4.7. Let $M = (|M|, \mathcal{G}_M)$ be a generalized analytic manifold. We say that M is **standardizable** if it is isomorphic to the enrichment of an standard analytic manifold; that is, if there exists a standard analytic manifold with boundary and corners A and an isomorphism $\phi^e : M \to A^e$ of $\mathcal{G}$-manifolds where A^e is the enrichment of A. Notice that then the composition $\phi = id \circ \phi^e : M \to A$:

$$\begin{array}{ccc} M & \xrightarrow{\phi^e} & A^e \\ & \overset{\phi}{\searrow} & \downarrow{\scriptstyle id} \\ & & A \end{array}$$

is a morphism of locally ringed spaces whose underlying continuous map $\phi : |M| \to |A|$ is a homeomorphism.

In this situation we say that the pair $(A, \phi : M \to A)$ is a **standardization** of M.

Notice that if $M = (|M|, \mathcal{G}_M)$ is a generalized analytic manifold, $A = (|A|, \mathcal{O}_A)$ a standard analytic manifold and $\phi : M \to A$ a morphism whose underlying continuous map $\phi : |M| \to |A|$ is a homeomorphism, then, in general $\phi^{-1} : |A^e| \to |M|$ does not induce a morphism from A^e to M. Consider for instance $M = A = \mathbb{R} = (\mathbb{R}, \mathcal{O}_\mathbb{R})$ and $\phi : x \mapsto x^3$. We have, however the following:

Proposition 2.4.8. Let $M = (|M|, \mathcal{G}_M)$ be a generalized analytic manifold, $A = (|A|, \mathcal{O}_A)$ a standard analytic manifold and $\phi : M \to A$ a morphism whose underlying continuous map $\phi : |M| \to |A|$ is a homeomorphism. Then, if $\phi^{-1} : |A^e| \to |M|$ induces a morphism from A^e to M it is in fact an isomorphism so that (A, ϕ) is a standardization of M.

Proof. Since ϕ is an homeomorphism, the dimension of M, A and A^e is the same, k. Let $p \in |A|$. Put $\psi = \phi^{-1}$, $m = m_p$ and $m' = m_{\psi(p)}$. Since $\psi = (\psi_1, \ldots, \psi_k)$ induces a morphism from A^e to M, by proposition 2.3.15 there are local coordinates (x, y) centered at p and (z, w) at $\psi(p)$ such that the components of ψ are expressed in those coordinates as

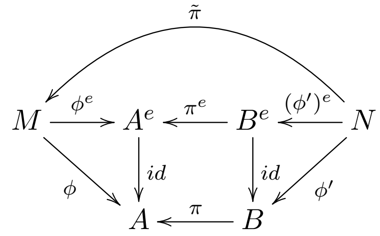
$$\psi_j(x, y) = x^{\alpha^j} g_j(x, y)$$

with $\alpha^j \in [0, \infty)^m$, g_j $\mathcal{G}$-analytic at $\underline{0}$ and $g_j(0, 0) > 0$ for any $j \in \{1, \ldots, m'\}$ and for $j \in \{m' + 1, \ldots, k\}$, ψ_j is $\mathcal{G}$-analytic at 0 and the map $y \mapsto (\psi_{m'+1}(0, y), \ldots, \psi_k(0, y))$ induces an analytic morphism from $\mathbb{R}^{k-m}$ to $\mathbb{R}^{k-m'}$. Since $\phi : M \to A$ is a morphism, in particular $y \mapsto (\phi_{m+1}(0, w), \ldots, \phi_k(0, w))$ induces an analytic morphism from $\mathbb{R}^{k-m'}$ to $\mathbb{R}^{k-m}$. Then, $m = m'$ and $y \mapsto (\psi_{m'+1}(0, y), \ldots, \psi_k(0, y))$ induces an analytic isomorphism from $\mathbb{R}^{k-m}$ to $\mathbb{R}^{k-m}$. As ψ is an homeomorphism, if there exists $j \in \{1, \ldots, m\}$ with $\alpha^j_i \neq 0 \neq \alpha^j_l$ for $i \neq l$, $\psi(\{x_i = 0 = x_l\}) \subseteq \{z_j = 0\}$, against the Invariance of domain theorem. Thus, by proposition 2.3.15, ψ induces an isomorphism. $\square$

As it is the case for enrichments, standardizations do not behave always functorially; i.e. morphisms between standard manifolds which are standardizations of generalized manifolds do not "lift" to a morphism between these generalized manifolds. But, using Proposition 2.4.6, we can say

Proposition 2.4.9. - Let M, N be standardizable $\mathcal{G}$-manifolds and let (A, ϕ), (B, ϕ') be standardizations of M and N respectively. Given a morphism $\pi : B \to A$, there exists a morphism

$\tilde{\pi} : N \to M$ such that $\phi \circ \tilde{\pi} = \pi \circ \phi'$



if and only if π is locally monomial. We say that in this case also that π **lifts** to N and that ϕ' is the lifting (notice that it is unique).

Proof. Using Proposition 2.4.6, π lifts to the enrichments $\pi^e : B^e \to A^e$ iff π is locally monomial. But π^e exists iff $\tilde{\pi} = (\phi^e)^{-1} \circ \pi^e \circ \phi'^e$ exists. □

2.4.2 An example of a non-standardizable manifold.

We want to prove here finally that there are generalized analytic manifolds which are not standardizable. In fact,

Proposition 2.4.10. The (exotic) cylinder $\mathcal{C}_\alpha$ constructed in 2.3.5 is standarizable if and only if $\alpha = 1$.

Proof. Assume the same notations as in 2.3.5. Fix $\alpha > 0$ and suppose that there exists a standardization (A, ϕ) of the $\mathcal{G}$-manifold $\mathcal{C}_\alpha$. Denote also by ϕ the underlying homeomorphism $\phi : C_\alpha = |\mathcal{C}_\alpha| \to |A|$. Let $V_i = \phi(U_{\alpha,i})$ for $i = 1, 2$, an open subset of $|A|$ homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Let $D_i = V_i \cap \partial|A|$ be the boundary inside this open set. The proof is finished once we show the two following claims.

Claim 1.- For $i = 1, 2$ there exists an analytic function h_i in a neighborhood of D_i in V_i (thus a $\mathcal{G}$-analytic function for the structure of the enrichment A^e) such that D_i is the zero locus of h_i and, in the intersection $V_1 \cap V_2$, the quotients h_1/h_2 and h_2/h_1 (both defined outside the boundary) remain bounded in a neighborhood of any point of the boundary $\partial|A| \cap V_1 \cap V_2$, except possibly for a discrete subset of points.

Claim 2.- If $\alpha \neq 1$, the analogous claim 1 for $\mathcal{C}_\alpha$ is not true: there are no $\mathcal{G}$-analytic functions h_i on a neighborhood of $\partial C_\alpha \cap U_{\alpha,i}$ for $i = 1, 2$, having the boundary ∂C_α as the zero locus and such that h_1/h_2 and h_2/h_1 remain bounded in a neighborhood of each point of the boundary except for a discrete subset of them.

Proof of claim 1.- We take an analytic coordinate chart (x_i, y_i) centered at some point $q_i \in D_i$ such that $D_i = \{x_i = 0\}$ and we consider h_i as the analytic continuation in V_i (simply connected domain) of the coordinate function x_i, locally defined and analytic in a neighborhood of q_i. Given a point $q \in V_i$, the function h_i writes in analytic coordinates (x, y) at q for which $x = 0$ is the boundary as

$$h_i(x, y) = x^{\gamma(q)} H(x, y), \text{ where } \gamma_i(q) \in \mathbb{N}_{\geq 1} \text{ and } H(0, y) \not\equiv 0.$$

The fact that the change of coordinates (x, y) and (x', y') centered at two points in the boundary satisfies $x' = xU(x, y)$, where U is a unity (see Proposition 2.1.18), implies that the exponent $\gamma_i(q)$ is well defined independently of the chosen coordinates at q. Moreover, it is locally constant with respect to q and thus constant for every $q \in V_i$. Since $\gamma_i(q_i) = 1$ we have the same exponent, 1, for every point of the whole boundary $\partial|A|$. This gives the desired condition about the quotients h_1/h_2 and h_2/h_1 in the intersection $V_1 \cap V_2$.

Proof of Claim 2.- Suppose that h_i is a $\mathcal{G}$-analytic function in a neighborhood of $\partial C_\alpha \cap U_{\alpha,i}$ for

$i = 1, 2$ whose zero locus is equal to the boundary. Consider the coordinates $(x_{\alpha,i}, y_{\alpha,i})$ globally defined in $U_{\alpha,i}$ (see the notations in 2.3.5). Then we can write h_i globally in its domain of definition as:

$$h_i(x_{\alpha,i}, y_{\alpha,i}) = x_{\alpha,i}^{\beta_i} H_i,$$

where $\beta_i \in \mathbb{R}_{>0}$ and H_i is a $\mathcal{G}$-analytic function in a neighborhood of $\partial C_\alpha \cap U_{\alpha,i}$ such that the restriction $H_i|_{\partial C_\alpha}$ to the boundary does not vanishes identically (thus, since this restriction is analytic, its zero locus is a discrete subset of $\partial C_\alpha \cap U_{\alpha,i}$). Now, consider an open set Ω^ϵ, for $\epsilon = +$ or $-$, contained in $\partial C_\alpha \cap \{\epsilon y_{\alpha,1} > 0\}$ where neither H_1 or H_2 vanishes. Taking into account the expression of the change of variables between $(x_{\alpha,1}, y_{\alpha,1})$ and $(x_{\alpha,2}, y_{\alpha,2})$, we can write

$$h_1 = x_{\alpha,1}^{\beta} H_1 = x_{\alpha,2}^{\beta_1} H_1 \text{ in } \Omega^+,$$

$$h_1 = x_{\alpha,1}^{\beta} H_1 = x_{\alpha,2}^{\alpha\beta_1} H_1 \text{ in } \Omega^-.$$

If the condition about the quotients h_1/h_2 and h_2/h_1 is true in both open sets Ω^+ and Ω^- then we must have $\beta_2 = \beta_1 = \alpha\beta_1$, which is impossible if $\alpha \neq 1$. □

Chapter 3

Local monomialisation.

We attack in this chapter the main result in this work: to transform a given $\mathcal{G}$-analytic function in a neighborhood of a point into locally monomial functions by means of local blowing-ups with admissible centers. This is a kind of result that can be untitled as **Local Monomialisation of generalized analytic functions**, since this is the name of the analogous (well known) result on real analytic functions (see [5] or [2] for instance). It can be considered inside the frame of the theory of reduction of singularities in the category of generalized analytic manifolds. In order to state correctly the Theorem of Local Monomialisation, we need first to define what a blowing-up morphism is.

The plan is as follows. First we define the kind of "admissible" centers to be blown-up, both in the category of standard and generalized analytic manifolds. These centers are closed "subvarieties" locally given at any point by the annihilation of several coordinate functions.

Second, we recall what a blowing-up morphism is in the category of (standard) real analytic manifolds with boundary and corners. This a quite well known notion in the category of analytic manifolds without boundary. In our point of view, since the analytic manifolds that we consider have boundary and corners, we follow the suitable approach of considering the so called *oriented real blowing-up*, in contrast with the (relatively more usual) *projective real blowing-up*. The main difference is that, in the former case, points of the center of blowing-ups are replaced by the set of half-lines, normal to the center, defined by means of a system of coordinates; while for the projective blowing-up, points are replaced by the set of normal lines through them. At boundary points, we have no entire but half-lines, thus showing the convenience of the use of oriented blowing-up.

As a consequence, the exceptional divisor (the inverse image of the center) always becomes a new boundary component to the blown-up space even if the center of blowing-up is contained in the interior of the standard analytic manifold (where normal entire lines are defined). The choice for this kind of blowing-up also at interior points is based only on consideration of coherency.

In compensation, we do not alter the properties of orientability of the manifold, although in these pages, where we only use local blowing-ups (that is, whose center is just a closed "subvariety" on some open domain), this point does not give us an advantage.

Third, we introduce the concept of blowing-up morphism in the category of generalized analytic manifolds. This notion has a (a priori unexpectable) peculiarity that does not occur in the standard case: if we proceed defining directly the blowing-up for the local model (as we may do in the standard case) by "gluing" the local charts of a standard blowing-up and then take the enrichments, we could obtain different (non-isomorphic) blowing-up morphisms for different choices of local coordinates. Thus, our concept of blowing-up morphism is not only attached to an admissible center of blowing-up, but relative also to a standardization of the manifold.

With this peculiarity in mind, no good notion of blowing-up is possible when the center to be blown-up has not a neighborhood which admits a standardization. A concrete example of this situation can be constructed using the example of the exotic cylinder $\mathcal{C}_\alpha$ with $\alpha \neq 1$ (cf. 2.3.5): put $\mathcal{D}_\alpha = \mathcal{C}_\alpha \times \mathbb{L}^1$, whose boundary $\partial C_\alpha \times \{0\}$ is a curve isomorphic to the circle $\mathbb{S}^1$ (an admissible center of codimension 2) with no open standardizable neighborhood. The geometric interpretation of this pathological example is that this center has not a good "global normal bundle" of half-lines: once you start at a point with half-lines in some given coordinates you return, after a turn in the circle, with a "non-compatible" family of half-lines with respect to another system of coordinates.

3.1 Admissible centers.

We give here the definition of regular submanifold both in the category $\mathcal{O}$ of standard real analytic manifolds with boundary and corners and in the category $\mathcal{G}$ of generalized real analytic manifolds. Admissible centers to be considered below for blowing-up are among regular submanifolds of a very specific nature (those having also normal crossings with the boundary of the manifold).

The lack of differentiability of a morphism in the later case prevents to define immersions in the usual way. However, as it is defined in the book of Gunning & Rossi [10], the immersion condition is replaced by the fact that the morphism induced on the stalks is surjective.

In this section, the notation $\mathcal{A}$ stands either for the standard category $\mathcal{A} = \mathcal{O}$ or for the generalized one $\mathcal{A} = \mathcal{G}$.

3.1.1 Submanifolds and regular subsubmanifolds.

Definition 3.1.1. Let $M = (|M|, \mathcal{A}_M)$ and $N = (|N|, \mathcal{A}_N)$ be $\mathcal{A}$-manifolds. A morphism $\varphi : N \to M$ is a **submanifold** if

i) φ is injective

ii) for each $p \in |N|$, the induced homomorphism in the ring of germs

$$\varphi_p^\sharp : \mathcal{G}_{M,\varphi(p)} \to \mathcal{G}_{N,p}$$

is surjective.

If in addition, $\varphi(|N|)$ is a closed subset of $|M|$ we say that the submanifold $\varphi : N \to M$ is **closed**. A submanifold $\varphi : N \to M$ is said to be a **regular submanifold of** M if moreover $\varphi : |N| \to \varphi(|N|)$ is an homeomorphism.

Remark 3.1.2. Notice that the condition that φ is injective and continuous implies (by the Theorem of Invariance of the Domain) that $\dim(N) \leq \dim(M)$. On the other hand, it can be shown, though we will not make use of, that in the standard category $\mathcal{A} = \mathcal{O}$, the condition *ii)* is equivalent to the usual condition for immersions, that is, that the differential $d\varphi_p$ at the point p is injective.

Examples 3.1.3. *i)* The morphism induced by the map

$$t \in \mathbb{R}^1_{\geq 0} \mapsto (t, t, 0) \in \mathbb{R}^3_{\geq 0}$$

is a closed regular submanifold (of $\mathbb{A}^3_+$ in the standard category $\mathcal{O}$ and of $\mathbb{L}^3$ in the $\mathcal{G}$ category).

ii) Let $m, n \in \mathbb{N}$. Consider the $\mathcal{O}$-manifold product $\mathbb{A}^m_+ \times \mathbb{R}^n$. Then the morphism $i : \mathbb{A}^m_+ \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ induced by the inclusion mapping $i : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{m+n}$ is a regular submanifold.

iii) Let $m, n \in \mathbb{N}$. Consider the $\mathcal{G}$-manifold product $\mathbb{L}^m \times \mathbb{R}^n$. Then the morphism $i : \mathbb{L}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ induced by the inclusion mapping $i : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{m+n}$ is a regular submanifold.

3.1.2 Admissible centers.

Definition 3.1.4. A regular submanifold $\varphi : N \to M$ between $\mathcal{A}$-manifolds is said to be an **admissible center** if for every $p \in |N|$, there exist $\mathcal{A}$-coordinates x and y, centered at p and at $\varphi(p)$, respectively, such that, up to permutation of the target variables y, φ writes locally as

$$\varphi(x) = (x, 0).$$

Example 3.1.5. *i)* The morphisms induced by the map

$$t \in \mathbb{R}^1_{\geq 0} \mapsto (t, t, 0) \in \mathbb{R}^3_{\geq 0}$$

are not admissible (neither in the standard or generalized category).

ii) For any $a \geq 0$ the morphism induced by

$$t \in [0, 1] \mapsto (a, t, 1 - t) \in \mathbb{R}^3_{\geq 0}$$

is a closed admissible center.

Proposition 3.1.6. Let $M = (|M|, \mathcal{A}_M)$ be a $\mathcal{A}$-manifold and $|Y|$ a connected subset of $|M|$. Suppose that for any $p \in |Y|$ there exists $(U_p, \varphi_p = (x_1, \ldots, x_k))$ a $\mathcal{A}$-local chart at p and $J_p \subseteq \{1, \ldots, k\}$ such that

$$\varphi_p(|Y| \cap U_p) = \{q \in U_p : x_j(q) = x_j(p) \ \text{ for any } \ j \in J_p\} \tag{3.1}$$

Then there exists a unique structure of $\mathcal{A}$-manifold over $|Y|$, say $Y = (|Y|, \mathcal{G}_Y)$ such that the morphism induced by the inclusion map $i : |Y| \hookrightarrow |M|$ is an admissible center. Reciprocally, if $\varphi : N \to M$ is an admissible center, then $|Y| = \varphi(|N|)$ has the above property.

Proof. For any $p \in |Y|$ put $l_p = k - \#J_p$, $V_p := |Y| \cap U_p$, $\pi_p : (x_1, \ldots, x_k) \in \mathbb{R}^k \mapsto (x_j)_{j \notin J_p} \in \mathbb{R}^{l_p}$ and $\psi_p = \pi_p \circ \varphi_p$

$$\begin{array}{ccc} V_p = |Y| \cap U_p & \xrightarrow{\ \varphi_p\ } & \mathbb{R}^k_{\geq 0} \\ & \searrow^{\psi_p} & \downarrow \pi_p \\ & & \mathbb{R}^{l_p}_{\geq 0} \end{array} \tag{3.2}$$

Since l_p is locally constant on $|Y|$ and $|Y|$ is connected l_p is constantly equals to l. We check easily that $\{(V_p, \psi_p)\}_{p \in |Y|}$ is a $\mathcal{A}$-atlas of $|Y|$ (the change of variables between two such local charts comes from considering some components of a change of variables $x \to y$ for two local charts of the ambient manifold M where some of the variables between the x are substituted by a constant). We can consider $Y = (|Y|, \mathcal{A}_Y)$ the $\mathcal{A}$-manifold associated to this atlas as in 2.3.8.

Now we prove that $i : Y \hookrightarrow M$ is a closed regular submanifold. Let $p \in |Y|$ and $i^\sharp_p : \mathcal{G}_{M,p} \to \mathcal{G}_{Y,p}$ the induced homomorphism in the stalks. Taking local coordinates $(U_p, \varphi_p = (x_1 \ldots, x_k))$ at $i(p)$

and $(V_p, \psi_p = (x_j)_{j \notin J_p})$ at p we have the isomorphisms F and G, (as in Proposition 2.2.6 for the generalized category and the analogous for the standard category):

$$\begin{array}{ccc} \mathcal{G}_{M,p} & \xrightarrow{i_p^\sharp} & \mathcal{G}_{Y,p} \\ F \uparrow & & \uparrow G \\ \mathbb{R}\{X^\square\} & \xrightarrow{\phi} & \mathbb{R}\{(X(p))^\square\} \end{array}$$

where $\square$ is equal to an asterisque $*$ in the generalized category (and nothing in the standard one), $X(p) = (X_j)_{j \notin J_p}$. We have that $\phi = G^{-1} \circ i_p^\sharp \circ F$ is given by substituting those variables X_j such that $j \in J_p$ by a constant (equal to zero if X_j is a boundary variable). Thus, ϕ is surjective and consequently $i_p^\sharp$ too.

Uniqueness of the structure $\mathcal{A}_Y$ comes from the following observation: if (U_p, φ) is a local chart of M at p satisfying the condition (3.1) and if π_p stands for the same meaning as in the diagram (3.2) then $\psi = \pi_p \circ \varphi$ is a local chart of the $\mathcal{A}$-manifold Y. $\square$

In the sequel, we will just use the expression "Y is an admissible center of M" or "$Y \subset M$ is an admissible center" if $Y = (|Y|, \mathcal{A}_Y)$ is in the conditions of Proposition 3.1.6 with $|Y| \subset |M|$.

With the notations of proposition 3.1.6 above, for each $p \in |Y|$, there exists a local chart (U_p, x) with $p \in U_p$ such that $M|_{U_p}$ is isomorphic to $Y|_{U_p} \times \mathbb{L}^{m'_p} \times \mathbb{R}^{n'_p}$ in the generalized case (and to $Y|_{U_p} \times \mathbb{A}_+^{m'_p} \times \mathbb{R}^{n'_p}$ in the standard case) where $m'_p = |\{j \notin J_p : j \in A(\varphi_p(p))\}|$ and $n'_p = |\{j \notin J_p : j \notin A(\varphi_p(p))\}|$. We call U_p together with the isomorphism $M|_{U_p} \cong Y|_{U_p} \times \mathbb{L}^{m'_p} \times \mathbb{R}^{n'_p}$ a **normalizing chart** for Y. We have that m'_p and n'_p does not depend on the normalizing chart and that $\dim(Y) = k - m'_p - n'_p$. In a normalizing chart, we have that Y is described as the zeros of the last $m'_p + n'_p$ coordinates and that the restriction of the first $k - m'_p - n'_p$ coordinates to Y gives a chart for the structure $\mathcal{A}_Y$ of Y as a regular subvariety of M.

Example 3.1.7. The numbers m'_p and n'_p may depend on the point $p \in |Y|$ (although its sum, equal to the codimension of Y in M is independent of p). Take for instance for $M = \mathbb{L}^2$, with coordinates (x, y), the admissible center whose underline space is

$$|Y| = \{(x, y) \in \mathbb{R}^2_{\geq 0} : x + y = 1\}$$

3.1.3 Standardizable admissible centers.

Let A be a standard analytic manifold and let $Y \subset A$ be an admissible center. By its very definition, the inclusion $i : Y \hookrightarrow A$ is a morphism which is locally of monomial type. Thus, using 2.4.6, it lifts to a morphism $i^e : Y^e \hookrightarrow A^e$, which is, moreover, an admissible center.

We have not, however, the reciprocal of the above situation.

Example 3.1.8. Consider the $\mathcal{G}$-manifold $\mathbb{L}^1 \times \mathbb{R}$ with coordinates (x, y). Let $Y \hookrightarrow \mathbb{L}^1 \times \mathbb{R}$ be the regular submanifold where $|Y| = \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : y = x^\alpha\}$ where $\alpha > 0$ is not rational. Then $Y \subset \mathbb{L}^1 \times \mathbb{R}$ is an admissible center (in the category $\mathcal{G}$). However, if we consider the standardization $\phi : \mathbb{L}^1 \times \mathbb{R} \to \mathbb{A}_+^1 \times \mathbb{R}$ induced by the identity (that is, so that $\mathbb{L}^1 \times \mathbb{R}$ is the enrichment of $\mathbb{A}_+^1 \times \mathbb{R}$), then the image $\phi(|Y|) = |Y|$ does not satisfy the property (3.1) in Proposition 3.1.6.

In view of this example we give the following definition.

Definition 3.1.9. Let M be a $\mathcal{G}$-manifold and let $Y \subset M$ be an admissible center. We say that Y is **standardizable inside** M or that **the pair** (M, Y) **is standardizable** if there exists a standardization $\phi : M \to A$ of M such that $|Z| = \phi(|Y|) \subset |A|$ has the property (**??**); thus $|Z|$ is the underlying space of an admissible center $Z \subset A$. If such a standardization ϕ exists, we will say that ϕ is a standardization of the pair (M, Y).

As we have seen in Example 2.4.2, the regular subvariety ∂C_α of the exotic cylinder $\mathcal{C}_\alpha$ is an admissible center of $\mathcal{C}_\alpha$ but the pair $(\mathcal{C}_\alpha, \partial C_\alpha)$ is not standardizable if $\alpha \neq 1$ (in fact there exists no open neighborhood of ∂C_α which is a standardizable $\mathcal{G}$-manifold. This is an example of a non standardizable admissible center of codimension 1, but we can construct similar examples of any codimension just by making the product of $\mathcal{C}_\alpha$ by the local models $\mathbb{L}^k$.

This pathology occurs only on the global setting, the local counterpart being always simpler (the proof comes easily from the definitions):

Proposition 3.1.10. Let M be a $\mathcal{G}$-manifold and let $Y \subset M$ be an admissible center. Given a point $p \in |Y|$, there exists an open neighborhood U_p of p in $|M|$ such that $Y|_{U_p}$ is a standardizable admissible center inside U_p.

3.2 Blowing-up on standard analytic manifolds.

In this section we recall the notion of blowing-up with a closed admissible center in a standard analytic manifold (with boundary and corners). We will proceed by defining explicit models and explicit charts of blowing-up morphisms, although the notion could be given in categorical terms as a solution of a universal problem inside the category of these manifolds (this is the way the blowing-up morphisms are defined for instance in Hironaka's paper [15] for the category of complex analytic spaces).

We start with the very well known case of the (polar) blowing-up of a point in the model $\mathbb{R}^k$ of analytic manifold without boundary.

Theorem 3.2.1. Let $\widetilde{\mathbb{R}^k} := \mathbb{A}^1_+ \times \mathbb{S}^{k-1}$ the product in the category $\mathcal{O}$ of $\mathbb{A}^1_+$ and $\mathbb{S}^{k-1}$. We define

$$\pi_{\underline{0}}^{\mathbb{R}^k} : \mathbb{R}_{\geq 0} \times \mathbb{S}^{k-1} \longrightarrow \mathbb{R}^k$$
$$\pi_{\underline{0}}^{\mathbb{R}^k}(r, (x_1, \ldots, x_k)) = (rx_1, \ldots, rx_k)$$

Then, the map $\pi_{\underline{0}}^{\mathbb{R}^k}$ is continuous and proper. Moreover, it induces a morphism form $\widetilde{\mathbb{R}^k}$ to $\mathbb{R}^k$, $(\pi_{\underline{0}}^{\mathbb{R}^k})^{-1}(\underline{0}) = \{\underline{0}\} \times \mathbb{S}^1$ is a closed regular submanifold of $\widetilde{\mathbb{R}^k}$ (in fact an admissible center), and the restriction $\pi_{\underline{0}}^{\mathbb{R}^k}|_{\widetilde{\mathbb{R}^k} \setminus (\pi_{\underline{0}}^{\mathbb{R}^k})^{-1}(\underline{0})}$ induces an isomorphism between $\widetilde{\mathbb{R}^k} \setminus (\pi_{\underline{0}}^{\mathbb{R}^k})^{-1}(\underline{0})$ and $\mathbb{R}^k \setminus \{\underline{0}\}$. Moreover, the morphism $\pi_0^{\mathbb{R}^k}$ is locally of monomial type.

The pair $(\widetilde{\mathbb{R}^k}, \pi_0^{\mathbb{R}^k})$ will be called the blowing up of $\mathbb{R}^k$ with center the origin. If $p \in \mathbb{R}^k$ is any point and $T_p : \mathbb{R}^k \to \mathbb{R}^k$ is the translation of the point p to the origin, totally analogous properties as above are true for the morphism $\pi_p^{\mathbb{R}^k} = T_p \circ \pi_0^k : \widetilde{\mathbb{R}^k} = \mathbb{A}^1_+ \times \mathbb{S}^{k-1} \to \mathbb{R}^k$. The pair $(\widetilde{\mathbb{R}^k}, \pi_p^{\mathbb{R}^k})$ in this case is called the **blowing-up in $\mathbb{R}^k$ with center the point** p.

In all the cases, we call, as usual, $(\pi_p^{\mathbb{R}^k})^{-1}(p)$ the **exceptional divisor** of the (corresponding) blowing-up.

Now we can define the blowing up at any point in each of the mixed models $\mathbb{A}^m_+ \times \mathbb{R}^n$ for analytic manifolds with boundary and corners.

Let $m, n \in \mathbb{N}$. Consider the $\mathcal{O}$-manifold product $\mathbb{A}^m_+ \times \mathbb{R}^n$ as a regular submanifold of $\mathbb{R}^{m+n}$ by the set-theoretic inclusion (see Example 3.1.3)

Theorem-definition 3.2.2. Let $p \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$ and let $(\widetilde{\mathbb{R}^{m+n}}, \pi_p^{\mathbb{R}^{m+n}})$ be the blowing-up in $\mathbb{R}^{m+n}$ with center p. Then $\widetilde{\mathbb{R}_p^{m,n}} = (\pi_p^{\mathbb{R}^{m+n}})^{-1}(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n) \subset \mathbb{R}_{\geq 0} \times \mathbb{S}^{m+n-1}$ is a regular subvariety of $\widetilde{\mathbb{R}^{m+n}}$ (by set-theoretic inclusion) and the restriction induces an analytic morphism

$$\pi_p^{m,n} = \pi_p^{\mathbb{R}^{m+n}}|_{\widetilde{\mathbb{R}_p^{m,n}}} : \widetilde{\mathbb{R}_p^{m,n}} \to \mathbb{A}^m_+ \times \mathbb{R}^n$$

which is proper and a local isomorphism at any point except for those in $(\pi_p^{m,n})^{-1}(p)$, which is a regular subvariety of $\widetilde{\mathbb{R}_p^{m,n}}$ of codimension 1 (in fact an admissible center). The pair $(\widetilde{\mathbb{R}_p^{m,n}}, \pi_p^{m,n})$ is called the **blowing-up of $\mathbb{A}^m_+ \times \mathbb{R}^n$ at the point** p and $(\pi_p^{m,n})^{-1}(p)$ is called the **exceptional divisor of the blowing-up**.

The definition of blowing-up at a point as we have stated above gives explicitly the blown-up space together with the blowing-up morphism. But it is one of the possibilities to consider a blowing-up morphism. We need not to be attached to a concrete form of a blowing-up, mostly if we have the aim to define it in general analytic manifolds. So we define:

Definition 3.2.3. Let $p \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$. A **blowing-up of $\mathbb{A}^m_+ \times \mathbb{R}^n$ at the point** p is any pair (B, π_p) where B is a standard analytic manifold with boundary and corners and $\pi_p : B \to \mathbb{A}^m_+ \times \mathbb{R}^n$ is an analytic morphism such that there exists an analytic isomorphism $\theta : B \to \widetilde{\mathbb{R}^{m,n}}$ with $\pi_p = \pi_p^{m,n} \circ \theta$

$$\begin{array}{ccc} B & \xrightarrow{\pi_p} & \\ \downarrow \theta & & \searrow \\ \widetilde{\mathbb{R}^{m,n}} & \xrightarrow{\pi_p^{m,n}} & \mathbb{A}^m_+ \times \mathbb{R}^n \end{array}$$

Examples 3.2.4. *i)* Consider two copies of $\mathbb{R}_{\geq 0}$ with coordinates x_1 and x_2 respectively. Let B be the disjoint union $\mathbb{R}_{\geq 0} \sqcup \mathbb{R}_{\geq 0}$. Then the two copies of $\mathbb{R}_{\geq 0}$ embed as open coordinate domains of B giving rise to a structure of $\mathcal{O}$-manifold to B. Together with the map $\pi : B \to \mathbb{R}$ which is well defined in these charts as

$$\pi(x_1) = x_1; \pi(x_2) = -x_2$$

the pair (B, π) is a blowing-up of $\mathbb{R}$ at the origin.

ii) The pair $(\mathbb{A}_+, id_{\mathbb{A}_+})$ is a blowing-up of $\mathbb{A}_+$ at the origin.

iii) If $m = 1 = n$, we take two copies of $\mathbb{R}^2_{\geq 0}$ and $\mathbb{R}^1_{\geq 0} \times \mathbb{R}$ with coordinates (x_1, y_1), (x_3, y_3) and (x_2, y_2) respectively. Let B be the quotient space obtained from the disjoint union $(\mathbb{R}^2_{\geq 0}) \sqcup (\mathbb{R}_{\geq 0} \times \mathbb{R}) \sqcup (\mathbb{R}^2_{\geq 0})$ by the relation

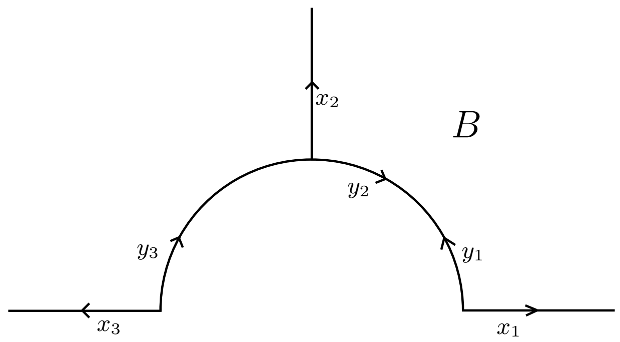
$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow y_1 y_2 \neq 0, y_2 > 0, x_1 y_1 = x_2, \text{ and } x_1 = x_2 y_2$$

$$(x_2, y_2) \sim (x_3, y_3) \Leftrightarrow y_2 y_3 \neq 0, y_2 < 0, x_3 y_3 = x_2 \text{ and } -x_3 = x_2 y_2$$

Then the two copies of $\mathbb{R}^2_{\geq 0}$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}$ embed as open coordinate domains of B giving rise to a structure of $\mathcal{O}$-manifold to this quotient topological space. Together with the map $\pi : B \to \mathbb{R}_{\geq 0} \times \mathbb{R}$ which is well defined in these charts as

$$\pi(x_1, y_1) = (x_1 y_1, x_1); \pi(x_2, y_2) = (x_2, x_2 y_2); \pi(x_3, y_3) = (x_3 y_3, -x_3)$$

the pair (B, π) is a blowing-up of $\mathbb{A}^1_+ \times \mathbb{R}^1$ at the origin.



iv) If $m = 0, n = 2$, we take four copies of $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) respectively. Let B be the quotient space obtained from the disjoint union $(\mathbb{R}_{\geq 0} \times \mathbb{R}) \sqcup (\mathbb{R}_{\geq 0} \times \mathbb{R}) \sqcup (\mathbb{R}_{\geq 0} \times \mathbb{R}) \sqcup (\mathbb{R}_{\geq 0} \times \mathbb{R})$ by the relation

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow y_1 > 0, y_2 < 0, x_1 = -x_2 y_2 \text{ and } x_1 y_1 = x_2$$

$$(x_1, y_1) \sim (x_4, y_4) \Leftrightarrow y_1 < 0, y_4 > 0, x_1 = x_4 y_4 \text{ and } x_1 y_1 = -x_4$$

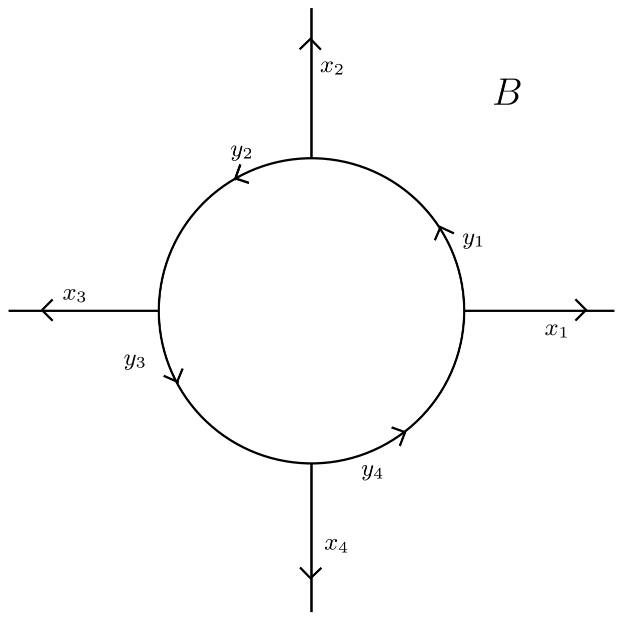
$$(x_2, y_2) \sim (x_3, y_3) \Leftrightarrow y_2 > 0, y_3 < 0, x_2 y_2 = x_3 \text{ and } x_2 = -x_3 y_3$$

$$(x_3, y_3) \sim (x_4, y_4) \Leftrightarrow y_3 > 0, y_4 < 0, -x_3 = x_4 y_4 \text{ and } x_3 y_3 = x_4$$

Then the four copies of $\mathbb{R}_{\geq 0} \times \mathbb{R}$ embed as open coordinate domains of B giving rise to a structure of $\mathcal{O}$-manifold to this quotient topological space. Together with the map $\pi : B \to \mathbb{R}^2$ which is well defined in these charts as

$$\pi(x_1, y_1) = (x_1, x_1 y_1); \pi(x_2, y_2) = (-x_2 y_2, x_2); \pi(x_3, y_3) = (-x_3, -x_3 y_3); \pi(x_4, y_4) = (x_4 y_4, -x_4)$$

the pair (B, π) is a blowing-up of $\mathbb{R}^2$ at the origin.



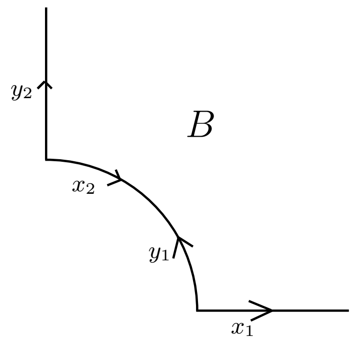
v) Take $m = 2, n = 0$. Consider two copies of $\mathbb{A}^2_+$ with coordinates (x_1, y_1) and (x_2, y_2) respectively. Let B be the quotient space obtained from the disjoint union $\mathbb{R}^2_{\geq 0} \sqcup \mathbb{R}^2_{\geq 0}$ by the relation

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 y_2 \neq 0, x_1 = x_2 y_2 \text{ and } y_2 = x_1 y_1$$

Then the two copies of $\mathbb{R}^2_{\geq 0}$ embed as open coordinate domains of B giving rise to a structure of $\mathcal{O}$-manifold to this quotient topological space. Together with the map $\pi : B \to \mathbb{R}^2_{\geq 0}$ which is well defined in these charts as

$$\pi(x_1, y_1) = (x_1, x_1 y_1),\ \pi(x_2, y_2) = (x_2 y_2, y_2),$$

the pair (B, π) is a blowing-up of $\mathbb{A}^2_+$ at the origin.



3.2.1 Blowing up points in analytic manifolds.

Now we want to define the blowing-up at a point in any standard analytic manifold with boundary and corners. In a natural way, we use the fact that any point has a neighborhood U which is isomorphic to one of the models $\mathbb{A}^m_+ \times \mathbb{R}^n$ and then consider the blowing-up as defined in this model which can be carried to the blowing-up on U. But this involves the ambiguity of the chosen isomorphism. So we need to prove first the following result:

Proposition 3.2.5. Let $\theta : \mathbb{A}^m_+ \times \mathbb{R}^n \to \mathbb{A}^m_+ \times \mathbb{R}^n$ an isomorphism sending the origin to the origin. Then there exists an isomorphism $\widetilde{\theta} : \widetilde{\mathbb{R}^{m,n}} \to \widetilde{\mathbb{R}^{m,n}}$ such that $\pi^{m,n}_{\underline{0}} \circ \widetilde{\theta} = \theta \circ \pi^{m,n}_{\underline{0}}$

$$\begin{array}{ccc} \widetilde{\mathbb{R}^{m,n}} & \xrightarrow{\widetilde{\theta}} & \widetilde{\mathbb{R}^{m,n}} \\ \Big\downarrow{\scriptstyle \pi^{m,n}_{\underline{0}}} & & \Big\downarrow{\scriptstyle \pi^{m,n}_{\underline{0}}} \\ \mathbb{A}^m_+ \times \mathbb{R}^n & \xrightarrow{\theta} & \mathbb{A}^m_+ \times \mathbb{R}^n \end{array}$$

Proof. Considering $\mathbb{A}^m_+ \times \mathbb{R}^n$ as a regular submanifold of $\mathbb{R}^{m+n}$, and taking into ac account that the blowing-up morphism $\pi^{m,n}_{\underline{0}}$ is defined as the restriction of the blowing-up of $\mathbb{R}^{m+n}$ at $\underline{0}$ to the corresponding spaces, it is enough to prove the case where $m = 0$.

This is a quite well known result: the isomorphism $\widetilde{\theta}$ is unambiguously determined at any point outside the exceptional divisor $D_0 = (\pi^{0,n}_{\underline{0}})^{-1}(\underline{0}) = \{0\} \times \mathbb{S}^{n-1} \subseteq \mathbb{A}^1_+ \times \mathbb{S}^{n-1} = \widetilde{\mathbb{R}^{0,n}}$. If we write $\theta = (\theta_1, \ldots, \theta_n)$ the components of the map $\theta : \mathbb{R}^n \to \mathbb{R}^n$, the explicit expression is, for $(r, \underline{x}) \in \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$ with $r \neq 0$:

$$\widetilde{\theta} : (r, \underline{x}) \longmapsto (\rho = \sqrt{\theta_1(r\underline{x})^2 + \cdots + \theta_n(r\underline{x})^2}, (\frac{\theta_1(r\underline{x})}{\rho}, \ldots, \frac{\theta_n(r\underline{x})}{\rho}))$$

Use now the Taylor expansion of θ of order 1 at the origin

$$\theta(\underline{w}) = L(\underline{w}) + O(\|\underline{w}\|^2)$$

where L is a linear isomorphism, to conclude that the expression above extends to a local isomorphism at any point of the exceptional divisor. $\square$

Now we can extend the Theorem-Definition 3.2.2 to general analytic manifolds.

Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold with boundary and corners and let $p \in |A|$. Take a local chart $\varphi : U \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ centered at p. Consider the two open immersions

$$U \setminus \{p\} \xrightarrow{\psi_1} A$$

$$U \setminus \{p\} \xrightarrow{\psi_2} \widetilde{\mathbb{R}^{m,n}}$$

where ψ_1 is the set-theoretic inclusion and $\psi_2 = i \circ (\pi^{m,n}_{\underline{0}})^{-1} \circ \varphi^{-1}$

$$U \setminus \{p\} \xrightarrow{\varphi^{-1}} \mathbb{A}^m_+ \times \mathbb{R}^n \setminus \{0\} \to \widetilde{\mathbb{R}^{m,n}} \setminus (\pi^{m,n}_{\underline{0}})^{-1}(0) \hookrightarrow \widetilde{\mathbb{R}^{m,n}}$$

Let $\widetilde{A}(\varphi)$ be the gluing manifold associated to these immersions and $\pi^A_p(\varphi) : \widetilde{A}(\varphi) \to A$ the corresponding projection onto A.

Theorem-definition 3.2.6. *i)* The morphism $\pi^A_p(\varphi)$ is proper and surjective and it induces an isomorphism from the open submanifold $|\widetilde{A}(\varphi)| \setminus \pi^A_p(\varphi)^{-1}(p)$ to $|A| \setminus \{p\}$ and $\pi^A_p(\varphi)^{-1}(p)$ is a regular submanifold of $\widetilde{A}(\varphi)$ of codimension 1 (in fact an admissible center.)

ii) If $\varphi' : U' \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ is another local chart centered at p then there exists an isomorphism $\theta_{\varphi,\varphi'} : \widetilde{A}(\varphi) \to \widetilde{A}(\varphi')$ such that $\pi^A_p(\varphi') \circ \theta_{\varphi,\varphi'} = \pi^A_p(\varphi)$.

A **blowing up of** A **at** p is any pair $(\widetilde{A}, \pi^A_p)$ where $\widetilde{A}$ is an $\mathcal{O}$-manifold and $\pi^A_p(\varphi) : \widetilde{A} \to A$ is a morphism such that there exists an isomorphism $\theta : \widetilde{A} \to \widetilde{A}(\varphi)$.

3.2.2 Blowing-up an admissible center.

In the previous paragraph we have defined the blowing-up of a standard analytic manifold at a point (an admissible center of dimension zero). Here we define the blowing-up with center a closed admissible center of any dimension.

Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold with boundary and corners and let $Z = (|Z|, \mathcal{O}_Z) \subset A$ be a closed (connected) admissible center of A. Recall from the paragraph 3.1.2 that for any point $p \in |A|$ we have an open neighborhood U of p in A which is a normalizing domain for the subvariety Y, that is, that

$$A|_U \simeq Y|_U \times \mathbb{A}^{m'_p}_+ \times \mathbb{R}^{n'_p}.$$

(If $p \notin |Y|$, since $|Y|$ is closed, we take U that does not intersect $|Y|$ so that $Y|_U = \emptyset$). We can moreover assume that the isomorphism above restrict to the identity between $U \cap Y$ and $Y|_U \times \{0\}$. The natural numbers may depend on the point p but not in the neighborhood U (cf. Example 3.1.7).

The following Lemma generalizes Lemma 3.2.6

Lemma 3.2.7. Let U and V be two normalizing domains of Y:

$$\varphi_U : A|_U \cong Z|_U \times \mathbb{A}^{m(U)}_+ \times \mathbb{R}^{n(U)}$$

and

$$\varphi_V : A|_V \cong Z|_V \times \mathbb{A}^{m(V)}_+ \times \mathbb{R}^{n(V)}$$

Define $\pi_Z^A(U) := (id, \pi_{\underline{0}}^{\mathbb{A}_+^{m(U)}\times\mathbb{R}^{n(U)}}) : Z|_U \times \widetilde{\mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}} \to Z|_U \times \mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}$ (and analogously $\pi_Z^A(V)$), where the second component is the blowing-up of $\mathbb{A}_+^{m'(U)} \times \mathbb{R}^{n'(U)}$ at the origin. Assume that $U \cap V \neq \emptyset$. Then $m'(U) = m'(V)$, $n'(U) = n'(V)$. In this case, there exists a unique isomorphism

$$\tilde{\theta} : (\pi_Z^A(U))^{-1}(\varphi_U(U \cap V)) \to (\pi_Z^A(V))^{-1}(\varphi_V(U \cap V))$$

such that $\varphi_U^{-1} \circ \pi_Z^A(V) \circ \tilde{\theta} = \varphi_V^{-1} \circ \pi_Z^A(U)$.

Proof. The first claim follows from the fact that if $q \in U \cap V$ then $m'(U) = m'_p = m'(V)$ and $n'(U) = n'_p = n'(V)$ because of the invariance of the number of boundary components of an $\mathcal{O}$-manifold at a point.

The change of normalizations $\theta = \varphi_V \circ \varphi_U^{-1}$ is an isomorphism between an open submanifold W_1 of $Y|_U \times \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}$ and an open submanifold W_2 of $Y|_V \times \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}$ such that $Y|_{W_1} = Y|_{W_2} = Y|_{U\cap V}$. Using Proposition 2.1.18, the isomorphism θ writes (with evident notations) as

$$\theta : (q, (x', y')) \mapsto (q, (z'(q, x', y'), w'(q, x', y'))),$$

where $q \in |Y| \cap U \cap V$, z', w' are analytic in all arguments, each component of z' is divisible by some of the variables x' and, moreover, for any fixed q, the jacobian matrix of (z', w') with respect to the variables (x', y') is non singular. We proceed similarly as in the proof of Lemma 2.4.6 (this time as a parametric version with parameter $q \in Y$) to lift the isomorphism θ to an isomorphism $\tilde{\theta}$ to the blown-up spaces. □

Theorem-definition 3.2.8. Let $A = (|A|, \mathcal{O}_A)$ be a standard analytic manifold and $Z = (|Z|, \mathcal{O}_Z)$ an admissible center. Consider the topological space

$$|\widetilde{A}| = \bigsqcup_{U \text{ normalizing chart}} |U \cap Z| \times \widetilde{\mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}} \Big/ \sim$$

where the equivalence relation is defined for $p = (a, x) \in |U \cap Z| \times \widetilde{\mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}}$ and $q = (b, y) \in |V \cap Z| \times \widetilde{\mathbb{A}_+^{m(V)} \times \mathbb{R}^{n(V)}}$ as

$$p \sim q \Leftrightarrow \tilde{\theta}(a, x) = (b, y)$$

where $\tilde{\theta} : Z_{U\cap V} \times \widetilde{\mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}} \to Z_{U\cap V} \times \widetilde{\mathbb{A}_+^{m(V)} \times \mathbb{R}^{n(V)}}$ is the isomorphism given in lemma 3.2.7.

For any U normalizing chart of Z, let $\mathcal{U}(U)$ be an $\mathcal{O}$-atlas of $\widetilde{\mathbb{A}_+^{m(U)} \times \mathbb{R}^{n(U)}}$. Then, $\{|Z \cap U| \times \mathcal{U}(U)\}_{U \text{ normalizing chart}}$ is an $\mathcal{O}$-atlas of $|\widetilde{A}|$ because the change of charts are analytic. Then we can endow $|\tilde{A}|$ with an structure of $\mathcal{O}$-manifold $\tilde{A} = (|\tilde{A}|, \mathcal{O}_{\tilde{A}})$.

The map $\pi_Z^A : |\tilde{A}| \to |A|$ defined by the restriction of the blowing-up morphism $\pi_Z^A(U)$ for any normalizing domain is a well defined continuous, surjective, proper map that induces a morphism from $\tilde{A}$ to A. Moreover, it restricts to an isomorphism from the open submanifold $\tilde{A} \setminus (\pi_Z^A)^{-1}(Z)$ onto $A \setminus Z$.

Any pair (B, π) where B is an $\mathcal{O}$-manifold and $\pi : B \to A$ is a morphism for which there exists an isomorphism $\theta : B \to \tilde{A}$ such that $\pi_Z^A \circ \theta = \pi$ will be called a **blowing-up of A with center Z**. The inverse image $\pi^{-1}(Z)$ will be called the **exceptional divisor of the blowing-up π**. It is a regular subvariety of B of codimension one (in fact an admissible center).

The definition above of blowing-up with an admissible center in an analytic manifold is of global nature. In fact, the Theorem above shows that for any closed admissible center $Y \subset A$ there always exists a blowing-up of A with center Y. This is a result which will be untrue in the category of generalized manifolds.

For our purposes later, we will not need to make blowing-ups repeatedly with global closed admissible centers in the whole manifold, but only with centers that are locally closed, i.e., closed in some open submanifold.

Definition 3.2.9. Let A be a standard analytic manifold with boundary and corners. A **local blowing-up on A (with locally closed admissible center)** is a pair (B, π) where B is an $\mathcal{O}$-manifold and $\pi : B \to A$ is a morphism obtained as the composition

$$\pi = i \circ \tau : B \to U \hookrightarrow A,$$

where $i : U \hookrightarrow A$ is an open submanifold and $\tau : B \to U$ is a blowing-up on U with an admissible center $Y \subset U$ closed in U.

Example 3.2.10. As an example, if $Y \subset A$ is an admissible center of an analytic manifold A, $\varphi : U \simeq Y|_U \times \mathbb{A}_+^{m'} \times \mathbb{R}^{n'}$ is a normalizing chart and $(\mathbb{R}^{\widetilde{m',n'}}, \pi^{m',n'})$ is the blowing up on $\mathbb{A}_+^{m'} \times \mathbb{R}^{n'}$ at the origin, the composition

$$\pi = i_U \circ \varphi^{-1} \circ (id, \pi^{m'n'}) : Y|_U \times \mathbb{R}^{\widetilde{m',n'}} \to A$$

is a local blowing-up.

All of this kind of examples with codimension of Y less or equal than two (that is, $m'+n' \leq 2$) can be made explicit with the use of Examples 3.2.4 with the role of m, n there as m', n' here. In order to give precise expressions in local charts, we just take the expressions already presented in those corresponding examples and take the cartesian product with the identity for local coordinates on the subvariety Y.

Recall once more that blowing-up a center of codimension one may produce some effect, contrary to the case of standard projective blowing-up in analytic manifolds without boundary: if for instance we have $m' = 0$, $n' = 1$, the local blowing-up writes as

$$\pi : Y|_U \times \mathbb{A}_+^1 \sqcup Y|_U \times \mathbb{A}_+^1 \to Y|_U \times \mathbb{R}; \pi(q, y) = (q, \pm y), \text{ for } (q, y) \in Y|_U \times \mathbb{A}_+^1,$$

where the sign $+$ or $-$ depends if the point is in the first or the second of the copies $Y|_U \times \mathbb{A}_+^1$. Geometrically, we add a new boundary component, $\{y = 0\}$, of codimension one so that the non-boundary normal-to-Y variable y becomes a boundary variable after the blowing-up.

3.3 Blowing-up on generalized analytic manifolds.

In this paragraph, we define the notion of blowing-up generalized manifolds with admissible centers.

The same approach as in the case of standard manifolds (i.e. define the blowing-up of a point in the local models and then use coordinates in a general manifold) does not work. The problem is that the analogous of Proposition 3.2.5, that permits to define the blowing-up independently of the used coordinates, does not hold.

Example 3.3.1. Let $M = \mathbb{L}^2$ be the quadrant in the plane, as the local model of $\mathcal{G}$-manifold, with coordinates (x, y). A (a priori) good candidate for the blowing-up of M can be constructed as follows (analogously as in example *iv*) in 3.2.4): consider the $\mathcal{G}$-manifold $\tilde{M} = (|\tilde{M}|, \mathcal{G}_{\tilde{M}})$ where $|\tilde{M}|$ is the quotient space from the disjoint union of two copies of $\mathbb{L}^2$ with coordinates (x_1, y_1) and (x_2, y_2), respectively, by the equivalence relation

$$x_1 \neq 0, y_2 \neq 0, x_1 = x_2 y_2 \text{ and } y_2 = x_1 y_1,$$

and where the sheaf $\mathcal{G}_{\tilde{M}}$ is obtained by the consideration of the two systems of coordinates (x_1, y_1) and (x_2, y_2) as a $\mathcal{G}$-atlas. Then we consider $\pi_0 : \tilde{M} \to M$ as the $\mathcal{G}$-morphism induced by the map defined by $\pi_0(x_1, y_1) = (x_1, y_1)$ and $\pi_0(x_2, y_2) = (x_2 y_2, y_2)$. This morphism π_0 has the required properties analogous to those in Definition-Theorem 3.2.6. Consider now the isomorphism

$$\theta : M \to M,\ \theta(x, y) = (x^\alpha, y),$$

where $\alpha > 0$ is different from 1. Then there is no isomorphism $\tilde{\theta} : \tilde{M} \to \tilde{M}$ with the property

$$\pi_0 \circ \tilde{\theta} = \theta \circ \pi_0. \tag{3.3}$$

(There is no even a local isomorphism defined in a neighborhood of the exceptional divisor $\pi_0^{-1}(0)$ satisfying (3.3)).

The reason is obvious: the isomorphism θ gives a correspondence between the family of "half-lines" $\{y = \lambda x\}_\lambda$ inside the quadrant into the family of curves (regular submanifolds) $\{y = \lambda x^\alpha\}_\lambda$. The morphism π_0 has the effect of "opening" the family of half-lines so that each element accumulate to a single point in the exceptional divisor, whereas it does not open the later family so that the inverse image of each member of that family accumulate to a unique point in the exceptional divisor. As a consequence, any morphism $\tilde{\theta}$ satisfying (3.3) would not be $1:1$ in restriction to the exceptional divisor.

The above example makes necessary in the category of generalized manifolds to speak, not about a blowing-up with an admissible center, but about a blowing-up with an admissible center *relatively to some coordinates.*

When we want to precise what does it mean *relatively to some coordinates* we find out that a more convenient terminology is that of the standardizations.

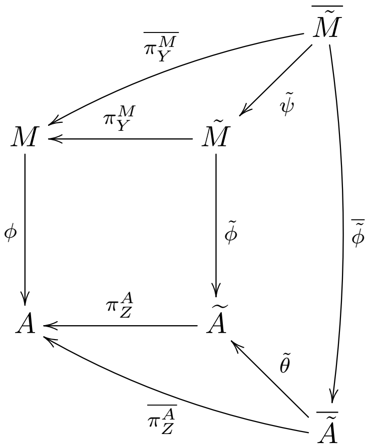
Theorem-definition 3.3.2. Let M be a $\mathcal{G}$-manifold and let $Y \subset M$ be a closed (connected) admissible center in M such that the pair (M, Y) is standardizable by means of a standardization $\phi : M \to A$. Let $Z = \phi(Y) \subset A$, by definition of standardization, a (closed and connected) admissible center in A. Let $(\tilde{A}, \pi_Z^A)$ be a blowing-up on A with center Z. Then there exists a triple $(\tilde{M}, \pi_Y^M, \tilde{\phi})$ where $\tilde{M}$ is a $\mathcal{G}$-manifold, $\pi_Y^M : \tilde{M} \to M$ is a morphism of $\mathcal{G}$-manifolds and $\tilde{\phi} : \tilde{M} \to \tilde{A}$ is a standardization of $\tilde{M}$ such that $(\tilde{A}, \pi_Z^A)$ is a blowing-up of A with center $Z = \phi(Y)$ and the diagram

$$\begin{array}{ccc} Y \subseteq M & \xleftarrow{\pi_Y^M} & \tilde{M} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\tilde{\phi}} \\ Z = \phi(Y) \subseteq A & \xleftarrow{\pi_Z^A} & \tilde{A} \end{array}$$

commutes.

If $(\overline{\tilde{A}}, \overline{\pi_Z^A})$ is another blowing-up on A with center Z and $(\overline{\tilde{M}}, \overline{\pi_Y^M}, \overline{\tilde{\phi}})$ is the corresponding triple, then there exist isomorphisms $\tilde{\theta} : \overline{\tilde{A}} \to \tilde{A}$ and $\tilde{\psi} : \overline{\tilde{M}} \to \tilde{M}$ and a standardization $\overline{\tilde{\phi}} : \overline{\tilde{M}} \to \overline{\tilde{A}}$

making the whole diagram



(3.4)

commuting. Any such triple so constructed will be called a **blowing-up of M with center Y relatively to the standardization ϕ**. For any such blowing-up, the inverse image $D = (\pi_Y^M)^{-1}(Y)$ is a regular submanifold of codimension 1, called the **exceptional divisor of the blowing-up** and π_Y^M is a proper, surjective morphism which restricts to an isomorphism from $\tilde{M} \setminus D$ to $M \setminus Y$.

Proof. The existence of such a triple $(\tilde{M}, \pi_Y^M, \tilde{\phi})$ is given as follows. Given a blowing-up $(\tilde{A}, \pi_Z^A)$ with center Z, we consider just $\tilde{M}$ as the enrichment $\tilde{A}^e$ of the analytic manifold $\tilde{A}$ and $\tilde{\phi} : \tilde{M} \to \tilde{A}$ as the morphism induced by the identity map in the underlying spaces. The morphism of blowing-up $\pi_Y^M : \tilde{M} \to M$ is given by using Proposition 2.4.6: the blowing-up morphism π_Z^A is locally monomial by Theorem-Definition 3.2.8.

The second claim about the commutativity of the diagram (3.4) is proved similarly: the existence of the isomorphism $\tilde{\theta} : \overline{\tilde{A}} \to \tilde{A}$ is guaranteed by Theorem-Definition 3.2.8. This isomorphism, being locally monomial, lifts to an isomorphism $\tilde{\psi} : \overline{\tilde{M}} \to \tilde{M}$ with the required properties of commutativity.

The rest of the properties come easily from the corresponding properties on the bottom row of the standard analytic manifolds: In one hand, any topological property of the underlying map, π_Y^A, of the blowing-up of A with center Z is directly translated to the map π_Y^M since the standardizations ϕ and $\tilde{\phi}$ are homeomorphisms. On the other hand, if p is a point of $\tilde{M}$ not in the exceptional divisor D of π_Y^M then $\tilde{\phi}(p)$ is not in the exceptional divisor of π_Z^A and, since the later is a local isomorphism at that point, the same occurs for π_Y^M at p. □

Definition 3.3.3. Let M be a generalized analytic manifold. A **local blowing-up on A (with locally closed admissible center)** is a pair (N, π, ϕ) where N is a G-manifold, $\pi : N \to M$ is a morphism obtained as the composition

$$\pi = i \circ \tau : N \to U \hookrightarrow M,$$

where $i : U \hookrightarrow M$ is an open submanifold and $\tau : N \to U$ is a blowing-up morphism on U with an admissible center $Y \subset U$ closed in U with respect to some standardization $\phi : U \to V$ of the pair (U, Y).

Example 3.3.4. Using Proposition 3.1.10, if $Y \subset M$ is an admissible center of generalized manifold M and $\varphi : U \simeq Y|_U \times \mathbb{L}^{m'} \times \mathbb{R}^{n'}$ is a normalizing chart, then we can assume that $Y|_U \subset U$ is standardizable.

A particular case that we will use repeatedly is when Y is of codimension two $(m' + n' = 2)$.

Denote by (x_i, x_j) the variables of $\mathbb{L}^{m'} \times \mathbb{R}^{n'}$ they can be generalized or analytic variables). Consider the closed admissible center $Y = \{x_i = x_j = 0\}$ inside U. Let $\gamma > 0$ and consider the standardization of the pair (U, Y) given by

$$\phi_\gamma : U \to \mathbb{A}_+^m \times \mathbb{R}^n, \phi_\gamma(x,y) = (x_1, \ldots, x_i^\gamma, \ldots, x_j, y).$$

Let $\pi : \tilde{M} \to U \hookrightarrow M$ the local blowing-up with center Y relatively to the standardization ϕ_γ. Then $\tilde{M}$ is covered by two charts (x', y') and (x'', y''), both with values in $\mathbb{R}_{\geq 0}^{m'} \times \mathbb{R}^{n'}$, such that the expression of the blowing-up morphism is

$$\pi(x', y') = (x_1', \ldots, x_i', \ldots, (x_i')^\gamma x_j', \ldots, y'),$$

$$\pi(x'', y'') = (x_1', \ldots, x_j', \ldots, (x_j')^{1/\gamma} x_i', \ldots, y'),$$

We notice again that the definition of blowing-up on a generalized manifold with a closed admissible center requires also to specify a standardization of the manifold (or at least of an open submanifold containing the center). If such a standardization does not exists then, a priori, we have not the possibility to blow-up this center.

The example of the exotic cylinder (cf. Example 2.3.5) gives an example. Consider $\mathcal{C}_\alpha$ one of those exotic cylinders with $\alpha \neq 1$ and put $M = \mathcal{C}_\alpha \times \mathbb{L}^1$. Then $Y = \partial\mathcal{C}_\alpha \times \{0\}$ is an admissible center of M of codimension two. It has no standardizable open neighborhood in M so it can no be used as a center of blowing-up with the meaning of Theorem-Definition 3.3.2. Geometrically, there is no good "generalized normal bundle along Y". Very roughly speaking, if we start at some point $p \in |Y|$ with a family of (local) regular surfaces of the form $\{x = z^\gamma\}_\gamma$, where $(x, y, z) \in \mathcal{C}_\alpha \times \mathbb{L}^1$ are coordinates at p, then the exponent γ transforms into another one and the corresponding surfaces locally defined do not match.

3.4 Local Monomialisation Theorem.

Before the statement of the main result, we consider the following useful definition.

Definition 3.4.1. Let M be a generalized analytic manifold and p a point in M. A **proper *étoilé*-neighborhood (or é-neighborhood) of** p (the name is taken from what Hironaka calls "voûte étoilé") is a finite family

$$\Sigma = \{\pi_j : W_j \to M, L_j\}_{j \in J}$$

where

1. each π_j is the composition of a sequence of finitely many local blowing-ups (with admissible centers)
$$\pi_j : W_j = W_{j,n_j} \overset{\pi_{j,n_j}}{\to} W_{j,n_j-1} \overset{\pi_{j,n_j-1}}{\to} W_{j,n_j-2} \cdots \overset{\pi_{j,1}}{\to} W_{j,0} = M$$
2. each L_j is a compact subset of $|W_j|$ such that $\cup_{j\in J}\pi_j(L_j)$ is a compact neighborhood of p in $|M|$.

Theorem 3.4.2. (Local Monomialisation of $\mathcal{G}$-analytic functions) Let M be a generalized analytic manifold and $f \in \mathcal{G}(M)$ a $\mathcal{G}$-analytic function. Given $p \in |M|$ there exists a proper é-neighborhood $\Sigma = \{\pi_j : W_j \to M, L_j\}_{j\in J}$ of p such that for all $j \in J$, $f \circ \pi_j : W_j \to \mathbb{R}$ is locally monomial at any point of L_j. We can furthermore take such a proper é-neighborhood such that any of the local blowing-ups involved in it is with an admissible center of codimension ≤ 2.

The following result about "composition" of proper é-neighborhoods is an easy consequence of the definitions

Remark 3.4.3. Let $p \in M$ and $\Sigma = \{\pi_j : W_j \to M, L_j\}_{j\in J}$ a proper é-neighborhood of p. Suppose that for every $q \in L = \cup_{j\in J} L_j$ there exists a proper é-neighborhood $\Sigma_q = \{\pi_{q,j} : W_{q,j} \to W_j, L_{q,j}\}_{j\in J(q)}$ of q. Then since $V_q := \cup_{j\in J(q)}\pi_{q,j}(L_{q,j})$ is a neighborhood of q, by compactness of L (in the disjoint union of the topological spaces $|W_j|$), there exists finitely many points $q_1, q_2, \ldots, q_l \in L$, such that $L \subseteq V_{q_1} \cup V_{q_2} \cup \ldots \cup V_{q_l}$. Then, the set

$$\bigcup_{i=1}^{l}\{\pi_j \circ \pi_{q_i,j} : W_{q_i,j} \to M, L_{q_i,j}\}_{j\in J(q_i)}$$

is a proper é-neighborhood.

We will make use of the remark above several times during the proof of Theorem 3.4.2. Notably in order to reduce the proof to every point of the exceptional divisor after a local blowing-up with an admissible center that passes through the point p. More precisely:

Lemma 3.4.4. Let U be a neighborhood of p in $|M|$ and let $Y \subset U$ a standardizable admissible center such that $p \in |Y|$. Let $\pi_Y^U : \tilde{U} \to U \hookrightarrow M$ be the local blowing-up of M with center Y with respect to a given standardization ϕ of $Y \subset U$. Let $D = (\pi_Y^U)^{-1}(Y)$ be the exceptional divisor of the blowing-up and $D_p = (\pi_Y^U)^{-1}(p)$ the fiber over p. If Theorem 3.4.2 holds at any point $q \in |D_p|$ then it holds at the point p.

Proof. For any $q \in |D_p|$, let $\Sigma_q = \{\pi_{q,j} : W_{q,j} \to \tilde{U}, L_{q,j}\}$ be a é-neighborhood of q for which Theorem 3.4.2 is true. Denote by $V_q = \cup_j \pi_{q,j}(L_{q,j})$, a (compact) neighborhood of q in $\tilde{U}$, and consider a smaller compact neighborhood $\tilde{V}_q$ of q such that $\tilde{V}_q$ is contained in the interior $int(V_q)$ of V_q. Notice that Σ_q is also a proper é-neighborhood of any point in $\tilde{V}_q$. Since the blowing-up is a proper mapping (see 3.3.2), there exists finitely many points $q_1, ..., q_n \in |D_p|$ such that $D_p \subset \cup_l \tilde{V}_{q_l}$. Then $\Sigma = \{\pi_Y^U : int(V_{q_l}) \to U, \tilde{V}_{q_l}\}_l$ is a proper é-neighborhood of p. The result follows from the Remark 3.4.3 above. □

The rest of this section is devoted to the proof of the Main result Theorem 3.4.2. We prove it by induction on the dimension of M. Notice that if $k = \dim M = 1$ the result is immediate. For $k \geq 2$ the proof is given in several steps.

3.4.1 The case of a Weierstrass polynomial

Proposition 3.4.5. Let $p \in M$, $\dim M = k \geq 2$ and $f \in G(M)$. Assume that theorem 3.4.2 is true for $\dim M < k$. Assume that the number $m(p)$ of boundary components of M at p is strictly smaller than k and that there exists a local chart $(U, \varphi = (x, y))$ where y is an analytic variable such that

$$f(\underline{x}, y) = y^d + a_1(\underline{x})y^{d-1} + a_2(\underline{x})y^{d-2} + \ldots + a_d(\underline{x})$$

with $a_i \in \mathcal{G}(U)$ is independent of the coordinate y and $a_i(p) = 0$ for all i. Then Theorem 3.4.2 is true for f at p.

Proof. If $d = 1$ the change of coordinates $x_1 = x, y_1 = y - a_1(\underline{x})$ gives a new local chart (see Proposition 2.3.15) for which f is monomial at p. If $d > 1$, we make first the Tschirnhausen transformation: $y \rightsquigarrow y - \frac{a_1(\underline{x})}{d}$ so that in these new coordinates we write

$$f(\underline{x}, y) = y^d + b_2(\underline{x})y^{d-2} + b_3(\underline{x})y^{d-3} + \ldots + b_d(\underline{x})$$

with $b_i \in \mathcal{G}(U)$ does not depend on the coordinate y and and $b_i(p) = 0$ for all i. Remark that we can consider the b_i as $\mathcal{G}$-analytic functions on the (admissible) subvariety $M' = \{(x, y) \in U : y = 0\}$ of U which is a $\mathcal{G}$-manifold of dimension $k - 1$.

Special case.- We consider first the special case where we can furthermore write

$$f(\underline{x},y) = y^d + x^{\alpha_2}u_2(x)y^{d-2} + \ldots + x^{\alpha_d}u_d(x)$$

such that the set of vectors $\{\alpha_l/l\}_{l=2,\ldots,d}$ is totally ordered (by the division order). Take r such that $\alpha_r/r \leq \alpha_j/j$ for all j, $2 \leq j \leq d$. Take l such that $\alpha_{r,l} \neq 0$. Consider the admissible center $Y = \{y = x_l = 0\} \subset U$, closed in U and of codimension 2, together with the standardization of the pair (U,Y) given by

$$\phi : U \to \mathbb{R}^{k-1}_{\geq 0} \times \mathbb{R}, \phi = (x_1, \ldots, x_{l-1}, x_l^{\alpha_{r,l}/r}, x_{l+1}, \ldots, x_{-1}, y).$$

The corresponding (local) blowing-up $\pi^U_Y : \tilde{U} \to M$ with center Y and with respect to this standardization is such that $\tilde{U}$ is covered by two charts (x',y') and (x'',y''), both with values in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, so that the exceptional divisor $(\pi^U_Y)^{-1}(Y)$ has equations $\{x'_l = 0\}$ and $\{y'' = 0\}$ and such that the morphism π^U_Y writes

$$\pi^U_Y(x',y') = (x', (x'_l)^{\alpha_{r,l}/r}y'),$$

$$\pi^U_Y(x'',y'') = (x''_1, \ldots, (y'')^{r/\alpha_{r,l}}x''_l, \ldots, y'').$$

Let $q \in (\pi^U_Y)^{-1}(Y)$. There are 3 cases

1. <u>q is the origin of the chart (x'',y'').</u>

   We obtain

   $$(y'')^d + \ldots + (x'')^{\alpha_j}u_j(x'')(y'')^{d-j+\alpha_{j,l}\frac{r}{\alpha_{r,l}}} + \ldots$$

   As $\alpha_r/r \leq \alpha_j/j$ for all $2 \leq j \leq d$, $\frac{r\alpha_{j,l}}{\alpha_{r,l}} - j \geq 0$ and we can factor out $(y'')^d$

   $$(y'')^d(1 + \ldots + (x'')^{\alpha_j}u_j(x'')(y'')^{\alpha_{j,l}\frac{r}{\alpha_{r,l}}-j} + \ldots)$$

   and the expression in brackets is a unit

2. <u>q is the domain of the chart (x',y') but it is not the origin of this chart.</u>

   In order to simplify notations, put (x,y) instead of (x',y'). Then, locally around q we have coordinates $(x, \lambda + y)$ where $\lambda = -y(q)$. We obtain

   $$x_l^{d\frac{\alpha_{r,l}}{r}}(\lambda+y)^d + \ldots + x_l^{(d-j)\frac{\alpha_{r,l}}{r}+\alpha_{j,l}}x'^{\alpha'_j}u_j(x)(\lambda+y)^{d-j} + \ldots$$

   (notice that here $x'^{\alpha'_j}$ means $x_1^{\alpha_{1,j}} \cdots x_{l-1}^{\alpha_{j,l-1}}x_{l+1}^{\alpha_{j,l+1}} \cdots x_m^{\alpha_{j,m}}$)). As $\alpha_r/r \leq \alpha_j/j$ for all $2 \leq j \leq d$, $\alpha_{j,l} - j\alpha_{r,l}/r \geq 0$ and we can factor out $x_l^{d\alpha_{r,l}/r}$

   $$x_l^{d\frac{\alpha_{r,l}}{r}}((\lambda+y)^d + \ldots + x_l^{\alpha_{j,l}-j\frac{\alpha_{r,l}}{r}}x'^{\alpha'_j}u_j(x)(\lambda+y)^{d-j} + \ldots)$$

   by the Tschirnhausen transformation the coefficient of y^{d-1} is $\lambda \neq 0$ so the expression in brackets is regular of order less or equal to $d-1$ in y and by Weierstrass preparation we can assume that is a Weierstrass polynomial of degree less or equal to $d-1$.

3. <u>q is the origin of the chart (x',y').</u>

   Again, we put (x,y) instead of (x',y'). We obtain

   $$x_l^{d\frac{\alpha_{r,l}}{r}}y^d + \ldots + x_l^{(d-j)\frac{\alpha_{r,l}}{r}+\alpha_{j,l}}x'^{\alpha'_j}u_j(x)y^{d-j} + \ldots$$

As $\alpha_r/r \leq \alpha_j/j$ for all $2 \leq j \leq d$, $\alpha_{j,l} - j\alpha_{r,l}/r \geq 0$ and we can factor out $x_l^{d\alpha_{r,l}/r}$

$$x_l^{d\frac{\alpha_{r,l}}{r}}(y^d + \ldots + x_l^{\alpha_{j,l}-j\frac{\alpha_{r,l}}{r}} x'^{\alpha'_j} u_j(x) y^{d-j} + \ldots)$$

If $\alpha_{r,i} = 0$ for all $i \neq l$ then the expression in brackets is regular of order $d-l$, and by Weierstrass preparation we can assume that is a W. polynomial of degree $d-r$. If there exists $i \neq l$ such that $\alpha_{r,i} \neq 0$ we proceed by making a local blowing-up with the corresponding center of codimension two relatively to a suitable standardization such that the morphism has the expression in two charts (x', y'), (x'', y'')

$$\pi_Y^U(x', y') = (x', (x'_l)^{\alpha_{r,i}/r} y'),$$

$$\pi_Y^U(x'', y'') = (x''_1, \ldots, (y'')^{r/\alpha_{r,i}} x''_l, \ldots, y'').$$

This works analogously because we have chosen r such that $\alpha_r/r \leq \alpha_j/j$ for all j, $2 \leq j \leq d$, which means that $\alpha_{r,s}/r \leq \alpha_{j,s}/j$ for all j, $2 \leq j \leq d$ and $1 \leq s \leq k-1$. The "bad" case will be again at the origin of the chart (x', y') but we will have the same polynomial with less variables x appearing in the monomial of the coefficient of y^{d-r}. After at most $k-1$ steps we have finished.

General case.- Let $b \in \mathcal{G}(M')$ denote the $\mathcal{G}$-analytic function obtained as the product of all non-zero functions among the b_i as well as their non-zero differences. By the hypothesis that Theorem 3.4.2 is true for dimension smaller than k, there exists a proper é-neighborhood $\sigma' = \{\pi'_j : W'_j \to M', L'_j\}$ of p in M' (where the centers for the local blowing-ups involved are of codimension ≤ 2) such that $b \circ \pi'_j$ is monomial at any point of L_j, for any j. Fix some $\delta > 0$ such that $(-\delta, \delta)$ is contained in the range of values of the coordinate y. Consider, for each j, the morphism obtained by "fibering" π'_j on the variable y; precisely:

$$\pi_j : W_j = W'_j \times (-\delta, \delta) \to M, \ \pi_j(q, t) = \varphi^{-1}(x(\pi'_j(q)), t).$$

Then π_j is a composition of local blowing-ups with admissible centers (as in 3.3.3). We conclude that

$$\Sigma = \{\pi_j : W_j \to M, L_j = L'_j \times [-\delta/2, \delta/2]\}$$

is an é-neighborhood of the point p (with centers of codimension ≤ 2). Using Lemma 3.4.4, it suffices to prove Theorem 3.4.2 for the transform, $f \circ \pi_j \in \mathcal{G}(W_j)$, of f by π_j at any (fixed but arbitrary) point in L_j, in fact, taking δ sufficiently small, at any point of $L'_j \times \{0\} \subset W_j$. Fix some of these points $(q, 0)$. By construction, there exist local coordinates x' at $q \in W'_j$ such that $b \circ \pi'_j$ is locally monomial at q with respect to x'. Consequently, using *iv*) of Remark 2.3.17, the transformation $b_l \circ \pi_j$ of each of the coefficients b_l is locally monomial with respect to the same coordinates at q. Moreover, considering b also as a function in M and, since $b \circ \pi_j$ does not depend on the second component of $W_j = W'_j \times (-\delta, \delta)$, we conclude that it is locally monomial at the point $(q, 0) \in L_j$ with respect to the coordinates (x', t) in $W_j = W'_j \times (-\delta, \delta)$ (where t is the usual coordinate in $\mathbb{R}$). Write locally at $(q, 0)$, $b_l \circ \pi_j = (x')^{\alpha_l} u'_l(x', t)$ where u'_l does not vanish at $(q, 0)$. Then we have a local expression of $f \circ \pi_j$ at $(q, 0)$ as

$$f \circ \pi_j = t^d + (x')^{\alpha_2} u'_2 t^{d-2} + \cdots + (x')^{\alpha_d} u'_d.$$

The proposition is finished, thanks to the special case considered above, once we prove the following

Claim.- Up to a further composition of local blowing-ups with admissible centers of codimension ≤ 2, we can suppose that the set of vectors $\{\alpha_l/l\}_{l=2,\ldots,d}$ is totally ordered (by the division order).

Proof of the Claim.- First, after performing blowing-ups with centers at the coordinate planes

$x'_i = 0$ (of codimension 1), we can suppose that the number of boundary components of q in W'_j is maximal, equal to $m(q) = k - 1$. In this case, $(b_l \circ \pi'_j)^{1/l} = (x')^{\alpha_l/l}(u')_l^{1/l}$ is a $\mathcal{G}$-analytic function. Now, consider the function $\tilde{b}$ obtained as the product of all non-zero functions and of the non-zero differences among the family $\{(b_l \circ \pi'_j)^{1/l}\}_l$. Repeating the argument, Theorem [main] being by hypothesis true for $\tilde{b}$, up to further local blowing-ups, $\tilde{b}$ and all its factors can be considered as a locally monomial function (with respect to the same system of coordinates). Now $v)$ of Remark 2.3.17 gives the desired result about the exponents α_l/l. □

3.4.2 The b invariant for a $\mathcal{G}$-analytic function.

Let M be a $\mathcal{G}$-manifold, $p \in |M|$ and f a $\mathcal{G}$-analytic function at p. Let $(U, \varphi = (x, y))$ be a local chart of M centered at p ($x = (x_1, \ldots, x_{m_p})$ and $y = (y_1, \ldots, y_{n_p})$). Assume the notations of paragraph 1.1.4. By 2.3.7 and, we can define $b(f, p, (U, \varphi)) := b(s) \in \mathbb{N}^2$ where $s \in \mathbb{R}\{X^*, Y\} \subseteq (\mathbb{R}[[Y]])[[X^*]]$ is the Taylor expansion of f at p with respect to the coordinates (x, y) (notice that s is defined up to a permutation $\sigma \in G_{m,n}$, but for such a σ we have that $b(s) = b(\sigma s)$).

Proposition 3.4.6. $b(f, p, (U, \varphi))$ does not depend on the local chart (U, φ).

Proof. Let $(V, \psi = (z, w))$ be another local chart at p. Denote by $s_\varphi \in \mathbb{R}\{X^*, Y\}$ and $s_\psi \in \mathbb{R}\{Z^*, W\}$ respectively the Taylor expansion of f at p with respect to the coordinates (x, y) and (z, w). Denote by $\phi : \mathbb{R}\{Z^*, W\} \to \mathbb{R}\{X^*, Y\}$ the isomorphism induced by the change of coordinates as in 2.3.11. Notice that up to a permutation $\sigma \in G_{m_p, n_p}$, we can suppose $\phi(s_\psi) = s_\varphi$. By 2.3.15, for $1 \le j \le m_p$, $z_j = x_{i(j)}^{a_j} g_j(x, y)$ with $a_j > 0$, $g_j(0,0) > 0$ and i a permutation of the index $\{1, \ldots, m\}$. Thus, $\phi(Z_j) = X_{i(j)}^{a_j} G_j$ where $G_j \in \mathbb{R}\{X^*, Y\}$ is such that $G_j(0,0) > 0$. Then for any exponent $\alpha \in [0, \infty)^m$, $\phi(Z^\alpha) = X^{a(\alpha)} G$ where $G \in \mathbb{R}\{X^*, Y\}$ is such that $G_j(0,0) > 0$ and $a(\alpha) := (a_{i^{-1}(1)}\alpha_{i^{-1}(1)}, \ldots, a_{i^{-1}(m)}\alpha_{i^{-1}(m)})$. Since $(a_{i^{-1}(1)}, \ldots, a_{i^{-1}(m)}) \in (0, \infty)^m$, for any $\alpha, \beta \in [0, \infty)^m$, $a(\alpha) \le a(\beta) \Leftrightarrow \alpha \le \beta$, which in turn implies $d(\alpha, \beta) = d(a(\alpha), a(\beta))$. Then, as $\phi(s_\psi) = s_\varphi$, $b(s_\varphi) = b(s_\psi)$ and so $b(f, p, (U, \varphi)) = b(f, p, (V, \psi))$. □

Then, we let $b(f, p) = (b_1(f, p), b_2(f, p)) \in \mathbb{N}^2$ denote the invariant of a $\mathcal{G}$-analytic function in a point p of M. As a corollary, the numerical data $I(f, p) = (m_p, n_p, b(f, p))$ is a well defined invariant in $\mathbb{N}^4$ depending only on f and the point p. In fact, the first two components depends only on p and M. Notice that with this definition, $b_2(f, p) > 0$ implies $m_p > 1$.

The following result permits to prove theorem 3.4.2 by induction in the invariant $I(f, p)$ in lexicographic order when $b_2(f, p) > 0$.

Theorem 3.4.7. Let $f \in \mathcal{G}(M)$ and $p \in |M|$ and assume that $b_2(f, p) > 0$. Then there exists a local blowing-up $\pi : \tilde{M} \to M$ with admissible center Y through p, of codimension 2, such that for any point q in the fiber $\pi^{-1}(p)$ of the exceptional divisor, the transformed function $\tilde{f} = f \circ \pi \in \mathcal{G}(\tilde{M})$ satisfies

$$I(\tilde{f}, q) < I(f, p).$$

Proof. Consider a local chart $(U, (x, y))$ centered at p where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_n)$, where $m = m_p$, $n = n_p$, and let $s \in \mathbb{R}\{X^*, Y\}_{m,n}$ be the Taylor expansion of f at p with respect to these coordinates. Then $b(f, p) = b(s)$, considered s as a series in $\mathbb{R}\{Y\}[[X^*]]$. By Proposition 1.1.23, there exists $\gamma > 0$ and two different indices $i, j \in \{1, \ldots, m\}$ such that the transformations ς_{ij}^{γ} y $\varsigma_{ji}^{1/\gamma}$ (defined in 3.3.4) applied to s gives series with smaller b-invariant. Consider the closed admissible center $Y = \{x_i = x_j = 0\}$ inside U and the standardization of the pair (U, Y) given by

$$\phi : U \to \mathbb{A}_+^m \times \mathbb{R}^n, \phi(x, y) = (x_1, \ldots, x_i^\gamma, \ldots, x_j, y).$$

Let $\pi : \tilde{M} \to U \hookrightarrow M$ the local blowing-up with center Y and associated to the standardization ϕ. Then, as in the example 3.3.4, $\tilde{M}$ is covered by two charts (x', y') and (x'', y''), both with values in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, such that the expression of the blowing-up morphism is

$$\pi(x', y') = (x'_1, \ldots, x'_i, \ldots, (x'_i)^\gamma x'_j, \ldots, y'),$$

$$\pi(x'', y'') = (x''_1, \ldots, x''_j, \ldots, (x''_j)^{1/\gamma} x''_i, \ldots, y'').$$

Thus, we see that the Taylor expansion of $\tilde{f}$ at the origin p_1 of the first (respectively p_2 the origin of the second) chart with respect to (x', y') (respectively with respect to (x'', y'')) is just $\varsigma^{\gamma}_{ij}(s)$ (respectively $\varsigma^{1/\gamma}_{ji}(s)$). Moreover, $m_{p_1} = m_{p_2} = m$ and thus the Theorem is proved at those two points by our choice of the admissible center using Lemma 1.1.23.

Finally, for any point $q \in \pi^{-1}(p)$ different from p_2 in in the domain of the first chart we can use the local chart $((x')^q, y')$ centered at q where $(x')^q_l = x'_l$ if $l \neq j$ and $(x')^q_j = x'_j - x'_j(q)$. Assuming that $q \neq p_1$ we have $x'_j(q) \neq 0$ and thus $(x')^q_j$ becomes an analytic variable (it takes positive an negative values in a neighborhood of the point q). The rest of coordinates remaining unchanged, we obtain that $m_q = m_p - 1$ and thus also $I(\tilde{f}, q) < I(f, p)$ and we are done. □

In order to finish the proof of theorem 3.4.2 it remains the case $b_2(f, p) = 0$. In this situation, there are two possibilities:

1. $n_p = 0$. Then f is already locally monomial at p.

2. $n_p > 0$. Then there exists a local chart $(U, \varphi = (x, y))$ centered at p, $\alpha \in [0, \infty)^{m_p}$ and $g \in \mathcal{G}_M(U)$ such that
$$f(x, y) = x^\alpha g(x, y)$$
with $g(0, y) \not\equiv 0$ in $\varphi(U)$. Then, there exist a suitable change of coordinates involving only the y variables making y_{n_p} regular in g. By Weierstrass Preparation Theorem, g is under the hypothesis of Proposition 3.4.5 and the result follows.

Appendix A

Sheaves and ringed spaces.

We reproduce the definitions and results from the sheaves theory that we need as they appear in [8].

Definition A.0.8. Let X be a topological space. A *presheaf* $\mathcal{F}$ of rings on X consists of the data

(a) for every open subset $U \subseteq X$, a ring $\mathcal{F}(U)$ and

(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of rings $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

subject to the conditions

(0) $\mathcal{F}(\emptyset) = \{0\}$,

(1) ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and

(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

As a matter of terminology, if $\mathcal{F}$ is a presheaf on X, we refer to $\mathcal{F}(U)$ as the *sections* of the presheaf $\mathcal{F}$ over the open set U, and we sometimes use the notation $\Gamma(U, \mathcal{F})$ to denote the ring $\mathcal{F}(U)$. We call the maps ρ_{UV} *restrictions* maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$ if $s \in \mathcal{F}(U)$.

A sheaf is roughly speaking a presheaf whose sections are determinated by local data. To be precise, we give the following definition.

Definition A.0.9. A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions:

(3) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;

(4) if U is an open set, if $\{V_i\}$ is an open covering of U and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Note condition (3) implies that s is unique).

Example A.0.10. Let X be a topological space. For each open set $U \subseteq X$, let $\mathcal{C}^0(U;\mathbb{R})$ be the ring of continuous real-valued functions on U, and for each $V \subseteq U$, let $\rho_{UV} : \mathcal{C}^0(U;\mathbb{R}) \to \mathcal{C}^0(V;\mathbb{R})$ be the restriction map (in the usual sense). Then the assignment $U \mapsto \mathcal{C}^0(U;\mathbb{R})$ for any U open subset of X together with the restriction of maps as restrictions morphisms is a sheaf of rings on X that we call the *sheaf of continuous functions* on X and denote by $\mathfrak{C}_{C(X)}$. It is clear that $\mathfrak{C}_{C(X)}$ is a presheaf of rings. To verify the conditions (3) and (4), we note that a function which is 0 locally is 0, and a function which is continuous locally is continuous. In the same way we can define the sheaf of real analytic functions on a real analytic manifold. If M is a real analytic manifold we denote by $\mathcal{O}_M$ the sheaf of real analytic functions on M.

A *directed set* is a partially ordered set $(I, \leq)$ with the additional property that every pair of elements has a lower bound. Let $(I, \leq)$ be a directed set. Let $\{A_i : i \in I\}$ be a family of rings indexed by I and $f_{ji} : A_j \to A_i$ be a ring homomorphism for all $i \leq j$ with the following properties:

1. $f_{ii} : A_i \to A_i$ is the identity of A_i for all $i \in I$, and
2. $f_{ki} = f_{ji} \circ f_{kj}$ for all $i \leq j \leq k$.

Then the pair $\langle A_i, f_{ij} \rangle$ is called a *direct system* over I. The underlying set of the *direct limit*, A, of the direct system $\langle A_i, f_{ij} \rangle$ is defined as the disjoint union of the A_i's modulo an equivalence relation $\sim$:

$$A = \varinjlim A_i = \coprod_i A_i / \sim$$

Here, if $x_i \in A_i$ and $x_j \in A_j$, $x_i \sim x_j$ if there is some $k \in I$, $k \leq i$ and $k \leq j$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. One naturally obtains from this definition canonical morphisms $\phi_i : A_i \to A$ sending each element to its equivalence class. The ring operations on A are defined via these maps in the obvious manner.

Let $\mathcal{F}$ be a presheaf on X, and p a point of X. Let $\mathcal{E}_X(p)$ be the set of all open neighborhoods of p on X. We order partially $\mathcal{E}_X(p)$ with the inclusion order. Actually $(\mathcal{E}_X(p), \subseteq)$ is a directed set, because if $U, V \in \mathcal{E}_X(p)$, $U \cap V \in \mathcal{E}_X(p)$ and is a lower bound of U and V. We construct a direct system over $\mathcal{E}_X(p)$ considering the family of rings $\{\mathcal{F}(U) : U \in \mathcal{E}_X(p)\}$ and the ring homomorphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$.

Definition A.0.11. If $\mathcal{F}$ is a presheaf on X, and if p is a point of X, we define the *stalk* $\mathcal{F}_p$ of $\mathcal{F}$ at p to be the direct limit of the rings $\mathcal{F}(U)$ for all open sets U containing p via the restriction maps ρ.

Thus an element of $\mathcal{F}_p$ is represented by a pair $\langle U, s \rangle$, where U is an open neighborhood of p, and s is an element of $\mathcal{F}(U)$. Two such pairs $\langle U, s \rangle$ and $\langle V, t \rangle$ define the same element of $\mathcal{F}_p$ if and only if there is an open neighborhood W of p with $W \subseteq U \cap V$, such that $s|_W = t|_W$. Thus we may speak of elements of the stalk $\mathcal{F}_p$ as *germs* of sections of $\mathcal{F}$ at the point p. In the case of topological space X and its sheaf of continuous functions (example A.0.10) $\mathfrak{C}_{C(X)}$, the stalk $\mathfrak{C}_{C(X),p}$ at a point p is just the local ring of germs of continuous functions at p.

Definition A.0.12. If $\mathcal{F}$ and $\mathcal{H}$ are presheaves on X, a *morphism* $\varphi^\sharp : \mathcal{F} \to \mathcal{H}$ consists of a morphism of rings $\varphi^\sharp(U) : \mathcal{F}(U) \to \mathcal{H}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi^\sharp(U)} & \mathcal{H}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi^\sharp(V)} & \mathcal{H}(V) \end{array}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathcal{F}$ and $\mathcal{H}$. If $\mathcal{F}$ and $\mathcal{H}$ are sheaves on X, we use the same definition for a morphism of sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

Note that a morphism $\varphi^\sharp : \mathcal{F} \to \mathcal{H}$ of presheaves on X induces a morphism $\varphi^\sharp_p : \mathcal{F}_p \to \mathcal{H}_p$ on the stalks, for any point $p \in X$, given by $\varphi^\sharp_p(\langle U, s \rangle) = \langle U, \varphi^\sharp(U)(s) \rangle$ such that if U is an open neighborhood of p the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi^\sharp(U)} & \mathcal{H}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_p & \xrightarrow{\varphi^\sharp_p} & \mathcal{H}_p \end{array}$$

The following proposition (which would be false for presheaves) illustrates the local nature of a sheaf.

Proposition A.0.13. Let $\varphi^\sharp : \mathcal{F} \to \mathcal{H}$ be a morphism of sheaves on a topological space X. Then $\varphi^\sharp$ is an isomorphism if and only if the induced map on the stalk $\varphi^\sharp_p : \mathcal{F}_p \to \mathcal{H}_p$ is an isomorphism for every $p \in X$.

Proof. If $\varphi^\sharp$ is an isomorphism it is clear that each $\varphi^\sharp_p$ is an isomorphism. Conversely, assume $\varphi^\sharp_p$ is an isomorphism for all $p \in X$. To show that $\varphi^\sharp$ is an isomorphism, it will be sufficient to show that $\varphi^\sharp(U) : \mathcal{F}(U) \to \mathcal{H}(U)$ is an isomorphism for all U, because then we can define an inverse morphism $\psi^\sharp$ by $\psi^\sharp(U) = (\varphi^\sharp)^{-1}(U)$ for each U. First we show $\varphi^\sharp(U)$ is injective. Let $s \in \mathcal{F}(U)$, and suppose $\varphi^\sharp(U)(s) \in \mathcal{H}(U)$ is 0. Then for every point $p \in U$, the image $\varphi^\sharp(s)_p$ of $\varphi^\sharp(s)$ in the stalk $\mathcal{H}_p$ is 0. Since $\varphi^\sharp_p$ is injective for each p, we deduce that $s_p = 0$ in $\mathcal{F}_p$ for each $p \in U$. To say that $s_p = 0$ means that s and s_p have the same image in $\mathcal{F}_p$, which means that there is an open neighborhood W_p of p, with $W_p \subseteq U$, such that $s|_{W_p} = 0$. Now U is covered by the neighborhoods W_p of all its points, so by the sheaf property (3), s is 0 on U. Thus $\varphi^\sharp(U)$ is injective.

Next we show that $\varphi^\sharp(U)$ is surjective. Suppose we have a section $t \in \mathcal{H}(U)$. For each $p \in U$, let t_p be its germ at p. Since $\varphi^\sharp_p$ is surjective, we can find $s_p \in \mathcal{F}_p$ such that $\varphi^\sharp_p(s_p) = t_p$. Let s_p be represented by a section $s(p)$ on a neighborhood V_p of p. Then $\varphi^\sharp(s(p))$ and $t|_{V_p}$ are two elements of $\mathcal{H}(V_p)$, whose germs at p are the same. Hence, replacing V_p by a smaller neighborhood of p if necessary, we may assume that $\varphi^\sharp(s(p)) = t|_{V_p}$ in $\mathcal{H}(V_p)$. Now U is covered by the open sets V_p, and on each V_p we have a section $s(p) \in \mathcal{F}(V_p)$. If p, q are two points, then $s(p)|_{V_p \cap V_q}$ and $s(q)|_{V_p \cap V_q}$ are two sections of $\mathcal{F}(V_p \cap V_q)$, which are both sent by $\varphi^\sharp$ to $t|_{V_p \cap V_q}$. Hence, by the injectivity of $\varphi^\sharp$ proved above they are equal. Then by the sheaf property (4), there is a section $s \in \mathcal{F}(U)$ such that $s|_{V_p} = s(p)$ for each p. Finally we have to check that $\varphi(s) = t$. Indeed, $\varphi(s)$ and t are two sections of $\mathcal{H}(U)$, and for each p, $\varphi^\sharp(s)|_{V_p} = t|_{V_p}$, hence by the sheaf property (3) applied to $\varphi^\sharp(s) - t$, we conclude that $\varphi\sharp(s) = t$. □

Definition A.0.14. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subring of $\mathcal{F}(U)$, and the restrictions maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that for any point p, the stalk $\mathcal{F}'_p$ is a subring of $\mathcal{F}_p$.

Example A.0.15. Let $M = (|M|, \mathcal{O}_M)$ be a real analytic manifold. Then, $\mathcal{O}_M$ is the sheaf of real analytic functions on $|M|$. We can consider $\mathfrak{C}_{C(|M|)}$, the sheaf of continuous functions on $|M|$ (as a topological space). Then we can see that $\mathcal{O}_M$ is a subsheaf of $\mathfrak{C}_{C(|M|)}$, because any analytic function in an open set of M, U, is continuous on U. The second condition is verified because we are dealing with sheaves of functions and the restrictions maps are the restrictions in the usual sense.

So far we have talked only about sheaves on a single topological space. Now we define some operations on sheaves, associated with a continuous map from one topological space to another.

Definition A.0.16. Let $\varphi : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the *direct image* sheaf $\varphi_*\mathcal{F}$ on Y by $(\varphi_*\mathcal{F})(V) = \mathcal{F}(\varphi^{-1}(V))$ for any open set $V \subseteq Y$, and the restrictions maps $\rho'_{VW} := \rho_{\varphi^{-1}(V)\varphi^{-1}(W)}$ for any open set $W \subseteq V$ where ρ is the restriction map of $\mathcal{F}$. For any sheaf $\mathcal{H}$ on Y, we define the *inverse image* sheaf $\varphi^{-1}\mathcal{H}$ on X to be the sheaf associated to the presheaf $U \to \lim_{V \supseteq \varphi(U)} \mathcal{H}(U)$, where U is any open set in X, and the limit is taken over all open set V of Y containing $\varphi(U)$. As a particular case, if Z is a subset of X, regarded as a topological subspace with the induced topology, $i : Z \to X$ is the inclusion map, and if $\mathcal{F}$ is a sheaf on X, then we call $i^{-1}\mathcal{F}$ the *restriction* of $\mathcal{F}$ to Z, and we often denote it by $\mathcal{F}|_Z$. Note that the stalk of $\mathcal{F}|_Z$ at any point $p \in Z$ is just $\mathcal{F}_p$.

Definition A.0.17. A **ringed space** is a pair $(X, \mathcal{F}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{F}_X$ on X.

Then, if X is a topological space the pair $(X, \mathfrak{C}_{C(X)})$ is a ringed space.

A **morphism** of ringed spaces from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$ is a pair $(\varphi, \varphi^\sharp)$ of a continuous map $\varphi : X \to Y$ and a morphism $\varphi^\sharp : \mathcal{F}_Y \to \varphi_*\mathcal{F}_X$ of sheaves of rings on Y.

The ringed space $(X, \mathcal{F}_X)$ is a **locally ringed space** if for each point $p \in X$, the stalk $\mathcal{F}_{X,p}$ is a local ring. Notice that if X is a topological space the pair $(X, \mathfrak{C}_{C(X)})$ is in fact a locally ringed space, because for any $p \in X$, $\mathfrak{M}_p := \{\mathbf{f}_p \in \mathfrak{C}_{C(X),p} : f(p) = 0\}$ is the unique maximal ideal of $\mathfrak{C}_{C(X),p}$.

A **morphism** of locally ringed spaces is a morphism $(\varphi, \varphi^\sharp)$ of ringed spaces, such that for each point $p \in X$, the induced map (see below) of local rings $\varphi^\sharp_p : \mathcal{F}_{Y,\varphi(P)} \to \varphi_*\mathcal{F}_{X,p}$ is a *local homomorphism* of local rings. We explain this last condition. First of all, given a point $p \in X$, the morphism of sheaves $\varphi^\sharp : \mathcal{F}_Y \to \varphi_*\mathcal{F}_X$ induces a homomorphism of rings $\mathcal{F}_Y(V) \to \mathcal{F}_X(\varphi^{-1}(V))$, for every open set V in Y. As V ranges over all open neighborhoods of $\varphi(p)$, $\varphi^{-1}(V)$ ranges over a subset of the neighborhoods of p. Taking direct limits we obtain a map

$$\mathcal{F}_{Y,\varphi(p)} = \varinjlim_V \mathcal{F}_Y(V) \to \varinjlim_V \mathcal{F}_X(\varphi^{-1}(V))$$

and the latter limit maps to the stalk $\mathcal{F}_{X,p}$.

Thus we have an induced homomorphism $\varphi^\sharp_p : \mathcal{F}_{Y,\varphi(p)} \to \mathcal{F}_{X,p}$. We require that this be a local homomorphism: if A and B are local rings with maximal ideals $\mathfrak{M}_A$ and $\mathfrak{M}_B$ respectively, a homomorphism $\phi : A \to B$ is called a local homomorphism if $\phi^{-1}(\mathfrak{M}_B) =$, or equivalently, if $\phi(\mathfrak{M}_A) \subseteq \mathfrak{M}_B$. An **isomorphism** of local ringed spaces is a morphism with a two-sided inverse. Thus a morphism $(\varphi, \varphi^\sharp)$ is an isomorphism if and only if φ is a homeomorphism of the underlying topological spaces, and $\varphi^\sharp$ is an isomorphism of sheaves.

Example A.0.18. Let X and Y be topological spaces and consider the locally ringed spaces $(X, \mathfrak{C}_{C(X)})$ and $(Y, \mathfrak{C}_{C(Y)})$. Let $\varphi : X \to Y$ a continuous function. Define, for any V open subset of Y, $\varphi^\sharp(V) : f \in \mathfrak{C}_{C(Y)}(V) \mapsto f \circ \varphi \in \mathfrak{C}_{C(X)}(\varphi^{-1}(V))$. Notice that $\varphi^\sharp(V)$ is a well defined ring homomorphism because $\varphi^{-1}(V)$ is an open subset of X and $f \circ \varphi$ is a continuous function on $\varphi^{-1}(V)$ whenever f is a continuous function on V since φ is continuous, and for any $f, g \in \mathfrak{C}_{C(Y)}(V)$, $(f+g) \circ \varphi = (f \circ \varphi) + (g \circ \varphi)$, $(fg) \circ \varphi = (f \circ \varphi)(g \circ \varphi)$, and the diagram

$$\begin{array}{ccc}
\mathfrak{C}_{C(Y)}(V) & \xrightarrow{\varphi^\sharp(V)} & \mathfrak{C}_{C(X)}(\varphi^{-1}(V)) \\
\Big\downarrow \rho_{VW} & & \Big\downarrow \rho_{\varphi^{-1}(V)\varphi^{-1}(W)} \\
\mathfrak{C}_{C(Y)}(W) & \xrightarrow{\varphi^\sharp(W)} & \mathfrak{C}_{C(X)}(\varphi^{-1}(W))
\end{array}$$

is commutative for every open $W \subseteq V$.

Hence $\varphi^\sharp : \mathfrak{C}_{C(Y)} \to \varphi_*\mathfrak{C}_{C(X)}$ given by $\varphi^\sharp(V) : f \in \mathfrak{C}_{C(Y)}(V) \mapsto f \circ \varphi \in \mathfrak{C}_{C(X)}(\varphi^{-1}(V))$ for any

V open subset of Y is a morphism of sheaves and thus the pair $(\varphi, \varphi^\sharp)$ is a morphism of ringed spaces. Even more, if $p \in X$, the induced homomorphism on the stalk $\varphi_p^\sharp : \mathfrak{C}_{C(Y),\varphi(p)} \to \mathfrak{C}_{C(X),p}$ is given by $\varphi_p^\sharp : \mathbf{f}_p \in \mathfrak{C}_{C(Y),\varphi(p)} \mapsto (\mathbf{f} \circ \varphi)_p \in \mathfrak{C}_{C(X),p}$. Recall that $\mathfrak{C}_{C(X),p}$ and $\mathfrak{C}_{C(Y),\varphi(p)}$ are local rings with maximal ideals respectively $\mathfrak{M}_{\mathfrak{C}_{C(X),p}} = \{\mathbf{g}_p \in \mathfrak{C}_{C(X),p} : g(p) = 0\}$ and $\mathfrak{M}_{\mathfrak{C}_{C(Y),\varphi(p)}} = \{\mathbf{f}_{\varphi(p)} \in \mathfrak{C}_{C(Y),\varphi(p)} : f(\varphi(p)) = 0\}$. Let $\mathbf{f}_{\varphi(p)} \in \mathfrak{M}_{\mathfrak{C}_{C(Y),\varphi(p)}}$. Then, $\varphi_p^\sharp(\mathbf{f}_{\varphi(p)}) = (\mathbf{f} \circ \varphi)_p$ and $f \circ \varphi(p) = f(\varphi(p)) = 0$. Thus $\varphi_p^\sharp(\mathfrak{M}_{\mathfrak{C}_{C(Y),\varphi(p)}}) \subseteq \mathfrak{M}_{\mathfrak{C}_{C(X),p}}$ which implies that $(\varphi, \varphi^\sharp)$ is a morphism of locally ringed spaces.

Appendix B

Locally ringed spaces on $\mathbb{R}$-algebras of continuous functions.

Let $\mathfrak{C}$ denote the category whose objects are locally ringed spaces on $\mathbb{R}$-algebras of continuous functions, and the morphisms between two objects are the morphisms as locally ringed spaces. Put

$$\mathfrak{C} = (\mathrm{Objets}(\mathfrak{C}), \mathrm{Morphisms}(\mathfrak{C})) = (\mathrm{Obj}(\mathfrak{C}), \mathrm{Morph}(\mathfrak{C}))$$

An objet on $\mathfrak{C}$ is a locally ringed space $X = (|X|, \mathfrak{C}_X)$, where $|X|$ is a topological space and $\mathfrak{C}_X$ is a sheaf of $\mathbb{R}$-algebras of continuous functions over $|X|$; that is, if U is an open subset of $|X|$ the sections over U, $\mathfrak{C}_X(U)$, is an $\mathbb{R}$-subalgebra of the algebra of continuous real-valued functions on U. Notice that this implies that the inclusion morphism $\mathbb{R} \hookrightarrow \mathfrak{C}_X(U)$ sends a constant $a \in \mathbb{R}$ to the function constantly equal to a in $\mathfrak{C}_X(U)$ for any open subset U of $|X|$.

Example B.0.19. Given $|X|$ a topological space we define the object $C(X) = (|X|, \mathfrak{C}_{C(X)})$ in $\mathfrak{C}$ where $\mathfrak{C}_{C(X)}$ is the sheaf of continuous real-valued functions on $|X|$. Then, for any $p \in |X|$, the stalk at p, $\mathfrak{C}_{C(X),p}$ is a local $\mathbb{R}$-algebra whose maximal ideal is

$$\mathfrak{M}_{\mathfrak{C}_{C(X),p}} = \{\mathbf{f}_p \in \mathfrak{C}_{C(X),p} : f(p) = 0 \text{ for any representant } f : U \to \mathbb{R} \text{ of } \mathbf{f}_p\}$$

It is the unique maximal ideal because every $\mathbf{g}_p \notin \mathfrak{M}_{\mathfrak{C}_{C(X),p}}$ is a unit of $\mathfrak{C}_{C(X),p}$, i.e. has an inverse in $\mathfrak{C}_{C(X),p}$. In fact, the assignment $|X| \mapsto C(X)$ is a functor from the category of topological spaces and continuous maps to the category $\mathfrak{C}$.

Notice that if $X = (|X|, \mathfrak{C}_X)$ is another object in $\mathfrak{C}$ whose underlying topological space is $|X|$, the sheaf $\mathfrak{C}_X$ is by definition a subsheaf of $\mathfrak{C}_{C(X)}$ and hence for any $p \in |X|$ the stalk at p, $\mathfrak{C}_{X,p}$, is naturally considered as a local $\mathbb{R}$-subalgebra of $\mathfrak{C}_{C(X),p}$. Notice also that for any U open subset of $|X|$ the following diagram is commutative

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & \mathfrak{C}_{C(X)}(U) \\ {\scriptstyle id_{\mathbb{R}}}\downarrow & & \uparrow{\scriptstyle \subseteq} \\ \mathbb{R} & \hookrightarrow & \mathfrak{C}_X(U) \end{array}$$

where the hooked arrow $\mathfrak{C}_X(U) \hookrightarrow \mathfrak{C}_{C(X)}(U)$ is the set inclusion $\mathfrak{C}_X(U) \subseteq \mathfrak{C}_{C(X)}(U)$. This implies that the induced inclusion morphism on the stalks $\mathfrak{C}_{X,p} \xrightarrow{i} \mathfrak{C}_{C(X),p}$ makes

$$\begin{array}{ccc} & & \mathfrak{C}_{C(X),p} \\ & \nearrow & \\ \mathbb{R} & & \uparrow{\scriptstyle i} \\ & \searrow & \\ & & \mathfrak{C}_{X,p} \end{array} \tag{B.1}$$

commutative.

Proposition B.0.20. Let $X = (|X|, \mathfrak{C}_X)$ be an object in $\mathfrak{C}$, and for any point $p \in |X|$ let $\mathfrak{M}_{\mathfrak{C}_{X,p}}$ denote the maximal ideal of $\mathfrak{C}_{X,p}$ and $\mathfrak{C}_{X,p} \xrightarrow{i} \mathfrak{C}_{C(X),p}$ the morphism in (B.1). They are equivalent

i) The morphism $\mathfrak{C}_{X,p} \xrightarrow{i} \mathfrak{C}_{C(X),p}$ is local (i.e. $i(\mathfrak{M}_{\mathfrak{C}_{X,p}}) \subseteq \mathfrak{M}_{\mathfrak{C}_{C(X),p}}$), and the morphism $\mathbb{R} \hookrightarrow \mathfrak{C}_{X,p}/\mathfrak{M}_{\mathfrak{C}_{X,p}}$ induced by the inclusion morphism $\mathbb{R} \hookrightarrow \mathfrak{C}_{X,p}$ is an isomorphism.

ii) $\mathfrak{M}_{\mathfrak{C}_{X,p}} = \mathfrak{M}_{\mathfrak{C}_{C(X),p}} \cap \mathfrak{C}_{X,p} = \{\mathbf{f}_p \in \mathfrak{C}_{X,p} : f(p) = 0 \text{ for any representant } f : U \to \mathbb{R} \text{ of } \mathbf{f}_p\}$

iii) Any section of $\mathfrak{C}_X$ which does not vanish at p is locally invertible at p in $\mathfrak{C}_X$

Proof. Consider the diagram

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & \mathfrak{C}_{C(X),p}\big/\mathfrak{M}_{\mathfrak{C}_{C(X),p}} \\ {\scriptstyle id_{\mathbb{R}}}\downarrow & & \uparrow{\scriptstyle \bar{i}} \\ \mathbb{R} & \hookrightarrow & \mathfrak{C}_{X,p}\big/\mathfrak{M}_{\mathfrak{C}_{X,p}} \end{array} \tag{B.2}$$

If we suppose $i)$, every arrow in $(B.2)$ is an isomorphism. We have that $\mathfrak{M}_{\mathfrak{C}_{X,p}} \subseteq \mathfrak{M}_{\mathfrak{C}_{C(X),p}} \cap \mathfrak{C}_{X,p}$ since i is local. Conversely, if $\mathbf{f}_p \in \mathfrak{C}_{X,p}$ is such that $f(p) = 0$, $\bar{i}(\mathbf{f}_p) \in \mathfrak{M}_{\mathfrak{C}_{C(X),p}}$. Following the diagram anticlockwise, $\mathbf{f}_p \in \mathfrak{M}_{\mathfrak{C}_{X,p}}$.

Suppose $ii)$. Then, $\mathfrak{C}_{X,p} \xrightarrow{i} \mathfrak{C}_{C(X),p}$ is local and the injective homomorphism $\mathbb{R} \hookrightarrow \mathfrak{C}_{X,p}$ is surjective because given $\mathbf{f}_p + \mathfrak{M}_{\mathfrak{C}_{X,p}}$ either $f(p) = 0$, which implies by $ii)$ that $\mathbf{f}_p \in \mathfrak{M}_{\mathfrak{C}_{X,p}}$ or $f(p) \neq 0$, so it is the germ of the function constantly equals to $f(p)$. □

Given $X = (|X|, \mathfrak{C}_X)$ and $Y = (|Y|, \mathfrak{C}_Y)$ two objects in $\mathfrak{C}$ we denote by $\mathrm{Morph}_{\mathfrak{C}}(X, Y)$ the set of morphisms of $\mathfrak{C}$ from X to Y. A morphism $(\varphi, \varphi^\sharp) \in \mathrm{Morph}_{\mathfrak{C}}(X, Y)$ is given by a continuous map $\varphi : |X| \to |Y|$ between the underlying topological spaces, and a morphism of sheaves $\varphi^\sharp : \mathfrak{C}_Y \to \varphi_* \mathfrak{C}_X$, such that for any $p \in |X|$ the induced morphism in the stalks $\varphi^\sharp_p : \mathfrak{C}_{Y,\varphi(p)} \to \mathfrak{C}_{X,p}$ is a local homomorphism of $\mathbb{R}$-algebras (recall that given two local algebras A and B with maximal ideal respectively $\mathfrak{M}_A$ and $\mathfrak{M}_B$ an homomorphism $\varphi^\sharp : A \to B$ is called local if $\varphi^\sharp(\mathfrak{M}_A) \subseteq \mathfrak{M}_B$.)

Proposition B.0.21. Let $X = (|X|, \mathfrak{C}_X), Y = (|Y|, \mathfrak{C}_Y)$ be objects of $\mathfrak{C}$ and $(\varphi, \varphi^\sharp) \in \mathrm{Morph}_{\mathfrak{C}}(X, Y)$. If X, Y satisfy one of the equivalent conditions of proposition B.0.20, then the morphism of sheaves $\varphi^\sharp$ is given by composition with φ; that is, for any V open subset of $|Y|$,

$$\varphi^\sharp(V) : f \in \mathfrak{C}_Y(V) \mapsto f \circ \varphi \in \mathfrak{C}_X(\varphi^{-1}(V))$$

Proof. Let V be an open subset of $|Y|$. We have to prove that $\varphi^\sharp(V)(f) = f \circ \varphi$ for any $f \in \mathfrak{C}_Y(V)$. So let $f \in \mathfrak{C}_Y(V)$ and $p \in \varphi^{-1}(V)$. We want the equality $\varphi^\sharp(V)(f)(p) = (f \circ \varphi)(p)$. Put $a = f(\varphi(p))$ and define $g : V \to \mathbb{R}$ by $g(q) = f(q) - a$. Then $g = f - a \in \mathfrak{C}_Y(V)$ and $g(\varphi(p)) = 0$ so $\mathbf{g}_{\varphi(p)} = (\mathbf{f} - \mathbf{a})_{\varphi(p)} \in \mathfrak{M}_{\mathfrak{C}_{Y,\varphi(p)}}$. As the induced homomorphism on the stalks $\varphi^\sharp_p : \mathfrak{C}_{Y,\varphi(p)} \to \mathfrak{C}_{X,p}$ is a local homomorphism, $\varphi^\sharp_p(\mathbf{g}_p) \in \mathfrak{M}_{\mathfrak{C}_{X,p}}$. Thus, by lemma B.0.20, $0 = \varphi^\sharp_p(\mathbf{g}_{\varphi(p)})(p) = \varphi^\sharp_p((\mathbf{f} - \mathbf{a})_{\varphi(p)})(p) = \varphi^\sharp_p(\mathbf{f}_p)(p) - a$. □

As a question of notation, as the morphism of sheaves $\varphi^\sharp$ is completely determined by the continuous mapping φ, we will use frequently the same letter φ for the underlying continuous mapping and the morphism itself, so saying simply that $\varphi : X \to Y$ is a morphism of locally ringed spaces or in the category $\mathfrak{C}$.

We have that a morphism $(\varphi, \varphi^\sharp) \in \mathrm{Morph}_{\mathfrak{C}}(X, Y)$ is an isomorphism if and only if φ is a homeomorphism of the underlying topological spaces, and $\varphi^\sharp$ is an isomorphism of sheaves, or equivalently the homomorphisms induced on the stalks are isomorphisms for any point.

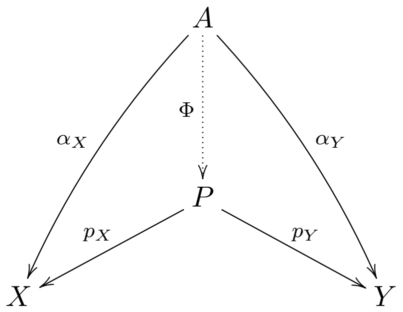
Definition B.0.22. Let $\mathfrak{S} = (\mathrm{Obj}(\mathfrak{S}),\mathrm{Morph}(\mathfrak{S}))$ be a subcategory of $\mathfrak{C}$. If for any $X, Y \in \mathfrak{S}$, $\mathrm{Morph}_{X,Y}(\mathfrak{S}) = \mathrm{Morph}_{X,Y}(\mathfrak{C})$ we say that $\mathfrak{S}$ is a **full** subcategory of $\mathfrak{C}$.

Product.

Definition B.0.23. Let $\mathfrak{S}$ be a subcategory of $\mathfrak{C}$. Given two objets of the category, $X, Y \in \mathrm{Obj}(\mathfrak{S})$, a **product of** X**,** Y in $\mathfrak{S}$ is a triple (P, p_X, p_Y) where P is an object $P \in \mathrm{Obj}(\mathfrak{S})$ and $p_X : P \to X$ and $p_Y : P \to Y$ are morphisms such that for every triplet

$$A = (A, \alpha_X : A \to X, \alpha_Y : A \to Y)$$

with $A \in \mathrm{Obj}(\mathfrak{S})$ and $\alpha_X, \alpha_Y \in \mathrm{Morph}(\mathfrak{S})$ there exists an unique morphism, $\Phi : A \to P$ such that $\alpha_X = p_X \circ \Phi$ and $\alpha_Y = p_Y \circ \Phi$



By definition the product is unique up to unique isomorphism in the category (in the sense that if (A, α_X, α_Y) is another product then Φ in the diagram above is an isomorphism).

If for any $X, Y \in \mathrm{Obj}(\mathfrak{S})$ there exists a product of X, Y in $\mathfrak{S}$, we say that $\mathfrak{S}$ **is a category with product**.

Similar definitions apply for the product of a finite family of objects in the category. We use it without any more description of the details.

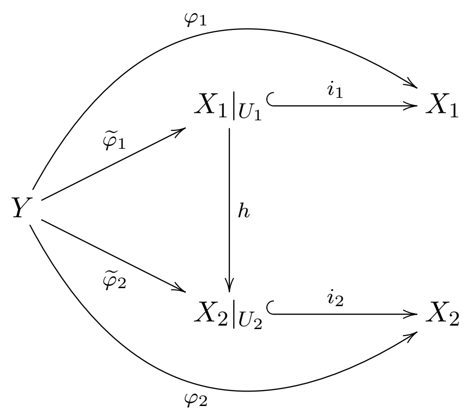
Gluing.

Definition B.0.24. Let $\mathfrak{S}$ be a subcategory of $\mathfrak{C}$. Let $X, Y \in \mathrm{Obj}(\mathfrak{S})$. An **open immersion** on X is a morphism $\varphi : Y \to X$ for which there exists an open set $U \subset |X|$ such that φ decompose in

$$\varphi = i \circ \widetilde{\varphi} : Y \xrightarrow{\widetilde{\varphi}} X|_U \overset{i}{\hookrightarrow} X$$

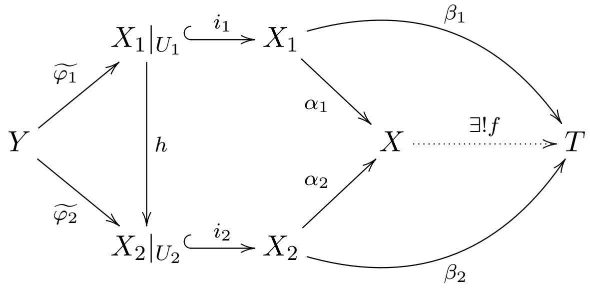
where $\widetilde{\varphi}$ is an isomorphism.

Remark B.0.25. Let $Y, X_1, X_2 \in \mathrm{Obj}(\mathfrak{S})$ and $\varphi_i : Y \to X_i$ open immersions decomposing in



Then $h = \widetilde{\varphi_2} \circ \widetilde{\varphi_1}^{-1} : X_1|_{U_1} \to X_2|_{U_2}$ is an isomorphism.

Definition B.0.26. Given two objets of the category, $X, Y \in \mathrm{Obj}(\mathfrak{S})$ and open immersions $\varphi_1 : Y \to X_1$, $\varphi_2 : Y \to X_2$, we define the **gluing of X_1 and X_2 with respect to the open immersions φ_1 and φ_2** as a triplet (X, α_1, α_2) where $X \in \mathrm{Obj}(\mathfrak{S})$, $\alpha_i : X_i \to X$ are open immersions for $i = 1, 2$ satisfying $\alpha_1 \circ \varphi_1 = \alpha_2 \circ \varphi_2$ and such that for any other triplet (T, β_1, β_2) where $T \in \mathrm{Obj}(\mathfrak{S})$ and $\beta_i : X_i \to T$ are open immersions such that $\beta_1 \circ \varphi_1 = \beta_2 \circ \varphi_2$ there exists an unique morphism $f : X \to T$ such that $\beta_i = f \circ \alpha_i$ for $i = 1, 2$.



If for any $X, Y \in \mathrm{Obj}(\mathfrak{S})$ and open immersions $\varphi_1 : Y \to X_1$, $\varphi_2 : Y \to X_2$, there exists the gluing of X_1 and X_2 with respect to the open immersions φ_1 and φ_2, we say that the category **$\mathfrak{S}$ is a category with gluing**.

Similar definitions apply for the gluing of a finite (or even more generally infinite) family of open immersions $\{\varphi_i : Y \to X_i\}_{i \in I}$. However, we do not use it in this text so we omit the details.

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