UNIVERSITÉ DE BOURGOGNE
Institut de Mathématiques de Bourgogne

UNIVERSITÉ DE SCIENCE DE HO CHI MINH VILLE
Faculté de Mathématiques - Informatique

THÈSE

présentée par

NGUYEN Le Hoang Anh

en vue d'obtenir le titre de

DOCTEUR

DE L'UNIVERSITÉ DE BOURGOGNE
ET DE L'UNIVERSITÉ DE SCIENCE DE HO CHI MINH VILLE

Discipline: MATHÉMATIQUES

QUELQUES THÈMES EN L'ANALYSE VARIATIONNELLE ET OPTIMISATION

23 Février 2014

Directeurs de thèse

Szymon DOLECKI et **PHAN Quoc Khanh**

Membres du jury

Szymon DOLECKI,	Université de Bourgogne
HUYNH Quang Vu,	Université de Science de Ho Chi Minh ville
Abderrahim JOURANI,	Université de Bourgogne
NGUYEN Dinh,	Université Internationale de Ho Chi Minh ville
NGUYEN Hoi Nghia,	Université Nationale du Vietnam à Ho Chi Minh ville
PHAN Quoc Khanh,	Université de Science de Ho Chi Minh ville
Jean-Jacques STRODIOT,	Université de Namur

Résumé

Dans cette thèse, j'étudie d'abord la théorie des Γ-limites. En dehors de quelques propriétés fondamentales des Γ-limites, les expressions de Γ-limites séquentielles généralisant des résultats de Greco sont présentées. En outre, ces limites nous donnent aussi une idée d'une classification unifiée de la tangence et la différentiation généralisée. Ensuite, je développe une approche des théories de la différentiation généralisée. Cela permet de traiter plusieurs dérivées généralisées des multi-applications définies directement dans l'espace primal, tels que des ensembles variationnels, des ensembles radiaux, des dérivées radiales, des dérivées de Studniarski. Finalement, j'étudie les règles de calcul de ces dérivées et les applications liées aux conditions d'optimalité et à l'analyse de sensibilité.

Abstract

In this thesis, we first study the theory of Γ-limits. Besides some basic properties of Γ-limits, expressions of sequential Γ-limits generalizing classical results of Greco are presented. These limits also give us a clue to a unified classification of derivatives and tangent cones. Next, we develop an approach to generalized differentiation theory. This allows us to deal with several generalized derivatives of set-valued maps defined directly in primal spaces, such as variational sets, radial sets, radial derivatives, Studniarski derivatives. Finally, we study calculus rules of these derivatives and applications related to optimality conditions and sensitivity analysis.

Acknowledgements

Completion of this doctoral dissertation was possible with the support of several people. I would like to express my sincere gratitude to all of them.

First, I want to express my deepest gratitude to my supervisors, Professor PHAN Quoc Khanh and Professor Szymon DOLECKI, for their valuable guidance, scholarly inputs, and consistent encouragement I received throughout the research work. From finding an appropriate subject in the beginning to the process of writing thesis, professors offer their unreserved help and lead me to finish my thesis step by step. People with an amicable and positive disposition, they have always made themselve available to clarify my doubts despite their busy schedules and I consider it as a great opportunity to do my doctoral programme under their guidance and to learn from their research expertise. Their words can always inspire me and bring me to a higher level of thinking. Without their kind and patient instructions, it is impossible for me to finish this thesis.

Second, I am very pleased to extend my thanks to reviewers of this thesis. Their comments, observations and questions have truly improved the quality of this manuscript. I would like to thank professors who have also agreed to participate in my jury.

To my colleagues, I would like to express my thankfulness to Dr. NGUYEN Dinh Tuan and Dr. LE Thanh Tung who have all extended their support in a very special way, and I gained a lot from them, through their personal and scholarly interactions, their suggestions at various points of my research programme.

Next, I also would like to give my thankfulness to Quang, Hanh, Thoai, Ha, Hung and other Vietnamese friends in Dijon for their help, warmness and kindness during my stay in France.

In addition, I am particularly grateful to The Embassy of France in Vietnam and Campus France for their aid funding and accommodation during my staying in Dijon. My thanks are also sent to Faculty of Mathematics and Computer Science at the University of Science of Ho Chi Minh City and the Institute of Mathematics of Burgundy for their support during the period of preparation of my thesis.

Finally, I owe a lot to my parents and my older sister who support, encourage and help me at every stage of my personal and academic life, and long to see this achievement come true. They always provide me with a carefree enviroment, so that I can concentrate on my study. I am really lucky to have them be my family.

Contents

Preface

Variational analysis is related to a broad spectrum of mathematical theories that have grown in connection with the study of problems of optimization and variational convergence.

To my knowledge, many concepts of convergence for sequences of functions have been introduced in mathematical analysis. These concepts are designed to approach the limit of sequences of variational problems and are called variational convergence. Introduced by De Giorgi in the early 1970s, Γ-convergence plays an important role among notions of convergences for variational problems. Moreover, many applications of this concept have been developed in other fields of variational analysis such as calculus of variations and differential equations.

Recently, nonsmoothness has become one of the most characteristic features of modern variational analysis. In fact, many fundamental objects frequently appearing in the frame work of variational analysis (e.g., the distance function, value functions in optimization and control problems, maximum and minimum functions, solution maps to perturbed constraint and variational systems, etc.) are inevitably of nonsmooth and/or set-valued structures. This requires the development of new forms of analysis that involve generalized differentiation.

The analysis above motivates us to study some topics on Γ-limits, generalized differentiation of set-valued maps and their applications.

Chapter 1

Motivations

1.1 Γ-limits

Several last decades have seen an increasing interest for variational convergences and for their applications to different fields, like approximation of variational problems and nonsmooth analysis, see [22, 32, 113, 120, 131, 133, 150]. Among variational convergences, definitions of Γ-convergence, introduced in [48] by Ennio De Giorgi and Franzoni in 1975, have become commonly-recognizied notions (see [37] of Dal Maso for more detail introduction). Under suitable conditions, Γ-convergence implies stability of extremal points, while some other convergences, such as pointwise convergence, do not. Moreover, almost all other variational convergences can be easily expressed in the language of Γ-convergence. As explained in [16, 57, 169], this concept plays a fundamental role in optimization theory, decision theory, homogenization problems, phase transitions, singular perturbations, the theory of integral functionals, algorithmic procedures, and in many others.

In 1983 Greco introduced in [82] a concept of limitoid and noticed that all the Γ-limits are special limitoids. Each limitoid defines its support, which is a family of subsets of the domain of the limitoid, which in turn determines this limitoid. Besides, Greco presented in [82, 84] a representation theorem for which each relationship of limitoids corresponds a relationship in set theory. This theorem enabled a calculus of supports and was instrumental in the discovery of a limitation of equivalence between Γ-limits and sequential Γ-limits, see [83].

Recently, a lot of research has been carried in the realm of tangency and differentiation and their applications, see [2, 4, 8, 14, 15, 50, 77, 101, 109, 134]. We propose a unified approach to approximating tangency cones and generalized derivatives based on the theory of Γ-limits. This means that most of them can be expressed in terms of Γ-limits.

The analysis above motivates us to study the theory of Γ-limits.

1.2 Sensitivity analysis

Stability and sensitivity analyses are of great importance for optimization from both the theoretical and practical view points. As usual, stability is understood as a qualitative analysis, which concerns mainly studies of various continuity (or semicontinuity) properties of solution maps and optimal-value maps. Sensitivity means a quantitative analysis, which can be expressed in terms of various derivatives of the mentioned maps. For sensitivity results in nonlinear programming using classical derivatives, we can see the book [64] of Fiacco. However, practical optimization problems are often nonsmooth. To cope with this crucial difficulty, most of approaches to studies of optimality conditions and sensitivity analysis are based on generalized derivatives.

Nowadays, set-valued maps (also known as multimaps or multifunctions) are involved frequently in optimization-related models. In particular, for vector optimization, both perturbation and solution maps are set-valued. One of the most important derivatives of a multimap is the contingent derivative. In [107–109, 153, 154, 162, 163], behaviors of perturbation maps for vector optimization were investigated quantitatively by making use of contingent derivatives. Results on higher-order sensitivity analysis were studied in [158, 167], applying kinds of contingent derivatives. To the best of our knowledge, no other kinds of generalized derivatives have been used in contributions to this topic, while so many notions of generalzed differentiability have been introduced and applied effectively in investigations of optimality conditions, see books [11] of Aubin and Frankowska, [129, 130] of Mordukhovich, and [147] of Rockafellar and Wets.

We mention in more detail only several recent papers on generalized derivatives of set-valued maps and optimality conditions. Radial epiderivatives were used to get optimality conditions for nonconvex vector optimization in [66] by Flores-Bazan and for set-valued optimization in [102] by Kasimbeyli. Variants of higher-order radial derivatives for establishing higher-order conditions were proposed by Anh et al. in [2, 4, 8]. The higher-order lower Hadamard directional derivative was the tool for set-valued vector optimization presented by Ginchev in [71, 72]. Higher-order variational sets of a multimap were proposed in [105, 106] by Khanh and Tuan in dealing with optimality conditions for set-valued optimization.

We expect that many generalized derivatives, besides the contingent ones, can be employed effectively in sensitivity analysis. Thus, we choose variational sets for higher-order considerations of perturbation maps, since some advantages of this generalized differentiability were shown in [7, 105, 106], e.g., almost no assumptions are required for variational sets to exist (to be nonempty); direct calculating of these sets is simply a computation of a set limit; extentions to higher orders are direct; they are bigger than corresponding sets of most derivatives (this property is decisively advantageous in establishing necessary optimality conditions by separation techniques), etc. Moreover, Anh et al. established calculus rules for variational sets in [7] to ensure the applicability of variational sets.

1.3 Optimality conditions

Various problems encountered in the areas of engineering, sciences, management science, economics and other fields are based on the fundamental idea of mathematical formulation. Optimization is an essential tool for the formulation of many such problems expressed in the form of minimization/maximization of a function under certain constraints like inequalities, equalities, and/or abstract constraints. It is thus rightly considered a science of selecting the best of the many possible decisions in a complex real-life environment.

All initial theories of optimization theory were developed with differentiability assumptions of functions involved. Meanwhile, efforts were made to shed the differentiability hypothesis, there by leading to the development of nonsmooth analysis as a subject in itself. This added a new chapter to optimization theory, known as nonsmooth optimization. Optimality conditions in nonsmooth problems have been attracting increasing efforts of mathematicians around the world for half a century. For systematic expositions about this topic, including practical applications, see books [11] of Aubin and Frankowska, [29] of Clarke, [92] of Jahn, [129, 130] of Mordukhovich, [142] of Penot, [146] of Rockafellar and Wets, and [149] of Schirotzek. A significant number of generalized derivatives have been introduced to replace the Fréchet and Gâteaux derivatives which do not exist for studying optimality conditions in nonsmooth optimization.

One can roughly separate the wide range of methods for nonsmooth problems into two groups : the primal space and the dual space approaches. The primal space approach has been more developed, since it exhibits a clear geometry, originated from the famous works of Fermat and Lagrange. Most derivatives in this stream are based on kinds of tangency/linear approximations. Among tangent cones, contingent cone plays special roles, both in direct uses as derivatives/linear approximations and in combination with other ideas to provide kinds of generalized derivatives (contingent epiderivatives by Jahn and Rauh in [96], contingent variations by Frankowska and Quincampoix in [68], variational sets by Khanh et al. in [7, 105, 106], generalized (adjacent) epiderivatives by Li et al. in [27, 166, 168], etc).

Similarly as for generalized derivatives defined based on kinds of tangent cones, the radial derivative was introduced by Taa in [160]. Coupling the idea of tangency and epigraphs, like other epiderivatives, radial epiderivatives were defined and applied to investigating optimality conditions in [65–67] by Flores-Bazan and in [102] by Kasimbeyli. To include more information in optimality conditions, higher-order derivatives should be defined.

The discussion above motivates us to define a kind of higher-order radial derivatives and use them to obtain higher-order optimality conditions for set-valued vector optimization.

1.4 Calculus rules and applications

The investigation of optimality conditions for nonsmooth optimization problems has implied many kinds of generalized derivatives (introduced in above subsections). However, to the best of our knowledge, there are few research on their calculus rules. We mention in more detail some recent papers on generalized derivatives of set-valued maps and their calculus rules. In [94], some calculus rules for contingent epiderivatives of set-valued maps were given by Jahn and Khan. In [116], Li et al. obtained some calculus rules for intermediate derivative-like multifunctions. Similar ideas had also been utilized for the calculus rules for contingent derivatives of set-valued maps and for generalized derivatives of single-valued nonconvex functions in [11, 164, 165]. Anh et al. developed elements of calculus of higher-order variational sets for set-valued mappings in [7].

In [156], Studniarski introduced another way to get higher-order derivatives (do not depend on lower orders) for extended-real-valued functions, known as Studniarski derivatives, and obtained necessary and sufficient conditions for strict minimizers of order greater than 2 for optimization problems with vector-valued maps as constraints and objectives. Recently, these derivatives have been extended to set-valued maps and applied to optimality conditions for set-valued optimization problems in [1, 117, 159]. However, there are no results on their calculus rules.

The analysis above motivates us to study on calculus rules of Studniarski derivatives and their applications.

Chapter 2

Preliminaries

2.1 Some definitions in set theory

Definition 2.1.1. ([23, 24]) Let S be a subset of a topological sapce X.

(i) A family $\mathscr{F}$ of subsets of S is called a *non-degenerate family* on S if $\emptyset \notin \mathscr{F}$.

(ii) A non-degenerate family $\mathscr{F}$ on S is called a *semi-filter* if

$$G \supseteq F \in \mathscr{F} \Longrightarrow G \in \mathscr{F}.$$

(iii) A semi-filter $\mathscr{F}$ on S is called a *filter* if

$$F_0, F_1 \in \mathscr{F} \Longrightarrow F_0 \cap F_1 \in \mathscr{F}.$$

The *set of filters* and the *set of semi-filters* on S are denoted by $\mathbb{F}(S)$ and $\mathbb{SF}(S)$, respectively. If $\mathscr{A}, \mathscr{B}$ are two families, then $\mathscr{B}$ is called *finer* than $\mathscr{A}$ (denoted by $\mathscr{A} \leq \mathscr{B}$) if for each $A \in \mathscr{A}$ there exists $B \in \mathscr{B}$ such that $B \subseteq A$. We say that $\mathscr{A}$ and $\mathscr{B}$ are equivalent ($\mathscr{A} \approx \mathscr{B}$) if $\mathscr{A} \leq \mathscr{B}$ and $\mathscr{B} \leq \mathscr{A}$. A subfamily $\mathscr{B}$ of a non-degenerate family $\mathscr{F}$ is said a *base* of $\mathscr{F}$ (or $\mathscr{B}$ generates $\mathscr{F}$) if $\mathscr{F} \leq \mathscr{B}$. We say that $\mathscr{A}$ and $\mathscr{B}$ *mesh* (denoted by $\mathscr{A} \# \mathscr{B}$) if $A \cap B \neq \emptyset$ for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

The *grill* of a family $\mathscr{A}$ on S, denoted by $\mathscr{A}^{\#}$, is defined by

$$\mathscr{A}^{\#} := \{A \subseteq S : \underset{F \in \mathscr{A}}{\forall} A \cap F \neq \emptyset\}.$$

Therefore $\mathscr{A} \# \mathscr{B}$ is equivalent to $\mathscr{A} \subseteq \mathscr{B}^{\#}$ and to $\mathscr{B} \subseteq \mathscr{A}^{\#}$.

If $\mathscr{F}$ is a filter, then $\mathscr{F} \subseteq \mathscr{F}^{\#}$. In $\mathbb{SF}(S)$, the operation of grills is an *involution*, i.e., the following equalities hold (see [55])

$$\mathscr{A}^{\#\#} = \mathscr{A}, \quad \left(\bigcup_i \mathscr{A}_i\right)^{\#} = \bigcap_i \left(\mathscr{A}_i^{\#}\right), \quad \left(\bigcap_i \mathscr{A}_i\right)^{\#} = \bigcup_i \left(\mathscr{A}_i^{\#}\right). \tag{2.1}$$

Semi-filters, filters, and grills were thoroughly studied in [53] by Dolecki.

Definition 2.1.2. ([18]) (i) A set S with a binary relation ($\leq$) satisfying three properties : reflexity, antisymmetry, and transitivity is called an *ordered set S* (also called a *poset*).

(ii) Let S be a subset of a poset P. An element $a \in P$ is called an upper bound (or lower bound) of S if $a \geq s$ ($a \leq s$, respectively) for all $s \in S$.

(iii) An upper bound a (lower bound, respectively) of a subset S is called the *least upper bound* (or the *greatest lower bound*) of S, denoted by $\sup S$ or $\bigvee S$ ($\inf S$ or $\bigwedge S$, respectively) if $a \leq b$ ($a \geq b$, respectively) for all b be another upper bound (lower bound, respectively) of S.

Definition 2.1.3. ([18, 82]) (i) A poset L is called a *lattice* if each couple of its elements has a least upper bound or "join" denoted by $x \vee y$, and a greatest lower bound or "meet" denoted by $x \wedge y$.

(ii) A lattice L is called *complete* if each of its subsets S has a greatest lower bound and a least upper bound in L.

(iii) A complete lattice L is called *completely distributive* if

$$\text{(a)} \bigvee_{j\in J} \bigwedge_{i\in A_j} f(j,i) = \bigwedge_{\varphi\in \prod\limits_{j\in J} A_j} \bigvee_{j\in J} f(j,\varphi(j)),$$

$$\text{(b)} \bigwedge_{j\in J} \bigvee_{i\in A_j} f(j,i) = \bigvee_{\varphi\in \prod\limits_{j\in J} A_j} \bigwedge_{j\in J} f(j,\varphi(j)),$$

for each non-empty family $\{A_j\}_{j\in J}$ of non-empty sets and for each function f defined on $\{(j,i) \in J \times I : i \in A_j\}$ with values in L, and $\prod\limits_{j\in J} A_j := \{\varphi \in (\bigcup_{j\in J} A_j)^J : \underset{j\in J}{\forall}\ \varphi(j) \in A_j\}$, where $(\bigcup_{j\in J} A_j)^J$ denotes the set of functions from J into $\bigcup_{j\in J} A_j$.

(iv) A non-empty subset S of a lattice L is called a *sublattice* if for every pair of elements a, b in S both $a \wedge b$ and $a \vee b$ are in S.

(v) A sublattice S of a complete lattice L is called *closed* if for every non-empty subset A of S both $\bigwedge A$ and $\bigvee A$ are in L.

2.2 Some definitions in set-valued analysis

Let X, Y be *vector spaces*, C be a non-empty cone in Y, and $A \subseteq Y$. We denote sets of positive integers, of real numbers, and of non-negative real numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively. We often use the following notations

$$\operatorname{cone} A := \{\lambda a : \lambda \geq 0,\ a \in A\}, \quad \operatorname{cone}_+ A := \{\lambda a : \lambda > 0,\ a \in A\},$$

$$C^* := \{y^* \in Y^* \; : \; \langle y^*, c\rangle \geq 0, \, \forall c \in C\}, \quad C^{+i} := \{y^* \in Y^* \; : \; \langle y^*, c\rangle > 0, \, \forall c \in C \setminus \{0\}\}.$$

A subset B of a cone C is called a *base of* C if and only if $C = \operatorname{cone} B$ and $0 \notin \operatorname{cl} B$.

For a *set-valued map* $F : X \to 2^Y$, $F + C$ is called the *profile map* of F with respect to C defined by $(F+C)(x) := F(x) + C$. The domain, graph, epigraph and hypograph of F are denoted by $\operatorname{dom} F$, $\operatorname{gr} F$, $\operatorname{epi} F$, and $\operatorname{hypo} F$, respectively, and defined by

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}, \quad \operatorname{gr} F := \{(x,y) \in X \times Y : y \in F(x)\},$$

$$\operatorname{epi} F := \operatorname{gr}(F + C), \quad \operatorname{hypo} F := \operatorname{gr}(F - C).$$

A subset $M \subseteq X \times Y$ can be considered as a set-valued map M from X into Y, called a *relation* from X into Y. The image of a singleton $\{x\}$ by Mx is denoted by $Mx := \{y \in Y : (x,y) \in M\}$, and of a subset S of X is denoted by $MS := \bigcup_{x \in S} Mx$. The preimage of a subset K of Y by M is denoted by $M^{-1}K := \{x : Mx \cap K \neq \emptyset\}$.

Definition 2.2.1. Let C be a convex cone, $F : X \to 2^Y$ and $(x_0, y_0) \in \operatorname{gr} F$.

(i) F is called a *convex map* on a convex set $S \subseteq X$ if, for all $\lambda \in [0,1]$ and $x_1, x_2 \in S$,

$$(1-\lambda)F(x_1) + \lambda F(x_2) \subseteq F((1-\lambda)x_1 + \lambda x_2).$$

(ii) F is called a *C-convex map* on a convex set S if, for all $\lambda \in [0,1]$ and $x_1, x_2 \in S$,

$$(1-\lambda)F(x_1) + \lambda F(x_2) \subseteq F((1-\lambda)x_1 + \lambda x_2) + C.$$

Definition 2.2.2. Let $F : X \to 2^Y$ and $(x_0, y_0) \in \operatorname{gr} F$.

(i) F is called a *lower semicontinuous* map at (x_0, y_0) if for each $V \in \mathcal{N}(y_0)$ there is a neighborhood $U \in \mathcal{N}(x_0)$ such that $V \cap F(x) \neq \emptyset$ for each $x \in U$.

(ii) Suppose that X, Y are normed spaces. The map F is called a m-th order *locally pseudo-Hölder calm* map at x_0 for $y_0 \in F(x_0)$ if $\exists \lambda > 0$, $\exists U \in \mathcal{N}(x_0)$, $\exists V \in \mathcal{N}(y_0)$, $\forall x \in U$,

$$(F(x) \cap V) \subseteq \{y_0\} + \lambda ||x - x_0||^m B_Y,$$

where B_Y stands for the closed unit ball in Y.

For $m = 1$, the word "Hölder" is replaced by "Lipschitz". If $V = Y$, then "locally pseudo-Hölder calm" becomes "locally Hölder calm".

Example 2.2.3. (i) For $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \{y : -x^2 \leq y \leq x^2\}$ and $(x_0, y_0) = (0,0)$, F is the second order locally pseudo-Hölder calm map at x_0 for y_0.

(ii) Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} \{0, 1/x\}, & \text{if } x \neq 0, \\ \{0, (1/n)_{n\in\mathbb{N}}\}, & \text{if } x = 0, \end{cases}$$

and $(x_0, y_0) = (0,0)$. Then, for all $m \geq 1$, F is not m-th order locally pseudo-Hölder calm at x_0 for y_0.

Observe that if F is m-th order locally (pseudo-)Hölder calm at x_0 for y_0, it is also n-th order locally (pseudo-)Hölder calm at x_0 for y_0 for all $m > n$. However, the converse may not hold. The following example shows the case.

Example 2.2.4. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and $(x_0, y_0) = (0,0)$. Obviously, F is second order locally Hölder calm x_0 for y_0, but F is not third order locally Hölder calm at x_0 for y_0.

In the rest of this section, we introduce some definitions in vector optimization. Let $C \subseteq Y$, we consider the following relation $\leq_C$ in Y, for $y_1, y_1 \in Y$,

$$y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C.$$

Recall that a cone K in Y is called *pointed* if $K \cap -K \subseteq \emptyset$.

Proposition 2.2.5. *If C is a cone, then $\leq_C$ is*

(i) *reflexive if and only if* $0 \in C$*,*

(ii) *antisymmetric if and only if C is pointed,*

(iii) *transitive if and only if C is convex.*

Proof. (i) Suppose that $\leq_C$ is reflexive, then $y \leq_C y$ for all $y \in Y$. This means $0 = y - y \in C$. Conversely, since $0 \in C$, $y - y \in D$ for all $y \in Y$. Thus, $y \leq_C y$.

(ii) Suppose that $\leq_C$ is antisymmetric. If $C \cap -C$ is empty, we are done. Assume that $y \in C \cap -C$, then $0 \leq_C y$, $y \leq_C 0$. This implies $y = 0$. Conversely, let $y_1, y_2 \in Y$ such that $y_1 \leq_C y_2$ and $y_2 \leq_C y_1$. Then, $y_2 - y_1 \in C \cap -C$. Since C is pointed, $y_2 = y_1$.

(iii) Suppose that $\leq_C$ is transitive. Let $y_1, y_2 \in C$ and $\lambda \in (0,1)$. Since C is cone, $\lambda y_1 \in C$ and $(1-\lambda)y_2 \in C$. It follows from $\lambda y_1 \in C$ that $0 \leq_C \lambda y_1$. Similarly, $-(-(1-\lambda)y_2) = (1-\lambda)y_2 \in C$ means $-(1-\lambda)y_2 \leq_C 0$. This implies $-(1-\lambda)y_2 \leq_C \lambda y_1$. Thus, $\lambda y_1 + (1-\lambda)y_2 \in C$.

Conversely, let $y_1, y_2, y_3 \in Y$ such that $y_1 \leq_C y_2$ and $y_2 \leq_C y_3$. It means that $y_2 - y_1 \in C$ and $y_3 - y_2 \in C$. Since C is cone, $\frac{1}{2}(y_2 - y_1) \in C$ and $\frac{1}{2}(y_3 - y_2) \in C$. It follows from the convexity of C that $\frac{1}{2}(y_3 - y_2) + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_3 - y_1) \in C$. Thus, $y_1 \leq_C y_3$.

□

A relation $\leq_C$ satisfying three properties in the proposition above is called an order (or order structure) in Y. Proposition 2.2.5 gives us conditions for which a cone C generates an order in Y.

We now recall some conditions on C, introduced in [28] by Choquet, to ensure that $(Y, \leq_C)$ is a lattice. Recall that in $\mathbb{R}^n$, a *n-simplex* is the convex hull of $n+1$ (affinely) independent points.

Proposition 2.2.6. ([28]) *Suppose that C is a convex cone in $\mathbb{R}^n$. Then $(Y, \leq_C)$ is a lattice if and only if there exists a base of C which is a $(n-1)$-simplex in $\mathbb{R}^{n-1}$.*

Proof. It follows from Proposition 28.3 in [28]. □

By the proposition above, $(\mathbb{R}^2, \leq_C)$ is a lattice if and only if C has a base which is a line segment. In $\mathbb{R}^3$, the base of C must be triangle to ensure that $(\mathbb{R}^3, \leq_C)$ is a lattice.

Let C be a convex cone in Y. A main concept in vector optimization is Pareto efficiency. $A \subseteq Y$, recall that a_0 is a *Pareto efficient point* of A with respect to C if

$$(A - a_0) \cap (-C \setminus l(C)) = \emptyset, \tag{2.2}$$

where $l(C) := C \cap -C$. We denote the *set of all Pareto efficient points* of A by $\mathrm{Min}_{C \setminus l(C)} A$.

If, addtionally, C is closed and pointed, then (2.2) becomes $(A - a_0) \cap (-C \setminus \{0\}) = \emptyset$ and is denoted by $a_0 \in \mathrm{Min}_{\mathrm{C} \setminus \{0\}} \mathrm{A}$.

Next, we are concerned also with the other concepts of efficiency as follows.

Definition 2.2.7. ([88]) Let $A \subseteq Y$.

(i) Supposing $\mathrm{int}\, C \neq \emptyset$[1], $a_0 \in A$ is a *weak efficient point* of A with respect to C if $(A - a_0) \cap -\mathrm{int}\, C = \emptyset$.

(ii) $a_0 \in A$ is a *strong efficient point* of A with respect to C if $A - a_0 \subseteq C$.

[1] $\mathrm{int}\, C$ denotes the interior of C.

(iii) Supposing $C^{+i} \neq \emptyset$, $a_0 \in A$ is a *positive-proper efficient point* of A with respect to C if there exists $\varphi \in C^{+i}$ such that $\varphi(a) \geq \varphi(a_0)$ for all $a \in A$.

(iv) $a_0 \in A$ is a *Geoffrion-proper efficient point* of A with respect to C if a_0 is a Pareto efficient point of A and there exists a constant $M > 0$ such that, whenever there is $\lambda \in C^*$ with norm one and $\lambda(a_0 - a) > 0$ for some $a \in A$, one can find $\mu \in C^*$ with norm one such that $\langle \lambda, a_0 - a \rangle \leq M \langle \mu, a - a_0 \rangle$.

(v) $a_0 \in A$ is a *Henig-proper efficient point* of A with respect to C if there exists a pointed convex cone K with $C \setminus \{0\} \subseteq \operatorname{int} K$ such that $(A - a_0) \cap (-\operatorname{int} K) = \emptyset$.

(vi) Supposing C has a convex base B, $a_0 \in A$ is a *strong Henig-proper efficient point* of A with respect to C if there is $\varepsilon > 0$ such that $\operatorname{cl}\operatorname{cone}(A - a_0) \cap (-\operatorname{cl}\operatorname{cone}(B + \varepsilon B_Y)) = \{0\}$[2].

Note that Geoffrion originally defined the properness notion in (iv) for $\mathbb{R}^n$ with the ordering cone $\mathbb{R}^n_+$. The above general definition of Geoffrion properness is taken from [103].

To unify the notation of these above efficiency (with Pareto efficiency), we introduce the following definition. Let $Q \subseteq Y$ be a nonempty cone, different from Y, unless otherwise specified.

Definition 2.2.8. ([88]) We say that a_0 is a *Q-efficient point* of A if

$$(A - a_0) \cap -Q = \emptyset.$$

We define the set of Q-efficient points by $\operatorname{Min}_{\mathrm{Q}} A$.

Recall that a cone in Y is said to be a *dilating cone* (or a dilation) of C, or dilating C if it contains $C \setminus \{0\}$. Let B be, as before, a convex base of C. Setting $\delta := \inf\{||b|| : b \in B\} > 0$, for $\varepsilon \in (0, \delta)$, we associate to C a pointed convex cone $C_\varepsilon(B) := \operatorname{cone}(B + \varepsilon B_Y)$. For $\varepsilon > 0$, we also associate to C another cone $C(\varepsilon) := \{y \in Y : d_C(y) < \varepsilon d_{-C}(y)\}$.

Any kind of efficiency in Definition 2.2.7 is in fact a Q- efficient point with Q being appropriately chosen as follows.

Proposition 2.2.9. ([88]) (i) *Supposing* $\operatorname{int} C \neq \emptyset$*, a_0 is a weak efficient point of A with respect to C if and only if $a_0 \in \operatorname{Min}_{\mathrm{Q}} A$ with $Q = \operatorname{int} C$.*

(ii) *a_0 is a strong efficient point of A with respect to C if and only if $a_0 \in \operatorname{Min}_{\mathrm{Q}} A$ with $Q = Y \setminus (-C)$.*

(iii) *Supposing $C^{+i} \neq \emptyset$, a_0 is a positive-proper efficient point of A with respect to C if and only if $a_0 \in \operatorname{Min}_{\mathrm{Q}} A$ with $Q = \{y \in Y : \varphi(y) > 0\}$ (denoted by $Q = \{\varphi > 0\}$), φ being some functional in C^{+i}.*

[2]Let E be a set, then $\operatorname{cl} E$ denotes the closure of E.

(iv) *a_0 is a Geoffrion-proper efficient point of A with respect to C if and only if $a_0 \in \mathrm{Min}_{\mathrm{Q}} A$ with $Q = C(\varepsilon)$ for some $\varepsilon > 0$.*

(v) *a_0 is a Henig-proper efficient point of A with respect to C if and only if $a_0 \in \mathrm{Min}_{\mathrm{Q}} A$ with Q being pointed open convex, and dilating C.*

(vi) *Supposing C has a convex base B, a_0 is a strong Henig-proper efficient point of A with respect to C if and only if $a_0 \in \mathrm{Min}_{\mathrm{Q}} A$ with $Q = \mathrm{int}\, C_\varepsilon(B)$, ε satisfying $0 < \varepsilon < \delta$.*

The above proposition gives us a unified way to denote sets of efficient points by the following table

Sets of	**Notations**
C-efficiency	$\mathrm{Min}_{C\setminus\{0\}}$
weak C-efficiency	$\mathrm{Min}_{\mathrm{int}C}$
strong C-efficiency	$\mathrm{Min}_{Y\setminus(-C)}$
positive-proper C-efficiency	$\bigcup_{\varphi \in C^{+i}} \mathrm{Min}_{\{\varphi>0\}}$
Geoffrion-proper C-efficiency	$\bigcup_{\varepsilon>0} \mathrm{Min}_{C(\varepsilon)}$
Henig-proper C-efficiency	Min_Q where Q is pointed open convex, and dilating C
strong Henig-proper C-efficiency	$\mathrm{Min}_{\mathrm{int}C_\varepsilon(B)}$ ε satisfying $0 < \varepsilon < \delta$, where $\delta := \inf\{\|b\| : b \in B\}$

For relations of the above properness concepts and also other kinds of efficiency see, e.g., [87, 88, 103, 104, 125]. Some of them are collected in the diagram below as examples, see [88].

$$
\begin{tikzcd}
& \text{Geoffrion-proper } C\text{-efficiency} \arrow[d] & \\
\text{strong } C\text{-efficiency} \arrow[r] & C\text{-efficiency} \arrow[r] & \text{weak } C\text{-efficiency} \\
\text{positive-proper } C\text{-efficiency} \arrow[r] & \text{Henig-proper } C\text{-efficiency} \arrow[u] \arrow[d, shift left, "C \text{ has a compact convex base}"] & \\
& \text{strong Henig-proper } C\text{-efficiency} \arrow[u, shift left] &
\end{tikzcd}
$$

Let us observe that

Proposition 2.2.10. *Suppose that Q is any cone given in Proposition 2.2.9. Then*

$$Q + C \subseteq Q.$$

Proof. It is easy to prove the assertion, when $Q = \operatorname{int} C$, $Q = Y \setminus (-C)$, $Q = \{y \in Y : \varphi(y) > 0\}$ for $\varphi \in C^{+i}$, or Q is a pointed open convex cone dilating C.

Now let $Q = C(\varepsilon)$ for some $\varepsilon > 0$, $y \in Q$ and $c \in C$. We show that $y + c \in Q$. It is easy to see that $d_C(y+c) \leq d_C(y)$ and $d_{-C}(y) \leq d_{-C}(y+c)$. Because $y \in Q$, we have $d_C(y) < \varepsilon d_{-C}(y)$. Thus, $d_C(y+c) < \varepsilon d_{-C}(y+c)$ and hence $y + c \in Q$.

For $Q = \operatorname{int} C_\varepsilon(B)$, it is easy to see that $C \subseteq Q$ for any ε satisfying $0 < \varepsilon < \delta$. So, $Q + C \subseteq Q + Q \subseteq Q$. $\square$

Chapter 3

The theory of Γ-limits

3.1 Introduction

Γ-convergence were introduced by Ennio De Giorgi in a series of papers published between 1975 and 1985. In the same years, De Giorgi developed the theoretical framework of Γ-convergence and explored multifarious applications of this tool. We now give a brief on the development of Γ-convergence in this peroid.

In 1975, a formal definition of Γ-convergence for a sequence of functions on a topological vector space appeared in [48] by De Giorgi and Franzoni. It included the old notion of G-convergence (introduced in [155] by Spagnolo for elliptic operators) as a particular case, and provided a unified framework for the study of many asymptotic problems in the calculus of variations.

In 1977, De Giorgi defined in [38] the so called multiple Γ-limits, i.e., Γ-limits for functions depending on more than one variable. These notions have been a starting point for applications of Γ-convergence to the study of asymptotic behaviour of saddle points in min-max problems and of solutions to optimal control problems.

In 1981, De Giorgi formulated in [40, 41] the theory of Γ-limits in a very general abstract setting and also explored a possibility of extending these notions to complete lattices. This project was accomplished in [43] by De Giorgi and Buttazzo in the same year. The paper also contains some general guide-lines for the applications of Γ-convergence to the study of limits of solutions of ordinary and partial differential equations, including also optimal control problems.

Other applications of Γ-convergence was considered in [39, 45] by De Giorgi et al. in 1981. These papers deal with the asymptotic behaviour of the solutions to minimum problems for the Dirichlet integral with unilateral obstacles. In [44], De Giorgi and Dal Maso gave an account of main results on Γ-convergence and of its most significant applications to the calculus of variations.

In 1983, De Giorgi proposed in [42] several notions of convergence for measures defined on the space of lower semicontinuous functions, and formulated some problems whose solutions would be useful to identify the most suitable notion of convergence for the study of Γ-limits of random functionals. This notion of convergence was pointed out and studied in detail by De Giorgi et al. in [46, 47].

In 1983 in [82] Greco introduced limitoids and showed that all the Γ-limits are special limitoids. In a series of papers published between 1983 and 1985, he developed many applications of this tool in the general theory of limits. The most important result regarding limitoids, presented in [82, 84], is the representation theorem for which each relationship of limitoids becomes a relationship of their supports in set theory. In 1984, by applying this theorem, Greco stated in [83] important results on sequential forms of De Giorgi's Γ-limits via a decomposition of their supports in the setting of completely distributive lattice. These results simplify calculation of complicated Γ-limits. This enabled him to find many errors in the literature.

In this chapter, we first introduce definitions and some basic properties of Γ-limits. Greco's results on sequential forms of Γ-limits are also recalled. Finally, we give some applications of Γ-limits to derivatives and tangent cones.

Consider n sets $S_1,...,S_n$ and a function f from $S_1 \times ... \times S_n$ into $\overline{\mathbb{R}}$. Given non-degenerate families $\mathscr{A}_1,...,\mathscr{A}_n$ on $S_1,...,S_n$, respectively, and $\alpha_1,...,\alpha_n \in \{+,-\}$.

Definition 3.1.1. ([38]) Let

$$\Gamma(\mathscr{A}_1^{\alpha_1},...,\mathscr{A}_n^{\alpha_n})\lim f := \operatorname*{ext}^{-\alpha_n}_{A_n\in\mathscr{A}_n} ... \operatorname*{ext}^{-\alpha_1}_{A_1\in\mathscr{A}_1} \operatorname*{ext}^{\alpha_1}_{x_1\in A_1} ... \operatorname*{ext}^{\alpha_n}_{x_n\in A_n} f(x_1,...,x_n),$$

where $\mathrm{ext}^+ = \sup$ and $\mathrm{ext}^- = \inf$.

The expression above, called a Γ-*limit* of f, is a (possibly infinite) number. It is obvious that

$$\Gamma(\mathscr{A}_1^{\alpha_1},...,\mathscr{A}_n^{\alpha_n})\lim f = -\Gamma(\mathscr{A}_1^{-\alpha_1},...,\mathscr{A}_n^{-\alpha_n})\lim(-f). \tag{3.1}$$

Given topologies $\tau_1,...,\tau_n$ on $S_1,...,S_n$, we write

$$\left(\Gamma(\tau_1^{\alpha_1},...,\tau_n^{\alpha_n})\lim f\right)(x_1,...,x_2) := \Gamma(\mathscr{N}_{\tau_1}(x_1)^{\alpha_1},...,\mathscr{N}_{\tau_n}(x_n)^{\alpha_n})\lim f^1. \tag{3.2}$$

Notice that $\Gamma(\tau_1^{\alpha_1},...,\tau_n^{\alpha_n})\lim f$ is a function from $S_1 \times ... \times S_n$ into $\overline{\mathbb{R}}$.

[1]If (X,τ) is a topological space, then $\mathscr{N}_\tau(x)$ stands for the set of all neighborhoods of x.

Proposition 3.1.2. ([50]) (i) *If* $\mathscr{A}_k \leq \mathscr{B}_k$, *then*

$$\Gamma(\dots,\mathscr{A}_k^-,\dots)\lim f \leq \Gamma(\dots,\mathscr{B}_k^-,\dots)\lim f,$$

$$\Gamma(\dots,\mathscr{A}_k^+,\dots)\lim f \geq \Gamma(\dots,\mathscr{B}_k^+,\dots)\lim f.$$

(ii) Suppose that $\mathscr{A}_i$, $i = 1,\dots,n$, *are filters. Then*

$$\Gamma(\dots,\mathscr{A}_k^-,\dots)\lim f \leq \Gamma(\dots,\mathscr{A}_k^+,\dots)\lim f,$$

$$\Gamma(\dots,\mathscr{A}_k^+,\mathscr{A}_{k+1}^-,\dots)\lim f \leq \Gamma(\dots,\mathscr{A}_{k+1}^-,\mathscr{A}_k^+,\dots)\lim f.$$

It is a simple observation that "sup" and "inf" operations are examples of Γ-limits:

$$\inf_B f(x) = \Gamma(N_\iota(B)^-)\,f, \quad \sup_B f(x) = \Gamma(N_\iota(B)^+)\,f,$$

where ι stands for the *discrete topology*, $N_\iota(B)$ is the filter of all supersets of the set B. If B is the whole space, we may also use the *chaotic topology* o.

3.2 Γ-limits in two variables

Let $\mathfrak{f} : I \times X \to \overline{\mathbb{R}}$ defined by $\mathfrak{f}(i,x) := f_i(x)$, where $\{f_i\}_{i\in I}$ is a family of functions from X into $\overline{\mathbb{R}}$ and filtered by a filter $\mathscr{F}$ on I. Thus, results on Γ-limits of $\mathfrak{f}$ implies those on limits of $\{f_i\}_{i\in I}$.

From Definition 3.1.1, we get for $x \in X$,

$$\begin{aligned}
(\Gamma(\mathscr{F}^+;\tau^-)\lim\mathfrak{f})(x) &= \Gamma(\mathscr{F}^+,\mathscr{N}_\tau(x)^-)\lim\mathfrak{f} \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \inf_{F\in\mathscr{F}} \sup_{i\in F} \inf_{y\in U} \mathfrak{f}(i,y) \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \limsup_{\mathscr{F}} \inf_{y\in U} f_i(y), \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \Gamma(\mathscr{F}^+) \inf_{y\in U} f_i(y),
\end{aligned}$$

$$\begin{aligned}
(\Gamma(\mathscr{F}^-;\tau^-)\lim\mathfrak{f})(x) &= \Gamma(\mathscr{F}^-,\mathscr{N}_\tau(x)^-)\lim\mathfrak{f} \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \sup_{F\in\mathscr{F}} \inf_{i\in F} \inf_{y\in U} \mathfrak{f}(i,y) \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \liminf_{\mathscr{F}} \inf_{y\in U} f_i(y), \\
&= \sup_{U\in\mathscr{N}_\tau(x)} \Gamma(\mathscr{F}^-) \inf_{y\in U} f_i(y),
\end{aligned}$$

$$\begin{aligned}
(\Gamma(\mathscr{F}^+;\tau^+)\lim \mathfrak{f})(x) &= \Gamma(\mathscr{F}^+,\mathscr{N}_\tau(x)^+)\lim \mathfrak{f}\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \inf_{F\in\mathscr{F}} \sup_{i\in F} \sup_{y\in U} \mathfrak{f}(i,y)\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \limsup{}_{\mathscr{F}} \sup_{y\in U} f_i(y),\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \Gamma(\mathscr{F}^+) \sup_{y\in U} f_i(y),\\
(\Gamma(\mathscr{F}^-;\tau^+)\lim \mathfrak{f})(x) &= \Gamma(\mathscr{F}^-,\mathscr{N}_\tau(x)^+)\lim \mathfrak{f}\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \sup_{F\in\mathscr{F}} \inf_{i\in F} \sup_{y\in U} \mathfrak{f}(i,y)\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \liminf{}_{\mathscr{F}} \sup_{y\in U} f_i(y),\\
&= \inf_{U\in\mathscr{N}_\tau(x)} \Gamma(\mathscr{F}^-) \sup_{y\in U} f_i(y).
\end{aligned}$$

Based on these limits above, we can show that some well-known limits are special cases of Γ-limits as follows.

Remark 3.2.1. (i) If functions $f_i(x)$ are independent of x, i.e., for every i there exists a constant $a_i \in \overline{\mathbb{R}}$ such that $f_i(x) = a_i$ for every $x \in X$, then

$$\left(\Gamma(\mathscr{F}^+;\tau^-)\lim \mathfrak{f}\right)(x) = \left(\Gamma(\mathscr{F}^+;\tau^+)\lim \mathfrak{f}\right)(x) = \mathrm{limsup}_{\mathscr{F}}\, a_i,$$

$$\left(\Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f}\right)(x) = \left(\Gamma(\mathscr{F}^-;\tau^+)\lim \mathfrak{f}\right)(x) = \mathrm{liminf}_{\mathscr{F}}\, a_i.$$

(ii) If functions $f_i(x)$ are independent of i, i.e., there exists $g : X \to \overline{\mathbb{R}}$ such that $f_i(x) = g(x)$ for every $x \in X$, $i \in I$, then

$$\left(\Gamma(\mathscr{F}^-;\tau^+)\lim \mathfrak{f}\right)(x) = \left(\Gamma(\mathscr{F}^+;\tau^+)\lim \mathfrak{f}\right)(x) = \limsup_{y\to_\tau x} g(y),$$

$$\left(\Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f}\right)(x) = \left(\Gamma(\mathscr{F}^+;\tau^-)\lim \mathfrak{f}\right)(x) = \liminf_{y\to_\tau x} g(y).$$

In [36], $\Gamma(\mathscr{F}^+;\tau^-)\lim \mathfrak{f}$ and $\Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f}$ are called, by Dal Maso, the Γ-*upper limit* and the Γ-*lower limit* of the family f_i and are denoted by $\Gamma\text{-}\limsup_{\mathscr{F}}^{\tau} f_i$ and $\Gamma\text{-}\liminf_{\mathscr{F}}^{\tau} f_i$, respectively. If there exists a function f_0 such that for all $x \in X$,

$$(\Gamma(\mathscr{F}^+;\tau^-)\lim \mathfrak{f})(x) \le f_0(x) \le (\Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f})(x),$$

then we say that $\{f_i\}$ Γ-*convergences* to f_0 or f_0 is a Γ-limit of $\{f_i\}$.

The following examples show that, in general, Γ-convergence and pointwise convergence are independent.

Example 3.2.2. ([36]) Let $X=\mathbb{R}$ (with a usual topology ν on $\mathbb{R}$) and $\{f_n\}$ be defined by

(i) $f_n(x)=\sin(nx)$. Then, $\{f_n\}$ Γ-converges to the constant function $f=-1$, whereas $\{f_n\}$ does not converge pointwise in $\mathbb{R}$.

(ii) $f_n(x)=\begin{cases} nxe^{-2n^2x^2}, & \text{if n is even,} \\ 2nxe^{-2n^2x^2}, & \text{if n is odd.}\end{cases}$

By calculating, $\{f_n\}$ converges pointwise to 0, but $\{f_n\}$ does not Γ-converge since

$$\big(\Gamma(\mathscr{I}^-;\nu^-)\lim \mathfrak{f}\big)(x)=\begin{cases} -e^{-1/2}, & \text{if } x=0, \\ 0, & \text{if } x\neq 0,\end{cases}$$

$$\big(\Gamma(\mathscr{I}^+;\nu^-)\lim \mathfrak{f}\big)(x)=\begin{cases} -\frac{1}{2}e^{-1/2}, & \text{if } x=0, \\ 0, & \text{if } x\neq 0,\end{cases}$$

where $\mathfrak{f}(n,x):=f_n(x)$ and $\mathscr{I}:=\{I_m\}_{m\in\mathbb{N}}$ with $I_m:=\{n\in\mathbb{N}:n\geq m\}$.

We now compare the notion of Γ-limits with some classical notions of convergence.

Definition 3.2.3. A family $\{f_i\}_{i\in I}$ is said to be *continuously convergent* to a function $g:X\to\overline{\mathbb{R}}$ if for every $x\in X$ and for every neighborhood V of $g(x)$, there exist $F\in\mathscr{F}$ and $U\in\mathscr{N}_\tau(x)$ such that $f_i(y)\in V$ for every $i\in F$ and for every $y\in U$.

It follows immediately from the definitions that $\{f_i\}$ is continuously convergence to g if and only if

$$\Gamma(\mathscr{F}^+;\tau^+)\lim \mathfrak{f}\leq g\leq \Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f}.$$

Definition 3.2.4. ([111]) Let $\{A_i\}_{i\in I}$ be a family of subsets in (X,τ) filtered by $\mathscr{F}$.

(i) The *K-upper limit* of $\{A_i\}_{i\in I}$ is defined by

$$\operatorname{Limsup}^\tau_{\mathscr{F}} A_i=\bigcap_{F\in\mathscr{F}}\operatorname{cl}_\tau\bigcup_{i\in F}A_i.$$

(ii) The *K-lower limit* of $\{A_i\}_{i\in I}$ is defined by

$$\operatorname{Liminf}^\tau_{\mathscr{F}} A_i=\bigcap_{F\in\mathscr{F}^\#}\operatorname{cl}_\tau\bigcup_{i\in F}A_i.$$

(iii) If there exists a subset A in X such that

$$\operatorname{Limsup}^\tau_{\mathscr{F}} A_i\subseteq A\subseteq \operatorname{Liminf}^\tau_{\mathscr{F}} A_i,$$

then we say that $\{A_i\}$ *K-converges* to A.

It follows from the above definition that $x \in \mathrm{Limsup}^{\tau}_{\mathscr{F}} A_i$ if and only if for every $U \in \mathscr{N}_{\tau}(x)$ every $F \in \mathscr{F}$, there is $i \in F$ such that $U \cap A_i \neq \emptyset$; due to the duality of filters and their grills, if for every $U \in \mathscr{N}_{\tau}(x)$ there is $H \in \mathscr{F}^{\#}$ such that $U \cap A_i \neq \emptyset$ for each $i \in H$.

A point $x \in \mathrm{Liminf}^{\tau}_{\mathscr{F}} A_i$ if and only if for every $U \in \mathscr{N}_{\tau}(x)$ and every $H \in \mathscr{F}^{\#}$, there is $i \in H$ such that $U \cap A_i \neq \emptyset$. Dually, if for every $U \in \mathscr{N}_{\tau}(x)$ there is $F \in \mathscr{F}$ such that for each $i \in F$, $U \cap A_i \neq \emptyset$.

When X is equipped with the discrete topology ι, the discussed limits become set-theoretical, that is

$$\mathrm{Limsup}^{\iota}_{\mathscr{F}} A_i = \bigcap_{F \in \mathscr{F}} \bigcup_{i \in F} A_i,$$

and

$$\mathrm{Liminf}^{\iota}_{\mathscr{F}} A_i = \bigcap_{H \in \mathscr{F}^{\#}} \bigcup_{i \in H} A_i = \bigcup_{F \in \mathscr{F}} \bigcap_{i \in F} A_i.$$

Remark 3.2.5. It is generally admitted among those who study optimization, that modern definition of limits of sets by Painlevé and Kuratowski (see [110, 172, 173]). However, Peano already introduced them in 1887. Indeed, in [135], Peano defined the lower limit of a family, indexed by the reals, of a subsets A_λ of an affine Euclidean space A by

$$\mathrm{Liminf}_{\lambda \to +\infty} A_\lambda := \{y \in X : \lim_{\lambda \to +\infty} \mathrm{d}(\mathrm{y}, \mathrm{A}_\lambda) = 0\}.$$

In [137], he also defined the upper limits of $\{A_\lambda\}$

$$\mathrm{Limsup}_{\lambda \to +\infty} A_\lambda := \{y \in X : \mathrm{liminf}_{\lambda \to +\infty} \mathrm{d}(\mathrm{y}, \mathrm{A}_\lambda) = 0\}$$

that he also expresses as

$$\mathrm{Limsup}_{n \to +\infty} A_\lambda = \bigcap_{n \in \mathbb{N}} \mathrm{cl} \bigcup_{\mathrm{k} \geq \mathrm{n}} \mathrm{A}_{\mathrm{k}}.$$

In 1948, Kuratowski, by his work (see [111]), has definitely propagated the concept of limits of variable sets and established the use of them in mathematics, that are called today upper and lower Kuratowski limits.

Recall that, for every $A \subseteq X$, the *characteristic function* of A is defined by

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let $\Omega \subseteq (S_1, \tau_1) \times ... \times (S_n, \tau_n)$. We define

$$\chi_{G(\tau_1^{\alpha_1},...,\tau_n^{\alpha_n})\Omega} := \Gamma(\tau_1^{\alpha_1}, ..., \tau_n^{\alpha_n}) \lim \chi_\Omega.$$

Then $G(\tau_1^{\alpha_1}, ..., \tau_n^{\alpha_n})\Omega$ is called *G-limit* of Ω, see [38].

The following proposition shows that K-limits of a family of subsets can be expressed in terms of G-limits.

Proposition 3.2.6. ([36]) *Let* $\{A_i\}_{i\in I}$ *be a family of subsets of* X*. Then*

$$G(\mathscr{F}^-;\tau^+)\mathbb{A} = \mathrm{Liminf}^\tau_{\mathscr{F}} A_i, \quad G(\mathscr{F}^+;\tau^+)\mathbb{A} = \mathrm{Limsup}^\tau_{\mathscr{F}} A_i,$$

where $\mathbb{A}$ *is a relation from* I *into* X*, i.e.* $\mathbb{A} \subseteq I \times X$*, defined by* $\mathbb{A}(i) := A_i$*.*

In particular, $\{A_i\}$ *K-converges to* A *if and only if* $\{\chi_{A_i}\}$ Γ*-converges to* χ_A*.*

Proof. We prove only the first equality, the other one being analogous. Since $\chi_{G(\mathscr{F}^-;\tau^+)\mathbb{A}} = \Gamma(\mathscr{F}^-;\tau^+)\lim \chi_{\mathbb{A}}$ takes only the values 0 and 1, it is enough to show that

$$(\Gamma(\mathscr{F}^-;\tau^+)\lim \chi_{\mathbb{A}})(x) = 1 \iff x \in \mathrm{Liminf}^\tau_{\mathscr{F}} A_i. \tag{3.3}$$

By (3.2), $(\Gamma(\mathscr{F}^-;\tau^+)\lim \chi_{\mathbb{A}})(x) = 1$ if and only if

$$\inf_{U\in\mathscr{N}_\tau(x)} \sup_{F\in\mathscr{F}} \inf_{i\in F} \sup_{x'\in U} \chi_{\mathbb{A}}(i,x') = 1.$$

It means that for every $U \in \mathscr{N}_\tau(x)$ there is $F \in \mathscr{F}$ such that for each $i \in F$, there is $x' \in U$ satisfying $x' \in A_i$, i.e., $U \cap A_i \neq \emptyset$. Thus, $x \in \mathrm{Liminf}^\tau_{\mathscr{F}} A_i$. This prove (3.3) and concludes the proof of the proposition. □

The next result shows a connection between Γ-convergence of functions and K-convergence of their epigraphs or hypographs.

Proposition 3.2.7. ([36]) *Let* $\{f_i\}_{i\in I}$ *be a family of extended-real-valued functions, and let*

$$f^- := \Gamma(\mathscr{F}^-;\tau^-)\lim \mathfrak{f}, \quad f^+ := \Gamma(\mathscr{F}^+;\tau^-)\lim \mathfrak{f}.$$

Then

$$\mathrm{epi}(f^-) = \mathrm{Limsup}^\tau_{\mathscr{F}} \mathrm{epi}(f_i), \quad \mathrm{epi}(f^+) = \mathrm{Liminf}^\tau_{\mathscr{F}} \mathrm{epi}(f_i),$$

$$\mathrm{hypo}(f^-) = \mathrm{Liminf}^\tau_{\mathscr{F}} \mathrm{hypo}(f_i), \quad \mathrm{hypo}(f^+) = \mathrm{Limsup}^\tau_{\mathscr{F}} \mathrm{hypo}(f_i).$$

In particular, $\{f_i\}$ Γ*-converges to* f *if and only if* $\{\mathrm{epi}(f_i)\}$ *(or* $\{\mathrm{hypo}(f_i)\}$*) K-convergences to* $\mathrm{epi}(f)$ *(*$\{\mathrm{hypo}(f)\}$*, respectively).*

Proof. By the similarity, we prove only the first equality. A point $(x,t) \in X \times \mathbb{R}$ belongs to epi(f^-) if and only if $f^-(x) \leq t$. By the defininition of f^-, this happens if and only if for every $\varepsilon > 0$, and for every $U \in \mathscr{N}_\tau(x)$ we have

$$\operatorname{liminf}_{\mathscr{F}} \inf_{y \in U} f_i(y) < t + \varepsilon,$$

and this is equivalent to say that for every $\varepsilon > 0, U \in \mathscr{N}_\tau(x), F \in \mathscr{F}$ there exists $i \in F$ such that $\inf\limits_{y \in U} f_i(y) < t + \varepsilon$. Since this inequality is equivalent to

$$\mathrm{epi}(f_i) \cap (\mathrm{U} \times (\mathrm{t} - \varepsilon, \mathrm{t} + \varepsilon)) \neq \emptyset,$$

and the sets of the form $U \times (t - \varepsilon, t + \varepsilon)$, with $U \in \mathscr{N}_\tau(x)$ and $\varepsilon > 0$, are a base for the neighborhood systems of (x,t) in $X \times \mathbb{R}$, we have proved that $(x,t) \in \mathrm{epi}(f^-)$ if and only if $(x,t) \in \operatorname{Limsup}^\tau_{\mathscr{F}} \mathrm{epi}(f_i)$. □

3.3 Γ-limits valued in completely distributive lattices

This section presents some results related to Γ-limits given by Greco in [82, 83]. More precisely, he defined functionals called limitoids and proved that Γ-limits are special limitoids. Then he proved representation theorem showing that for which each relationship of limitoids corresponds a relationship in set theory.

3.3.1 Limitoids

Let S be a set with at least two elements and L be a complete lattice with the minimum element 0_L and the maximum element 1_L ($0_L \neq 1_L$).

Definition 3.3.1. ([82]) A function $T : L^S \to L$ is said *L-limitoid* in S (or *limitoid* , in short) if for every $f, g \in L^S$ and for each *complete homomorphism* φ of L in L,

(i) $g \leq f \Longrightarrow T(g) \leq T(f)$,

(ii) $T(\varphi \circ g) = \varphi(T(g))$,

(iii) $T(g) \in \overline{g(S)}^L$,

where $\overline{g(S)}^L$ is the smallest closed sublattice of L containing $g(S)$, and L^S denotes the set of functions from S into L .

We recall that a *complete homomorphism* $\varphi : L \to L'$ between two complete lattices is a function verifying two equalities $\varphi(\bigvee A) = \bigvee \varphi(A)$ and $\varphi(\bigwedge A) = \bigwedge \varphi(A)$ for each non-empty subset A of L, see [82].

Simple examples of limitoids are limit inferior and limit superior. Let f be a function from S into L and $\mathscr{A}$ be a non-degenerate family of subsets of S, the *limit inferior* and *limit superior* of f along $\mathscr{A}$ are defined, respectively,

$$\operatorname{liminf}_{\mathscr{A}} f = \bigvee_{A \in \mathscr{A}} \bigwedge_{x \in A} f(x) \left(= \sup_{A \in \mathscr{A}} \inf_{x \in A} f(x) \right),$$

$$\operatorname{limsup}_{\mathscr{A}} f = \bigwedge_{A \in \mathscr{A}} \bigvee_{x \in A} f(x) \left(= \inf_{A \in \mathscr{A}} \sup_{x \in A} f(x) \right).$$

The $\Gamma-$limit introduced in Definition 3.1.1 is another example of the limitoid.

It is evident that the limit inferior, limit superior and Γ-limit do not change if we use equivalent families. Since $\mathscr{A}^{\#\#} = \mathscr{A}$ for each family $\mathscr{A}$, we have

$$\operatorname{liminf}_{\mathscr{A}} f = \operatorname{liminf}_{\mathscr{A}^{\#\#}} f, \quad \operatorname{limsup}_{\mathscr{A}} f = \operatorname{limsup}_{\mathscr{A}^{\#\#}} f.$$

The following result characterises of a completely distributive lattice L.

Proposition 3.3.2. ([82]) *A complete lattice L is completely distributive if and only if*

$$\operatorname{liminf}_{\mathscr{A}} f = \operatorname{limsup}_{\mathscr{A}^{\#}} f, \tag{3.4}$$

for each non-degenerate family $\mathscr{A}$ of subsets of S and for each function f from S into L.

Proof. It follows from Proposition D.3 in [82]. □

Definition 3.3.3. ([83]) The *support of a limitoid T in* S, denoted by $\mathbf{st}(T)$, is the family of sets defined by

$$\mathbf{st}(T) := \{A \subseteq S : T(\chi_A^L) = 1_L\},$$

where $\chi_A^L : S \to L$ is to 1_L on A and to 0_L on $S \setminus A$.

A support of a limitoid T in S is a semi-filter, i.e., $\mathbf{st}(T) \in \mathbb{SF}(S)$. Recall that, $\mathbb{SF}(S)$ is a completely distributive lattice with respect to inclusion, see [83], with its operations $\bigwedge$ and $\bigvee$ be the intersection and the union of sets, respectively.

The support of Γ-limit in Definition 3.1.1 is indicated with $(\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_n^{\alpha_n})$. In [83], Greco proved recursively that

$$(\mathscr{A}^-)\approx\mathscr{A}^{\#\#},\quad(\mathscr{A}^+)\approx\mathscr{A}^{\#},$$
$$(\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_{n-1}^{\alpha_{n-1}},\mathscr{A}_n^-)\approx(\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_{n-1}^{\alpha_{n-1}})\times\mathscr{A}_n,$$
$$(\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_{n-1}^{\alpha_{n-1}},\mathscr{A}_n^+)\approx((\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_{n-1}^{\alpha_{n-1}})^{\#}\times\mathscr{A}_n)^{\#},$$

where $\mathscr{A}\times\mathscr{B}$ indicates the family generated by $\{A\times B: A\in\mathscr{A}, B\in\mathscr{B}\}$. Some special cases are

$$\begin{aligned}&(\mathscr{A}^-,\mathscr{B}^-)\approx\mathscr{A}\times\mathscr{B}, && (\mathscr{A}^+,\mathscr{B}^-)\approx\mathscr{A}^{\#}\times\mathscr{B},\\ &(\mathscr{A}^+,\mathscr{B}^+)\approx(\mathscr{A}\times\mathscr{B})^{\#}, && (\mathscr{A}^-,\mathscr{B}^+)\approx(\mathscr{A}^{\#}\times\mathscr{B})^{\#},\\ &(\mathscr{A}^+,\mathscr{B}^-,\mathscr{C}^-)\approx\mathscr{A}^{\#}\times\mathscr{B}\times\mathscr{C}, && (\mathscr{A}^-,\mathscr{B}^+,\mathscr{C}^-)\approx(\mathscr{A}^{\#}\times\mathscr{B})^{\#}\times\mathscr{C}.\end{aligned}$$

Then we get the following property

$$(\mathscr{A}_1^{-\alpha_1},\dots,\mathscr{A}_n^{-\alpha_n})\approx(\mathscr{A}_1^{\alpha_1},\dots,\mathscr{A}_n^{\alpha_n})^{\#}. \tag{3.5}$$

3.3.2 Representation theorem

The representation theorem of Greco showed that each limitoid valued in a completely distributive lattice is a limit inferior.

Theorem 3.3.4. ([82]) (The representation of limitoids) *Let L be a completely distributive lattice and T be a limitoid in S. Then, for each $f\in L^S$,*

$$T(f)=\operatorname{liminf}_{\mathbf{st}(T)}f, \tag{3.6}$$

where $\mathbf{st}(T)$ is the support of T.

Proof. First, we check that for each $f\in L^S$,

$$\operatorname{liminf}_{\mathbf{st}(T)}f\le T(f)\le\operatorname{limsup}_{(\mathbf{st}(T))^{\#}}f, \tag{3.7}$$

where $(\mathbf{st}(T))^{\#}:=\{A\subseteq S: A\cap F\neq\emptyset, \forall F\in\mathbf{st}(T)\}$. For each $A\in\mathbf{st}(T)$, we put $g:=\chi_A^L\wedge(\bigwedge f(A))$. Since L is completely distributive, the function $\varphi(x):=x\wedge(\bigwedge f(A))$ is a complete homomorphism of L in L. So, from condition (ii) in Definition 3.3.1 and the definition of carriers, we get $T(g)=\bigwedge f(A)$. Since $g\le f$, it follows from condition (i) in Definition 3.3.1 that $\bigwedge f(A)\le T(f)$. Therefore,

$$\operatorname{liminf}_{\mathbf{st}(T)}f\le T(f).$$

On the other hand, let $A \in (\mathbf{st}(T))^{\#}$ and $g := \chi^L_{S\setminus A} \vee (\bigvee f(A))$. From the definition of $(\mathbf{st}(T))^{\#}$, we have $(S\setminus A) \notin \mathbf{st}(T)$, so $T(\chi^L_{S\setminus A}) = 0_L$. Since the function φ defined by $\varphi(x) := x \vee (\bigvee f(A))$ is a complete homomorphism of L in L, so $T(g) = \bigvee f(A)$. Finally, since $g \geq f$, we obtain $\bigvee f(A) \geq T(f)$, which implies

$$T(f) \leq \operatorname{limsup}_{(\mathbf{st}(T))^{\#}} f.$$

Thus, we have proved (3.7). (3.6) is obtained by Proposition 3.3.2. □

It follows from (3.6) that

$$\Gamma(\mathscr{A}_1^{\alpha_1},\ldots,\mathscr{A}_n^{\alpha_n}) \lim f = \liminf_{(\mathscr{A}_1^{\alpha_1},\ldots,\mathscr{A}_n^{\alpha_n})} f = \sup_{A\in(\mathscr{A}_1^{\alpha_1},\ldots,\mathscr{A}_n^{\alpha_n})} \inf_{(x_1,\ldots,x_n)\in A} f(x_1,\ldots,x_n).$$

By virtue of (3.4) and (3.5), we get

$$\Gamma(\mathscr{A}_1^{-\alpha_1},\ldots,\mathscr{A}_n^{-\alpha_n}) \lim f = \liminf_{(\mathscr{A}_1^{\alpha_1},\ldots,\mathscr{A}_n^{\alpha_n})^{\#}} f = \inf_{A\in(\mathscr{A}_1^{\alpha_1},\ldots,\mathscr{A}_n^{\alpha_n})} \sup_{(x_1,\ldots,x_n)\in A} f(x_1,\ldots,x_n).$$

The representation of limitoids allows us to describe the structure of lattice $\mathbf{Lim}(S,L)$ of limitoids in S valued in L, where L is a completely distibutive lattice. The set $\mathbf{Lim}(S,L)$ of limitoids in S valued in L is the complete lattice with respect to the order defined by $T \leq T'$ if and only if $T(g) \leq T'(g)$ for each $g \in L^S$. In $\mathbf{Lim}(S,L)$, the limitoids $\bigvee_i T_i, \bigwedge_i T_i$ are defined by for each $g \in L^S$,

$$\left(\bigvee\nolimits_i T_i\right)(g) = \bigvee\nolimits_i (T_i(g)), \quad \left(\bigwedge\nolimits_i T_i\right)(g) = \bigwedge\nolimits_i (T_i(g)).$$

Furthermore, the function $\mathbf{st} : \mathbf{Lim}(S,L) \to \mathbb{SF}(S)$ is a complete homomorphism of $\mathbf{Lim}(S,L)$ on $\mathbb{SF}(S)$, see [82], since for each semi-filter $\mathscr{A}$ in S,

$$\mathbf{st}\left(\bigwedge\nolimits_i T_i\right) = \bigcap\nolimits_i \mathbf{st}(T_i), \quad \mathbf{st}\left(\bigvee\nolimits_i T_i\right) = \bigcup\nolimits_i \mathbf{st}(T_i), \quad \mathbf{st}(\operatorname{liminf}_{\mathscr{A}}) = \mathscr{A}.$$

On the other hand, if L is completely distributive, then two limitoids in S with the same supports are equal (by the representation of limitoids). Therefore

Theorem 3.3.5. ([82]) (The structure of lattice of limitoids) *If L is a completely distributive lattice, then the function that associates each limitoid in S to its support is a complete isomorphism, i.e., a bijective complete homomorphism, from* $\mathbf{Lim}(S,L)$ *into* $\mathbb{SF}(S)$.

The function $\operatorname{liminf} : \mathbb{SF}(S) \to \mathbf{Lim}(S,L)$ that associates each semi-filter $\mathscr{A}$ in S to the limit inferior with respect to $\mathscr{A}$ is the inverse isomorphism. Therefore, for a completely distributive lattice L, we have

$$\operatorname{liminf}_{\bigcap_i \mathscr{A}_i} f = \bigwedge\nolimits_i \operatorname{liminf}_{\mathscr{A}_i} f, \tag{3.8}$$

$$\operatorname{liminf}_{\bigcup_i \mathscr{A}_i} f = \bigvee_i \operatorname{liminf}_{\mathscr{A}_i} f, \tag{3.9}$$

for each $f \in L^S$ and $\{\mathscr{A}_i\}_i \subseteq \mathbb{SF}(S)$.

These analyses above mean that with a completely distributive lattice L, each theorem in $\mathbf{Lim}(S,L)$ becomes a theorem in set theory in $\mathbb{SF}(S)$, and vice versa.

3.4 Sequential forms of Γ-limits for extended-real-valued functions

In this section, we extend Greco's results to more general filters related to sequentiality, like Fréchet, strongly Fréchet, and productively Fréchet filters.

Let $\mathscr{F}$ be a filter on X. We recall that, see [53]

• $\mathscr{F}$ is called a *principal filter* if there exists a nonempty subset A of X such that $\mathscr{F} = \{B \subseteq X : A \subseteq B\}$. The set of principal filters on X is denoted by $\mathbb{F}_0(X)$.

• $\mathscr{F}$ is called a *sequential filter* if there exists a sequence $\{x_n\}_n$ in X such that the family $\{\{x_n : n \geq m\} : m \in \mathbb{N}\}$ is a base of $\mathscr{F}$. Then, we denote $\mathscr{F} \approx \{x_n\}_n$. The set of sequential filters on X is denoted by $\mathbb{F}_{\text{seq}}(X)$.

• $\mathscr{F}$ is called a *countably based filter* if it admits a countable base. The set of countably based filters on X is denoted by $\mathbb{F}_1(X)$.

Principal filters and sequential filters are special cases of countably based filters. If $\mathscr{F} \approx \{x_n\}_n$, then (see [83])

$$\operatorname{liminf}_{\mathscr{F}} f = \liminf_{n \to +\infty} f(x_n), \quad \operatorname{liminf}_{\mathscr{F}^\#} f = \limsup_{n \to +\infty} f(x_n).$$

To facilitate for results in the sequel, we denote $\operatorname{Seq}(\mathscr{F}) := \{\mathscr{E} \in \mathbb{F}_{\text{seq}}(X) : \mathscr{E} \geq \mathscr{F}\}$.

Definition 3.4.1. ([62]) A topological space X is called

(i) *Fréchet* if whenever $A \subseteq X$, and $x \in \operatorname{cl} A$, there exists a sequence $\{x_n\}_n$ on A such that $x = \lim_{n \to +\infty} x_n$.

(ii) *strongly Fréchet* if for each a decreasing sequence $\{A_n\}_n$ of subsets of X and $x \in \bigcap_n \operatorname{cl}(A_n)$, there exists a sequence $\{x_n\}_n$ such that $x_n \in A_n$ and $x = \lim_{n \to +\infty} x_n$.

(iii) *first countable* if for all $x \in X$, $\mathscr{N}(x)$ is a countably based filter.

Definitions of Fréchet and strongly Fréchet spaces can be rephrased in terms of filters as follows, see [52, 100].

• A space X is Fréchet if and only if for all $x \in X$, $\mathscr{N}(x)$ is a Fréchet filter on X in the following sense: a filter $\mathscr{F}$ is *Fréchet* if

$$\forall \mathscr{G} \in \mathbb{F}_0(X) : \mathscr{G} \# \mathscr{F} \Longrightarrow \exists \mathscr{H} \in \mathbb{F}_{\text{seq}}(X) : \mathscr{H} \geq \mathscr{F} \vee \mathscr{G}, \tag{3.10}$$

where $\mathscr{F} \vee \mathscr{G} := \{F \cap G : F \in \mathscr{F}, G \in \mathscr{G}\}$ is the supremum of $\mathscr{F}$ and $\mathscr{G}$.

• A space X is strongly Fréchet if and only if for all $x \in X$, $\mathscr{N}(x)$ is a strongly Fréchet filter on X in the following sense: a filter $\mathscr{F}$ is *strongly Fréchet* if

$$\forall \mathscr{G} \in \mathbb{F}_1(X) : \mathscr{G} \# \mathscr{F} \Longrightarrow \exists \mathscr{H} \in \mathbb{F}_{\text{seq}}(X) : \mathscr{H} \geq \mathscr{F} \vee \mathscr{G}. \tag{3.11}$$

Based on definitions above, Jordan and Mynard introduced a productively Fréchet space by using a new filter, called productively Fréchet filter, as follows

Definition 3.4.2. ([100]) A space X is productively Fréchet if and only if for all $x \in X$, $\mathscr{N}(x)$ is a productively Fréchet filter on X in the following sense: a filter $\mathscr{F}$ is *productively Fréchet* if

$$\forall \mathscr{G} \in \mathbb{F}_{\text{sF}}(X) : \mathscr{G} \# \mathscr{F} \Longrightarrow \exists \mathscr{H} \in \mathbb{F}_1(X) : \mathscr{H} \geq \mathscr{F} \vee \mathscr{G},$$

where $\mathbb{F}_{\text{sF}}(X)$ denotes the set of strongly Fréchet filters on X.

In [100], Jordan and Mynard showed that

first countable filter $\Longrightarrow$ productively Fréchet filter $\Longrightarrow$ strongly Fréchet filter $\Longrightarrow$ Fréchet filter,

i.e.,

first countable space $\Longrightarrow$ productively Fréchet space $\Longrightarrow$ strongly Fréchet space $\Longrightarrow$ Fréchet space.

Proposition 3.4.3. ([100]) *A filter $\mathscr{F}$ is productively Fréchet if and only if $\mathscr{F} \times \mathscr{G}$ is a Fréchet filter (equivalently a strongly Fréchet filter) for every strongly Fréchet filter $\mathscr{G}$.*

Proof. It follows from Theorem 9 in [100]. □

Remark 3.4.4. ([53]) (i) For every semi-filter $\mathscr{F}$, we have

$$H \notin \mathscr{F}^{\#} \Longleftrightarrow H^c \in \mathscr{F}, \tag{3.12}$$

where H^c denotes the complement of H. In fact, by definition, $H \notin \mathscr{F}^{\#}$ whenever there is $F \in \mathscr{F}$ such that $H \cap F = \emptyset$, equivalently $F \subseteq H^c$, that is, $H^c \in \mathscr{F}$ since $\mathscr{F}$ is filter.

(ii) If $\mathscr{F}$ is a filter on a set X, $\mathscr{G}$ is a filter on a set Y, and $\mathscr{H}$ is a filter on $X \times Y$, we denote by $\mathscr{H}\mathscr{F}$ the filter on Y generated by the sets

$$HF = \{y : \exists x \in F, (x,y) \in H\},$$

for $H \in \mathscr{H}$ and $F \in \mathscr{F}$, and by $\mathscr{H}^{-}\mathscr{G}$ the filter on X generated by the sets

$$H^{-}G = \{x : \exists y \in G, (x,y) \in H\},$$

for $H \in \mathscr{H}$ and $G \in \mathscr{G}$. Notice that

$$\mathscr{H}\#(\mathscr{F} \times \mathscr{G}) \Longleftrightarrow (\mathscr{H}\mathscr{F})\#\mathscr{G} \Longleftrightarrow \mathscr{F}\#(\mathscr{H}^{-1}\mathscr{G}).$$

We now recall some definitions introduced in [83] by Greco as follows

Definition 3.4.5. ([83]) Let $\mathscr{N}$ be a sequential filter associated with a sequence $\{n\}_n$, and let $\mathscr{A}_1, ..., \mathscr{A}_k$ be filters.

$$(\mathscr{N}^{\alpha_0}, \mathscr{A}_1^{\alpha_1}, ..., \mathscr{A}_k^{\alpha_k})_{\text{seq}} := \underset{\{x_n^1\}_n \in \text{Seq}(\mathscr{A}_1)}{\text{ext}^{\alpha_1}} \cdots \underset{\{x_n^k\}_n \in \text{Seq}(\mathscr{A}_k)}{\text{ext}^{\alpha_k}} \underset{m \in \mathbb{N}}{\text{ext}^{-\alpha_0}} \underset{n \geq m}{\text{ext}^{\alpha_0}} \{(n, x_n^1, ..., x_n^k)\}_n,$$

where $\alpha_0, \alpha_1, ..., \alpha_k$ are signs of $+, -$ and $\text{ext}^- = \bigcap, \text{ext}^+ = \bigcup$.

From Definition 3.4.5, the sequential form of the Γ-limit is defined as follows

Definition 3.4.6. ([83]) Let $\mathscr{N}$ be the sequential filter associated with a sequence $\{n\}_n$, $\mathscr{A}_1, ..., \mathscr{A}_k$ be filters in $S_1, ..., S_k$, respectively, and $f : \mathbb{N} \times S_1 \times ... \times S_k \to \overline{\mathbb{R}}$. The *sequential* Γ*-limit* is defined by

$$\Gamma_{\text{seq}}(\mathscr{N}^{\alpha_0}, \mathscr{A}_1^{\alpha_1}, ..., \mathscr{A}_k^{\alpha_k}) \lim f := \underset{\{x_n^1\}_n \in \text{Seq}(\mathscr{A}_1)}{\text{ext}^{\alpha_1}} \cdots \underset{\{x_n^k\}_n \in \text{Seq}(\mathscr{A}_k)}{\text{ext}^{\alpha_k}} \underset{m \in \mathbb{N}}{\text{ext}^{-\alpha_0}} \underset{n \geq m}{\text{ext}^{\alpha_0}} f(n, x_n^1, ..., x_n^k),$$

where $\text{ext}^- = \inf, \text{ext}^+ = \sup$, $\alpha_0, \alpha_1, ..., \alpha_k$ are signs of $+, -$.

The Γ_{seq}-limit is a limitoid and its support is the family $(\mathscr{N}^{\alpha_0}, \mathscr{A}_1^{\alpha_1}, ..., \mathscr{A}_k^{\alpha_k})_{\text{seq}}$.

3.4.1 Two variables

Proposition 3.4.7. *Suppose that* $\mathscr{F}$ *is a Fréchet filter. Then*

$$\mathscr{F} = \bigcap_{\mathscr{E} \in \text{Seq}(\mathscr{F})} \mathscr{E} = \bigcap_{\mathscr{E} \in \text{Seq}(\mathscr{F})} \mathscr{E}^{\#}, \tag{3.13}$$

$$\mathscr{F}^{\#} = \bigcup_{\mathscr{E} \in \text{Seq}(\mathscr{F})} \mathscr{E} = \bigcup_{\mathscr{E} \in \text{Seq}(\mathscr{F})} \mathscr{E}^{\#}. \tag{3.14}$$

Proof. Since (3.13) implies (3.14) (by (2.1)), we prove only (3.13). Let $F \in \mathscr{F}$ and an arbitrary $\mathscr{E} \in \mathrm{Seq}(\mathscr{F})$. Since $\mathscr{E} \geq \mathscr{F}$, there exists $E \in \mathscr{E}$ such that $E \subseteq F$, so $F \in \mathscr{E}$. This implies $F \in \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}$. Thus, $\mathscr{F} \subseteq \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}$. Since $\mathscr{E} \subseteq \mathscr{E}^{\#}$, we get

$$\mathscr{F} \subseteq \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E} \subseteq \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}^{\#}.$$

We now prove $\bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}^{\#} \subseteq \mathscr{F}$. It follows from the definition of Fréchet filters that

$$\forall\, A \in \mathscr{F}^{\#} \Longrightarrow \exists \mathscr{E} \in \mathrm{Seq}(\mathscr{F}) : A \in \mathscr{E}. \tag{3.15}$$

Suppose that $H \notin \mathscr{F}$. This implies $H^c \in \mathscr{F}^{\#}$. From (3.15), there is $\mathscr{E} \in \mathrm{Seq}(\mathscr{F})$ such that $H^c \in \mathscr{E}$, i.e., $H \notin \mathscr{E}^{\#}$. Thus, $H \notin \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}^{\#}$. □

Proposition 3.4.8. *Let $\mathscr{F}, \mathscr{G}$ be filters.*

(i) *Suppose that $\mathscr{F}$ is Fréchet. Then*

$$(\mathscr{F}^{+}, \mathscr{G}^{-}) = \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{-}, \mathscr{G}^{-}) = \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{+}, \mathscr{G}^{-}), \tag{3.16}$$

$$(\mathscr{F}^{-}, \mathscr{G}^{+}) = \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{+}, \mathscr{G}^{+}) = \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{-}, \mathscr{G}^{+}). \tag{3.17}$$

(ii) *Suppose that $\mathscr{F}$ is strongly Fréchet and $\mathscr{G}$ is productively Fréchet (or vice versa). Then*

$$(\mathscr{F}^{-}, \mathscr{G}^{-}) = \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{-}, \mathscr{G}^{-}) = \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{+}, \mathscr{G}^{-}), \tag{3.18}$$

$$(\mathscr{F}^{+}, \mathscr{G}^{+}) = \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{+}, \mathscr{G}^{+}) = \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{-}, \mathscr{G}^{+}). \tag{3.19}$$

Proof. (i) Since (3.16) implies (3.17), we prove only (3.16). Because $(\mathscr{E}^{-}, \mathscr{G}^{-}) \approx \mathscr{E} \times \mathscr{G}$, $(\mathscr{E}^{+}, \mathscr{G}^{-}) \approx \mathscr{E}^{\#} \times \mathscr{G}$ and $\mathscr{E} \subseteq \mathscr{E}^{\#}$, we get

$$\bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{-}, \mathscr{G}^{-}) \subseteq \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E}^{+}, \mathscr{G}^{-}).$$

It follows from (3.14) that

$$(\mathscr{F}^{+}, \mathscr{G}^{-}) \approx \mathscr{F}^{\#} \times \mathscr{G} = \Big(\bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \mathscr{E}\Big) \times \mathscr{G} \subseteq \bigcup_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} (\mathscr{E} \times \mathscr{G}).$$

Thus,

$$(\mathscr{F}^+,\mathscr{G}^-) \subseteq \bigcup_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^-,\mathscr{G}^-) \subseteq \bigcup_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^+,\mathscr{G}^-).$$

Let $A \in \bigcup_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^+,\mathscr{G}^-)$, then there exists $\mathscr{E} \in \mathrm{Seq}(\mathscr{F})$ such that $A \in \mathscr{E}^\# \times \mathscr{G}$. It means that there exist $A_1 \in \mathscr{E}^\#$ and $A_2 \in \mathscr{G}$ such that $A \supseteq A_1 \times A_2$. Suppose that $A \notin (\mathscr{F}^+,\mathscr{G}^-) \approx \mathscr{F}^\# \times \mathscr{G}$. It follows from $A_2 \in \mathscr{G}$ that $A_1 \notin \mathscr{F}^\#$. By (3.12), we get $A_1^c \in \mathscr{F}$. Because $\mathscr{E} \geq \mathscr{F}$, we get $A_1^c \in \mathscr{E}$, which contradicts to $A_1 \in \mathscr{E}^\#$.

(ii) Similarly, we prove only (3.18). It follows from Proposition 3.4.3 that $\mathscr{F} \times \mathscr{G}$ is a Fréchet filter. We can see that

$$\begin{aligned}
(\mathscr{F}^-,\mathscr{G}^-) &\approx \mathscr{F} \times \mathscr{G} \\
&= \Big(\bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \mathscr{E}\Big) \times \mathscr{G} \text{ (by (3.13))} \\
&\subseteq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E} \times \mathscr{G}) \\
&\subseteq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^\# \times \mathscr{G}) \\
&= \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \Big(\mathscr{E}^\# \times \Big(\bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})} \mathscr{B}\Big)\Big) \\
&\subseteq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})} (\mathscr{E}^\# \times \mathscr{B}) \\
&\subseteq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})} (\mathscr{E} \times \mathscr{B})^\# \\
&\subseteq \bigcap_{\mathscr{D}\in\mathrm{Seq}(\mathscr{F}\times\mathscr{G})} \mathscr{D}^\# \\
&= \mathscr{F} \times \mathscr{G} \text{ (by (3.13))}.
\end{aligned}$$

□

Remark 3.4.9. In [100] Jordan and Mynard showed that if $\mathscr{F}$ is not a strongly Fréchet filter (or not a productively Fréchet filter), then there exists a countably based filter $\mathscr{G}$ (a strongly Fréchet filter $\mathscr{G}$, respectively) such that $\mathscr{F} \times \mathscr{G}$ is not a Fréchet filter. On the other hand, by Proposition 3.4.3, $\bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^-,\mathscr{G}^-)$ is a Fréchet filter. Thus

$$(\mathscr{F}^-,\mathscr{G}^-) \neq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^-,\mathscr{G}^-).$$

Proposition 3.4.10. *Let* $\mathscr{E} \approx \{x_n\}_n$*.*

(i) *If* $\mathscr{G}$ *is strongly Fréchet filter, then*

$$(\mathscr{E}^-,\mathscr{G}^-) = \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n, \quad (\mathscr{E}^+,\mathscr{G}^+) = \bigcup_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\#.$$

(ii) *If* $\mathscr{G}$ *is countably based filter, then*

$$(\mathscr{E}^+,\mathscr{G}^-)=\bigcap_{\{y_n\}_n\geq\mathscr{G}}\{(x_n,y_n)\}_n^{\#},\quad (\mathscr{E}^-,\mathscr{G}^+)=\bigcup_{\{y_n\}_n\geq\mathscr{G}}\{(x_n,y_n)\}_n.$$

Proof. (i) We prove only $(\mathscr{E}^-,\mathscr{G}^-)=\bigcap_{\{y_n\}_n\in\mathrm{Seq}(\mathscr{G})}\{(x_n,y_n)\}_n$ since this formula implies the other one by properties of grills. It follows from Proposition 3.4.3 that $\mathscr{E}\times\mathscr{G}$ is Fréchet filter. We can see that

$$\begin{aligned}
(\mathscr{E}^-,\mathscr{G}^-) &\approx \mathscr{E}\times\mathscr{G}\\
&= \mathscr{E}\times(\bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})}\mathscr{B}) \text{ (by (3.13))}\\
&\subseteq \bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})}(\mathscr{E}\times\mathscr{B})\\
&= \bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})}((\bigcap_{\mathscr{D}\in\mathrm{Seq}(\mathscr{E})}\mathscr{D})\times\mathscr{B}) \text{ (by (3.13))}\\
&\subseteq \bigcap_{\mathscr{B}\in\mathrm{Seq}(\mathscr{G})}\bigcap_{\mathscr{D}\in\mathrm{Seq}(\mathscr{E})}(\mathscr{D}\times\mathscr{B})\\
&\subseteq \bigcap_{\mathscr{L}\in\mathrm{Seq}(\mathscr{E}\times\mathscr{G})}\mathscr{L}\\
&= \mathscr{E}\times\mathscr{G} \text{ (by (3.13))}.
\end{aligned}$$

Because $\mathscr{E}\approx\{x_n\}_n$, we get

$$(\mathscr{E}^-,\mathscr{G}^-)=\bigcap_{\{y_n\}_n\geq\mathscr{G}}\{(x_n,y_n)\}_n.$$

(ii) We prove only $(\mathscr{E}^-,\mathscr{G}^+)=\bigcup_{\{y_n\}_n\geq\mathscr{G}}\{(x_n,y_n)\}_n$. Let $\mathscr{G}\approx\{G_n\}_n$, where $\{G_n\}_n$ is a decreasing sequence of subsets, and $A\in(\mathscr{E}^-,\mathscr{G}^+)\approx(\mathscr{E}^{\#}\times\mathscr{G})^{\#}$. In other words, for each n and $H\in\mathscr{E}^{\#}$,

$$(H\times G_n)\cap A\neq\emptyset\Longleftrightarrow AG_n\cap H\neq\emptyset, \tag{3.20}$$

that is, $AG_n\in\mathscr{E}^{\#\#}=\mathscr{E}$ for each n. This means that for every n there is k_n such that $\{x_k:k\geq k_n\}\subseteq AG_n$, hence there exists $\{y_k^n:k\geq k_n\}\subseteq G_n$ with

$$\{(x_k,y_k^n):k\geq k_n\}\subseteq A.$$

Using induction, we can get a strictly increasing sequence $\{k_n\}_n$ with this property. Let

$$y_k:=y_k^n \quad\text{if}\quad k_n\leq k<k_{n+1}.$$

Then $\{y_k\}_k\geq\mathscr{G}$ and $\{(x_k,y_k)\}\subseteq A$ for each $k\geq k_1$.

Conversely, let $A \in \bigcup_{\{y_n\}_n \geq \mathscr{G}} \{(x_n, y_n)\}_n$, then there is $D \approx \{y_k\}_k \geq \mathscr{G}$ such that $(x_k, y_k) \subseteq A$. We can check that (3.20) holds, then $A \in (\mathscr{E}^-, \mathscr{G}^+)$.

□

Theorem 3.4.11. *Let* $f : \mathbb{N} \times X \to \overline{\mathbb{R}}$.

(i) *If* $\mathscr{G}$ *is a strongly Fréchet filter on* X*, then*

$$\Gamma(\mathscr{N}^-, \mathscr{G}^-) = \Gamma_{\mathrm{seq}}(\mathscr{N}^-, \mathscr{G}^-), \quad \Gamma(\mathscr{N}^+, \mathscr{G}^+) = \Gamma_{\mathrm{seq}}(\mathscr{N}^+, \mathscr{G}^+).$$

(ii) *If* $\mathscr{G}$ *is a countably based filter on* X*, then*

$$\Gamma(\mathscr{N}^-, \mathscr{G}^+) = \Gamma_{\mathrm{seq}}(\mathscr{N}^-, \mathscr{G}^+), \quad \Gamma(\mathscr{N}^+, \mathscr{G}^-) = \Gamma_{\mathrm{seq}}(\mathscr{N}^+, \mathscr{G}^-).$$

Proof. We check only for $\Gamma(\mathscr{N}^+, \mathscr{G}^+) = \Gamma_{\mathrm{seq}}(\mathscr{N}^+, \mathscr{G}^+)$. The other cases are analogous. Let $\mathscr{E} = \mathscr{N}$, from (3.4.10), we get

$$(\mathscr{N}^+, \mathscr{G}^+) = \bigcup_{\{y_n\}_n \in \mathrm{Seq}(\mathscr{G})} \{(n, y_n)\}_n^\#.$$

It follows from (3.6) and (3.9) that for all $f : \mathbb{N} \times X \to \overline{\mathbb{R}}$,

$$\begin{aligned} \Gamma(\mathscr{N}^+, \mathscr{G}^+) \lim f &= \operatorname{liminf}_{(\mathscr{N}^+, \mathscr{G}^+)} f \\ &= \sup_{\{y_n\} \in \mathrm{Seq}(\mathscr{G})} \operatorname{liminf}_{\{(n, y_n)\}_n^\#} f \\ &= \sup_{\{y_n\} \in \mathrm{Seq}(\mathscr{G})} \limsup_{n \to +\infty} f(n, x_n) \\ &= \Gamma_{\mathrm{seq}}(\mathscr{N}^+, \mathscr{G}^+) \lim f. \end{aligned}$$

□

The following example shows that the strongly Fréchetness of $\mathscr{G}$ in Theorem 3.4.11(i) is necessary.

Example 3.4.12. Let a sequential fan $S_{\mathbb{N}} := \{\bigcup_{n \in \mathbb{N}} X_n\} \cup \{x_\infty\}$, where $X_n := \{x_{n,k} : k \in \mathbb{N}\}$ and $X_n \cap X_m = \emptyset$ for all $n \neq m$, be equipped with a topology defined as follows

- each point $x_{n,k}$ is isolated;
- a basic open neighborhood of x_∞ in the form

$$O_f(x_\infty) := \{x_\infty\} \cup \{x_{n,k} : k \geq f(n)\},$$

for each function $f \in \mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}} := \{f : \mathbb{N} \to \mathbb{N}\}$.

First, we prove that $S_{\mathbb{N}}$ is not strongly Fréchet, i.e., there exists $\{A_n\}$ be a decreasing sequence of subsets in $S_{\mathbb{N}}$ such that $x_\infty \in \bigcap_n \mathrm{cl}(A_n)$, but there is no $x_n \in A_n$ with $x_\infty = \lim\limits_{n\to\infty} x_n$.

Setting $A_n := \{X_m : m \geq n\}$. It is easy to check that $A_{n+1} \subseteq A_n$ and $x_\infty \in \bigcap_n \mathrm{cl}(A_n)$. For each n, choose any finite set F_n of A_n and denote $F := \bigcup_n F_n$. We define a function $h \in \mathbb{N}^{\mathbb{N}}$ by

$$h(n) := \begin{cases} 1 + \max\{k : x_{n,k} \in F\}, & \text{if } F \cap X_n \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Then, $O_h(x_\infty) \cap F = \emptyset$, i.e., $x_\infty \notin \mathrm{cl}\, F$. Since F_n is arbitrary for all n, there is no $x_n \in A_n$ with $x_\infty = \lim\limits_{n\to+\infty} x_n$.

Let $\mathfrak{g} : \mathbb{N} \times S_{\mathbb{N}} \to [0,1]$ be defined by

$$\mathfrak{g}(n,x) = \begin{cases} \dfrac{1}{\sqrt[n]{m}}, & \text{if } x \in X_m, \\ 1, & \text{otherwise.} \end{cases}$$

By calculating, we get

$$\Gamma(\mathcal{N}^-, \mathcal{N}_\tau(x_\infty)^-) \lim \mathfrak{g} = 0.$$

Let $\{x^m\}$ be a sequence converging to x_∞. If $x^m = x_\infty$ for infinitely many $m \in \mathbb{N}$, we get $\lim\limits_{m\to+\infty} \mathfrak{g}(m, x^m) = 1$. We now consider $x^m \neq x_\infty$ for all $m \in \mathbb{N}$. Then, there exists $n_0 \in \mathbb{N}$ such that $\{x^m\} \subseteq \bigcup_{n=1}^{n_0} X_n$. Indeed, if not, for all $n \in \mathbb{N}$, there exists m_n such that $x^{m_n} \notin \bigcup_{l=1}^{n} X_l$. This implies that $x^{m_n} \in A_{n+1}$, where $\{A_n\}$ is the decreasing sequence of subsets defined above. By the previous analysis, $\{x^{m_n}\}_n$ does not converge to x_∞, which is a contradiction. Thus, each sequence $\{x^m\}$ ($\neq \{x_\infty\}$) converging to x_∞ is in type of $\{x_{p,k} : \exists n_0 \in \mathbb{N}, p \in [1, n_0], k \to +\infty\}$. By calculating, we get

$$\liminf_{m\to+\infty} \mathfrak{g}(m, x^m) = \liminf_{k\to+\infty} \mathfrak{g}(k, x_{p,k}) = 1.$$

This implies that

$$\Gamma_{\mathrm{seq}}(\mathcal{N}^-, \mathcal{N}_\tau(x_\infty)^-) \lim \mathfrak{g} = 1.$$

Thus

$$\Gamma(\mathcal{N}^-, \mathcal{N}_\tau(x_\infty)^-) \lim \mathfrak{g} < \Gamma_{\mathrm{seq}}(\mathcal{N}^-, \mathcal{N}_\tau(x_\infty)^-) \lim \mathfrak{g}.$$

3.4.2 Three variables

Proposition 3.4.13. *Let $\mathscr{F}, \mathscr{G}$ be filters.*

(i) *If* $\mathscr{F}$ *is strongly Fréchet and* $\mathscr{G}$ *is productively Fréchet (or vice versa), then*

$$(\mathscr{F}^-,\mathscr{G}^-) = \bigcap_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n = \bigcap_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\#,$$

(ii) *If* $\mathscr{F}$ *is Fréchet and* $\mathscr{G}$ *is countably based, then*

$$(\mathscr{F}^+,\mathscr{G}^-) = \bigcup_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n = \bigcup_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\#.$$

Proof. (i) It follows from Proposition 3.4.3 that $\mathscr{F}\times\mathscr{G}$ is a Fréchet filter. We can see that

$$\begin{aligned}
(\mathscr{F}^-,\mathscr{G}^-) &= \bigcap_{\{x_n\}_n \geq \mathscr{F}} (\{x_n\}_n^-,\mathscr{G}^-) \text{ (by Proposition 3.4.8)}\\
&= \bigcap_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n \text{ (by Proposition 3.4.10)}\\
&\subseteq \bigcap_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\#\\
&\subseteq \bigcap_{\mathscr{L}\in \mathrm{Seq}(\mathscr{F}\times\mathscr{G})} \mathscr{L}^\#\\
&= \mathscr{F}\times\mathscr{G} \text{ (by (3.13))}.
\end{aligned}$$

(ii) We can see that

$$\begin{aligned}
(\mathscr{F}^+,\mathscr{G}^-) &= \bigcup_{\{x_n\}_n \geq \mathscr{F}} (\{x_n\}_n^-,\mathscr{G}^-) \text{ (by Proposition 3.4.8)}\\
&= \bigcup_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n \text{ (by Proposition 3.4.10)}.
\end{aligned}$$

Besides,

$$\begin{aligned}
(\mathscr{F}^+,\mathscr{G}^-) &= \bigcup_{\{x_n\}_n \geq \mathscr{F}} (\{x_n\}_n^+,\mathscr{G}^-) \text{ (by Proposition 3.4.8)}\\
&= \bigcup_{\{x_n\}_n \geq \mathscr{F}} \bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\# \text{ (by Proposition 3.4.10)}.
\end{aligned}$$

□

Proposition 3.4.14. *Let* $\mathscr{E}\approx\{x_n\}_n$*, and* $\mathscr{G},\mathscr{H}$ *are filters.*

(i) *Suppose that* $\mathscr{G}$ *is strongly Fréchet and* $\mathscr{H}$ *is productively Fréchet (or vice versa). Then*

$$(\mathscr{E}^-,\mathscr{G}^-,\mathscr{H}^-) = \bigcap_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n.$$

(ii) *Suppose that* $\mathscr{G}$ *is countably based and* $\mathscr{H}$ *is strongly Fréchet. Then*

$$(\mathscr{E}^-,\mathscr{G}^+,\mathscr{H}^-) = \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n.$$

(iii) *Suppose that* $\mathscr{G},\mathscr{H}$ *are countably based. Then*

$$(\mathscr{E}^+,\mathscr{G}^-,\mathscr{H}^-) = \bigcap_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n^\#.$$

(iv) *Suppose that* $\mathscr{G}$ *is strongly Fréchet and* $\mathscr{H}$ *is countably based. Then*

$$(\mathscr{E}^+,\mathscr{G}^+,\mathscr{H}^-) = \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n^\#.$$

Proof. We only prove (iii). The other ones are analogous. It follows from Proposition 3.4.3 that $\mathscr{E}\times\mathscr{G}\times\mathscr{H}$ is a Fréchet filter. We have

$$\begin{aligned}
(\mathscr{E}^+,\mathscr{G}^-,\mathscr{H}^-) &\approx \mathscr{E}^\# \times \mathscr{G} \times \mathscr{H} \\
&= \Big(\bigcap_{\{y_n\}_n \geq \mathscr{G}} \{(x_n,y_n)\}_n^\#\Big) \times \mathscr{H} \text{ (by Proposition 3.4.10)} \\
&\subseteq \bigcap_{\{y_n\}_n \geq \mathscr{G}} (\{(x_n,y_n)\}_n^\# \times \mathscr{H}) \\
&= \bigcap_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n^\# \text{ (by Proposition 3.4.10)} \\
&\subseteq \bigcap_{\mathscr{B} \in \mathrm{Seq}(\mathscr{E}\times\mathscr{G}\times\mathscr{H})} \mathscr{B}^\# \\
&= \mathscr{E}\times\mathscr{G}\times\mathscr{H} \text{ (by (3.13))} \\
&\subseteq \mathscr{E}^\# \times \mathscr{G} \times \mathscr{H}.
\end{aligned}$$

□

Corollary 3.4.15. *Let* $\mathscr{E} \approx \{x_n\}_n$, *and* $\mathscr{G},\mathscr{H}$ *are filters.*

(i) *Suppose that* $\mathscr{G}$ *is strongly Fréchet and* $\mathscr{H}$ *is productively Fréchet (or vice versa). Then*

$$(\mathscr{E}^+,\mathscr{G}^+,\mathscr{H}^+) = \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcup_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n^\#.$$

(ii) *Suppose that* $\mathscr{G}$ *is countably based and* $\mathscr{H}$ *is strongly Fréchet. Then*

$$(\mathscr{E}^+,\mathscr{G}^-,\mathscr{H}^+) = \bigcap_{\{y_n\}_n \geq \mathscr{G}} \bigcup_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n^\#.$$

(iii) *Suppose that* $\mathscr{G},\mathscr{H}$ *are countably based. Then*

$$(\mathscr{E}^-,\mathscr{G}^+,\mathscr{H}^+) = \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcup_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n.$$

(iv) *Suppose that* $\mathscr{G}$ *is strongly Fréchet and* $\mathscr{H}$ *is countably based. Then*

$$(\mathscr{E}^-,\mathscr{G}^-,\mathscr{H}^+) = \bigcap_{\{y_n\}_n \geq \mathscr{G}} \bigcup_{\{z_n\}_n \geq \mathscr{H}} \{(x_n,y_n,z_n)\}_n.$$

Proof. It follows from Proposition 3.4.14. □

Theorem 3.4.16. *Let $\mathscr{G},\mathscr{H}$ be filters on X,Y, respectively, and $f:\mathbb{N}\times X\times Y\to\overline{\mathbb{R}}$.*

(i) *Suppose that $\mathscr{G}$ is strongly Fréchet and $\mathscr{H}$ is productively Fréchet (or vice versa). Then*

$$\Gamma(\mathscr{N}^-,\mathscr{G}^-,\mathscr{H}^-)=\Gamma_{\mathrm{seq}}(\mathscr{N}^-,\mathscr{G}^-,\mathscr{H}^-),$$

$$\Gamma(\mathscr{N}^+,\mathscr{G}^+,\mathscr{H}^+)=\Gamma_{\mathrm{seq}}(\mathscr{N}^+,\mathscr{G}^+,\mathscr{H}^+).$$

(ii) *Suppose that $\mathscr{G}$ is countably based and $\mathscr{H}$ is strongly Fréchet. Then*

$$\Gamma(\mathscr{N}^-,\mathscr{G}^+,\mathscr{H}^-)=\Gamma_{\mathrm{seq}}(\mathscr{N}^-,\mathscr{G}^+,\mathscr{H}^-),$$

$$\Gamma(\mathscr{N}^+,\mathscr{G}^-,\mathscr{H}^+)=\Gamma_{\mathrm{seq}}(\mathscr{N}^+,\mathscr{G}^-,\mathscr{H}^+).$$

(iii) *Suppose that $\mathscr{G},\mathscr{H}$ are countably based. Then*

$$\Gamma(\mathscr{N}^+,\mathscr{G}^-,\mathscr{H}^-)=\Gamma_{\mathrm{seq}}(\mathscr{N}^+,\mathscr{G}^-,\mathscr{H}^-),$$

$$\Gamma(\mathscr{N}^-,\mathscr{G}^+,\mathscr{H}^+)=\Gamma_{\mathrm{seq}}(\mathscr{N}^-,\mathscr{G}^+,\mathscr{H}^+).$$

(iv) *Suppose that $\mathscr{G}$ is strongly Fréchet and $\mathscr{H}$ is countably based. Then*

$$\Gamma(\mathscr{N}^+,\mathscr{G}^+,\mathscr{H}^-)=\Gamma_{\mathrm{seq}}(\mathscr{N}^+,\mathscr{G}^+,\mathscr{H}^-).$$

$$\Gamma(\mathscr{N}^-,\mathscr{G}^-,\mathscr{H}^+)=\Gamma_{\mathrm{seq}}(\mathscr{N}^-,\mathscr{G}^-,\mathscr{H}^+).$$

Proof. Based on Proposition 3.4.14 and Corollary 3.4.15, the proof is similar to that of Theorem 3.4.11. □

3.4.3 More than three variables

Let T and T_i, $i\in I$, be $\overline{\mathbb{R}}$-limitoids in S, and $\mathbf{st}(T)=\bigcap_{i\in I}\mathbf{st}(T_i)$. It follows from (3.6) and (3.8) that for all $f:S\to\overline{\mathbb{R}}$,

$$T(f)=\inf\nolimits_{i\in I}T_i(f). \tag{3.21}$$

The following lemma give us a condition for which "inf" in (3.21) can be attained.

Lemma 3.4.17. *The following properties are equivalent*

(i) *for each $f:S\to\overline{\mathbb{R}}$,*

$$T(f)=\min\nolimits_{i\in I}T_i(f),$$

(ii) *for each countably based filter $\mathscr{F}$,*

$$\left(\mathbf{st}(T)^-,\mathscr{F}^-\right)=\bigcap\nolimits_{i\in I}\left(\mathbf{st}(T_i)^-,\mathscr{F}^-\right).$$

Proof. Suppose that (ii) holds. By setting $\mathscr{F} := \{S\}$, we get $\mathbf{st}(T) = \bigcap_{i\in I} \mathbf{st}(T_i)$. This implies

$$T(f) = \inf_{i\in I} T_i(f).$$

Let $\mathscr{H} \approx \{\{r : r < r_n\} : n \in \mathbb{N}\}$, where $\{r_n\}_n$ is a strictly decreasing sequence converging to $T(f)$ such that $T(f) < r_n$ for all n. It follows from (3.6) that

$$T(f) = \liminf_{\mathbf{st}(T)} f = \sup_{A\in\mathbf{st}(T)} \inf_{x\in A} f < r_n,$$

that is, $\{x \in S : f(x) < r_n\} \in \mathbf{st}(T)^{\#}$ for all n, that is, $(f^{-1}(\mathscr{H}))\#(\mathbf{st}(T))$ or else $\mathrm{gr}(f) \in (\mathbf{st}(T) \times \mathscr{H})^{\#}$. By (ii), there is $i \in I$ such that $\mathrm{gr}(f) \in (\mathbf{st}(T_i) \times \mathscr{H})^{\#}$, equivalently for all n, $T_i(f) = \liminf_{\mathbf{st}(T_i)} f < r_n$. Then,

$$T_i(f) \leq \lim_{n\to+\infty} r_n = T(f),$$

which implies (i).

Conversely, suppose that (i) holds and $\mathscr{F} \approx \{F_n : n \in \mathbb{N}\}$, where $\{F_n\}_n$ is a decreasing sequence of subsets. We prove that

$$\left(\mathbf{st}(T)^{-}, \mathscr{F}^{-}\right)^{\#} = \bigcup_{i\in I} \left(\mathbf{st}(T_i)^{-}, \mathscr{F}^{-}\right)^{\#}.$$

Let $H \in (\mathbf{st}(T)^{-}, \mathscr{F}^{-})^{\#} \approx (\mathbf{st}(T) \times \mathscr{F})^{\#}$. By Remark 3.4.4(ii), $H \in (\mathbf{st}(T) \times \mathscr{F})^{\#}$ if and only if $\mathbf{st}(T)\#(H^{-1}\mathscr{F})$, that is, $H^{-1}F_n \in (\mathbf{st}(T))^{\#}$ for all n. Let

$$f_H(x) := \begin{cases} \inf\left\{\frac{1}{n} : x \in H^{-1}F_n\right\}, & \text{if } \left\{\frac{1}{n} : x \in H^{-}F_n\right\} \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Then if $n > \frac{1}{r}$, that is, $r > \frac{1}{n}$, and $x \in H^{-1}F_n$, then $f_H(x) < r$, so that $H^{-1}F_n \subseteq \{x : f_H(x) < r\}$. It follows from

$$T(f_H) = \liminf_{\mathbf{st}(T)} f = \sup_{A\in\mathbf{st}(T)} \inf_{x\in A} f = \inf_{B\in(\mathbf{st}(T))^{\#}} \sup_{x\in B} f$$

that $T(f_H) < r$ for each $r > 0$. By (i), there exists $i \in I$ such that $T_i(f_H) = 0$, i.e., for each $r > 0$,

$$T_i(f_H) = \liminf_{\mathbf{st}(T_i)} f_H = \sup_{A\in\mathbf{st}(T_i)} \inf_{x\in A} f_H < r,$$

equivalently, $\{x : f_H(x) < r\} \in \mathbf{st}(T_i)^{\#}$ for all $r > 0$. Therefore, if $\frac{1}{n} < r < \frac{1}{n-1}$, then

$$H^{-}F_n \subseteq \{x : f_H(x) < r\} \subseteq H^{-}F_{n-1},$$

so that $H^{-}F_{n-1} \in \mathbf{st}(T_i)^{\#}$ for each $n > 1$, that is $H \in (\mathbf{st}(T_i) \times \mathscr{F})^{\#}$.

$\square$

Proposition 3.4.18. *Suppose that $\mathscr{F}$ is a strongly Fréchet filter, $\mathscr{G},\mathscr{H}$ are countably based filters. Then*

$$(\mathscr{F}^-,\mathscr{G}^+,\mathscr{H}^-) = \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^-,\mathscr{G}^+,\mathscr{H}^-). \tag{3.22}$$

However, there exist countably based filters $\mathscr{F},\mathscr{G},\mathscr{H}$ such that

$$(\mathscr{F}^-,\mathscr{G}^+,\mathscr{H}^-) \neq \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^+,\mathscr{G}^+,\mathscr{H}^-).$$

Proof. It follows from Lemma 3.4.17 that (3.22) will be proved if for each extended-real-valued function f,

$$\Gamma(\mathscr{F}^-,\mathscr{G}^+)\lim f = \min_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \Gamma(\mathscr{E}^-,\mathscr{G}^+)\lim f. \tag{3.23}$$

It follows from (3.17) that $(\mathscr{F}^-,\mathscr{G}^+) = \bigcap_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} (\mathscr{E}^-,\mathscr{G}^+)$. This implies

$$\Gamma(\mathscr{F}^-,\mathscr{G}^+)\lim f = \inf_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \Gamma(\mathscr{E}^-,\mathscr{G}^+)\lim f.$$

Setting $b := \inf_{\mathscr{E}\in\mathrm{Seq}(\mathscr{F})} \Gamma(\mathscr{E}^-,\mathscr{G}^+)\lim f$. Let $\{\mathscr{E}_n\}_n$ be a sequence of sequential filters such that $\mathscr{E}_n \geq \mathscr{F}$ for all n and $b = \inf_n \Gamma(\mathscr{E}_n^-,\mathscr{G}^+)\lim f$. Let $\{r_n\}_n$ be a strictly decreasing sequence converging to b and $\Gamma(\mathscr{E}_n^-,\mathscr{G}^+)\lim f < r_n$ for all n. By the definition of Γ-limits, for all n,

$$\inf_{G\in\mathscr{G}} \Gamma(\mathscr{E}_n^-) \sup_{y\in G} f(x,y) < r_n.$$

It means that for every n, there exists $G_n \in \mathscr{G}$ such that

$$\Gamma(\mathscr{E}_n^-) \sup_{y\in G_n} f(x,y) < r_n,$$

that is,

$$\left\{x : \sup_{y\in G_n} f(x,y) < r_n\right\} \in \mathscr{E}_n^{\#}.$$

Hence

$$\left\{x : \sup_{y\in G_n} f(x,y) < r_n\right\}_n \subseteq \mathscr{F}^{\#}.$$

Since $\mathscr{F}$ is a strongly Fréchet filter, there is a sequential $\mathscr{E}_0 \approx \{x_n\}_n \geq \mathscr{F}$ such that

$$\sup_{y\in G_n} f(x_n,y) < r_n.$$

This implies

$$\inf_{G_n \in \mathscr{G}} \Gamma(\mathscr{E}_0{}^-) \sup_{y \in G_n} f(x,y) < r_n.$$

Thus, $\Gamma(\mathscr{E}_0{}^-, \mathscr{G}^+) \lim f \leq b = \bigcap_{\mathscr{E} \in \mathrm{Seq}(\mathscr{F})} \Gamma(\mathscr{E}^-, \mathscr{G}^+) \lim f$, which implies (3.23).

For the second part of (iii), we consider the following example. □

Example 3.4.19. ([83]) Let $\mathscr{F} = \mathscr{G} = \mathscr{H} = \mathscr{N}(0)$, where $\mathscr{N}(0)$ is a filter of neighborhoods of 0 on $S = [0,1]$, and $g : [0,1] \times [0,1] \to \overline{\mathbb{R}}$ be defined by

$$g(x,y) := \begin{cases} 2^{-m}, & \text{if } \exists\, n,m \in \mathbb{N} \text{ such that } x = 2^{-n}(1-2^{-m}),\ 0 \leq y \leq 2^{-m}, \\ 1, & \text{otherwise.} \end{cases}$$

Setting $\mathscr{E}_m := \{2^{-n}(1-2^{-m})\}_n$. It is evident that $\Gamma(\mathscr{E}_m^+, \mathscr{N}(0)^+) \lim g = 2^{-m}$. By virtue of (3.8) and Proposition 3.4.18(ii), we get

$$0 = \inf_{\mathscr{E} \geq \mathscr{N}(0)} \Gamma(\mathscr{E}^+, \mathscr{N}(0)^+) \lim g = \Gamma(\mathscr{N}(0)^-, \mathscr{N}(0)^+) \lim g.$$

Recall that $\mathrm{hypo}(g) \in (\mathscr{N}(0)^-, \mathscr{N}(0)^+, \mathscr{N}(0)^-)$ if and only if $\Gamma(\mathscr{N}(0)^-, \mathscr{N}(0)^+) \lim g > 0$, see Lemma 3.4 in [83]. This implies $\mathrm{hypo}(g) \notin (\mathscr{N}(0)^-, \mathscr{N}(0)^+, \mathscr{N}(0)^-)$. Let $\mathscr{E}$ be an sequential filter associated with a sequence $\{x_n\} \subseteq [0,1]$ converging to 0. If there exists m such that $\{x_n\}_n \cap \{2^{-n}(1-2^{-m})\}_n$ is infinite, we have $\Gamma(\mathscr{E}^+, \mathscr{N}(0)^+) \lim g \geq 2^{-m}$; otherwise, $\Gamma(\mathscr{E}^+, \mathscr{N}(0)^+) \lim g = 1$. Then, $\mathrm{hypo}(g) \in (\mathscr{E}^+, \mathscr{N}(0)^+, \mathscr{N}(0)^-)$ for each sequential filter $\mathscr{E}$ finer than $\mathscr{N}(0)$. In conclusion, we demonstrate that $\mathrm{hypo}(g) \notin (\mathscr{N}(0)^-, \mathscr{N}(0)^+, \mathscr{N}(0)^-)$ and $\mathrm{hypo}(g) \in \bigcap_{\mathscr{E} \geq \mathscr{N}(0)} (\mathscr{E}^+, \mathscr{N}(0)^+, \mathscr{N}(0)^-)$.

Proposition 3.4.20. *Suppose that $\mathscr{F}$ is a strongly Fréchet filter, and $\mathscr{G}, \mathscr{H}$ are countably based filters. Then*

$$(\mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) = \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(u_n, y_n, z_n)\}_n.$$

However, there exist countably based filters $\mathscr{F}, \mathscr{G}, \mathscr{H}$ such that

$$(\mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \neq \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(u_n, y_n, z_n)\}_n^{\#}.$$

Proof. It follows from Propositions 3.4.14(ii) and 3.4.18. □

Proposition 3.4.21. *Let* $\mathscr{E} \approx \{x_n\}_n$*,* $\mathscr{F}$ *be a strongly Fréchet filter,* $\mathscr{G}, \mathscr{H}$ *be countably based filters. Then*

$$(\mathscr{E}^-, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) = \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n, u_n, y_n, z_n)\}_n.$$

However, there exist countably based filters $\mathscr{F}, \mathscr{G}, \mathscr{H}$ *such that*

$$(\mathscr{E}^+, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \neq \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(x_n, u_n, y_n, z_n)\}_n^{\#}.$$

Proof. It is implied from Propositions 3.4.10 and 3.4.20. □

The results above define the possibility to express filters. They are mainstay for De Giorgi's Γ-limits, for example, if $\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{C}$ are countably based filters then the following extensions of previous results are not true in general

$$(\mathscr{C}^+, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) = \bigcup_{\{w_n\}_n \geq \mathscr{C}} \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(w_n, u_n, y_n, z_n)\}_n$$

and

$$(\mathscr{C}^+, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) = \bigcup_{\{w_n\}_n \geq \mathscr{C}} \bigcap_{\{u_n\}_n \geq \mathscr{F}} \bigcup_{\{y_n\}_n \geq \mathscr{G}} \bigcap_{\{z_n\}_n \geq \mathscr{H}} \{(w_n, u_n, y_n, z_n)\}_n^{\#}.$$

Theorem 3.4.22. *Let* $\mathscr{F}$ *be a strongly Fréchet filter on* X*, and* $\mathscr{G}, \mathscr{H}$ *be countably based filters on* Y, Z*, respectively. Then, for every* $f : \mathbb{N} \times X \times Y \times Z \to \overline{\mathbb{R}}$*,*

$$\Gamma(\mathscr{N}^-, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \lim f = \Gamma_{\text{seq}}(\mathscr{N}^-, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \lim f.$$

However, there exist countably based filters $\mathscr{F}, \mathscr{G}, \mathscr{H}$ *and an extended-real-valued function* f *such that*

$$\Gamma(\mathscr{N}^+, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \lim f \neq \Gamma_{\text{seq}}(\mathscr{N}^+, \mathscr{F}^-, \mathscr{G}^+, \mathscr{H}^-) \lim f.$$

Proof. It follows from Theorem 3.3.5, Proposition 3.4.21 with $\mathscr{E} = \mathscr{N}$. □

Corollary 3.4.23. *Let* $\mathscr{F}$ *be a strongly Fréchet filter on* X*, and* $\mathscr{G}, \mathscr{H}$ *be countably based filters on* Y, Z*, respectively. Then, for every* $f : \mathbb{N} \times X \times Y \times Z \to \overline{\mathbb{R}}$*,*

$$\Gamma(\mathscr{N}^+, \mathscr{F}^+, \mathscr{G}^-, \mathscr{H}^+) \lim f = \Gamma_{\text{seq}}(\mathscr{N}^+, \mathscr{F}^+, \mathscr{G}^-, \mathscr{H}^+) \lim f.$$

However, there exist countably based filters $\mathscr{F}, \mathscr{G}, \mathscr{H}$ *and an extended-real-valued function* f *such that*

$$\Gamma(\mathscr{N}^-, \mathscr{F}^+, \mathscr{G}^-, \mathscr{H}^+) \lim f \neq \Gamma_{\text{seq}}(\mathscr{N}^-, \mathscr{F}^+, \mathscr{G}^-, \mathscr{H}^+) \lim f.$$

Proof. It follows from Theorem 3.4.22. $\square$

From the theorem and corollary above, we see that for $k \geq 3$, regardless of the sign α_0 there are countably based filters $\mathscr{F}_1, ..., \mathscr{F}_k$ and $\alpha_1, ..., \alpha_k$ such that

$$\Gamma(\mathscr{N}^{\alpha_0}, \mathscr{F}_1^{\alpha_1}, \mathscr{F}_2^{\alpha_2}, ..., \mathscr{F}_k^{\alpha_k})\lim \neq \Gamma_{\text{seq}}(\mathscr{N}^{\alpha_0}, \mathscr{F}_1^{\alpha_1}, \mathscr{F}_2^{\alpha_2}, ..., \mathscr{F}_k^{\alpha_k})\lim.$$

3.5 Applications

In this section, applications of Γ-limits to generalized derivatives and tangency are given.

3.5.1 Generalized derivatives

It is generally admitted that definition of differentiability of functions in Euclidean spaces was introduced in [69] by Fréchet in 1911. However, the definition of derivative at a point x of a real-valued function defined on a subset of Euclidean space was already given in [135] by Peano in 1887 and generalized in 1908 in [138] to function valued in Euclidean space.

Let A be a subset in $\mathbb{R}^m$ and x be an accumulation point of A. A function $f : A \to \mathbb{R}^n$ is said to be differentiable at x if there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - L(y-x)}{||y-x||} = 0. \tag{3.24}$$

It should be stressed that Peano's definition appeared in a rigorous modern form (3.24), that is used nowadays in contrast to the standard language of mathematical definition in that epoch, was usually informal and often vague. Even if, in giving this definition, Peano referred to the concepts of Grassmann in [81] and of Jacobi in [91], those however were more rudimentary. As A is not the whole of Euclidean space, in general the linear operator L in (3.24) is not unique; if it is, it is called the derivative of f at x and is denoted by $Df(x)$.

$Df(x)$ is called nowadays the Fréchet derivative of f at x, although Fréchet gave its informal (geometric) definition only in [69] in 1911. Fréchet was apparently unaware of Peano's definition, because one month later, he published in [70] another note, acknowledging contributions of some authors, such as Stolz, Pierpoint and Young, but not that of Peano.

In 1892, Peano introduced the strict differentiability of f at x, see [136], that is, if (3.24) is strengthened to

$$\lim_{y,u \to x, y \neq u} \frac{f(y) - f(u) - L(y-u)}{||y-u||} = 0.$$

He also noticed that strict differentiability amounts to continuous differentiability. This definition is frequently referred to Leach in [114], where it is called strong differentiability, and to Bourbaki in [21].

Not only recently, a prominent role that nonsmooth analysis plays in connection with optimization theory is widely recognized, especially since the latter has natural mechanisms that generate nonsmoothness: duality theory, sensitivity and stability analysis, decomposition techniques, etc. Therefore, theories of generalized differentiability have been started. One of its important applications is the topic of optimality conditions for nonsmooth and nonconvex problems. References [19, 129, 130, 147] are recent books that contain systematic expositions and references on generalized differentiation and their applications to optimization-related problems, including optimality conditions. Also, [89, 141] are also detailed treatments on the issues.

Although a whole spectrum of denitions of differentiability can be given in analytical and/or geometrical ways, we can observe that using of kinds of directional derivatives is often a first step for a differential construction; see e.g. [89, 97, 147] for often-met notions of directional derivatives. Therefore, applying directional derivatives is a simple way to deal with optimization-related problems in general and optimality conditions in particular, see [71, 73–76, 157].

In this subsection, by using Γ-limits, we introduce a unified notation of derivatives. Let $\mathscr{N}_+(0) := \mathscr{N}(0) \cap (0, +\infty)$ be a filter on $(0, +\infty)$, τ be a topology on X, and $f : X \to \overline{\mathbb{R}}$.

Let ϑ_f be the supremum of τ and of the coarsest topology in X for which f is continuous. An unified notation of derivatives of f at x_0 is defined by

$$D(\mathscr{N}_+(0)^{\alpha_1}; \vartheta_f^{\alpha_2}, \tau^{\alpha_3}) f(x_0)(h) := \left(\Gamma(\mathscr{N}_+(0)^{\alpha_1}; \vartheta_f^{\alpha_2}, \tau^{\alpha_3}) \lim \frac{f(x+tu) - f(x)}{t} \right)(x_0, h), \tag{3.25}$$

with $\alpha_1 \in \{+, -\}$, $\alpha_2, \alpha_3 \in \{+, -, *\}$ where $\alpha_2 = *$ (or $\alpha_3 = *$) means that x (u, respectively) is fixed and equal to x_0 (h, respectively).

The formula (3.25) can be abbreviated as follows

$$D^{(\alpha_1, \alpha_2, \alpha_3)} f(x_0)(h) = \Gamma\left((t \to 0^+)^{\alpha_1}, (x \to x_0)^{\alpha_2}, (u \to h)^{\alpha_3}\right) \lim \frac{f(x+tu) - f(x)}{t}.$$

By using (3.25), we get the following definition.

Definition 3.5.1. Let $f : (X, \tau) \to \overline{\mathbb{R}}$ and $x_0 \in X$.

(i) The *upper directional derivative* of f at x_0 in direction $h \in X$ is

$$D^{(+,*,*)} f(x_0)(h) = \Gamma((t \to 0^+)^+) \lim \frac{f(x_0 + th) - f(x_0)}{t}.$$

(ii) The *lower directional derivative* of f at x_0 in direction $h \in X$ is

$$D^{(-,*,*)} f(x_0)(h) = \Gamma\left((t \to 0^+)^-\right) \lim \frac{f(x_0 + th) - f(x_0)}{t}.$$

(iii) The *upper tangent derivative* of f at x_0 in direction $h \in X$ is

$$D^{(+,*,+)} f(x_0)(h) = \Gamma\left((t \to 0^+)^+, (u \to h)^+\right) \lim \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(iv) The *lower tangent derivative* of f at x_0 in direction $h \in X$ is

$$D^{(-,*,-)} f(x_0)(h) = \Gamma\left((t \to 0^+)^-, (u \to h)^-\right) \lim \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(v) The *upper paratangent derivative* of f at x_0 in direction $h \in X$ is

$$D^{(+,+,+)} f(x_0)(h) = \Gamma\left((t \to 0^+)^+, (x \to x_0)^+, (u \to h)^+\right) \lim \frac{f(x + tu) - f(x)}{t}.$$

(vi) The *lower paratangent derivative* of f at x_0 in direction $h \in X$ is

$$D^{(-,-,-)} f(x_0)(h) = \Gamma\left((t \to 0^+)^-, (x \to x_0)^-, (u \to h)^-\right) \lim \frac{f(x + tu) - f(x)}{t}.$$

In short, the results above can be expressed as follows

Unifying terminology	**Notation**	**Traditional terminology**
upper directional derivative	$D^{(+,*,*)} f(x_0)(h)$	upper Dini derivative
lower directional derivative	$D^{(-,*,*)} f(x_0)(h)$	lower Dini derivative
upper tangent derivative	$D^{(+,*,+)} f(x_0)(h)$	upper Hadamard derivative
lower tangent derivative	$D^{(-,*,-)} f(x_0)(h)$	lower Hadamard derivative
upper paratangent derivative	$D^{(+,+,+)} f(x_0)(h)$	paratangent derivative
lower paratangent derivative	$D^{(-,-,-)} f(x_0)(h)$	Clarke derivative

3.5.2 Tangent cones

In this subsection, we recall some well-known tangent cones in optimization theory and express them in terms of Γ-limits.

Recently, various types of tangent cones have been studied in the literature. Their definitions depend on variants of the limiting process. The most known contribution to the investigation of these concepts is due to Bouligand in 1932, see [20]. One can find a mention about other contributors in papers of Guareschi [85, 86], Saks [148], Severi [151, 152], Federer [63] and Whitney [170]. We can say that, for tangent cones, main references are Bouligand in optimization theory, Ferderer in geometric measure theory and calculus of variations, and Whitney in differential geometry.

However, by [54], we discover that tangent cones were already known by Peano at the end of 19th century. Indeed, in 1887, Peano gave in [135] a metric definition of tangent straight line and tangent plane, then reaches, in a natural way, a unifying notion as follows

$$\operatorname{tang}(A,x) := x + \operatorname*{Liminf}_{\lambda\to+\infty} \lambda(A-x).$$

Later, in 1908, he introduced in [138] another types of tangent cone, namely

$$\operatorname{Tang}(A,x) := x + \operatorname*{Limsup}_{\lambda\to+\infty} \lambda(A-x).$$

To distinguish two above notions, we shall call the first lower tangent cone and the second upper tangent cone. As usual, after abstract investigation of a notion, Peano considered significant special cases and calculated the upper tangent cone in several basic figures (closed ball, curves and surfaces parametrized in a regular way).

Let S be a subset of X. The *homothety* of S, see [50, 139, 145], is the set-valued map from $(0,+\infty)\times X$ into X defined by

$$\mathscr{H}_S(t,x) := \frac{1}{t}(S-x). \tag{3.26}$$

$\mathscr{H}_S$ can be considered as a relation in $(0,+\infty)\times X\times X$. If x_0 is fixed, (3.26) is called the homothety of S at x_0 and is denoted by $\mathscr{H}_{S,x_0}$.

Let τ,θ be topologies on X. Based on the above homothety, we now give a unified notation of cones as follows

$$v\in T_S(\mathscr{N}_+(0)^{\alpha_1};\theta_S^{\alpha_2},\tau^{\alpha_3})(x_0) \Longleftrightarrow \left(\Gamma(\mathscr{N}_+(0)^{\alpha_1};\theta_S^{\alpha_2},\tau^{\alpha_3})\lim\chi_{(\mathscr{H}_S)}\right)(x_0,v)=1, \tag{3.27}$$

with θ_S be a topology induced on S by θ, $\alpha_1,\alpha_3\in\{+,-\}$, $\alpha_2\in\{+,-,*\}$, where $\alpha_2=*$ means that x is fixed and equal to x_0.

The formula (3.27) can be abbreviated as follows

$$v\in T_S^{(\alpha_1,\alpha_2,\alpha_3)}(x_0)\Longleftrightarrow\Gamma\left((t\to0^+)^{\alpha_1},(x\to x_0)^{\alpha_2},(u\to v)^{\alpha_3}\right)\lim\chi_{(\mathscr{H}_S)}=1.$$

By (3.27), we introduce the following definition.

Definition 3.5.2. Let $S \subseteq X$ and $x_0 \in \operatorname{cl} S$.

(i) The *upper tangent cone* of S at x_0 is defined by

$$v \in T_S^{(+,*,+)}(x_0) \Longleftrightarrow \Gamma((t \to 0^+)^+, (u \to v)^+) \lim \chi_{(\mathscr{H}_{S,x_0})} = 1.$$

(ii) The *lower tangent cone* of S at x_0 is defined by

$$v \in T_S^{(-,*,+)}(x_0) \Longleftrightarrow \Gamma((t \to 0^+)^-, (u \to v)^+) \lim \chi_{(\mathscr{H}_{S,x_0})} = 1.$$

(iii) The *upper paratangent cone* of S at x_0 is defined by

$$v \in T_S^{(+,+,+)}(x_0) \Longleftrightarrow \Gamma((t \to 0^+)^+, (x \to x_0)^+, (u \to v)^+) \lim \chi_{(\mathscr{H}_S)} = 1.$$

(iv) The *lower paratangent cone* of S at x_0 is defined by

$$v \in T_S^{(-,-,+)}(x_0) \Longleftrightarrow \Gamma((t \to 0^+)^-, (x \to x_0)^-, (u \to v)^+) \lim \chi_{(\mathscr{H}_S)} = 1.$$

Proposition 3.5.3. (i) *$v \in T_S^{(+,*,+)}(x_0)$ if and only if for every $Q \in \mathscr{N}_\tau(v)$, $t > 0$, there are $t' \leq t$ and $v' \in Q$ such that $x_0 + t'v' \in S$.*

(ii) *$v \in T_S^{(-,*,+)}(x_0)$ if and only if for every $Q \in \mathscr{N}_\tau(v)$ there is $t > 0$ such that for all $t' \leq t$ there is $v' \in Q$ satisfying $x_0 + t'v' \in S$.*

(iii) *$v \in T_S^{(+,+,+)}(x_0)$ if and only if for every $Q \in \mathscr{N}_\tau(v)$, $W \in \mathscr{N}_{\theta_S}(x_0)$, $t > 0$ there are $v' \in Q$, $x' \in W$, $t' > 0$ such that $x' + t'v' \in S$.*

(iv) *$v \in T_S^{(-,-,+)}(x_0)$ if and only if for every $Q \in \mathscr{N}_\tau(v)$ there are $W \in \mathscr{N}_{\theta_S}(x_0)$ and $t > 0$ such that for all $t' \leq t$, $x' \in W$ there is $v' \in Q$ satisfying $x' + t'v' \in S$.*

Proof. By the similarity, we prove only (iv). It follows from (3.27) that $v \in T_S^{(-,-,+)}(x_0)$ if and only if $\left(\Gamma(\mathscr{N}_+(0)^-; \theta_S^-, \tau^+) \lim \chi_{(\mathscr{H}_S)}\right)(x_0, v) = 1$, i.e.,

$$\inf_{Q \in \mathscr{N}_\tau(v)} \sup_{W \in \mathscr{N}_{\theta_S}(x_0)} \sup_{t > 0} \inf_{t' \leq t} \inf_{x' \in W} \sup_{v' \in Q} \chi_{(\mathscr{H}_S)}(t', x', v') = 1.$$

It means that for every $Q \in \mathscr{N}_\tau(v)$ there are $W \in \mathscr{N}_{\theta_S}(x_0)$ and $t > 0$ such that for all $t' \leq t$, $x' \in W$ there is $v' \in Q$ satisfying $x' + t'v' \in S$. $\square$

Remark 3.5.4. When X is a normed space. We get the sequential form of these tangent cones as follows

(i) $T_S^{(+,*,+)}(x_0) = \{v \in X : \exists t_n \to 0^+, \exists v_n \to v, x_0 + t_n v_n \in S\}$.

(ii) $T_S^{(-,*,+)}(x_0) = \{v \in X : \forall t_n \to 0^+, \exists v_n \to v, x_0 + t_n v_n \in S\}$.

(iii) $T_S^{(+,+,+)}(x_0) = \{v \in X : \exists t_n \to 0^+, \exists x_n \in S : x_n \to x_0, \exists v_n \to v, x_n + t_n v_n \in S\}$.

(iv) $T_S^{(-,-,+)}(x_0) = \{v \in X : \forall t_n \to 0^+, \forall x_n \in S : x_n \to x_0, \exists v_n \to v, x_n + t_n v_n \in S\}$.

These cones, also called the contingent cone, the adjacent cone, the paratangent cone, and the Clarke cone, respectively, were thoroughly studied in [11–13, 17, 50, 60, 61, 77–79, 161] in detail.

These results above give us a unified way to denote cones as follows

Unifying terminology	**Notation**	**Traditional terminology**
upper paratangent cone	$T_S^{(+,+,+)}(x_0)$	paratangent cone
upper tangent cone	$T_S^{(+,*,+)}(x_0)$	contingent cone
lower tangent cone	$T_S^{(-,*,+)}(x_0)$	adjacent cone
lower paratangent cone	$T_S^{(-,-,+)}(x_0)$	Clarke cone

The tangent cones above play an important role in the study of various mathematical problems, including optimization, viability theory, and control theory. Moreover, once one has a concept of tangent cone, one can construct a corresponding derivative of a set-valued map. Some of them are presented in the table below as examples, see book [11] of Aubin and Frankowska. Let $F : X \to 2^Y$ and $(x_0, y_0) \in \text{gr}F$.

Traditional terminology	**Unifying notation**	**Definition**
paratangent derivative	$DF^{(+,+,+)}(x_0,y_0)$	$\text{gr}(DF^{(+,+,+)}(x_0,y_0)) = T_{\text{gr}F}^{(+,+,+)}(x_0,y_0)$
contingent derivative	$DF^{(+,*,+)}(x_0,y_0)$	$\text{gr}(DF^{(+,*,+)}(x_0,y_0)) = T_{\text{gr}F}^{(+,*,+)}(x_0,y_0)$
adjacent derivative	$DF^{(-,*,+)}(x_0,y_0)$	$\text{gr}(DF^{(-,*,+)}(x_0,y_0)) = T_{\text{gr}F}^{(-,*,+)}(x_0,y_0)$
circatangent derivative	$DF^{(-,-,+)}(x_0,y_0)$	$\text{gr}(DF^{(-,-,+)}(x_0,y_0)) = T_{\text{gr}F}^{(-,-,+)}(x_0,y_0)$

Inspired by them, many kinds of generalized derivatives have been defined and applied to optimization. Some of them, e.g., variational sets, radial sets, radial derivatives, and Studniarski derivatives, will be introduced in Chapters 4, 5, and 6. They can also be expressed in terms of Γ-limits.

Chapter 4

Variational sets and applications to sensitivity analysis for vector optimization problems

4.1 Introduction

First-order derivatives (of various types, classical or generalized) of a map are used to approximate a given map to simplify a problem under consideration. To have better approximations, higher-order derivatives are applied. For generalized derivatives and their applications in variational analysis, see books [19] of Bonnans and Shapiro, [129, 130] of Mordukhovich, [147] of Rockafellar and Wets, and long papers [89] of Ioffe, and [141] of Penot. Examining the existing optimality conditions, we can observe that the key argument is included in a separation of suitable sets. To explain the idea, let us take the well-known scheme of Dubovitskii-Milyutin in [58] for first-order optimality conditions in single-valued scalar optimization problems : the intersection of the cone of decrease directions of the objective function and the cone of feasible directions defined by constraints must be empty at a local minimizer. Here, the cone of decrease directions is defined by a kind of derivatives. For other theories of optimality conditions, especially of higher-order conditions, we may have separations of sets, not cones. An important point for a necessary optimality condition of this type is that the larger the separated sets, the stronger the result. This was a motivation for Khanh and Tuan in [105, 106] to propose variational sets replacing derivatives so that they are bigger than sets defined by known derivatives and can be used in the mentioned separation. Some advantages of this generalized differentiability were shown by Khanh et al. in [7, 105, 106], e.g., almost no assumptions are required for variational sets to exist; extentions to higher orders are direct; they are bigger than corresponding sets of most derivatives, which implies advantageous results in establishing necessary optimality conditions by separation techniques, etc.

In this chapter, we will present our results, published in [3], on variational sets in sensitivity analysis. We study properties of perturbation maps, in terms of higher-order variational sets. Regarding solutions of vector optimization, we restrict ourselves to basic notions of (Pareto) efficient points and weak efficient points. Correspondingly, our concern is to deal with perturbation maps and weak perturbation maps. We employ variational sets in both assumptions and conclusions. We also show cases where our results can be employed but some existing results cannot. Examples are provided to ensure the essentialness of each imposed assumption.

4.2 Variational sets of set-valued maps

In this section, we introduce the concept of variational sets of set-valued maps and establish some results on the relationship between variational sets of F and its profile map.

4.2.1 Definitions

Let X and Y be normed spaces, C be a pointed closed convex cone in Y. To approximate multivalued map $F : X \to 2^Y$ at $(x_0, y_0) \in \mathrm{gr}F$, we recall two types of higher-order variational sets as follows.

Definition 4.2.1. ([105, 106]) Let $v_1, ..., v_{m-1} \in Y$.

(i) The first, second, and higher-order *variational sets of type 1* are the following

$$V^1(F,x_0,y_0) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t}(F(x) - y_0),$$

$$V^2(F,x_0,y_0,v_1) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t^2}(F(x) - y_0 - tv_1),$$

$$V^m(F,x_0,y_0,v_1,\cdots,v_{m-1}) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0,\, t \to 0^+} \frac{1}{t^m}(F(x) - y_0 - tv_1 - \cdots - t^{m-1}v_{m-1}),$$

where $x \xrightarrow{F} x_0$ means that $x \in \mathrm{dom}\, F$ and $x \to x_0$.

(ii) The first, second, and higher-order *variational sets of type 2* are the following

$$W^1(F,x_0,y_0) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0\ t \to 0^+} \mathrm{cone}_+(F(x) - y_0),$$

$$W^2(F,x_0,y_0,v_1) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0\ t\to 0^+} \frac{1}{t}(\mathrm{cone}_+(F(x)-y_0)-v_1),$$

$$W^m(F,x_0,y_0,v_1,\cdots,v_{m-1}) := \operatorname*{Limsup}_{x \xrightarrow{F} x_0\ t\to 0^+} \frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x)-y_0)-v_1-\cdots-t^{m-2}v_{m-1}).$$

(iii) If the upper limits in (i) are equal to the lower ones, then these limits are called the first, second, and higher-order *proto-variational sets* of type 1 of F at (x_0,y_0). Similar terminology is defined for type 2.

By using equivalent formulations for the upper limit of a set-valued map (see [11]), i.e.,

$$\operatorname*{Limsup}_{x \xrightarrow{F} x_0} F(x) = \{y \in Y : \exists x_n \in \operatorname{dom} F : x_n \to x_0, \exists y_n \in F(x_n), y_n \to y\},$$

we easily obtain the following formulae of two types of variational sets.

Proposition 4.2.2. ([7]) (Equivalent formulations of V^m) $V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$ *is equal to all of the following sets*

(i) $\{y \in Y : \liminf_{x \xrightarrow{F} x_0,\, t\to 0^+} \frac{1}{t^m} d(y_0+tv_1+\ldots+t^{m-1}v_{m-1}+t^m y, F(x)) = 0\}$,

(ii) $\{y \in Y : \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists r(t_n^m) = 0(t_n^m), \forall n, y_0+t_nv_1+\ldots+t_n^{m-1}v_{m-1}+t_n^m y+r(t_n^m) \in F(x_n)\}$,

(iii) $\{y \in Y : \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists v_n \to y, \forall n, y_0+t_nv_1+\ldots+t_n^{m-1}v_{m-1}+t_n^m v_n \in F(x_n)\}$,

(iv) $\{y \in Y : \exists t_n \to 0^+, \exists x_n \xrightarrow{F} x_0, \exists y_n \in F(x_n), \lim_{n\to\infty} \frac{1}{t_n^m}(y_n-y_0-t_nv_1-\ldots-t_n^{m-1}v_{m-1}) = y\}$,

(v) $\bigcap_{\varepsilon>0} \bigcap_{\substack{\alpha>0\\ \beta>0}} \bigcup_{\substack{0<t\le\alpha\\ \|x-x_0\|\le\beta}} (\frac{1}{t^m}(F(x)-y_0-tv_1-\ldots-t^{m-1}v_{m-1})+\varepsilon B_Y)$,

(vi) $\bigcap_{\substack{\alpha>0\\ \beta>0}} \mathrm{cl} \bigcup_{\substack{0<t\le\alpha\\ \|x-x_0\|\le\beta}} \frac{1}{t^m}(F(x)-y_0-tv_1-\ldots-t^{m-1}v_{m-1})$.

Proposition 4.2.3. ([7]) (Equivalent formulations of W^m) $W^m(F,x_0,y_0,v_1,...,v_{m-1})$ *has the following equivalent expressions*

(i) $\{y \in Y : \liminf_{x \xrightarrow{F} x_0,\, t\to 0^+} \frac{1}{t^{m-1}} d(v_1+\ldots+t^{m-2}v_{m-1}+t^{m-1}y, \mathrm{cone}_+(F(x)-y_0)) = 0\}$,

(ii) $\{y\in Y:\exists t_n\to 0^+,\exists x_n\xrightarrow{F}x_0,\exists r(t_n^{m-1})=0(t_n^{m-1}),\forall n,v_1+...+t^{m-2}v_{m-1}+t_n^{m-1}y+r(t_n^{m-1})\in \text{cone}_+(F(x_n)-y_0)\}$,

(iii) $\{y\in Y:\exists t_n\to 0^+,\exists x_n\xrightarrow{F}x_0,\exists v_n\to y,\forall n,v_1+...+t^{m-2}v_{m-1}+t_n^{m-1}v_n\in \text{cone}_+(F(x_n)-y_0)\}$,

(iv) $\{y\in Y:\exists t_n\to 0^+,\exists x_n\xrightarrow{F}x_0,\exists y_n\in\text{cone}_+(F(x_n)-y_0),\lim\limits_{n\to\infty}\dfrac{1}{t_n^{m-1}}(y_n-v_1-...-t_n^{m-2}v_{m-1})=y\}$,

(v) $\displaystyle\bigcap_{\varepsilon>0}\bigcap_{\substack{\alpha>0\\ \beta>0}}\bigcup_{\substack{0<t\le\alpha\\ \|x-x_0\|\le\beta}}(\frac{1}{t^{m-1}}(\text{cone}_+(F(x)-y_0)-v_1-...-t^{m-2}v_{m-1})+\varepsilon B_Y)$,

(vi) $\displaystyle\bigcap_{\substack{\alpha>0\\ \beta>0}}\text{cl}\bigcup_{\substack{0<t\le\alpha\\ \|x-x_0\|\le\beta}}\frac{1}{t^{m-1}}(\text{cone}_+(F(x)-y_0)-v_1-...-t^{m-2}v_{m-1})$.

Remark 4.2.4. For all $m\geq 1$, we have

(i) $V^m(F,x_0,y_0,v1,...,v_{m-1})\subseteq W^m(F,x_0,y_0,v_1,...,v_{m-1})$.

(ii) $V^m(F,x_0,y_0,0,...,0)=V^1(F,x_0,y_0)$, $W^m(F,x_0,y_0,0,...,0)=W^1(F,x_0,y_0)$.

(iii) If $v_1\notin V^1(F,x_0,y_0)$ then $V^2(F,x_0,y_0,v_1)=\emptyset$. If one of the conditions $v_1\in V^1(F,x_0,y_0)$, ..., $v_{m-1}\in V^{m-1}(F,x_0,y_0,v_1,...,v_{m-2})$ is violated, then $V^m(F,x_0,y_0,v_1,...,v_{m-1})=\emptyset$. The variational sets of type 2 have the same property.

(iv) Variational sets can be expressed in terms of Γ-limits as follows

$$\begin{aligned}
&y\in V^m(F,x_0,y_0,v_1,...,v_{m-1})\\
\iff\ &\inf_{Q\in\mathscr{N}(y)}\ \inf_{W\in\mathscr{N}(x_0,\text{dom}F)}\ \inf_{t>0}\ \sup_{0<t'<t}\ \sup_{x'\in W}\ \sup_{y'\in Q}\ \chi_{\text{gr}(\mathscr{L}_{F,y_0,v_1,...,v_{m-1}})}(t',x',y')=1\\
\iff\ &\Gamma(\mathscr{N}_+(0)^+,\mathscr{N}(x_0,\text{dom}F)^+,\mathscr{N}(y)^+)\lim\chi_{\text{gr}(\mathscr{L}_{F,y_0,v_1,...,v_{m-1}})}=1,
\end{aligned}$$

where $\mathscr{N}(x_0,\text{dom}F):=\mathscr{N}(x_0)\cap\text{dom}F$, and $\mathscr{L}_{F,y_0,v_1,...,v_{m-1}}:(0,+\infty)\times X\to 2^Y$ is defined by

$$\mathscr{L}_{F,y_0,v_1,...,v_{m-1}}(t',x'):=\frac{1}{t'^m}(F(x')-y_0-t'v_1-...-t'^{m-1}v_{m-1}).$$

$$\begin{aligned}
&y\in W^m(F,x_0,y_0,v_1,...,v_{m-1})\\
\iff\ &\inf_{Q\in\mathscr{N}(y)}\ \inf_{W\in\mathscr{N}(x_0,\text{dom}F)}\ \inf_{t>0}\ \sup_{0<t'<t}\ \sup_{x'\in W}\ \sup_{y'\in Q}\ \chi_{\text{gr}(\mathscr{H}_{F,y_0,v_1,...,v_{m-1}})}(t',x',y')=1\\
\iff\ &\Gamma(\mathscr{N}_+(0)^+,\mathscr{N}(x_0,\text{dom}F)^+,\mathscr{N}(y)^+)\lim\chi_{\text{gr}(\mathscr{H}_{F,y_0,v_1,...,v_{m-1}})}=1,
\end{aligned}$$

where $\mathscr{H}_{F,y_0,v_1,...,v_{m-1}} : (0,+\infty)\times X \to 2^Y$ is defined by

$$\mathscr{H}_{F,y_0,v_1,...,v_{m-1}}(t',x') := \frac{1}{t'^m}(\mathrm{cone}_+(\mathrm{F(x')-y_0)-t'v_1-...-t'^{m-1}v_{m-1}}).$$

The inclusion in Remark 4.2.4(i) may be a strict inclusion or an equality as shown by the following examples.

Example 4.2.5. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$ and, for $n=1,2,...$,

$$F(x)=\begin{cases} \{(0,0)\}, & \text{if } x=0,\\ \{(-n,n)\}, & \text{if } x=\frac{1}{n},\\ \left\{\left(\frac{1}{n},0\right)\right\}, & \text{if } x=\ln\left(1+\frac{1}{n}\right),\\ \left\{\left(1,\frac{1}{n^2}\right)\right\}, & \text{if } x=\sin\left(\frac{1}{n}\right),\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then, for $(x_0,y_0)=(0,(0,0))\in \mathrm{gr}F$ and $v_1=(1,0)\in Y$, one has

$$V^1(F,x_0,y_0)=\{(y_1,0)\in Y: y_1\geq 0\},$$

$$V^2(F,x_0,y_0,v_1)=\{(y_1,0)\in Y: y_1\in\mathbb{R}\},$$

$$W^1(F,x_0,y_0)=\{(y_1,0)\in Y: y_1\geq 0\}\cup\{(-y_1,y_1)\in Y: y_1\geq 0\},$$

$$W^2(F,x_0,y_0,v_1)=\{(y_1,y_2)\in Y: y_2\geq 0\}.$$

Example 4.2.6. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$ and, for $n=1,2,...$,

$$F(x)=\begin{cases} \{(0,0)\}, & \text{if } x=0,\\ \{(-n,n)\}, & \text{if } x=\frac{1}{n},\\ \left\{\left(\frac{1}{n},0\right)\right\}, & \text{if } x=\ln\left(1+\frac{1}{n}\right),\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then, for $(x_0,y_0)=(0,(0,0))\in \mathrm{gr}F$ and $v_1=(1,0)\in Y$, one has

$$V^1(F,x_0,y_0)=\{(y_1,0)\in Y: y_1\geq 0\},$$

$$W^1(F,x_0,y_0)=\{(y_1,0)\in Y: y_1\geq 0\}\cup\{(-y_1,y_1)\in Y: y_1\geq 0\},$$

$$V^2(F,x_0,y_0,v_1)=W^2(F,x_0,y_0,v_1)=\{(y_1,0)\in Y: y_1\in\mathbb{R}\}.$$

Remark 4.2.7. Recall that the *higher-order contingent derivative* of F at (x_0, y_0) (relative to (u_1, v_1), $\cdots$, (u_{m-1}, v_{m-1})) is the map $D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}) : X \to 2^Y$ (see [11]) defined by

$$\begin{aligned} D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(u) \quad := \quad & \operatorname*{Limsup}_{u' \to u, t \to 0^+} \frac{1}{t^m}(F(x_0 + tu_1 + \cdots + t^{m-1}u_{m-1} + t^m u') - \\ & -y_0 - tv_1 - \cdots - t^{m-1}v_{m-1}). \end{aligned}$$

We can say roughly that the contingent derivative is a directional variant of variational set V^m. Similarly, most of generalized derivatives (e.g., the (upper) Dini derivative, Hadamard derivative, adjacent derivative, etc) are also based on directional rates, while for the variational sets we allow the flexibility $x_n \xrightarrow{F} x_0$. That is why these sets are big

$$D^m F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})X \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}).$$

More comparisons between variational sets with well-known derivatives were stated in Proposition 4.1 in [105] by Khanh and Tuan.

4.2.2 Relationships between variational sets of F and those of its profile map

The first simple result about a relation between variational sets of the two maps F and its profile map is as follows.

Proposition 4.2.8. (i) $V^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + C \subseteq V^m(F + C, x_0, y_0, v_1, \cdots, v_{m-1})$,
(ii) $W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + C \subseteq W^m(F + C, x_0, y_0, v_1, \cdots, v_{m-1})$.

Proof. By the similarity, we present only a proof for (ii). Let $y \in W^m(F, x_0, y_0, v_1, \cdots, v_{m-1}) + C$, i.e., there exist $v \in W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$ and $c \in C$ such that $y = v + c$. Then, there are $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$ and $v_n \to v$ such that

$$h_n(v_1 + \cdots + t_n^{m-2}v_{m-1} + t_n^{m-1}(v_n + c)) \in F(x_n) + C - y_0.$$

So, $v + c \in W^m(F + C, x_0, y_0, v_1, \cdots, v_{m-1})$. $\square$

The inclusions opposite to those in Proposition 4.2.8 may not hold as the following example shows.

Example 4.2.9. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $(x_0, y_0) = (0, (0,0))$, and

$$F(x) = \begin{cases} \{(0,0)\}, & \text{if } x = 0, \\ \{(-1,-1)\}, & \text{if } x \neq 0. \end{cases}$$

Then, we have $V^1(F,x_0,y_0) = \{(0,0)\}$ and $V^1(F+C,x_0,y_0) = \mathbb{R}^2$. Thus, $V^1(F+C,x_0,y_0) \not\subseteq V^1(F,x_0,y_0)+C$. Let $v_1 = (0,1) \in V^1(F+C,x_0,y_0)$. Then, $V^2(F+C,x_0,y_0,v_1) \neq \emptyset$ and $V^2(F,x_0,y_0,v_1) = \emptyset$. Consequently,

$$V^2(F+C,x_0,y_0,v_1) \not\subseteq V^2(F,x_0,y_0,v_1)+C.$$

For variational sets of type 2, one has $W^1(F,x_0,y_0)+C = \mathbb{R}^2 = W^1(F+C,x_0,y_0)$ and $v_1 \in W^1(F+C,x_0,y_0)$. But, $W^2(F+C,x_0,y_0,v_1) = \mathbb{R}^2$ and $W^2(F,x_0,y_0,v_1) = \emptyset$. Hence, $W^2(F+C,x_0,y_0,v_1) \not\subseteq W^2(F,x_0,y_0,v_1)+C$.

Proposition 4.2.10. *Suppose C have a compact base. Then*

(i) $\mathrm{Min}_{C\setminus\{0\}} V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1}) \subseteq V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$,

(ii) $\mathrm{Min}_{C\setminus\{0\}} W^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1}) \subseteq W^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$.

Proof. We prove only (i). Let $v \in \mathrm{Min}_{C\setminus\{0\}} V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})$. Then, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$, $y_n \in F(x_n)$, and $c_n \in C$ such that, for all n, $\overline{v_n} := t_n^{-m}(y_n + c_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}) \to v$. Then,

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \overline{v_n} - c_n \in F(x_n). \tag{4.1}$$

We claim that $c_n/t_n^m \to 0$ (for a subsequence). For a compact base Q of C, one has $c_n = \alpha_n b_n$ for some $\alpha_n \geq 0$, $b_n \in Q$. If $\alpha_n = 0$ for infinitely many $n \in \mathbb{N}$, we are done. Hence, let $\alpha_n > 0$ for all n, we may assume that $b_n \to b \in Q$. Then, $c_n/t_n^m = \alpha_n b_n/t_n^m \to 0$ if and only if $\alpha_n/t_n^m \to 0^+$. Suppose that α_n/t_n^m does not converge to 0. Then, nothing is lost by assuming that $\alpha_n/t_n^m \geq \varepsilon$ for some $\varepsilon > 0$. Let $\overline{c_n} := (\varepsilon t_n^m/\alpha_n)c_n$. Then, $\overline{c_n} - c_n \in -C$ and

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \overline{v_n} - \overline{c_n} \in F(x_n) + C.$$

Since $\overline{c_n}/t_n^m \to \varepsilon b \neq 0$, one has $\overline{v_n} - t_n^{-m}\overline{c_n} \to v - \varepsilon b$, and hence $v - \varepsilon b \in V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})$. Thus,

$$-\varepsilon b \in (V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1}) - v) \cap (-C\setminus\{0\}),$$

contradicting the efficiency of v. Therefore, $c_n/t_n^m \to 0$. It follows from (4.1) that

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m \left(\overline{v_n} - \frac{c_n}{t_n^m} \right) \in F(x_n),$$

and $\overline{v_n} - t_n^{-m} c_n \to v$. So, $v \in V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$. □

For weak efficiency, we do not have a similar result, as indicated by the following example.

Example 4.2.11. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $(x_0, y_0) = (0, (0,0))$, and

$$F(x) = \begin{cases} \{(0,0)\}, & \text{if } x = 0, \\ \{(0,-1)\}, & \text{if } x = \dfrac{1}{n}, \\ \{(\dfrac{1}{n}, \dfrac{-1}{n})\}, & \text{if } x = \sin\dfrac{1}{n} \text{ for } n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, we have

$$V^1(F, x_0, y_0) = \{(x,y) \in Y : y = -x, x \geq 0\},$$

$$W^1(F, x_0, y_0) = \{(0,y) \in Y : y \leq 0\} \cup \{(x,y) \in Y : y = -x, x \geq 0\},$$

$$V^1(F + C, x_0, y_0) = W^1(F + C, x_0, y_0) = \mathbb{R}_+ \times \mathbb{R}.$$

Consequently, $\mathrm{Min}_{\mathrm{int}C} V^1(F + C, x_0, y_0) = \mathrm{Min}_{\mathrm{int}C} W^1(F + C, x_0, y_0) = \{0\} \times \mathbb{R}$. Therefore,

$$\mathrm{Min}_{\mathrm{int}C} V^1(F + C, x_0, y_0) \not\subseteq V^1(F, x_0, y_0),\ \mathrm{Min}_{\mathrm{int}C} W^1(F + C, x_0, y_0) \not\subseteq W^1(F, x_0, y_0).$$

If $\mathrm{int}C \neq \emptyset$, for weak efficiency, we have the following analogous properties.

Proposition 4.2.12. *Suppose $\widehat{C} \subseteq \mathrm{int}C \cup \{0\}$ be a closed convex cone with a compact base. Then*

(i) $\mathrm{Min}_{\mathrm{int}C} V^m(F + \widehat{C}, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq V^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$,

(ii) $\mathrm{Min}_{\mathrm{int}C} W^m(F + \widehat{C}, x_0, y_0, v_1, \cdots, v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, \cdots, v_{m-1})$.

Proof. We prove only (ii). Since $\widehat{C} \subseteq \mathrm{int}C \cup \{0\}$, any $v \in \mathrm{Min}_{\mathrm{int}C} W^m(F + \widehat{C}, x_0, y_0, v_1, \cdots, v_{m-1})$ satisfies

$$v \in W^m(F + \widehat{C}, x_0, y_0, v_1, \cdots, v_{m-1}) \cap \mathrm{Min}_{\widehat{C}\setminus\{0\}} W^m(F + \widehat{C}, x_0, y_0, v_1, \cdots, v_{m-1}), \tag{4.2}$$

where $\mathrm{Min}_{\widehat{C}\setminus\{0\}} W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})$ is the set of efficient points of $W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})$ with respect to the cone $\widehat{C}$. Hence, there exist $t_n \to 0^+$, $x_n \xrightarrow{F} x_0$, $v_n \to v$, $c_n \in \widehat{C}$, and $h_n > 0$ such that, for all n,

$$\frac{v_1+\cdots+t_n^{m-2}v_{m-1}+t_n^{m-1}v_n}{h_n} - c_n \in F(x_n)-y_0. \tag{4.3}$$

For a compact base $\widehat{Q}$ of $\widehat{C}$, there exist $\alpha_n \geq 0$ and $q_n \in \widehat{Q}$ such that $c_n = \alpha_n q_n$. We may assume that $q_n \to q \in \widehat{Q}$. We claim that $h_n\alpha_n/t_n^{m-1} \to 0^+$ (for a subsequence). This is true if $\alpha_n = 0$ for infinitely many $n \in \mathbb{N}$. Now, suppose to the contrary that $\alpha_n > 0$, and $h_n\alpha_n/t_n^{m-1}$ does not converge to 0. Then, we may assume that $h_n\alpha_n/t_n^{m-1} \geq \varepsilon$ for some $\varepsilon > 0$. Let $\overline{c_n} := (\varepsilon t_n^{m-1}/h_n\alpha_n)c_n \in \widehat{C}$. Then, we have $c_n - \overline{c_n} \in \widehat{C}$. By (4.3), we obtain

$$\frac{v_1+\cdots+t_n^{m-2}v_{m-1}+t_n^{m-1}v_n}{h_n} - \overline{c_n} \in F(x_n)+\widehat{C}-y_0.$$

As $h_n\overline{c_n}/t_n^{m-1} \to \varepsilon q \neq 0$, this implies that $v-\varepsilon q \in W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})$. Therefore,

$$-\varepsilon q \in (W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})-v)\cap(-\widehat{C}\setminus\{0\}),$$

which contradicts (4.2). Hence, $h_n\alpha_n/t_n^{m-1} \to 0^+$ and $v_n - t_n^{-(m-1)}h_nc_n \to v$. It follows from (4.3) that $v \in W^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$. □

To get the equalities in Proposition 4.2.8, we need the following new notions.

Definition 4.2.13. Let $(x_0,y_0) \in \mathrm{gr}F$, $v_1,\cdots,v_{m-1} \in Y$, and $m \in \mathbb{N}$. The m-th order *singular variational set* of type 1 (type 2, respectively) of F at (x_0,y_0) is defined by

$$\begin{aligned} V^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1}) &:= \{y \in Y : \exists x_n \xrightarrow{F} x_0, \exists t_n \to 0^+, \exists \lambda_n \to 0^+,\\ &\qquad \exists y_n \in \frac{F(x_n)-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1}}{t_n^m}, \lambda_n y_n \to y\} \end{aligned}$$

$$\begin{aligned} (W^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1}) &:= \{y \in Y : \exists x_n \xrightarrow{F} x_0, \exists t_n \to 0^+, \exists \lambda_n \to 0^+,\\ &\qquad \exists y_n \in \frac{\mathrm{cone}_+(F(x_n)-y_0)-v_1-\cdots-t_n^{m-2}v_{m-1}}{t_n^{m-1}}, \lambda_n y_n \to y\}). \end{aligned}$$

Definition 4.2.14. Let $A \subseteq Y$.

(i) A is said to have the *domination property* if and only if $A \subseteq \mathrm{Min}_{C\setminus\{0\}}A + C$.

(ii) When $\mathrm{int}C \neq \emptyset$, we say that A has the *weak domination property* with respect to $\widehat{C}$ if and only if $A \subseteq \mathrm{Min}_{\mathrm{int}C}A+\widehat{C}$, where $\widehat{C} \subseteq \mathrm{int}C\cup\{0\}$ is a closed convex cone.

Proposition 4.2.15. *Let C have a compact base.*

(i) *Let either of the following conditions hold:*

(i$_1$) $V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})$ *has the domination property,*

(i$_2$) $V^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1})\cap(-C)=\{0\}$.

Then

$$V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})=V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})+C, \tag{4.4}$$

$$\mathrm{Min}_{C\setminus\{0\}}V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})=\mathrm{Min}_{C\setminus\{0\}}V^m(F,x_0,y_0,v_1,\cdots,v_{m-1}). \tag{4.5}$$

(ii) *Let either of the following two conditions hold:*

(ii$_1$) $W^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})$ *has the domination property,*

(ii$_2$) $W^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1})\cap(-C)=\{0\}$.

Then

$$W^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})=W^m(F,x_0,y_0,v_1,\cdots,v_{m-1})+C,$$

$$\mathrm{Min}_{C\setminus\{0\}}W^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})=\mathrm{Min}_{C\setminus\{0\}}W^m(F,x_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. We prove only (i). First, we check (4.4). By Proposition 4.2.8(i), we need simply to verify that

$$V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})\subseteq V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})+C.$$

If (i$_1$) holds, then $V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})\subseteq\mathrm{Min}_{C\setminus\{0\}}V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})+C$. Hence, (4.4) is satisfied since we have (by Proposition 4.2.10)

$$\mathrm{Min}_{C\setminus\{0\}}V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})+C\subseteq V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})+C.$$

If (i$_2$) holds and $v\in V^m(F+C,x_0,y_0,v_1,\cdots,v_{m-1})$, then there exist $t_n\to0^+$, $x_n\xrightarrow{F}x_0$, $y_n\in F(x_n)$, and $c_n\in C$ such that $t_n^{-m}(y_n+c_n-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1})\to v$. If one has n_0 such that $c_n=0$ for all $n\geq n_0$, then $v\in V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$. If there is a subsequence, denoted again by $\{c_n\}$ with $c_n\neq0$, we claim that $\{||c_n||/t_n^m\}$ be bounded. Indeed, otherwise we may assume that $||c_n||/t_n^m\to\infty$ and $c_n/||c_n||\to\overline{c}\in C\setminus\{0\}$. Setting

$$v_n:=\frac{y_n+c_n-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1}}{t_n^m},\ \ \lambda_n:=\frac{t_n^m}{||c_n||},$$

we get

$$\lambda_n\frac{y_n-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1}-t_n^mv_n}{t_n^m}\to-\overline{c}\in-C\setminus\{0\}.$$

As $\lambda_n \to 0^+$, this means $-\overline{c} \in V^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1}) \cap -C \setminus \{0\}$, contradicting ($\mathrm{i}_2$). So, $\{||c_n||/t_n^m\}$ is bounded and $||c_n||/t_n^m \to a \geq 0$. With $\overline{v_n} := ||c_n||^{-1}(y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1}v_{m-1} - t_n^m v_n)$, one has

$$y_0 + t_n v_1 + \cdots + t_n^{m-1}v_{m-1} + t_n^m \left(v_n + \frac{||c_n||\overline{v_n}}{t_n^m}\right) = y_n \in F(x_n).$$

It easy to see that $v_n + t_n^{-m}||c_n||\overline{v_n} \to v - a\overline{c}$. Thus, $v - a\overline{c} \in V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})$ and (4.4) is satisfied. (4.5) is implied directly from (4.4). $\square$

The following example shows that conditions in Proposition 4.2.15 are essential.

Example 4.2.16. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $x_0 = 0$, $y_0 = (0,0)$, and

$$F(x) = \begin{cases} \{(0,0)\}, & \text{if } x \leq 0, \\ \{(0,0),(-1,-1)\}, & \text{if } x > 0. \end{cases}$$

Then,

$$(F+C)(x) = \begin{cases} \mathbb{R}^2_+, & \text{if } x \leq 0, \\ \{(y_1,y_2) : y_1 \geq -1, y_2 \geq -1\}, & \text{if } x > 0. \end{cases}$$

By calculating, we get $V^1(F,x_0,y_0) = \{(0,0)\}$ and $V^1(F+C,x_0,y_0) = \mathbb{R}^2$. So the conclusions in Proposition 4.2.15(i) does not hold. The reason is that both conditions in Proposition 4.2.15(i) are not satisfied.

Obviously, $V^1(F+C,x_0,y_0)$ does not have the domination property. Next, we show that $V^{\infty(1)}(F,x_0,y_0) \cap (-C) \neq \{(0,0)\}$. Indeed, by chosing $x_n = \frac{1}{n} \to x_0$, $t_n = \frac{1}{n} \to 0^+$, $\lambda_n = \frac{1}{n} \to 0^+$ and $y_n = (-n,-n)$, it is easy to check that

$$y_n \in \frac{F(x_n) - y_0}{t_n} \text{ and } \lambda_n y_n \to (-1,-1).$$

Thus, $(-1,-1) \in V^{\infty(1)}(F,x_0,y_0) \cap (-C)$.

The following result for weak efficiency can be proved similarly as Proposition 4.2.15.

Proposition 4.2.17. *Let $\widehat{C} \subseteq \operatorname{int} C \cup \{0\}$ be a closed convex cone with a compact base.*

(i) *Impose either of the following two conditions:*

(i_1) $V^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})$ *has the weak domination property with respect to* $\widehat{C}$, (i$_2$) $V^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1})\cap(-\widehat{C})=\{0\}$.

Then

$$\mathrm{Min}_{\mathrm{int}\,C}V^m(F,x_0,y_0,v_1,\cdots,v_{m-1})=\mathrm{Min}_{\mathrm{int}\,C}V^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1}).$$

(ii) *Let either of the following conditions hold:*

(ii$_1$) $W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1})$ *has the weak domination property with respect to* $\widehat{C}$*,*

(ii$_2$) $W^{\infty(m)}(F,x_0,y_0,v_1,\cdots,v_{m-1})\cap(-\widehat{C})=\{0\}$.

Then

$$\mathrm{Min}_{\mathrm{int}\,C}W^m(F,x_0,y_0,v_1,\cdots,v_{m-1})=\mathrm{Min}_{\mathrm{int}\,C}W^m(F+\widehat{C},x_0,y_0,v_1,\cdots,v_{m-1}).$$

4.3 Variational sets of perturbation maps

In this section, we apply results of subsection 4.2.2 to set-valued optimization. Let U be a normed space of perturbation parameters, Y be an objective (normed) space ordered by a pointed closed convex cone C and $F:U\to 2^Y$. One aims at finding the set of efficient points or the set of weak efficient points of $F(u)$ for a given parameter value u. Hence, we define set-valued maps G and S from U to Y by, for $u\in U$,

$$G(u):=\mathrm{Min}_{C\setminus\{0\}}F(u),\ S(u):=\mathrm{Min}_{\mathrm{int}\,C}F(u).$$

As it is well-known, G and S are called the *perturbation map* and *weak perturbation map*, respectively. The purpose of this section is to investigate relationships between variational sets of F and that of G and S, including relations between the set of efficient points or the set of weak efficient points of these variational sets.

A map F is said to have the *domination property* around u_0 if and only if there exists a neighborhood V of u_0 such that $F(u)$ has the domination property for all $u\in V$. The map F is said to have the *weak domination property* around u_0 with respect to $\widehat{C}$ if and only if there exists a neighborhood V of u_0 such that $F(u)$ has the weak domination property with respect to $\widehat{C}$ for all $u\in V$, where $\widehat{C}\subseteq \mathrm{int}\,C\cup\{0\}$ is a closed convex cone.

Remark 4.3.1. (i) Suppose $y_0\in G(u_0)$ and F have the domination property around u_0. Then

$$V^m(G+C,u_0,y_0,v_1,\cdots,v_{m-1})=V^m(F+C,u_0,y_0,v_1,\cdots,v_{m-1}),$$

$$W^m(G+C,u_0,y_0,v_1,\cdots,v_{m-1}) = W^m(F+C,u_0,y_0,v_1,\cdots,v_{m-1}).$$

(ii) Suppose $y_0 \in S(u_0)$ and F have the weak domination property around u_0 with respect to $\widehat{C}$. Then

$$V^m(S+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1}) = V^m(F+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1}),$$

$$W^m(S+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1}) = W^m(F+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1}).$$

The first result on efficiency is as follows.

Theorem 4.3.2. *Let $(u_0,y_0) \in \mathrm{gr}\, G$ and $v_1,\cdots,v_{m-1} \in Y$. Suppose F have the domination property around u_0 and C have a compact base.*

(i) *Assume further either of the following two conditions:*

(i$_1$) $V^m(F+C,u_0,y_0,v_1,\cdots,v_{m-1})$ *has the domination property,*

(i$_2$) $V^{\infty(m)}(F,u_0,y_0,v_1,\cdots,v_{m-1}) \cap (-C) = \{0\}$.

Then

$$\mathrm{Min}_{C\setminus\{0\}} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{C\setminus\{0\}} V^m(G,u_0,y_0,v_1,\cdots,v_{m-1}).$$

(ii) *Impose either of the following conditions:*

(ii$_1$) $W^m(F+C,u_0,y_0,v_1,\cdots,v_{m-1})$ *has the domination property,*

(ii$_2$) $W^{\infty(m)}(F,u_0,y_0,v_1,\cdots,v_{m-1}) \cap (-C) = \{0\}$.

Then

$$\mathrm{Min}_{C\setminus\{0\}} W^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{C\setminus\{0\}} W^m(G,u_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. We prove only assertion (i). Remark 4.3.1(i) yields that $V^m(G+C,u_0,y_0,v_1,\cdots,v_{m-1})$ also has the domination property. Because either (i$_1$) or (i$_2$) holds, from Proposition 4.2.15 we get

$$\mathrm{Min}_{C\setminus\{0\}} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{C\setminus\{0\}} V^m(F+C,u_0,y_0,v_1,\cdots,v_{m-1})$$
$$= \mathrm{Min}_{C\setminus\{0\}} V^m(G+C,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{C\setminus\{0\}} V^m(G,u_0,y_0,y_1,\cdots,y_{m-1}).$$

□

The following example illustrates Theorem 4.3.2.

Example 4.3.3. Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0,0)$, and $F(u) = \{(y_1,y_2) \in Y : y_1 = u,\ y_2 \geq |y_1|\}$ for $u \in U$. Then, $G(u) = \{(y_1,y_2) \in Y : y_1 = u,\ y_2 = |y_1|\}$. Let $v_i = (-1,1)$ for $i = 1,\cdots,m-1$. Direct calculations give

$$\begin{aligned} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) &= W^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) \\ &= \begin{cases} \{(y_1,y_2) \in Y : y_2 \geq |y_1|\}, & \text{if } m = 1, \\ \{(y_1,y_2) \in Y : y_1 + y_2 \geq 0\}, & \text{if } m > 1. \end{cases} \\ V^m(G,u_0,y_0,v_1,\cdots,v_{m-1}) &= W^m(G,u_0,y_0,v_1,\cdots,v_{m-1}) \\ &= \begin{cases} \{(y_1,y_2) \in Y : y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1,y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

We can check that assumptions of Theorem 4.3.2 are satisfied for all m. Direct checking yields

$$\begin{aligned} \mathrm{Min}_{C\setminus\{0\}} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) &= \mathrm{Min}_{C\setminus\{0\}} V^m(G,u_0,y_0,v_1,\cdots,v_{m-1}) \\ &= \mathrm{Min}_{C\setminus\{0\}} W^m(G,u_0,y_0,v_1,\cdots,v_{m-1}) \\ &= \begin{cases} \{(y_1,y_2) \in Y : y_1 \leq 0,\ y_2 = |y_1|\}, & \text{if } m = 1, \\ \{(y_1,y_2) \in Y : y_1 + y_2 = 0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

Similarly, by Remark 4.3.1(ii) and Proposition 4.2.17, we have the following for weak efficiency.

Theorem 4.3.4. *Let $(u_0,y_0) \in \mathrm{gr}\, S$ and $v_1,\cdots,v_{m-1} \in Y$. Suppose F have the weak domination property around u_0 with respect to $\widehat{C}$, where $\widehat{C} \subseteq \mathrm{int}C \cup \{0\}$ is a closed convex cone having a compact base.*

(i) *Let either of the following two conditions hold:*

(i$_1$) $V^m(F+\widehat{C},u_0,y_0,v_1\cdots,v_{m-1})$ *has the weak domination property with respect to $\widehat{C}$,*

(i$_2$) $V^{\infty(m)}(F,u_0,y_0,v_1,\cdots,v_{m-1}) \cap (-\widehat{C}) = \{0\}$.

Then

$$\mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{\mathrm{int}C} V^m(S,u_0,y_0,v_1,\cdots,v_{m-1}).$$

(ii) *Impose one of the following two conditions:*

(ii$_1$) $W^m(F+\widehat{C}, u_0, y_0, v_1, \cdots, v_{m-1})$ *has the weak domination property with respect to* $\widehat{C}$,

(ii$_2$) $W^{\infty(m)}(F, u_0, y_0, v_1, \cdots, v_{m-1}) \cap (-\widehat{C}) = \{0\}$.

Then

$$\mathrm{Min}_{\mathrm{int\,C}} W^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \mathrm{Min}_{\mathrm{int\,C}} W^m(S, u_0, y_0, v_1, \cdots, v_{m-1}).$$

Example 4.3.5. Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0,0)$, and

$$F(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u,\ y_2 \geq -y_1\}, & \text{if } u \leq 0, \\ \{(y_1, y_2) \in Y : 0 \leq y_1 \leq u,\ y_2 \geq 0\}, & \text{if } u > 0. \end{cases}$$

Then,

$$S(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u,\ y_2 \geq -y_1\}, & \text{if } u \leq 0, \\ \{(y_1, y_2) \in Y : 0 \leq y_1 \leq u,\ y_2 = 0\}, & \text{if } u > 0. \end{cases}$$

Let $v_i = (1,0)$ for $i = 1, \cdots, m-1$. Direct computations yield that

$$V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) = \begin{cases} \mathbb{R}^2_+ \cup \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 \geq -y_1\}, & \text{if } m = 1, \\ \mathbb{R} \times \mathbb{R}_+, & \text{if } m > 1, \end{cases}$$

and

$$V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) = \begin{cases} \{(y_1, y_2) \in Y : y_1 \leq 0,\ y_2 \geq -y_1\} \cup (\mathbb{R}_+ \times \{0\}), & \text{if } m = 1, \\ \mathbb{R} \times \{0\}, & \text{if } m > 1. \end{cases}$$

For each of F and S, variational sets of two types coincide for all $m \geq 1$. We can check that assumptions of Theorem 4.3.4 are fulfilled for all m (for an arbitrary closed convex cone $\widehat{C}$ such that $\widehat{C} \in \mathrm{int}\,\mathbb{R}^2_+ \cup \{(0,0)\}$). Direct verifying gives

$$\begin{aligned} \mathrm{Min}_{\mathrm{int\,C}} V^m(F, u_0, y_0, v_1, \cdots, v_{m-1}) &= \mathrm{Min}_{\mathrm{int\,C}} V^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) \\ &= \mathrm{Min}_{\mathrm{int\,C}} W^m(S, u_0, y_0, v_1, \cdots, v_{m-1}) \\ &= \begin{cases} \{(y_1, y_2) : y_1 \leq 0,\ y_2 = -y_1\} \cup (\mathbb{R}_+ \times \{0\}), & \text{if } m = 1, \\ \mathbb{R} \times \{0\}, & \text{if } m > 1. \end{cases} \end{aligned}$$

Note that the set of (Pareto) efficient points is much smaller than that of weak efficient points

$$G(u) = \begin{cases} \{(y_1, y_2) \in Y : y_1 = u,\ y_2 = -y_1\}, & \text{if } u \leq 0, \\ \{(0,0)\}, & \text{if } u > 0. \end{cases}$$

For $v_i = (1,0)$, $i = 1,\cdots,m-1$, we have $V^1(G,u_0,y_0) = W^m(G,u_0,y_0) = \{(y_1,y_2) \in Y : y_1 \leq 0,\ y_2 = -y_1\}$, and they are empty for $m > 1$. We can check that assumptions of Theorem 4.3.2 are satisfied for $m = 1$ and

$$\begin{aligned}\mathrm{Min}_{C\setminus\{0\}} V^1(F,u_0,y_0) &= \mathrm{Min}_{C\setminus\{0\}} V^1(G,u_0,y_0)\\ &= \mathrm{Min}_{C\setminus\{0\}} W^1(G,u_0,y_0)\\ &= \{(y_1,y_2) \in Y : y_1 \leq 0,\ y_2 = -y_1)\}.\end{aligned}$$

In the following case, Theorems 4.3.2 and 4.3.4 can be used, but some recent existing results cannot.

Example 4.3.6. Let $U = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $u_0 = 0$, $y_0 = (0,0)$, and

$$F(u) = \begin{cases} \{(0,0)\}, & \text{if } u = 0,\\ \left\{(0,0);(\frac{1}{n^3},\frac{-1}{n^3});(\frac{-1}{n^3},\frac{1}{n^3})\right\}, & \text{if } u = \frac{1}{n} \text{ for } n \in \mathbb{N},\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then, $S(u) = G(u) = F(u)$. For $v_1 = (1,-1)$, $v_2 = (-1,1)$. Calculations give

$$V^1(F,u_0,y_0) = W^1(F,u_0,y_0) = \{(y_1,y_2) \in Y : y_1 + y_2 = 0\},$$

$$V^2(F,u_0,y_0,v_1) = W^2(F,u_0,y_0,v_1) = \{(y_1,y_2) \in Y : y_1 + y_2 = 0\},$$

$$V^3(F,u_0,y_0,v_1,v_2) = W^3(F,u_0,y_0,v_1,v_2) = \{(y_1,y_2) \in Y : y_1 + y_2 = 0\}.$$

We can check that assumptions of Theorems 4.3.2 and 4.3.4 are satisfied. Calculating the lower Studniarski derivative of F at (u_0,y_0) (see [158] for the definition), we have $\underline{d}^m F(u_0,y_0)(u)$ is empty for all $u \in \mathbb{R}$. Hence, Theorems 4.1-4.3 and Corollaries 4.1-4.3 of [158] cannot be in use.

Since $D^2F(u_0,y_0,u_1,v_1)(u) = \emptyset$, for all $u \in \mathbb{R}$, Theorems 4.3, 4.7, and 4.10 of [167] in terms of second-order contingent derivatives cannot be applied either.

Proposition 4.3.7. *Let $(u_0,y_0) \in \mathrm{gr}\, S$ and $v_1,\cdots,v_{m-1} \in Y$. Suppose F have a proto-variational set of order m of type 1 at (u_0,y_0). Then*

$$V^m(S,u_0,y_0,v_1,\cdots,v_{m-1}) \subseteq \mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. Let $y \in V^m(S,u_0,y_0,v_1,\cdots,v_{m-1})$, i.e., there exist $t_n \to 0^+$, $u_n \xrightarrow{S} u_0$, and $y_n \to y$ such that

$$y_0 + t_n v_1 + \cdots + t_n^m y_n \in S(u_n) \subseteq F(u_n), \tag{4.6}$$

so $y \in V^m(F,u_0,y_0,v_1,\cdots,v_{m-1})$. Suppose $y \notin \mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1})$, i.e., there exists some y' in $V^m(F,u_0,y_0,v_1,..., v_{m-1})$ such that $y-y' \in \mathrm{int}C$. For the above sequences t_n and u_n, there exists $y'_n \to y'$ such that $y_0+t_nv_1+\cdots+t_n^m y'_n \in F(u_n)$, and $y_n-y'_n \in \mathrm{int}C$ for large n. Consequently,

$$(y_0+t_nv_1+\cdots+t_n^m y_n)-(y_0+t_nv_1+\cdots+t_n^m y'_n) = t_n^m(y_n-y'_n) \in \mathrm{int}C,$$

i.e., $y_0+t_nv_1+\cdots+t_n^m y_n \notin \mathrm{Min}_{\mathrm{int}C} F(u_n) = S(u_n)$, which contradicts to (4.6).

$\square$

Unfortunately, the similar result is not true for W^m, as indicated by the next example.

Example 4.3.8. Let $U=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, and

$$F(u)=\begin{cases} (\{0\}\times\mathbb{R})\cup(\mathbb{R}\times\{0\})\cup\{(x,y)\in Y : x^2+y^2=1\}, & \text{if } u=0,\\ \emptyset, & \text{if } u\neq 0.\end{cases}$$

Then,

$$S(u)=\begin{cases} ((-\infty,-1)\times\{0\})\cup(\{0\}\times(-\infty,-1))\cup\{(x,y) : x^2+y^2=1, x\leq 0, y\leq 0\}, & \text{if } u=0,\\ \emptyset, & \text{if } u\neq 0.\end{cases}$$

The map F has a proto-variational set at $(0,(-1,0))$ and $W^1(F,0,(-1,0)) = (\mathbb{R}_+\times\mathbb{R})\cup(\mathbb{R}_-\times\{0\})$. However, we have

$$\mathrm{Min}_{\mathrm{int}C} W^1(F,0,(-1,0)) = (\{0\}\times\mathbb{R}_-)\cup(\mathbb{R}_-\times\{0\}),$$

$$W^1(S,0,(-1,0)) = (\mathbb{R}_-\times\{0\})\cup\{(x,y)\in Y : y\leq -x, x\geq 0\},$$

and hence $W^1(S,0,(-1,0)) \not\subseteq \mathrm{Min}_{\mathrm{int}C} W^1(F,0,(-1,0))$.

Can we get a similar result when "G"and "$\mathrm{Min}_{C\setminus\{0\}}$" replacing "$S$"and "$\mathrm{Min}_{\mathrm{int}C}$" in Proposition 4.3.7? The following example gives a negative answer.

Example 4.3.9. Let $U=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, and

$$F(u)=\begin{cases} \{(x,y)\in Y : x>0, y<-x\}\cup\{(x,y)\in Y : y=-x\}, & \text{if } u=0,\\ \emptyset, & \text{if } u\neq 0.\end{cases}$$

Then, $G(u) \equiv \mathrm{Min}_{C\setminus\{0\}} F(u)$ is defined by $G(0) = \{(x,y) \in Y : y = -x, x \leq 0\}$ and $G(u) = \emptyset$ for any $u \neq 0$. We see that F has the following proto-variational set

$$V^1(F,0,(0,0)) = \{(x,y) \in Y : x \geq 0, y < -x\} \cup \{(x,y) \in Y : y = -x\}.$$

Since $\mathrm{Min}_{C\setminus\{0\}} V^1(F,0,(0,0)) = \{(x,y) \in Y : y = -x, x < 0\}$ and $V^1(G,0,(0,0)) = \{(x,y) \in Y : y = -x,\, x \leq 0\}$, one has

$$V^1(G,0,(0,0)) \not\subseteq \mathrm{Min}_{C\setminus\{0\}} V^1(F,0,(0,0)).$$

Theorem 4.3.10. *Let* $(u_0,y_0) \in \mathrm{gr}S$*,* $v_1,\cdots,v_{m-1} \in Y$*, and* $\widehat{C}$ *be a closed convex cone contained in* $\mathrm{int}C \cup \{0\}$ *and have a compact base. Suppose the following conditions be satisfied:*

(i) *either of the following holds*

(i_1) $V^m(F+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1})$ *has the weak domination property with respect to* $\widehat{C}$*,*

(i_2) $V^{\infty(m)}(F,u_0,y_0,v_1,\cdots,v_{m-1}) \cap (-\widehat{C}) = \{0\}$,

(ii) F *has the weak domination property around* u_0 *with respect to* $\widehat{C}$*,*

(iii) F *has a proto-variational set of order m of type 1 at* (u_0,y_0)*.*

Then

$$V^m(S,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. Obviously, by Proposition 4.3.7, we need to prove only that

$$V^m(S,u_0,y_0,v_1,\cdots,v_{m-1}) \supseteq \mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}).$$

Propositions 4.2.12, 4.2.17 and Remark 4.3.1(iii) together imply that

$$\mathrm{Min}_{\mathrm{int}C} V^m(F,u_0,y_0,v_1,\cdots,v_{m-1}) = \mathrm{Min}_{\mathrm{int}C} V^m(F+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1})$$
$$= \mathrm{Min}_{\mathrm{int}C} V^m(S+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1}) \subseteq V^m(S,u_0,y_0,v_1,\cdots,v_{m-1}).$$

□

4.4 Sensitivity analysis for vector optimization problems

In this section, we consider the following two constrained vector optimization problems, where both the objective map and the constraint set depend on a perturbation parameter,

$$\mathrm{Min}_{C\setminus\{0\}} F(x,u), \text{ subject to } x \in X(u), \tag{4.7}$$

$$\mathrm{Min}_{\mathrm{int}C}\, F(x,u), \text{ subject to } x \in X(u). \tag{4.8}$$

Here, as before, U, W, Y are normed spaces, C is a pointed closed convex ordering cone in Y, F is a set-valued objective map from $W \times U$ to Y, and X is a set-valued map from U to W. We define a set-valued map H from U to Y by

$$H(u) := F(X(u),u) = \{y \in Y : y \in F(x,u), x \in X(u)\}.$$

So, $H(u)$ is the parameterized feasible set in the objective space. In problems (4.7) and (4.8), we aim to obtain efficient points and weak efficient points of $H(u)$, respectively. Solution sets in Y of problems (4.7) and (4.8) are denoted by $\mathrm{Min}_{C\setminus\{0\}} H(u)$ and $\mathrm{Min}_{\mathrm{int}C} H(u)$, respectively. Like in Section 4.3, we define

$$G(u) := \mathrm{Min}_{C\setminus\{0\}} H(u), \;\; S(u) := \mathrm{Min}_{\mathrm{int}C} H(u).$$

We need the following new definition.

Definition 4.4.1. Let W, U, Y be normed spaces, $F : W \times U \to 2^Y$, $((x_0,u_0),y_0) \in \mathrm{gr}F$, $x \in W$, and $(w_i, v_i) \in W \times Y$ for $i = 1, \cdots, m-1$.

(i) The m-th order *upper (lower, respectively) variation* of F at $((x_0,u_0),y_0)$ with respect to x is

$$V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}) := \{v \in Y : \exists t_n \to 0^+, \exists h_n \to 0^+, \exists x_n \to x, \exists u_n \to u_0,$$

$$\exists v_n \to v, \forall n, y_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m v_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\}$$

$$(\underline{V}_q^m(F,(x_0[x],u_0),y_0,w_1,v_1 ..., w_{m-1},v_{m-1}) := \{v \in Y : \forall t_n \to 0^+, \forall x_n \to x, \forall u_n \to u_0, \exists v_n \to v, \forall n,$$

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m x_n, u_n)\}).$$

(ii) F is said to have a m-th order *proto variation* of F at $((x_0,u_0),y_0)$ if and only if, for all x,

$$V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}) = \underline{V}_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}).$$

We recall that a map $M : X \to 2^Y$ is said to be *calm* around $x_0 \in \mathrm{dom}M$ if and only if there exist a neighborhood V of x_0 and $L > 0$ such that $\forall x \in V$,

$$M(x) \subseteq M(x_0) + L||x - x_0||B_Y.$$

We now investigate connections of a proto variation of F and a variational set of X to the corresponding variational set of H.

Proposition 4.4.2. *Let $u_0 \in U$, $x_0 \in X(u_0)$, and $y_0 \in F(x_0,u_0)$. If F has a m-th order proto variation at $((x_0,u_0),y_0)$, then*

$$\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}) \subseteq V^m(H,u_0,y_0,v_1,\cdots,v_{m-1}). \tag{4.9}$$

Moreover, if W is finite dimensional, $\widetilde{X}(u,y) := \{x \in \mathbb{R}^n : x \in X(u), y \in F(x,u)\}$ is calm around (u_0,y_0), $\widetilde{X}(u_0,y_0) = \{x_0\}$, and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\}$, then the inclusion opposite to (4.9) is valid.

Proof. Let $x \in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})$ such that there exists v satisfying

$$v \in V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1}).$$

Since $x \in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})$, there exist $t_n \to 0^+$, $u_n \to u_0$, $x_n \xrightarrow{X} x$ such that, for all n,

$$x_0+t_nw_1+\cdots+t_n^{m-1}w_{m-1}+t_n^m x_n \in X(u_n).$$

Then,

$$F(x_0+t_nw_1+\cdots+t_n^{m-1}w_{m-1}+t_n^m x_n,u_n) \subseteq H(u_n). \tag{4.10}$$

Because $v \in V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})$, with the above t_n,u_n,x_n, there exists $y_n \in F(x_0+t_nw_1+\cdots+t_n^{m-1}w_{m-1}+t_n^m x_n,u_n)$ such that $t_n^{-m}(y_n-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1}) \to v$. So, we have

$$y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m\left(\frac{y_n-y_0-t_nv_1-\cdots-t_n^{m-1}v_{m-1}}{t_n^m}\right) = y_n \in$$
$$\in F(x_0+t_nw_1+\cdots+t_n^{m-1}w_{m-1}+t_n^m x_n,u_n).$$

It follows from (4.10) that $v \in V^m(H,u_0,y_0,v_1,\cdots,v_{m-1})$.

Next, we prove the inclusion reverse to (4.9). Let $v \in V^m(H,u_0,y_0,v_1,\cdots,v_{m-1})$, i.e., there exist $t_n \to 0^+$, $u_n \to u_0$, and $v_n \xrightarrow{H} v$ such that $y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m v_n \in H(u_n)$ for all n. Then, there exists $x_n \in X(u_n)$ such that $y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m v_n \in F(x_n,u_n)$. Hence, $x_n \in \widetilde{X}(u_n,y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m v_n)$. The calmness of $\widetilde{X}$ yields $M>0$ such that

$$||x_n-x_0|| \le M||(u_n,y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m v_n)-(u_0,y_0)||.$$

Then, $x_n \to x_0$ and hence $(x_n-x_0-t_nw_1-\cdots-t_n^{m-1}w_{m-1}) \to 0$. We claim that $\{t_n^{-m}(x_n-x_0-t_nw_1-\cdots-t_n^{m-1}w_{m-1})\}$ is bounded. Indeed, we have

$$x_0+||x_n-x_0||\frac{(x_n-x_0)}{||x_n-x_0||} = x_n \in \widetilde{X}(u_n,y_0+t_nv_1+\cdots+t_n^{m-1}v_{m-1}+t_n^m v_n). \tag{4.11}$$

We may assume that $a_n := (x_n - x_0)/||x_n - x_0|| \to a$ with norm one. Setting $r_n^m := ||x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1}||$, we have $r_n^m \to 0^+$ and

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n = y_0 + r_n \frac{t_n}{r_n} v_1 + \cdots + r_n^{m-1} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^m \frac{t_n^m}{r_n^m} v_n$$

$$= y_0 + r_n \left(\frac{t_n}{r_n} v_1 + \cdots + r_n^{m-2} \frac{t_n^{m-1}}{r_n^{m-1}} v_{m-1} + r_n^{m-1} \frac{t_n^m}{r_n^m} v_n \right) =: y_0 + r_n q_n.$$

For $h_n := ||x_n - x_0||$, (4.11) is written equivalently as $x_0 + h_n a_n \in \widetilde{X}(u_n, y_0 + r_n q_n)$. If $t_n^m / r_n^m \to 0^+$, then $q_n \to 0$ and $a \in V_q^1(\widetilde{X}, (u_0, y_0[0]), x_0)$, which is impossible. Thus, $\{t_n^{-m}(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1})\}$ is bounded and

$$\overline{x_n} := \frac{1}{t_n^m}(x_n - x_0 - t_n w_1 - \cdots - t_n^{m-1} w_{m-1})$$

converges to some $\bar{x} \in \mathbb{R}^n$. Since $x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m \overline{x_n} \in X(u_n)$, one has

$$y_0 + t_n v_1 + \cdots + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n w_1 + \cdots + t_n^{m-1} w_{m-1} + t_n^m \overline{x_n}, u_n).$$

Therefore, $\bar{x} \in V^m(X, u_0, x_0, w_1, \cdots, w_{m-1})$ and $v \in V_q^m(F, (x_0[\bar{x}], u_0), y_0, w_1, v_1, \cdots, w_{m-1}, v_{m-1})$. □

The following four examples ensure the essentialness of each assumption of Proposition 4.4.2.

Example 4.4.3. ($\widetilde{X}(u_0, y_0) = \{x_0\}$ is needed)

Let $U = W = Y = \mathbb{R}$, $F(x,u) = \{x(x-1)\}$, $u_0 = 0$, $x_0 = 1$, $y_0 = 0 \in F(x_0, u_0)$, and

$$X(u) = \begin{cases} \{x \in W : 0 \le x \le 1\}, & \text{if } u = 0, \\ \{x \in W : -u \le x \le 1\}, & \text{if } u = \dfrac{1}{n}, n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,

$$\widetilde{X}(u,y) = \begin{cases} \left\{ \dfrac{1 - \sqrt{1+4y}}{2}, \dfrac{1 + \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \{0\} \cup \{\dfrac{1}{n} : n \in \mathbb{N}\},\ \dfrac{-1}{4} \le y \le 0, \\ \left\{ \dfrac{1 - \sqrt{1+4y}}{2} \right\}, & \text{if } u \in \{\dfrac{1}{n} : n \in \mathbb{N}\},\ 0 < y \le u(u+1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$H(u)=\begin{cases}\{y\in Y:\dfrac{-1}{4}\le y\le 0\}, & \text{if } u=0,\\ \{y\in Y:\dfrac{-1}{4}\le y\le u(u+1)\}, & \text{if } u=\dfrac{1}{n},\ n\in\mathbb{N},\\ \emptyset, & \text{otherwise.}\end{cases}$$

The map $\widetilde{X}$ is clearly calm around (u_0,y_0) and we can obtain by direct calculations that

$$V_q^1(\widetilde{X},(u_0,y_0[0]),x_0)=\{0\},\ V^1(X,u_0,x_0)=-\mathbb{R}_+,$$

$$V_q^1(F,(x_0[x],u_0),y_0)=\{x\},\ V^1(H,u_0,y_0)=\mathbb{R}.$$

So,

$$\bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0)=-\mathbb{R}_+.$$

Thus, since $\widetilde{X}(u_0,y_0)=\{0,1\}\neq\{x_0\}$, we have

$$V^1(H,u_0,y_0)\not\subseteq \bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0).$$

Example 4.4.4. (the calmness around (u_0,y_0) cannot be dropped)

Let $U=W=Y=\mathbb{R}$, $F(x,u)=\{x(x-1)\}$, $u_0=0$, $x_0=1$, $y_0=0\in F(x_0,u_0)$, and

$$X(u)=\begin{cases}\{x\in W:0<x\le 1\}, & \text{if } u=0,\\ \{x\in W:-u<x\le 1\}, & \text{if } u=\dfrac{1}{n},\ n\in\mathbb{N},\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then,

$$\widetilde{X}(u,y)=\begin{cases}\{1\}, & \text{if } u\in\{0\}\cup\{\dfrac{1}{n}:n\in\mathbb{N}\},\ y=0,\\ \left\{\dfrac{1-\sqrt{1+4y}}{2},\dfrac{1+\sqrt{1+4y}}{2}\right\}, & \text{if } u\in\{0\}\cup\{\dfrac{1}{n}:n\in\mathbb{N}\},\ \dfrac{-1}{4}\le y<0,\\ \left\{\dfrac{1-\sqrt{1+4y}}{2}\right\}, & \text{if } u\in\{\dfrac{1}{n}:n\in\mathbb{N}\},\ 0<y<u(u+1),\\ \emptyset, & \text{otherwise,}\end{cases}$$

and $H(u)$ is as in Example 4.4.3 with only "$y\le u(u+1)$" replaced by the strict inequality. Hence,

$$\widetilde{X}(u_0,y_0)=\{x_0\},\ V_q^1(\widetilde{X},(u_0,y_0[0]),x_0)=\{0\},$$

$$V^1(X,u_0,x_0) = -\mathbb{R}_+,\ V^1_q(F,(x_0[x],u_0),y_0) = \{x\},\ V^1(H,u_0,y_0) = \mathbb{R}.$$

Consequently, because $\widetilde{X}$ is not calm around (u_0,y_0), we really have

$$\bigcup_{x\in V^1(X,u_0,x_0)} V^1_q(F,(x_0[x],u_0),y_0) = -\mathbb{R}_+,$$

$$V^1(H,u_0,y_0) \not\subseteq \bigcup_{x\in V^1(X,u_0,x_0)} V^1_q(F,(x_0[x],u_0),y_0).$$

Example 4.4.5. ($V_q(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\}$ is essential)

Let $U = Y = \mathbb{R}$, $W = \mathbb{R}^2$, $X(u) = \{x \in W : x_1 = u,\ x_2 = 0\}$, $F(x,u) = \{x_1^2(x_1-1)\}$, $u_0 = 0$, $x_0 = (0,0)$, and $y_0 = f(x_0,u_0) = 0$. Then,

$$\widetilde{X}(u,y) = \begin{cases} \{(u,0)\}, & \text{if } u\in\mathbb{R},\ y = u^2(u-1),\\ \emptyset, & \text{otherwise,}\end{cases}$$

and $H(u) = \{u^2(u-1)\}$. Hence, $\widetilde{X}(u_0,y_0) = \{x_0\}$ and $\widetilde{X}$ is calm around (u_0,y_0). Direct calculations give $V^1(X,u_0,x_0) = \mathbb{R}\times\{0\}$, $V^1_q(F,(x_0[x],u_0),y_0) = \{0\}$. Therefore,

$$\bigcup_{x\in V^1(X,u_0,x_0)} V^1_q(F,(x_0[x],u_0),y_0) = \{0\}.$$

By taking $t_n = \dfrac{1}{n}$, $u_n = \dfrac{1}{\sqrt{n}}$, $x_n = (u_n,0) \in X(u_n)$, $v_n = \dfrac{1}{\sqrt{n}} - 1 \to -1$, we can check that $y_0 + t_n v_n \in H(u_n)$. Thus, $-1 \in V^1(H,u_0,y_0)$. Consequently,

$$V^1(H,u_0,y_0) \not\subseteq \bigcup_{x\in V^1(X,u_0,x_0)} V^1_q(F,(x_0[x],u_0),y_0).$$

To see the cause, let $t_n = \dfrac{1}{n}$, $u_n = \dfrac{1}{n}$, $y_n = \dfrac{1}{n}(\dfrac{1}{n} - 1) \to 0$, $x_n = (1,0)$ to have that $x_0 + t_n x_n \in \widetilde{X}(u_n, y_0 + t_n y_n)$, and so $(1,0) \in V^1_q(\widetilde{X},(u_0,y_0[0]),x_0)$.

Example 4.4.6. (W needs be finite dimensional)

Let $U = Y = \mathbb{R}$, and $W = l_1$, the space of all real sequences $x = (x^i)_{i\in\mathbb{N}}$ with $\sum\limits_{i=1}^{\infty} \left|x^i\right| < \infty$. Let

$$X(u) = \begin{cases} \{0\}, & \text{if } u = 0,\\ \{x = (x^i)_{i\in\mathbb{N}} \in W : x^i = u \text{ if } i = n;\ x^i = 0 \text{ if } i \neq n\}, & \text{if } u = \dfrac{1}{n},\ n\in\mathbb{N},\\ \emptyset, & \text{otherwise,}\end{cases}$$

$F(x,u) = \{||x||(||x||-1)\}$, $u_0 = 0$, $x_0 = 0 \in X(u_0)$, and $y_0 = 0 \in F(x_0,u_0)$. Then,

$$\widetilde{X}(u,y) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \{(x^i)_{i\in\mathbb{N}} \in W : x^i = u \text{ if } i = n;\ x^i = 0 \text{ if } i \neq n\}, & \text{if } u = \dfrac{1}{n}, n \in \mathbb{N},\ y = |u|(|u|-1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$H(u) = \begin{cases} \{0\}, & \text{if } u = 0, \\ \{|u|(|u|-1)\}, & \text{if } u = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, $\widetilde{X}$ is calm around (u_0,y_0). We can compute directly that

$$\widetilde{X}(u_0,y_0) = \{x_0\},\ V_q^1(\widetilde{X},(u_0,y_0[0]),x_0) = \{0\},$$

$$V_q^1(F,(x_0[x],u_0),y_0) = \{-||x||\},\ V^1(X,u_0,x_0) = \{0\}.$$

Therefore,

$$\bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0) = \{0\}.$$

By taking $t_n = \dfrac{1}{n}$, $u_n = \dfrac{1}{n}$, $x_n = (x_n^i)_{i\in\mathbb{N}} \in X(u_n)$ satisfying $x_n^i = u_n$ if $i = n$ and $x_n^i = 0$ if $i \neq n$, and $v_n = \dfrac{1}{n} - 1 \to -1$, we can check that $y_0 + t_n v_n \in H(u_n)$. Hence, $-1 \in V^1(H,u_0,y_0)$. Thus,

$$V^1(H,u_0,y_0) \not\subseteq \bigcup_{x\in V^1(X,u_0,x_0)} V_q^1(F,(x_0[x],u_0),y_0).$$

Finally, invoking to Proposition 4.4.2 and results of Section 4.3, we easily establish relations between the set of efficient points and the set of weak efficient points of the mentioned variational sets stated in the following theorems.

Theorem 4.4.7. *Let $(u_0,y_0) \in \operatorname{gr} G$, $x_0 \in X(u_0)$, $y_0 \in F(x_0,u_0)$, W be finite dimensional, and C have a compact base. Suppose that*

(i) *H has the domination property around u_0,*

(ii) *either of the following two conditions holds:*

(ii$_1$) *$V^m(H+C,u_0,y_0,v_1,\cdots,v_{m-1})$ has the domination property,*

(ii$_2$) *$V^{\infty(m)}(H,x_0,y_0,v_1,\cdots,v_{m-1}) \cap (-C) = \{0\}$,*

(iii) *F has a m-th order proto variation at $((x_0,u_0),y_0)$,*

(iv) *$\widetilde{X}$ is calm around (u_0,y_0),*

(v) *$\widetilde{X}(u_0,y_0)=\{x_0\}$ and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0)=\{0\}$.*

Then

$$\mathrm{Min}_{C\setminus\{0\}}\left(\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right)=$$

$$\mathrm{Min}_{C\setminus\{0\}}V^m(G,x_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. This follows from Theorem 4.3.2 and Proposition 4.4.2. □

Theorem 4.4.8. *Let $(u_0,y_0)\in \mathrm{gr}S$, $x_0\in X(u_0)$, $y_0\in F(x_0,u_0)$, W be finite dimensional, and $\widehat{C}$ be a closed convex cone contained in* $\mathrm{int}C\cup\{0\}$, *and have a compact base. Suppose that*

(i) *Y has the weak domination property around u_0 with respect to $\widehat{C}$,*

(ii) *either of the following two conditions is satisfied:*

(ii$_1$) *$V^m(H+\widehat{C},u_0,y_0,v_1,\cdots,v_{m-1})$ has the weak domination property with respect to $\widehat{C}$,*

(ii$_2$) *$V^{\infty(m)}(H,x_0,y_0,v_1,\cdots,v_{m-1})\cap(-\widehat{C})=\{0\}$,*

(iii) *F has a m-th order proto variation at $((x_0,u_0),y_0)$,*

(iv) *$\widetilde{X}$ is calm around $((u_0,y_0),x_0)$,*

(v) *$\widetilde{X}(u_0,y_0)=\{x_0\}$ and $V_q^1(\widetilde{X},(u_0,y_0[0]),x_0)=\{0\}$.*

Then

$$\mathrm{Min}_{\mathrm{int}C}\left(\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right)=$$

$$\mathrm{Min}_{\mathrm{int}C}V^m(S,x_0,y_0,v_1,\cdots,v_{m-1}).$$

Proof. Theorem 4.3.4(i) and Proposition 4.4.2 together imply this theorem. □

Theorem 4.4.9. *Let $(u_0,y_0)\in \mathrm{gr}S$, the assumptions of Theorem 4.4.8 be satisfied, and H have a proto-variational set of order m of type 1 at (u_0,y_0). Then*

$$V^m(S,u_0,y_0,v_1,\cdots,v_{m-1})=$$

$$\mathrm{Min}_{\mathrm{int}C}\left(\bigcup_{x\in V^m(X,u_0,x_0,w_1,\cdots,w_{m-1})} V_q^m(F,(x_0[x],u_0),y_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})\right).$$

Proof. Applying Theorem 4.3.10 and Proposition 4.4.2, we are done. □

Remark 4.4.10. Though there have been several contributions to analysis of perturbation map G and weak perturbation map S for unconstrained feasible map F (defined in Section 4.3), we see only Tanino in [162] dealing with this topic for a map F in a set-constrained smooth single-valued problem. That paper was limited to first-order results in terms of gradients of F. The present section is the first attempt of higher-order considerations of F for a set-constrained nonsmooth multivalued problem. The extension has been performed in several aspects. Furthermore, we have extended successfully almost directly Theorem 4.1 of Tanino in [162]. However, a drawback here is that the results are technically complicated. We hope that, excluding inevitable complexity, e.g., with higher-order derivatives (at least because of long expressions) and a high level of nonsmoothness, improvements can be obtained in future. In this section, we restrict ourselves to making sure that the relatively complicated assumptions imposed in the results cannot be avoided by showing (in examples) their essentialness.

Chapter 5

Radial sets, radial derivatives and applications to optimality conditions for vector optimization problems

5.1 Introduction

On optimality conditions for nonsmooth problems, to meet the increasing diversity of practical situations, a broad spectrum of generalized derivatives has been developed to replace the Fréchet and Gâteaux derivatives. Each of them is suitable for several models, and none is universal. Note that the wide range of methods in nonsmooth optimization can be roughly separated into the primal and the dual space approaches. Almost notions of generalized derivatives in the primal space approach are based on corresponding tangency concepts and hence carry only local information. In other words, such derivatives are local linear approximations of a considered map.

Until now, only few concepts of such derivatives have been extended to orders greater than two, which are naturally understood to be inevitable for higher-order optimality conditions, such as contingent, adjacent and Clarke derivatives (see [26, 59, 67, 80, 87, 95, 99, 103, 115, 118, 119, 125], variational sets (see [7, 105, 106]), etc. (The derivatives constructed in the dual space approach are hardly extended to orders greater than two.) The radial derivative, introduced by Taa in [160], is in the first approach, but encompasses the idea of a conical hull, and hence contains global information of a map as its conical hull, and is closed and can be used to obtain optimality conditions for global solutions without convexity assumptions.

We recall that, for a subset S of a normed space X and $x_0 \in \mathrm{cl}\, S$, the radial cone of S at x_0 is defined by Taa in [160] as follows

$$R_S(x_0) := \{u \in X : \exists t_n > 0, \exists u_n \to u, \forall n, x_0 + t_n u_n \in S\}.$$

For $F : X \to 2^Y$, X, Y be normed spaces, $(x_0, y_0) \in \mathrm{gr}F$ and $u \in X$, the first-order radial derivative of F at (x_0, y_0) is defined in [160] by $\mathrm{gr}(D_R F(x_0, y_0)) = R_{\mathrm{gr}F}(x_0, y_0)$. A kind of higher-order radial derivatives was proposed by Anh et al. in [8] as follows. The m-th order outer radial derivative of F at $(x_0, y_0) \in \mathrm{gr}F$ is

$$\overline{D}_R^m F(x_0, y_0)(u) := \{v \in Y : \exists t_n > 0, \exists (u_n, v_n) \to (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}, \quad (5.1)$$

the m-th order inner radial derivative of F at $(x_0, y_0) \in \mathrm{gr}F$ is

$$\overline{D}_R^{\flat(m)} F(x_0, y_0)(u) := \{v \in Y : \forall t_n > 0, \exists (u_n, v_n) \to (u, v), \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

Observe that the graph of the above higher-order radial derivatives is not a higher-order tangent set of the graph of the map, because the rates of change of the points under consideration in X and Y are different (t_n and t_n^m). The graph of many other higher-order derivatives is such a corresponding graph. For instance, for $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $(u_i, v_i) \in X \times Y$, $i = 1, ..., m-1$, recall that the m-th order contingent derivative of F at (x_0, y_0) with respect to $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is

$$D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(u) := \{v \in Y : \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v),$$

$$\forall n, y_0 + t_n v_1 + ... + t_n^m v_n \in F(x_0 + t_n u_1 + ... + t_n^m u_n)\}.$$

Its graph is just the m-th order contingent set of the graph of F. In some sense, this property has a better geometry and is more natural.

The discussion above motivates the aim of this chapter: to define another kind of higher-order radial derivatives based on (higher-order) radial sets and use it to obtain higher-order optimality conditions for set-valued vector optimization. It turns out that, in general this kind of radial derivatives is incomparable with our previous definitions in [8], but it provides a tool for establishing new optimality conditions, which also sharpen or improve a number of the existing results in the literature. Note further that the obtained optimality conditions have global characters and do not need any convexity assumptions. The content of this chapter is also our research published in [2, 4].

5.2 Radial sets and radial derivatives

Definitions of higher-order radial sets and corresponding derivatives are introduced in this section, followed by their properties and basic calculus rules like those for a sum or composition of mappings.

5.2.1 Definitions and properties

Definition 5.2.1. Let X be a normed space, $x_0 \in S \subseteq X$, and $u_1, ..., u_{m-1} \in X$ with $m \geq 1$.

(i) The m-th order *upper radial set* of S at x_0 with respect to $u_1, ..., u_{m-1}$ is defined as

$$T_S^{r(m)}(x_0, u_1, ..., u_{m-1}) := \{y \in X : \exists t_n > 0, \exists y_n \to y, \forall n, x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m y_n \in S\}.$$

(ii) The m-th order *lower radial set* of S at x_0 with respect to $u_1, ..., u_{m-1}$ is

$$T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1}) := \{y \in X : \forall t_n > 0, \exists y_n \to y, \forall n, x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m y_n \in S\}.$$

Definition 5.2.2. Let X, Y be normed spaces, $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr} F$, and $(u_i, v_i) \in X \times Y$, $i = 1, ..., m-1$ with $m \geq 1$.

(i) The m-th order *upper radial derivative* of F at (x_0, y_0) with respect to (u_1, v_1) , ..., (u_{m-1}, v_{m-1}) is the multimap $D_R^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) : X \to 2^Y$ whose graph is

$$\mathrm{gr} D_R^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) := T_{\mathrm{gr} F}^{r(m)}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(ii) The m-th order *lower radial derivative* of F at (x_0, y_0) with respect to $(u_1, v_1), ...,$ (u_{m-1}, v_{m-1}) is the multimap $D_R^{\flat(m)} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) : X \to 2^Y$ with

$$\mathrm{gr} D_R^{\flat(m)} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) := T_{\mathrm{gr} F}^{r\flat(m)}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

We easily obtain the following formulae, for $x \in X$,

$$D_R^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) = \{v \in Y : \exists t_n > 0, \exists x_n \to x, \exists v_n \to v, \forall n,$$

$$= y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m x_n)\}$$

$$D_R^{\flat(m)} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) = \{v \in Y : \forall t_n > 0, \exists x_n \to x, \exists v_n \to v, \forall n,$$

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m x_n)\}.$$

Remark 5.2.3. (i) It follows that

$$D_R^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(X) \subseteq T_{F(X)}^{r(m)}(y_0, v_1, ..., v_{m-1}).$$

(ii) Radial sets are especial cases of Γ-limits. Indeed,

$$\begin{aligned} y \in T_S^{r(m)}(x_0, u_1, ..., u_{m-1}) &\iff \inf_{Q \in \mathscr{N}(y)} \sup_{t>0} \sup_{y' \in Q} \chi_{\mathrm{gr}(\mathscr{H}_{S,x_0,u_1,...,u_{m-1}})}(t, y') = 1 \\ &\iff \Gamma(\mathscr{R}^+, \mathscr{N}(y)^+) \lim \chi_{\mathrm{gr}(\mathscr{H}_{\mathrm{S,x_0,u_1,...,u_{m-1}}})} = 1, \\ y \in T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1}) &\iff \inf_{Q \in \mathscr{N}(y)} \inf_{t>0} \sup_{y' \in Q} \chi_{\mathrm{gr}(\mathscr{H}_{S,x_0,u_1,...,u_{m-1}})}(t, y') = 1 \\ &\iff \Gamma(\mathscr{R}^-, \mathscr{N}(y)^+) \lim \chi_{\mathrm{gr}(\mathscr{H}_{\mathrm{S,x_0,u_1,...,u_{m-1}}})} = 1, \end{aligned}$$

where $\mathscr{R} := \{(0, +\infty)\}$ be a filter on $(0, +\infty)$, and $\mathscr{H}_{S,x_0,u_1,...,u_{m-1}} : (0, +\infty) \to 2^X$ is defined by

$$\mathscr{H}_{S,x_0,u_1,...,u_{m-1}}(t) := \frac{1}{t^m}(S - x_0 - tu_1 - ... - t^{m-1}u_{m-1}).$$

(iii) Radial derivatives can be expressed in terms of Γ-limits as follows

$$\begin{aligned} & y \in D_R^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \\ \iff & \inf_{Q \in \mathscr{N}(y)} \inf_{W \in \mathscr{N}(x)} \sup_{t>0} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\mathrm{gr}(\mathscr{H}_{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})})}(t, x', y') = 1 \\ \iff & \Gamma(\mathscr{R}^+, \mathscr{N}(x)^+, \mathscr{N}(y)^+) \lim \chi_{\mathrm{gr}(\mathscr{H}_{\mathrm{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})}})} = 1, \\ & y \in D_R^{\flat(m)} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \\ \iff & \inf_{Q \in \mathscr{N}(y)} \inf_{W \in \mathscr{N}(x)} \inf_{t>0} \sup_{x' \in W} \sup_{y' \in Q} \chi_{\mathrm{gr}(\mathscr{H}_{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})})}(t, x', y') = 1 \\ \iff & \Gamma(\mathscr{R}^-, \mathscr{N}(x)^+, \mathscr{N}(y)^+) \lim \chi_{\mathrm{gr}(\mathscr{H}_{\mathrm{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})}})} = 1, \end{aligned}$$

where $\mathscr{H}_{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})} : (0, +\infty) \times X \to 2^Y$ is defined by

$$\begin{aligned} \mathscr{H}_{F,(x_0,y_0),(u_1,v_1),...,(u_{m-1},v_{m-1})}(t, x') \quad := \quad & \frac{1}{t^m}(F(x_0 + tu_1 + ... + t^{m-1}u_{m-1} + t^m x') - y_0 - \\ & - tv_1 - ... - t^{m-1}v_{m-1}. \end{aligned}$$

(iv) Definitions 5.2.1, 5.2.2 correspond to the following known definition of the contingent objects

$$T_S^m(x_0, u_1, ..., u_{m-1}) := \{y \in X : \exists t_n \to 0, \exists y_n \to y, \forall n, x_0 + t_n u_1 + ... + t_n^{m-1}u_{m-1} + t_n^m y_n \in S\},$$

$$\mathrm{gr}\, D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) := T_{\mathrm{gr} F}^m(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

Observe another fact, which makes the radial set and derivative different from the contingent set and derivative, and hence also from other tangency notions and derivatives in variational analysis. Let us explain this difference only between $T_S^{r(2)}(x_0,u)$ and $T_S^2(x_0,u)$ for simplicity. It is known that if $u \notin T_S(x_0)$, then $T_S^2(x_0,u) = \emptyset$. But, the simple example with $X = \mathbb{R}^2$, $S = \{0_{\mathbb{R}^2}, e_1\}$, $x_0 = 0_{\mathbb{R}^2}$, $e_1 = (1,0)$, $e_2 = (0,1)$ shows that this is not valid for radial cones: $u = e_1 - e_2 \notin T_S^r(x_0)$, but $e_2 \in T_S^{r(2)}(x_0,u)$, i.e., the last cone is nonempty.

(v) The objects defined in Definitions 5.2.1, 5.2.2 are called upper and lower radial sets and derivatives, respectively. However, though being applied to establish necessary optimality conditions similarly as the upper radial concept is, this lower radial object yields weaker results, and it is not convenient for dealing with sufficient optimality conditions. Therefore, in this chapter we develop only the upper radial concepts and thus omit the term "upper".

In the following example, we compute a radial derivative and a contingent derivative in an infinite dimensional case.

Example 5.2.4. Let $X = \mathbb{R}$ and $Y = l^2$, the Hilbert space of the numerical sequences $x = (x_i)_{i\in\mathbb{N}}$ with $\sum_{i=1}^{\infty} x_i^2$ being convergent. By $(e_i)_{i\in\mathbb{N}}$ we denote standard unit basis of l^2. Consider $F : X \to 2^Y$ defined by

$$F(x) := \begin{cases} \left\{\dfrac{1}{n}(-e_1+2e_n)\right\}, & \text{if } x = \dfrac{1}{n}, \\ \{y = (y_i)_{i=1}^{\infty} \in l^2 : y_1 \geq 0, y_1^2 \geq \sum_{i=2}^{\infty} y_i^2\}, & \text{if } x = n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $C := \{y = (y_i)_{i=1}^{\infty} \in l^2 : y_1 \geq 0, y_1^2 \geq \sum_{i=2}^{\infty} y_i^2\}$ is a closed, convex, and pointed cone. For $(x_0,y_0) = (0,0)$, we compute $T_{\mathrm{gr}F}^{r(1)}(x_0,y_0)$. If $(u,v) \in T_{\mathrm{gr}F}^{r(1)}(x_0,y_0)$, by definition, there exist $t_n > 0$ and $(u_n,v_n) \to (u,v)$ such that

$$t_n v_n \in F(t_n u_n). \tag{5.2}$$

If $t_n u_n \notin \{1/n, n\}$, then from (5.2) a direct computation gives $v = 0$. Now assume that $t_n u_n \in \{1/n, n\}$. Consider the first case with $t_n u_n = 1/n$. From (5.2), one has

$$t_n v_n = \frac{1}{n}(-e_1+2e_n). \tag{5.3}$$

We have two subcases. If $u = 0$, then $u_n = 1/(nt_n) \to 0$, and hence (5.3) implies that $v_n = u_n(-e_1+2e_n) \to 0$, i.e., $v = 0$. In the second subcase with $u > 0$, we claim that there is no v such that $(u,v) \in T^{r(1)}_{\mathrm{gr}F}(x_0,y_0)$. Suppose there exists such a (u,v) (with $u>0$). Then, from (5.3), the sequence $(nt_n)^{-1}(-e_1+2e_n) = u_n(-e_1+2e_n)$ converges. Hence, as $u_ne_1 \to ue_1$, the sequence $d_n := 2u_ne_n$ is also convergent. Suppose that d_n converges to d. By a direct computation, one has

$$||2u_ne_n - d||^2 = (2u_n)^2 + ||d||^2 + 2\langle 2u_ne_n, -d\rangle \to 0.$$

Therefore, $4(u)^2 + ||d||^2 = 0$ (since $\{e_n\}$ converges weakly to 0, $\langle e_n, d\rangle \to 0$), which is a contradiction.

Now consider the second case with $t_nu_n = n$. From (5.2) we get $t_nv_n \in C$. Thus, $v \in C$ since C is a closed cone. It follows from $u_n = n/t_n$ that $u \geq 0$. Consequently, we have proved that

$$T^{r(1)}_{\mathrm{gr}F}(x_0,y_0) \subseteq ([0,+\infty)\times C) \cup ((-\infty,0)\times\{0\}).$$

We now show the reverse inclusion. Let $(u,v) \in ([0,+\infty)\times C)\cup((-\infty,0)\times\{0\})$. We prove that there exist $t_n > 0$, $u_n \to u$, and $v_n \to v$ such that (5.2) holds for all n. Indeed, depending on u and v, such t_n, u_n, and v_n can be chosen as follows.

- For $(0,v)$ such that $v \in C$, we take $t_n = n^2$, $u_n = 1/n$, $v_n \equiv v$.
- For $(u,v) \in (0,+\infty)\times C$, we take $t_n = n/u$, $u_n \equiv u$, $v_n \equiv v$.
- For $(u,v) \in (-\infty,0)\times\{0\}$, we take $t_n = n/|u|$, $u_n \equiv u$, $v_n \equiv 0$.

So,

$$T^{r(1)}_{\mathrm{gr}F}(x_0,y_0) = ([0,+\infty)\times C)\cup((-\infty,0)\times\{0\}).$$

Therefore,

$$D^1_RF(x_0,y_0)(u) = \begin{cases} C, & \text{if } u \geq 0,\\ \{0\}, & \text{if } u < 0.\end{cases}$$

By a similar way, with simpler calculations, we get $T^1_{\mathrm{gr}F}(x_0,y_0) = \mathbb{R}\times\{0\}$, and hence for all $u \in \mathbb{R}$, $D^1F(x_0,y_0)(u) = \{0\}$.

The next example highlights detailed differences between (5.1) and the radial derivative introduced in Definition 5.2.2(i).

Example 5.2.5. Let $X = Y = \mathbb{R}$ and $F(x) = \{x^2\}$ and $(x_0,y_0) = (0,0)$. Direct calculations yield

$$\overline{D}^1_RF(x_0,y_0)(x) = D^1_RF(x_0,y_0)(x) = \mathbb{R}_+.$$

Without any information, we have $\overline{D}^2_RF(x_0,y_0)(x) = x^2$. Now let $(u_1,v_1) = (0,0)$ be given, then $D^2_RF(x_0,y_0,u_1,v_1)(x) = \mathbb{R}_+$. For another given direction $(u_1,v_1) = (1,0)$, $D^2_RF(x_0,y_0,u_1,v_1)(x) = \{1 + a^2x^2 + 2ax : a \geq 0\}$.

Definition 5.2.6. Let $F : X \to 2^Y$, $(x_0, y_0) \in \operatorname{gr} F$ and $(u_i, v_i) \in X \times Y$, $i = 1, ..., m-1$ with $m \geq 1$.

(i) If $T^{r(m)}_{F(X)}(y_0, v_1, ..., v_{m-1}) = T^{r\flat(m)}_{F(X)}(y_0, v_1, ..., v_{m-1})$, then this set is called a m-th order *proto-radial set* of $F(X)$ at y_0 with respect to $v_1, ..., v_{m-1}$.

(ii) If $D^m_R F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) = \{v \in Y : \forall t_n > 0, \forall x_n \to x, \exists v_n \to v : y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(y_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m x_n), \forall n\}$, for any $x \in \operatorname{dom} D^m_R F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$, then this derivative is called a m-th order *radial semiderivative* of F at (x_0, y_0) with respect to $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$.

Note that, following strictly Definition 5.2.2, we would define that $D^m_R F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ is a m-th order radial semiderivative if

$$\operatorname{gr} D^m_R F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^{r\flat(m)}_{\operatorname{gr} F}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

But, this last condition is equivalent to

$$D^m_R F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) = \{v \in Y : \forall t_n > 0, \exists x_n \to x, \exists v_n \to v, \forall n, y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in F(x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m x_n)\},$$

which is weaker than Definition 5.2.6(ii). (This weaker condition was used to define proto-contingent derivatives in many papers in the literature.) Definition 5.2.6 is restrictive. However, it may be satisfied as shown in the following.

Example 5.2.7. Let $X = Y = \mathbb{R}$, $F : X \to 2^Y$ be defined by $F(x) = \{y \in Y : y \geq x\}$, and $(x_0, y_0) = (0, 0)$. Then, direct calculations yield $T^{r(1)}_{F(X)}(y_0) = T^{r\flat(1)}_{F(X)}(y_0) = \mathbb{R}$ and

$$\begin{aligned} D^1_R F(x_0, y_0)(x) &= \{v \in Y : \forall t_n > 0, \forall x_n \to x, \exists v_n \to v, \forall n, y_0 + t_n v_n \in F(x_0 + t_n x_n)\} \\ &= \{v \in Y : v \geq x\}. \end{aligned}$$

So, $T^{r(1)}_{F(X)}(y_0)$ is a first-order proto-radial set of $F(X)$ at y_0, and $D^1_R F(x_0, y_0)$ is a first-order radial semiderivative of F at (x_0, y_0).

For examples of second-orders, with any $v_1 \in \mathbb{R}$, direct computations show that

$$T^{r(2)}_{F(X)}(y_0, v_1) = T^{r\flat(2)}_{F(X)}(y_0, v_1) = \mathbb{R}.$$

Thus, $T^{r(2)}_{F(X)}(y_0, v_1)$ is a second-order proto-radial set of $F(X)$ at y_0 with respect to $v_1 \in \mathbb{R}$.

Passing to derivatives, let $(u_1, v_1) = (1, 1)$. Direct computations indicate that both $D^2_R F(x_0, y_0, u_1, v_1)(x)$ and the set on the right-hand side of the equality in Definition 5.2.6(ii) are equal to

$\{v \in Y : v \geq x\}$. Thus, F has a second-order radial semiderivative at (x_0, y_0) with respect to $(1,1)$. However, for $(u_1, v_1) = (0, -1)$, the derivatives are "worse":

$$D_R^2 F(x_0, y_0, u_1, v_1)(x) = \{v \in Y : v \geq x\}$$

and the other mentioned set is empty. So, F does not have a second-order radial semiderivative at (x_0, y_0) with respect to $(0, -1)$.

Proposition 5.2.8. *If S is convex, then so is $T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1})$.*

Proof. If $T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1}) = \emptyset$ or is a singleton, the result holds trivially. Now assume that there are $v_1, v_2 \in T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1})$ and $\lambda \in (0,1)$. It follows from the definition that, for any $t_n > 0$, there exist sequences v_n^1 and v_n^2 such that $v_n^1 \to v^1$, $v_n^2 \to v^2$ and for $i = 1, 2$,

$$x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m v_n^i \in S.$$

From the convexity of S, we have

$$x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m \left(\lambda v_n^1 + (1-\lambda) v_n^2\right) \in S.$$

Thus, $\lambda v_1 + (1-\lambda) v_2 \in T_S^{r\flat(m)}(x_0, u_1, ..., u_{m-1})$ and the proof is complete. □

Proposition 5.2.9. *If S is convex and $u_1, ..., u_{m-1} \in S$, then*

$$T_S^{r\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0) \subseteq T_S^{\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0) =$$
$$= T_S^m(x_0, u_1 - x_0, ..., u_{m-1} - x_0) = T_S^{r(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0)$$

Proof. From the definitions, we have

$$T_S^{r\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0) \subseteq T_S^{\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0)$$

and

$$T_S^{r(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0) = \mathrm{cl}\left(\bigcup_{t>0} \frac{S - x_0 - t(u_1 - x_0) - ... - t^{m-1}(u_{m-1} - x_0)}{t^m}\right).$$

Since S is convex, Proposition 3.1 of [118] says that the right side of the last equality is equal to $T_S^m(x_0, u_1 - x_0, ..., u_{m-1} - x_0) = T_S^{\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0)$ and we are done □

The inclusion in Proposition 5.2.9 may be strict as for $S = [0,1]$ and $x_0 = 0$, since we have $T_S^{r\flat(1)}(x_0) = \{0\}$ and $T_S^{\flat(1)}(x_0) = T_S^1(x_0) = T_S^{r(1)}(x_0) = \mathbb{R}_+$.

It is known (Corollary 3.1 of [118]) that if S is convex and $u_1,...,u_{m-1} \in S$, then $T_S^{\flat(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0)$ is convex. Therefore, Proposition 5.2.9 implies the following.

Corollary 5.2.10. *If S is convex and $u_1,...,u_{m-1} \in S$, then $T_S^{r(m)}(x_0, u_1 - x_0, ..., u_{m-1} - x_0)$ is convex.*

Proposition 5.2.11. *Let $S = \mathrm{dom} F$ and $(x_0, y_0) \in \mathrm{gr} F$. Then, for all $x \in S$,*

(i) $F(x) - y_0 \subseteq D_R^1 F(x_0, y_0)(x - x_0)$,

(ii) $F(x) - y_0 \subseteq T_{F(S)}^{r(1)}(y_0)$.

Proof. Let $x \in S$, $y \in F(x) - y_0$, then $y_0 + y \in F(x)$. Therefore, there exist $t_n = 1, y_n = y$ and $x_n = x - x_0$, for all n, such that $y_0 + t_n^m y_n \in F(x_0 + t_n x_n)$ for all n. Hence, $y \in D_R^1 F(x_0, y_0)(x - x_0)$.

(ii) Let $x \in S$, $y \in F(x) - y_0$, then $y_0 + y \in F(x)$. Therefore, there exist $t_n = 1, y_n = y$ and $x_n = x - x_0$, $\forall n$ such that $y_0 + t_n^m y_n \in F(x_0 + t_n x_n) \subseteq F(S)$. So, $y \in T_{F(S)}^{r(1)}(y_0)$. □

Note that these assertions say, in particular, that radial sets and derivatives possess global properties without any (relaxed) convexity assumption. To make this clear, recall that $F : X \to 2^Y$ is termed pseudoconvex at $(x_0, y_0) \in \mathrm{gr}F$ if $\mathrm{epi}F - (x_0, y_0) \subseteq T_{\mathrm{epi}F}(x_0, y_0)$. Furthermore, if F is pseudoconvex at (x_0, y_0), then, $\forall x \in \mathrm{dom}F$, $F(x) - y_0 \subseteq V^1(F_+, x_0, y_0)$ (see Proposition 2.1 of Khanh and Tuan in [105]). Roughly speaking, that is why, in the sufficient condition (see Theorem 5.3.7), no convexity assumption is needed.

5.2.2 Sum rule and chain rule

Proposition 5.2.12. (Sum rule) *Let $F_i : X \to 2^Y$, $x_0 \in \Omega := \mathrm{dom} F_1 \cap \mathrm{dom} F_2$, $y_i \in F_i(x_0)$ for $i = 1, 2$.*

(i) *If either $F_1(\Omega)$ or $F_2(\Omega)$ has a m-th order proto-radial set at y_1 with respect to $v_{1,1},...,v_{1,m-1}$ or at y_2 with respect to $v_{2,1},...,v_{2,m-1}$, respectively, then*

$$T_{F_1(\Omega)}^{r(m)}(y_1, v_{1,1}, ..., v_{1,m-1}) + T_{F_2(\Omega)}^{r(m)}(y_2, v_{2,1}, ..., v_{2,m-1})$$

$$\subseteq T_{(F_1+F_2)(\Omega)}^{r(m)}(y_1 + y_2, v_{1,1} + v_{2,1}, ..., v_{1,m-1} + v_{2,m-1}).$$

(ii) *If either F_1 or F_2 has a m-th order radial semiderivative at (x_0,y_1) with respect to $(u_1,v_{1,1}),...,(u_{m-1},v_{1,m-1})$ or at (x_0,y_2) with respect to $(u_1,v_{2,1}),...,(u_{m-1},v_{2,m-1})$, respectively, then for any $u\in X$,*

$$D^m_RF_1(x_0,y_1,u_1,v_{1,1},...,u_{m-1},v_{1,m-1})(u)+D^m_RF_2(x_0,y_2,u_1,v_{2,1},...,u_{m-1},v_{2,m-1})(u)$$

$$\subseteq D^m_R(F_1+F_2)(x_0,y_1+y_2,u_1,v_{1,1}+v_{2,1},...,u_{m-1},v_{1,m-1}+v_{2,m-1})(u).$$

Proof. (i) Let $u_i\in T^{r(m)}_{F_i(\Omega)}(y_i,v_{i,1},...,v_{i,m-1})$ for $i=1,2$. Then, there exist $t_n>0$ and $u^1_n\to u_1$ such that

$$y_1+t_nv_{1,1}+...+t_n^{m-1}v_{1,m-1}+t_n^mu^1_n\in F_1(\Omega).$$

Suppose that $F_2(\Omega)$ has a m-th order proto-radial set at y_2. Then, with t_n above, there exists $u^2_n\to u_2$ such that

$$y_2+t_nv_{2,1}+...+t_n^{m-1}v_{2,m-1}+t_n^mu^2_n\in F_2(\Omega).$$

Thus,

$$(y_1+y_2)+t_n(v_{1,1}+v_{2,1})+...+t_n^{m-1}(v_{1,m-1}+v_{2,m-1})+t_n^m(u^1_n+u^2_n)\in(F_1+F_2)(\Omega).$$

Hence, $u_1+u_2\in T^{r(m)}_{(F_1+F_2)(\Omega)}(y_1+y_2,v_{1,1}+v_{2,1},...,v_{1,m-1}+v_{2,m-1})$.

(ii) If u does not belong to the intersection of $\mathrm{dom}\,D^m_RF_1(x_0,y_1,u_1,v_{1,1},...,u_{m-1},v_{1,m-1})$ and $\mathrm{dom}\,D^m_RF_2(x_0,y_1,u_1,v_{2,1},...,u_{m-1},v_{2,m-1})$, the conclusion is trivial. Suppose that $u\in\mathrm{dom}\,D^m_RF_i$ $(x_0,y_i,u_1,v_{i,1},...,u_{m-1},v_{i,m-1})$ for $i=1,2$ and let $v_i\in D^m_RF_i(x_0,y_i,u_1,v_{i,1},...,u_{m-1},v_{i,m-1})(u)$ for $i=1,2$. Then, there exist $t_n>0$ and $(u_n,v^1_n)\to(u,v_1)$ such that

$$y_1+t_nv_{1,1}+...+t_n^{m-1}v_{1,m-1}+t_n^mv^1_n\in F_1(x_0+t_nu_1+...+t_n^{m-1}u_{m-1}+t_n^mu_n).$$

For v_2, suppose that F_2 has a m-th order radial semiderivative at (x_0,y_2), with t_n,u_n above, there exists $v^2_n\to v_2$ such that

$$y_2+t_nv_{2,1}+...+t_n^{m-1}v_{2,m-1}+t_n^mv^2_n\in F_2(x_0+t_nu_1+...+t_n^{m-1}u_{m-1}+t_n^mu_n).$$

Thus,

$$(y_1+y_2)+t_n(v_{1,1}+v_{2,1})+...+t_n^{m-1}(v_{1,m-1}+v_{2,m-1})+t_n^m(v^1_n+v^2_n)$$

$$\in(F_1+F_2)(x_0+t_nu_1+...+t_n^{m-1}u_{m-1}+t_n^mu_n).$$

Hence, $v_1+v_2\in D^m_R(F_1+F_2)(x_0,y_1+y_2,u_1,v_{1,1}+v_{2,1},...,u_{m-1},v_{1,m-1}+v_{2,m-1})(u)$. □

The following example shows that the assumption about proto-radial sets in Proposition 5.2.12 cannot be dropped.

Example 5.2.13. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, and $F_1, F_2 : X \to 2^Y$ be given by

$$F_1(x) = \begin{cases} \{1\}, \text{ if } x = \dfrac{1}{n}, n \in \mathbb{N}, \\ \{0\}, \text{ if } x = 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{0\}, \text{ if } x = \dfrac{1}{n}, n \in \mathbb{N}, \\ \{1\}, \text{ if } x = 0. \end{cases}$$

It is easy to see that F_1 and F_2 have neither first order proto-radial set (of $F_1(\Omega)$ and $F_2(\Omega)$) nor first order radial semiderivative at (0,0) and (0,1), respectively. We have

$$T^{r(1)}_{F_1(\Omega)}(0) = \mathbb{R}_+,\ T^{r(1)}_{F_2(\Omega)}(1) = \mathbb{R}_-,$$

where $\Omega := \operatorname{dom} F_1 = \operatorname{dom} F_2 = \left\{0, \dfrac{1}{n}\right\}_{n \in \mathbb{N}}$, and

$$D^1_R F_1(0,0)(0) = \mathbb{R}_+,\ D^1_R F_2(0,1)(0) = \mathbb{R}_-.$$

On the other hand,

$$(F_1 + F_2)(x) = \begin{cases} \{1\}, \text{ if } x = \dfrac{1}{n}, n \in \mathbb{N}, \\ \{1\}, \text{ if } x = 0. \end{cases}$$

Direct calculations yield

$$T^{r(1)}_{(F_1+F_2)(\Omega)}(1) = \{0\},\ D^1_R(F_1 + F_2)(0,1)(0) = \{0\}.$$

Thus,

$$T^{r(1)}_{F_1(\Omega)}(0) + T^{r(1)}_{F_2(\Omega)}(1) \not\subseteq T^{r(1)}_{(F_1+F_2)(\Omega)}(1)$$

and

$$D^1_R F_1(0,0)(0) + D^1_R F_2(0,1)(0) \not\subseteq D^1_R(F_1 + F_2)(0,1)(0).$$

Proposition 5.2.14. (Chain rule) *Let $G : X \to 2^Y$, $F : Y \to 2^Z$ with $\operatorname{Im} G \subseteq \operatorname{dom} F$, $(x_0, y_0) \in \operatorname{gr} G$, $(y_0, z_0) \in \operatorname{gr} F$ and $(u_1, v_1, w_1), ..., (u_{m-1}, v_{m-1}, w_{m-1}) \in X \times Y \times Z$. Suppose that F has a m-th order radial semiderivative at (y_0, z_0) with respect to $(v_1, w_1), ..., (v_{m-1}, w_{m-1})$. Then*

(i) $D^m_R F(y_0, z_0, v_1, w_1, ..., v_{m-1}, w_{m-1})[T^{r(m)}_{G(X)}(y_0, v_1, ..., v_{m-1})] \subseteq T^{r(m)}_{(F \circ G)(X)}(z_0, w_1, ..., w_{m-1})$,

(ii) $D^m_R F(y_0, z_0, v_1, w_1, ..., v_{m-1}, w_{m-1})[D^m_R G(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(X)]$
$$\subseteq T^{r(m)}_{(F \circ G)(X)}(z_0, w_1, ..., w_{m-1}).$$

Proof. (i) Let $z \in D_R^m F(y_0, z_0, v_1, w_1, ..., v_{m-1}, w_{m-1})[T_{G(X)}^{r(m)}(y_0, v_1, ..., v_{m-1})]$. There exists $v \in T_{G(X)}^{r(m)}(y_0, v_1, ..., v_{m-1})$ such that $z \in D_R^m F(y_0, z_0, v_1, w_1, ..., v_{m-1}, w_{m-1})(v)$. Hence, there exist $t_n > 0$ and $v_n \to v$ such that

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n \in G(X).$$

Because F has a m-th order radial semiderivative of F at (y_0, z_0) with respect to $(v_1, w_1), ...,$ (v_{m-1}, w_{m-1}), there exists $z_n \to z$ such that

$$z_0 + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m z_n \in F(y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m v_n) \subseteq F(G(X)).$$

Thus, $z \in T_{(F \circ G)(X)}^{r(m)}(z_0, w_1, ..., w_{m-1})$.

(ii) This follows from (i) and Remark 5.2.3(i). □

These rules will be applied in the sequel since they are simple (at least their formulations are). However, being a proto-radial set or radial semiderivative is a restrictive condition. Hence, we develop now another sum rule and another chain rule for possible better applications. For a sum $M + N$ of two multimaps $M, N : X \to 2^Y$, we express it as a composition as follows. Define $G : X \to 2^{X \times Y}$ and $F : X \times Y \to 2^Y$ by, for the identity map I on X,

$$G = I \times M \qquad \text{and} \qquad F(x, y) = y + N(x). \tag{5.4}$$

Then, clearly $M + N = F \circ G$. So, we will apply a chain rule. The chain rule given in Proposition 5.2.14, though simple and relatively direct, is not suitable for dealing with this composition $F \circ G$, since the intermediate space (Y there and $X \times Y$ here) is little involved. We develop another chain rule as follows. Let general multimaps $G : X \to 2^Y$ and $F : Y \to 2^Z$ be considered, where X, Y, Z be normed spaces. The so-called resultant multimap $C : X \times Z \to 2^Y$ is defined by $C(x, z) := G(x) \cap F^{-1}(z)$. Then, $\operatorname{dom} C = \operatorname{gr}(F \circ G)$.

We can obtain a general chain rule suitable for dealing with a sum expressed as a composition as above, and without assumption about radial semiderivatives, as follows.

Proposition 5.2.15. *Let* $(x_0, z_0) \in \operatorname{gr}(F \circ G)$, $y_0 \in C(x_0, z_0)$, *and* $(u_i, v_i, w_i) \in X \times Y \times Z$.

(i) *If, for all* $w \in Z$, *one has*

$$T_{G(X)}^{r(m)}(y_0, v_1, ..., v_{m-1}) \cap D_R^m F^{-1}(z_0, y_0, w_1, v_1, ..., w_{m-1}, v_{m-1})(w)$$
$$\subseteq D_R^m C_X(z_0, y_0, w_1, v_1, \cdots, w_{m-1}, v_{m-1})(w), \tag{5.5}$$

where $C_X : Z \to 2^Y$ is defined by $C_X(z) := C(X,z)$, then

$$D_R^m F(y_0,z_0,v_1,w_1...,v_{m-1},w_{m-1})[T^{r(m)}_{G(X)}(y_0,v_1,...,v_{m-1})] \subseteq T^{r(m)}_{(F\circ G)(X)}(z_0,w_1,...,w_{m-1}).$$

(ii) *If, for all $(u,w) \in X \times Z$, one has*

$$D_R^m G(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(u) \cap D_R^m F^{-1}(z_0,y_0,w_1,v_1,...,w_{m-1},v_{m-1})(w)$$

$$\subseteq D_R^m C((x_0,z_0),y_0,(u_1,w_1),v_1,...,(u_{m-1},w_{m-1}),v_{m-1})(u,w), \tag{5.6}$$

then

$$D_R^m F(y_0,z_0,v_1,w_1...,v_{m-1},w_{m-1})[D_R^m G(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(u)]$$

$$\subseteq D_R^m (F\circ G)(x_0,z_0,u_1,w_1,...,u_{m-1},w_{m-1})(u).$$

Proof. By the similarity, we prove only (i). Let $w \in D_R^m F(y_0,z_0,v_1,w_1...,v_{m-1},w_{m-1})[T^{r(m)}_{G(X)}(y_0,v_1, ...,v_{m-1})]$, i.e., there exists some $y \in T^{r(m)}_{G(X)}(y_0,v_1,...,v_{m-1})$ such that $y \in D_R^m F^{-1}(z_0,y_0,w_1,v_1,..., w_{m-1},v_{m-1})(w)$. Then, (5.5) ensures that $y \in D_R^m C_X(y_0,z_0,w_1,v_1,\cdots,w_{m-1},v_{m-1})(w)$. This means the existence of $t_n > 0$ and $(y_n,w_n) \to (y,w)$ such that, for all $n \in \mathbb{N}$,

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m y_n \in C(X, z_0 + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m w_n).$$

From the definition of the map C, we get $z_0 + t_n w_1 + ... + t_n^{m-1} w_{m-1} + t_n^m w_n \in (F\circ G)(X)$. So, $w \in T^{r(m)}_{(F\circ G)(X)}(z_0,w_1,...,w_{m-1})$. $\square$

We now show the essentialness of the assumption (5.6) in Proposition 5.2.15 (it is similar for (5.5)) by the following.

Example 5.2.16. Let $X = Y = Z = \mathbb{R}$, $G : X \to 2^Y$ and $F : Y \to 2^Z$ be defined by

$$G(x) = \begin{cases} \{1,2\}, & \text{if } x = 1, \\ \{0\}, & \text{if } x = 0, \end{cases} \quad F(y) = \begin{cases} \{0\}, & \text{if } y = 1, \\ \{1\}, & \text{if } y = 0. \end{cases}$$

Then,

$$(F\circ G)(x) = \begin{cases} \{0\}, & \text{if } x = 1, \\ \{1\}, & \text{if } x = 0, \end{cases} \quad F^{-1}(z) = \begin{cases} \{0\}, & \text{if } z = 1, \\ \{1\}, & \text{if } z = 0, \end{cases}$$

$$C(x,z) = G(x) \cap F^{-1}(z) = \begin{cases} \{1\}, & \text{if } (x,z) = (1,0), \\ \{0\}, & \text{if } (x,z) = (0,1). \end{cases}$$

Let $(x_0,z_0)=(0,1)$ and $y_0=0\in C(x_0,z_0)$. Direct calculations give

$$D_R^1G(x_0,y_0)(1/2)=\{1/2,1\},\ D_R^1F(y_0,z_0)(1)=\{-1\},$$

$$D_R^1F(y_0,z_0)(1/2)=\{-1/2\},\ D_R^1(F\circ G)(x_0,z_0)(1/2)=\{-1/2\}.$$

So, the conclusion of Proposition 5.2.15(ii) does not hold. The reason is the violence of (5.6): for $(u,w)=(1/2,-1)$, $D_R^1F^{-1}(z_0,y_0)(-1)=\{1\}$, $D_R^1C((x_0,z_0),y_0)(u,w)=\emptyset$, and hence

$$D_R^1G(x_0,y_0)(u)\cap D_R^1F^{-1}(z_0,y_0)(w)\not\subseteq D_R^1C((x_0,z_0),y_0)(u,w).$$

Now we apply the preceding composition rule to establish a sum rule for $M,N:X\to 2^Y$. For this purpose, we use $G:X\to 2^{X\times Y}$ and $F:X\times Y\to 2^Y$ defined in (5.4). For $(x,z)\in X\times Y$, we set $H(x,z):=M(x)\cap(z-N(x))$. Then, the resultant multimap $C:X\times Y\to 2^{X\times Y}$ associated to these F and G is $C(x,z)=\{x\}\times H(x,z)$.

Proposition 5.2.17. *Let $(x_0,z_0)\in \mathrm{gr}\,(M+N)$, $y_0\in H(x_0,z_0)$ and $(u_i,v_i,w_i)\in X\times Y\times Y$.*

(i) *If, for all $w\in Y$, one has*

$$T_{M(X)}^{r(m)}(y_0,v_1,...,v_{m-1})\cap[w-T_{N(X)}^{r(m)}(z_0-y_0,w_1,...,w_{m-1})]$$
$$\subseteq D_R^mH_X(z_0,y_0,v_1+w_1,v_1,\cdots,v_{m-1}+w_{m-1},v_{m-1})(w), \tag{5.7}$$

where $H_X:Y\to 2^Y$ is defined by $H_X(y):=H(X,y)$, then

$$T_{M(X)}^{r(m)}(y_0,v_1,...,v_{m-1})+T_{N(X)}^{r(m)}(z_0-y_0,w_1,...,w_{m-1})$$
$$\subseteq T_{(M+N)(X)}^{r(m)}(z_0,v_1+w_1,\cdots,v_{m-1}+w_{m-1}).$$

(ii) *If, for all $(u,w)\in X\times Y$, one has*

$$D_R^mM(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(u)\cap[w-D_R^mN(x_0,z_0-y_0,u_1,w_1,...,u_{m-1},w_{m-1})(u)$$
$$\subseteq D_R^mH((x_0,z_0),y_0,(u_1,v_1+w_1),v_1,...,(u_{m-1},v_{m-1}+w_{m-1}),v_{m-1})(u,w), \tag{5.8}$$

then

$$D_R^mM(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(u)+D_R^mN(x_0,z_0-y_0,u_1,w_1,...,u_{m-1},w_{m-1})(u)$$
$$\subseteq D_R^m(M+N)(x_0,z_0,u_1,v_1+w_1,...,u_{m-1},v_{m-1}+w_{m-1})(u).$$

Proof. By the similarity, we prove only (ii). Let $w \in D_R^m M(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(u) + D_R^m N(x_0, z_0 - y_0, u_1, w_1, ..., u_{m-1}, w_{m-1})(u)$, i.e., there exists $y \in D_R^m M(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(u)$ such that $y \in w - D_R^m N(x_0, z_0 - y_0, u_1, w_1, ..., u_{m-1}, w_{m-1})(u)$. Then, (5.8) ensures that $y \in D_R^m H((x_0, z_0), y_0, (u_1, v_1 + w_1), v_1, ..., (u_{m-1}, v_{m-1} + w_{m-1}), v_{m-1})(u, w)$. This means the existence of $t_n > 0$ and $(u_n, y_n, w_n) \to (u, y, w)$ such that

$$y_0 + t_n v_1 + ... + t_n^{m-1} v_{m-1} + t_n^m y_n \in$$

$$H(x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m u_n, z_0 + t_n(v_1 + w_1) + ... + t_n^{m-1}(v_{m-1} + w_{m-1}) + t_n^m w_n).$$

From the definition of H, we get

$$z_0 + t_n(v_1 + w_1) + ... + t_n^{m-1}(v_{m-1} + w_{m-1}) + t_n^m w_n \in (M + N)(x_0 + t_n u_1 + ... + t_n^{m-1} u_{m-1} + t_n^m u_n).$$

So, $w \in D_R^m (M + N)(x_0, z_0, u_1, v_1 + w_1, ..., u_{m-1}, v_{m-1} + w_{m-1})(u)$. □

The following example shows that assumptions (5.7) and (5.8) cannot be dispensed and are not difficult to check.

Example 5.2.18. Let $X = Y = \mathbb{R}$ and $M, N : X \to 2^Y$ be given by

$$M(x) = \begin{cases} \{1\}, \text{ if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \{0\}, \text{ if } x = 0, \end{cases} \quad N(x) = \begin{cases} \{0\}, \text{ if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \{1\}, \text{ if } x = 0. \end{cases}$$

Then,

$$H(x, z) = M(x) \cap (z - N(x)) = \begin{cases} \{0\}, \text{ if } (x, z) = (0, 1), \\ \{1\}, \text{ if } (x, z) = \left(\dfrac{1}{n}, 1\right),\ n \in \mathbb{N}, \\ \emptyset, \quad \text{otherwise}, \end{cases}$$

$$(M + N)(x) = \begin{cases} \{1\}, \text{ if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \{1\}, \text{ if } x = 0. \end{cases}$$

Choose $x_0 = 0$, $z_0 = 1$, $y_0 = 0 \in \operatorname{cl} H(x_0, z_0)$ and $u = w = 0$. Then, one can easily show

$$T_{M(X)}^{r(1)}(y_0) = \mathbb{R}_+,\ T_{N(X)}^{r(1)}(z_0 - y_0) = \mathbb{R}_-,\ D_R^1 H_X(z_0, y_0)(w) = \{0\},$$

$$D_R^1 M(x_0, y_0)(u) = \mathbb{R}_+,\ D_R^1 N(x_0, z_0 - y_0)(u) = \mathbb{R}_-,\ D_R^1 H((x_0, z_0), y_0)(u, w) = \{0\}.$$

Thus, (5.7) and (5.8) are violated:

$$T^{r(1)}_{M(X)}(y_0)\cap[w-T^{r(1)}_{N(X)}(z_0-y_0)]\not\subseteq D^1_RH_X(z_0,y_0)(w),$$

$$D^1_RM(x_0,y_0)(u)\cap[w-D^1_RN(x_0,z_0-y_0)(u)]\not\subseteq D^1_RH((x_0,z_0),y_0)(u,w).$$

Direct computations show that conclusions of Proposition 5.2.17 do not hold:

$$T^{r(1)}_{M(X)}(y_0)+T^{r(1)}_{N(X)}(z_0-y_0)\not\subseteq T^{r(1)}_{(M+N)(X)}(z_0),$$

$$D^1_RM(x_0,y_0)(u)+D^1_RN(x_0,z_0-y_0)(u)\not\subseteq D^1_R(M+N)(x_0,z_0)(u),$$

since $T^{r(1)}_{(M+N)(X)}(z_0)=\{0\}$ and $D^1_R(M+N)(x_0,z_0)(u)=\{0\}$.

5.3 Optimality conditions

Let X, Y and Z be normed spaces, $C\subseteq Y$ and $D\subseteq Z$ be pointed closed convex cones, not the entire space, $S\subseteq X$ nonempty, and $F:S\to 2^Y$, $G:S\to 2^Z$. Our problem is

$$\text{(P)}\qquad\qquad \mathrm{Min}_Q F(x),\ \text{s.t. } x\in S,\ G(x)\cap -D\neq\emptyset.$$

We denote $A:=\{x\in S: G(x)\cap -D\neq\emptyset\}$ (the feasible set).

In this section, both necessary and sufficient optimality conditions for the mentioned efficient solutions of the problem (P) are established. As Q-efficiency (see Definition 2.2.7) includes many other kinds of solutions as particular cases (see Proposition 2.2.9), we first prove necessary optimality conditions of this notion.

Proposition 5.3.1. *Let $(x_0,y_0)\in \mathrm{gr}F$ be a Q-efficient solution of* (P)*, $(u_i,v_i,w_i)\in X\times(-C)\times(-D)$, $i=1,...,m-1$, and $z_0\in G(x_0)\cap -D$. Suppose that the open cone Q satisfies $Q+C\subseteq Q$. Then, the following separations hold*

$$T^{r(m)}_{(F,G)_+(S)}(y_0,z_0,(v_1,w_1),...,(v_{m-1},w_{m-1}))\cap(-Q\times-\mathrm{int}D)=\emptyset, \tag{5.9}$$

$$D^m_R(F,G)_+(x_0,y_0,z_0,(u_1,v_1,w_1),...,(u_{m-1},v_{m-1},w_{m-1}))(X)\cap(-Q\times-\mathrm{int}D)=\emptyset. \tag{5.10}$$

Proof. Suppose (5.9) does not hold. Then, there exists (y,z) such that

$$(y,z)\in T^{r(m)}_{(F,G)_+(S)}(y_0,z_0,(v_1,w_1),...,(v_{m-1},w_{m-1})), \tag{5.11}$$

$$(y,z)\in(-Q\times-\mathrm{int}D). \tag{5.12}$$

It follows from (5.11) and the definition of m-th order radial sets that there exist sequences $t_n > 0$, $x_n \in S$, and $(y_n, z_n) \in (F,G)(x_n) + C \times D$ such that

$$\frac{(y_n, z_n) - (y_0, z_0) - t_n(v_1, w_1) - \dots - t_n^{m-1}(v_{m-1}, w_{m-1})}{t_n^m} \to (y, z). \tag{5.13}$$

From (5.12) and (5.13), one has, for large n,

$$y_n - y_0 - t_n v_1 - \dots - t_n^{m-1} v_{m-1} \in -Q,\ z_n \in -\mathrm{int} D. \tag{5.14}$$

As $v_1, \dots, v_{m-1} \in -C$, (5.14) implies that, for large n,

$$y_n - y_0 \in -Q. \tag{5.15}$$

Because $z_n \in G(x_n) + D$, there exist $\overline{z_n} \in G(x_n)$ and $d_n \in D$ such that $z_n = \overline{z_n} + d_n$. (5.14) implies also that $\overline{z_n} \in G(x_n) \cap (-D)$ for large n, and then $x_n \in A$. Because $y_n \in F(x_n) + C$, there exist $\overline{y_n} \in F(x_n)$ and $c_n \in C$ such that $y_n = \overline{y_n} + c_n$. Then, (5.15) implies that $\overline{y_n} - y_0 \in -Q$ for large n. Therefore, $\overline{y_n} - y_0 \in (F(A) - y_0) \cap (-Q)$, which contradicts the Q-efficiency of (x_0, y_0). Thus, (5.9) holds. (5.10) follows from (5.9) and the evident fact that

$$D_R^m F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(X) \subseteq T_{F(X)}^{r(m)}(y_0, v_1, \dots, v_{m-1}).$$

□

Propositions 5.3.1 and 2.2.9 together yield the following result.

Theorem 5.3.2. *Let $(x_0, y_0) \in \mathrm{gr} F$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, \dots, m-1$, and $z_0 \in G(x_0) \cap -D$. Then, (5.9) and (5.10) hold in each of the following cases*

(i) (x_0, y_0) *is a weak efficient solution of* (P) *and* $Q = \mathrm{int} C$,

(ii) (x_0, y_0) *is a strong efficient solution of* (P) *and* $Q = Y \setminus -C$,

(iii) (x_0, y_0) *is a positive-proper efficient solution of* (P) *and* $Q = \{y : \varphi(y) > 0\}$ *for some functional* $\varphi \in C^{+i}$,

(iv) (x_0, y_0) *is a Geoffrion-proper efficient solution of* (P) *and* $Q = C(\varepsilon)$ *for* $\varepsilon > 0$,

(v) (x_0, y_0) *is a Henig-proper efficient solution of* (P) *and* $Q = K$ *for some pointed open convex cone* K *dilating* C,

(vi) (x_0, y_0) *is a strong Henig-proper efficient solution of* (P) *and* $Q = \mathrm{int} C_\varepsilon(B)$ *for* ε *satisfying* $0 < \varepsilon < \delta$.

The next example illustrates Theorem 5.3.2(vi).

Example 5.3.3. Let $X = Y = Z = \mathbb{R}$, $S = X$, $C = D = \mathbb{R}_+$, $G(x) \equiv \mathbb{R}$, and $F(x) = \{y \in Y : y \geq |x|\}$. Choose the base $B = \{1\}$. Then, $\delta = 1$ and $C_\varepsilon(B) = \mathbb{R}_+$ for all $\varepsilon \in (0, \delta)$. Let $(x_0, y_0) = (0,0)$ and $z_0 = 0$. It is easy to see that (x_0, y_0) is a strong Henig-proper efficient solution. For any $(v_1, w_1) \in -(C \times D)$, direct computations give

$$T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) = \mathbb{R}_+ \times \mathbb{R},\ \ T^{r(2)}_{(F,G)_+(S)}(y_0, z_0, v_1, w_1) = \mathbb{R}_+ \times \mathbb{R},$$

and hence

$$T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) \cap -\mathrm{int}(C_\varepsilon(B) \times D) = \emptyset,\ T^{r(2)}_{(F,G)_+(S)}(y_0, z_0, v_1, w_1) \cap -\mathrm{int}(C_\varepsilon(B) \times D) = \emptyset,$$

i.e., the necessary optimality condition of Theorem 5.3.2(vi) holds.

We now compare Theorem 5.3.2 with some known results. Because variational sets (see Section 4.2), played the role of generalized derivatives (see [105, 106]), are bigger than the image of X through most of kinds of generalized derivatives (see Remark 2.1 and Proposition 4.1 in [105]), optimality conditions (obtained by separating sets as usual) in terms of these variational sets are strong. So, we will compare our results with those using variational sets. However, in general, these sets are incomparable with radial sets as follows

Example 5.3.4. Suppose that $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $(x_0, y_0) = (0, (0,0))$, and $F : X \to 2^Y$ be defined by

$$F(x) = \begin{cases} \{(0,0)\}, & \text{if } x = 0, \\ \{(1,0)\}, & \text{otherwise.} \end{cases}$$

Then,

$$W^1(F, x_0, y_0) = T^{r(1)}_{F(X)}(x_0, y_0) = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_2 = 0\}.$$

Let $v_1 = (1,0)$. Then,

$$W^2(F, x_0, y_0, v_1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0\},$$

$$T^{r(2)}_{F(X)}(y_0, v_1) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2^2 + y_2, y_2 \leq 0\}.$$

Hence, the latter may be more advantageous in cases as ensured by the following.

Example 5.3.5. Let $X = Y = Z = \mathbb{R}$, $S = \{0, 1\}$, $C = D = \mathbb{R}_+$, $(x_0, y_0) = (0,0)$, G and F be

$$G(x) = F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \{-1\}, & \text{if } x = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Choose the base $B = \{1\}$. Then, $\delta = 1$ and $C_\varepsilon(B) = \mathbb{R}_+$ for all $\varepsilon \in (0, \delta)$. Let $z_0 = 0$. Let us try to use optimality conditions given in [106] by Khanh and Tuan in terms of variational sets to eliminate (x_0, y_0) as a candidate for a strong Henig-proper efficient solution. We can compute directly that $V^1((F,G)_+, x_0, y_0, z_0) = \mathbb{R}_+ \times \mathbb{R}_+$. Let $(v_1, w_1) \in V^1((F,G)_+, x_0, y_0, z_0) \cap -\mathrm{bd}(C_\varepsilon(B) \times D(z_0)), \cdots, (v_{m-1}, w_{m-1}) \in V^{m-1}((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-2}, w_{m-2}) \cap -\mathrm{bd}(C_\varepsilon(B) \times D(z_0))$, for $m \geq 2$, where bd A means the boundary of A. It is easy to check that $(v_1, w_1) = \cdots = (v_{m-1}, w_{m-1}) = (0, 0)$ and

$$V^m((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-1}, w_{m-1}) = \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, for all $m \geq 1$, we get

$$V^m((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-1}, w_{m-1}) \cap -\mathrm{int}(C_\varepsilon(B) \times D(z_0)) = \emptyset.$$

For variational sets of type 2, by direct calculating we get

$$W^1((F,G)_+, x_0, y_0, z_0) = \mathbb{R}_+ \times \mathbb{R}_+,$$

$$W^1((F,G)_+, x_0, y_0, z_0) \cap -\mathrm{int}(C_\varepsilon(B) \times D) = \emptyset.$$

Let $(v_1, w_1) \in W^1((F,G)_+, x_0, y_0, z_0) \cap -\mathrm{bd}(C_\varepsilon(B) \times D)$, $(v_2, w_2) \in W^2((F,G)_+, x_0, y_0, z_0, v_1, w_1) \cap -\mathrm{bd}(C_\varepsilon(B) \times D(w_1)), \cdots, (v_{m-1}, w_{m-1}) \in W^{m-1}((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-2}, w_{m-2}) \cap -\mathrm{bd}(C_\varepsilon(B) \times D(w_1))$, $m \geq 3$. We have $(v_1, w_1) = \cdots = (v_{m-1}, w_{m-1}) = (0, 0)$ and

$$W^m((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-1}, w_{m-1}) = \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, for all $m \geq 2$, we get

$$W^m((F,G)_+, x_0, y_0, z_0, v_1, w_1, \cdots, v_{m-1}, w_{m-1}) \cap -\mathrm{int}(C_\varepsilon(B) \times D(w_1)) = \emptyset.$$

So, Theorems 3.4 and 3.5 of Khanh and Tuan in [106] say nothing about (x_0, y_0) being strong Henig-proper efficient or not. By virtue of Remark 3.3 in [106] and Proposition 2.2 in [106], we can see that Theorems 4.1, 4.2, 5.1, and 5.2 of Li and Chen in [115], Theorem 3.1, Proposition 3.1 of Gong et al. in [80] and Theorem 1 of Liu and Gong in [119] cannot be in use to reject

(x_0, y_0) either. On the other hand, since $T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) = \mathbb{R} \times \mathbb{R}$, Theorem 5.3.2(vi) rejects the candidate (x_0, y_0).

Moreover, Theorem 3.2(i) of Khanh and Tuan in [105] says nothing about (x_0, y_0) being weak efficiency for (P). By Proposition 4.1 in [105], we can see that Theorem 7 of Jahn and Rauh in [96], Theorem 5 of Chen and Jahn in [26], Theorem 4.1 of Corley in [33], Proposition 3.1 of Taa in [160], Theorem 2.7(a) of Jahn and Khan in [93], Theorem 2 of Crepsi et al. in [34], Theorem 4.1 of Crepsi et al. in [35] and Theorem 3.1 of Jahn at al. in [95] cannot be in use to reject (x_0, y_0) either. On the other hand, by using Theorem 5.3.2(i), (x_0, y_0) is not a weak efficiency for (P).

Finally we discuss sufficient conditions for the mentioned efficient solutions of problem (P).

Proposition 5.3.6. *Let $(x_0, y_0) \in \operatorname{gr} F$ and $x_0 \in A$, the feasible set. Suppose that there exists $z_0 \in G(x_0) \cap (-D)$ such that, for $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, ..., m-1$, and $x \in S$, either of the following separations holds*

$$T^{r(m)}_{(F,G)_+(S)}((y_0, z_0), (v_1, w_1), ..., (v_{m-1}, w_{m-1})) \cap -(Q \times D(z_0)) = \emptyset, \tag{5.16}$$

$$D^m_R(F,G)_+(x_0, y_0, z_0, u_1, v_1, w_1, ..., u_{m-1}, v_{m-1}, w_{m-1})(x - x_0) \cap -(Q \times D(z_0)) = \emptyset. \tag{5.17}$$

Then, (x_0, y_0) is a Q-efficient solution of (P)*, for any non-empty open cone Q.*

Proof. By the similarity, we prove only (5.16). Note that (5.16) is required to be satisfied also for $v_i = 0 \in -C$ and $w_i = 0 \in -D$, $i = 1, ..., m-1$. Therefore, $T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) \cap -(Q \times D(z_0)) = \emptyset$. It follows from Proposition 5.2.11 that $(y - y_0, z - z_0) \in T^{r(1)}_{(F,G)_+(S)}(y_0, z_0)$ for all $y \in F(S)$, $z \in G(S)$. Then,

$$(F,G)(S) - (y_0, z_0) \cap -(Q \times D(z_0)) = \emptyset.$$

Suppose the existence of $x \in A$ and $y \in F(x)$ such that $y - y_0 \in -Q$. Then, there exists $z \in G(x) \cap -D$ such that $(y, z) - (y_0, z_0) \in -(Q \times D(z_0))$, a contradiction. $\square$

From Propositions 5.3.6 and 2.2.9, we obtain immediately the following result.

Theorem 5.3.7. *Let $(x_0, y_0) \in \operatorname{gr} F$ and $x_0 \in A$. Suppose that there exists $z_0 \in G(x_0) \cap (-D)$ such that, for $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, ..., m-1$, and $x \in S$, either of (5.16) or (5.17) holds. Then, one has the following assertions*

(i) *(x_0, y_0) is a weak efficient solution of (P) and $Q = \operatorname{int} C$,*

(ii) (x_0, y_0) *is a strong efficient solution of* (P) *and* $Q = Y \setminus -C$*,*

(iii) (x_0, y_0) *is a positive-proper efficient solution of* (P) *and* $Q = \{y : \varphi(y) > 0\}$ *for some functional* $\varphi \in C^{+i}$*,*

(iv) (x_0, y_0) *is a Geoffrion-proper efficient solution of* (P) *and* $Q = C(\varepsilon)$ *for* $\varepsilon > 0$*,*

(v) (x_0, y_0) *is a Henig-proper efficient solution of* (P) *and* $Q = K$ *for some pointed open convex cone* K *dilating* C*,*

(vi) (x_0, y_0) *is a strong Henig-proper efficient solution of* (P) *and* $Q = \mathrm{int} C_\varepsilon(B)$ *for* ε *satisfying* $0 < \varepsilon < \delta$.

In the following example, Theorems 5.3.7(vi) works, while several existing results do not.

Example 5.3.8. Let $X = Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$, $G(x) \equiv \{0\}$, and

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ \left\{ \dfrac{1}{n^2} \right\}, & \text{if } x = n,\ n \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Choose the base $B = \{1\}$. Then, $\delta = 1$ and $C_\varepsilon(B) = \mathbb{R}_+$ for all $\varepsilon \in (0, \delta)$. Let $(x_0, y_0) = (0, 0)$ and $z_0 = 0$. It is easy to see that $A = \mathbb{N} \cup \{0\}$. Then, $T^{r(1)}_{(F,G)_+(A)}(y_0, z_0) = \mathbb{R}_+ \times \mathbb{R}_+$. It follows from Theorem 5.3.7(vi) that (x_0, y_0) is a strong Henig-proper efficient solution. It is easy to see that $\mathrm{dom} F = \mathbb{N} \cup \{0\}$ is not convex and F is not pseudoconvex at (x_0, y_0). So, Theorem 3.6 of Khanh and Tuan in [106], Theorems 5.3, 5.4 of Li and Chen in [115], and Theorem 2 of Liu and Gong in [119] cannot be applied.

Moreover, it is easy to see that (x_0, y_0) is also a weak efficient solution. But, Theorem 8 of Jahn and Rauh in[96], Theorem 6 of Chen and Jahn in [26] and Theorem 3.3 of Khanh and Tuan in [105] cannot be applied.

A natural question now arises: can we replace D by $D(z_0)$ in the necessary condition given by Theorem 5.3.2 to obtain a smaller gap with the sufficient one expressed by Theorem 5.3.7? Unfortunately, a negative answer is supplied by the following example.

Example 5.3.9. Suppose that $X = Y = Z = \mathbb{R}$, $S = X$, $C = D = \mathbb{R}_+$, and $F : X \to 2^Y$, $G : X \to 2^Z$ are given by

$$F(x) = \begin{cases} \{y : y \geq x^2\}, & \text{if } x \in [-1, 1], \\ \{-1\}, & \text{if } x \notin [-1, 1], \end{cases}$$

$$G(x) = \{z \in Z : z = x^2 - 1\}.$$

We see that $(x_0, y_0) = (0,0)$ is a weak efficient pair of (P). Take $z_0 = -1 \in G(x_0) \cap (-D)$. We have $T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) = \mathbb{R} \times \mathbb{R}_+$ and hence

$$T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) \cap -\mathrm{int}(C \times D) = \emptyset.$$

On the other hand, $D(z_0) = \mathbb{R}$. Thus,

$$T^{r(1)}_{(F,G)_+(S)}(y_0, z_0) \cap -\mathrm{int}(C \times D(z_0)) \neq \emptyset.$$

5.4 Applications in some particular problems

In this section, we apply calculus rules to establish necessary conditions for some kinds of efficient solutions of several particular optimization problems. We first prove a simple characterization of this notion.

Proposition 5.4.1. *Let X, Y and Q as before, $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}\, F$ and $(u_i, v_i) \in X \times (-C)$, $i = 1, ..., m-1$. Suppose that the open cone Q satisfies $Q + C \subseteq Q$. Then, y_0 is a Q-efficient point of $F(X)$ if and only if one of the following separations holds*

$$T^{r(m)}_{F_+(X)}(y_0, v_1, ..., v_{m-1}) \cap (-Q) = \emptyset, \tag{5.18}$$

$$D^m_R F_+(x_0, y_0, (u_1, v_1), ..., (u_{m-1}, v_{m-1}))(X) \cap (-Q) = \emptyset. \tag{5.19}$$

Proof. It follows from Propositions 5.3.1 and 5.3.6.

□

Let $F : X \to 2^Y$ and $G : X \to 2^X$. Consider

$$(\mathrm{P}_1) \qquad \mathrm{Min}_{\mathrm{Q}} F(x') \text{ s.t. } x \in X \text{ and } x' \in G(x).$$

This problem can be restated as the unconstrained problem: $\mathrm{Min}_{\mathrm{Q}}\,(F \circ G)(x)$. Recall that (x_0, y_0) is called a Q-efficient solution if $y_0 \in (F \circ G)(x_0)$ and $((F \circ G)(X) - y_0) \cap (-Q) = \emptyset$.

Proposition 5.4.2. *Assume for* (P_1) *that* $\mathrm{Im}\, G \subseteq \mathrm{dom}\, F$*,* $(x_0, z_0) \in \mathrm{gr}\, G$*,* $(z_0, y_0) \in \mathrm{gr}\, F$*, and* $(u_1, v_1, w_1), ..., (u_{m-1}, v_{m-1}, w_{m-1}) \in X \times X \times (-C)$*. Suppose that an open cone Q satisfies $Q + C \subseteq Q$ and (x_0, y_0) is a Q-efficient solution of* (P_1).

(i) *If either F_+ has a m-th order radial semiderivative at (z_0,y_0) with respect to $(v_1,w_1),...,$ (v_{m-1},w_{m-1}) or (5.5) holds for F_+ and G, then*

$$D_R^m F_+(z_0,y_0,v_1,w_1,...,v_{m-1},w_{m-1})[T^{r(m)}_{G(X)}(z_0,v_1,...,v_{m-1})]\cap(-Q)=\emptyset.$$

(ii) *If either F_+ has a m-th order radial semiderivative at (z_0,y_0) with respect to $(v_1,w_1),...,$ (v_{m-1},w_{m-1}) or (5.6) holds for F_+ and G, then*

$$D_R^m F_+(z_0,y_0,v_1,w_1,...,v_{m-1},w_{m-1})[D_R^m G(x_0,z_0,u_1,v_1,...,u_{m-1},v_{m-1})(X)]\cap(-Q)=\emptyset.$$

Proof. We prove (i). By Proposition 5.4.1, $T^{r(m)}_{(F\circ G)_+(X)}(y_0,w_1,...,w_{m-1})\cap(-Q)=\emptyset$. Propositions 5.2.14(i) and 5.2.15(i) say that

$$D_R^m F_+(z_0,y_0,v_1,w_1,...,v_{m-1},w_{m-1})[T^{r(m)}_{G(X)}(z_0,v_1,...,v_{m-1})]\subseteq T^{r(m)}_{(F\circ G)_+(X)}(y_0,w_1,...,w_{m-1}).$$

□

From Propositions 5.4.2 and 2.2.9, we obtain immediately the following result for (P_1).

Theorem 5.4.3. *Assume for* (P_1) *that* $\mathrm{Im}\,G\subseteq\mathrm{dom}\,F$, $(x_0,z_0)\in\mathrm{gr}\,G$, $(z_0,y_0)\in\mathrm{gr}\,F$, *and* (u_1,v_1,w_1) $,...,(u_{m-1},v_{m-1},w_{m-1})\in X\times X\times(-C)$. *Then, assertions (i) and (ii) in Proposition 5.4.2 hold in each of the following cases*

(i) (x_0,y_0) *is a weak efficient solution of* (P_1) *and* $Q=\mathrm{int}C$,

(ii) (x_0,y_0) *is a strong efficient solution of* (P_1) *and* $Q=Y\setminus -C$,

(iii) (x_0,y_0) *is a positive-proper efficient solution of* (P_1) *and* $Q=\{y:\varphi(y)>0\}$ *for some functional* $\varphi\in C^{+i}$,

(iv) (x_0,y_0) *is a Geoffrion-proper efficient solution of* (P_1) *and* $Q=C(\varepsilon)$ *for* $\varepsilon>0$,

(v) (x_0,y_0) *is a Henig-proper efficient solution of* (P_1) *and* $Q=K$ *for some pointed open convex cone K dilating C,*

(vi) (x_0,y_0) *is a strong Henig-proper efficient solution of* (P_1) *and* $Q=\mathrm{int}C_\varepsilon(B)$ *for* ε *satisfying* $0<\varepsilon<\delta$.

To compare with a result of Jahn and Khan in [94], we recall the definition of *contingent epiderivatives*. For a multimap F between normed spaces X and Y, Y being partially ordered by a pointed convex cone C and a point $(x,y)\in\mathrm{gr}F$, a single-valued map $EDF(x,y):X\to Y$ satisfying $\mathrm{epi}(EDF(x,y))=T_{\mathrm{epi}F}(x,y)\equiv T_{\mathrm{gr}F_+}(x,y)$ is said to be the contingent epiderivative of F at (x,y).

Example 5.4.4. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $G(x) = \{-|x|\}$, and F be defined by

$$F(x) = \begin{cases} \mathbb{R}_-, & \text{if } x \leq 0, \\ \emptyset, & \text{if } x > 0. \end{cases}$$

Since G is single-valued we can try to make use of Proposition 5.2 of [94]. By a direct computation we have $T_{\mathrm{epi}F}(G(0),0) = \mathbb{R}^2$. So, the contingent epiderivative $EDF(G(0),0)(h)$ does not exist for any $h \in X$. Hence, the necessary condition in the mentioned Proposition 5.2 says nothing about the candidate point $(0,0)$ for weak efficiency. However, F_+ has the first order radial semiderivative at $(G(0),0)$ and $T^{r(1)}_{G(X)}(G(0)) = \mathbb{R}_-$. Furthermore, $D^m_R F_+(G(0),0)[T^{r(1)}_{G(X)}(G(0))] = \mathbb{R}$, which meets $-\mathrm{int}C$, and hence Theorem 5.4.3(i) above rejects this candidate.

Our sum rule can be applied directly to the following problem

$$(\mathrm{P}_2) \quad \mathrm{Min}_{\mathrm{Q}}\, F(x) \text{ s.t. } g(x) \leq 0,$$

where X, Y are as for problem (P_1), $F : X \to 2^Y$ and $g : X \to Y$. Denote $A := \{x \in X : g(x) \leq 0\}$ (the feasible set). Define $G : X \to 2^Y$ by $G(x) := \{0\}$ if $x \in A$ and $G(x) := \{g(x)\}$ otherwise. Consider the following unconstrained set-valued optimization problem, for arbitrary $s > 0$,

$$(\mathrm{P}_3) \quad \mathrm{Min}_{\mathrm{Q}}\, (F + sG)(x).$$

In the particular case, when $Y = R$ and F is single-valued, (P_3) is used to approximate (P_2) in penalty methods (see [147]). We will apply our calculus rules for radial sets to get the following necessary condition for a Q-minimal solution of (P_3).

Proposition 5.4.5. *Let $y_0 \in F(x_0)$, $x_0 \in \Omega = \mathrm{dom}F \cap \mathrm{dom}G$, and $(u_1, v_{i,1}), \ldots, (u_{m-1}, v_{i,m-1}) \in X \times (-C)$ for $i = 1,2$. Suppose that an open cone Q satisfies $Q + C \subseteq Q$ and (x_0, y_0) is a Q-efficient solution of* (P_3). *Then*

(i) *if either $F_+(\Omega)$ (or $sG_+(\Omega)$) has a m-th order proto-radial set at y_0 with respect to $v_{1,1}, \ldots, v_{1,m-1}$ (at 0 with respect to $v_{2,1}, \ldots, v_{2,m-1}$, respectively) or (5.7) holds for F_+ and sG_+, then*

$$(T^{r(m)}_{F_+(\Omega)}(y_0, v_{1,1}, \ldots, v_{1,m-1}) + sT^{r(m)}_{G_+(\Omega)}(0, v_{2,1}/s, \ldots, v_{2,m-1}/s)) \cap (-Q) = \emptyset.$$

(ii) *if either F_+ (or sG_+) has a m-th order radial semiderivative at (x_0, y_0) with respect to $(u_1, v_{1,1}), \ldots, (u_{m-1}, v_{1,m-1})$ (at $(x_0, 0)$ with respect to $(u_1, v_{2,1}), \ldots, (u_{m-1}, v_{2,m-1})$, respectively) or (5.8) holds for F_+ and sG_+, then, for any $u \in X$,*

$(D^m_R F_+(x_0, y_0, u_1, v_{1,1}, \ldots, u_{m-1}, v_{1,m-1})(u) + sD^m_R G_+(x_0, 0, u_1, v_{2,1}/s, \ldots, u_{m-1}, v_{2,m-1}/s)(u)) \cap (-Q) = \emptyset$.

Proof. We prove (i). It follows from Propositions 5.2.12(i) and 5.2.17(i) that

$$T^{r(m)}_{F_+(\Omega)}(y_0, v_{1,1}, ..., v_{1,m-1}) + T^{r(m)}_{sG_+(\Omega)}(0, v_{2,1}, ..., v_{2,m-1}) \subseteq$$

$$T^{r(m)}_{(F+sG)_+(\Omega)}(y_0, v_{1,1} + v_{2,1}, ..., v_{1,m-1} + v_{2,m-1}).$$

It is easy to see that

$$T^{r(m)}_{sG_+(\Omega)}(0, v_{2,1}, ..., v_{2,m-1}) = sT^{r(m)}_{G_+(\Omega)}(0, v_{2,1}/s, ..., v_{2,m-1}/s),$$

$$T^{r(m)}_{F_+(\Omega)}(y_0, v_{1,1}, ..., v_{1,m-1}) + sT^{r(m)}_{G_+(\Omega)}(0, v_{2,1}/s, ..., v_{2,m-1}/s)$$
$$\subseteq T^{r(m)}_{(F+sG)_+(\Omega)}(y_0, v_{1,1} + v_{2,1}, ..., v_{1,m-1} + v_{2,m-1}).$$

By Proposition 5.4.1, one gets

$$T^{r(m)}_{(F+sG)_+(\Omega)}(y_0, v_{1,1} + v_{2,1}, ..., v_{1,m-1} + v_{2,m-1}) \cap (-Q) = \emptyset,$$

and hence the proof is complete. □

From Propositions 5.4.5 and 2.2.9, we obtain immediately the following statement for (P_3).

Theorem 5.4.6. *Let* $y_0 \in F(x_0)$, $x_0 \in \Omega = \mathrm{dom}F \cap \mathrm{dom}G$, *and* $(u_1, v_{i,1}), ..., (u_{m-1}, v_{i,m-1}) \in X \times (-C)$ *for* $i = 1,2$. *Then, assertions (i) and (ii) in Proposition 5.4.5 hold in each of the followiing cases*

(i) (x_0, y_0) *is a weak efficient solution of* (P_3) *and* $Q = \mathrm{int}C$,

(ii) (x_0, y_0) *is a strong efficient solution of* (P_3) *and* $Q = Y \setminus -C$,

(iii) (x_0, y_0) *is a positive-proper efficient solution of* (P_3) *and* $Q = \{y : \varphi(y) > 0\}$ *for some functional* $\varphi \in C^{+i}$,

(iv) (x_0, y_0) *is a Geoffrion-proper efficient solution of* (P_3) *and* $Q = C(\varepsilon)$ *for* $\varepsilon > 0$,

(v) (x_0, y_0) *is a Henig-proper efficient solution of* (P_3) *and* $Q = K$ *for some pointed open convex cone K dilating C*,

(vi) (x_0, y_0) *is a strong Henig-proper efficient solution of* (P_3) *and* $Q = \mathrm{int}C_\varepsilon(B)$ *for* ε *satisfying* $0 < \varepsilon < \delta$.

The next example illustrates a case, Theorem 5.4.6 is more advantageous than earlier existing results.

Example 5.4.7. Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $g(x) = x^4 - 2x^3$, and $F(x) = \mathbb{R}_-$ for all $x \in X$. Then, $S = [0,2]$ and $G(x) = \max\{0, g(x)\}$. Furthermore, since $T_{\mathrm{epi}F}(0,0) = \mathbb{R}^2$ and $T_{\mathrm{epi}G}(0,0) = \{(x,y) : \ y \geq 0\}$, the contingent epiderivative $EDF(0,0)(h)$ does not exist for any $h \in X$ and Proposition 5.1 of Jahn and Khan in [94] cannot be applied. But we have proto-radial sets $T^{r(1)}_{F_+(X)}(0) = \mathbb{R}$ and $T^{r(1)}_{G_+(X)}(0) = \mathbb{R}_+$. So,

$$(T^{r(1)}_{F_+(X)}(0) + sT^{r(1)}_{G_+(X)}(0)) \cap (-\mathrm{int}C) \neq \emptyset.$$

By Theorem 5.4.6(i), (x_0, y_0) is not a weak efficient solution of (P_3). This fact can be checked directly too.

Chapter 6

Calculus rules and applications of Studniarski derivatives to sensitivity and implicit function theorems

6.1 Introduction

In set-valued analysis, one of the most popular and useful higher-order derivatives is the contingent derivative introduced in [10] by Aubin in 1981. However, the set $D^mF(x_0,y_0,u_1,v_1,\ldots,u_{m-1},v_{m-1})(u)$ (its definition is given in Remark 4.2.7 in Chapter 4) is non-empty only if $v_1 \in DF(x_0,y_0)(u_1)$, ..., $v_{m-1} \in D^{m-1}F(x_0,y_0,u_1,v_1,\ldots,u_{m-2},v_{m-2})(u_{m-1})$. In applications, even the need of having these $m-1$ points may lead to inconvenience. In 1986, the following modification was proposed in [156] by Studniarski, without the "intermediate" orders in definition and hence without the need of these $m-1$ points,

$$D^mF(x_0,y_0)(x) := \{v \in Y : \exists t_n \to 0^+, \exists (x_n,v_n) \to (x,v),\ \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n x_n)\}.$$

We can write the following two equivalent formulations as follows

$$D^mF(x_0,y_0)(u) = \operatorname*{Limsup}_{(t,u')\to(0^+,u)} \frac{F(x_0+tu')-y_0}{t^m},$$

and, by setting $(x_n,y_n) := (x_0+t_nu_n, y_0+t_n^m v_n)$, $\gamma_n = t_n^{-1}$,

$$D^mF(x_0,y_0)(u) = \{v \in Y :\ \exists \gamma_n > 0,\ \exists (x_n,y_n) \in \mathrm{gr}F :\ (x_n,y_n) \to (x_0,y_0),$$

$$(\gamma_n(x_n-x_0), \gamma_n^m(y_n-y_0)) \to (u,v)\}.$$

This object is called, by several authors, the Studniarski derivative. In nonsmooth optimization, it was applied in obtaining optimality conditions, e.g., in [97, 99, 117, 124, 144, 156, 159] and in discussing sensitivity analysis in [158] by Sun and Li.

The idea of omitting "intermediate" orders in defining higher-order derivatives was continued in [8, 49]. Namely, several notions of higher-order derivatives were developed, combining the idea of extending to higher-order the radial derivative proposed by Taa in [160] (for the first-order) with this omitting. In that way, global (not local as with the above two derivatives) higher-order optimality conditions were established for nonconvex optimization. (The main technical change in the above definitions is replacing $\exists t_n \to 0^+$ by $\exists t_n > 0$.) The possibility for global consideration is good for optimality conditions, but for some other topics like sensitivity analysis or implicit function theorems, this may be inconvenient.

Thus, in this chapter, we return to the Studniarski derivative. Namely, we are concerned with two topics. First we develop calculus rules for this derivative, observing that these rules have not been studied, but a kind of derivatives is significant only if it enjoys enough calculus rules. Later, we use the Studniarski derivative just to the two mentioned topics of sensitivity analysis and implicit function theorems to ensure that in this paper we can investigate what is difficult for the derivatives considered in [8]. The content of this chapter is also our results in [6].

6.2 The Studniarski derivative

Let X, Y be normed spaces, $F : X \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$, $u \in X$, and $m \geq 1$.

Definition 6.2.1. The m-th order *Studniarski derivative* of F at (x_0, y_0) is defined by

$$D^m F(x_0, y_0)(x) := \operatorname*{Limsup}_{(t,x') \to (0^+,x)} \frac{F(x_0 + tx') - y_0}{t^m},$$

or, equivalently,

$$D^m F(x_0, y_0)(x) = \{v \in Y : \exists t_n \to 0^+, \exists (x_n, v_n) \to (x, v),\ \forall n, y_0 + t_n^m v_n \in F(x_0 + t_n x_n)\}.$$

The m-th order Studniarski derivative can be expressed as the Γ-limit as follows

$$\begin{aligned} v \in D^m F(x_0, y_0)(x) \quad &\Longleftrightarrow \quad \inf_{Q \in \mathcal{N}(v)} \inf_{W \in \mathcal{N}(x)} \inf_{t>0} \sup_{0<t'<t} \sup_{x' \in W} \sup_{v' \in Q} \chi_{\mathrm{gr}(\mathcal{L}_{F,(x_0,y_0)})}(t', x', v') = 1 \\ &\Longleftrightarrow \quad \Gamma(\mathcal{N}_+(0)^+, \mathcal{N}(x)^+, \mathcal{N}(v)^+)\ \chi_{\mathrm{gr}(\mathcal{L}_{F,(x_0,y_0)})} = 1, \end{aligned}$$

where $\mathscr{L}_{F,(x_0,y_0)} : (0,+\infty)\times X \to 2^Y$ is defined by

$$\mathscr{L}_{F,(x_0,y_0)}(t',x') := \frac{1}{t'^m}(F(x_0+t'x')-y_0).$$

If the upper limit in Definition 6.2.1 is a full limit, i.e., the upper limit coincides with the lower limit for all u, then the map F is called to have a m-th order *proto-Studniarski derivative* at (x_0,y_0).

Example 6.2.2. Let $X=Y=\mathbb{R}$ and $F_n : X \to 2^Y$, $n\in\mathbb{N}$, be defined by, for all $x\in X$, $F_n(x)=\{y\in Y : y\geq x^n\}$. By calculating, we can find the m-th order Studniarski derivatives of F_n at $(x_0,y_0)=(0,0)$ as follows

If $m=n$, then for all $u\in X$, $D^mF_n(x_0,y_0)(u)=\{y\in Y : y\geq u^n\}$.

If $m<n$, then for all $u\in X$, $D^mF_n(x_0,y_0)(u)=\mathbb{R}_+$.

If $m>n$, then

$$D^mF_n(x_0,y_0)(u)=\begin{cases}\mathbb{R}, & \text{if n} = 2\text{k}-1 \text{ (k} = 1,2,..\text{) and u} \leq 0,\\ \mathbb{R}_+, & \text{if n} = 2\text{k (k} = 1,2,..\text{) and u} = 0,\\ \emptyset, & \text{otherwise.}\end{cases}$$

In the following example, we compute the Studniarski derivative of a map into an infinite dimensional case.

Example 6.2.3. Let $X=\mathbb{R}$ and $Y=l^2$, the Hilbert space of the numerical sequences $x=(x_i)_{i\in\mathbb{N}}$ with $\sum_{i=1}^{\infty}x_i^2$ being convergent. By $(e_i)_{i\in\mathbb{N}}$ we denote standard unit basis of l^2. Let $f : X\to Y$ be defined by

$$f(x) := \begin{cases}\frac{1}{n}(-e_1+2e_n), & \text{if } x=\frac{1}{n},\\ 0, & \text{otherwise}\end{cases}$$

and $(x_0,y_0)=(0,0)$. We find the higher-order Studniarski derivatives of f at (x_0,y_0). It follows from Definition 2.1 that $v\in D^mf(x_0,y_0)(u)$ means the existence of $t_k\to 0^+$, $u_k\to u$, and $v_k\to v$ such that

$$y_0+t_k^mv_k\in f(x_0+t_ku_k). \tag{6.1}$$

For all $u\in X$, we can choose $t_k\to 0^+$, $u_k\to u$ such that $t_ku_k\neq 1/k$. So, for all $u\in X$,

$$\{0\}\subseteq D^mf(x_0,y_0)(u).$$

We now prove that, for each $v \in Y \setminus \{0\}$, $v \notin D^m f(x_0,y_0)(u)$ for all $u \in X$. Suppose, on the contrary, there exist $u \in U$ and $v \in Y \setminus \{0\}$ such that $v \in D^m f(x_0,y_0)(u)$, i.e., there are $t_k \to 0^+$, $u_k \to u$, $v_k \to v$ such that (6.1) holds. If $t_k u_k \neq 1/k$ for infinitely many $k \in \mathbb{N}$, we get a contradiction easily. Hence, assume that $t_k u_k = 1/k$. Then (6.1) becomes $v_k = \frac{1}{k.t_k^m}(-e_1 + 2e_k)$. If $1/(k.t_k^m) \to +\infty$, we get a contradiction to the convergence of the sequence $(-e_1+2e_k)/(k.t_k^m)$. Suppose $1/(k.t_k^m) \to a \geq 0$. As $e_1/(k.t_k^m) \to ae_1$, the sequence $e_k/(k.t_k^m)$ converges to some c, i.e.,

$$||\frac{2}{k.t_k^m}e_k - c||^2 \to 0,$$

that is,

$$||\frac{2}{k.t_k^m}e_k - c||^2 = (\frac{2}{k.t_k^m})^2 + ||c||^2 + 2\left\langle \frac{2}{k.t_k^m}e_k, -c \right\rangle \to 0. \tag{6.2}$$

Since (e_k) converges to 0 with respect to the weak topology, then $\langle e_k, -c\rangle \to 0$. From (6.2), we get $4a^2 + ||c||^2 = 0$. If $a = 0$, then $c = v(\neq \emptyset)$ since $(-e_1+2e_k)/(k.t_k^m) \to v$. If $a > 0$, then $4a^2 + ||c||^2 \neq 0$. Therefore, we always have a contradiction. Thus, for all $u \in X$, $D^m f(x_0,y_0)(u) = \{0\}$.

We now present a condition for a m-th order Studniarski derivative to be nonempty.

Proposition 6.2.4. *Let* $\dim Y < +\infty$, $(x_0,y_0) \in \operatorname{gr} F$, *and* $x_0 \in \operatorname{int}(\operatorname{dom} F)$. *Suppose that*

(i) *F is lower semicontinuous at (x_0,y_0),*

(ii) *F is m-th order locally pseudo-Hölder calm at x_0 for y_0.*

Then, $D^m F(x_0,y_0)(x) \neq \emptyset$ for all $x \in X$.

Proof. For $x = 0$, this is trivial because we always have $0 \in D^m F(x_0,y_0)(0)$. By assumption (ii), there exist $\lambda > 0$, $U_1 \in \mathscr{N}(x_0)$ and $V \in \mathscr{N}(y_0)$ such that $\forall x' \in U_1$,

$$(F(x') \cap V) \subseteq \{y_0\} + \lambda||x' - x_0||^m B_Y.$$

By assumption (i), with V above, there exists $U_2 \in \mathscr{N}(x_0)$ such that $\forall \widehat{x} \in U_2$, $V \cap F(\widehat{x}) \neq \emptyset$. It follows from $x_0 \in \operatorname{int}(\operatorname{dom} F)$ that there exists $U_3 \in \mathscr{N}(x_0)$ such that $U_3 \in \operatorname{dom} F$. Setting $\widehat{U} = U_1 \cap U_2 \cap U_3$, we get $\widehat{U} \in \mathscr{N}(x_0)$. Let an arbitrary $x \in X \setminus \{0\}$ and $t_n \to 0^+$. Because $x_0 + t_n x \to x_0$, we get $x_0 + t_n x \in \widehat{U}$ for large n. Hence, there exists $y_n \in F(x_0 + t_n x) \cap V$ such that

$$t_n^{-m}||y_n - y_0|| \leq \lambda||x||^m.$$

So, $t_n^{-m}(y_n - y_0)$ is a bounded sequence and hence has a convergent subsequence. By Definition 6.2.1, the limit of this subsequence is an element of the set $D^m F(x_0,y_0)(x)$. □

Example 6.2.5. (assumption (ii) is essential) Let $F:\mathbb{R}\to 2^{\mathbb{R}}$ be defined by

$$F(x)=\begin{cases}\{x^{1/3}\}, & \text{if } 0\le x\le 1,\\ \{x\}, & \text{if } x>1,\\ \{-x\}, & \text{if } -1\le x<0,\\ \{-x^{1/3}\}, & \text{if } x<-1.\end{cases}$$

Direct computations yield that $D^mF(0,0)(1)=\emptyset$ for all $m\ge 1$. Here, F is lower semicontinuous at $(0,0)$, but the m-th order locally pseudo-Hölder calmness fails.

Example 6.2.6. (assumption (i) cannot be dropped) Let $F:\mathbb{R}\to 2^{\mathbb{R}}$ be defined by

$$F(x)=\begin{cases}\{1\}, & \text{if } x=0,\\ \{y:y\le x\}, & \text{if } x\ne 0.\end{cases}$$

Then, assumption (ii) is satisfied at $(0,1)$. Direct calculations give that $D^mF(0,1)(1)=\emptyset$ for all $m\ge 1$. The cause is that F is not lower semicontinuous at $(0,1)$, since F is mth-order locally pseudo-Holder calm at 0 for 1. Indeed, pick $\lambda=1, U=\{x\in\mathbb{R}:-1/2<x<1/2\}$, $V=\{y\in\mathbb{R}:1/2<y<3/2\}$. Then, $F(x)=\{y\in\mathbb{R}:y\le x\}\subset(-\infty,1/2)$ for all $x\in U\setminus\{0\}$. Therefore, $F(x)\cap V=\emptyset$ for all $x\in U\setminus\{0\}$, and

$$F(0)\cap V=\{1\}\subset\{y_0\}+||x||^mB_Y$$

for all $m\ge 1$.

A map F is said to have a *strict Studniarski derivative* at $(x_0,y_0)\in \mathrm{gr}F$ if

$$D^mF(x_0,y_0)(u)=\{v\in Y:\forall t_n\to 0^+,\exists(u_n,v_n)\to(u,v),\ \forall n, y_0+t_n^mv_n\in F(x_0+t_nu_n)\}.$$

Proposition 6.2.7. *Let $F:X\to 2^Y$, $(x_0,y_0)\in \mathrm{gr}F$, and F be a convex map and have a strict Studniarski derivative at (x_0,y_0). Then, $D^mF(x_0,y_0)$ is convex.*

Proof. Let $x^1,x^2\in X$ and $y^i\in D^mF(x_0,y_0)(x^i)$, $i=1,2$, i.e., for any $t_n\to 0^+$, there exists $(x_n^i,y_n^i)\to(x^i,y^i)$ such that, for all n, $y_n^i\in t_n^{-m}(F(x_0+t_nx_n^i)-y_0)$. Since F is convex, for all $\lambda\in[0,1]$,

$$\lambda\left(\frac{F(x_0+t_nx_n^1)-y_0}{t_n^m}\right)+(1-\lambda)\left(\frac{F(x_0+t_nx_n^2)-y_0}{t_n^m}\right)\subseteq\frac{F(\lambda(x_0+t_nx_n^1)+(1-\lambda)(x_0+t_nx_n^2))-y_0}{t_n^m}.$$

Therefore,

$$\lambda y_n^1 + (1-\lambda) y_n^2 \in \frac{F(x_0 + t_n(\lambda x_n^1 + (1-\lambda)x_n^2)) - y_0}{t_n^m}.$$

Hence, $\lambda y^1 + (1-\lambda)y^2 \in D^m F(x_0, y_0)(\lambda x^1 + (1-\lambda)x^2)$. □

The next statement is a relation between the Studniarski derivative of F and that of the profile map.

Proposition 6.2.8. *Let $F : X \to 2^Y$, and $(x_0, y_0) \in \operatorname{gr} F$. Then, for all $x \in X$,*

$$D^m F(x_0, y_0)(x) + C \subseteq D^m (F + C)(x_0, y_0)(x). \tag{6.3}$$

If $\dim Y < +\infty$ and F is m-th order locally Hölder calm at x_0 for y_0, then (6.3) becomes an equality.

Proof. Let $w \in D^m F(x_0, y_0)(x) + C$, i.e., there exists $v \in D^m F(x_0, y_0)(x)$ and $c \in C$ such that $w = v + c$. We then have sequences $t_n \to 0^+$, $x_n \to x$, and $v_n \to v$ such that, for all n,

$$y_0 + t_n^m (v_n + c) \in F(x_0 + t_n x_n) + t_n^m c \subseteq F(x_0 + t_n x_n) + C.$$

So, $v + c \in D^m (F + C)(x_0, y_0)(x)$.

Let $w \in D^m (F + C)(x_0, y_0)(x)$, i.e., there exist $t_n \to 0^+$, $x_n \to x$, $w_n \to w$ such that $y_0 + t_n^m w_n \in F(x_0 + t_n x_n) + C$. Then, there exist $y_n \in F(x_0 + t_n x_n)$ and $c_n \in C$ satisfying

$$w_n = t_n^{-m}(y_n - y_0) + t_n^{-m} c_n. \tag{6.4}$$

Because F is m-th order locally Hölder calm at x_0 for y_0, there exists $\lambda > 0$ such that, for large n,

$$y_n \in F(x_0 + t_n x_n) \subseteq \{y_0\} + \lambda ||t_n x_n||^m B_Y.$$

So,

$$t_n^{-m} ||y_n - y_0|| \leq \lambda ||x_n||^m.$$

Since $\dim Y < +\infty$, $t_n^{-m}(y_n - y_0)$ (using a subsequence, if necessary) converges to some v and $v \in D^m F(x_0, y_0)(x)$. From (6.4), the sequence c_n / t_n^m converges to some $c \in C$ and $w = v + c$. Thus $w \in D^m F(x_0, y_0)(x) + C$. □

6.3 Calculus rules

Proposition 6.3.1. (Sum rule) *Let $F_1, F_2 : X \to 2^Y$, $x_0 \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, $y_i \in F(x_i)$ (i=1,2) and $u \in X$. Suppose either F_1 or F_2 has a m-th order proto-Studniarski derivative at (x_0, y_1) or (x_0, y_2), respectively. Then*

$$D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u) \subseteq D^m(F_1 + F_2)(x_0, y_1 + y_2)(u). \tag{6.5}$$

If, additionally, $\dim Y < +\infty$ *and either F_1 or F_2 is m-th order locally Hölder calm at x_0 for y_1 or y_2, respectively, then (6.5) becomes an equality.*

Proof. Consider $v^i \in D^m F_i(x_0, y_i)(u)$. For v^1, there exist $t_n \to 0^+$, $u_n \to u$, and $v_n^1 \to v^1$ such that, for all n, $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$. For v_2, supposing that F_2 has the m-th order proto-Studniarski derivative at (x_0, y_2), with t_n, u_n above, there exists $v_n^2 \to v^2$ such that $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$. Hence, $y_1 + y_2 + t_n^m(v_n^1 + v_n^2) \in (F_1 + F_2)(x_0 + t_n u_n)$ and $v^1 + v^2 \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$.

Let $v \in D^m(F_1 + F_2)(x_0, y_1 + y_2)(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that

$$y_1 + y_2 + t_n^m v_n \in (F_1 + F_2)(x_0 + t_n u_n) = F_1(x_0 + t_n u_n) + F_2(x_0 + t_n u_n).$$

This means that there exist $y_n^i \in F_i(x_0 + t_n u_n)$, i = 1, 2, such that

$$v_n = t_n^{-m}(y_n^1 - y_1) + t_n^{-m}(y_n^2 - y_2). \tag{6.6}$$

Suppose F_1 is m-th order locally Hölder calm at x_0 for y_1, i.e., there exists $L > 0$ such that, for large n,

$$y_n^1 \in F_1(x_0 + t_n^m u_n) \subseteq \{y_1\} + L||t_n u_n||^m B_Y.$$

Because $\dim Y < +\infty$, $t_n^{-m}(y_n^1 - y_1)$ (using a subsequence, if necessary) converges to some v^1 and hence $v^1 \in D^m F_1(x_0, y_1)(u)$. From (6.6), the sequence $t_n^{-m}(y_n^2 - y_2)$ also converges to some v^2 such that $v^2 = v - v^1$, and $v^2 \in D^m F_2(x_0, y_2)(u)$. Thus, $v \in D^m F_1(x_0, y_1)(u) + D^m F_2(x_0, y_2)(u)$. □

Proposition 6.3.2. (Chain rule) *Let $F : X \to 2^Y$, $G : Y \to 2^Z$, $(x_0, y_0) \in \operatorname{gr} F$, $(y_0, z_0) \in \operatorname{gr} G$, and* $\operatorname{Im} F \subseteq \operatorname{dom} G$.

(i) *Suppose G has a m-th order proto-Studniarski derivative at (y_0, z_0). Then, for all $u \in X$,*

$$D^m G(y_0, z_0)(D^1 F(x_0, y_0)(u)) \subseteq D^m(G \circ F)(x_0, z_0)(u). \tag{6.7}$$

If, additionally, $\dim Y < +\infty$ *and* F *is locally Lipschitz calm at* x_0 *for* y_0*, then (6.7) becomes an equality.*

(ii) *Suppose* G *has a first-order proto-Studniarski derivative at* (y_0, z_0)*. Then, for all* $u \in X$*,*

$$D^1G(y_0,z_0)(D^mF(x_0,y_0)(u)) \subseteq D^m(G \circ F)(x_0,z_0)(u). \tag{6.8}$$

If, additionally, $\dim Y < +\infty$ *and* F *is m-th order locally Hölder calm at* x_0 *for* y_0*, then (6.8) becomes an equality.*

Proof. By the similarity, we prove only (i). Let $w \in D^mG(y_0,z_0)(D^1F(x_0,y_0)(u))$, i.e., there exists $v \in D^1F(x_0,y_0)(u)$ such that $w \in D^mG(y_0,z_0)(v)$. There exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that $y_0 + t_nv_n \in F(x_0 + t_nu_n)$. With t_n, v_n above, we have $w_n \to w$ such that $z_0 + t_n^m w_n \in G(y_0 + t_nv_n)$. So, $z_0 + t_n^m w_n \in G(F(x_0 + t_nu_n))$. Thus, $w \in D^m(G \circ F)(x_0,z_0)(u)$.

Let $w \in D^m(G \circ F)(x_0,z_0)(u)$, i.e., there exists $t_n \to 0^+$, $u_n \to u$, and $w_n \to w$ such that $z_0 + t_n^m w_n \in G(F(x_0 + t_nu_n))$. Then, there exists $y_n \in F(x_0 + t_nu_n)$ such that $z_0 + t_n^m w_n \in G(y_n)$. Due to the local Lipschitz calmness of F and the finiteness of $\dim Y$, the sequence $v_n := t_n^{-1}(y_n - y_0)$, or a subsequence, converges to some v and $v \in D^1F_1(x_0,y_0)(u)$. This implies that $z_0 + t_n^m w_n \in G(y_0 + t_nv_n)$ and hence $w \in D^mG(y_0,z_0)(v)$. □

We next discuss calculus rules for the following operations.

Definition 6.3.3. (i) For $F_1, F_2 : X \to 2^{\mathbb{R}^k}$, $\mathbb{R}^k$ being an Euclidean space, the product of F_1 and F_2 is the set-valued map $\langle F_1, F_2 \rangle : X \to 2^{\mathbb{R}}$ defined by $\langle F_1, F_2 \rangle (x) := \{\langle y_1, y_2 \rangle : y_1 \in F_1(x), y_2 \in F_2(x)\}$.

(ii) For $F_1, F_2 : X \to 2^{\mathbb{R}}$, the quotient of F_1 and F_2 is the set-valued map $F_1/F_2 : X \to 2^{\mathbb{R}}$ defined by $(F_1/F_2)(x) := \{y_1/y_2 : y_1 \in F_1(x), y_2 \in F_2(x), y_2 \neq 0\}$.

Proposition 6.3.4. (Product rule) *Let* $F_1, F_2 : X \to 2^{\mathbb{R}^k}$, $x_0 \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, $y_i \in F_1(x_0)$ *(i=1,2). Suppose either* F_1 *or* F_2 *has a m-th order proto-Studniarski derivative at* (x_0,y_1) *or* (x_0,y_2)*, respectively. Then, for all* $u \in X$*,*

$$\langle y_2, D^mF_1(x_0,y_1)(u)\rangle + \langle y_1, D^mF_2(x_0,y_2)(u)\rangle \subseteq D^m(\langle F_1,F_2\rangle)(x_0,\langle y_1,y_2\rangle)(u). \tag{6.9}$$

If, additionally, F_i *are m-th order locally Hölder calm at* x_0 *for* y_i*,* $i = 1,2$*, then (6.9) becomes an equality.*

Proof. Consider $v^i \in D^mF_i(x_0,y_i)(u)$. There exist $t_n \to 0^+$, $u_n \to u$, and $v_n^1 \to v^1$ such that $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_nu_n)$. Supposing F_2 has the m-th order proto-Studniarski derivative at (x_0,y_2), with t_n, u_n above, there exists $v_n^2 \to v^2$ such that $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$. We have

$$\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2 \rangle = \langle y_1, y_2 \rangle + t_n^m(\langle y_1, v_n^2 \rangle + \langle y_2, v_n^1 \rangle + t_n^m \langle v_n^1, v_n^2 \rangle),$$

and

$$\langle y_1 + t_n^m v_n^1, y_2 + t_n^m v_n^2 \rangle \in \langle F_1, F_2 \rangle (x_0 + t_n u_n).$$

This implies that $\langle y_1, v^2 \rangle + \langle y_2, v^1 \rangle \in D^m(\langle F_1, F_2 \rangle)(x_0, \langle y_1, y_2 \rangle)(u)$.

Let $v \in D^m(\langle F_1, F_2 \rangle)(x_0, \langle y_1, y_2 \rangle)(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that $\langle y_1, y_2 \rangle + t_n^m v_n \in \langle F_1, F_2 \rangle (x_0 + t_n u_n)$. Then, there exist $y_n^i \in F_i(x_0 + t_n u_n)$ such that $\langle y_1, y_2 \rangle + t_n^m v_n = \langle y_n^1, y_n^2 \rangle$. We have

$$\langle y_n^1, y_n^2 \rangle = \langle y_n^1 - y_1 + y_1, y_n^2 - y_2 + y_2 \rangle = \langle y_n^1 - y_1, y_n^2 - y_2 \rangle + \langle y_n^1 - y_1, y_2 \rangle + \langle y_n^2 - y_2, y_1 \rangle + \langle y_1, y_2 \rangle.$$

This implies that

$$v_n = \left\langle \frac{y_n^1 - y_1}{t_n^m}, y_2 \right\rangle + \left\langle \frac{y_n^2 - y_2}{t_n^m}, y_1 \right\rangle + t_n^m \left\langle \frac{y_n^1 - y_1}{t_n^m}, \frac{y_n^2 - y_2}{t_n^m} \right\rangle. \tag{6.10}$$

Because F_i is m-th order locally Hölder calm at x_0 for y_i, there exists $L_i > 0$ such that, for large n,

$$y_n^i \in F_i(x_0 + t_n^m u_n) \subseteq \{y_i\} + L_i ||t_n u_n||^m B_Y.$$

This implies that there are two subsequence (the subscripts of the second one are taken among those of the first), denoted by the same notations $t_n^{-m}(y_n^i - y_i)$, converging to some $v^i \in \mathbb{R}^k$ and $v^i \in D^m F_i(x_0, y_i)(u)$, $i = 1, 2$. Thus, from (6.10), $v \in \langle D^m F_1(x_0, y_1)(u), y_2 \rangle + \langle D^m F_2(x_0, y_2)(u), y_1 \rangle$.

□

Proposition 6.3.5. (Quotient rule) *Let* $F_1, F_2 : X \to 2^{\mathbb{R}}$, $x_0 \in \mathrm{dom} F_1 \cap \mathrm{dom} F_2$, *and* $y_i \in F_i(x_0)$ *(i=1,2) with* $y_2 \neq 0$. *Suppose either* F_1 *or* F_2 *has a m-th order proto-Studniarski derivative at* (x_0, y_1) *or* (x_0, y_2), *respectively. Then, for all* $u \in X$,

$$\frac{1}{y_2^2}(y_2 D^m F_1(x_0, y_1)(u) - y_1 D^m F_2(x_0, y_2)(u)) \subseteq D^m((F_1/F_2)(x_0, y_1/y_2)(u). \tag{6.11}$$

If, in addition, F_2 *is m-th order locally Hölder calm at* x_0 *for* y_2, *then (6.11) becomes an equality.*

Proof. Consider $v^i \in D^m F_i(x_0, y_i)(u)$. There exist $t_n \to 0^+$, $u_n \to u$, and $v_n^1 \to v^1$ such that $y_1 + t_n^m v_n^1 \in F_1(x_0 + t_n u_n)$. Supposing F_2 has the m-th order proto-Studniarski derivative at (x_0, y_2), with t_n, u_n above, there exists $v_n^2 \to v^2$ such that $y_2 + t_n^m v_n^2 \in F_2(x_0 + t_n u_n)$. We have

$$\frac{y_1 + t_n^m v_n^1}{y_2 + t_n^m v_n^2} = \frac{y_1}{y_2} + t_n^m \left(\frac{y_2 v_n^1 - y_1 v_n^2}{y_2^2 + t_n^m v_n^2 y_2} \right) \in (F_1/F_2)(x_0 + t_n u_n).$$

This implies that $y_2^{-2}(y_2v^1 - y_1v^2) \in D^m((F_1/F_2)(x_0, y_1/y_2)(u)$.

Let $v \in D^m(F_1/F_2)(x_0,(y_1/y_2))(u)$, i.e., there exist $t_n \to 0^+$, $u_n \to u$, and $v_n \to v$ such that $(y_1/y_2)+t_n^m v_n \in (F_1/F_2)(x_0+t_nu_n)$. So, there exist $y_n^i \in F_i(x_0+t_nu_n)$ such that $(y_1/y_2)+t_n^m v_n = y_n^1/y_n^2$. We get

$$\frac{y_n^1}{y_n^2} = \frac{y_1}{y_2} + \frac{y_2(y_n^1 - y_1) - y_1(y_n^2 - y_2)}{y_2^2 + y_2(y_n^2 - y_2)},$$

and hence

$$v_n = \frac{y_2(y_n^1 - y_1)/t_n^m - y_1(y_n^2 - y_2)/t_n^m}{y_2^2 + t_n^m . y_2(y_n^2 - y_2)/t_n^m}. \tag{6.12}$$

Because F_2 is m-th order locally Hölder calm at x_0 for y_2, $(y_n^2 - y_2)/t_n^m$ converges to some v^2 with $v^2 \in D^mF_2(x_0,y_2)(u)$. From (6.12), the sequence $(y_n^1 - y_1)/t_n^m$ also converges to v^1 such that $v^1 \in D^mF_1(x_0,y_1)(u)$ and $v = y_2^{-2}(y_2v^1 - y_1v^2)$. Thus, $v \in y_2^{-2}(y_2D^mF_1(x_0,y_1)(u) - y_1D^mF_2(x_0,y_2)(u))$. □

Corollary 6.3.6. (Reciprocal rule) *Let $F : X \to 2^{\mathbb{R}}$, $y_0 \in F(x_0)$ with $y_0 \neq 0$. Then, for all $u \in X$,*

$$-y_0^{-2}(D^mF(x_0,y_0)(u) \subseteq D^m(1/F)(x_0,1/y_0)(u). \tag{6.13}$$

If, in addition, F is m-th order locally Hölder calm at x_0 for y_0, then (6.13) becomes an equality.

In the rest of this section, we discuss other sum and chain rules, which may be more useful in some cases (see, e.g., Section 6.4). Let X,Y,Z be normed spaces. To investigate the sum $M+N$ of multifunctions $M,N : X \to 2^Y$, we express $M+N$ as a composition as follows. Define $F : X \to 2^{X\times Y}$ and $G : X\times Y \to 2^Y$ by, for I being the identity map on X and $(x,y) \in X\times Y$,

$$F = I \times M \qquad \text{and} \qquad G(x,y) = y + N(x). \tag{6.14}$$

Then, clearly $M+N = G\circ F$.

First, we develop a chain rule. Let general multimaps $F : X \to 2^Y$ and $G : Y \to 2^Z$ be considered. The so-called resultant set-valued map $C : X\times Z \to 2^Y$ is defined by

$$C(x,z) := F(x) \cap G^{-1}(z).$$

Then, $\mathrm{dom}\, C = \mathrm{gr}(G\circ F)$. We need the following compactness properties.

Definition 6.3.7. Let $H : X \to 2^Y$ be a set-valued map.

(i) H is said to be *compact*, see [140], at $x \in \mathrm{cl}(\mathrm{dom}\, H)$ if any sequence $y_n \in H(x_n)$ satisfying $x_n \to x$ has a convergent subsequence.

(ii) H is said to be *closed* at x if $(\mathrm{cl}\, H)(x) = H(x)$, where $\mathrm{cl}\, H$ is the closure map of H defined by $\mathrm{gr}(\mathrm{cl}\, H) = \mathrm{cl}(\mathrm{gr}\, H)$.

Note that when H is compact at x, the image $H(x)$ still may be not closed. Simply think of $H : \mathbb{R} \to 2^{\mathbb{R}}$ equal to $(0,1)$ if $x = 0$, and to $\{0\}$ if $x \neq 0$. Then, H is compact at 0, but $H(0) = (0,1)$ is not closed.

We define other kinds of m-th order Studniarski derivatives of $G \circ F$ with respect to variable y as follows.

Definition 6.3.8. Let $((x,z),y) \in \mathrm{gr}\, C$.

(i) The m-th order *y-Studniarski derivative* of $G \circ F$ at $((x,z),y)$ is defined as, for $u \in X$,

$$D^m(G \circ_y F)(x,z)(u) := \{w \in Z : \exists t_n \to 0^+, \exists (u_n, y_n, w_n) \to (u, y, w), \forall n, y_n \in C(x + t_n u_n, z + t_n^m w_n)\}.$$

(ii) For an integer k, the m-th order *pseudo-Studniarski derivative* of the map C at (x,z) with respect to k is defined as, for $(u,w) \in X \times Z$,

$$\begin{aligned} D_p^{m(k)} C((x,z),y)(u,w) \quad := \quad & \{\overline{y} \in Y : \exists t_n \to 0^+, \exists (u_n, \overline{y_n}, w_n) \to (u, \overline{y}, w), \forall n, \\ & y + t_n^k \overline{y_n} \in C(x + t_n u_n, z + t_n^m w_n)\}. \end{aligned}$$

If $k = m$, the set in Definition 6.3.8(ii) is denoted shortly by $D_p^m C((x,z),y)(u,w)$. One has a relationship between $D^m(G \circ_y F)(x,z)(u)$ and $D^m(G \circ F)(x,z)(u)$ in the following statement.

Proposition 6.3.9. *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $u \in X$.

(i) *For* $y \in C(x,z)$*, one has*

$$D^m(G \circ_y F)(x,z)(u) \subseteq D^m(G \circ F)(x,z)(u).$$

(ii) *If* C *is compact and closed at* (x,z)*, then*

$$\bigcup_{y \in C(x,z)} D^m(G \circ_y F)(x,z)(u) = D^m(G \circ F)(x,z)(u).$$

Proof. (i) This follows immediately from the definitions.

(ii) "$\subseteq$" follows from (i). For "$\supseteq$", let $w \in D^m(G \circ F)(x,z)(u)$, i.e., there exist sequences $t_n \to 0^+$ and $(u_n, w_n) \to (u, w)$ such that $z + t_n^m w_n \in (G \circ F)(x + t_n u_n)$. So, there exists $y_n \in Y$ with $y_n \in C(x + t_n u_n, z + t_n^m w_n)$. Since C is compact at (x,z), y_n (or a subsequence) has a limit y. Since $(x + t_n u_n, z + t_n^m w_n, y_n) \to (x, z, y)$, one has $y \in (\mathrm{cl}\, C)(x,z)$. It follows from the closedness of C at (x,z) that $y \in C(x,z)$.

□

The first chain rule for $G \circ F$ using these Studniarski derivatives is

Proposition 6.3.10. *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $y \in C(x,z)$. *Suppose, for all* $(u,w) \in X \times Z$,

$$D^m F(x,y)(u) \cap (D^1 G(y,z))^{-1}(w) \subseteq D^m_p C((x,z),y)(u,w). \tag{6.15}$$

Then

$$D^1 G(y,z)[D^m F(x,y)(u)] \subseteq D^m (G \circ_y F)(x,z)(u).$$

Proof. Let $v \in D^1 G(y,z)[D^m F(x,y)(u)]$, i.e., there exists $\overline{y} \in D^m F(x,y)(u)$ such that $\overline{y} \in (D^1 G(y,z))^{-1}(v)$. Then, (6.15) ensures that $\overline{y} \in D^m_p C((x,z),y)(u,v)$. This means the existence of $t_n \to 0^+$ and $(u_n, \overline{y}_n, v_n) \to (u, \overline{y}, v)$ such that $y + t_n^m \overline{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$. We have $y_n := y + t_n^m \overline{y}_n \in C(x + t_n u_n, z + t_n^m v_n)$. So, $v \in D^m (G \circ_y F)(x,z)(u)$ and we are done.

□

Proposition 6.3.11. *Let* $(x,z) \in \mathrm{gr}(G \circ F)$ *and* $y \in C(x,z)$. *Suppose, for all* $(u,w) \in X \times Z$,

$$D^1 F(x,y)(u) \cap (D^m G(y,z))^{-1}(w) \subseteq D^{m(1)}_p C((x,z),y)(u,w). \tag{6.16}$$

Then

$$D^m G(y,z)[D^1 F(x,y)(u)] \subseteq D^m (G \circ_y F)(x,z)(u).$$

Proof. The proof is similar to that of Proposition 6.3.10. □

Note that, when $m = 1$, we have $(D^1 G(y,z))^{-1} = D^1 G^{-1}(z,y)$, however this is not true for $m \geq 2$ as shown in the following example.

Example 6.3.12. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x^2$. Then,

$$F^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

Direct computations yield that $D^1 F(0,0)(u) = \{0\}$ for all $u \in \mathbb{R}$, which implies that $(D^1 F(0,0))^{-1}(0) = \mathbb{R}$ and $(D^1 F(0,0))^{-1}(v) = \emptyset$ for $v \neq 0$. It is easy to check that $D^1 F^{-1}(0,0)$ coincides with $(D^1 F(0,0))^{-1}$.

For $m = 2$, $D^2 F(0,0)(u) = \{u^2\}$ for all $u \in \mathbb{R}$, which implies

$$(D^2 F(0,0))^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & \text{if } y \geq 0, \\ \emptyset, & \text{if } y < 0. \end{cases}$$

However,

$$D^2F^{-1}(0,0)(v) = \begin{cases} \mathbb{R}, & \text{if } v = 0, \\ \emptyset, & \text{if } v \neq 0. \end{cases}$$

To get a chain rule for Studniarski derivatives in the form of equalities, we first prove the inclusions reverse to those in Propositions 6.3.10(i), 6.3.11(i) under additional assumptions as follows.

Proposition 6.3.13. *Let $y \in C(x,z)$ and Y be finite dimensional.*

(i) *If*

$$D_p^m C((x,z),y)(0,0) = \{0\}, \tag{6.17}$$

then

$$D^m(G \circ_y F)(x,z)(u) \subseteq D^1G(y,z)[D^mF(x,y)(u)].$$

(ii) *If*

$$D_p^{m(1)} C((x,z),y)(0,0) = \{0\}, \tag{6.18}$$

then

$$D^m(G \circ_y F)(x,z)(u) \subseteq D^mG(y,z)[D^1F(x,y)(u)].$$

Proof. By the similarity, we prove only (i). Let $w \in D^m(G \circ_y F)(x,z)(u)$, i.e., there exist $t_n \to 0^+$ and $(u_n, y_n, w_n) \to (u, y, w)$ such that $y_n \in C(x + t_n u_n, z + t_n^m w_n)$. If $y_k = y$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D^mF(x,y)(u)$, $w \in D^1G(y,z)(0)$ and we are done. Thus, suppose $y_n \neq y$ for all n and, for $s_n := ||y_n - y||^{1/m}$, the sequence $v_n := s_n^{-m}(y_n - y)$ or some subsequence has a limit v of norm one. If $t_n/s_n \to 0$, since

$$y + s_n^m v_n = y_n \in C\left(x + s_n(\frac{t_n u_n}{s_n}), z + s_n^m(\frac{t_n^m w_n}{s_n^m})\right),$$

one sees that $v \in D_p^m C((x,z),y)(0,0)$, contradicting (6.17). Consequently, $t_n^{-1}s_n$ has a bounded subsequence and one may assume that $t_n^{-1}s_n$ tends to $q \in R_+$. So,

$$y + t_n^m(s_n^m v_n / t_n^m) = y_n \in C(x + t_n u_n, z + t_n^m w_n)$$

and then one gets $q^m v \in D_p^m C((x,z),y)(u,w)$. It follows from the definition of $D_p^m C((x,z),y)(u,w)$ that $q^m v \in D^mF(x,y)(u)$ and $w \in D^1G(y,z)(q^m v)$. $\square$

Combining Propositions 6.3.9 - 6.3.13, we arrive at the following chain rule.

Proposition 6.3.14. *Suppose Y is finite dimensional and $(x,z) \in \mathrm{gr}(G \circ F)$ is such that C is compact and closed at (x,z).*

(i) *Assume that (6.17) holds for every $y \in C(x,z)$. Then*

$$D^m(G \circ F)(x,z)(u) \subseteq \bigcup_{y \in C(x,z)} D^1G(y,z)[D^mF(x,y)(u)]. \tag{6.19}$$

If, additionally, (6.15) holds for every $y \in C(x,z)$, then (6.19) is an equality.

(ii) *Assume that (6.18) holds for every $y \in C(x,z)$. Then*

$$D^m(G \circ F)(x,z)(u) \subseteq \bigcup_{y \in C(x,z)} D^mG(y,z)[D^1F(x,y)(u)]. \tag{6.20}$$

If, additionally, (6.16) holds for every $y \in C(x,z)$, then (6.20) is an equality.

Now we apply the preceding chain rules to establish sum rules for $M, N : X \to 2^Y$. For this purpose we use $F : X \to 2^{X \times Y}$ and $G : X \times Y \to 2^Y$ defined in (6.14). For $(x,z) \in X \times Y$, set

$$S(x,z) := M(x) \cap (z - N(x)).$$

Then, the resultant map $C : X \times Y \to 2^{X \times Y}$ associated to these F and G is

$$C(x,z) = \{x\} \times S(x,z).$$

Given $((x,z),y) \in \mathrm{gr}\, S$, the m-th order y-Studniarski derivative of $M+N$ at (x,z) is defined as, for $u \in X$,

$$D^m(M +_y N)(x,z)(u) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n, y_n, w_n) \to (u,y,w), \forall n, y_n \in S(x + t_n u_n, z + t_n^m w_n)\}.$$

Observe that

$$D^m(M +_y N)(x,z)(u) = D^m(G \circ_y F)(x,z)(u). \tag{6.21}$$

One has a relationship between $D^m(M +_y N)(x,z)(u)$ and $D^m(M+N)(x,z)(u)$ as noted in the next statement.

Proposition 6.3.15. *Let $(x,z) \in \mathrm{gr}(M+N)$ and $y \in S(x,z)$.*

(i) $D^m(M +_y N)(x,z)(u) \subseteq D^m(M+N)(x,z)(u)$.

(ii) *If S is compact and closed at (x,z), then*

$$\bigcup_{y \in S(x,z)} D^m(M +_y N)(x,z)(u) = D^m(M+N)(x,z)(u).$$

Proof. (i) This is an immediate consequence of the definitions.

(ii) When S is compact and closed at (x,z), C is compact and closed at (x,z). Hence, the equality in Proposition 6.3.9(ii) holds. In view of (6.21), this relation implies the required equality. □

For higher-order sum rules, we have

Proposition 6.3.16. *Let* $(x,z) \in \mathrm{gr}(M+N)$ *and* $y \in S(x,z)$.

(i) *Suppose, for all* $(u,v) \in X \times Y$,

$$D^m M(x,y)(u) \cap [v - D^m N(x,z-y)(u)] \subseteq D_p^m S((x,z),y)(u,v). \tag{6.22}$$

Then

$$D^m M(x,y)(u) + D^m N(x,z-y)(u) \subseteq D^m(M +_y N)(x,z)(u).$$

(ii) *If (6.22) holds for all* $y \in S(x,z)$, *then*

$$\bigcup_{y \in S(x,z)} (D^m M(x,y)(u) + D^m N(x,z-y)(u)) \subseteq D^m(M+N)(x,z)(u).$$

Proof. (i) Let $w \in D^m M(x,y)(u) + D^m N(x,z-y)(u)$, i.e., there exists $\bar{y} \in D^m M(x,y)(u)$ such that $\bar{y} \in w - D^m N(x,z-y)(u)$. Hence, (6.22) ensures that $\bar{y} \in D_p^m S((x,z),y)(u,w)$. Therefore, there exist $t_n \to 0^+$ and $(u_n, \bar{y}_n, w_n) \to (u, \bar{y}, w)$ such that $y + t_n^m \bar{y}_n \in S(x + t_n u_n, z + t_n^m w_n)$. Setting $y_n = y + t_n^m \bar{y}_n$, we have $y_n \in S(x + t_n u_n, z + t_n^m w_n)$. Consequently, $w \in D^m(M +_y N)(x,z)(u)$.

(ii) This follows from (i) and Proposition 6.3.15(i). □

We can impose an additional condition to get equalities in the above sum rules as follows.

Proposition 6.3.17. *Let* Y *be finite dimensional and* $(x,z) \in \mathrm{gr}(M+N)$.

(i) *Suppose, for* $y \in S(x,z)$,

$$D_p^m S((x,z),y))(0,0) = \{0\}. \tag{6.23}$$

Then

$$D^m(M +_y N)(x,z)(u) \subseteq D^m M(x,y)(u) + D^m N(x,z-y)(u).$$

(ii) *If* S *is compact and closed at* (x,z) *and (6.23) holds for every* $y \in S(x,z)$, *then one has*

$$D^m(M+N)(x,z)(u) \subseteq \bigcup_{y \in S(x,z)} (D^m M(x,y)(u) + D^m N(x,z-y)(u)). \tag{6.24}$$

If, additionally, (6.22) holds for every $y \in S(x,z)$, *then (6.24) becomes an equality.*

Proof. (i) Let $w \in D^m(M +_y N)(x,z)(u)$, i.e., there exist $t_n \to 0^+$ and $(u_n, y_n, w_n) \to (u, y, w)$ such that $y_n \in S(x + t_n u_n, z + t_n^m w_n)$. If $y_k = y$ for infinitely many $k \in \mathbb{N}$ one has $0 \in D^m M(x,y)(u)$ and $w \in D^m N(x, z - y)(u)$, and we are done. Thus, suppose $y_n \neq y$ for all n and, for $s_n := ||y_n - y||^{1/m}$, the sequence $v_n := s_n^{-m}(y_n - y)$ converges to v of norm one. If $t_n/s_n \to 0$, since

$$y + s_n^m v_n = y_n \in S\left(x + s_n(\frac{t_n u_n}{s_n}), z + s_n^m(\frac{t_n^m w_n}{s_n^m})\right),$$

one sees that $v \in D_p^m S((x,z),y)(0,0)$, contradicting (6.23). Consequently, s_n/t_n has a bounded subsequence and we may assume that s_n/t_n tends to $q \in R_+$. So,

$$y + t_n^m \left(\frac{s_n^m}{t_n^m} v_n\right) = y_n \in S(x + t_n u_n, z + t_n^m w_n)$$

and then $q^m v \in D_p^m S((x,z),y)(u,w)$. It follows from the definition of $D_p^m S((x,z),y)(u,w)$ that $q^m v \in D^m M(x,y)(u)$ and $w - q^m v \in D^m N(x, z - y)(u)$.

(ii) This follows from (i) and Propositions 6.3.15 and 6.3.16. □

Next, we define two other m-th order Studniarski derivatives, which are slight modifications of those in the above definitions and suitable for applications to variational inequalities in Section 6.4. Let P be also a normed space, $F : P \times X \to 2^Y$ and $N : P \times X \to 2^Y$. Let $\widehat{S} : P \times X \times Y \to 2^Y$ be given by

$$\widehat{S}(p,x,y) := F(p,x) \cap (y - N(p,x)).$$

Definition 6.3.18. Given $y_0 \in \widehat{S}(p,x,y)$ and $(u,v) \in P \times X$, we define

$$D^m(F +_{y_0} N)((p,x),y)(u,v) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n, v_n, y_n, w_n) \to (u, v, y_0, w),$$
$$y_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m w_n)\},$$

and

$$D_p^m \widehat{S}((p,x,y),y_0)(u,v,s) := \{w \in Y : \exists t_n \to 0^+, \exists (u_n, v_n, s_n, w_n) \to (u, v, s, w),$$
$$y_0 + t_n^m w_n \in \widehat{S}(p + t_n u_n, x + t_n^m v_n, y + t_n^m s_n)\}.$$

Proposition 6.3.19. *Let Y be finite dimensional and* $((p,x),y) \in \mathrm{gr}(F+N)$.

(i) *Suppose, for* $y_0 \in \widehat{S}(p,x,y)$,

$$D_p^m \widehat{S}((p,x,y),y_0))(0,0,0) = \{0\}. \tag{6.25}$$

Then

$$D^m(F+_{y_0}N)((p,x),y)(u,v) \subseteq D_p^m F((p,x),y_0)(u,v) + D_p^m N((p,x),y-y_0)(u,v).$$

(ii) *If $\widehat{S}$ is compact and closed at (p,x,y) and (6.25) holds for every $y_0 \in \widehat{S}(p,x,y)$, then one has*

$$D_p^m(F+N)((p,x),y)(u,v) \subseteq \bigcup_{y_0\in\widehat{S}(p,x,y)} (D_p^m F((p,x),y_0)(u,v) + D_p^m N((p,x),y-y_0)(u,v)).$$

Proof. (i) Let $w \in D^m(F+_{y_0}N)((p,x),y)(u,v)$, i.e., there exist $t_n \to 0^+$ and $(u_n,v_n,y_n,w_n) \to (u,v,y_0,w)$ such that $y_n \in \widehat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)$. If $y_k = y_0$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D_p^m F((p,x),y_0)(u,v)$ and $w \in D_p^m N((p,x),y-y_0)(u,v)$, and we are done. Now suppose $y_n \neq y_0$ for all n and, for $s_n := ||y_n - y_0||^{1/m}$, the sequence $l_n := s_n^{-m}(y_n - y_0)$ converges to l of norm one. If $t_n/s_n \to 0$, since

$$y_0 + s_n^m l_n = y_n \in \widehat{S}\left(p + s_n\frac{t_nu_n}{s_n}, x + s_n(\frac{t_n^m v_n}{s_n}), y + s_n^m(\frac{t_n^m w_n}{s_n^m})\right),$$

one sees that $l \in D_p^m\widehat{S}((p,x,y),y_0)(0,0,0)$, contradicting (6.25). Consequently, one may assume that s_n/t_n tends to $q \in R_+$. So,

$$y_0 + t_n^m\left(\frac{s_n^m}{t_n^m}l_n\right) = y_n \in \widehat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)$$

and thus $q^m l \in D_p^m\widehat{S}((p,x,y),y_0)(u,v,w)$. By the definition of $D_p^m\widehat{S}((p,x,y),y_0)(u,v,w)$, one has $q^m l \in D_p^m F((p,x),y_0)(u,v)$ and $w - q^m l \in D_p^m N((p,x),y-y_0)(u,v)$.

(ii) We need to prove that, if $\widehat{S}$ is compact and closed at (p,x,y), then

$$D_p^m(F+N)((p,x),y)(u,v) = \bigcup_{y_0\in\widehat{S}(p,x,y)} D^m(F+_{y_0}N)((p,x),y)(u,v).$$

The containment "$\supseteq$" follows from definitions. For "$\subseteq$", let $w \in D_p^m(F+N)((p,x),y)(u,v)$. There exist $t_n \to 0^+$ and $(u_n,v_n,w_n) \to (u,v,w)$ such that $y + t_n^m w_n \in F(p+t_nu_n, x+t_n^m v_n) + N(p+t_nu_n, x+t_n^m v_n)$. Then, one can find $y_n \in F(p+t_nu_n, x+t_n^m v_n)$ such that $y + t_n^m w_n - y_n \in N(p+t_nu_n, x+t_n^m v_n)$. Therefore, $y_n \in \widehat{S}(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n)$. Since $\widehat{S}$ is compact at (p,x,y), one may assume that y_n converges to y_0. As $(p+t_nu_n, x+t_n^m v_n, y+t_n^m w_n) \to (p,x,y)$, one has $y_0 \in (\mathrm{cl}\,\widehat{S})(p,x,y)$. It follows from the closedness of $\widehat{S}$ that $y_0 \in \widehat{S}(p,x,y)$. □

6.4 Applications

6.4.1 Studniarski derivatives of solution maps to inclusions

Let $M : P \times X \to 2^Z$ be a set-valued map between normed spaces. Then, the map S defined by

$$S(p) := \{x \in X : 0 \in M(p,x)\}, \tag{6.26}$$

is said to be the solution map of the *parametrized inclusion* $0 \in M(p,x)$.

Theorem 6.4.1. *For a solution map S defined by (6.26) and $\overline{x} \in S(\overline{p})$, we have, for $p \in P$,*

$$D^m S(\overline{p},\overline{x})(p) \subseteq \{x \in X : 0 \in D^m_p M((\overline{p},\overline{x}),0)(p,x)\}.$$

Proof. Let $(p,x) \in \mathrm{gr} D^m S(\overline{p},\overline{x})$, i.e., there exist sequences $p_n \to p$, $x_n \to x$, and $t_n \to 0^+$ such that $\overline{x}+t_n^m x_n \in S(\overline{p}+t_n p_n)$. This implies that 0 is an element of the set $M(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n)$. Hence, for $z_n = 0$, the inclusion $0+t_n^m z_n \in M(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n)$ holds, i.e., $0 \in D^m_p M((\overline{p},\overline{x}),0)(p,x)$. □

In parameterized optimization, we frequently meet M of the form

$$M(p,x) = F(p,x) + N(p,x), \tag{6.27}$$

where $F : P \times X \to 2^Z$ and $N : P \times X \to 2^Z$. Let $\widehat{S} : P \times X \times Z \to 2^Z$ be defined by

$$\widehat{S}(p,x,z) := F(p,x) \cap (z - N(p,x)).$$

The following theorem gives an approximation of the m-th order Studniarski derivative of S when M is defined by (6.27).

Theorem 6.4.2. *For the solution map $S(p) = \{x \in X : 0 \in F(p,x) + N(p,x)\}$ and $\overline{x} \in S(\overline{p})$ with Z being finite dimentional, suppose either of the following conditions hold*

(i) *$\widehat{S}$ is compact and closed at $(\overline{p},\overline{x},0)$ and $D^m_p \widehat{S}((\overline{p},\overline{x},0),y)(0,0,0) = \{0\}$ for all $y \in \widehat{S}(\overline{p},\overline{x},0)$;*

(ii) *there exists $y \in \widehat{S}(\overline{p},\overline{x},0)$ such that either F or N is m-th order locally Hölder calm at $(\overline{p},\overline{x})$ for y or $-y$, respectively.*

Then

$$D^m S(\overline{p},\overline{x})(p) \subseteq \{x \in X : 0 \in \bigcup_{y \in \widehat{S}(\overline{p},\overline{x},0)} (D^m_p F((\overline{p},\overline{x}),y)(p,x) + D^m_p N((\overline{p},\overline{x}),0-y)(p,x))\}.$$

Proof. We first prove that

$$D_p^m M((\overline{p},\overline{x}),0)(p,x) \subseteq \bigcup_{y\in\widehat{S}(\overline{p},\overline{x},0)} (D_p^m F((\overline{p},\overline{x}),y)(p,x) + D_p^m N((\overline{p},\overline{x}),0-y)(p,x)).$$

If (i) holds, the above inclusion is followed by Proposition 6.3.19. For the case (ii) and $y \in \widehat{S}(\overline{p},\overline{x},0)$, we see that $y \in F(\overline{p},\overline{x})$ and $-y \in N(\overline{p},\overline{x})$. Let $v \in D_p^m M((\overline{p},\overline{x}),0)(p,x)$, i.e., there exist $t_n \to 0^+$, $(p_n,x_n) \to (p,x)$, and $v_n \to v$ such that

$$0 + t_n^m v_n \in M(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n) = F(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n) + N(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n).$$

Then, there exist $y_n^1 \in F(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n)$ and $y_n^2 \in N(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n)$ such that

$$v_n = t_n^{-m}(y_n^1 - y) + t_n^{-m}(y_n^2 - (-y)). \tag{6.28}$$

Suppose F is m-th order locally Hölder calm at at $(\overline{p},\overline{x})$ for y. Then, there exists $L > 0$ such that for large n,

$$y_n^1 \in F(\overline{p}+t_n p_n, \overline{x}+t_n^m x_n) \subseteq \{y\} + L||(t_n p_n, t_n^m x_n)||^m B_Z.$$

Because $\dim Z < +\infty$, $t_n^{-m}(y_n^1 - y)$, or a subsequence, converges to some $v^1 \in Z$ and so $v^1 \in D_p^m F((\overline{p},\overline{x}),y)\ (p,x)$. From (6.28), the sequence $t_n^{-m}(y_n^2 - (-y))$ also converges to some v^2 such that $v^2 = v - v^1$, and $v^2 \in D_p^m N((\overline{p},\overline{x}),-y)(p,x)$. Thus,

$$v \in D_p^m F((\overline{p},\overline{x}),y)(p,x) + D_p^m N((\overline{p},\overline{x}),-y)(p,x).$$

Now applying Theorem 6.4.1 completes the proof. □

6.4.2 Implicit multifunction theorems

Let $M : P\times X \to Z$ and $S(p) := \{x \in X : M(p,x) = 0\}$, be the set of solutions to the *parametrized equation* $M(x,p) = 0$. We impose the condition

$$(*) \begin{cases} \exists\overline{x} \in X \text{ such that } M(0,\overline{x}) = 0 \text{ and} \\ M_p \text{ is continuous in a neighborhood } (U,V) \in \mathcal{N}(0)\times\mathcal{N}(\overline{x}), \end{cases}$$

where M_p denotes the partial Fréchet derivative with respect to p. Let $H = V \cap M(0,.)^{-1}$, i.e.,

$$H(z) = \{x \in V : M(0,x) = z\}.$$

Under the hypotheses of usual implicit function theorems for $M \in C^1$, S and H are single-valued and smooth (with derivatives DS, DH), and there holds

$$DS(0) = -DH(0)M_p(0,\bar{x}) = -M_x(0,\bar{x})^{-1}M_p(0,\bar{x}).$$

Now we are interested in a similar formula of the m-th order Studniarski derivative $D^mS(0,\bar{x})(.)$ of the map S under the assumption $(*)$. For (p,x) near $(0,\bar{x})$, we consider the map

$$r(p,x) := M(p,x) - M(0,x) - M_p(0,\bar{x})p.$$

By the mean-value theorem, one obtains

$$r(p,x) = \int_0^1 [M_p(\theta p,x) - M_p(0,\bar{x})]p d\theta,$$

where

$$\alpha(p,x,\theta) := ||M_p(\theta p,x) - M_p(0,\bar{x})||$$

can be estimated (uniformly for $0 < \theta < 1$) by

$$\alpha(p,x,\theta) \leq 0(p,x) \text{ with } 0(p,x) \to 0^+ \text{ as } x \to \bar{x} \text{ and } ||p|| \to 0^+.$$

Due to $||r(p,x)|| \leq 0(p,x)||p||$, one easily sees that $||p||^{-1}||r(p,x)|| \to 0^+$ as $x \to \bar{x}$ and $||p|| \to 0^+$, and

$$r(p(t),x(t)) = o_2(t) \text{ if } x(t) \to \bar{x} \text{ and } p(t) = tq + o_1(t) \text{ with some } q \in P,$$

where $o_k(t)$ means that $\frac{||o_k(t)||}{t} \to 0^+$ as $t \to 0^+$.

For (p,x) near $(0,\bar{x})$, we have

$$M(p,x) = 0 \text{ if and only if } M(0,x) = -M_p(0,\bar{x})p - r(p,x),$$

i.e.,

$$x \in S(p) \text{ if and only if } x \in H(-M_p(0,\bar{x})p - r(p,x)).$$

Let $\widehat{M} : P \times X \to Z$ be defined by

$$\widehat{M}(p,x) := -M_p(0,\bar{x})(p) - r(p,x).$$

Then, $x \in S(p)$ if and only if $x \in H(\widehat{M}(p,x))$. Set $C(p,x) := \widehat{M}(p,x) \cap H^{-1}(x)$. It is easy to see that $C(0,\bar{x}) = \{0\}$.

The following result is a slight modification of that in Proposition 6.3.13(ii).

Lemma 6.4.3. *Let Z be finite dimentional and either of the following conditions hold*

(i) $\widehat{M}$ *is locally Lipschitz calm at* $(0,\bar{x})$ *for* 0*;*

(ii) C *is compact and closed at* $(0,\bar{x})$ *and*

$$D_p^{m(1)}C((0,\bar{x}),0)(0,0) = \{0\}. \tag{6.29}$$

Then, $x \in D^m S(0,\bar{x})(q)$ *implies that* $x \in D^m H(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$.

Proof. Let $x \in D^m S(0,\bar{x})(q)$, i.e., there exist $t_n \to 0^+$, $q_n \to q$, and $x_n \to x$ such that $\bar{x} + t_n^m x_n \in S(0+t_n q_n)$. This implies that

$$\bar{x} + t_n^m x_n \in H(\widehat{M}(0+t_n q_n, \bar{x}+t_n^m x_n)). \tag{6.30}$$

Then, there exists $y_n \in \widehat{M}(0+t_n q_n, \bar{x}+t_n^m x_n)$ such that $\bar{x}+t_n^m x_n \in H(y_n)$. Suppose (i) hold. Because $\widehat{M}$ is locally Lipschitz calm at $(0,\bar{x})$ for 0, there exists $L > 0$ such that, for large n,

$$y_n \in \widehat{M}(0+t_n q_n, \bar{x}+t_n^m x_n) \subseteq \{0\} + L||(t_n q_n, t_n^m x_n)||B_Z.$$

Since $\dim Z < +\infty$, $v_n := (t_n)^{-1}(y_n - 0)$ converges to some $v \in Z$. So, $v \in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)$. It implies that $\bar{x}+t_n^m x_n \in H(0+t_n v_n)$. Thus, $x \in D^m H(0,\bar{x})(v)$.

If assumption (ii) holds, it follows from (6.30) that there exists

$$y_n \in \widehat{M}(0+t_n q_n, \bar{x}+t_n^m x_n) \cap H^{-1}(\bar{x}+t_n^m x_n) = C(0+t_n q_n, \bar{x}+t_n^m x_n).$$

Since C is compact at $(0,\bar{x})$, y_n (or a subsequence) has a limit y. Since $(0+t_n q_n, \bar{x}+t_n^m x_n, y_n) \to (0,\bar{x},y)$, ones has $y \in (\mathrm{cl}C)(0,\bar{x})$. It follows from the closedness of C at $(0,\bar{x})$ that $y \in C(0,\bar{x}) = 0$.

If $y_k = 0$ for infinitely many $k \in \mathbb{N}$, one has $0 \in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,w)$ and $w \in D^m H(0,\bar{x})(0)$, and we are done. Thus, one may suppose, for $s_n := ||y_n||$, the sequence $v_n := y_n/s_n$ has a limit v of norm one. If $t_n/s_n \to 0$, since

$$0 + s_n v_n = y_n \in C\left(0 + s_n(\frac{t_n q_n}{s_n}), \bar{x} + s_n^m(\frac{t_n^m x_n}{s_n^m})\right),$$

one sees that $v \in D_p^{m(1)}C((0,\bar{x}),0)(0,0)$, contradicting (6.29). Consequently, one may assume that s_n/t_n converges to $q \in R_+$. So,

$$0 + t_n\left(\frac{s_n}{t_n}v_n\right) = y_n \in C(0+t_n q_n, \bar{x}+t_n^m x_n)$$

and thus $qv \in D_p^{m(1)}C((0,\bar{x}),0)(q,x)$. It follows from the definition of $D_p^{m(1)}C((0,\bar{x}),0)(q,x)$ that $qv \in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)$ and $x \in D^m H(y,z)(qv)$.

□

Lemma 6.4.4. *Let Z be finite dimentional, the asumptions of Lemma 6.4.3 be satisfied and*

$$D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)\cap(D^mH(0,\bar{x}))^{-1}(x)\subseteq D_p^{m(1)}C((0,\bar{x}),0)(q,x). \tag{6.31}$$

Then, $x\in D^mS(0,\bar{x})(q)$ if and only if $x\in D^mH(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$.

Proof. By Lemma 6.4.3, we need to prove that $x\in D^mH(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$ implies $x\in D^mS(0,\bar{x})(q)$. $x\in D^mH(0,\bar{x})[D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)]$ means the existence of $v\in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)\cap(D^mH(0,\bar{x}))^{-1}(x)$. Then, (6.31) ensures that $v\in D_p^{m(1)}C((0,\bar{x}),0)(q,x)$. This means the existence of $t_n\to 0^+$ and $(q_n,x_n,v_n)\to(q,x,v)$ such that

$$0+t_nv_n\in C(0+t_nq_n,\bar{x}+t_n^mx_n).$$

From the definition of the map C, we get $0+t_nv_n\in\widehat{M}(0+t_nq_n,\bar{x}+t_n^mx_n)$ and $\bar{x}+t_n^mx_n\in H(0+t_nv_n)$, which imply that $\bar{x}+t_n^mx_n\in H(\widehat{M}(0+t_nq_n,\bar{x}+t_n^mx_n))$. Thus, we have $\bar{x}+t_n^mx_n\in S(0+t_nq_n)$ and $x\in D^mS(0,\bar{x})(q)$.

□

Theorem 6.4.5. *Impose the assumptions of Lemma 6.4.3. Then,*

$$D^mS(0,\bar{x})(q)\subseteq D^mH(0,\bar{x})[-M_p(0,\bar{x})(q)]. \tag{6.32}$$

If, additionally, (6.31) holds, then (6.32) becomes an equality.

Proof. By Lemmas 6.4.3 and 6.4.4, we need to prove that $D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)=-M_p(0,\bar{x})(q)$. Let $v\in D_p^{m(1)}\widehat{M}((0,\bar{x}),0)(q,x)$. There exist $t_n\to 0^+$ and $(q_n,x_n,v_n)\to(q,x,v)$ such that

$$0+t_nv_n=\widehat{M}(0+t_nq_n,\bar{x}+t_n^mx_n)=-M_p(0,\bar{x})(0+t_nq_n)-r(0+t_nq_n,\bar{x}+t_n^mx_n).$$

Therefore,

$$v_n=-M_p(0,\bar{x})(q_n)-t_n^{-1}r(t_nq_n,\bar{x}+t_n^mx_n)\to -M_p(0,\bar{x})(q).$$

Thus, $v=-M_p(0,\bar{x})(q)$ and we are done. □

Conclusions

In this thesis, we have presented results related to some topics of variational analysis. First, we have stated definitions and basic properties of Γ-limits. Then, several generalized results on sequential forms of Γ-limits have been given. We have also got important applications of Γ-limits to tangency and generalized differentiation theory. It turns out that most of generalized derivatives can be expressed in terms of Γ-limits. Finally, we have introduced some kinds of generalized derivatives and their applications. In detail,

• We have discussed higher-order analysis for quantitative properties of perturbation maps of nonsmooth vector optimization in terms of variational sets, a kind of generalized derivatives which is suitable for a high level of nonsmoothness and relatively easy to compute. We have established relations between variational sets of a perturbation map and weak perturbation map or the efficiency/weak efficiency of these sets and the corresponding ones of the feasible-set map to the objective space. These results have been applied to sensitivity analysis for set-constrained vector optimization. As some results look complicated, we have tried to confirm the essentialness of each imposed assumption, as well as to illustrate advantages of our results by a number of examples, which indicate also that computing variational sets is not a hard work.

• Realizing advantages in some aspects of radial sets and derivatives, we have aimed to establish both necessary and sufficient higher-order conditions in terms of radial sets and derivatives for various kinds of efficiency concepts in set-valued vector optimization. We have chosen the Q-efficiency defined in [88] to unify these concepts. Thus, we have first discussed optimality conditions for Q-efficiency and then rephrase the results for the other kinds of solutions. Besides, we have also discussed properties and basic calculus rules of radial sets and derivatives like those for a sum or composition of maps. Furthermore, direct applications of these rules in proving optimality conditions for some particular problems have been given.

• Some calculus rules for Studniarski derivatives have been given to ensure that it can be used in practice. Most of the usual rules, from the sum and chain rules to various operations in analysis, have been investigated. It turns out that Studniarski derivatives possesses many fundamental and comprehensive calculus rules. Although this construction is not comparable with objects in the dual approach like Mordukhovich's coderivatives (see books [129, 130] and papers [89, 90, 128]) in enjoying rich calculus, it may be better in dealing with higher-order properties. We have paid attention also on relations between the established calculus rules and applications of some rules to get others. As such applications we have provided a direct employment of sum rules to establishing an explicit formula for the Studniarski derivative of the solution map to a parametrized inclusion in terms of Studniarski derivatives of the data. Furthermore, chain rules have been also used to get implicit multifunction theorems.

Further works

1) Studying on stability theories with respect to solution sets under perturbations of the data has been of great interest in variational convergence. Some papers presented the convergence of solution sets and efficient sets of perturbed problems in general convergence spaces, by using some kinds of convergence, such as adherence and persistence, see [51, 56]. For vector problems, the notion of continuous convergence of the sequence of perturbed vector valued objective functions has usually been used, see [51, 56, 122, 146, 169, 171]. Extending this convergence, in [132, 133], by using Γ-limits, Oppezzi and Rossi introduced the notion of Γ-convergence for sequences of vector valued functions and investigated stability results for convex vector optimization problems. Some recent papers dealing with this convergence are [9, 112, 113, 120, 123]. For possible developments of Chapter 3, we think that, by using Γ-limits, ones can get stability results to vector programming problem, where the objective function holds some conditions on convexity as the inequality constraints, while the equality constraints are linear.

A well established practice to solve a vector optimization problem is through an associated scalar optimization problem. There are various approaches, see [121, 126, 127, 143], to scalarize a vector optimization and obtain a complete characterization for several types of solution sets of the vector optimization in terms of solution sets of the scalarized problem. Thus, we can establish the convergence of solution sets of scalarized problems by using Γ-limits.

2) For possible developments of Chapter 4, we think that, besides going deeper in relations for perturbation and weak perturbation maps in terms of variational sets for set-constrained problems, one can consider the most important case of optimization problems with constraints defined by inequalities and equalities. Furthermore, sensitivity analysis in terms of generalized derivatives other than contingent ones and variational sets is a promising research direction. In fact, first results of this kind have been obtained very recently by Diem et al. in [49], using the so-called contingent-radial derivatives. This generalized derivative is defined by a combination of the ideas of contingent derivatives and radial ones (for the latter derivative see recent works

of Anh and Khanh in [2, 5]).

3) Note that separation theorems are a basic tool for proving optimality conditions and duality statements. With the development of nonconvex optimization theory, there has come into existence a need for nonconvex separation theorems, see [25]. So, for possible developments of Chapter 5, we can derive optimality conditions of several kinds of efficient solutions for set-valued optimization problems by using a nonconvex separation function given by Certh and Weidner in [25]. Then, we can use these results to derive duality results for vector optimization when the objectives and the constraints are nonconvex. Besides, motivated by [14, 30, 31], we can obtain some calculations of higher-order radial sets in certain functional spaces.

Publications

1) N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Variational sets : calculus and applications to nonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 2358 - 2379.

2) N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Higher-order radial derivatives and optimality conditions innonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 7365 - 7379.

3)* N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives*, J. Global Optim. **56** (2013), 519 - 536.

4)* N.L.H. Anh and P.Q. Khanh, *Variational sets of perturbation maps and applications to sensitivity analysis for constrained vector optimization*, J. Optim. Theory Appl. **158** (2013), 363 - 384.

5)* N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions for proper efficiency in nonsmooth vector optimization using radial sets and radial derivatives*, J. Global Optim., DOI10.1007/s10898-013-0077-7 (2013), Online First.

6) N.L.H. Anh, *Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality*, Positivity, DOI10.1007/s11117-013-0254-4 (2013), Online First.

7) N.L.H. Anh and P.Q. Khanh, *Higher-order radial epiderivatives and optimality conditions in nonsmooth vector optimization*, submitted to Math. Meth. Oper. Res.

8)* N.L.H. Anh and P.Q. Khanh, *Calculus and applications of Studniarski derivatives to sensitivity and implicit function theorems*, Control and Cybernetics, Accepted.

*: The contents of these papers are presented in the thesis.

Bibliography

[1] N.L.H. Anh, *Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality*, Positivity, DOI10.1007/s11117-013-0254-4 (2013), Online First. ↑4

[2] N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives*, J. Global Optim. **56** (2013), 519 – 536. ↑1, 2, 72, 122

[3] N.L.H. Anh and P.Q. Khanh, *Variational sets of perturbation maps and applications to sensitivity analysis for constrained vector optimization*, J. Optim. Theory Appl. **158** (2013), 363–384. ↑46

[4] N.L.H. Anh and P.Q. Khanh, *Higher-order optimality conditions for proper efficiency in nonsmooth vector optimization using radial sets and radial derivatives*, J. Global Optim., DOI10.1007/s10898-013-0077-7 (2013), Online First. ↑1, 2, 72

[5] N.L.H. Anh and P.Q. Khanh, *Higher-order radial epiderivatives and optimality conditions in nonsmooth vector optimization*, submitted to Math. Meth. Oper. Res. ↑122

[6] N.L.H. Anh and P.Q. Khanh, *Calculus and applications of Studniarski derivatives to sensitivity and implicit function theorems*, Control and Cybernetics, Accepted. ↑98

[7] N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Variational sets : calculus and applications to nonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 2358 – 2379. ↑3, 4, 45, 47, 71

[8] N.L.H. Anh, P.Q. Khanh, and L.T. Tung, *Higher-order radial derivatives and optimality conditions innonsmooth vector optimization*, Nonlinear Anal. TMA **74** (2011), 7365 – 7379. ↑1, 2, 72, 98

[9] H. Attouch, *Variational convergence for functions and operators*, Pitman, Boston, 1984. ↑121

[10] J.-P.H. Aubin, *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, in "Advances in Mathematics, Supplementary studies ", E. Nachbin (ed), Pitagora, Bologna, (1981), 160 – 232. ↑97

[11] J.-P. H. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990. ↑2, 3, 4, 44, 47, 50

[12] A. Auslender and M. Teboulle, *Asymptotic cones and functions in optimization and variational inequalities*, Springer-Verlag, New York, 2003. ↑44

[13] M.S. Bazaraa and C.M. Shetty, *Foundations of optimization*, Springer-Verlag, Berlin, 1976. ↑44

[14] E.M. Bednarczuk, M. Pierre, E. Rouy, and J Sokolowski, *Tangent sets in some functional spaces*, Nonlinear Anal. **42** (2000), 871 – 886. ↑1, 122

[15] E.M. Bednarczuk and W. Song, *Contingent epiderivative and its applications to set-valued maps*, Control Cybern. **27** (1998), 375 – 386. ↑1

[16] G. Beer, R.T. Rockafellar, and R.J.-B. Wets, *A characterization of epi-convergence in terms of convergence of level sets*, Proc. Am. Math. Soc. **116** (1992), 753 – 761. ↑1

↑: cited on page

[17] F. Bigolin and G.H. Greco, *Geometric characterizations of C1 manifolds in Euclidean spaces by tangent cones*, J. Math. Anal. Appl. **396** (2012), 145 – 163. ↑44

[18] G. Birkhoff, *Lattice theory*, Am. Math. Soc. Coll. Publ., Providence, 1973. ↑6

[19] J.F. Bonnans and A. Shapiro, *Perturbation analysis of optimization problems*, Springer, New York, 2000. ↑40, 45

[20] G. Bouligand, *Introdution à la géométrie infinitésimale directe*, Gauthier-Villars, Paris, 1932. ↑41

[21] N. Bourbaki, *Variétés Différentielles et Analytiques*, Éléments de Mathématique 13, Fascicule de Résultats, Hermann, Paris, 1967. ↑40

[22] G. Buttazzo and G. Dal Maso, Γ-*convergence and optimal control problems*, J. Optim. Theory Appl. **38** (1982), 385 – 407. ↑1

[23] H. Cartan, *Thorie des filtres*, CR Acad. Paris **205** (1937), 595 – 598. ↑5

[24] H. Cartan, *Filtres et ultrafiltres*, CR Acad. Paris **205** (1937), 777 – 779. ↑5

[25] C. Certh and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl. **67** (1990), 297 – 320. ↑122

[26] Y. Chen and J. Jahn, *Optimality conditions for set-valued optimization problems*, Math. Methods Oper. Res. **48** (1998), 187 – 200. ↑71, 90, 91

[27] C.R. Chen, S.J. Li, and K.L. Teo, *Higher order weak epiderivatives and applications to duality and optimality conditions*, Comp. Math. Appl. **57** (2009), 1389 – 1399. ↑4

[28] G. Choquet, *Lectures on Analysis. Vol. II: Representation Theory*, W. A. Benjamin, New York-Amserdam, 1969. ↑9

[29] F.H. Clarke, *Optimization and nonsmooth analysis*, New York, Wiley, 1983. ↑3

[30] R. Cominetti and J.-P. Penot, *Tangent sets to unilateral sets*, CRAS série I **321** (1995), 1631 – 1636. ↑122

[31] R. Cominetti and J.-P. Penot, *Tangent sets of order one and two to the positive cones of some functional spaces*, Appl. Math. Optim. **36** (1997), 291 – 312. ↑122

[32] A. Contreras and P. Sternberg, *Gamma-convergence and the emergence of vortices for GinzburgLandau on thin shells and manifolds*, Calculus of Variations and Partial Differential Equations **38** (2010), 243 – 274. ↑1

[33] W. Corley, *Optimality conditions for maximization of set-valued functions*, J. Optim. Theory Appl. **58** (1988), 1 – 10. ↑90

[34] G. P. Crespi, I. Ginchev, and M. Rocca, *First-order optimality conditions in set-valued optimization*, Math. Methods Oper. Res. **63** (2006), 87 – 106. ↑90

[35] G. P. Crespi, I. Ginchev, and M. Rocca, *First-order optimality conditions for constrained set-valued optimization*, Pac. J. Optim. **2** (2006), 225 – 239. ↑90

[36] G. Dal Maso, *An introduction to* $\Gamma-$*convergence. Progress in Nonlinear Differential Equations and their Application*, Vol. 8, Birkhäuser, Boston, 1993. ↑16, 17, 19

[37] G. Dal Maso, *Ennio De Giorgi and* $\Gamma-$*convergence*, Dis. Con. Dyn. Sys. **31** (2011), 1017 – 1021. ↑1

[38] E. De Giorgi, $\Gamma-$*convergenza e G*$-$*convergenza*, Boll. Un. Mat. Ital. **14** (1977), 213 – 220. ↑13, 14, 19

[39] E. De Giorgi, $\Gamma-$*limiti di ostacoli*, in "Recent Methods in Nonlinear Analysis and Applications, "Proc. SAFA IV (Napoli, 1980), A. Canfora, S. Rionero, C. Sbordone, and G. Trombetti (eds.), Liguori, Napoli, (1981), 51 – 84. ↑13

[40] E. De Giorgi, *Generalized limits in calculus of variations*, in "Topics in Functional Analysis " (1980-81), Quad. Sc. Norm. Sup. Pisa, Pisa, (1981), 117 – 148. ↑13

[41] E. De Giorgi, *Operatori elementari di limite ed applicazioni al Calcolo delle Variazioni*, Atti del convegno "Studio di problemi limite della Analisi Funzionale " (Bressanone, 1981), Pitagora, Bologna, (1982), 101 – 116. ↑13

[42] E. De Giorgi, *$G-$operators and $\Gamma-$convergence*, in "Proc. Internat. Congr. Math., Vol. 1,2 " (Warsaw, 1983), PWN, Warsaw, (1984), 1175 – 1191. ↑14

[43] E. De Giorgi and G. Buttazzo, *Limiti generalizzati e loro applicazioni alle equazioni differenziali*, Matematiche **36** (1981), 53 – 64. ↑13

[44] E. De Giorgi and G. Dal Maso, *$\Gamma-$convergence and the calculus of variations*, in "Mathematical Theory of Optimization " (Genova, 1981), T. Zolezzi (eds.), Lecture Notes in Math., **979**, Springer, Berlin, (1983), 121 – 143. ↑13

[45] E. De Giorgi, G. Dal Maso, and P. Longo, *$\Gamma-$limiti di ostacoli*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **68** (1980), 481 – 487. ↑13

[46] E. De Giorgi, G. Dal Maso, and L. Modical, *Convergenza debole di misure su spazi di funzioni semicontinue*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **79** (1985), 98 – 106. ↑14

[47] E. De Giorgi, G. Dal Maso, and L. Modical, *Weak convergence of measures on spaces of semicontinuous functions*, Proc. of the Int. Workshop on Integral Functionals in the Calculus of Variations (Trieste, 1985), Rend. Circ. Mat. Palermo **15** (1987), 59 – 100. ↑14

[48] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **58** (1975), 842 – 850. ↑1, 13

[49] H.T.H. Diem, P.Q. Khanh, and L.T. Tung, *On higher-order sensitivity analysis in nonsmooth vector optimization*, J. Optim. Theory Appl., DOI10.1007/s10957-013-0424-3 (2013), Online First. ↑98, 121

[50] S. Dolecki, *Tangency and differentiation : some applications of convergence theory*, Ann. Mat. Pura Appl. **130** (1982), 223 – 255. ↑1, 15, 42, 44

[51] S. Dolecki, *Convergence of minima in convergence spaces*, Optimization **17** (1986), 553 – 572. ↑121

[52] S. Dolecki, *Active boundaries of upper semi-continuous and compactoid relations closed and inductively perfect maps*, Rostock Math. Coll. **54** (2000), 51 – 68. ↑24

[53] S. Dolecki, *An initiation into convergence theory*, in Beyond topology (F. Mynardand, and E. Pearl, eds.), Contemporary Mathematics Series A.M.S **486** (2009), 115 – 162. ↑6, 24, 25

[54] S. Dolecki and G.H. Greco, *Towards historical roots of necessary conditions of optimality: Regula of Peano*, Control and Cybernetics **36** (2007), 491 – 518. ↑42

[55] S. Dolecki, G.H. Greco, and A. Lechicki, *Compactoid and compact filters*, Pacific J. Math. **117** (1985), 69 – 98. ↑5

[56] S. Dolecki and C. Malivert, *Stability of efficient sets: continuity of mobile polarites*, Nonlinear Anal. TMA **12** (1988), 1461 – 1486. ↑121

[57] A.V. Dovzhenko, P.I. Kogut, and R. Manzo, *On the concept of $\Gamma-$convergence for locally compact vector-valued mappings*, Far. East J. Appl. Math. **60** (2011), 1 – 39. ↑1

[58] A.J. Dubovitskii and A.A. Milyutin, *Extremal problems with constraints*, USSR Comp. Math. Math. Phys. **5** (1965), 395 – 453. ↑45

[59] M. Durea, *Global and local optimality conditions in set-valued optimization problems*, Int. J. Math. Math. Sci. **11** (2005), 1693 – 1711. ↑71

[60] K.H. Elser and J. Thierfelder, *The general concept of cone aproximations in nondifferentiable optimization*, Lecture Notes in Econom. and Math. Systems **255** (1984), 170 – 189. ↑44

[61] K.H. Elser and J. Thierfelder, *Abstract cone aproximations and generalized differentiability in nonsmooth optimization*, Optimization **19** (1988), 315 – 341. ↑44

[62] R. Engelking, *General Topology*, Warszawa, Poland, 1977. ↑24

[63] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418 – 491. ↑42

[64] A.V. Fiacco, *Introduction to sensitivity and stability analysis in nonlinear programming*, Academic Press, New York, 1983. ↑2

[65] F. Flores-Bazan, *Optimality conditions in non-convex set-valued optimization*, Math. Meth. Oper. Res. **53** (2001), 403 – 417. ↑4

[66] F. Flores-Bazan, *Radial epiderivatives and asymptotic function in nonconvex vector optimization*, SIAM J. Optim. **14** (2003), 284 – 305. ↑2, 4

[67] F. Flores-Bazan and B. Jimenez, *Strict efficiency in set-valued optimization*, SIAM J. Control Optim. **48** (2009), 881 – 908. ↑4, 71

[68] H. Frankowska and M. Quincampoix, *Hölder metric regularity of set-valued maps*, Math. Prog. **132** (2012), 333 – 354. ↑4

[69] M. Fréchet, *Sur la notion de différentielle*, C.R.A.Sc. Paris **152** (1911), 845 – 847. ↑39

[70] M. Fréchet, *Sur la notion de différentielle*, C.R.A.Sc. Paris **152** (1911), 1950 – 1951. ↑39

[71] I. Ginchev, *Higher-order optimality conditions in nonsmooth optimization*, Optimization **51** (2002), 47 – 72. ↑2, 40

[72] I. Ginchev, *Higher-order optimality conditions in nonsmooth vector optimization*, in "Generalized Convexity, Generalized Monotonicity, Optimality Conditions and Duality in Scalar and Vector Optimization ", A. Cambini, B.K. Dass, and L. Martein (eds.), J. Stat. Manag. Syst., **5** (2002), 321 – 339. ↑2

[73] I. Ginchev and A. Guerraggio, *Second order optimality conditions in nonsmooth unconstrained optimization*, Pliska Stud. Math. Bulg. **12** (1998), 39 – 50. ↑40

[74] I. Ginchev, A. Guerraggio, and M. Rocca, *First order conditions for $C^{0,1}$ constrained vector optimization*, in "Variational Analysis and Applications ", F. Giannessi, and A. Maugeri (eds.), Kluwer Academic, London (2005), 427 – 450. ↑40

[75] I. Ginchev, A. Guerraggio, and M. Rocca, *Second order conditions for $C^{1,1}$ constrained vector optimization*, Math. Program. Ser. B **104** (2005), 389 – 405. ↑40

[76] I. Ginchev, A. Guerraggio, and M. Rocca, *From scalar to vector optimization*, Appl. Math. **51** (2006), 5 – 36. ↑40

[77] G. Giorgi, B. Jiménez, and V. Novo, *An overview of second order tangent sets and their application to vector optimization*, Bol. Soc. Esp. Mat. Apl. (2010), 73 – 96. ↑1, 44

[78] G. Giorgi and A. Guerraggio, *On the notion of tangent cone in mathematical programming*, Optimization **25** (1992), 11 – 23. ↑44

[79] G. Giorgi and A. Guerraggio, *Characterizations, comparisons, algebraic and topological properties of tangent cones*, J. Stat. Manag. Sci. **5** (2002), 275 – 294. ↑44

[80] X. H. Gong, H.B. Dong, and S.Y. Wang, *Optimality conditions for proper efficient solutions of vector set-valued optimization*, J. Math. Anal. Appl. **284** (2003), 332 – 350. ↑71, 89

[81] H.G. Grassmann, *Lehrbuch der Arithmetik für höhere Lehranstalten*, Berlin, 1861. ↑39

[82] G. H. Greco, *Limitoidi e reticoli completi*, Ann. Univ. Ferrara **29** (1983), 153 – 164. ↑1, 6, 14, 20, 21, 22, 23

[83] G. H. Greco, *Decomposizioni di semifiltri e $\Gamma-$limiti sequenziali in reticoli completamente distributivi*, Ann. Mat. Pura Appl. **137** (1984), 61 – 81. ↑1, 14, 20, 21, 22, 24, 26, 37

[84] G. H. Greco, *Operatori di tipo G su reticoli completi*, Rend. Sem. Mat. Univ. Padova **72** (1984), 277 – 288. ↑1, 14

[85] G. Guareschi, *Un concetto di derivazione delle funzioni di più variabili reali più ampio di quello della derivazione parziale*, Mem. Reale Accad. d'Italia **5** (1934), 173 – 207. ↑42

[86] G. Guareschi, *Sulla differenziabilità delle funzioni di una o più variabili complesse*, Mem. Accad. Sci. Torino **68** (1936), 169 – 186. ↑42

[87] A. Guerraggio, E. Molho, and A. Zaffaroni, *On the notion of proper efficiency in vector optimization*, J. Optim. Theory Appl. **82** (1994), 1 – 21. ↑12, 71

[88] T.X.D. Ha, *Optimality conditions for several types of efficient solutions of set-valued optimization problems*, in "Nonlinear Analysis and Variational Problems ", P. Pardalos, Th.M. Rassis and A.A. Khan (eds.), Springer, Berlin (2009), 305 – 324. ↑9, 10, 12, 119

[89] A. D. Ioffe, *Metric regularity and subdifferential calculus*, Russ. Math. Surv. **55** (2000), 501 – 558. ↑40, 45, 120

[90] A. D. Ioffe and J.-P. Penot, *Subdifferentials of performance functions and calculus od set-valued mappings*, Serdica Math. J. **22** (1996), 359 – 384. ↑120

[91] C. Jacobi, *De determinantibus functionalibus*, Issue 78 of Ostwald's Klassiker der exacten Wissenschaften, 1841. ↑39

[92] J. Jahn, *Vector Optimization - Theory, Applications, Extensions*, Springer, Berlin, 2004. ↑3

[93] J. Jahn and A.A. Khan, *Generalized contingent epiderivatives in set valued optimization : optimality conditions*, Numer. Funct. Anal. Optim. **23** (2002), 807 – 831. ↑90

[94] J. Jahn and A.A. Khan, *Some calculus rules for contingent epiderivatives*, Optimization **52** (2003), 113 – 125. ↑4, 93, 94, 96

[95] J. Jahn, A.A. Khan, and P. Zeilinger, *Second-order optimality conditions in set optimization*, J. Optim.Theory Appl. **125** (2005), 331 – 347. ↑71, 90

[96] J. Jahn and R. Rauh, *Contingent epiderivatives and set-valued optimization*, Math. Meth. Oper. Res. **46** (1997), 193 – 211. ↑3, 90, 91

[97] B. Jimenez, *Strict minimality conditions in nondifferentiable multiobjective programming*, J. Optim. Theory Appl. **116** (2003), 99– 116. ↑40, 98

[98] B. Jimenez and V. Novo, *First and second-order sufcient conditions for strict minimality in nonsmooth vector optimization*, J. Math. Anal. Appl. **284** (2003), 496– 510. ↑40, 98

[99] B. Jimenez and V. Novo, *Higher-order optimality conditions for strict local minima*, Ann. Oper. Res. **157** (2008), 183– 192. ↑71, 98

[100] F. Jordan and F. Mynard, *Productively Fréchet spaces*, Czechoslovak Math. J. **54** (2004), 981– 990. ↑24, 25, 28

[101] A. Jourani, *Tangency conditions for multivalued mappings*, Set-valued analysis **4** (1996), 157– 172. ↑1

[102] R. Kasimbeyli, *Radial epiderivatives and set-valued optimization*, Optimization **58** (2009), 521– 534. ↑2, 4

[103] P. Q. Khanh, *Proper solutions of vector optimization problems*, J. Optim. Theory Appl. **74** (1992), 105 – 130. ↑10, 12, 71

[104] P. Q. Khanh, *Optimality conditions via norm scalarization in vector optimization*, SIAM J. Control Optim. **31** (1993), 646 – 658. ↑12

[105] P. Q. Khanh and N.D. Tuan, *Variational sets of multivalued mappings and a unified study of optimality conditions*, J. Optim. Theory Appl. **139** (2008), 45 – 67. ↑2, 3, 4, 45, 46, 50, 71, 79, 88, 90, 91

[106] P. Q. Khanh and N.D. Tuan, *Higher-order variational sets and higher-order optimality conditions for proper efficiency in set-valued nonsmooth vector optimization.*, J. Optim. Theory Appl. **139** (2008), 243 – 261. ↑2, 3, 4, 45, 46, 71, 88, 89, 91

[107] H. Kuk, T. Tanino, and M. Tanaka, *Sensitivity analysis in vector optimization*, J. Optim. Theory Appl. **89** (1996), 713 – 730. ↑2

[108] H. Kuk, T. Tanino, and M. Tanaka, *Sensitivity analysis in parametrized convex vector optimization*, J. Math. Anal. Appl. **202** (1996), 511 – 522. ↑2

[109] J. Klose, *Sensitivity analysis using the tangent derivative*, Numer. Funct. Anal. Optim. **13** (1992), 143 – 153. ↑1, 2

[110] K. Kuratowski, *Sur les décompositions semi-continues d'espaces métriques compacts*, Fund. Math. **11** (1928), 167 – 185. ↑18

[111] K. Kuratowski, *Topologie*, Vol. I, Panstwowe Wydawnictwo Naukowe, Warszawa, 1948. ↑17, 18

[112] C. S. Lalitha and P. Chatterjee, *Stability for properly quasi convex vector optimization problem*, J. Optim. Theory Appl. **155** (2012), 492 – 506. ↑121

[113] C. S. Lalitha and P. Chatterjee, *Stability and scalarization of weak efficient, efficient and Henig proper efficient sets using generalized quasiconvexities*, J. Optim. Theory Appl. **155** (2012), 941 – 961. ↑1, 121

[114] E. B. Leach, *A note on inverse function theorems*, Proc. Amer. Math. Soc. **12** (1961), 694 – 697. ↑40

[115] S.J. Li and C.R. Chen, *Higher-order optimality conditions for Henig efficient solutions in set-valued optimization*, J. Math. Anal. Appl. **323** (2006), 1184 – 1200. ↑71, 89, 91

[116] S.J. Li, K.W. Meng, and J.-P. Penot, *Calculus rules for derivatives of multimaps*, Set-Valued Anal. **17** (2009), 21 – 39. ↑4

[117] S.J. Li, X.K. Sun, and S.K. Zhu, *Higher-order optimality conditions for strict minimality in set-valued optimization*, J. Nonlinear and Covex Analysis. **13** (2012), 281 – 291. ↑4, 98

[118] S.J. Li, K.L. Teo, and X.Q. Yang, *Higher-order optimality conditions for set-valued optimization*, J. Optim. Theory Appl. **137** (2008), 533 – 553. ↑71, 78, 79

[119] W. Liu and X. Gong, *Proper efficiency for set-valued vector optimization problems and vector variational inequalities*, Math. Meth. Oper. Res. **51** (2000), 443 – 457. ↑71, 89, 91

[120] R. López, *Variational convergence for vector-valued functions and its applications to convex multiobjective optimization*, Math. Meth. Oper. Res. **78** (2013), 1-34. ↑1, 121

[121] D.T. Luc, *Scalarization of vector optimization problems*, J. Optim. Theory Appl. **55** (1987), 85 – 102. ↑121

[122] D.T. Luc, *Theory of vector optimization. Lecture notes in Economics and Mathematical Systems*, Springer, Berlin, 1989. ↑121

[123] R.E. Lucchetti and E. Miglierina, *Stability for convex vector optimization problems*, Optimization **53** (2004), 517 – 528. ↑121

[124] D.V. Luu, *Higher-order necessary and sufficient conditions for strict local Pareto minima in terms of Studniarski's derivatives*, Optimization **57** (2008), 593 – 605. ↑98

[125] E.K. Makarov and N.N. Rachkovski, *Unified representation of proper efficiency by means of dilating cones*, J. Optim. Theory Appl. **101** (1999), 141 – 165. ↑12, 71

[126] E. Miglierina, *Characterization of solutions of multiobjective optimization problem*, Rend. Circ. Mat. Palermo **50** (2001), 153 – 164. ↑121

[127] E. Miglierina and E. Molho, *Scalarization and stability in vector optimization*, J. Optim. Theory Appl. **114** (2002), 657 – 670. ↑121

[128] B.S. Mordukhovich, *Generalized differential calculus for nonsmooth ans set-valued mappings*, J. Math. Anal. Appl. **183** (1994), 250 – 288. ↑120

[129] B.S. Mordukhovich, *Variational analysis and generalized Differentiation, vol. I : Basic Theory*, Springer, Berlin, 2006. ↑2, 3, 40, 45, 120

[130] B.S. Mordukhovich, *Variational analysis and generalized Differentiation, vol. II : Applications*, Springer, Berlin, 2006. ↑2, 3, 40, 45, 120

[131] S. Müller and M. Ortiz, *On the* Γ*-convergence of discrete dynamics and variational integrators*, J. Nonlinear Sci. **14** (2004), 279 – 296. ↑1

[132] P. Oppezzi and A.M. Rossi, *Existence and convergence of Pareto minima*, J. Optim. Theory Appl. **128** (2006), 653 – 664. ↑121

[133] P. Oppezzi and A.M. Rossi, *A convergence for vector valued functions*, Optimization **57** (2008), 435 – 448. ↑1, 121

[134] M. Pappalardo, *Tangent cones and Dini derivatives*, J. Optim. Theory Appl. **70** (1991), 97 – 107. ↑1

[135] G. Peano, *Applicazioni Geometriche*, Fratelli Bocca, Torino, 1887. ↑18, 39, 42

[136] G. Peano, *Sur la définition de la dérivée*, Mathesis 2 **2** (1892), 12– 14. ↑39

[137] G. Peano, *Formulaire Mathématique*, Fratelli Bocca, Torino, 1903. ↑18

[138] G. Peano, *Formulairo Mathematico*, Fratelli Bocca, Torino, 1908. ↑39, 42

[139] J.-P. Penot, *The use of generalized subdifferential calculus in optimization theory*, Methods of Operations Res. **31** (1979), 495– 511. ↑42

[140] J.-P. Penot, *Compact nets, filters and relation*, J. Math. Anal. Appl. **93** (1983), 400– 417. ↑106

[141] J.-P. Penot, *Are generalized derivatives useful for generalized convex function ?*, in "Generalized Convexity, Generalized Monotonicity : Recent Results " , J. P. Crouzeix, J.E. Martinez-Legaz, and M. Volle (eds.), Kluwer Academic, Dordrecht **27** (1998), 3 – 59. ↑40, 45

[142] J.-P. Penot, *Calculus without derivatives*, Springer, New York, 2013. ↑3

[143] J.H. Qiu and Y. Hao, *Scalarization of Henig properly efcient points in locally convex spaces*, J. Optim. Theory Appl. **147** (2010), 71 – 92. ↑121

[144] E.-D. Rahmo and M. Studniarski, *Higher-order conditions for strict local Pareto minima in terms of generalized lower and upper directional derivatives*, J. Math. Anal. Appl. **393** (2012), 211 – 221. ↑98

[145] R.T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Can. J. Math. **32** (1980), 257 – 280. ↑42

[146] R.T. Rockafellar and R.J.-B. Wets, *Theory of vector optimization. Lecture notes in Economics and Mathematical Systems*, Springer, New York, 2009. ↑3, 121

[147] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis,3rdedn*, Springer, Berlin, 2009. ↑2, 40, 45, 94

[148] S. Saks, *Theory of the Integral*, Hafner, New York, 1937. ↑42

[149] W. Schirotzek, *Nonsmooth analysis*, Springer, Berlin, 2007. ↑3

[150] B. Schmidt, S. Leyendecker, and M. Ortiz, Γ*-convergence of variational integrators for constrained systems*, J. Nonlinear Sci. **19** (2009), 153 – 177. ↑1

[151] F. Severi, *Su alcune questioni di topologia infinitesimale*, Ann. Soc. Polonaise **9** (1931), 97 – 108. ↑42

[152] F. Severi and B. Segre, *Un paradosso topologico*, Rend. Acc. Naz. Lincei **9** (1929), 1 – 35. ↑42

[153] D.S. Shi, *Contingent derivative of the perturbation map in multiobjective optimization*, J. Optim. Theory Appl. **70** (1991), 385 – 396. ↑2

[154] D.S. Shi, *Sensitivity analysis in convex vector optimization*, J. Optim. Theory Appl. **77** (1993), 145 – 159. ↑2

[155] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore*, Ann. Scuola Norm. Sup. Pisa **21** (1967), 657 – 699. ↑13

[156] M. Studniarski, *Necessary and sufficient conditions for isolated local minima of nonsmooth functions*, SIAM J. Control Optim. **24** (1986), 1044 – 1049. ↑4, 97, 98

[157] M. Studniarski, *Second order necessary conditions for optimality in nonsmooth nonlinear programming*, J. Math. Anal. Appl. **154** (1991), 303 – 317. ↑40

[158] X.K. Sun and S.J. Li, *Lower Studniarski derivative of the perturbation map in parametrized vector optimization*, Optim. Lett. **5** (2011), 601 – 614. ↑2, 60, 98

[159] X.K. Sun and S.J. Li, *Weak lower Studniarski derivative in set-valued optimization*, Pac. J. Optim. **8** (2012), 307 – 320. ↑4, 98

[160] A. Taa, *Set-valued derivatives of multifunctions and optimality conditions*, Num. Funct. Anal. Optim. **19** (1998), 121 – 140. ↑4, 71, 72, 90, 98

[161] T. Tanaka, *Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions*, J. Optim. Theory Appl. **81** (1994), 355 – 377. ↑44

[162] T. Tanino, *Sensitivity analysis in multiobjective optimization*, J. Optim. Theory Appl. **56** (1988), 479 – 499. ↑2, 70

[163] T. Tanino, *Stability and sensitivity analysis in convex vector optimization*, SIAM J. Control Optim. **26** (1988), 521 – 536. ↑2

[164] D. Ward, *Convex subcones of the contingent cone in nonsmooth calculus and optimization*, Trans. Amer. Math. Soc. **302** (1987), 661 – 682. ↑4

[165] D.E. Ward and J.M. Borwein, *Nonsmooth calculus in finite dimensions*, SIAM J. Control Optim. **25** (1987), 1312 – 1340. ↑4

[166] Q.L. Wang and S.J. Li, *Higher-order weakly generalized adjacent epiderivatives and applications to duality of set-valued optimization*, J. Inequal. Appl. **2009** (2009), 462637. ↑4

[167] Q.L. Wang and S.J. Li, *Second-order contingent derivative of the perturbation map in multiobjective optimization*, Fixed Point Theory Appl. **2011** (2011), 857520. ↑2, 60

[168] Q.L. Wang, S.J. Li, and K.L. Teo, *Higher-order optimality conditions for weakly efcient solutions in nonconvex set-valued optimization*, Optim. Lett. **4** (2010), 425-437. ↑4

[169] R.J.-B. Wets, *Convergence of convex functions, variational inequalities, and convex optimization*, in "Variational inequalities and complementarity problems" , R. Cottle, F. Gianessi, and J.-L. Lions (eds.), Wiley, Chichester, (1980), 256 – 268. ↑1, 121

[170] H. Whitney, *Complex Analytic Varieties*, Addison-Wesley Publ.Co, Reading, 1972. ↑42

[171] T. Zolezzi, *On the stability analysis in mathematical programming*, Math. Prog. Studies **21** (1984), 227 – 242. ↑121

[172] L. Zoretti, *Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers*, J. Math. Pures Appl. **1** (1905), 1 – 51. ↑18

[173] L. Zoretti, *Sur les ensembles de points*, Encyclopédie des Sciences Mathématiques **1** (1912), 113 – 170. ↑18

Index