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Présentée par

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Contribution à l'analyse variationnelle : Stabilité des cônes tangents et normaux et convexité des ensembles de Chebyshev

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Իմ ծնողներին՝ Աշոտին եւ Սուսաննային

Abstract

The aim of this thesis is to study the following three problems:
1) We are concerned with the behavior of normal cones and subdifferentials with respect to two types of convergence of sets and functions: Mosco and Attouch-Wets convergences. Our analysis is devoted to proximal, Fréchet, and Mordukhovich limiting normal cones and subdifferentials. The results obtained can be seen as extensions of Attouch theorem to the context of non-convex functions on locally uniformly convex Banach space.
2) For a given bornology β on a Banach space X we are interested in the validity of the following "lim inf" formula

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x).$$

Here $T_\beta(C; x)$ and $T_c(C; x)$ denote the β-tangent cone and the Clarke tangent cone to C at x. We proved that it holds true for every closed set $C \subset X$ and any $x \in C$, provided that the space $X \times X$ is ∂_β-trusted. The trustworthiness includes spaces with an equivalent β-differentiable norm or more generally with a Lipschitz β-differentiable bump function. As a consequence, we show that for the Fréchet bornology, this "lim inf" formula characterizes in fact the Asplund property of X.
3) We investigate the convexity of Chebyshev sets. It is well known that in a smooth reflexive Banach space with the Kadec-Klee property every weakly closed Chebyshev subset is convex. We prove that the condition of the weak closedness can be replaced by the local weak closedness, that is, for any $x \in C$ there is $\epsilon > 0$ such that $C \cap \mathbb{B}(x, \varepsilon)$ is weakly closed. We also prove that the Kadec-Klee property is not required when the Chebyshev set is represented by a finite union of closed convex sets.

Keywords: Mosco (Attouch-Wets) convergence, proximal normal cone, Fréchet (Mordukhovich limiting) subdifferential, subsmooth sets (functions), Clarke tangent (normal) cone, contingent cone, bornology, Asplund space, trustworthiness, Chebyshev set, metric projection, minimizing sequence.

Résumé

Le but de cette thèse est d'étudier les trois problèmes suivantes :
1) On s'intéresse à la stabilité des cônes normaux et des sous-différentiels via deux types de convergence d'ensembles et de fonctions : La convergence au sens de Mosco et celle d'Attouch-Wets. Les résultats obtenus peuvent être vus comme une extension du théorème d'Attouch aux fonctions non nécessairement convexes sur des espaces de Banach localement uniformément convexes.
2) Pour une bornologie β donnée sur un espace de Banach X, on étudie la validité de la formule suivante

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x).$$

Ici $T_\beta(C; x)$ et $T_c(C; x)$ désignent le β-cône tangent et le cône tangent de Clarke à C en x. On montre que si, $X \times X$ est ∂_β-"trusted" alors cette formule est valable pour tout ensemble fermé non vide $C \subset X$ et $x \in C$. Cette classe d'espaces contient les espaces ayant une norme équivalent β-différentiable, et plus généralement les espaces possédant une fonction "bosse" lipschitzienne et β-différentiable). Comme conséquence, on obtient que pour la bornologie de Fréchet, cette formule caractérise les espaces d'Asplund.
3) On examine la convexité des ensembles de Chebyshev. Il est bien connu que, dans un espace normé réflexif ayant la propriété Kadec-Klee, tout ensemble de Chebyshev faiblement fermé est convexe. On démontre que la condition de faible fermeture peut être remplacée par la fermeture faible locale, c'est-à-dire pour tout $x \in C$ il existe $\epsilon > 0$ tel que $C \cap \mathbb{B}(x, \epsilon)$ est faiblement fermé. On montre aussi que la propriété Kadec-Klee n'est plus exigée lorsque l'ensemble de Chebyshev est représenté comme une union d'ensembles convexes fermés.

Mots-clés : Convergence au sens de Mosco (d'Attouch-Wets), cône normal proximal, sous-différentiel de Fréchet (de Mordukhovich), ensembles sous-réguliers, fonctions sous-régulières cône normal (tangent) de Clarke, cône tangent de Bouligand, bornologie, espace d'Asplund, ensemble de Chebyshev, projection metrique, suite minimisante.

Table des matières

Chapitre 1

Présentation générale

Ce travail est composé de trois parties indépendantes. Dans la première nous présentons l'article "*Convergence of subdifferentials and normal cones in locally uniformly convex Banach space*" écrit en collaboration avec Lionel Thibault. Dans la deuxième partie nous présentons l'article "*The validity of the "lim inf" formula and a characterization of Asplund spaces*" écrit en collaboration avec Abderrahim Jourani. Et dans la dernière partie nous présentons l'article "*New conditions ensuring the convexity of Chebyshev sets*".

1.1 Présentation des outils d'analyse non lisse

Soient X un espace de Banach et X^* son dual topologique avec un crochet de dualité $\langle \cdot, \cdot \rangle$. Une bornologie β sur X est une famille d'ensembles bornés et centralement symétriques de X dont l'union est X, et telle que l'union de deux éléments de β est un élément de β. Les bornologies les plus importantes sont la bornologie de Gâteaux qui consiste en tous les ensembles finis symétriques de X, la bornologie de Hadamard qui consiste en tous les ensembles compacts symétriques, la bornologie faible de Hadamard qui consiste en tous les ensembles faiblement compacts symétriques et enfin la bornologie de Fréchet qui consiste en tous les ensembles bornés et symétriques.

Chaque bornologie β génère un β-sous-différentiel qui est à son tour engendre un β-cône normal, et en polarisant on obtient aussi le β-cône tangent.

Definition 1.1.1 *Soient $f : X \to \mathbb{R} \cup \{\pm\infty\}$ une fonction finie en x et β une bornologie sur X.*

(a) f est β-différentiable en x s'il existe $x^* \in X^*$ tel que pour tout ensemble $S \in \beta$

$$\lim_{t\to 0^+} t^{-1} \sup_{h\in S} |f(x+th) - f(x) - \langle x^*, th\rangle\,| = 0,$$

(b) $x^* \in X^*$ est appelé β-sous-gradient de f en x, si pour tout $\varepsilon > 0$ et tout ensemble $S \in \beta$ il existe $\delta > 0$ tel que pour tout $0 < t < \delta$ et tout $h \in S$

$$t^{-1}\big(f(x+th) - f(x)\big) - \langle x^*, h\rangle \geq -\varepsilon.$$

On note $\partial_\beta f(x)$ l'ensemble de tous β-sous-gradients de f en x.

En appliquant la définition 1.1.1(a) à la bornologie de Fréchet et à la Gâteaux bornologie, on obtient les définitions classiques suivantes :

- Fréchet-différentiabilité : il existe $x^* \in X^*$ tel que

$$\lim_{h\to 0} \|h\|^{-1}(f(x+h) - f(x) - \langle x^*, h\rangle) = 0.$$

- Gâteaux-différentiabilité : il existe $x^* \in X^*$ tel que

$$\forall h \in X, \quad \lim_{t\to 0^+} t^{-1}(f(x+th) - f(x)) = \langle x^*, h\rangle.$$

De la même manière la définition 1.1.1(b) aboutit dans le cas de la bornologie de Fréchet (c.-à-d. $\beta = F$) à la définition classique du *sous-différentiel Fréchet* de f en x :

$$\partial_F f(x) = \left\{ x^* \in X^* : \liminf_{y\to x} \frac{f(y) - f(x) - \langle x^*, y-x\rangle}{\|y-x\|} \geq 0 \right\}.$$

Nous allons noter par $N_\beta(C;x)$ le *β-cône normal* de C en x :

$$N_\beta(C;x) = \partial_\beta \psi_C(x)$$

où ψ_C est la fonction indicatrice de C, c'est à dire

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C \end{cases}$$

et par $T_\beta(C;x)$ le *β-cône tangent* qui est défini comme le cône polaire négatif du β-cône normal intersecté avec X, c'est à dire

$$T_\beta(C,x) = (N_\beta(C,x))^\circ \cap X.$$

Definition 1.1.2 *Soit X un espace de Banach et β une bornologie sur X. On dit que X est ∂_β-"trusted", si la règle floue souivante de minimisation est verifiée : soit f une fonction sur X finie en $\bar{x} \in X$, et soit g une fonction lipschitzienne sur X. Supposons que $f + g$ atteint un minimum local en $\bar{x}$. Alors pour tout $\varepsilon > 0$ ils existent $x, u \in X$ et $x^* \in \partial_\beta f(x)$, $u^* \in \partial_\beta g(u)$ tels que*

$$\|x - \bar{x}\| < \varepsilon, \quad \|u - \bar{x}\| < \varepsilon, \quad |f(x) - f(\bar{x})| < \varepsilon, \quad \textit{et} \quad \|x^* + u^*\| < \varepsilon.$$

Le *cône tangent de Bouligand* $K(C; x)$ de C en x est défini par :

$$K(C, x) = \{h \in X : \exists t_n \to 0^+, \exists h_n \to h, \text{ tel que } x + t_n h_n \in C \ \forall n \in \mathbb{N}\}$$

Le *cône tangent de Clarke* $T_c(C; x)$ de C en x est défini par :

$$T_c(C; x) = \{h \in X : \forall x_n \xrightarrow{C} x, \forall t_n \to 0^+, \exists h_n \to h \text{ tel que } x_n + t_n h_n \in C \ \forall n \in \mathbb{N}\}$$

Soient $(C_n)_n$ une suite d'ensembles d'un espace normé X. Notons

$$\operatorname*{Lim\,inf}_{n\to\infty} C_n := \big\{x \in X : \exists \text{ une suite } (x_n)_n \text{ convergeant vers } x$$
$$\text{avec } x_n \in C_n \text{ pour tout entier } n \text{ suffisamment grand}\},$$

$$\operatorname*{Lim\,sup}_{n\to\infty} C_n := \big\{x \in X : \exists \text{ une suite } (k(n))_n \text{ dans } \mathbb{N} \text{ et}$$
$$(x_n)_n \text{ convergeant vers } x \text{ avec } x_n \in C_{k(n)} \text{ pour tout } n \in \mathbb{N}\big\},$$

$${}^w\operatorname*{Lim\,sup}_{n\to\infty} C_n := \big\{x \in X : \exists \text{ une suite } (k(n))_n \text{ dans } \mathbb{N} \text{ et } (x_n)_n$$
$$\text{convergeant faiblement vers } x \text{ avec } x_n \in C_{k(n)} \text{ pour tout } n \in \mathbb{N}\big\}.$$

Definition 1.1.3 *On dit que la suite $(C_n)_n$ converge au sens de Painlevé-Kuratowski vers l'ensemble C de X si*

$$C = \operatorname*{Lim\,inf}_{n\to\infty} C_n = \operatorname*{Lim\,sup}_{n\to\infty} C_n.$$

Definition 1.1.4 *On dit la suite $(C_n)_n$ converge au sens de Mosco vers l'ensemble C de X si*

$$C = \operatorname*{Lim\,inf}_{n\to\infty} C_n = {}^w \operatorname*{Lim\,sup}_{n\to\infty} C_n.$$

L'excès de l'ensemble A sur l'ensemble A' et Hausdorff ρ-semi-distance entre A et A' sont définis de la manière suivante

$$e(A, A') := \sup_{a\in A} \operatorname{dist}(a, A'),$$

$$\operatorname{Haus}_\rho(A, A') := \max\left\{e(A \cap \rho\mathbb{B}_X, A'), e(A' \cap \rho\mathbb{B}_X, A)\right\}.$$

Definition 1.1.5 *On dit que la suite $(C_n)_n$ converge au sens d'Attouch-Wets vers l'ensemble C de X, si pour tout $\rho > 0$ suffisamment grand*

$$\operatorname{Haus}_\rho(C_n, C) \xrightarrow[n\to\infty]{} 0.$$

1.2 Convergence des sous-différentiels et des cônes normaux dans un espace de Banach localement uniformément convexe

Dans cette partie nous étudions les questions suivantes : supposons que la suite $(C_n)_n$ des ensembles fermés dans un espace de Banach converge au sens de Mosco ou d'Attouch-Wets vers un ensemble C, que peut-on dire sur la convergence de la suite des graphes des cônes normaux ? Nous allons nous concentrer sur les cônes normaux proximaux, Fréchet et de Mordukhovich. Et le même type de questions se pose en remplaçant les ensembles par des fonctions et les cônes normaux par des sous-différentiels.

H. Attouch [3] était le premier à s'intéresser à des questions de ce type. Il a établi qu'une suite de fonctions à valeurs réelles étendues, semi-continues inférieurement, convexes et propres sur un espace normé réflexif converge au sens de Mosco si et seulement si la suite des graphes des sous-différentiels converge au sens de Painlevé-Kuratowski vers le graphe du sous-différentiel de la fonction limite à une constante additive près. Concernant la convergence à la Attouch-Wets, H. Attouch, J.L. Ndoutoume et M. Théra [4] ont montré qu'une famille de fonctions convexes propres semi-continues inférieurement (définies sur un espace normé super-réflexif) converge au sens d'Attouch-Wets si et seulement si la suite des graphes des sous-différentiels converge au sens d'Attouch-Wets à une constante additive près.

R.A. Poliquin [20] a étendu le théorème d'Attouch aux fonctions non-convexes dites"primal lower nice" (pln) dans un espace de dimension finie. A. Levy, R.A. Poliquin et L. Thibault [18] ont prouvé que dans un espace d'Hilbert, si $(f_n)_n$ est une suite de fonctions minorées par la même constante autour de x, avec $\big(f_n(x)\big)_n$ bornées, et "equi-primal-lower-nice" en x, alors la convergence au sens de Mosco vers f entraîne que la suite des graphes des sous-différentiels converge au sens de Painlevé-Kuratowski vers le graphe du sous-diffrentiel de f. Ils ont aussi montré que la convergence au sens d'Attouch-Wets des $(f_n)_n$ implique la convergence au sens de Painlevé-Kuratowski des graphes des sous-différentiel aussi bien que la quasi Attouch-Wets convergence des sous-gradients dans le même sens. A. Jourani [16] a montré que le sous-différentiel approché de Ioffe d'une fonction semi-continue inférieurement sur un espace de Banach est contenu dans la limite supérieur des sous-différentiels approchés de Ioffe d'une famille de fonctions semi-continues infé-rieurement convergeant uniformément vers cette fonction.

Récemment X.Y. Zheng and Z. Wei [25] ont considéré la convergence des cônes normaux pour une suite d'ensembles sous-réguliers sur un espace d'Hilbert. Comme conséquence, ils ont obtenu une généralisation de [18] : Si la suite $(f_n)_n$ de fonctions à valeurs réelles propres semi-continues inférieurement sur un espace d'Hilbert H converge au sens de Mosco vers une fonction propre f et $(f_n)_n$ est uniformément sous-réguliers en $\bar{x}$, alors un élément $\zeta \in H$ appartient au sous-différentiel de Mordukhovich de la fonction f en $\bar{x}$ si et seulement si il existe une suite $\big((x_n, \zeta_n)\big)_n$ dans $H \times H$ et une suite strictement décroissante $\big(k(n)\big)_n$ dans $\mathbb{N}$ telles que

$$\zeta_n \in \partial_P f_{k(n)}(x_n) \quad \text{et} \quad \big(x_n, f_{k(n)}(x_n)\big) \to \big(\bar{x}, f(\bar{x})\big)$$

et $(\zeta_n)_n$ converge faiblement vers ζ. Ici on désigne par $\partial_P f_{k(n)}$ le sous-différentiel proximal de $f_{k(n)}$. On dit que $(f_n)_n$ est uniformément sous-régulière en $\overline{x}$ si pour tous $\epsilon > 0$ et $M > 0$ il existe $\delta > 0$ tel que

$$\langle \zeta_n, x' - x \rangle \leq f_n(x') - f_n(x) + \epsilon \|x' - x\|$$

pour tous $n \in \mathbb{N}$, $x', x \in \mathbb{B}(\overline{x}, \delta)$ (la boule ouverte de centre $\overline{x}$ et de rayon δ) et $\zeta_n \in \partial_P f_n(x)$ avec $\|\zeta_n\| \leq M$.

Dans ce travail nous avons étendu le résultat ci-dessus à un espace normé réflexif locallement uniformément convexe avec une norme Fréchet différentiable (sauf en zéro). Nous établissons aussi un résultat similaire pour la convergence au sens d'Attouch-Wets : Si une suite de fonctions semi-continues inférieurement propres à valeurs réelles étendues converge au sens d'Attouch-Wets vers une fonction propre f et la suite $(f_n)_n$ est sous-régulière en $\overline{x} \in \operatorname{dom} f$ avec une indéxation compatible, alors un élément x^* de X^* appartient au sous-différentiel de Mordukhovich de f au point $\overline{x}$ si et seulement si il existe une suite $\big((x_n, x_n^*)\big)_n$ dans $X \times X^*$ et une suite strictement croissante $\big(k(n)\big)_n$ dans $\mathbb{N}$ telles que

$$x_n^* \in \partial_P f_{k(n)}(x_n) \quad \text{et} \quad \big(x_n, f_{k(n)}(x_n)\big) \to \big(\overline{x}, f(\overline{x})\big)$$

et $(x_n^*)_n$ converge faiblement vers x^*. Notre définition de l'indéxation compatible est la suivante : pour chaque $\epsilon > 0$ il existe $\delta > 0$ et $N \in \mathbb{N}$ tels que

$$\langle x^*, x' - x \rangle \leq f_n(x') - f_n(x) + \epsilon(1 + \|x^*\|)\|x' - x\|$$

pour tous $n \geq N$, $x' \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \operatorname{dom} \partial_P f_n$ et $x^* \in \partial_P f_n(x)$. Lorsque la suite $(f_n)_n$ est sous-régulière en tout point de $\operatorname{dom} f$ avec une indéxation compatible, nous obtenons que le graphe du sous-différentiel de Mordukhovich de f est une limite supérieur des graphes des sous-différentiels proximaux des fonctions f_n.

Les applications de nos resultats sont nombreuses, on peut citer, par exemple, les processus de Rafle de Moreau qui jouent un rôle important en mécanique du contact, la stabilité des solutions de viscosités, l'existence de la proto-dérivée au sens de Poliquin-Rockafellar. En effet, pour cette dernière le théorème d'Attouch avait été utilisé pour relier diverses dérivées généralisées. On rappelle la définition de la proto-dérivée et l'épi-dérivée. Supposons que f est une fonction à valeur réelle définie sur un espace de Banach X et ∂f son sous-différentiel de Fréchet. On considère la multiaplication suivante

$$\Delta_\varepsilon[\partial f](\overline{x}|x^*) : x' \mapsto \frac{1}{\varepsilon}\Big[\partial f(\overline{x} + \varepsilon x') - x^*\Big] \quad \text{avec} \quad x^* \in \partial f(\overline{x}).$$

Lorsque ces multiapplications convergent graphiquement, alors la multiapplication limite est la proto-dérivée de ∂f en $\overline{x}$ pour x^*,

$$D[\partial f](\overline{x}, x^*) : X \rightrightarrows X^*.$$

Maintenant on considère le quotient d'ordre deux de f

$$\Delta_\epsilon^2 f(\overline{x}|x^*) : x' \mapsto \frac{1}{\varepsilon^2}\Big[f(\overline{x}+\varepsilon x') - f(\overline{x}) - \varepsilon\,\langle x^*, x'\rangle\Big] \text{ avec } x^* \in \partial f(\overline{x}).$$

Si ces fonctions épi-convergent lorsque $\varepsilon \to 0$, alors la fonction limite notée par $d^2 f(\overline{x}|x^*)$, est appelée épi-dérivée d'ordre deux de f en $\overline{x}$ pour x^*.

Supposons que $X = \mathbb{R}^n$ et f est de classe C^2. Alors on obtient que

$$D[\partial f](\overline{x}, \overline{v}) = d^2 f(\overline{x}|\overline{v}) = \nabla^2 f(\overline{x})x'$$

où $\nabla^2 f(\overline{x})$ est le hessien et $\overline{v} = \nabla f(\overline{x})$.

Ceci nous suggère la possibilité d'une certaine relation entre ces deux approches différentes de la différentiation d'ordre deux. De plus par des outils du calcul sous-différentiel on obtient que

$$\partial[\Delta_\varepsilon^2 f(\overline{x}|x^*)](x') = 2\Delta_\varepsilon[[\partial f](\overline{x}, x^*)](x') \text{ pourtout } x' \in X.$$

Est-ce que l'égalité est préservée à la limite lorsque $\epsilon \to 0$? Grâce aux Théorèmes mentionnés ci-dessus nous pouvons avoir l'égalité sous certaines hypothèses sur f.

1.3 La validité de la formule de "$\liminf$" et une caractérisation des espaces d'Asplund

Dans cette partie nous nous sommes intéressés aux conditions suffisantes sur l'espace de Banach X assurant la formule suivante

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x) \tag{1.3.1}$$

pour tout ensemble fermé $C \subset X$ et $x \in C$. Ici $T_\beta(C;x)$ et $T_c(C;x)$ désignent le β-cône tangent et le cône tangent de Clarke de C en x et pour une multiapplication $F : C \rightrightarrows X$, $h \in \liminf_{u \xrightarrow{C} x} F(u)$ ssi pour toute suite $x_n \xrightarrow{C} x$ il existe $h_n \to h$, telle que pour tout n suffisamment grand, $h_n \in F(x_n)$.

Plusieurs auteurs se sont intéressés à ce type de questions. En particulier Cornet [14] a démontré que si $C \subset \mathbb{R}^n$, alors

$$T_c(C; x) = \liminf_{x' \xrightarrow{C} x} K(C; x'),$$

où $K(C,x)$ désigne le cône tangent de Bouligand ou le cône contingent à C en x. Treiman [22, 23] a démontré que sur un espace de Banach on a

$$\liminf_{x' \xrightarrow{C} x} K(C;x') \subset T_c(C;x),$$

et de plus l'inclusion devient égalité si C est épi-Lipschitzien en x au sens de Rockafellar [21].

Borwein et Ioffe [7] ont démontré la validité de la formule (1.3.1) dans le cas où X admet un norme équivalente β-différentiable.

Dans ce travail nous prouvons que pour un espace X tel que $X \times X$ soit ∂_β-"trusted" ou autrement dit vérifiant ce qu'on appelle "basic fuzzy principle" est satisfait sur $X \times X$ (ceci inclut les espaces avec une norme équivalente β-différentiable, et plus généralement les espaces possédant une fonction "bosse" lipschitzienne et β-différentiable) alors la formule "lim inf" a lieu. Comme conséquence, nous avons montré que pour la bornologie de Fréchet, la formule (1.3.1) caractérise les espaces d'Asplund.

Ce sujet a de diverses applications (voir [9]) dans divers domaines en mathématiques appliquées. En particulier il intervient dans les problèmes de viabilité.

Supposons que X est un espace de Banach et C un ensemble fermé de X et $F : X \rightrightarrows X$. On considère la problème de *viabilité* suivant

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0, \quad x(t) \in C \quad \text{for all} \quad t \in [0,T), \tag{1.3.2}$$

qui consiste à trouver une fonction absolument continue $x : [0,T) \to X$ avec $x(0) = x_0$, $x(t) \in C$ et $\dot{x}(t) \in F(x(t))$ presque partout sur l'intervalle $[0,T)$. Ce type de problèmes a été étudié par divers auteurs (voir en particulier [11, 12, 13, 15]), nous considérons le resultat classique suivant :

Theorem 1.3.1 *(A. Bressan [11]) Supposons que C est compact et $F : X \rightrightarrows X$ une multiapplication semi-continue inférieurement tel que $F(x) \subset K(C;x)$ pour tout $x \in X$. Alors pour tout $x_0 \in C$, le problème (1.3.2) a une solution x definie sur* $[0,+\infty)$.

La condition $F(x) \subset K(C;x)$ s'appelle la condition de tangence. Comme un corollaire (voir (3.4.5)) nous avons obtenu que $K(C;x)$ peut être remplacé par d'autres cônes :

Theorem 1.3.2 *Supposons que X est un espace de Banach et β une bornologie sur X contenant la bornologie d'Hadamard tels que $X \times X$ soit ∂_β-"trusted" et C un ensemble fermé de X. Supposons que $F : C \rightrightarrows X$ est semi-continue inférieurement sur C. Alors les assertions suivantes sont équivalentes.*

(*i*) $F(x) \subset T_c(C; x)$, *pour tout* $x \in C$,

(*ii*) $F(x) \subset K(C; x)$, *pour tout* $x \in C$,

(*iii*) $F(x) \subset T_\beta(C; x)$, *pour tout* $x \in C$.

1.4 Nouvelles conditions assurant la convexité des ensembles de Chebyshev

Soit C un ensemble non-vide d'un espace de Banach $(X, \| \|)$. La projection métrique de x sur C est définie par

$$P_C(x) = \{y \in C : \|x - y\| = d_C(x)\},$$

où $d_C(x)$ est la fonction distance, i.e., $d_C(x) = \inf\{\|x - y\| : y \in C\}$. On dit que C est de Chebyshev si $P_C(x)$ est un singleton pour tout $x \in X$. C'est facile de voir que les ensembles de Chebyshev sont fermés. Le premier resultat positif sur la convexité des ensembles de Chebyshev dans les espaces Euclidien de dimension finie, est dû indépendamment à Bunt [10] et Motzkin [19]. Plus tard, dans [17, 24] il a été démontré que chaque ensemble de Chebyshev d'un espace normé, lisse, de dimension finie est convexe. L'une des conjectures la plus célèbre en théorie d'approximation est la suivante : dans un espace normé lisse réflexif (ou même dans un espace d'Hilbert) un ensemble de Chebyshev est-il nécessairement convexe ? Bien que le problème reste ouvert (voir [2, 6]), plusieurs conditions suffisantes ont été données, jusqu'à maintenant. Voici le premier résultat important :

Theorem 1.4.1 *(Vlasov [15]) Soit X un espace de Banach dont l'espace dual est strictement convexe. Alors tout sous-ensemble de Chebyshev de X dont la projection métrique est continue, est convexe.*

Ce théorème était précédemment obtenu par Asplund [1] dans un espace d'Hilbert.

Supposons maintenant que C est un ensemble faiblement fermé d'un espace normé réflexif X. Considérons un point $x \in X$ et une suite $(x_n)_n$ dans X convergeant vers x et notons que

$$\|x_n - P_C(x_n)\| = d_C(x_n) \to d_C(x) = \|x - P_C(x)\|. \tag{1.4.1}$$

Ceci nous montre que la suite $(P_C(x_n))_n$ est borné, donc il admet une sous-suite convergeant faiblement vers $y \in X$ en tenant compte de la réflexivité de X et de la faible fermeture de C. Utilisant (1.4.1) et la faible semi-continuité inférieure de le norme on voit que $\|x - y\| \leq d_C(x)$ et par conséquent $y = P_C(x)$. Cela donne le résultat suivant (voir aussi [8, p. 193]) :

Theorem 1.4.2 *Soit X un espace normé réflexif avec la propriété Kadec-Klee. Alors tout ensemble de Chebyshev faiblement fermé a une projection continue.*

Dans ce travail nous considérons deux conditions suffisantes afin que l'ensemble de Chebyshev soit convexe. Premièrement nous allons regarder la fermeture faible locale au sens suivant : pour tout $x \in C$ il existe $\varepsilon > 0$ tel que $C \cap \mathbb{B}(x, \varepsilon)$ est faiblement fermé. Nous prouvons que tout ensemble de Chebyshev localement faiblement fermé d'un espace normé réflexif avec la propriété Kadec-Klee a une projection continue. Comme corollaire, on obtient que tout ensemble de Chebyshev localement faiblement fermé d'un espace normé lisse réflexif avec la propriété Kadec-Klee est convexe. Deuxièment nous allons regarder les ensembles de Chebyshev qui peuvent être représentés comme une union d'ensembles convexes fermés. Nous prouvons que ces ensembles sont convexes dans un espace normé réflexif lisse. L'intérêt de ce résultat est que la propriété Kadec-Klee n'est plus nécessaire.

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Chapitre 2

Convergence of subdifferentials and normal cones in locally uniformly convex Banach space

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Abstract. In this paper we study the behaviour of normal cones and subdifferentials with respect to two types of convergence of sets and functions : Mosco and Attouch-Wets convergences. Our analysis is devoted to proximal, Fréchet, and Mordukhovich limiting normal cones and subdifferentials. The results obtained can be seen as extensions of Attouch to the context of non-convex functions on locally uniformly convex Banach space. They also generalize, to sequences of subsmooth sets or functions, various results in the literature.

2.1 Introduction

This paper is concerned with the behaviour of normal cones and subdifferentials with respect to two types of convergence of sets and functions : Mosco and Attouch-Wets convergences. More precisely, given a sequence $\{C_n\}_{n\in\mathbb{N}}$ of closed sets of a Banach space converging to a set C of this space in the sense of Mosco or Attouch-Wets, we study how the graphs of the normal cones of C_n converge to the graph of the normal cone of C. We focus the analysis to proximal, Fréchet, and Mordukhovich limiting normal cones. The study for subdifferentials of extended real-valued lower semicontinuous functions is then derived through the epigraphs of the functions.

Such a study of convergence of subdifferentials began in the 70s when H. Attouch (see [1]) established that a sequence of extended real-valued lower semicontinuous proper convex functions on a reflexive Banach space, converges in the sense of Mosco if and only if the graphs of the subdifferentials Painleve-Kuratowski converge to the graph of the subdifferential of the limit function and a condition which fixes the constant of integration holds ; we also refer to [3], [6], [16], [37] and [38] for other results in this line under Mosco convergence of convex functions. Concerning the Attouch-Wets convergence, H. Attouch, J.L Ndoutoume and M. Théra in [2] showed that a family of lower semicontinuous proper convex functions (defined on a super reflexive Banach space) Attouch-Wets converges if and only if the graphs of the subdifferentials Attouch-Wets converge plus a condition fixing the constant.

Poliquin [31] extended Attouch's Theorem to possibly nonconvex primal-lower-nice functions in a finite-dimensional setting. A. Levy, R.A. Poliquin and L. Thibault [26] proved, in the Hilbert space setting, that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of equi-primal-lower-nice functions at x (see the definition in Section 5), equibounded below near x with $\{f_n(x)\}$ bounded, then the convergence in the sense of Mosco to f entails that the graphs of the subdifferentials Painlevé-Kuratowski converge to the graph of the subdifferential of f. They also showed, in the same Hilbert space setting, that the convergence in the sense of Attouch-Wets of $\{f_n\}_{n\in\mathbb{N}}$ implies the Painlevé-Kuratowski convergence of the graphs of the subdifferentials, as well as almost Attouch-Wets convergence of the subgradients in some sense. A. Jourani [15] showed that the Ioffe (geometric) approximate subdifferential of a lower semicontinuous function on a Banach space is contained in the limit superior of the Ioffe approximate subdifferential of lower semicontinuous uniformly convergent family to this function. Through the latter result, the approximate subdifferential of a lower semicontinuous function f (bounded from below on the Banach space by a quadratic function) is described in [15] in terms of the subdifferentials of the Moreau envelopes ; see also [24] where the Mordukhovich limiting subdifferential of f is obtained, in the Asplund space setting, as some limit superior of the Fréchet subdifferentials of Moreau envelopes.

Recently X.Y. Zheng and Z. Wei [39] considered the convergence of normal cones for sequences of subsmooth sets of a Hilbert space. As a consequence, for sequences of extended real-valued functions, they obtained the following generalization of [26] : If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of extended real-valued proper lower semicontinuous functions on a Hilbert space H Mosco converges to a proper function f and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly subsmooth at $\bar{x}$, then an element $\zeta \in H$ belongs to the Mordukhovich limiting subdifferential of the function f at $\bar{x}$ if and only if there exist a sequence $\{(x_n, \zeta_{n\in\mathbb{N}})\}_n$ in $H \times H$ and a strictly increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\zeta_n \in \partial_P f_{k(n)}(x_n) \quad \text{and} \quad (x_n, f_{k(n)}(x_n)) \to (\bar{x}, f(\bar{x}))$$

and such that $\{\zeta_n\}_{n\in\mathbb{N}}$ weakly converges to ζ. Above $\partial_P f_{k(n)}$ denotes the proximal subdifferential of $f_{k(n)}$, and the uniform subsmoothness of $\{f_n\}_{n\in\mathbb{N}}$ at $\bar{x}$ means that for any reals $\varepsilon > 0$ and $M > 0$ there exists $\delta > 0$ such that

$$\langle \zeta_n, x' - x \rangle \leq f_n(x') - f_n(x) + \varepsilon \|x' - x\|$$

whenever $n \in \mathbb{N}$, $x', x \in B(\bar{x}, \delta)$ (the open ball around $\bar{x}$) and $\zeta_n \in \partial_P f_n(x)$ with $\|\zeta_n\| \leq M$.

In the present paper, we extend the latter result to a reflexive locally uniformly convex Banach space X with a norm Fréchet differentiable off zero. We also establish a similar result for the Attouch-Wets convergence : If a sequence of extended real-valued proper lower semicontinuous functions $\{f_n\}_{n\in\mathbb{N}}$ converges in the sense of Attouch-Wets to a proper function f and the sequence is subsmooth at $\bar{x} \in \operatorname{dom} f$ with a compatible indexation, then a continuous linear functional $x^* \in X^*$ belongs to the Mordukhovich limiting subdifferential of the function f at $\bar{x}$ if and only if there exist a sequence $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ in $X \times X^*$ and a strictly increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$x_n^* \in \partial_P f_{k(n)}(x_n) \quad \text{and} \quad (x_n, f_{k(n)}(x_n)) \to (\bar{x}, f(\bar{x}))$$

and such that $\{x_n^*\}_{n\in\mathbb{N}}$ converges weakly to x^*. Our definition of subsmoothness with a compatible indexation is the following : for any $\varepsilon > 0$ there exist some real $\delta > 0$ and some $N \in \mathbb{N}$ satisfying for each integer $n \geq N$

$$\langle x^*, x' - x \rangle \leq f_n(x') - f_n(x) + \varepsilon(1 + \|x^*\|)\|x' - x\|$$

for all $x' \in B(\bar{x}, \delta)$, $x \in B(\bar{x}, \delta) \cap \operatorname{Dom} \partial_P f_n$ and $x^* \in \partial_P f_n(x)$. When the sequence $\{f_n\}_{n\in\mathbb{N}}$ is subsmooth at every point of $\operatorname{dom} f$ with a compatible indexation, we obtain that the graph of the Mordukhovich limiting subdifferential of f is a certain limit superior of the graphs of the proximal subdifferentials of the functions f_n.

The paper is organized as follows. In Section 2 we recall some properties of uniformly convex/smooth norm and some concepts of subgradients and normals ; we also establish a result of approximation of horizontal proximal normals to the epigraph of a function by nonhorizontal ones in the context of a reflexive locally uniformly convex Banach space with a norm Fréchet differentiable off zero. The latter result is involved in several places of the paper. Section 3 studies the convergence of the graphs of normal cones of Mosco convergent sequences of sets and Section 4 deals with Attouch-Wets convergence of sequences of sets. The results in both sections are obtained for (non-Hilbert) reflexive locally uniformly convex Banach space with a norm Fréchet differentiable off zero ; a particular attention is paid to the case when the sequence of sets is subsmooth with compatible indexation. Section 5 provides the aforementioned extensions of [31, 26, 39] to subdifferentials of Mosco and Attouch-Wets convergent sequences of subsmooth functions with compatible indexation in the setting of reflexive locally uniformly convex Banach space with a norm Fréchet differentiable off zero.

2.2 Notation and Preliminaries

Recall that a norm $\|\cdot\|$ on a vector space X is *strictly convex* whenever for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ one has $\|\frac{x+y}{2}\| < 1$. One is often interested in the case when the latter inequality holds in a uniform way.

The norm $\|\cdot\|$ of X is *locally uniformly convex at* $x \in X$ with $\|x\| = 1$ if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending on both x and ε) such that, for every $y \in X$ with $\|y\| = 1$ and $\|y - x\| \geq \varepsilon$, the inequality $\|\frac{1}{2}(x+y)\| \leq 1-\delta$ is fulfilled. When the norm $\|\cdot\|$ is locally uniformly convex at any point of the unit sphere, one says that it is locally uniformly convex. Obviously, the norm $\|\cdot\|$ is stricly convex whenever it is locally uniformly convex.

Another important concept is that of uniform convexity. The norm $\|\cdot\|$ on X is *uniformly convex* when the real δ above depends merely on ε, that is, when for every $\varepsilon > 0$ there is some $\delta > 0$ so that for any two vectors $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ one has

$$\|\frac{1}{2}(x + y)\| \leq 1 - \delta.$$

Sometimes, instead of saying that the norm $\|\cdot\|$ is uniformly convex (resp. locally uniformly convex), it will be convenient as usual to say that the normed space $(X, \|\cdot\|)$ is uniformly convex (resp. locally uniformly convex).

Considering the modulus of uniform convexity $\delta_{\|\cdot\|}$ of the norm $\|\cdot\|$ defined for $\varepsilon \in [0,2]$ by

$$\delta_{\| \ \|}(\varepsilon) := \inf\{1 - \|\frac{x+y}{2}\| : x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\},$$

we see that the norm $\|\cdot\|$ is uniformly convex if and only if $\delta_{\| \ \|}(\varepsilon) > 0$ for all $\varepsilon \in]0,2]$.

Any uniformly convex normed space is obviously locally uniformly convex, and any uniformly convex Banach space is known to be reflexive (see, e.g., [27]). In [36] one can find an example of a norm $\|\cdot\|_L$ on the space $\ell^2(\mathbb{N})$, equivalent to the usual Hilbertian norm of $\ell^2(\mathbb{N})$, which is locally uniformly convex but is not uniformly convex. So, $(\ell^2(\mathbb{N}), \|\cdot\|_L)$ is an example of a reflexive Banach space which is not uniformly convex but is locally uniformly convex.

The norm $\|\cdot\|$ on X is uniformly smooth if its modulus of smoothness,

$$\rho_{\|\|}(\tau) := \sup\left\{\frac{1}{2}\|x+\tau y\| + \frac{1}{2}\|x-\tau y\| - 1 : \|x\| = 1, \|y\| = 1\right\} \quad \text{for } \ \tau \geq 0$$

satisfies $\lim_{\tau \downarrow 0} \frac{1}{\tau}\rho_{\| \ \|}(\tau) = 0$. A normed space $(X, \|\cdot\|)$ whose norm $\|\cdot\|$ is uniformly smooth is called a uniformly smooth space.

When the dual norm $\|\cdot\|_*$ in X^* is uniformly smooth (resp. uniformly convex), the norm $\|\cdot\|$ itself (of the space X) is uniformly convex (resp. uniformly smooth) (see [18, p. 35], [19, p. 38]).

It is possible to renorm any uniformly convex Banach space with an equivalent norm which is both uniformly convex and uniformly smooth. Then, the corresponding dual norm in X^* is both uniformly convex and uniformly smooth too.

When the dual norm $\|\cdot\|_*$ of the norm $\|\cdot\|$ of a vector space X is locally uniformly convex (resp. stricly convex), the norm $\|\cdot\|$ is Fréchet (resp. Gâteaux) differentiable off zero, see, for example, [18, p. 37], [19, p. 32].

The above properties of uniformly (resp. locally uniformly) convex spaces can be found in detail in [18, 19, 20, 27]. Let us recall some other properties.

It is well known (see, for example,[27]) that all Hilbert spaces H and the Banach spaces l^p, L^p, and $W^p_m (1 < p < \infty)$ are all (for their usual norms) uniformly convex and uniformly smooth.

Consider, for a normed space $(X, \|\cdot\|)$, the set-valued mapping $J : X \rightrightarrows X^*$ defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\| \cdot \|x\|, \quad \|x^*\| = \|x\|\}.$$

It is not difficult to see that the norm $\|\cdot\|$ is stricly convex if and only if $J(x_1) \cap J(x_2) = \emptyset$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$. It is generally called the normalized duality mapping associated with the norm $\|\cdot\|$. If X is reflexive, we have that J is surjective. The set-valued mapping J is the subdifferential of the convex function $\frac{1}{2}\|\cdot\|^2$, i.e., $J = \partial(\frac{1}{2}\|\cdot\|^2)$. If $(X, \|\cdot\|)$ is reflexive and the norm $\|\cdot\|$ is Fréchet differentiable off zero and stricly convex, then J is single-valued, norm-to-norm continuous and bijective. The inverse mapping J^{-1} (of J) will be denoted by J^*; it is the normalized duality mapping for the dual norm on X^*. So, according to what has been recalled above concerning locally uniformly convex norm and concerning differentiable norm, whenever $(X, \|\cdot\|)$ is a reflexive Banach space whose norm $\|\cdot\|$ is both locally uniformly convex and Fréchet differentiable off zero, then both duality mappings J and J^* are single-valued, bijective and norm-to norm continuous. It is worth mentioning (see, e.g., Corollary 3, page 167 in [19]) that *every reflexive Banach space X can be given an equivalent norm $\|\cdot\|$ such that both $\|\cdot\|$ and the dual norm $\|\cdot\|_*$ are simultaneously locally uniformly convex and Fréchet différentiable off zero.*

The space $X \times \mathbb{R}$ will be endowed with the norm $|||\ |||$ given by $|||(x, r)||| = \sqrt{\|x\|^2 + r^2}$. So, for the normalized duality mapping $J_{X\times\mathbb{R}} : X \times \mathbb{R} \to X^* \times \mathbb{R}$ associated with the norm $|||\cdot|||$, one has the equality

$$J_{X\times\mathbb{R}}(x, r) = J(x) \times \{r\}. \tag{2.2.1}$$

When there is no risk of confusion, $J_{X\times\mathbb{R}}$ will be simply denoted by J.

We will denote by $\mathbb{B}$ or $\mathbb{B}_X$ (resp. $\mathbb{B}^*$ or $\mathbb{B}_{X^*}$) the closed unit ball of X (resp. X^*) and by $B(x, \alpha)$ (resp. $B[x, \alpha]$) the open (resp. closed) ball centred at x with radius $\alpha > 0$.

For a closed set C of the normed space $(X, \|\cdot\|)$, a nonzero vector $p \in X$ is said to be a *primal proximal normal vector to* C at $x \in C$ (see [12]) if there are $u \notin C$ and $r > 0$ such that $p = r^{-1}(u - x)$ and $\|u - x\| = d_C(u)$. (Here $d_C(u)$ denotes the distance from u to the set C; sometimes it will be convenient to put $d(u, C)$ instead of $d_C(u)$). It is known, according to Lau's theorem [25] recalled below, that in any reflexive Banach space endowed with a stricly convex Kadec-Klee norm, the set of those points which have a nearest point in any fixed closed subset is a dense set. The norm $\|\cdot\|$ of a vector space has the (sequential) Kadec-Klee property provided the weak convergence of a sequence of the unit sphere of the space is equivalent to the norm convergence of this sequence. Hence, the Kadec-Klee property holds true whenever the norm is locally uniformly convex, as easily seen.

Equivalently, a nonzero $p \in X$ is a primal proximal normal vector to C at $x \in C$ if there exists $r > 0$ such that $x \in P_C(x + rp)$, where P_C denotes the metric projection on C, that is, for any $u \in X$,

$$P_C(u) := \{y \in C : \|u - y\| = d_C(u)\}.$$

Note that the inclusion $x \in P_C(x + rp)$ is equivalent to $P_C(x + r'p) = \{x\}$ for all $0 < r' < r$ whenever the norm $\|\cdot\|$ is stricly convex (as easily seen). We also take by convention the origin of X as a primal normal vector to C at x. The sets of all primal proximal normal vectors to C at x is obviously a cone. It will be denoted by $PN_C(x)$. The concept is local in the sense of the following proposition established in [8, p. 530].

Proposition 2.2.1 *([8]) Let $(X, \|\cdot\|)$ be a normed space and C be a nonempty closed set of X. For any $u \notin C$ and any closed ball $V := B[x, r]$ centred at $x \in C$ and such that $\|u - x\| = d(u, C \cap V)$, one has*

$$u - x \in PN_C(x).$$

□

A continuous linear functional $p^* \in X^*$ is said to be a *proximal normal functional to C* at $x \in C$ if there exists $p \in PN_C(x)$ such that $p^* \in J(p)$. This means for $p^* \neq 0$ (see [12]) that there is $r > 0$ such that $x \in P_C(x + rp)$ and $p^* \in J(p)$. The sets of all proximal normal functionals to C at x is a cone which will be denoted by $N_C^P(x)$ or $N^P(C; x)$. Of course $J(p) \subset N_C^P(x)$ whenever $p \in PN_C(x)$, and if in addition X is reflexive and the norm $\|\cdot\|$ of X is Fréchet differentiable outside zero (so J is bijective) one easily verifies that $J^*(p^*) \in PN_C(x)$ whenever $p^* \in N_C^P(x)$ (keep in mind that $J^* = J^{-1}$ is the normalized duality mapping for X^* endowed with the dual norm of $\|\cdot\|$). Hence, under the assumption that $(X, \|\cdot\|)$ is reflexive and the norm $\|\cdot\|$ is Fréchet differentiable off zero, $PN_C(x)$ and $N_C^P(x)$ completely determine each other.

We will also need in our development the concept of the Fréchet normal cone $N_C^F(x)$ or $N^F(C; x)$ of a set C of the normed space X. A continuous linear functional $x^* \in X^*$ is said to be a *Fréchet normal functional* (see,e.g., [28, 16]) to C at $x \in C$ if for any $\varepsilon > 0$ there exists a neighbourhood U of x such that the inequality $\langle x^*, x' - x\rangle \le \varepsilon\|x' - x\|$ holds for all $x' \in C \cap U$. We denote by $N_C^L(x)$ or $N^L(C; x)$ the *Mordukhovich limiting normal cone* of C at $x \in C$, that is,

$$N_C^L(x) = {}^{\text{seq}}\operatorname*{Lim\,sup}_{C\ni u\to x} N_C^F(u) := \left\{ \begin{array}{c} x^* \in X^* : \exists \text{ sequences } C \ni x_n \to x, \\ x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in N_C^F(x_n). \end{array} \right\}$$

More will be recalled in the next section concerning the concepts of limits superior and inferior of sets and set-valued mappings. By convention one defines $PN_C(x)$, $N_C^P(x)$, $N_C^F(x)$ and $N_C^L(x)$ as the empty set whenever $x \notin C$.

The above notation and concepts can be translated into the context of functions. Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function on the normed space X. By definition, the effective domain of f is the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ and the epigraph of f is the set $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$; the function f is *proper* when it does not take on the value $-\infty$ and $\operatorname{dom} f \neq \emptyset$. For a lower semicontinuous function f which is finite at x, we say that $p^* \in X^*$ is a *proximal subgradient* of f at x if $(p^*, -1)$ is a proximal normal functional to the epigraph of f at $(x, f(x))$. The *proximal subdifferential* of f at x, denoted by $\partial_P f(x)$, consists of all such functionals. Thus we have $p^* \in \partial_P f(x)$ if and only if $(p^*, -1) \in N^P_{\operatorname{epi} f}(x, f(x))$. Similarly, for a function f which is finite at x, the *Fréchet subdifferential* of f at x, denoted by $\partial_F f(x)$, consists of all functionals $x^* \in X^*$ such that $(x^*, -1) \in N^F_{\operatorname{epi} f}(x, f(x))$. If $x \notin \operatorname{dom} f$ then all subdifferentials of f at x are empty, by convention. It is known that, for an extended real-valued proper lower semicontinuous function f on a reflexive Banach space endowed with a Kadec-Klee and Fréchet differentiable norm, the (effective) domain of the set-valued mapping $\partial_P f : X \rightrightarrows X^*$

$$\operatorname{Dom}\ \partial_P f := \{x \in X : \partial_P f(x) \neq \emptyset\}$$

is dense in $\operatorname{dom} f$ (see [10, Theorem 7.1]). The Fréchet subgradients are known (see [28]) to have an analytical characterization in the sense the $x \in \partial_F f(x)$ if and only if

$$\liminf_{y\to x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|x - y\|} \ge 0.$$

When $\partial_F f(x) \neq \emptyset$, one says that f is Fréchet subdifferentiable at the point x. Similarly to the above definitions, the *Mordukhovich limiting subdifferential* of f at x, denoted by $\partial_L f(x)$, consists of all functionals $x^* \in X^*$ such that $(x^*, -1) \in N^L\big(\operatorname{epi} f; (x, f(x))\big)$. It is known that $x^* \in \partial_L f(x)$ if and only if there exists a sequence $\{(x_n, f(x_n))\}_{n\in\mathbb{N}}$ converging to $(x, f(x))$ and a sequence $\{x_n^*\}_{n\in\mathbb{N}}$ converging weakly star to x^* such that $x_n^* \in \partial_F f(x_n)$ (see [28]). That is,

$$\partial_L f(x) = {}^{\text{seq}}\operatorname*{Lim\,sup}_{u\to_f x} \partial^F f(u).$$

The last concepts that we need is the Clarke tangent and normal cones. For a subset C of X and a point $x \in C$, a vector $h \in X$ belongs to the *Clarke tangent cone* $T^{Cl}(C; x)$ of C at x provided that for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that $\big(u + tB(h, \varepsilon)\big) \cap C \neq \emptyset$ for all $u \in C \cap B(x, \delta)$ and $t \in]0, \delta[$, where we recall that $B(x, \delta)$ denotes the open ball centred at x and of radius δ. It is known that $h \in T^{Cl}(C; x)$ if and only if for any sequences $\{x_n\}_{n\in\mathbb{N}}$ in C converging to x and $\{t_n\}_{n\in\mathbb{N}}$ in $]0, +\infty[$ tending to 0 there is a sequence $\{h_n\}_{n\in\mathbb{N}}$ in X converging to h such that $x_n + t_n h_n \in C$ for all $n \in \mathbb{N}$. The *Clarke normal cone* is defined as the negative polar cone of the Clarke tangent cone, that is,

$$N^{Cl}(C; x) := \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \text{ for all } h \in T^{Cl}(C; x)\};$$

the *Clarke subdifferential* of the function f at $x \in \operatorname{dom} f$ is the set

$$\partial_{Cl} f(x) := \{x^* \in X^* :\ (x^*, -1) \in N^{Cl}\big(\operatorname{epi} f; (x, f(x))\big)\}.$$

As above $T^{Cl}(C; x)$ and $N^{Cl}(C; x)$ (resp. $\partial_{Cl} f(x)$) are defined to be empty whenever $x \notin C$ (resp. f is not finite at x).

If X is an Asplund space, we have (see [28])

$$N^{Cl}(C; x) = \overline{\text{co}}^*(N^L(C; x)) \tag{2.2.2}$$

where $\overline{\text{co}}^*$ denotes the weak star closed convex hull. Recall that a Banach space X is an Asplund space provided the toplogical dual of any separable subspace of X is separable.

We will use the result below concerning the proximal and Fréchet normal cones (see [10]). For the convenience of the reader, we sketch a proof.

Lemma 2.2.2 *([10]) Let $(X, \|\cdot\|)$ be a Banach space whose norm is Fréchet differentiable (off zero) and C be a closed subset of X. The following holds :*
(a) For all $x \in C$, we have $N_C^P(x) \subset N_C^F(x)$.
(b) The inclusion $\partial_P f(x) \subset \partial_F(x)$ holds for all $x \in X$.

Proof. (a) We take any $v \in PN(C;x)$, so there is $\sigma > 0$ such that

$$\|x-(x+\sigma v)\|^2 \le \|x+\sigma v - y\|^2 \qquad \text{for all} \;\; y \in C,$$

hence

$$\sigma^2\|v\|^2 \le \|x+\sigma v-y\|^2,$$
$$0 \le \sigma^2(\frac{1}{2}\|v+\sigma^{-1}(x-y)\|^2 - \frac{1}{2}\|v\|^2).$$

For each $y \in C$, this yields, according to the equality $J(v) = D(\frac{1}{2}\|\cdot\|^2)(v)$ (the Frechet differential of $\frac{1}{2}\| \, \|^2$), a mapping $\varepsilon : X \to \mathbb{R}$ with $\varepsilon(u) \to 0$ when $u \to 0$ and such that

$$0 \le \sigma^2\big(\langle J(v), \sigma^{-1}(x-y)\rangle + \sigma^{-1}\|x-y\|\varepsilon(y-x)\big)$$

$$\langle J(v), y-x\rangle \le \|y-x\|\varepsilon(y-x).$$

Thus $J(v) \in N^F(C;x)$, which implies that $N^P(C;x) \subset N^F(C;x)$.

(b) The assertion (b) follows directly from (a). □

We recall now the famous Lau theorem concerning the metric projection on closed sets which has been involved above. It easily ensures that the points of the set where the proximal normal cone is not reduced to zero are dense in the set.

Theorem 2.2.3 *(**Lau** [25]) Let X be a reflexive Banach space endowed with a stricly convex norm $\|\cdot\|$ satisfying the (sequential) Kadec-Klee property and let C be a nonempty (strongly) closed set of X. Then there exists a dense G_δ set of $X \backslash C$ with unique nearest points in C.*

We recall that any locally uniformly convex norm (in particular, any uniformly convex norm) fulfills the Kadec-Klee property.

The next proposition provides an approximation result of Fréchet normals by proximal normals. It appears in the paper [9] by F. Bernard, L. Thibault and N. Zlateva as an adaptation of the proof a similar result of A.D. Ioffe [13].

Proposition 2.2.4 *([9, 13]) Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable* *off zero. Let C be a closed subset of X with $x \in C$ and let $x^* \in N_C^F(x)$. Then for any $\varepsilon > 0$ there exist $u_\varepsilon^* \in N_C^P(u_\varepsilon)$ such that*

$$\|u_\varepsilon - x\| < \varepsilon \quad \textit{and} \quad \|u_\varepsilon^* - x^*\| < \varepsilon.$$

In fact the result uses only the Fréchet differentiability outside of zero of the norm $\|\cdot\|$ and of its dual norm.

Through the latter proposition we can approximate horizontal proximal normals to the epigraph of a function by nonhorizontal ones. Before proving that approximation property let us establish the following lemma which has its own interest.

Lemma 2.2.5 *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued proper lower semicontinuous function. The following hold :*
(a) For any $(p, -r) \in PN\big(\operatorname{epi} f; (x, s)\big)$ one has $r \geq 0$; further, if the norm $\|\cdot\|$ is stricly convex and if $r > 0$, then $s = f(x)$.
(b) If $\|\cdot\|$ is stricly convex, then, for any $(p, 0) \in PN\big(\operatorname{epi} f; (x, s)\big)$, one also has $(p, 0) \in PN\big(\operatorname{epi} f; (x, f(x))\big)$.
Similarly :
(c) For any $(x^*, -r) \in N^P\big(\operatorname{epi} f; (x, s)\big)$ one has $r \geq 0$; further, if the norm $\|\cdot\|$ is in addition stricly convex and if $r > 0$, then $s = f(x)$.
(d) If $\|\cdot\|$ is stricly convex, then, for any $(x^*, 0) \in N^P\big(\operatorname{epi} f; (x, s)\big)$, one also has $(x^*, 0) \in N^P\big(\operatorname{epi} f; (x, f(x))\big)$.

Proof. (a) Suppose that $(x, s) \in \operatorname{epi} f$ and $(p, -r) \in PN_{\operatorname{epi} f}(x, s)$. Then there is some $\sigma > 0$ such that

$$(x, s) \in P_{\operatorname{epi} f}\big((x, s) + \sigma(p, -r)\big), \tag{2.2.3}$$

hence

$$d_{\operatorname{epi} f}\big((x, s) + \sigma(p, -r)\big) = \sqrt{\sigma^2\|p\|^2 + \sigma^2 r^2}. \tag{2.2.4}$$

We want to show that $r \geq 0$. Suppose on the contrary $r < 0$ and fix $\alpha \in]\sigma, 2\sigma[$. Then $s - \alpha r > s$, so $(x, s - \alpha r)$ is also included in $\operatorname{epi} f$, and

$$\begin{aligned} \|\big((x, s) + \sigma(p, -r)\big) - (x, s - \alpha r)\| &= \sqrt{\sigma^2\|p\|^2 + (\alpha - \sigma)^2 r^2} \\ &< \sqrt{\sigma^2\|p\|^2 + \sigma^2 r^2} \\ &= d_{\operatorname{epi} f}\big((x, s) + \sigma(p, -r)\big). \end{aligned}$$

This contradiction ensures $r \geq 0$ as desired.

Now we suppose that $r > 0$ and the norm $\|\cdot\|$ of X is stricly convex. The norm $\|\cdot\|$ on $X \times \mathbb{R}$, given by $\|(u,t)\| := \sqrt{\|u\|^2 + t^2}$, is also stricly convex. Consequently, the real $\sigma > 0$ in (2.2.3) can be taken such that

$$P_{\operatorname{epi} f}\big((x,s) + \sigma(p,-r)\big) = \{(x,s)\}$$

(it is enough take as σ a positive real less than the one involved in (2.2.3)). Note that $f(x)$ is finite since $f(x) \leq s$. Let $0 \leq t = \min\{\sigma r,\ s - f(x)\}$. Therefore

$$0 \leq \sigma r - t \leq \sigma r \quad \text{and} \quad f(x) \leq s - t.$$

So we have that $(x, s-t) \in \operatorname{epi} f$. We consider the distance between the pairs $((x,s) + \sigma(p,-r))$ and $(x, s-t)$, and we write

$$\begin{aligned} \|\big((x,s) + \sigma(p,-r)\big) - (x, s-t)\| &= \sqrt{\sigma^2\|p\|^2 + (\sigma r - t)^2} \\ &\leq \sqrt{\sigma^2\|p\|^2 + \sigma^2 r^2} \\ &= d_{\operatorname{epi} f}\big((x,s) + \sigma(p,-r)\big), \end{aligned}$$

where the last equality is due to (2.2.4). By the uniqueness of (x,s) as the nearest point in $\operatorname{epi} f$ of $(x,s) + \sigma(p,-r)$ we deduce that $t = 0$ and we obtain that $s = f(x)$.

(b) Assume now that $(p,0) \in PN\big(\operatorname{epi} f; (x,s)\big)$ and the norm of X is stricly convex. Consider the norm $\|\cdot\|$ on $X \times \mathbb{R}$ defined as above. Taking σ as in (2.2.3) we have, for all $(x', s') \in \operatorname{epi} f$,

$$\|(x,s) + \sigma(p,0) - (x',s')\| \geq \sigma\|p\|$$

and since $(x', s' + s - f(x)) \in \operatorname{epi} f$ (because $s - f(x) \geq 0$) we also have

$$\|(x,s) + \sigma(p,0) - (x', s' + s - f(x))\| \geq \sigma\|p\|,$$

which yields, for all $(x', s') \in \operatorname{epi} f$,

$$\|(x, f(x)) + \sigma(p,0) - (x',s')\| \geq \sigma\|p\|.$$

The latter inequality means $(x, f(x)) \in P_{\operatorname{epi} f}\big((x, f(x)) + \sigma(p,0)\big)$. So $(p,0) \in PN_{\operatorname{epi} f}\big((x, f(x))\big)$ as required in (b).

Finally, the assertions (c) and (d) follow directly from (a) and (b) respectively.

□

Proposition 2.2.6 *Assume that* $(X, \|\cdot\|)$ *is a reflexive locally uniformly convex Banach space and that the norm* $\|\cdot\|$ *is Fréchet differentiable off zero. Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function and* $x \in \operatorname{dom} f$*, and let* $x^* \in X^*$ *with* $(x^*, 0) \in N^P_{\operatorname{epi} f}\big((x, f(x))\big)$*, where the proximal normal cone in* $X \times \mathbb{R}$ *is taken with respect to the product norm* $\|(u, r)\| = (\|u\|^2 + |r|^2)^{1/2}$*. Then for any* $\varepsilon > 0$ *there exist* $x_\varepsilon \in \operatorname{dom} f$ *and* $(x^*_\varepsilon, -r_\varepsilon) \in N^P_{\operatorname{epi} f}\big((x_\varepsilon, f(x_\varepsilon))\big)$ *with* $r_\varepsilon > 0$ *such that* $(x^*_\varepsilon, -r_\varepsilon) \in N^P_{\operatorname{epi} f}\big((x_\varepsilon, f(x_\varepsilon))\big)$ *along with*

$$\|x_\varepsilon - x\| + |f(x_\varepsilon) - f(x)| < \varepsilon \quad \textit{and} \quad \|(x^*_\varepsilon, -r_\varepsilon) - (x^*, 0)\| < \varepsilon.$$

Proof. We first observe that the norm $\|(u, r)\| = (\|u\|^2 + |r|^2)^{1/2}$ on $X \times \mathbb{R}$ is locally uniformly convex and Fréchet differentiable off zero. Fix $(x^*, 0) \in N^P_{\operatorname{epi} f}\big((x, f(x))\big)$. Then $(x^*, 0) \in N^F_{\operatorname{epi} f}\big((x, f(x))\big)$ by Lemma 2.2.2 above, so (see [28, Lemma 2.37]) we know that there is some $y_\varepsilon \in \operatorname{dom} f$ with $\|y_\varepsilon - x\| + |f(y_\varepsilon) - f(x)| < \varepsilon/2$ and $(y^*_\varepsilon, -s_\varepsilon) \in N^F_{\operatorname{epi} f}\big((y_\varepsilon, f(y_\varepsilon))\big)$ with $\|(y^*_\varepsilon, -s_\varepsilon) - (x^*, 0)\| < \varepsilon/2$ and $s_\varepsilon > 0$. Considering the positive real $\eta(\varepsilon) := \min\{s_\varepsilon, \varepsilon/2\}$, Proposition 2.2.4 furnishes $(x_\varepsilon, \rho_\varepsilon) \in \operatorname{epi} f$ with $\|x_\varepsilon - y_\varepsilon\| + |\rho_\varepsilon - f(y_\varepsilon)| < \eta(\varepsilon)$ and $(x^*_\varepsilon, -r_\varepsilon) \in N^P_{\operatorname{epi} f}\big((x_\varepsilon, \rho_\varepsilon)\big)$ with $\|(x^*_\varepsilon, -r_\varepsilon) - (y^*_\varepsilon, -s_\varepsilon)\| < \eta(\varepsilon)$. Since $|s_\varepsilon - r_\varepsilon| < s_\varepsilon$, we have $r_\varepsilon > 0$ hence $\rho_\varepsilon = f(x_\varepsilon)$ according to Lemma 2.2.5. Consequently,

$$\|x_\varepsilon - x\| + |f(x_\varepsilon) - f(x)| < \varepsilon \quad \text{and} \quad \|(x^*_\varepsilon, -r_\varepsilon) - (x^*, 0)\| < \varepsilon,$$

$r_\varepsilon > 0$ and $(x^*_\varepsilon, -r_\varepsilon) \in N^P_{\operatorname{epi} f}\big((x_\varepsilon, f(x_\varepsilon))\big)$, and this finishes the proof. □

2.3 Normal cones of Mosco convergent sequences of sets

Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of the normed space X. Prior to introduce the Mosco convergence, we will recall the Painlevé-Kuratowski convergence (see, e.g., [6, 34]). Given a topology τ on X, one defines the *sequential limit inferior* ${}^\tau\operatorname*{Lim\,inf}_{n\to\infty} C_n$ of the sequence $\{C_n\}_{n\in\mathbb{N}}$ with respect to the topology τ as the set of all τ-limits of sequences $\{x_n\}_n$ with $x_n \in C_n$ for all $n \in \mathbb{N}$ large enough. The *sequential limit superior* ${}^\tau\operatorname*{Lim\,sup}_{n\to\infty} C_n$ with respect to τ is defined as the set of all τ-limits of sequences $\{x_n\}_n$ with $x_n \in C_n$

for infinitely many $n \in \mathbb{N}$. Equivalently, $x \in {}^{\tau}\operatorname*{Lim\,sup}_{n\to\infty} C_n$ provided there are an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x with $x_n \in C_{k(n)}$ for all $n \in \mathbb{N}$. Clearly,

$$
{}^{\tau}\operatorname*{Lim\,inf}_{n\to\infty} C_n \subset {}^{\tau}\operatorname*{Lim\,sup}_{n\to\infty} C_n.
$$

One then says that the sequence $\{C_n\}_{n\in\mathbb{N}}$ τ-sequentially *Painlevé-Kuratowski converges* to a subset C of X whenever

$$
C = {}^{\tau}\operatorname*{Lim\,inf}_{n\to\infty} C_n = {}^{\tau}\operatorname*{Lim\,sup}_{n\to\infty} C_n.
$$

When τ is the topology associated with the norm of X, one just says that the sequence Painlevé-Kuratowski converges to C, and in that case, one can verify that

$$
{}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} C_n = \{x \in X : \limsup_{n\to\infty} d(x, C_n) = 0\},
$$

$$
{}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} C_n = \{x \in X : \liminf_{n\to\infty} d(x, C_n) = 0\}.
$$

For $C \subset {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} C_n$ (resp. $C \subset {}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} C_n$) it is known for any $x \in X$ that

$$
\limsup_{n\to\infty} d_{C_n}(x) \le d_C(x) \quad (\text{resp. } \liminf_{n\to\infty} d_{C_n}(x) \le d_C(x)\,). \tag{2.3.1}
$$

When the sequence $\{C_n\}_{n\in\mathbb{N}}$ (sequentially) Painlevé-Kuratowski converges to C with respect to both the norm convergence and the weak convergence, one says that it *converges in the sense of Mosco* to C. It is easily seen that this is equivalent to

$$
C = {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} C_n = {}^{w}\operatorname*{Lim\,sup}_{n\to\infty} C_n,
$$

where w stands here for the weak topology $w(X, X^*)$ of X. Note that, in this case, the subset C is weakly sequentially closed in the sense that the limit of any weakly convergent sequence of C belongs to C. Indeed, suppose without loss of generality that every C_n is nonempty, and take any sequence $\{x_m\}_{m\in\mathbb{N}}$ of C converging weakly to $x \in X$. For each $m \in \mathbb{N}$, from the equality $C = {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} C_n$ there is a sequence $\{x_{m,n}\}_{n\in\mathbb{N}}$ converging strongly to x_m

with $x_{m,n} \in C_n$ for all $n \in \mathbb{N}$. We can then choose an increasing sequence $\{k(m)\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|x_{m,k(m)} - x_m\| < 1/m$. So, for $x'_m := x_{m,k(m)}$, the sequence $\{x'_m\}_{m\in\mathbb{N}}$ converges weakly to x as $m \to \infty$ and $x'_m \in C_{k(m)}$ for all $m \in \mathbb{N}$. This and the equality $C = {}^w \operatorname*{Lim\,sup}_{m\to\infty} C_m$ justify the inclusion $x \in C$.

Let ${}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N^P_{C_n}$ denote the limit inferior (with respect to the norm topology in $X \times X^*$) of the sequence $\{\operatorname{gph} N^P_{C_n}\}_{n\in\mathbb{N}}$ of the graphs of the functional proximal normal cones, that is, the set of all (x, x^*) in $X \times X^*$ for which there exists a sequence $\{(x_n, x^*_n)\}_{n\in\mathbb{N}}$ in $X \times X^*$ such that

$$x_n \in C_n \quad \text{and} \quad x^*_n \in N^P_{C_n}(x_n) \quad \text{for } n \in \mathbb{N} \text{ large enough},$$

and such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{x^*_n\}_{n\in\mathbb{N}}$ converge to x and x^* with respect to the norm topology of X and X^* respectively. Similarly, we denote by ${}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{C_n}$ the sequential limit superior of $\{\operatorname{gph} N^P_{C_n}\}_{n\in\mathbb{N}}$ with respect to the $\|\ \| \times w(X^*, X)$ topology of $X \times X^*$, that is, the set of all (x, x^*) in $X \times X^*$ for which there exist a sequence $\{(x_n, x^*_n)\}_{n\in\mathbb{N}}$ in $X \times X^*$ and an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$x_n \in C_{k(n)} \quad \text{and} \quad x^*_n \in N^P_{C_{k(n)}}(x_n) \quad \text{for all } n \in \mathbb{N},$$

and such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{x^*_n\}_{n\in\mathbb{N}}$ converge to x and x^* with respect to the norm topology of X and the weak star topology of X^* respectively. It is evident that

$${}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N^P_{C_n} \subset {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{C_n}.$$

If they are equal we denote it by $\operatorname*{Lim}_{n\to\infty} \operatorname{gph} N^P_{C_n}$, then,

$$\operatorname*{Lim}_{n\to\infty} \operatorname{gph} N^P_{C_n} := {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N^P_{C_n} = {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{C_n}.$$

In the definition above, $\operatorname{gph} M$ denotes the graph of a set-valued mapping $M : U \rightrightarrows V$, that is,

$$\operatorname{gph}\ M := \{(u, v) \in U \times V :\ v \in M(u)\}.$$

We can now start with the lemma below. It has been proved by X.Y. Zheng and Z. Wei [39] for Hilbert spaces. Here with different techniques we establish the lemma in the context of reflexive Banach spaces.

Lemma 2.3.1 *Assume that $(X, \|\cdot\|)$ is a reflexive Banach space and that the norm $\|\cdot\|$ is strictly convex and has the Kadec-Klee property. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Mosco convergent to a closed subset C of X. Suppose that $x \in X$ and $y \in C$ satisfy $P_C(x) = \{y\}$. Then there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in $X \times X$ such that*

$$\lim_{n\to\infty} \big(\|x_n - x\| + \|y_n - y\|\big) = 0 \text{ and } P_{C_n}(x_n) = \{y_n\} \quad \text{for large } n \in \mathbb{N}.$$

Proof. The set C being nonempty, the convergence assumption of $\{C_n\}_{n\in\mathbb{N}}$ entails that $C_n \neq \emptyset$ for large n, so without loss of generality we may assume that all the sets C_n are nonempty. By Theorem 2.2.3, for any $n \in \mathbb{N}$ there exists $x_n \in B(x, \frac{1}{n})$ and $y_n \in C_n$ such that $P_{C_n}(x_n) = \{y_n\}$. Thus $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Further, it is not difficult to see that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Indeed, since $C \subset \operatorname*{Lim\,inf}_{n\to\infty} C_n$, we have (see (2.3.1)) that

$$\limsup_{n\to\infty} d_{C_n}(x) \leq d_C(x),$$

from which we deduce that

$$\begin{aligned}
\limsup_{n\to\infty} \|y_n\| &\leq \limsup_{n\to\infty} \|y_n - x_n\| + \lim_{n\to\infty} \|x_n\| \\
&= \limsup_{n\to\infty} d_{C_n}(x_n) + \|x\| \\
&\leq d_C(x) + \|x\|,
\end{aligned}$$

and so the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Consequently there exists a subsequence $\{y_{k(n)}\}_{n\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ converging weakly to some $z \in X$ and $z \in {}^w\operatorname*{Lim\,sup}_{n\to\infty} C_n \subset C$. Further, one has

$$\begin{aligned}
d_C(x) &\geq \limsup_{n\to\infty} d_{C_n}(x) = \limsup_{n\to\infty} (d_{C_n}(x) + \|x - x_n\|) \\
&\geq \limsup_{n\to\infty} d_{C_n}(x_n) = \limsup_{n\to\infty} \|x_n - y_n\| \\
&\geq \limsup_{n\to\infty} \|x_{k(n)} - y_{k(n)}\| = \limsup_{n\to\infty} \|x - y_{k(n)}\| \\
&\geq \liminf_{n\to\infty} \|x - y_{k(n)}\| \geq \|x - z\|,
\end{aligned}$$

the latter inequality being due to the weak lower semicontinuity of the norm $\|\cdot\|$. Therefore $z \in P_C(x)$, which entails $z = y$ according to the assumption $P_C(x) = \{y\}$. On the other hand, since

$$\liminf_{n\to\infty} \|x_{k(n)} - y_{k(n)}\| = \liminf_{n\to\infty} \|x - y_{k(n)}\| \geq \|x - y\|,$$

according again to the weak lower semicontinuity of the norm $\|\cdot\|$, and since

$$\begin{aligned}\limsup_{n\to\infty}\|x_{k(n)}-y_{k(n)}\| &= \limsup_{n\to\infty} d_{C_{k(n)}}(x_{k(n)})\\ &\le \limsup_{n\to\infty}(d_{C_{k(n)}}(x)+\|x-x_{k(n)}\|)\le d_C(x)=\|x-y\|,\end{aligned}$$

we see that $\lim_{k\to\infty}\|x_{k(n)}-y_{k(n)}\|=\|x-y\|$. We have also that

$$x_{k(n)}-y_{k(n)}\xrightarrow{w} x-y.$$

From the Kadec-Klee property of the norm $\|\cdot\|$ we obtain that

$$x_{k_n}-y_{k_n}\to x-y.$$

We deduce that $\{y_{k(n)}\}_{n\in\mathbb{N}}$ converges strongly to y, or equivalently, any weakly convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$ converges strongly to y. From this and the boundedness of $\{y_n\}_{n\in\mathbb{N}}$ it is easily seen (through the reflexivity of X) that the whole sequence converges strongly to y. □

The second lemma relax the equality assumption $P_C(x)=\{y\}$ in the lemma above into the inclusion $y\in P_C(x)$.

Lemma 2.3.2 *Assume that $(X,\|\cdot\|)$ is a reflexive Banach space and that the norm $\|\cdot\|$ is strictly convex and has the Kadec-Klee property. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Mosco convergent to a closed subset C of X. Suppose that $x\in X$ and $y\in C$ satisfy $y\in P_C(x)$. Then there exist an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ in $X\times X$ such that*

$$\lim_{n\to\infty}\big(\|x_n-x\|+\|y_n-y\|\big)=0 \quad \textit{and} \quad P_{C_{k(n)}}(x_n)=\{y_n\} \quad \textit{for all } n\in\mathbb{N}.$$

Proof. For each $n\in\mathbb{N}$ put $x'_n:=(1-\frac{1}{n})x+\frac{1}{n}y$. Then $P_C(x'_n)=\{y\}$ for all $n\in\mathbb{N}$. Choose by Lemma 2.3.1 some integer $k(1)$ and $x_1\in X$ with $\|x_1-x'_1\|<\frac{1}{1}$ such that $P_{C_{k(1)}}(x_1)=\{y_1\}$ with $\|y_1-y\|<\frac{1}{1}$. We can produce by induction an increasing sequence $\{k(n)\}_n$ in $\mathbb{N}$ and two sequences $\{x_n\}_n$ and $\{y_n\}_n$ in X such that $\|x_n-x'_n\|<\frac{1}{n}$, $\|y_n-y\|<\frac{1}{n}$ and $P_{C_{k(n)}}(x_n)=\{y_n\}$. Those sequences fulfill the desired properties. □

The next lemma is concerned with the assumption of the equality $P_{C_n}(x_n)=\{y_n\}$.

Lemma 2.3.3 *Let* $\{C_n\}_n$ *be a sequence of closed subsets of* X *and let* $C = \operatorname*{Lim\,sup}_{n\to\infty} C_n$. *Suppose that* $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ *is a sequence in* $X \times X$ *such that*

$$\lim_{n\to\infty} \big(\|x_n - x\| + \|y_n - y\|\big) = 0 \text{ and } P_{C_n}(x_n) = \{y_n\} \quad \text{for large } n \in \mathbb{N}.$$

Then one has $y \in P_C(x)$.

Proof. Since $\{y_n\}_{n\in\mathbb{N}}$ converges to y and $\operatorname*{Lim\,sup}_{n\to\infty} C_n \subset C$, we have $y \in C$. Further, since $C \subset \operatorname*{Lim\,sup}_{n\to\infty} C_n$, for any $u \in C$, there exists an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ converging to u with $u_n \in C_{k(n)}$ for all $n \in \mathbb{N}$. We then have

$$\begin{aligned} \|x - y\| &= \lim_{n\to\infty} \|x_{k(n)} - y_{k(n)}\| = \lim_{n\to\infty} d_{C_{k(n)}}(x_{k(n)}) \\ &\leq \liminf_{n\to\infty} \|x_{k(n)} - u_n\| = \|x - u\|. \end{aligned}$$

This translates the desired inclusion $y \in P_C(x)$. $\square$

Now we give the main results of this section. They establish connections between diverse limits of $\{\operatorname{gph} N^P_{C_n}\}_{n\in\mathbb{N}}$ and the sets $\operatorname{gph} N^P_C$ and $\operatorname{gph} N^L_C$ when the sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ Mosco converges to the set C.

Theorem 2.3.4 *Assume that* $(X, \|\cdot\|)$ *is a reflexive locally uniformly convex Banach space and that the norm* $\|\cdot\|$ *is Fréchet differentiable off zero. Let* $\{C_n\}_{n\in\mathbb{N}}$ *be a sequence of closed subsets of* X *Mosco converging to a nonempty closed subset* C *of* X. *Then the following assertions are equivalent :*
(a) one has $x^* \in N^P_C(x)$ *;*
(b) there exist $r > 0$, $n_0 \in \mathbb{N}$ *and* $\{x_n\}_n$ *strongly converging to* x, $\{x^*_n\}_{n\in\mathbb{N}}$ *strongly converging to* x^* *such that*

$$P_{C_n}(x_n + rJ^*(x^*_n)) = \{x_n\} \quad \text{for all } n \geq n_0.$$

(Note that for such x^*_n *one has* $x^*_n \in N^P_{C_n}(x_n)$*).*

Proof. Assume that, for x^*, the assertion (b) is satisfied, that is, there are $r > 0$, $n_0 \in \mathbb{N}$ and $\{x_n\}_{n\in\mathbb{N}}$ strongly converging to x, $\{x^*_n\}_{n\in\mathbb{N}}$ strongly converging to x^* such that

$$P_{C_n}(x_n + rJ^*(x^*_n)) = \{x_n\} \quad \text{for all } n \geq n_0.$$

The continuity of J^* (see the previous section) entails

$$\lim_{n\to\infty}(x_n + rJ^*(x_n^*)) = x + rJ^*(x^*),$$

and by Lemma 2.3.3

$$x \in P_C(x + rJ^*(x^*)),$$

which guarantees the inclusion $x^* \in N_C^P(x)$.

Now we assume that $x^* \in N_C^P(x)$. By definition there is $\sigma > 0$ such that

$$P_C(x + \sigma J^*(x^*)) = \{x\}.$$

By Lemma 2.3.1 there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging strongly to x and a sequence $\{z_n\}_{n\in\mathbb{N}}$ converging strongly to $x + \sigma J^*(x^*)$ such that $P_{C_n}(z_n) = \{x_n\}$, for large n, say $n \geq N$. For each such $n \geq N$, putting $x_n^* = J(\frac{1}{\sigma}(z_n - x_n))$ ensures $x_n^* \in N_{C_n}^P(x_n)$ (since $z_n - x_n \in PN_{C_n}(x_n)$), and by the continuity of J we also have

$$x_n^* = J(\frac{1}{\sigma}(z_n - x_n)) \xrightarrow[n\to\infty]{} J(\frac{1}{\sigma}(x + \sigma J^*(x^*) - x)) = J(J^*(x^*)) = x^*.$$

Further, since $z_n - x_n = \sigma J^*(x_n^*)$ we see that

$$\{x_n\} = P_{C_n}(z_n) = P_{C_n}(x_n + \sigma J^*(x_n^*)),$$

so the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_n^*\}_{n\in\mathbb{N}}$ fulfill the properties of the assertion (b) with $r = \sigma$. $\square$

Now we denote by $\Lambda_C^r : X \rightrightarrows X^*$ the set-valued mapping whose graph is given by

$$\begin{aligned}\operatorname{gph}\Lambda_C^r &:= \{(x, x^*) \in X \times X^* : P_C(x + rJ^*(x^*)) = \{x\}\} \\ &= \left\{(x, x^*) \in \operatorname{gph} N_C^P : P_C(x + rJ^*(x^*)) = \{x\}\right\},\end{aligned}$$

and we obtain the following corollary. The assertion (b) of the corollary is a generalization of Theorem 3.1 of X.Y. Zheng and Z. Wei [39] to uniformly smooth and uniformy convex Banach spaces which are not necessarily Hilbert spaces. The corollary is even obtained for some locally uniformly convex Banach spaces.

Corollary 2.3.5 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Mosco to a nonempty closed subset C of X. Then,*
(a) $\operatorname{gph} N_C^P = \bigcup_{r>0} {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} \Lambda_{C_n}^r$ *; (b)* $\operatorname{gph} N_C^P \subset {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N_{C_n}^P$ *;*

(c) $\operatorname{gph} N_C^F \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P$.

Proof. The first assertion is a direct consequence of Theorem 2.3.4. The second assertion obviously follows from the first since $\Lambda_r(C_n)$ is a subset of $\operatorname{gph} N_{C_n}^P$. For the third assertion, note that by Proposition 2.2.4 for each $(x, x^*) \in \operatorname{gph} N_C^F$ there is a sequence $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ in $\operatorname{gph} N_C^P$ such that

$$\|x_k - x\| < \frac{1}{k} \quad \text{and} \quad \|x_k^* - x^*\| < \frac{1}{k}.$$

From the second assertion, for each integer k, there is a sequence $\{(x_{k,n}, x_{k,n}^*)\}_{n\in\mathbb{N}}$ with $(x_{k,n}, x_{k,n}^*)$ in $\operatorname{gph} N_{C_n}^P$ such that

$$x_{k,n} \xrightarrow[n\to\infty]{} x_k \quad \text{and} \quad x_{k,n}^* \xrightarrow[n\to\infty]{} x_k^*.$$

Therefore there is a strictly increasing sequence $\{\nu(k)\}_k$ in $\mathbb{N}$ such that

$$\|x_{k,\nu(k)} - x_k\| < \frac{1}{k} \quad \text{and} \quad \|x_{k,\nu(k)}^* - x_k^*\| < \frac{1}{k}.$$

Thus we obtain that

$$x_{k,\nu(k)} \xrightarrow[k\to\infty]{} x \quad \text{and} \quad x_{k,\nu(k)}^* \xrightarrow[k\to\infty]{} x^*.$$

The proof is completed since $(x_{k,\nu(k)}, x_{k,\nu(k)}^*) \in \operatorname{gph} N_{C_{\nu(k)}}^P$. $\square$

Now, through Theorem 2.3.4 again, we reformulate the result above in a local way in the next corollary. In the statement of the corollary we denote by ${}^{\|\ \|}\operatorname*{Lim\,inf}_{C_n \ni x' \to x,\ n\to\infty} \Lambda_{C_n}^r(x')$ the set of all $x^* \in X^*$ such that for any sequence $\{x_n\}_n$ in X converging strongly to x with $x_n \in C_n$ for large n there exists a sequence $\{x_n^*\}_n$ in X^* converging strongly to x^* with $x_n^* \in \Lambda_{C_n}^r(x_n)$ for n large enough. We define similarly the set ${}^{\|\ \|}\operatorname*{Lim\,inf}_{C_n \ni x' \to x,\ n\to\infty} N_{C_n}^P(x')$ and ${}^{\|\ \|}\operatorname*{Lim\,sup}_{C_n \ni x' \to x,\ n\to\infty} N_{C_n}^P(x')$.

Corollary 2.3.6 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Mosco to a nonempty closed subset C of X and let $x \in C$. Then, the following hold :*

a) $N_C^P(x) = \bigcup_{r>0} {}^{\|\,\|}\operatorname*{Lim\,inf}_{\substack{C_n \ni x' \to x,\\ n\to\infty}} \Lambda_{C_n}^r(x')$; *(b)* $N_C^P(x) \subset {}^{\|\,\|}\operatorname*{Lim\,inf}_{\substack{C_n \ni x' \to x,\\ n\to\infty}} N_{C_n}^P(x')$;

(c) $N_C^F(x) \subset {}^{\|\,\|}\operatorname*{Lim\,sup}_{\substack{C_n \ni x' \to x,\\ n\to\infty}} N_{C_n}^P(x')$.

Theorem 2.3.7 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Mosco to a nonempty closed subset C of X. Then,*

$$\operatorname{gph} N_C^L \subset {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P.$$

Proof. Let $(x, x^*) \in \operatorname{gph} N_C^L(u)$. Then, by the definition of the Mordukhovich limiting normal cone and by Proposition 2.2.4, there exists a sequence $\{(z_n, z_n^*)\}_{n\in\mathbb{N}}$ in $X \times X^*$ such that

$$C \ni z_n \to x,\ z_n^* \xrightarrow{w^*} x^* \text{ and } z_n^* \in N_C^P(z_n)\ \forall n \in \mathbb{N}.$$

By Corollary 2.3.5, there exist an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ in $X \times X^*$ such that

$$\|x_n - z_n\| < \frac{1}{n},\ \|x_n^* - z_n^*\| < \frac{1}{n},\ x_n \in C_{k(n)},\ \text{and } x_n^* \in N_{C_{k(n)}}^P(x_n)$$

for all $n \in \mathbb{N}$. It follows that

$$x_n \to x \text{ and } x_n^* \xrightarrow{w^*} x^*.$$

The proof is completed. □

In the case of general reflexive Banach spaces, we have a similar result but the Fréchet normal functionals have to be involved in place of proximal normal functionals.

Theorem 2.3.8 *Let X be a reflexive Banach space and $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Mosco to a nonempty closed subset C of X. Then*
(a) $N_C^F(x) \subset {}^{\|\,\|}\operatorname*{Lim\,sup}_{\substack{C_n \ni x' \to x, \\ n\to\infty}} N_{C_n}^F(x')$ *; (b)* $\operatorname{gph} N_C^L \subset {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^F$.

Proof. Recall that any reflexive Banach space (see the section of preliminaries) can be given an equivalent norm $\|\cdot\|$ which is both unformly convex and Fréchet differentiable off zero. Endowing X with such a norm, it suffices to apply (c) of Corollary 2.3.6 and Theorem 2.3.7, and to use the inclusion $N_{C_n}^P(\cdot) \subset N_{C_n}^F(\cdot)$. $\square$

Now we recall the definition of subsmooth sets (see [5]).

Definition 2.3.9 *Assume that $(X, \|\cdot\|)$ is Banach space. A set C is called subsmooth at point $\bar{x} \in C$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that one has*

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|,$$

for all $x', x \in B(\bar{x}, \delta) \cap C$ and all $x^* \in N^{Cl}(C; x) \cap \mathbb{B}_{X^*}$. The set C is said to be submooth when it is submooth at any of its points.

The set is uniformly subsmooth if the inequality above holds in a uniform way, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|$$

for all $x', x \in C$ with $\|x' - x\| \leq \delta$ and all $x^* \in N^{Cl}(C; x) \cap \mathbb{B}_{X^*}$.

If the space X is an Asplund space and C is closed near $\bar{x} \in C$, then in the definition above we can replace $N^{Cl}(C; x) \cap \mathbb{B}_{X^*}$ by $N^F(C; x) \cap \mathbb{B}_{X^*}$ (see [5]). The next proposition says that for the large class of reflexive Banach spaces one can also replace $N^{Cl}(C; \cdot)$ by $N^P(C; \cdot)$ provided one endows X with an equivalent norm which is locally uniformly convex and Fréchet differentiable off zero, as guaranteed by the related renorming result recalled in the previous section.

Proposition 2.3.10 *Assume that the space X is a reflexive Banach space endowed with a locally uniformly convex norm $\|\cdot\|$ which is Fréchet differentiable off zero, and let C be a subset of X which is closed near $\bar{x} \in C$. Then C is subsmooth at $\bar{x}$ if and only if the inequality in the definition above holds true with $N^{Cl}(C; x)$ replaced by $N^P(C; x)$.*

Proof. Indeed, in such a space X assume that the property in the proposition is satisfied and, for any given $\varepsilon > 0$, choose $\delta > 0$ such that $\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|$ for all $x, x' \in C \cap B(\bar{x}, \delta)$ and $x^* \in N^P(C; x) \cap \mathbb{B}_{X^*}$. Fix any $x \in C \cap B(\bar{x}, \delta)$ and any nonzero $x^* \in N^F(C; x) \cap \mathbb{B}_{X^*}$. By Proposition 2.2.4 there are a sequence $\{x_k\}_{k \in \mathbb{N}}$ in $C \cap B(\bar{x}, \delta)$ converging to x and a sequence $\{x_k^*\}_{k \in \mathbb{N}}$ of nonzero vectors in X^* converging strongly to x^* with $x_k^* \in N^P(C; x_k)$. Since we have $\|x^*\| \frac{x_k^*}{\|x_k^*\|} \in N^P(C; x_k) \cap \mathbb{B}_{X^*}$, we may write, for every $x' \in C \cap B(\bar{x}, \delta)$,

$$\langle \|x^*\| \frac{x_k^*}{\|x_k^*\|}, x' - x_k \rangle \leq \varepsilon \|x' - x_k\|,$$

which gives as $k \to \infty$

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|,$$

hence the desired inequality is fulfilled with $N^F(C; x) \cap \mathbb{B}_{X^*}$. Consequently, the set C is subsmooth at $\bar{x}$ according to the result recalled above.

The converse implication is obvious since $N^P(C; \cdot) \subset N^F(C; \cdot)$. □

The concept of subsmooth sets has been introduced by Aussel, Daniilidis and Thibault in [5] as an adaptation of the hypomonotonicity property fulfilled by the truncated normal cone of a prox-regular set. Recall that a closed set C of a Hilbert space C is *prox-regular at* $x \in C$ provided there exists a neighbourhood of x over which the metric projection mapping P_C is single-valued and continuous. The hypomonotonicity characterization of prox-regularity says that C is prox-regular at x if and only if (see [33]) there exists a neighbourhood U of x and a real $r > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2 \tag{2.3.2}$$

for all $x_i \in U \cap C$ and $x_i^* \in N^{Cl}(C; x_i) \cap \mathbb{B}$ for $i = 1, 2$. The closed set C is called (uniformly) *prox-regular* when P_C is single-valued and continuous over some open r-enlargement

$$E_r(C) := \{x \in X : \mathrm{dist}(x, C) < r\};$$

it is proved in [33] (see also [14]) that C is uniformly prox-regular if and only if, for some $r > 0$, the inequality (2.3.2) holds for all $x_i \in C$ and $x_i^* \in N^{Cl}(C; x_i) \cap \mathbb{B}_{X^*}$ for $i = 1, 2$.

More generally, suppose that C is a closed set of a Banach space $(X, \|\cdot\|)$ and define, as above, the prox-regularity of C at $x \in C$ (resp. the uniform prox-regularity of C) by the single-valuedness and continuity of the metric projection mapping P_C on a neighbourhood of x (resp. on some r-enlargement $E_r(C)$). Assume that the norm $\|\cdot\|$ is both uniformly convex and uniformly smooth and that the moduli of convexity and of smoothness of the norm $\|\cdot\|$ are of power type, that is, there are constants $c, c' > 0$, $p > 0$ and $q > 1$ such that $\delta_{\|\,\|}(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in]0, 2]$ and $\rho_{\|\,\|}(\tau) \leq c'\tau^q$ for all $\tau \geq 0$ (see the preliminary section for the definitions of $\delta_{\|\,\|}(\cdot)$ and $\rho_{\|\,\|}(\cdot)$) ; under those assumptions it is proved in [8, Theorem 4.9] and [9, Theorem 3.2] that the set C is prox-regular at x if and only if there exist some real $r > 0$ and some neighbourhood U of x such that

$$\langle J[J^*(x_1^*) - (x_2 - x_1)] - J[J^*(x_2^*) - (x_1 - x_2)], x_2 - x_1 \rangle \leq 0 \qquad (2.3.3)$$

for all $x_i \in C \cap U$ and $x_i^* \in N^{Cl}(C; x_i) \cap r\mathbb{B}_{X^*}$ (resp. $x_i^* \in N^P(C; x_i) \cap r\mathbb{B}_{X^*}$) for $i = 1, 2$. Similarly, the set C is uniformly prox-regular if and only if (see [8, Proposition 5.6] and [9, Theorem 3.2]) for some $r > 0$ the latter inequality holds for all $x_i \in C$ and $x_i^* \in N^{Cl}(C; x_i) \cap r\mathbb{B}_{X^*}$ (resp. $x_i^* \in N^P(C; x_i) \cap r\mathbb{B}_{X^*}$). Further, if the modulus of smoothness $\rho_{\|\,\|}$ of $\|\,\|$ is of power type 2 (that is, $\rho_{\|\,\|}(\tau) \leq c'\tau^2$ for all $\tau \geq 0$), then C is prox-regular at $x \in C$ (resp. uniformly prox-regular) if and only if (see [9, Proposition 5.2]) for some $\sigma > 0$

$$\langle x_2^*, x_1 - x_2 \rangle \leq \sigma \|x_1 - x_2\|^2 \qquad (2.3.4)$$

for all $x_1, x_2 \in C \cap U$ (resp. $x_1, x_2 \in C$) and $x_2^* \in N^{Cl}(C; x_2) \cap \mathbb{B}_{X^*}$ (resp. $x_2^* \in N^P(C; x_2) \cap \mathbb{B}_{X^*}$) ; further, when C is r-prox-regular the constant σ depends only on r and the norm $\|\cdot\|$.

Definition 2.3.11 *Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed sets of a Banach space X. Then we say that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ with respect to the proximal normal cone with compatible indexation by $n \in \mathbb{N}$, if for any $\varepsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for each $n \geq N$*

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x_n\|,$$

for all $x, x' \in C_n \cap B(\bar{x}, \delta)$ and $x^* \in N^P_{C_n}(x) \cap \mathbb{B}_{X^*}$.

One defines in an obvious way the similar concept with respect to any normal cone. When the proximal normal cone is used, we will omit its name, that is, we will just say that the sequance $C_{n n\in\mathbb{N}}$ is subsmooth at $\bar{x}$ with compatible indexation.

Obviously the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ with compatible indexation by $n \in \mathbb{N}$ whenever it is equi-uniformly submooth in the following sense. A family $\{C_t\}_{t\in T}$ is called equi-uniformly subsmooth if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in T$

$$\langle x^*, x' - x\rangle \le \varepsilon \|x' - x\|$$

for all $x', x \in C_t$ with $\|x' - x\| \le \delta$ and all $x^* \in N^{Cl}(C_t; x) \cap \mathbb{B}_{X^*}$.

Similarly, if $(X, \| \ \|)$ is a Banach space whose modulus of uniform convexity of the norm is of power type and modulus of uniform smoothness is of power type 2, one sees from (2.3.4) that the sequence is subsmooth at any $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ with compatible indexation by $n \in \mathbb{N}$ whenever for some real $r > 0$ all the closed sets C_n are r-prox-regular. In fact, in such a case the sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ is, according to (2.3.4), even equi-uniformly subsmooth.

It is worth noting the following equivalent property.

Proposition 2.3.12 *Let X be a Banach space and $\mathcal{N}$ be a normal cone. A sequence of closed sets $\{C_n\}_{n\in\mathbb{N}}$ of X is subsmooth at $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ with respect to $\mathcal{N}$ with compatible indexation by $n \in \mathbb{N}$, if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x with $x_n \in C_n$ for large n there is some $N \in \mathbb{N}$ satisfying for each $n \ge N$*

$$\langle x^*, x' - x_n\rangle \le \varepsilon \|x' - x_n\|,$$

for all $x' \in C_n \cap B(x, \delta)$ and $x^* \in \mathcal{N}_{C_n}(x_n) \cap \mathbb{B}_{X^*}$.

Proof. The property is obviously implied by the statement of Definition 2.3.11. Suppose now that the statement of that definition fails. Then there exists some $\varepsilon_0 > 0$ such that for each integer $n \in \mathbb{N}$ there are an integer $k(n) \ge n$ with $k(n+1) > k(n)$, $u_n, x'_n \in C_{k(n)} \cap B(\bar{x}, 1/n)$ and $x^*_n \in \mathcal{N}(C_{k(n)}; u_n) \cap \mathbb{B}_{X^*}$ such that

$$\langle x^*_n, x'_n - u_n\rangle > \varepsilon_0 \|x'_n - u_n\|.$$

From the inclusion $\bar{x} \in {}^{\| \|}\operatorname*{Lim\,inf}_{n\to\infty} C_n$ there exits a sequence $\{v_n\}_{n\in\mathbb{N}}$ converging strongly to $\bar{x}$ with $v_n \in C_n$ for large n. Putting $x_n := v_n$ for every integer $n \notin k(\mathbb{N})$ and $x_{k(m)} := u_m$ for all $m \in \mathbb{N}$, we see that the whole sequence

$\{x_n\}_{n\in\mathbb{N}}$ converges srongly to $\bar{x}$ with $x_n \in C_n$ for all n large enough. Fix any real $\delta > 0$. There exists some integer K such that $x_n, x'_n \in B(\bar{x}, \delta)$ and $x_n \in C_n$ for all $n \geq K$. So, for the constructed sequence $\{x_n\}_{n\in\mathbb{N}}$, we see that it converges to $\bar{x}$ with $x_n \in C_n$ for large n and for each integer $N \geq K$ we have $k(N) \geq N$ with the point $x'_N \in C_{k(N)} \cap B(\bar{x}, \delta)$ and the vector $x^*_N \in \mathcal{N}(C_{k(N)}; x_{k(N)}) \cap \mathbb{B}_{X^*}$, but

$$\langle x^*_N, x'_N - x_{k(N)} \rangle > \varepsilon_0 \|x'_N - x_{k(N)}\|.$$

This means that the property of the proposition is not satisfied at $\bar{x}$ and the proof is completed. □

The next theorem shows that the inclusion of Theorem 2.3.7 is an equality whenever the sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth.

Theorem 2.3.13 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Mosco to a nonempty closed subset C of X. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point of C with a compatible indexation. Then*

$$\operatorname{gph} N^L_C = \operatorname{gph} N^F_C = {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{C_n}.$$

Proof. The theorem is a consequence of Theorem 2.3.7 and Proposition 2.3.14 below since the Mosco convergence implies the Painlevé-Kuratowski convergence of sets. □

Proposition 2.3.14 *Let $(X, \|\cdot\|)$ is a Banach space and $\mathcal{N}$ be a normal cone. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Painlevé-Kuratowski to a nonempty closed subset C of X. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point of C with respect to $\mathcal{N}$ with compatible indexation. Then*

$${}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{C_n} \subset \operatorname{gph} N^F_C.$$

Proof. We follow the main ideas of the proof of Theorem 3.3 by X.Y. Zheng and Z. Wei [39].

Fix any $(\bar{x}, x^*) \in {}^{\| \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{C_n}$. By the definition of that limit superior there exist a sequence $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ in $X \times X^*$ and an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$x_n \in C_{k(n)} \quad \text{and} \quad x_n^* \in \mathcal{N}_{C_{k(n)}}(x_n) \quad \text{for all } n \in \mathbb{N},$$

and such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_n^*\}_{n\in\mathbb{N}}$ converge to $\bar{x}$ and x^* with respect to the norm topology in X and the weak* topology in X^* respectively. This implies in particular that $\bar{x} \in \operatorname*{Lim\,sup}_{n\to\infty} C_n$, hence $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ according to the convergence assumption. Choose a real $\beta > 0$ such that

$$\|x_n^*\| \le \beta \quad \text{for all } n \in \mathbb{N}.$$

Take any real $\varepsilon > 0$. For $\varepsilon' := \beta^{-1}\varepsilon$, the inclusion $\bar{x} \in \operatorname*{Lim\,inf}_{n\to\infty} C_n$ and the subsmoothness property of the sequence $\{C_n\}_{n\in\mathbb{N}}$ furnishes some real $\delta > 0$ and some $N \in N$ such that for each $n \ge N$

$$\langle x^*, x' - x\rangle \le \varepsilon' \|x' - x\|$$

for all $x, x' \in C_n \cap B(\bar{x}, \delta)$ and $x^* \in \mathcal{N}(C_n; x) \cap \mathbb{B}_{X^*}$. Taking $N_0 \ge N$ such that $x_n \in B(\bar{x}, \delta)$ for all $n \ge N_0$, we see in particular that, for each $n \ge N_0$, we have for all $x' \in C_{k(n)} \cap B(\bar{x}, \delta)$

$$\langle \beta^{-1} x_n^*, x' - x_n\rangle \le \varepsilon' \|x' - x_n\|$$

or equivalently

$$\langle x_n^*, x' - x_n\rangle \le \varepsilon \|x' - x_n\|. \tag{2.3.5}$$

Consider any $x \in C \cap B(\bar{x}, \delta)$. Since $C = \operatorname*{Lim\,inf}_{n\to\infty} C_n$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $B(\bar{x}, \delta)$ such that

$$\|y_n - x\| \to 0 \quad \text{and} \quad y_n \in C_n \quad \text{for all } n \text{ large enough.}$$

This and (2.3.5) imply for all n large enough that

$$\langle x_n^*, y_{k(n)} - x_n\rangle \le \varepsilon \|y_{k(n)} - x_n\|,$$

hence letting $n \to \infty$, we obtain

$$\langle x^*, x - \bar{x}\rangle \le \varepsilon \|x - \bar{x}\| \quad \text{for all } x \in C \cap B(\bar{x}, \delta),$$

which translates that $(\bar{x}, x^*) \in \operatorname{gph} N_C^F$. The proof is completed. $\square$

In the case of a general reflexive Banach space, the limit superior needs to involve the cones of Fréchet normal functionals instead of the cones of proximal normal functionals.

Theorem 2.3.15 *Let X be a reflexive Banach space and let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Mosco to a nonempty closed subset C of X. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth with respect to the Fréchet normal cone at any point of C with a compatible indexation. Then*

$$\operatorname{gph} N_C^L = \operatorname{gph} N_C^F = {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^F.$$

Proof. The theorem follows from Theorem 2.3.8 and Proposition 2.3.14. $\square$

The Painlevé-Kuratowski convergence of the graphs of $N_{C_n}^P$ can be deduced as follows.

Corollary 2.3.16 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Mosco to a nonempty closed subset C of X satisfying $N_C^P(x) = N_C^F(x)$ for all $x \in C$. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point of C with a compatible indexation. Then the sequence $\{\operatorname{gph} N_{C_n}^P\}_{n\in\mathbb{N}}$ of graphs of the functional proximal normal cones of C_n Painlevé-Kuratowski converges (with respect to the norm of $X \times X^*$) to the graph $\operatorname{gph} N_C^P$ of the functional proximal normal cone of C.*

Proof. By Corollary 2.3.5 we have

$$\operatorname{gph} N_C^P \subset {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N_{C_n}^P \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P,$$

and by Theorem 2.3.13 and the normal regularity assumption on the set C we have

$${}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P \subset {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P = \operatorname{gph} N_C^F = \operatorname{gph} N_C^P.$$

We deduce that

$$\mathrm{gph}\, N_C^P = {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} \mathrm{gph}\, N_{C_n}^P = {}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \mathrm{gph}\, N_{C_n}^P,$$

and this translates the desired Painlevé-Kuratowski convergence. □

It is worth recalling that the assumption $N_C^P(\cdot) = N^F C(\cdot)$ used in the above corollary is fulfilled whenever the set C is prox-regular at any of its points (see [33] for Hilbert spaces and [9] for uniformly convex Banach spaces).

Theorem 2.3.17 *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space whose moduli of convexity and smoothness of the norm are of power type. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of r-prox-regular closed sets of X which converges in the sense of Mosco to a nonempty closed set C of X. The following hold :*
(a) The set C is r-prox-regular.
(b) If in addition the modulus of the norm $\|\cdot\|$ is of power type 2, then the sequence $\{\mathrm{gph}\, N_{C_n}^P\}_{n\in\mathbb{N}}$ of graphs of the functional proximal normal cones of C_n Painlevé-Kuratowski converges (with respect to the norm of $X \times X^*$) to the graph $\mathrm{gph}\, N_C^P$ of the functional proximal normal cone of C.

Proof. (a) Let $x_i \in C$ and $x_i^* \in N^P(C; x_i) \cap r\mathbb{B}_{X^*}$ with $i = 1, 2$. By Corollary 2.3.5 there exist sequences $x_{i,n} \in C$ and $x_{i,n}^* \in N^P(C_n; x_{i,n})$ (for large n) with $x_{i,n} \to x_i$ and $x_{i,n}^* \to x_i^*$ strongly as $n \to \infty$. From (2.3.3) we have for large n

$$\langle J[J^*(x_{1,n}^*) - (x_{2,n} - x_{1,n})] - J[J^*(x_{2,n}^*) - (x_{1,n} - x_{2,n})], x_{2,n} - x_{1,n}\rangle \le 0,$$

so using the continuity of J and J^* and taking the limit as $n \to \infty$ give

$$\langle J[J^*(x_1^*) - (x_2 - x_1)] - J[J^*(x_2^*) - (x_1 - x_2)], x_2 - x_1\rangle \le 0.$$

This translates, according to (2.3.3) again, the prox-regularity of the set C.
(b) The set C being prox-regular, we have $N^P(C; \cdot) = N^F(C; \cdot)$ (see [8, 9]). Further, from (2.3.4) we see that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point with compatible indexation. The assertion (b) then follows from Corollary 2.3.16. □

2.4 Normal cones of Attouch-Wets convergent sequences of sets

Although the Mosco convergence is generally easy to be checked, the Mosco convergence of a sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ to a closed set C requires that the closed set C need to be sequentially weakly closed (as it has been seen in the previous section). Such a sequential weak closedness is not required by the Attouch-Wets convergence of sequences of sets. Recall (see [4, 6, 34]) that the sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ of the normed space $(X, \|\cdot\|)$ converges in the sense of Attouch-Wets or (AW) converges to a set C of X, denoted by $C_n \xrightarrow{AW} C$, if for all reals $\rho > 0$ big enough

$$\mathrm{Haus}_\rho(C_n, C) \xrightarrow[n\to\infty]{} 0,$$

where $\mathrm{Haus}_\rho(C_n, C)$ denotes the Hausdorff ρ-semidistance between C_n and C, that is,

$$\mathrm{Haus}_\rho(C_n, C) := \max\{e(C_n \cap \rho\mathbb{B}_X, C), e(C \cap \rho\mathbb{B}_X, C_n)\},$$

where $e(A, A') := \sup_{a\in A} d(a, A')$ is the excess of the set A over the set A'.

The Attouch-Wets convergence implies the Painlevé-Kuratowski convergence (as easily seen), but the converse does not hold in infinite dimensional normed space even for closed convex sets.

One of the properties of Attouch-Wets convergence that we will use is the following.

Lemma 2.4.1 *Let X be a Banach space. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets converging to a nonempty closed subset C of X and let $x \in X$. Then for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ (depending only on x and ε) such that*

$$d_{C_n}(z) - d_C(x) \le \varepsilon \quad \text{for all } z \in B(x, \frac{\varepsilon}{3}) \text{ and } n \ge N.$$

Proof. Let $\varepsilon > 0$. Choose $v \in C$ such that

$$d_C(x) = \inf_{y\in C} \|x - y\| \ge \|x - v\| - \frac{\varepsilon}{3}.$$

Then by Attouch-Wets convergence (taking ρ big enough so that $v \in C \cap \rho\mathbb{B}_X$ and $\mathrm{Haus}_\rho(C_n, C) \to 0$) there exists $N \in \mathbb{N}$ (depending only on x and ε) such that, for each integer $n \geq N$, we can select some $w_n \in C_n$ satisfying

$$\|v - w_n\| \leq \frac{\varepsilon}{3}.$$

Therefore, for all $n \geq N$ and $z \in B(x, \frac{\varepsilon}{3})$, we have

$$\begin{aligned} d_{C_n}(z) - d_C(x) &\leq d_{C_n}(x) - d_C(x) + \|x - z\| \\ &\leq \inf_{y \in C_n} \|x - y\| - \|x - v\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\leq \|x - w_n\| - \|x - v\| + \frac{2\varepsilon}{3} \\ &\leq \|v - w_n\| + \frac{2\varepsilon}{3} \leq \varepsilon. \end{aligned}$$

The proof is completed. $\square$

The next lemma will be crucial in the analysis, as $n \to \infty$, of $P_{C_n}(x_n)$ under the Attouch-Wets convergence of the sequence of sets $\{C_n\}_{n \in \mathbb{N}}$. It will be convenient below to write for $x \neq y$ in the vector space X

$$]x, y[:= \{ty + (1 - t)x : t \in]0, 1[\}.$$

Lemma 2.4.2 *Let $(X, \|\cdot\|)$ be a locally uniformly convex Banach space and C be a closed set of X. Suppose that*
(i) $x \notin C$ and $y \in P_C(x)$;
(ii) $x' \in]y, x[$ and $y_n \in C$ with $\|x' - y_n\| \xrightarrow[n \to \infty]{} d_C(x')$.
Then

$$y_n \xrightarrow[n \to \infty]{} y.$$

Proof. As $y \in P_C(x)$ and $x' \in]x, y[$, we have on one hand $d_C(x') = \|x' - y\|$ and on the other hand

$$0 < \|x - x'\| < \|x - y\| \leq \|x - y_n\| \quad \text{for all } n \in \mathbb{N}.$$

So we can choose $z_n \in]y_n, x[$ in such a way that

$$\|x - z_n\| = \|x - x'\|. \tag{2.4.1}$$

Our aim is to show that $z_n \to x'$. Let us proceed by contradiction and suppose there exist $\varepsilon > 0$ such that

$$\|z_n - x'\| > \varepsilon, \tag{2.4.2}$$

for some subsequence that we do not relabel. Put $\varepsilon' := \varepsilon/\|x - x'\| > 0$ and put also $w_n := \frac{1}{2}(x' + z_n)$ for all $n \in \mathbb{N}$. By the local uniform convexity of the norm $\| \ \|$ at the point $\bar{u} := (x - x')/\|x - x'\|$ there exists $\delta' > 0$ such that $\|(u + \bar{u})/2\| \le 1 - \delta'$ for all $u \in X$ with $\|u\| = 1$ and $\|u - \bar{u}\| \ge \varepsilon'$. For each $n \in \mathbb{N}$, observe that $u_n := (x - z_n)/\|x - x'\|$ is a unit vector because of (2.4.1), and observe also that

$$\|u_n - \bar{u}\| = \frac{1}{\|x - x'\|}\|x' - z_n\| > \varepsilon',$$

where the last inequality is due to (2.4.2). Substituting u_n for u in the above inequality of local uniform convexity gives $\|x - w_n\| \le (1 - \delta')\|x - x'\|$, hence for $\delta := \delta'\|x - x'\| > 0$ we have

$$\|x - w_n\| \le \|x - x'\| - \delta. \tag{2.4.3}$$

From (2.4.1) again and from the inclusion $z_n \in]y_n, x[$ we have

$$\begin{aligned} \|x - x'\| + \|z_n - y_n\| = \|x - z_n\| + \|z_n - y_n\| &= \|x - y_n\| \\ &\le \|x - x'\| + \|x' - y_n\| \end{aligned}$$

so $\|z_n - y_n\| \le \|x' - y_n\|$. We deduce

$$\begin{aligned} \|w_n - y_n\| &= \|\frac{1}{2}\big((z_n - y_n) + (x' - y_n)\big)\| \le \frac{1}{2}(\|z_n - y_n\| + \|x' - y_n\|) \\ &\le \frac{1}{2}(\|x' - y_n\| + \|x' - y_n\|) = \|x' - y_n\|, \end{aligned}$$

therefore

$$\|w_n - y_n\| \le \|x' - y_n\|. \tag{2.4.4}$$

It follows from (2.4.3) and (2.4.4) that

$$\begin{aligned} \|x - y\| &\le \|x - y_n\| \le \|x - w_n\| + \|w_n - y_n\| \\ &\le \|x - x'\| - \delta + \|x' - y_n\|, \end{aligned}$$

thus according to the assumption (ii)

$$\begin{aligned}\|x-y\| &\leq \lim_{n\to\infty}(\|x-x'\|-\delta+\|x'-y_n\|)=\|x-x'\|-\delta+d_C(x')\\ &\leq \|x-x'\|-\delta+\|x'-y\|=\|x-y\|-\delta,\end{aligned}$$

where the last equality is due to the inclusion $x' \in]x, y[$. The contradiction $\|x-y\| \leq \|x-y\|-\delta$ justifies the convergence $z_n \to x'$ as $n \to \infty$.

To complete the proof we observe on the one hand that

$$y_n = x + \frac{z_n - x}{\|z_n - x\|}\|y_n - x\| \tag{2.4.5}$$

according to the inclusion $z_n \in]y_n, x[$. On the other hand, from the assumption (i) and the inclusion $y_n \in C$ we have

$$\|y-x\| = d_C(x) \leq \|y_n - x\| \leq \|y_n - x'\| + \|x' - x\|$$

and (since $d_C(x') = \|x'-y\|$ by the inclusions $y \in P_C(x)$ and $x' \in]x, y[$) we also have from the assumption (ii)

$$\|y_n - x'\| + \|x'-x\| \to \|y-x'\| + \|x'-x\| = \|y-x\| \quad \text{as } n \to \infty,$$

where the equality is due to the inclusion $x' \in]x, y[$. Consequently, we obtain $\|y_n - x\| \to \|y-x\|$ and hence (2.4.5) yields

$$y_n \xrightarrow[n\to\infty]{} x + \frac{x'-x}{\|x'-x\|}\|y-x\| = x + \frac{y-x}{\|y-x\|}\|y-x\| = x+y-x = y,$$

where the first equality is due to the inclusion $x' \in]x, y[$. The proof is completed. □

Remark. The proof of the lemma provided above is direct and self-contained. The lemma can also be proved through the following result of Fitzpatrick (see [21]) : *A Banach space $(X, \|\ \|)$ is locally uniformly convex at a point $z \in X$ with $\|z\| = 1$ if and only if for each closed set C and $x \notin C$, if $\limsup_{t\to 0+} \big(d_C(x+tz) - d_C(x)\big)/t = 1$ then every minimizing sequence for x and $d_C(x)$ converges to $x - d_C(x)z$ and P_C is continuous at x.*

Proof. Let x, y and x' as in the statement of the lemma, that is, $x \notin C$, $y \in P_C(x)$, $x' \in]y, x[$.

Putting $z = \dfrac{x'-y}{d_C(x')}$ we obviously have $\|z\| = 1$. If $0 < t < (d_C(x) - d_C(x'))$ hence $x' + t\dfrac{x'-y}{d_C(x')} \in]x', x[$. Indeed if $t = d_C(x) - d_C(x')$ then

$$x' + t\frac{x'-y}{d_C(x')} = x' + (d_C(x) - d_C(x'))\frac{x'-y}{d_C(x')} = x' + d_C(x)\frac{x'-y}{d_C(x')} - d_C(x')\frac{x'-y}{d_C(x')} =$$

$$x' + d_C(x)\frac{x'-y}{d_C(x')} - x' + y = y + d_C(x)\frac{x'-y}{d_C(x')} = y + d_C(x)\frac{x-y}{d_C(x)} = x.$$

Therefore for any real t with $0 < t < (d_C(x) - d_C(x'))$, it results that $P_C(x' + t\frac{x'-y}{d_C(x')}) = \{y\}$ and

$$\begin{aligned} d_C\big(x' + t\frac{x'-y}{d_C(x')}\big) &= \|x' + t\frac{x'-y}{d_C(x')} - y\| = \big(1 + \frac{t}{d_C(x')}\big)\|x' - y\| \\ &= \big(1 + \frac{t}{d_C(x')}\big)d_C(x') = d_C(x') + t. \end{aligned}$$

It then follows that

$$\limsup_{t\to 0+} \Big(d_C(x' + t\frac{x'-y}{d_C(x')}) - d_C(x')\Big)/t = 1,$$

so applying Fitzpatrick's result we obtain that every minimizing sequence for x' and $d_C(x')$ converges to $x' - d_C(x')\frac{x'-y}{d_C(x')} = y$ as desired. □

We can now prove the following theorem concerning the behaviour of the metric projection P_{C_n} when the sets $\{C_n\}_n$ Attouch-Wets converge as $n \to \infty$.

Theorem 2.4.3 *Assume that $(X, \|\cdot\|)$ is a locally uniformly convex Banach space. Let C and $\{C_n\}_{n\in\mathbb{N}}$ be closed subsets of X such that $C_n \xrightarrow{AW} C$ and let $x \notin C$. Suppose that*
(i) $y \in P_C(x)$, $x' \in]y, x[$;
(ii) $x_n \to x'$ and $y_n \in P_{C_n}(x_n)$.
Then one has

$$y_n \xrightarrow[n\to\infty]{} y.$$

Proof. We first show that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Indeed take $z \in C$ such that

$$\|x' - z\| \le 2d_C(x').$$

By Attouch-Wets convergence there is $z_n \in C_n$ such that $\{z_n\}_n$ converges strongly to z. This and the inclusions $y_n \in P_{C_n}(x_n)$ and $z_n \in C_n$ entail

$$\begin{aligned}\limsup_{n\to\infty} \|x' - y_n\| &\le \limsup_{n\to\infty}(\|x_n - y_n\| + \|x' - x_n\|) = \limsup_{n\to\infty} d_{C_n}(x_n) \\ &\le \limsup_{n\to\infty} \|x_n - z_n\| = \limsup_{n\to\infty} \|x' - z_n\| = \|x' - z\| \le 2d_C(x')\end{aligned}$$

This justifies the boundedness of the sequence $\{y_n\}_{n\in\mathbb{N}}$. So, considering the real $\rho := \sup_{n\in\mathbb{N}} \|y_n\|$ (for which $y_n \in C_n \cap \rho\mathbb{B}_X$ for all $n \in \mathbb{N}$), the Attouch-Wets convergence furnishes $v_n \in C$ such that

$$\|y_n - v_n\| \xrightarrow[n\to\infty]{} 0. \tag{2.4.6}$$

Let $\varepsilon > 0$. By Lemma 2.4.1 there is $N_0 \in \mathbb{N}$ (depending only on x' and ε) such that

$$d_{C_n}(z) \le d_C(x') + \varepsilon$$

for all $z \in B(x', \frac{\varepsilon}{3})$ and all $n \ge N_0$. Since (by assumption) $x_n \to x'$ there is an integer $N \ge N_0$ such that for all $n \ge N$ we have $x_n \in B(x', \frac{\varepsilon}{3})$, hence

$$d_{C_n}(x_n) \le d_C(x') + \varepsilon.$$

From the inclusion $y_n \in P_{C_n}(x_n)$ in the assumption (ii) and from the latter inequality we deduce for every $n \ge N$

$$\begin{aligned}\|x' - v_n\| &\le \|x' - x_n\| + \|x_n - y_n\| + \|y_n - v_n\| \\ &= \|x' - x_n\| + \|y_n - v_n\| + d_{C_n}(x_n) \\ &\le \|x' - x_n\| + \|y_n - v_n\| + d_C(x') + \varepsilon,\end{aligned}$$

which implies by (2.4.6) and by the convergence $x_n \to x'$ in the assumption (ii)

$$\limsup_{n\to\infty} \|x' - v_n\| \le d_C(x') + \varepsilon.$$

Therefore

$$\limsup_{n\to\infty} \|x' - v_n\| \le d_C(x').$$

Since $v_n \in C$, it follows that

$$\lim_{n\to\infty} \|x' - v_n\| = d_C(x').$$

From Lemma 2.4.2 we obtain that $v_n \to y$ as $n \to \infty$, which combined with (2.4.6) justifies the convergence $y_n \xrightarrow[n\to\infty]{} y$ of the theorem. □

For a sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ Attouch-Wets converging to C, the next theorem charaterizes proximal normal functionals of C through the metric projection to C_n for a large class of Banach spaces.

Theorem 2.4.4 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets converging to a nonempty closed subset C of X. Then the following assertions are equivalent :*
(a) one has $x^* \in N_C^P(x)$;
(b) there exist $r > 0$, $n_0 \in \mathbb{N}$ and $\{x_n\}_{n\in\mathbb{N}}$ strongly converging to x, $\{x_n^*\}_{n\in\mathbb{N}}$ strongly converging to x^* such that

$$P_{C_n}(x_n + rJ^*(x_n^*)) = \{x_n\} \quad \textit{for all } n \geq n_0.$$

Proof. We follow several ideas from the proof of Theorem 2.3.4. Assume first that for x^* the assertion (b) is satisfied, that is, there are $r > 0$, $n_0 \in \mathbb{N}$ and $\{x_n\}_n$ strongly converging to x, $\{x_n^*\}_{n\in\mathbb{N}}$ strongly converging to x^* such that

$$P_{C_n}(x_n + rJ^*(x_n^*)) = \{x_n\} \quad \text{for all } n \geq n_0.$$

The continuity of J^* (see the section of preliminaries) entails

$$\lim_{n\to\infty} (x_n + rJ^*(x_n^*)) = x + rJ^*(x^*),$$

and by Lemma 2.3.3

$$x \in P_C(x + rJ^*(x^*)),$$

which ensures $x^* \in N_C^P(x)$.

Now we assume that $x^* \in N_C^P(x)$. If $x^* = 0$, it suffices to take any $r > 0$, any sequence $\{x_n\}_{n\in\mathbb{N}}$ strongly converging to x with $x_n \in C_n$ for large n (thanks to the convergence assumption of $\{C_n\}_n$) and $x_n^* = 0$, and to note that for such choices $P_{C_n}(x_n + rJ^*(x_n^*)) = \{x_n\}$. So suppose that $x^* \neq 0$. By definition there is $\sigma > 0$ such that

$$P_C(x + \sigma J^*(x^*)) = \{x\}.$$

Note that $x+\sigma J^*(x^*) \notin C$ since $x+\sigma J^*(x^*) \neq x$. Take a positive real $\sigma' < \sigma$. The non-emptiness of C and the Attouch-Wets convergence of $\{C_n\}_{n\in\mathbb{N}}$ to C ensures that $C_n \neq \emptyset$ for large n, say $n \geq N$. By Theorem 2.2.3, for each integer $n \geq N$, there are $z_n \in B(x + \sigma' J^*(x^*), 1/n)$ and $x_n \in C_n$ such that $P_{C_n}(z_n) = \{x_n\}$. Obviously $z_n \to x + \sigma' J^*(x^*)$ and $x + \sigma' J^*(x^*) \in]x, x + \sigma J^*(x^*)[$. Theorem 2.4.3 then guarantess that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to x. Putting $x_n^* = J(\frac{1}{\sigma'}(z_n - x_n))$ ensures $x_n^* \in N^P_{C_n}(x_n)$ (since $z_n - x_n \in PN_{C_n}(x_n)$), and by the continuity of J we also have

$$x_n^* = J(\frac{1}{\sigma'}(z_n - x_n)) \xrightarrow[n\to\infty]{} J(\frac{1}{\sigma'}(x + \sigma' J^*(x^*) - x)) = J(J^*(x^*)) = x^*.$$

Further, since $z_n - x_n = \sigma' J^*(x_n^*)$ we see that

$$\{x_n\} = P_{C_n}(z_n) = P_{C_n}(x_n + \sigma' J^*(x_n^*)),$$

so the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_n^*\}_{n\in\mathbb{N}}$ fulfill the properties of the assertion (b) with $r = \sigma'$. □

The next two corollaries follow directly from Theorem 3.4 (as it has been above the case for Corollaries 2.3.5 and 2.3.6).

Corollary 2.4.5 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Attouch-Wets to a nonempty closed subset C of X. Then,*
(a) $\operatorname{gph} N^P_C = \bigcup_{r>0} {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} \Lambda^r_{C_n}$ *; (b)* $\operatorname{gph} N^P_C \subset {}^{\|\,\|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N^P_{C_n}$ *;*

(c) $\operatorname{gph} N^F_C \subset {}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{C_n}$.

Corollary 2.4.6 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X which converges in the sense of Attouch-Wets to a nonempty closed subset C of X and let $x \in C$. Then,*
(a) $N^P_C(x) = \bigcup_{r>0} {}^{\|\,\|}\operatorname*{Lim\,inf}_{\substack{C_n\ni x'\to x,\\ n\to\infty}} \Lambda^r_{C_n}(x')$ *; (b)* $N^P_C(x) \subset {}^{\|\,\|}\operatorname*{Lim\,inf}_{\substack{C_n\ni x'\to x,\\ n\to\infty}} N^P_{C_n}(x')$ *;*

(c) $N^F_C(x) \subset {}^{\|\,\|}\operatorname*{Lim\,sup}_{\substack{C_n\ni x'\to x,\\ n\to\infty}} N^P_{C_n}(x')$.

The proof of the next theorem is obtained with the same arguments as those used in the proof of Theorem 2.3.7 in refering to Corollary 2.4.5 in place of Corollary 2.3.5.

Theorem 2.4.7 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets convergent to a nonempty closed subset C of X. Then,*

$$\operatorname{gph} N_C^L \subset {}^{\|\ \|,*}\limsup_{n\to\infty} \operatorname{gph} N_{C_n}^P.$$

As for Theorem 2.3.8, in the general reflexive Banach setting we have :

Theorem 2.4.8 *Let X be a reflexive Banach space and $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets convergent to a nonempty closed subset C of X. Then*
(a) $N_C^F(x) \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{\substack{C_n\ni x'\to x,\\ n\to\infty}} N_{C_n}^F(x')$ *; (b)* $\operatorname{gph} N_C^L \subset {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^F$.

When, in addition to the assumptions in Theorem 2.4.7, the sequence $\{C_n\}_n$ is subsmooth with a compatible indexation, the inclusion in Theorem 2.4.7 is an equality.

Theorem 2.4.9 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Attouch-Wets to a nonempty closed subset C of X. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point of C with a compatible indexation. Then*

$$\operatorname{gph} N_C^L = \operatorname{gph} N_C^F = {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^P.$$

Proof. The theorem is a direct consequence of Theorem 2.4.7 and Proposition 2.3.14. □

A similar equality holds true in general reflexive Banach spaces for the limit superior of the graphs of Fréchet normal cones.

Theorem 2.4.10 *Let X be a reflexive Banach space and let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets convergent to a nonempty closed subset C of X. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth with respect to the Fréchet normal cone at any point of C with a compatible indexation. Then*

$$\operatorname{gph} N_C^L = \operatorname{gph} N_C^F = {}^{\|\ \|,*} \operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{C_n}^F.$$

The Painlevé-Kuratowski convergence of the graphs of $N_{C_n}^P$ follows from Corollary 2.4.5 and Theorem 2.4.9 exactly as we deduced Corollary 2.3.16 from Corollary 2.3.5 and Theorem 2.3.13. We state that in the following corollary :

Corollary 2.4.11 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Attouch-Wets to a nonempty closed subset C of X satisfying $N_C^P(x) = N_C^F(x)$ for all $x \in C$. Assume that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is subsmooth at any point of C with a compatible indexation. Then the sequence $\{\operatorname{gph} N_{C_n}^P\}_{n\in\mathbb{N}}$ of graphs of the proximal normal cones of C_n Painlevé-Kuratowski converges (with respect to the norm of $X \times X^*$) to the graph $\operatorname{gph} N_C^P$ of the proximal normal cone of C.*

2.5 Subdifferentials of Mosco and Attouch-Wets convergent sequences of functions

Let X be a normed space. Consider the topology τ on $X \times \mathbb{R}$, product of the norm topology of X and the usual topology of $\mathbb{R}$. Let $f, f_n : X \to \mathbb{R} \cup \{+\infty\}$ be extended real-valued lower semicontinuous functions with $n \in \mathbb{N}$. One says that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ *(sequentially) epi-converges* or Γ-*converges* to the function f if the sequence of sets $\{\operatorname{epi} f_n\}_{n\in\mathbb{N}}$ τ-sequentially Painlevé-Kuratowski converges to $\operatorname{epi} f$ in $X \times \mathbb{R}$. Similarly, one says that $\{f_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to f if the sequence of sets $\{\operatorname{epi} f_n\}_{n\in\mathbb{N}}$ *converges in the sense of Mosco* to $\operatorname{epi} f$ in $X \times \mathbb{R}$. In fact, the Painlevé-Kuratowski limit inferior (resp. superior) of the epigraphs $\{\operatorname{epi} f_n\}_n$ is the epigraph of an extended real-valued function called the Γ-limit or epi-limit superior (resp. inferior) of $\{f_n\}_n$; this function is denoted by

$$\Gamma - \limsup_{n\to\infty} f_n \quad (\text{resp. } \Gamma - \liminf_{n\to\infty} f_n).$$

There is an analytic description of sequential Γ-convergence and Mosco convergence. Indeed $\{f_n\}_{n\in\mathbb{N}}$ (sequentially) Γ-converges (resp. Mosco converges) to f if and only if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X strongly (resp. weakly) converging to x, we have (see [1, 17])

$$f(x) \le \liminf_{n\to\infty} f_n(x_n),$$

and, for every $x \in X$, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ strongly converging to x such that

$$f(x) \ge \limsup_{n\to\infty} f_n(x_n).$$

Theorem 2.5.1 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R}\cup\{+\infty\}$ which converges in the sense of Mosco to a function f. Then,*

$$(a) \quad \operatorname{gph} \partial_P f \subset {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} \partial_P f_n;$$

$$(b) \quad \operatorname{gph} \partial_F f \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \partial_P f_n.$$

Proof. Endow $X\times\mathbb{R}$ with the norm $\|\cdot\|$ given by $\|(x,s)\| = (\|x\|^2 + |s|^2)^{1/2}$ and note that this norm is locally uniformly convex and Fréchet differentiable off zero, and that $X\times\mathbb{R}$ is reflexive. Fix any $(x,x^*) \in \operatorname{gph} \partial_P f$ hence

$$\big((x, f(x)), (x^*, -1)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f}.$$

Since the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to f, the sequence of sets $\{\operatorname{epi} f_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $\operatorname{epi} f$, then by the assertion (b) of Corollary 2.3.5

$$\big((x, f(x)), (x^*, -1)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f} \subset {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} N^P_{\operatorname{epi} f_n}.$$

Thus there exists $\big((x_n, s_n), (x_n^*, -r_n)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f_n}$ such that

$$(x_n, s_n) \to (x, f(x)) \quad \text{and} \quad (x_n^*, -r_n) \to (x^*, -1).$$

Choose some integer $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $r_n > 0$ hence $s_n = f_n(x_n)$ according to Lemma 2.2.5. Putting $u_n^* := r_n^{-1} x_n^*$ for all $n \geq N$ and $u_n^* := 0$ for all $n < N$ we see that

$$(x_n, u_n^*) \to (x, x^*) \text{ strongly in } X \times X^* \tag{2.5.1}$$

and

$$\big((x_n, f(x_n)), (u_n^*, -1)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f_n} \quad \text{for all } n \geq N.$$

The latter inclusion is equivalent to

$$(x_n, u_n^*) \in \operatorname{gph} \partial_P f_n \quad \text{for all } n \geq N,$$

which combined with (2.5.1) completes the proof of (a). The proof of (b) is similar with the use of the assertion (c) of Corollary 2.3.5. □

In the theorem below we use the notation ${}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty, f_n} \operatorname{gph} \partial_P f_n$ to denote the set of all pairs (x, x^*) in $X \times X^*$ for which there exists an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ with $(x_n, x_n^*) \in \operatorname{gph} \partial_P f_{k(n)}$ and such that

$$x_n \xrightarrow{\|\,\|} x, \;\; x_n^* \xrightarrow{w^*} x^*, \;\; f_{k(n)}(x_n) \to f(x).$$

Theorem 2.5.2 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ which converges in the sense of Mosco to a function f. Then,*

$$\operatorname{gph} \partial_L f \subset {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_P f_n.$$

Proof. The arguments are quite similar to those of the proof of the previous theorem. As above, endow $X \times \mathbb{R}$ with the locally uniformly convex norm $\|\cdot\|$ given by $\|(x,s)\| = (\|x\|^2 + |s|^2)^{1/2}$ and note that this norm is also Fréchet differentiable off zero. Assume that $(x, x^*) \in \operatorname{gph} \partial_L f$, or equivalently

$$\big((x, f(x)), (x^*, -1)\big) \in \operatorname{gph} N^L_{\operatorname{epi} f}.$$

By the definition of Mosco convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$, the sequence of sets $\{\operatorname{epi} f_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $\operatorname{epi} f$ in $X \times \mathbb{R}$. From Theorem 2.3.7 we deduce

$$\big((x, f(x)), (x^*, -1)\big) \in \operatorname{gph} N^L_{\operatorname{epi} f} \subset {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{\operatorname{epi} f_n}.$$

This furnishes an increasing sequence $\{k(n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}$, a sequence $\{(x_n, s_n)\}_{n\in\mathbb{N}}$ in $X \times \mathbb{R}$ with $(x_n, s_n) \in \operatorname{epi} f_{k(n)}$, and a sequence $\{(x_n^*, r_n)\}_{n\in\mathbb{N}}$ in $X^* \times \mathbb{R}$ such that

$$(x_n, s_n) \to (x, f(x)) \quad \text{and} \quad (x_n^*, -r_n) \xrightarrow{w^*} (x^*, -1),$$

and $\big((x_n, s_n), (x_n^*, r_n)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f_{k(n)}}$. There exists some $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $r_n > 0$ thus $s_n = f_{k(n)}(x_n)$ according to Lemma 2.2.5. Putting $u_n^* := r_n^{-1} x_n^*$ for all $n \geq N$ and $u_n^* := 0$ for all $n < N$, we obtain for every $n \geq N$

$$\big((x_n, f_n(x_n)), (u_n^*, -1)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f_{k(n)}}$$

or equivalently $(x_n, u_n^*) \in \operatorname{gph} \partial_P f_{k(n)}$. Observing that the sequence $\{u_n^*\}_{n\in\mathbb{N}}$ converges weakly star to x^*, we see that

$$(x, x^*) \in {}^{\|\ \|,*} \operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_P f_n,$$

as required. □

From the previous theorem and Theorem 2.5.1(b) we deduce, for general reflexive Banach spaces, as in Theorem 2.3.8 the following :

Theorem 2.5.3 *Let X be a reflexive Banach space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ which converges in the sense of Mosco to a function f. Then,*

$$\operatorname{gph} \partial_L f \subset {}^{\|\ \|,*} \operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_F f_n.$$

The next theorem provides an inclusion in the opposite sense of that of the previous theorem for the limit superior of subdifferentials of a sequence of functions. The desired inclusion requires the introduction of a form of equi-subsmoothness property for sequences of functions.

Definition 2.5.4 *Assume that X is a reflexive Banach space endowed with a stricly convex norm $\|\cdot\|$ satisfying the (sequential) Kadec-Klee property. Let $f_n : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. Then we say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is subsmooth at $\bar{x} \in \operatorname{dom}(\Gamma - \operatorname*{Lim\,sup}_{n\to\infty} f_n)$ with* *compatible indexation by $n \in \mathbb{N}$, if for any $\varepsilon > 0$ there exist some real $\delta > 0$ and some $N \in \mathbb{N}$ satisfying for each integer $n \geq N$*

$$\langle x^*, x' - x \rangle \leq f_n(x') - f_n(x) + \varepsilon(1 + \|x^*\|)\|x' - x\|$$

for all $x' \in B(\bar{x}, \delta)$, $x \in B(\bar{x}, \delta) \cap \operatorname{Dom} \partial_P f_n$ and $x^* \in \partial_P f_n(x)$.

When, in place of $\partial_P f_n$, another subdifferential is involved over a general Banach space, we will say that the sequence is subsmooth at $\bar{x}$ with respect to this subdifferential with compatible indexation by $n \in \mathbb{N}$.

Among examples of such sequences we have of course any sequence of convex functions. Another one is any sequence of extended real-valued functions which are equi-subsmooth in the following sense.

Definition 2.5.5 *Let X be a Banach space. A family $\{f_t\}_{t \in T}$ of functions from X into $\mathbb{R} \cup \{+\infty\}$ is called equi-subsmooth at $\bar{x} \in X$ whenever for any real $\varepsilon > 0$ there exists some real $\delta > 0$ such that for each $t \in T$*

$$\langle x^*, x' - x \rangle \leq f_t(x') - f_t(x) + \varepsilon(1 + \|x^*\|)\|x' - x\|$$

for all $x' \in B(\bar{x}, \delta)$, $x \in B(\bar{x}, \delta) \cap \operatorname{Dom} \partial_{Cl} f_t$ and $x^* \in \partial_{Cl} f_t(x)$.

An important example of equi-subsmooth family of functions is that of equi-primal lower regular functions; such functions have been introduced by R.A. Poliquin [30] under the name of primal lower nice functions. Recall that a family $\{f_t\}_{t \in T}$ of functions from the Banach space X into $\mathbb{R} \cup \{+\infty\}$ is *equi-primal lower regular* at $\bar{x} \in X$ (see [35]) provided there exists some reals $\delta > 0$ and $c \geq 0$ such that for each $t \in T$

$$\langle x^*, x' - x \rangle \leq f_t(x') - f_t(x) + c(1 + \|x^*\|)\|x' - x\|^2$$

for all $x' \in B(\bar{x}, \delta)$, $x \in B(\bar{x}, \delta) \cap \operatorname{Dom} \partial_{Cl} f_t$ and $x^* \in \partial_{Cl} f_t(x)$.

Lemma 2.5.6 *Assume that X is a reflexive Banach space endowed with a locally uniformly convex norm $\| \cdot \|$ which is Fréchet differentiable off zero. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ which Γ-converges to a proper function f from X into $\mathbb{R} \cup \{+\infty\}$. If $\{f_n\}_{n \in \mathbb{N}}$ is subsmooth at $\bar{x} \in dom f$ with compatible indexation by $n \in \mathbb{N}$, then the sequence of sets $\{\operatorname{epi} f_n\}_{n \in \mathbb{N}}$ is subsmooth at $(\bar{x}, f(\bar{x}))$ with compatible indexation by $n \in \mathbb{N}$.*

Proof. We endow the reflexive Banach space $X \times \mathbb{R}$ with the norm $\|\cdot\|$ given by $\|(x,s)\| = (\|x\|^2 + |s|^2)^{1/2}$. Assume that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is subsmooth at $x \in \operatorname{dom} f$ with compatible indexation by $n \in \mathbb{N}$. Fix any real $\varepsilon > 0$ and take $\delta > 0$ and $N \in \mathbb{N}$ given by Definition 2.5.4 above, so for each integer $n \geq N$

$$\langle x^*, x' - x\rangle \leq f_n(x') - f_n(x) + \varepsilon(1 + \|x^*\|)\|x' - x\| \tag{2.5.2}$$

for all $x' \in B(\bar{x}, \delta)$, $x \in B(\bar{x}, \delta) \cap \operatorname{Dom} \partial_P f_n$ and $x^* \in \partial_P f_n(x)$. Fix any integer $n \geq N$ and consider any $(x', s') \in B\big((\bar{x}, f(\bar{x})), \delta\big)$ with $(x', s') \in \operatorname{epi} f_n$, that is, $s' \geq f(x')$, any $(x, s) \in B\big((\bar{x}, f(\bar{x})), \delta\big)$ with $(x, s) \in \operatorname{epi} f_n$, any $(x^*, -r) \in N^P_{\operatorname{epi} f_n}\big((x, s)\big)$ with $(x^*, -r) \in \mathbb{B}_{X^* \times \mathbb{R}}$. We know from Lemma 2.2.5 that $r \geq 0$. We will distinguish two cases : $r > 0$ and $r = 0$.
Case I : $\boldsymbol{r > 0}$. In this case we have $s = f_n(x)$ (see Lemma 2.2.5) hence

$$(x^*, -r) \in N^P_{\operatorname{epi} f_n}\big((x, f_n(x))\big), \text{ that is, } (r^{-1}x^*, -1) \in N^P_{\operatorname{epi} f_n}\big((x, f_n(x))\big).$$

This means $r^{-1}x^* \in \partial_P f_n(x)$, thus we have by (2.5.2)

$$\langle r^{-1}x^*, x' - x\rangle + (-1)(f_n(x') - f_n(x)) \leq \varepsilon(1 + r^{-1}\|x^*\|)\|x' - x\|,$$

and according to the inequality $s' \geq f_n(x')$

$$\langle r^{-1}x^*, x' - x\rangle + (-1)(s' - f_n(x)) \leq \varepsilon(1 + r^{-1}\|x^*\|)\|x' - x\|$$

or equivalently

$$\langle (r^{-1}x^*, -1), (x', s') - (x, f_n(x))\rangle \leq \varepsilon(1 + r^{-1}\|x^*\|)\|x' - x\|.$$

Multiplying by $r > 0$ and taking the inclusion $(x^*, -r) \in \mathbb{B}_{X^* \times \mathbb{R}}$ into account, we obtain

$$\langle (x^*, -r), (x', s') - (x, s)\rangle \leq \varepsilon(r + \|x^*\|)\|x' - x\| \leq 2\varepsilon\|x' - x\|.$$

Case II : $\boldsymbol{r = 0}$. We know in this case by Lemma 2.2.5 that we also have $(x^*, 0) \in N^P_{\operatorname{epi} f_n}\big((x, f_n(x))\big)$. Then by Proposition 2.2.6 there exist a sequence $\{(x_{k,n}, f_n(x_{k,n}))\}_k$ in $\operatorname{epi} f_n$ converging to $(x, f_n(x))$ as $k \to \infty$ and a sequence $\{(x^*_{k,n}, -r_{k,n})\}_k$ in $X^* \times \mathbb{R}$ converging strongly to $(x^*, 0)$ as $k \to \infty$ such that $(x^*_{k,n}, -r_{k,n}) \in N^P_{\operatorname{epi} f_n}\big((x_{k,n}, f_n(x_{k,n}))\big)$ and $r_{k,n} > 0$ for all $k \in \mathbb{N}$. From the latter inclusion into the normal cone we see that $r^{-1}_{k,n}x^*_{k,n} \in \partial_P f_n(x_{k,n})$. Since

$x \in B(\bar{x}, \delta)$, there exists some integer K_n such that $x_{k,n} \in B(\bar{x}, \delta)$ for all integers $k \geq K_n$. Then for any $k \geq K_n$, the inequality (2.5.2) yields

$$\langle r_{k,n}^{-1} x_{k,n}^*, x' - x_{k,n} \rangle \leq f_n(x') - f_n(x_{k,n}) + \varepsilon(1 + r_{k,n}^{-1} \|x_{k,n}^*\|) \|x' - x_{k,n}\|,$$

so as above we obtain

$$\langle (x_{k,n}^*, -r_{k,n}), (x', s') - (x_{k,n}, f_n(x_{k,n})) \rangle \leq \varepsilon(r_{k,n} + \|x_{k,n}^*\|) \|x' - x_{k,n}\|.$$

Taking the limit as $k \to \infty$ gives

$$\langle (x^*, 0), (x', s') - (x, f_n(x)) \rangle \leq \varepsilon \|x^*\| \|x' - x\| \leq \varepsilon \|x' - x\|$$

thus

$$\langle (x^*, 0), (x', s') - (x, s) \rangle \leq \varepsilon \|x' - x\|$$

Consequenty both cases furnish

$$\langle (x^*, -r), (x', s') - (x, s) \rangle \leq 2\varepsilon(\|x' - x\| + |s' - s|)$$

for all $(x', s') \in \operatorname{epi} f_n$ with $(x', s') \in B\big((\bar{x}, f(\bar{x})), \delta\big)$, $(x, s) \in \operatorname{epi} f_n$ with $(x, s) \in B\big((\bar{x}, f(\bar{x})), \delta\big)$ and $(x^*, -r) \in N_{\operatorname{epi} f_n}^P\big((x, s)\big)$ with $(x^*, -r) \in \mathbb{B}_{X^* \times \mathbb{R}}$. We then conclude that the sequence of sets $\{\operatorname{epi} f_n\}_n$ is subsmooth at $(\bar{x}, f(\bar{x}))$ with compatible indexation. $\square$

Remark 2.5.7 *It is not difficult to see that the lemma still holds if the sequence is assumed to be subsmooth with respect to any other of subdifferentials* ∂_F, ∂_L, ∂_{Cl}. $\square$

Theorem 2.5.8 *Assume that* $(X, \|\cdot\|)$ *is a reflexive locally uniformly convex Banach space. Let* $\{f_n\}_{n \in \mathbb{N}}$ *be a sequence of proper lower semicontinuous functions from* X *into* $\mathbb{R} \cup \{+\infty\}$ *which* Γ*-converges to a proper function* f. *Assume that the sequence* $\{f_n\}_{n \in \mathbb{N}}$ *is subsmooth at every point of* $\operatorname{dom} f$ *with a compatible indexation. Then,*

$$^{\|\,\|,*}\operatorname*{Lim\,sup}_{n \to \infty,\, f_n} \operatorname{gph} \partial_P f_n \subset \operatorname{gph} \partial_F f.$$

Proof. Consider any $(x, x^*) \in {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n \to \infty,\, f_n} \operatorname{gph} \partial_P f_n$. There exist a sequence $(\{x_n, x_n^*)\}$ in $X \times X^*$ and a strictly increasing sequence $\{k(n)\}$ in $\mathbb{N}$ such that

$$(x_n, x_n^*) \in \operatorname{gph} \partial_P f_{k(n)} \quad \text{and} \quad (x_n, f_{k(n)}(x_n)) \to (x, f(x))$$

and such that $\{x_n^*\}$ converges to x^* with respect to the weak star topology in X^*. Therefore

$$\big((x_n, f_{k(n)}(x_n)), (x_n^*, -1)\big) \in \operatorname{gph} N^P_{\operatorname{epi} f_{k(n)}},$$

so by Lemma 2.5.6 and Proposition 2.3.14

$$\big((x, f(x)), (x^*, -1)\big) \in {}^{\|\;\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N^P_{\operatorname{epi} f_n} \subset \operatorname{gph} N^F_{\operatorname{epi} f}.$$

We then obtain

$$(x, x^*) \in \operatorname{gph} \partial_F f.$$

and the proof is completed. □

Theorems 2.5.2 and 2.5.8 directly yield :

Theorem 2.5.9 *Assume that* $(X, \|\cdot\|)$ *is a reflexive locally uniformly convex Banach space and that the norm* $\|\;\|$ *is Fréchet differentiable off zero. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of proper lower semicontinuous functions from* X *into* $\mathbb{R}\cup\{+\infty\}$ *which converges in the sense of Mosco to a proper function* f. *Assume that the sequence* $\{f_n\}$ *is subsmooth at every point of* $\operatorname{dom} f$ *with a compatible indexation. Then,*

$$\operatorname{gph} \partial_L f = \operatorname{gph} \partial_F f = {}^{\|\;\|,*}\operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_P f_n.$$

Concerning the limit superior of the graphs of Fréchet subdifferentials (instead of proximal subdifferentials), the following results hold in a general reflexive Banach space :

Theorem 2.5.10 *Let* X *is a reflexive Banach space and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of proper lower semicontinuous functions from* X *into* $\mathbb{R}\cup\{+\infty\}$ *which* Γ*-converges to a proper function* f. *Assume that the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is subsmooth at every point of* $\operatorname{dom} f$ *with respect to the Fréchet subdifferential with a compatible indexation. Then,*

$${}^{\|\;\|,*}\operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_F f_n \subset \operatorname{gph} \partial_F f.$$

If in addition to the above subsmoothness property, the sequence $\{f_n\}_n$ *satisfies the stronger assumption of Mosco convergence to* f *instead of the* Γ*-convergence, then one has the equalities*

$$\operatorname{gph} \partial_L f = \operatorname{gph} \partial_F f = {}^{\|\;\|,*}\operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_F f_n.$$

Proof. Concerning the assertion (a) it is enough to apply the arguments of Theorem 2.5.8 using Remark 2.5.7 in place of Lemma 2.5.6. The assertion (b) follows from (a) above and Theorem 2.5.3. □

Now we turn on the case of Attouch-Wets convergence. Recall that the sequence of extended real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ on X is said to converge in the sense of Attouch-Wets to an extended real-valued function f provided that the sequence of sets $\{\operatorname{epi} f_n\}_{n\in\mathbb{N}}$ in $X \times \mathbb{R}$ converges in the sense of Attouch-Wets to the set $\operatorname{epi} f$. All the theorems that we have established in the previous part of this section for the Mosco convergence can be also obtained for the Attouch-Wets convergence and the proofs are similar and omitted. Theorems 2.5.11 and 2.5.13 are concerned with locally unformly convex Banach spaces and the limit superior of proximal subdifferentials, while Theorems 2.5.12 and 2.5.14 are stated for general reflexive Banach spaces but with the limit superior of Fréchet subdifferentials.

Theorem 2.5.11 *Assume that $(X, \|\cdot\|)$ is a reflexive locally uniformly convex Banach space and that the norm $\|\cdot\|$ is Fréchet differentiable off zero. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R}\cup\{+\infty\}$ which converges in the sense of Attouch-Wets to a proper function f. Then,*

$$(a) \quad \operatorname{gph} \partial_P f \subset {}^{\|\ \|}\operatorname*{Lim\,inf}_{n\to\infty} \operatorname{gph} \partial_P f_n;$$

$$(b) \quad \operatorname{gph} \partial_F f \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \partial_P f_n.$$

Theorem 2.5.12 *Let X be a reflexive Banach space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R}\cup\{+\infty\}$ which converges in the sense of Attouch-Wets to a proper function f. Then*

$$(a) \quad \operatorname{gph} \partial_F f \subset {}^{\|\ \|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \partial_F f_n;$$

$$(b) \qquad \operatorname{gph} \partial_L f \subset {}^{\|\ \|,*}\operatorname*{Lim\,sup}_{n\to\infty, f_n} \operatorname{gph} \partial_F f_n.$$

Theorem 2.5.13 *Assume that* $(X, \|\cdot\|)$ *is a reflexive locally uniformly convex Banach space and that the norm* $\|\cdot\|$ *is Fréchet differentiable off zero. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of proper lower semicontinuous functions from* X *into* $\mathbb{R}\cup\{+\infty\}$ *which converges in the sense of Attouch-Wets to a proper function* f*. Assume that the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is subsmooth at every point of* $\operatorname{dom} f$ *with a compatible indexation. Then one has*

$$\operatorname{gph} \partial_L f = \operatorname{gph} \partial_F f = {}^{\|\ \|,*} \operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_P f_n.$$

Theorem 2.5.14 *Let* X *is a reflexive Banach space and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of proper lower semicontinuous functions from* X *into* $\mathbb{R}\cup\{+\infty\}$ *which converges in the sense of Attouch-Wets to a proper function* f*. Assume that the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is subsmooth at every point of* $\operatorname{dom} f$ *with respect to the Fréchet subdifferential with a compatible indexation. Then one has*

$$\operatorname{gph} \partial_L f = \operatorname{gph} \partial_F f = {}^{\|\ \|,*} \operatorname*{Lim\,sup}_{n\to\infty,\, f_n} \operatorname{gph} \partial_F f_n.$$

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Chapitre 3

The validity of the "lim inf" formula and a characterization of Asplund spaces

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Abstract. We show that for a given bornology β on a Banach space X the following "lim inf" formula

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x)$$

holds true for every closed set $C \subset X$ and any $x \in C$, provided that the space $X \times X$ is ∂_β-trusted. Here $T_\beta(C; x)$ and $T_c(C; x)$ denote the β-tangent cone and the Clarke tangent cone to C at x. The trustworthiness includes spaces with an equivalent β-differentiable norm or more generally with a Lipschitz β-differentiable bump function. As a consequence, we show that for the Fréchet bornology, this "lim inf" formula characterizes in fact the Asplund property of X. We use our results to obtain new characterizations of T_β-pseudoconvexity of X.

3.1 Introduction

Let X be a real Banach space and X^* be its topological dual with pairing $\langle \cdot, \cdot \rangle$. A *bornology* β on X is a family of bounded and centrally symmetric subsets of X whose union is X, which is closed under multiplication by positive scalars and is directed upwards (i.e., the union of any two members of β is contained in some member of β). The most important bornologies are Gâteaux bornology consisting of all finite symetric subset of X, Hadamard bornology consisting of all norm compact symetric sets, weak Hadamard bornology consisting of all weakly compact symetric sets and Fréchet bornology consisting of all bounded symetric sets.

Each bornology β generates a β-subdifferential which in turn gives rise to the β-normal cone, and hence by making polars to the β-tangent cone.

In this paper, we are concerned with sufficient conditions on a Banach space X satisfying the following "lim inf" formula

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C;x') \subset T_c(C;x) \tag{3.1.1}$$

for each closed set $C \subset X$, and for each $x \in C$. Here $T_\beta(C;x)$ and $T_c(C;x)$ denote the β-tangent cone and the Clarke tangent cones to C at x and for a multivalued mapping $F : C \rightrightarrows X$ $h \in \liminf_{u \xrightarrow{C} x} F(u)$ iff for each sequence $(x_n) \subset C$ converging in norm to x there exists a sequence $h_n \to h$, such that for all sufficiently large n, $h_n \in F(x_n)$.

This kind of formulas has been studied by many authors in special situations. They started with the work by Cornet [6] who found a topological connection between the Clarke tangent cone and the contingent cone $K(C;x)$ to C at x. He has shown that if $C \subset \mathbb{R}^m$, then

$$T_c(C;x) = \liminf_{x' \xrightarrow{C} x} K(C;x').$$

Using his new characterization of Clarke tangent cone, Treiman [20, 21](see also [8] for an independent proof) showed that the inclusion

$$\liminf_{x' \xrightarrow{C} x} K(C;x') \subset T_c(C;x)$$

is true in any Banach space and equality holds whenever C is epi-Lipschitzian at x in the sense of Rockafellar [19]. In [4, 5], Borwein and Strojwas introduced the concept of compactly epi-Lipschitz sets to show that the previous equality holds for C in this class unifying the finite and infinite dimensional situations. In the case when the space in question is reflexive, these authors obtained the following equality

$$T_c(C,x) = \liminf_{x' \xrightarrow{C} x} WK(C,x')$$

where $WK(C,x)$ denotes the weak-contingent cone to C at x. They generalize the results of Penot [16] for finite dimensional and reflexive Banach spaces and of Cornet [6] for finite dimensional spaces. Aubin-Frankowska [2] obtained the following formula

$$T_c(C;x) = \liminf_{x' \xrightarrow{C} x} WK(C;x') = \liminf_{x' \xrightarrow{C} x} \operatorname{co}(WK(C;x'))$$

in the case when the space X is uniformly smooth and the norm of X^* is Fréchet differentiable off the origin.

The validity of the "lim inf" formula (3.1.1) has been accomplished in Borwein and Ioffe [3] in the case when the space X admits a β-differentiable equivalent norm.
Our aim in this paper is to show that if the space $X \times X$ is ∂_β-trusted or equivalently basic fuzzy principle is satisfied on $X \times X$ (this includes spaces with equivalent β-differentiable norm or more generaly spaces with Lipschitz β-differentiable bump function) then the "lim inf" formula (3.1.1) holds. As a consequence, we show that for the Fréchet bornology, the formula (3.1.1) characterizes in fact the Asplund property of X. We then use our results to obtain new characterizations of β-pseudoconvexity.
The plan of the present paper is as follows : After recalling some tools of nonsmooth analysis in the second section, we establish in the third one a connection between Gâteaux (Fréchet) differentiability of the norm and the regularity of the set $D = \overline{\mathbb{B}^c} = \{x \in X : \|x\| \geq 1\}$. For $\bar{x} \in D$, with $\|\bar{x}\| = 1$, Borwein and Strojwas [5] showed that Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to $\overline{\operatorname{co}} K(C;\bar{x}) \neq X$. We prove that Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to $K(D;\bar{x})$ equal to a half space which in turn is equivalent to the Clarke tangential regularity of D at $\bar{x}$. Similar results are obtained for Fréchet differentiability by means of the Fréchet normal cone to D. In the fourth section, we prove our main theorem and some of its consequences. In the fifth section, we give some corollaries, namely a new characterization of Asplund spaces : A Banach space is Asplund space if and only if the "lim inf" formula holds true with the Fréchet bornology for any closed set $C \subset X$. The last section concerns characterizations of T_β-pseudoconvex sets.

3.2 Notation and Preliminaries

Let X be a Banach space with a given norm $\|\cdot\|$, X^* be its topological dual space and $\langle \cdot, \cdot \rangle$ be the duality pairing between X and X^*. The sphere of X and the open ball in X centered at x and of radius δ are defined by $S_X = \{h \in X : \|h\| = 1\}$ and $B(x, \delta) = \{h \in X : \|h - x\| < \delta\}$.

Let C be a closed subset of X. The *contingent cone* $K(C; x)$ (resp. weak-*contingent cone* $WK(C; x)$) to C at x is the set of all $h \in X$ for which there are a sequence (h_n) in X converging strongly (resp. weakly) to h and a sequence of positive numbers (t_n) converging to zero such that

$$x + t_n h_n \in C,$$

for all $n \in \mathbb{N}$. A vector $h \in X$ belongs to the *Clarke tangent* cone $T_c(C; x)$ of C at x provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big(u + tB(h, \varepsilon)\big) \cap C \neq \emptyset,$$

for all $u \in C \cap B(x, \delta)$ and $t \in]0, \delta[$. It is known that $h \in T_c(C; x)$ if and only if for any sequences $(x_n) \subset C$ converging to x and every sequence (t_n) of positive numbers converging to zero there is a sequence (h_n) in X converging to h such that

$$x_n + t_n h_n \in C, \ \forall n \in \mathbb{N}.$$

It is obvious that $T_c(C; x) \subset K(C; x)$. The *Clarke normal cone* is defined as the *negative polar cone* of the Clarke tangent cone, that is,

$$N_c(C; x) := \{x^* \in X^* : \ \langle x^*, h \rangle \leq 0 \text{ for all } h \in T_c(C; x)\}.$$

Let us recall that the (negative) polar cone of a convex cone K is given by

$$K^\circ = \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \ \ \forall h \in K\}.$$

Definition 3.2.1 *Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function finite at x and β be a bornology on X.*
(a) f is said to be β-differentiable at x if there is $x^* \in X^*$ such that for each set $S \in \beta$

$$\lim_{t \to 0^+} t^{-1} \sup_{h \in S} |f(x + th) - f(x) - \langle x^*, th \rangle| = 0,$$

(b) $x^* \in X^*$ is called a β-subgradient of f at x, if for each $\varepsilon > 0$ and each set $S \in \beta$ there is $\delta > 0$ such that for all $0 < t < \delta$ and all $h \in S$

$$t^{-1}\big(f(x+th) - f(x)\big) - \langle x^*, h\rangle \geq -\varepsilon.$$

We denote by $\partial_\beta f(x)$ the set of all β-subgradients of f at x.

It follows from this definition that if $\beta_1 \subset \beta_2$, then $\partial_{\beta_2} f(x) \subset \partial_{\beta_1} f(x)$.

Applying Definition 3.2.1(a) to the bounded bornology and Gâteaux bornology, we obtain the following classical definitions of :

- Fréchet differentiability : There is $x^* \in X^*$ such that

$$\lim_{h\to 0} \|h\|^{-1}(f(x+h) - f(x) - \langle x^*, h\rangle) = 0.$$

- Gâteaux differentiability : There is $x^* \in X^*$ such that

$$\forall h \in X, \quad \lim_{t\to 0^+} t^{-1}(f(x+th) - f(x)) = \langle x^*, h\rangle.$$

While Definition 3.2.1(b) leads ([14]) in the case of the bounded bornology (e.g. $\beta = F$) to the following classical definition of *Fréchet-subdifferential* of f at x :

$$\partial_F f(x) = \left\{ x^* \in X^* : \liminf_{y\to x} \frac{f(y) - f(x) - \langle x^*, y-x\rangle}{\|y-x\|} \geq 0 \right\}.$$

We denote by ∂ the *Fenchel (or Moreau-Rockafeller)* subdifferential that is

$$\partial f(x) = \{x^* \in X^* : f(x+h) - f(x) \geq \langle x, h\rangle, \forall h \in X\}.$$

It is important to note that in case of lower semicontinuous convex function f, we have

$$\partial_\beta f(x) = \partial f(x).$$

We will denote by $N_\beta(C;x)$ the β-normal cone of C at x which is defined by

$$N_\beta(C;x) = \partial_\beta \psi_C(x)$$

where ψ_C is the indicator fonction of C, that is,

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C \end{cases}$$

and by $T_\beta(C;x)$ the β-tangent cone which is defined as the negative polar cone of the β-normal cone intersected with X, that is

$$T_\beta(C,x) = (N_\beta(C,x))^\circ \cap X.$$

Clearly, for an bornology β the following inclusions hold :

$$N_F(C;x) \subset N_\beta(C;x) \subset N_G(C;x), \quad T_G(C;x) \subset T_\beta(C;x) \subset T_F(C;x).$$

When β is the Fréchet bornology, then ([1],[17]) we obtain that

$$N_F(C;\bar{x}) = \big\{x^* \in X^* : \limsup_{u \xrightarrow{C} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq 0\big\}.$$

Definition 3.2.2 *Let X be a Banach space and let β be a bornology on it. We say that X is ∂_β trusted, if the following fuzzy minimization rule holds : let f be a lower semicontinuous function on X finite at $\bar{x} \in X$, and let g be a Lipschitz continuous function on X. Assume that $f+g$ attains a local minimum at x. Then for any $\varepsilon > 0$ there are $x, u \in X$ and $x^* \in \partial_\beta f(x)$, $u^* \in \partial_\beta g(u)$ such that*

$$\|x-\bar{x}\| < \varepsilon, \quad \|u-\bar{x}\| < \varepsilon, \quad |f(x) - f(\bar{x})| < \varepsilon, \quad and \quad \|x^* + y^*\| < \varepsilon.$$

We recall that a bump function on X is a real-valued function ϕ which has bounded nonempty support $supp(\phi) = \{x \in X : \phi(x) \neq 0\}$.

Proposition 3.2.3 *[14] If there is on X a β-differentiable Lipschitz bump function, then X is ∂_β−trusted,*

Proposition 3.2.4 *[9] A Banach space is trusted for the Fréchet subdifferential if and only if it is Asplund.*

3.3 Characterizations of Gâteaux and Fréchet differentiability of the norm

In this section, we study the connection between differentiability of the norm $\|\cdot\|$ on X and some property of the subset $D := \overline{\mathbb{B}^c} = \{x \in X : \|x\| \geq 1\}$. In [5] Borwein and Srojwas showed several properties of D in various

Banach spaces. In particular they showed that if $\|\bar{x}\| = 1$ then Gâteaux differentiability of the norm at $\bar{x}$ is equivalent to the P-properness of D et $\bar{x}$, i.e., $\overline{\text{co}}K(D;x) \neq X$. In this section we will show furthur properties for various norms. We denote by $P_C(x)$ the set of projections of x on a subset C of X, i.e.,

$$P_C(x) = \{y \in C : \|x - y\| = d_C(x)\}.$$

Proposition 3.3.1 *Assume that X is a Banach space with a given norm $\|\cdot\|$. Let $\bar{x} \in X$ with $\|\bar{x}\| = 1$. Then*

(a) $K(D;\bar{x})$ contains at least one closed half space,

(b) $\bar{x} + K(D;\bar{x}) \subset D$,

(c) $K(D;\bar{x}) \neq X$,

(d) $\forall \lambda \in]0,1[,\ D \cap B(\bar{x}, 1-\lambda) + tB(\bar{x},\lambda) \subset D, \quad$ for all $t > 0$,

(e) $B(\bar{x}, 1) \subset T_c(D;\bar{x})$,

(f) $\dfrac{x}{\|x\|} \in P_D(x)$ and $d_D(x) = 1 - \|x\| \quad$ for all $x \in \mathbb{B}\backslash\{0\}$.

Proof. (a) By Hahn-Banach theorem, find $x^* \in X^*$ such that

$$\|x^*\| = \langle x^*, \overline{x}\rangle = 1$$

Then, clearly, the closed half space $\overline{x} + \{h \in X : \ \langle x^*, h\rangle \geq 0\}$ lies in D. Therefore $K(D;\bar{x})$ contains at least a one half space.

(b) Take any h in $K(D,\overline{x})$ and let sequences (h_n) and (t_n) witness for that. The convexity of $\|\cdot\|$ implies that for all large $n \in \mathbb{N}$ we have

$$\|\overline{x}+h\|-1 \geq \frac{\|\overline{x}+t_nh\|-\|\overline{x}\|}{t_n} \geq \frac{\|\overline{x}+t_nh_n\|-\|\overline{x}\|}{t_n} - \|h-h_n\| \geq -\|h-h_n\|.$$

Hence, letting n go to ∞ here, we get the desired inclusion

(c) It is a direct consequence of (b).

(d) For any $x \in D$, $z \in X$ and $t > 0$

$$\|z-(1+t)x\| \leq t \implies (1+t)\|x\| - \|z\| \leq t \implies 1 \leq \|z\|.$$

Therefore $B\big((1+t)x, t\big) \subset D$ or equivalently $x + tB(x,1) \subset D$. Let $\lambda \in]0,1[$ and pick $x \in B(\bar{x}, 1-\lambda) \cap D$. Then $B(\bar{x}, \lambda) \subset B(x,1)$ and hence $x + tB(\bar{x}, \lambda) \subset D$. Finally we receive that

$$D \cap B(\bar{x}, 1-\lambda) + tB(\bar{x}, \lambda) \subset D.$$

(e) Take any $h \in B(\overline{x}, 1)$. Consider any sequence $(x_n) \subset C$ converging to $\overline{x}$ and any $t_n \downarrow 0$. For $n \in \mathbb{N}$ put $h_n := h + x_n - \overline{x}$; then $h_n \to h$, and as

$$\|x_n + t_n h_n\| = \|(1+t_n)x_n + t_n(h - \overline{x})\| \geq 1 + t_n - t_n = 1$$

for every $n \in \mathbb{N}$, we can conclude that $h \in T_c(C, \overline{x})$.

(f) Suppose that $x \in \mathbb{B}_X$ and $z \in D$, then

$$\|x - z\| \geq \|z\| - \|x\| \geq 1 - \|x\| = \left\| x - \frac{x}{\|x\|} \right\|$$

therefore $\dfrac{x}{\|x\|} \in P_D(x)$. □

The following proposition contains several characterizations of the Gâteaux differentiability of the norm.

Proposition 3.3.2 *Let X be a Banach space with a given norm $\|\cdot\|$. Assume that $\|\bar{x}\| = 1$. Then the following assertions are equivalent :*

(a) $\|\cdot\|$ is Gâteaux differentiable at $\bar{x}$,

(b) there is $x^* \in X^*$, $\|x^*\| = 1$ such that $K(D; \bar{x}) = \{h \in X : \langle x^*, h\rangle \geq 0\}$,

(c) $T_c(D; \bar{x}) = K(D; \bar{x})$.

Proof. (a) ⇒ (b). Suppose that $\|\cdot\|$ is Gâteaux differentiable at $\bar{x}$ with derivative x^*. By (a) of Proposition 4.4.1 the cone $K(D; \bar{x})$ contains at least one closed half space. If we show that $K(D; \bar{x}) \subset \{h \in X : \langle x^*, h\rangle \geq 0\}$ then this inclusion will become equality. Take $h \in K(D; \bar{x})$ and find (h_n) in X converging strongly to h and a sequence $(t_n)_n$ of positive numbers converging to zero such that for all $n \in \mathbb{N}$ large enough

$$\bar{x} + t_n h_n \in D.$$

Thus, as $\|\bar{x} + t_n h_n\| \geq 1$,

$$\begin{aligned} \frac{\|\bar{x} + t_n h\| - \|\bar{x}\|}{t_n} - \langle x^*, h \rangle &\geq \frac{\|\bar{x} + t_n h_n\| - \|\bar{x}\|}{t_n} - \langle x^*, h \rangle - \|h - h_n\| \\ &\geq -\langle x^*, h \rangle - \|h - h_n\|. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} \frac{\|\bar{x} + t_n h\| - \|\bar{x}\|}{t_n} - \langle x^*, h \rangle \geq -\langle x^*, h \rangle,$$

$$0 \geq -\langle x^*, h \rangle,$$

$$\langle x^*, h \rangle \geq 0.$$

(b)⇒(a) Assume that $K(D; \bar{x}) = \{h : \langle x^*, h \rangle \geq 0\}$ for some $x^* \in X^*$, with $\|x^*\| = 1$. Let $z^* \in \partial\| \cdot \|(\bar{x})$. Then $\|z^*\| = 1$ and

$$\|\overline{x} + v\| - \|\overline{x}\| \geq \langle z^*, v \rangle \quad \text{for every } v \in X.$$

Hence, if $h \in X$ is such that $\langle z^*, h \rangle \geq 0$, then we have for all $t > 0$ that $\|\overline{x} + th\| \geq 1$, and so $\overline{x} + th \in D$, which means that $h \in K(D, \overline{x})$. By Farkas Lemma ([11]), we conclude that $z^* = \lambda x^*$ with $\lambda > 0$. Thus

$$\lambda = \frac{\|z^*\|}{\|x^*\|} = 1 \quad \text{and} \quad z^* = x^*.$$

This asserts that $\partial\| \cdot \|(\bar{x}) = \{x^*\}$ or equivalently the norm $\| \cdot \|$ is Gâteaux differentiable at $\bar{x}$.

(a) ⇒ (c) Suppose that the norm $\| \cdot \|$ is Gâteaux differentiable at $\bar{x}$. It suffices to show that there exists a unique $x^* \in X^*$, with $\|x^*\| = 1$ such that

$$T_c(D; \bar{x}) = \{h \in X : \langle x^*, h \rangle \geq 0\}.$$

Assertions (c) and (d) of Proposition 4.4.1 ensure that 0 is a boundary point of $T_c(D; \bar{x})$ and $\mathrm{int}T_c(D, \bar{x}) \neq \emptyset$. So the separation theorem produces $x^* \in X^*$, with $\|x^*\| = 1$ such that

$$T_c(D; \bar{x}) \subset \{h \in X : \langle x^*, h \rangle \geq 0\}$$

and as $B(\bar{x}, 1) \subset T_c(D; \bar{x})$ (by (d) of Proposition 4.4.1), the assumption (a) implies that x^* is exactly the Gâteaux derivative of the norm $\| \cdot \|$ at $\bar{x}$. It remains to establish the reverse inclusion

$$T_c(D; \bar{x}) \supset \{h \in X : \langle x^*, h \rangle \geq 0\}.$$

Suppose that there exists $v \in X$ satisfying $\langle x^*, v\rangle \geq 0$ and $v \notin T_c(D;\bar{x})$. Once again, the separation theorem yields $u^* \in X^*$, with $\|u^*\| = 1$, such that

$$T_c(D;\bar{x}) \subset \{h \in X : \langle u^*, h\rangle \geq 0\} \quad \text{and} \quad \langle u^*, v\rangle < 0.$$

As before we show that u^* is also a Gâteaux derivative of the norm $\|\cdot\|$ at $\bar{x}$, and by (a), $x^* = u^*$ and this contradicts the relations

$$\langle x^*, v\rangle \geq 0 \quad \text{and} \quad \langle u^*, v\rangle < 0.$$

(c) $\Rightarrow$ (b) Suppose that $T_c(D;\bar{x}) = K(D;\bar{x})$. Then $T_c(D;\bar{x})$ contains at least one half space. By Proposition 4.4.1, $T_c(D,\bar{x}) \neq X$ and by the separation Theorem (recall that the Clarke cone is convex and closed) there is $x^* \in X^*$, $\|x^*\| = 1$ such that

$$T_c(D;x) \subset \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

By the Farkas lemma we have

$$T_c(D;x) = \{h \in X : \langle x^*, h\rangle \geq 0\}.$$

□

The following corollary on the density of points of Gâteaux differentiability of the norm is a consequence of Propositions 4.4.1 and 3.3.2.

Corollary 3.3.3 *Let $(X, \|\cdot\|)$ be a Banach space and put $D = \{u \in X : \|u\| \geq 1\}$. The following assertions are equivalent :*

(1) For each $x \in S_X$, $\liminf_{x' \xrightarrow{D} x} \overline{\text{co}} K(D;x') \neq X$.

(2) The norm $\|\cdot\|$ is Gâteaux differentiable at the points of a dense subset of X.

Proof. First, we remark that

$$\liminf_{x' \xrightarrow{D} x} \overline{\text{co}} K(D;x') \neq X \iff \liminf_{x' \xrightarrow{S_X} x} \overline{\text{co}} K(D;x') \neq X$$

(1) $\Rightarrow$ (2) : It suffices to show that $\|\cdot\|$ is Gâteaux differentiable on dense subset of S_X. Let $x \in S_X$. Then

$$\liminf_{x' \xrightarrow{D} x} \overline{\text{co}} K(D;x') \neq X.$$

Therefore for any $\varepsilon > 0$ there is $z \in B(x, \varepsilon) \cap D$ such that

$$\overline{\text{co}} K(D; z) \neq X.$$

That is the convex cone $\overline{\text{co}} K(D; z)$ belongs to a half space, thus $K(D; z)$ also belongs to a half space. Since by (a) of Proposition 4.4.1 we know that $K(D; z)$ contains at least one half space, then by Farkas Lemma we deduce that $K(D; z)$ is equal to the half space and $\| \cdot \|$ is Gâteaux differentiable at z according to Proposition 3.3.2.

(2) $\Rightarrow$ (1) : Let $x \in S_X$ and consider a sequence (x_n) in S_X converging to x, such that $\| \cdot \|$ is Gâteaux differentiable at each x_n. Proposition 3.3.2 asserts that there exists $x_n^* \in X^*$, $\|x_n^*\| = 1$, such that $K(D, x_n) = \{h \in X : \langle x_n^*, h \leq 0\}$, and hence $\overline{\text{co}} K(D; x_n) = K(D; x_n)$. Applying Proposition 4.4.1 (b), we get $\overline{\text{co}} K(D; x_n) \subset D - x_n$. Thus $\liminf_{x' \xrightarrow{D} x} \overline{\text{co}} K(D; x') \subset D - x$, and the proof is completed. □

Proposition 3.3.4 *Let X be a Banach space with a given norm $\| \cdot \|$. Consider $\overline{x} \in S_X$. Then the following assertions are equivalent :*

- *(a) $\| \cdot \|$ is Fréchet differentiable at $\bar{x}$,*
- *(b) $N_F(D; \bar{x}) \neq \{0\}$,*

Proof. (a) $\Rightarrow$ (b) If (a) holds then there is some $x^* \in X^*$, $\|x^*\| = 1$ which is the Fréchet derivative of $\| \cdot \|$ at $\bar{x}$, that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$-\varepsilon \leq \frac{\|y\| - \|\bar{x}\| - \langle x^*, y - \bar{x} \rangle}{\|y - \bar{x}\|} \leq \varepsilon,$$

for all $y \in B(\bar{x}, \delta)$. If $y \in D \cap B(\bar{x}, \delta)$ then $\|y\| \geq 1 = \|\bar{x}\|$ and so

$$\frac{\langle -x^*, y - \bar{x} \rangle}{\|y - \bar{x}\|} \leq \varepsilon.$$

This implies that $-x^* \in N_F(D; \bar{x})$.
(b) $\Rightarrow$ (a) Suppose that $x^* \in N_F(D; \bar{x})$ with $\|x^*\| = 1$. Since $N_F(D; \bar{x}) \subset \big(K(D; x)\big)^\circ$ then $x^* \in \big(K(D; x)\big)^\circ$ or equivalently

$$\langle x^*, h \rangle \leq 0 \, \forall h \in K(D, x)$$

and hence
$$K(D;\bar{x}) \subset \{h \in X : \langle x^*, h\rangle \leq 0\}.$$
As $K(D;\bar{x})^\circ$ contains at least one half space, we deduce by Farkas Lemma that $K(D;\bar{x})$ is a half space and therefore Proposition 3.3.2 asserts that $-x^*$ is a Gâteaux derivative of $\|\cdot\|$ at $\bar{x}$ and $\langle -x^*, \bar{x}\rangle = 1$. By the definition of $N_F(D;\bar{x})$, for any $\varepsilon > 0$ there is $\delta > 0$ (with $\delta \leq 1$) such that
$$\langle x^*, x - \bar{x}\rangle \leq \varepsilon\|x - \bar{x}\| \tag{3.3.1}$$
for all $x \in D \cap B(\bar{x}, \delta)$. We note that
$$\begin{aligned}\Big\|\frac{x}{\|x\|} - \bar{x}\Big\| = \frac{1}{\|x\|}\Big\|x - \|x\|\bar{x}\Big\| &\leq \frac{1}{\|x\|}\Big[\Big\|x - \|x\|x\Big\| + \Big\|\|x\|x - \|x\|\bar{x}\Big\|\Big]\\ &= \big|\|x\| - 1\big| + \|x - \bar{x}\|\\ &\leq 2\|x - \bar{x}\|.\end{aligned}$$
Thus if $x \in B(\bar{x}, \delta/2)$, then $\dfrac{x}{\|x\|} \in B(\bar{x}, \delta) \cap D$ and therefore by inequality (3.3.1)
$$\Big\langle x^*, \frac{x}{\|x\|} - \bar{x}\Big\rangle \leq \varepsilon\Big\|\frac{x}{\|x\|} - \bar{x}\Big\|,$$
$$1 + \Big\langle x^*, \frac{x}{\|x\|}\Big\rangle \leq 2\varepsilon\big\|x - \bar{x}\big\|,$$
$$\|x\| + \langle x^*, x\rangle \leq 2\varepsilon\|x\|\|x - \bar{x}\|,$$
$$\|x\| - 1 + \langle x^*, x - \bar{x}\rangle \leq 4\varepsilon\|x - \bar{x}\|,$$
$$\|x\| - \|\bar{x}\| + \langle x^*, x - \bar{x}\rangle \leq 4\varepsilon\|x - \bar{x}\|.$$
As $-x^*$ is the Gâteaux derivative of $\|\cdot\|$ at $\bar{x}$ we receive finally that
$$0 \leq \|x\| - \|\bar{x}\| + \langle x^*, x - \bar{x}\rangle \leq 4\varepsilon\|x - \bar{x}\|,$$
for all $x \in B(\bar{x}, \delta/2)$. Therefore $\|\cdot\|$ is Fréchet differentiable at $\bar{x}$. □

The following corollary on the density of points of Fréchet differentiability of the norm is a consequence of Propositions 4.4.1 and 3.3.4. Its proof is similar to that of Corollary 3.3.3.

Corollary 3.3.5 *Let $(X, \|\cdot\|)$ be a Banach space and put $D = \{u \in X : \|u\| \geq 1\}$. The following assertions are equivalent :*

(1) For each $x \in S_X$, $\liminf_{x' \xrightarrow{D} x} T_F(D; x') \neq X$.

(2) The norm $\|\cdot\|$ is Fréchet differentiable at the points of a dense subset of X.

3.4 The validity of the "lim inf" formula

Theorem 3.4.1 *Let $(X, \|\cdot\|)$ be a Banach space and β a bornology on X such that $X \times X$ is ∂_β-trusted. Then for any closed subset C of X and $\bar{x} \in C$*

$$\liminf_{x \xrightarrow{C} \bar{x}} T_\beta(C; x) \subset T_c(C; \bar{x}).$$

Proof. We follow the proof in [15]. Pick $w \in \liminf_{x \xrightarrow{C} \bar{x}} T_\beta(C, x)$. We want to show that $w \in T_c(C; x)$. Suppose that $w \notin T_c(C, x)$. Then by Lemma 1.2.1 in [20] there are a sequence (x_n) in C converging to x, a sequence (λ_n) in $(0, +\infty)$ converging to zero and $\varepsilon > 0$ such that

$$\big(x_n +]0, \lambda_n] B(w, \varepsilon)\big) \cap C = \emptyset, \quad \forall n \in \mathbb{N}.$$

Let us fix an integer $n \in \mathbb{N}$ and put $D := x_n + \left[0, \frac{\lambda_n}{2}\right] B(w, \varepsilon)$. Then $\big(D + \lambda_n^4 w\big) \cap C = \emptyset$. Define the function f by

$$f(x, y) = \|x - y - \lambda_n^4 w\|, \quad \forall (x, y) \in X \times X.$$

Thus $f(x_n, x_n) = \lambda_n^4$ and

$$\lambda_n^4 + \inf_{(x,y) \in C \times D} f(x, y) \geq f(x_n, x_n).$$

The Ekeland's variational principle provides $(u_n, v_n) \in C \times D$ satisfying

$$\|u_n - x_n\| + \|v_n - x_n\| < \lambda_n^2,$$

and

$$\forall u \in C, \forall v \in D, \quad f(u_n, v_n) \leq f(u, v) + \lambda_n^2(\|u - u_n\| + \|v - v_n\|).$$

Thus

$$f(u_n, v_n) \leq f(u, v) + \lambda_n^2(\|u - u_n\| + \|v - v_n\|) + \psi_C(u) + \psi_D(v), \quad (3.4.1)$$

for all $u, v \in X$. Since $\big(D + \lambda_n^4 w\big) \cap C = \emptyset$, we get

$$\|u_n - v_n - \lambda_n^4 w\| > 0$$

and so there is $\delta_n > 0$ such that

$$\|t - \tau - \lambda_n^4 w\| > 0,$$

for all $t \in B(u_n, \delta_n)$ and $\tau \in B(v_n, \delta_n)$.

Since $X \times X$ is ∂_β-trusted, 3.4.1 provides there are $u_n^1, u_n^2, v_n^1, v_n^2 \in X$ and $u_n^{*1}, u_n^{*2}, v_n^{*1}, v_n^{*2} \in X^*$ such that

$$\|u_n^1 - u_n\| + \|u_n^2 - u_n\| + \|v_n^1 - v_n\| + \|v_n^2 - v_n\| < \alpha_n = \min\{\delta_n, \lambda_n^4\},$$

$$\|u_n^{*1} + u_n^{*2}\| + \|v_n^{*1} + v_n^{*2}\| \leq \alpha_n = \min\{\delta_n, \lambda_n^4\} \tag{3.4.2}$$

and

$$(u_n^{*1}, v_n^{*1}) \in \partial_\beta\Big(f + \lambda_n^2\big(\|\cdot - u_n\| + \|\cdot - v_n\|\big)\Big)(u_n^1, v_n^1),$$

$$(u_n^{*2}, v_n^{*2}) \in \partial_\beta\big(\psi_C(\cdot) + \psi_D(\cdot)\big)(u_n^2, v_n^2).$$

By the convexity and the continuity of separate summands

$$\begin{aligned} & \partial_\beta\Big(f + \lambda_n^2\big(\|\cdot - u_n\| + \|\cdot - v_n\|\big)\Big)(u_n^1, v_n^1) \\ = \;& \partial\Big(f + \lambda_n^2\big(\|\cdot - u_n\| + \|\cdot - v_n\|\big)\Big)(u_n^1, v_n^1) \\ \subset \;& \partial f(u_n^1, v_n^1) + \lambda_n^2(B_{X^*} \times B_{X^*}). \end{aligned}$$

Since $\|u_n^1 - v_n^1 - \lambda_n^4 w\| \neq 0$ we receive that $\partial f(u_n^1, v_n^1)$ is included in $\{(x^*, -x^*) : \|x^*\| = 1\}$. That is there is $x_n^* \in X^*$ with $\|x_n^*\| = 1$ such that

$$\|u_n^{*1} - x_n^*\| \leq \lambda_n^2 \quad \text{and} \quad \|v_n^{*1} + x_n^*\| \leq \lambda_n^2.$$

By the inequality (3.4.2) we receive that

$$\|x_n^* + u_n^{*2}\| \leq \lambda_n^2 + \lambda_n^4 \quad \text{and} \quad \|v_n^{*2} - x_n^*\| \leq \lambda_n^2 + \lambda_n^4 \tag{3.4.3}$$

and thus

$$\|u_n^{*2} + v_n^{*2}\| \leq 2(\lambda_n^2 + \lambda_n^4) \tag{3.4.4}$$

It is evident that

$$\partial_\beta(\psi_C(\cdot)+\psi_D(\cdot))(u_n^2,v_n^2)=\partial_\beta\psi_C(u_n^2)\times\partial_\beta\psi_D(v_n^2).$$

Thus

$$\begin{gathered}
\left\langle v_n^{*2}, u-v_n^2\right\rangle \le 0 \qquad \forall u\in D,\\
\left\langle v_n^{*2}, x_n+\frac{\lambda_n}{2}(w+b)-v_n^2\right\rangle \le 0 \qquad \forall b\in B(0,\varepsilon),\\
\varepsilon\|v_n^{*2}\|\frac{\lambda_n}{2}+\left\langle v_n^{*2}, x_n-v_n^2+\frac{\lambda_n}{2}w\right\rangle \le 0,\\
\frac{\varepsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)\le\left\langle v_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle .
\end{gathered}$$

Using (3.4.3) and (3.4.4), we get

$$\begin{aligned}
\frac{\varepsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4) &\le \left\langle v_n^{*2}+u_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle+\left\langle -u_n^{*2}, v_n^2-x_n-\frac{\lambda_n}{2}w\right\rangle\\
&\le 2(\lambda_n^2+\lambda_n^4)\|v_n^2-x_n-\frac{\lambda_n}{2}w\|+\left\langle -u_n^{*2}, v_n^2-x_n\right\rangle+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle,
\end{aligned}$$

$$\frac{\varepsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)+\left\langle u_n^{*2}, v_n^2-x_n\right\rangle\le 2(\lambda_n^2+\lambda_n^4)\|v_n^2-x_n-\frac{\lambda}{2}w\|+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle,$$

$$\frac{\varepsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)-\|u_n^{*2}\|\|v_n^2-x_n\|\le 2(\lambda_n^2+\lambda_n^4)\|v_n^2-x_n-\frac{\lambda}{2}w\|+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle,$$

$$\frac{\varepsilon\lambda_n}{2}(1-\lambda_n^2-\lambda_n^4)-(1+\lambda_n^2+\lambda_n^4)(\lambda_n^2+\lambda_n^4)\le 2(\lambda_n^2+\lambda_n^4)\|v_n^2-x_n-\frac{\lambda_n}{2}w\|+\frac{\lambda_n}{2}\left\langle u_n^{*2}, w\right\rangle,$$

$$\varepsilon(1-\lambda_n^2-\lambda_n^4)-2(1+\lambda_n^2+\lambda_n^4)(\lambda_n+\lambda_n^3)\le 4(\lambda_n+\lambda_n^3)\|v_n^2-x_n-\frac{\lambda_n}{2}w\|+\left\langle u_n^{*2}, w\right\rangle .$$

Now remember that $u_n^{*2}\in\partial_\beta\psi_C(u_n^2)=N^\beta(C,u_n^2)$, $\{u_n^2\}_n$ converges to $\bar{x}$, $(\lambda_n)_n$ converges to zero and $w\in\liminf_{x\xrightarrow{C}\bar{x}} T_\beta(C;x)$. Therefore, there are $w_n\in T_\beta(C,u_n^2)$ converging to w. Thus we receive that

$$\begin{aligned}
\varepsilon-(2\varepsilon+8)\lambda_n &\le 4(\lambda_n+\lambda_n^3)\|v_n^2-x_n-\frac{\lambda_n}{2}w\|+\left\langle u_n^{*2}, w-w_n\right\rangle+\left\langle u_n^{*2}, w_n\right\rangle\\
&\le 4(\lambda_n+\lambda_n^3)\|v_n^2-x_n-\frac{\lambda_n}{2}w\|+\|u_n^{*2}\|\|w-w_n\|.
\end{aligned}$$

as $u^{*2} \in N_\beta(C; u_n^2)$. Passing to the limit on n and taking into account that $\|u_n^{*2}\| \leq 1 + \lambda_n^2 + \lambda_n^4$ and $v_n^2 - x_n - \frac{\lambda_n}{2} w$ converges to 0 we receive $\varepsilon \leq 0$ which is contradiction. $\square$

We know that if there is on X a β-differentiable Lipschitz bump function then there is also on $X \times X$ a β-differentiable Lipschitz bump function, therefore according to Proposition 3.2.3, $X \times X$ is ∂_β-trusted. So the following corollary is a direct consequence of Theorem 3.4.1.

Corollary 3.4.2 *Assume that there is on X a β-differentiable Lipschitz bump function. Then for any closed subset C of X containing x*

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x).$$

We recall that if X admits an equivalent β-differentiable norm (at all nonzero points), then there is on X a β-differentiable Lipschitz bump function [18]. Note that the reverse is not true. Haydon [12] constructed a nonseparable Banach space that has Fréchet differentiable Lipschitz bump function but does not admit an equivalent Gâteaux differentiable norm.

Corollary 3.4.3 *([3]) Let X be a Banach space with a norm which is β-differentiable away from the origin. Let C be a closed subset of X. Then for any $x \in C$ we have*

$$\liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x).$$

The following corollary is an extention of Theorem 3.4 in [4] from spaces with equivalent Fréchet differentiable norm away from the origin to Asplund spaces and without the weak compcatness assumption on the set C. We recall that $WK(C; x)$ denotes the weak-contingent cone to C at x.

Corollary 3.4.4 *([15]) Let X be Asplund space and C be a closed subset of X. Then for any $x \in C$ we have*

$$\liminf_{x' \xrightarrow{C} x} \overline{\text{co}}(WK(C; x')) \subset T_c(C; x).$$

Proof. Borwein and Strojwas [4] proved that for any closed subset C of X and $x \in C$

$$N_F(C; x) \subset (WK(C; x))^\circ.$$

Therefore

$$\overline{\text{co}}(WK(C;x)) \subset T_F(C;x).$$

On the other hand since X is Asplund, $X \times X$ is also Asplund and therefore according to the Proposition 3.2.3 trusted for the Fréchet subdifferetnial. By Theorem 3.4.1 we receive that

$$\liminf_{x' \xrightarrow{C} x} T_F(C;x) \subset T_c(C;x),$$

and therefore

$$\liminf_{x' \xrightarrow{C} x} \overline{\text{co}}(WK(C;x)) \subset T_c(C;x).$$

The proof is completed. □

To end up this section, we give an extention of Theorem 5.4 in [5] where lower semicontinuity (LSC) of a multivalued mapping is involved. A multivalued mapping $F : C \rightrightarrows X$ is said to be lower semicontinuous at $x \in C$ if

$$F(x) \subset \liminf_{x' \xrightarrow{C} x} F(x')$$

and is LSC on C if it is LSC at each point x in C.

Theorem 3.4.5 *Let $(X, \|\cdot\|)$ be a Banach space, β be a bornology on X containing the Hadamard bornology such that $X \times X$ is ∂_β-trusted and C be a closed subset of X. Suppose that $F : C \rightrightarrows X$ is LSC on C. Then the following statements are equivalent :*

(i) *$F(x) \subset T_c(C;x)$, for all $x \in C$,*

(ii) *$F(x) \subset T_\beta(C;x)$, for all $x \in C$.*

Proof $(ii) \Rightarrow (i)$ follows from the lower semicontinuity of F and Theorem 3.4.1.
$(i) \Rightarrow (ii)$: Since $T_c(C;x) \subset \overline{\text{co}}K(C;x)$, our hypothesis on the bornology β ensures that $T_c(C;x) \subset \overline{\text{co}}K(C;x) \subset T_\beta(C;x)$ and so (i) implies (ii). □

Remark 3.4.6

- Statement (2) in Theorem 5.4 in [5] is extended from refelexive Banach spaces to Asplund spaces.

- The weak compactness assumptions and the Gâteaux smoothness of an equivalent norm (resp. the Fréchet differentiability of an equivalent norm) off zero assumed in the statement (4) (resp. (5)) of Theorem 5.4 in [5] are weakened by assuming that the space admits a Gâteaux differentiable lipschitz bump function (resp. the space is Asplund) and the set is closed.

3.5 The "$\liminf$" formula as a characterization of Asplund spaces

We begin by recalling that X is an *Asplund* space if every continuous convex function on any open convex subset U of X is Fréchet differentiable at the points of a dense G_δ subset of U.
A well known theorem of Fabian and Mordukhovich [10] affirms that the space X is Asplund if and only if for every closed set $C \subset X$ and every $\bar{x} \in C$ one has the limiting representation

$$N(C; \bar{x}) = \limsup_{x \to \bar{x}} N^F(C; x)$$

where $N(\bar{x}; C)$ denotes the limiting normal cone of C at $\bar{x}$. Here, we give a characterization of Asplund spaces by mean of the "liminf" formula.

Theorem 3.5.1 *A Banach space X is Asplund if and only if for every closed set C in it and every $x \in C$, the following inclusion holds*

$$\liminf_{x' \xrightarrow{C} x} T_F(C; x') \subset T_c(C; x).$$

Proof. (a) $\Rightarrow$ (b) : We know that if X is an Asplund space then $X \times X$ is also an Asplund space. According to (c) of Proposition 3.2.4 $X \times X$ is trusted for Fréchet subdifferential. Theorem 3.4.1 asserts that

$$\liminf_{x' \xrightarrow{C} x} T_F(C; x') \subset T_c(C; x),$$

for any set $C \subset X$ and $x \in C$.

(b) $\Rightarrow$ (a) : Suppose that X is not an Asplund space. Then it is known [7, p. 27] (see also [18, p. 33]) that there is an equivalent norm on X which is nowhere Fréchet differentiable. Therefore by Proposition 3.3.4 $N_F(C_1;x) = \{0\}$ for all $x \in C_1$, where $C_1 = \{z \in X : \|z\| \geq 1\}$. Thus $T_F(C_1;x) = X$ for all $x \in C_1$ and

$$X = \liminf_{x' \xrightarrow{C_1} x} T_F(C_1;x') \subset T_c(C_1;x).$$

This is in contradiction with $T_c(C_1;x) \subset K(C_1,x) \neq X$ (see Proposition 4.4.1 (c)). □

3.6 Convexity of Pseudoconvex sets

Let C be a set in a Banach space X and let $x \in C$. Let $R(C;x)$ denotes one of the cones $T_c(C;x),\ T_\beta(C;x),\ K(C;x),\ \ldots$. We say that C is *R-pseudoconvex at x* if

$$C - x \subset R(C;x).$$

We say that C is *R-pseudoconvex* if the last inclusion holds for every $x \in C$. Concerning this notion, Borwein and Strojwas [5] established the following result on the equivalence between convexity and R-pseudoconvexity.

Theorem 3.6.1 *[5] For a closed set C in a Banach space X TFAE : (i) C is convex; (ii) C is K-pseudoconvex; (iii) C is T_c-pseudoconvex.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (i)$: By Treiman Theorem ([20]-[21]) we receive that T_c-pseudoconvexity coincide with K-pseudoconvexity. Suppose that C is T_c-pseudoconvex, that is $C - x \subset T_c(C;x)$ for all $x \in C$. If C is not convex, then there exist distinct $u, v \in C$ such that $]u,v[\cap C = \emptyset$. Let $w \in]u,v[$ and consider the function $f(x) = \|x - w\|$. For every $n \in \mathbb{N}$ find $u_n \in C$ such that

$$\|u_n - w\| \leq \inf_{x \in C} \|x - w\| + \frac{1}{n^2}. \qquad (3.6.1)$$

By Ekeland's variational principle, there exists $x_n \in C$ such that

$$\|x_n - u_n\| \leq \frac{1}{n} \qquad (3.6.2)$$

and

$$f(x_n) \leq f(x) + \frac{1}{n}\|x - x_n\| \quad \forall x \in C.$$

This later one ensures that x_n is a local minimum of the function

$$x \mapsto (1+\frac{1}{n})d_C(x) + \|x-w\| + \frac{1}{n}\|x-x_n\|$$

and hence

$$0 \in (1+\frac{1}{n})\partial d_C(x_n) + \partial\|\cdot - w\|(x_n) + \frac{1}{n}\partial\|\cdot - x_n\|(x_n).$$

Since $x_n \neq w$, there exists $x_n^* \in \partial\|\cdot - w\|(x_n)$ and $b_n^* \in \frac{1}{n}\partial\|\cdot - x_n\|(x_n)$ such that

$$\|x_n^*\| = 1, \quad \langle x_n^*, x_n - w\rangle = \|x_n - w\|, \quad -\frac{x_n^* + b_n^*}{1+\frac{1}{n}} \in \partial d_C(x_n) = N_c(C; x_n).$$

By T_c-pseudoconvexity, we get

$$\langle -\frac{x_n^* + b_n^*}{1+\frac{1}{n}}, x - x_n\rangle \leq 0 \quad \forall x \in C$$

or equivalently

$$\langle -\frac{x_n^* + b_n^*}{1+\frac{1}{n}}, w - x_n\rangle \leq \langle \frac{x_n^* + b_n^*}{1+\frac{1}{n}}, x - w\rangle \quad \forall x \in C. \tag{3.6.3}$$

Remark that

$$\begin{aligned}\langle -\frac{x_n^* + b_n^*}{1+\frac{1}{n}}, w - x_n\rangle &= \frac{1}{1+\frac{1}{n}}[\langle -x_n^*, w - x_n\rangle + \langle -b_n^*, w - x_n\rangle] \\ &= \frac{1}{1+\frac{1}{n}}\|x_n - w\| + \frac{1}{1+\frac{1}{n}}\langle -b_n^*, w - x_n\rangle \\ &\geq \frac{1}{1+\frac{1}{n}}d_C(w) + \frac{1}{1+\frac{1}{n}}\langle -b_n^*, w - x_n\rangle\end{aligned}$$

and, by (3.6.1) and (3.6.2), $\langle -b_n^*, w - x_n\rangle \to 0$. Thus extracting subnet, we may assume that $x_n^* \xrightarrow{w^*} x^*$, with $\|x^*\| \leq 1$, and, by relation (3.6.3), we obtain

$$d_C(w) \leq \langle x^*, x - w\rangle \quad \forall x \in C.$$

In particular this later one holds for $x = u$ and $x = v$, and hence on all the segment $[u, v]$ and particularly for $x = w$. Thus $d_C(w) \leq 0$ and the closeness of C ensures that $w \in C$ and this is in contradiction with $]u, w[\cap C = \emptyset$.□

Here we give another result in terms of the T_β-pseudoconvexity.

Theorem 3.6.2 *Let X be a Banach space and β be a bornology on X. If $X \times X$ is ∂_β-trusted then*

a closed set $C \subset X$ is T_β-pseudoconvex (if and) only if it is convex.

Proof. If C is T_β-pseudoconvex then

$$C - x \subset T_\beta(C; x), \quad \forall x \in C,$$

and hence by Theorem 3.4.1

$$C - x = \lim_{x' \to x} (C - x') \subset \liminf_{x' \xrightarrow{C} x} T_\beta(C; x') \subset T_c(C; x),$$

and therefore by Theorem 3.6.1 C is convex. □

Using Propositions 3.2.3 and 3.2.4, we obtain the following corollaries.

Corollary 3.6.3 *Let X be a Banach space and β be a bornology on X. If there is on X a β-differentiable Lipschitz bump function, then*

C is T_β-pseudoconvex (if and) only if C is convex.

Corollary 3.6.4 *Assume that X is an Asplund space and C is a closed subset of X. Then*

C is $\overline{\text{co}}WK$-pseudoconvex (if and) only if C is convex.

Proof. It follows from Theorem 3.6.2 and Proposition 3.2.3. □

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Chapitre 4

New conditions ensuring the convexity of Chebyshev sets

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Abstract. We investigate the convexity of Chebyshev sets. It is well known that in a smooth reflexive Banach space with the Kadec-Klee property every weakly closed Chebyshev subset is convex. We prove that the condition of the weak closedness can be replaced by the local weak closedness, that is, for any $x \in C$ there is $\epsilon > 0$ such that $C \cap \mathbb{B}(x, \varepsilon)$ is weakly closed. We also prove that the Kadec-Klee property is not required when the Chebyshev set is represented by a finite union of closed convex sets.

4.1 Introduction

Let C be a nonempty subset of a Banach space $(X, \| \ \|)$. The *metric projection* (or *set of nearest points*) of x onto C is defined by :

$$P_C(x) = \{y \in C : \|x - y\| = d_C(x)\},$$

where $d_C(\cdot)$ is the *distance function*, i.e., $d_C(x) = \inf\{\|x-y\| : y \in C\}$. We say that C is *Chebyshev* if $P_C(x)$ is a singleton for all $x \in X$. It is easy to see that Chebyshev sets are strongly closed. The first positive result for the convexity of Chebyshev sets was established, in Euclidean finite dimensional spaces, independently by Blunt [4] and Motzkin [11]. Later, in [10, 13] it was shown that every Chebyshev subset of a smooth, finite-dimensional normed linear space is convex. One of the most famous unsolved problems in approximation theory is : whether in a smooth reflexive Banach space (or even in a Hilbert space) every Chebyshev set is convex ? Although this problem is open (see [3] and [2] a recent survey), several sufficient conditions for a Chebyshev set to be convex have been obtained, until now. Here is a first important result :

Theorem 4.1.1 *(Vlasov [15]) Let X be a Banach space with rotund dual. Then any Chebyshev subset of X with continuous metric projection is convex.*

This theorem was previously obtained by Asplund [1] in Hilbert spaces.

Assume now that C is a weakly closed Chebyshev set of a reflexive Banach space X. Consider any $x \in X$ and any sequence $(x_n)_n$ in X converging to x and note that

$$\|x_n - P_C(x_n)\| = d_C(x_n) \to d_C(x) = \|x - P_C(x)\|. \tag{4.1.1}$$

This tells us in particular that the sequence $(P_C(x_n))_n$ is bounded, hence it admits a subsequence (that we do not relabel) converging weakly to some $y \in C$ according to the reflexivity of X and to the weak closedness of C. Using (4.1.1) and the weak lower semicontinuity of $\|\cdot\|$, we see that $\|x-y\| \leq d_C(x)$, and hence $y = P_C(x)$. This yields the following result (see also [5, p. 193]) :

Theorem 4.1.2 *Let X be a reflexive Banach space with the Kadec-Klee property. Then any weakly closed Chebyshev subset of X has continuous metric projection.*

In this paper we consider two sufficient conditions for Chebyshev set to be convex. First, we look at the *local weak closedness*, in the sense that for any $x \in C$ there is $\epsilon > 0$ such that $C \cap \mathbb{B}(x, \varepsilon)$ is weakly closed, $\mathbb{B}(x, \varepsilon)$ denotes the closed ball centered at x with radius ε. We prove that any locally weakly closed Chebyshev subset of a reflexive Banach space with the Kadec-Klee property has continuous metric projection. As a corollary we derive that any locally weakly closed Chebyshev subset of a smooth reflexive Banach space with the Kadec-Klee property is convex. Second, we look at the Chebyshev sets which can be represented as a finite union of closed convex sets. We proved that they are convex in a smooth reflexive Banach space. The interest of the latter result is that the Kadec-Klee property is not required.

4.2 Notation and Preliminaries

Let X be a normed space with a given norm $\|\cdot\|$, X^* be its topological dual and $\langle\cdot,\cdot\rangle$ be the duality pairing between X and X^*. A real-valued function f on X is *Gâteaux differentiable* at x if there is $x^* \in X^*$ such that

$$\forall h \in X, \quad \lim_{t\to 0^+} t^{-1}\big(f(x+th) - f(x)\big) = \langle x^*, h\rangle\,.$$

If the limit in the definition of Gâteaux differentiability exists uniformly in h on the unit sphere of X, we say that f is *Fréchet differentiable* at x.

The normed space $(X, \|\cdot\|)$
(i) is *rotund or strictly convex* whenever for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ one has $\|\frac{x+y}{2}\| < 1$,
(ii) has the (sequential) *Kadec-Klee property* provided the weak convergence of a sequence of the unit sphere of the space is equivalent to the norm convergence of this sequence,
(iii) is *smooth* (or has *Gâteaux differentiable* norm) if the norm $\|\cdot\|$ is Gâteaux differentiable off zero (or equivalently, on the unit sphere of X),
(iv) has *Fréchet differentiable* norm if the norm $\|\cdot\|$ is Fréchet differentiable off zero.

Let C be a closed subset of the normed space X. The set C is *connected* if there are no disjoint nonempty open sets A, B such that $C \subset A \cup B$ and $A \cap C \neq \emptyset$, $B \cap C \neq \emptyset$.
Recall that a sequence $(y_n)_n$ from C is a *minimizing sequence* for x if

$$\|x - y_n\| \to d_C(x).$$

Recall also that the metric projection P_C is said to be continuous at $x \in X$ provided P_C is single-valued at x and $y_n \to P_C(x)$ whenever $x_n \to x$ and $y_n \in P_C(x_n)$. If X is strictly convex, then $y \in P_C(x)$ and $z \in]y, x[$ ensure $P_C(z) = \{y\}$. The set C is a *sun* if, for each point $x \in X$ and $y \in P_C(x)$, every point on the ray $y + \mathbb{R}_+(x - y)$ has y as a nearest point in C, where $\mathbb{R}_+ := [0, +\infty)$. This notion was introduced by Klee [8, 9] and studied by Efimov, Steckin and Vlasov [7, 13, 14]. It is not difficult to see that every convex set is a sun. Indeed, let $x \in X$, $y \in P_C(x)$ and $\lambda > 0$, then for all $z \in C$

$$\begin{aligned}\|y + \lambda(x-y) - y\| &= \lambda\|x - y\| \\ &\leq \lambda\|x - \left(\frac{1}{\lambda}z + (1 - \frac{1}{\lambda})y\right)\| = \|y + \lambda(x-y) - z\|,\end{aligned}$$

thus $y \in P_C\big(y + \lambda(x - y)\big)$.

Klee [8] proved that in a finite-dimensional Euclidean space, sun sets are convex. There are some generalizations of this result to infinite dimensional spaces. The following Vlasov [14] result is the most general one.

Theorem 4.2.1 *Let X be a smooth Banach space. Then every proximinal sun subset of X is convex.*

Recall that the set C is *proximinal* if for every $x \notin C$ the set $P_C(x)$ is not empty.

To end up this section, we denote by $x_n \xrightarrow[n\to\infty]{w} x$ the weak convergence of the sequence $(x_n)_n \subset X$ to $x \in X$.

4.3 The case of locally weakly closed Chebyshev set

We announce our main result of this section :

Theorem 4.3.1 *Let X be a reflexive Banach space with the Kadec-Klee property. Let C be a locally weakly closed Chebyshev subset of X. Then P_C is continuous.*

To study the relationship between properties of a Chebyshev set C of X and its metric projection, Wulbert [17] introduced the notion of bounded connectedness : a subset of X is called boundedly connected if its intersection with every open ball in X is a connected set. To prove Theorem 4.3.1 we will use the following result on bounded connectedness of a Chebyshev subset.

Theorem 4.3.2 *(Tsarkov [12]) Let X be a reflexive Banach space with the Kadec-Klee property. Then every Chebyshev subset of X is boundedly connected.*

Proof of Theorem 4.3.1 Let $x \in X \backslash C$ and $P_C(x) = \{y\}$. From the local weak closedness there is $\varepsilon > 0$ such that $\mathbb{B}(y, \varepsilon) \cap C$ is weakly closed or equivalently weakly compact. Let $(x_n)_n$ be any sequence of $X \setminus C$ converging to x, and put $y_n := P_C(x_n)$; note that

$$\|x - y_n\| \to d_C(x) \quad \text{since } \|x_n - y_n\| = d_C(x_n) \to d_C(x).$$

We want to show that the sequence $(y_n)_n$ converges to y. Suppose the contrary, that is, without loss of generality there is some real $\delta \in]0, \varepsilon[$ such that

$$\|y_n - y\| > \delta, \quad \forall n \in \mathbb{N}.$$

Put

$$\alpha_n := 2\big(\|x - y_n\| - d_C(x)\big) > 0 \ \text{ and } \ A_n := \operatorname{int}\Big(\mathbb{B}\big(x, d_C(x) + \alpha_n\big)\Big) \cap C,$$

(where $\operatorname{int}(K)$ denote the interior of a set K). By Theorem 4.3.2 the set A_n is connected and obviously $y, y_n \in A_n$. We define two open disjoint sets B_1 and B_2 as follows :

$$B_1 = \Big\{z \in X : \|z - y\| < \frac{\delta}{2}\Big\} \ \text{ and } \ B_2 = \Big\{z \in X : \|z - y\| > \delta\Big\}.$$

It is evident that $y \in B_1 \cap A_n$ and $y_n \in B_2 \cap A_n$. Therefore by the connectedness of A_n there is $z_n \in A_n$ such that

$$z_n \notin B_1 \cup B_2$$

and thus

$$\frac{\delta}{2} \leq \|y - z_n\| \leq \delta < \varepsilon. \tag{4.3.1}$$

We deduce that $z_n \in \mathbb{B}(y, \varepsilon) \cap C$ for every $n \in \mathbb{N}$. By weak compactness of $\mathbb{B}(y, \varepsilon) \cap C$ there is $\bar{z} \in C$ such that some subsequence of $(z_n)_n$ (that we do not relabel) converges weakly to $\bar{z}$. Therefore

$$\begin{aligned} d_C(x) \leq \|x - \bar{z}\| &\leq \liminf_{n\to\infty} \|x - z_n\| \leq \limsup_{n\to\infty} \|x - z_n\| \\ &\leq \limsup_{n\to\infty} \big(d_C(x) + \alpha_n\big) = d_C(x). \end{aligned}$$

Finally we obtain that $\bar{z} \in P_C(x)$ and thus $\bar{z} = y$ (since C is Chebyshev) and

$$x - z_n \xrightarrow[n\to\infty]{w} x - y \ \text{ and } \ \|x - z_n\| \xrightarrow[n\to\infty]{} \|x - y\|,$$

which by the Kadec-Klee property implies that $z_n \to y$. This is in contradiction with (4.3.1) and the proof is completed. □

Theorem 4.3.3 *Let X be a smooth reflexive Banach space with the Kadec-Klee property. Then every locally weakly closed Chebyshev set is convex.*

Proof. It is well known that every smooth reflexive Banach space has rotund dual. Then Theorem 4.1.1 and Theorem 4.3.1 together imply the convexity of any locally weakly closed Chebyshev set of X. □

Remark. After I have completed this work, I received a very interesting paper by D. Zagrodny [16] dealing with the convexity of Chebyshev sets by using the local approximate weak compactness notion. A set $C \subset X$ is called locally approximately weakly compact if for every $u \notin C$ and $\overline{s} \in \mathrm{cl}C$ there is $\delta > 0$ such that we have the following implication

$$\left.\begin{array}{l}(s_n)_n \subset \big(\mathbb{B}(\overline{s}, \delta) \cap C\big), \\ \|s_n - u\| \to d_C(u) \text{ and } s_n \xrightarrow[n\to\infty]{w} s \in X\end{array}\right\} \Longrightarrow s \in C. \tag{4.3.2}$$

This notion is slightly weaker than the local weak closedness. Nevertheless Theorem 4.3.1 and Theorem 4.3.3 remain true if we replace condition of the local weak closedness by the local approximate weak compactness condition. Indeed within the proof of Theorem 4.3.1 we use the local weak closedness to ensure that the weak limit of the sequence $(z_n)_n$ belongs to C. This will be provided by assuming C is locally approximately weakly compact since $(z_n)_n$, x and y satisfy the assumptions on the left-hand side of the implication (4.3.2). In the framework of Hilbert spaces, Zagrodny's proof is completely different from the present one.

4.4 The case of the finite union of closed convex sets

The union of finitely many closed convex sets being weakly closed, we see in a smooth reflexive Banach space X with the Kadec-Klee property that a subset C of X is convex whenever $C = \bigcup_{i=1}^{n} C_i$ where C_i are closed convex sets. Our aim in this section is to remove for such a set C the Kadec-Klee assumption of the norm.

We start with some properties of sets which can be represented as a finite union of closed convex sets.

Proposition 4.4.1 *Let X be a normed space and let $C = \bigcup_{i=1}^{m} C_i$ be a union of finitely many closed subsets of X. Then for any $x \in X$,* *(a)* $P_C(x) = \bigcup_{i \in J} P_{C_i}(x)$, *where* $J = \{i : 1 \leq i \leq m,\ d_C(x) = d_{C_i}(x)\}$,

(b) there is $\delta > 0$ *such that for all* $u \in \mathbb{B}(x, \delta)$

$$\max_{i \in J} d_{C_i}(u) < \min_{i \in J^c} d_{C_i}(u),$$

where $J^c = \{1, 2, ..., m\} \backslash J$.

Proof. (a) It is evident that $d_C(x) = \min_{1 \leq i \leq m} d_{C_i}(x)$ and therefore $J \neq \emptyset$. Let $j \in J$ and $z \in P_{C_j}(x)$. By the definition of J and C we have

$$d_C(x) = d_{C_j}(x) = \|x - z\| \text{ and } z \in C,$$

which means that $z \in P_C(x)$. Now let $z \in P_C(x)$, then $z \in C = \bigcup_{i=1}^{m} C_i$ and consequently $z \in C_j$ for some j, $1 \leq j \leq m$. We deduce that

$$d_C(x) = \min_{1 \leq i \leq m} d_{C_i}(x) \leq d_{C_j}(x) \leq \|x - z\| = d_C(x),$$

and thus $j \in J$ and $z \in P_{C_j}(x)$.

(b) By the definition of J we have that

$$\max_{i \in J} d_{C_i}(x) = d_C(x) < \min_{i \in J^c} d_{C_i}(x). \quad (4.4.1)$$

The continuity of $u \mapsto \max_{i \in J} d_{C_i}(u)$ and $u \mapsto \min_{i \in J^c} d_{C_i}(u)$ and (4.4.1) ensure the existence of $\delta > 0$ satisfying

$$\max_{i \in J} d_{C_i}(u) < \min_{i \in J^c} d_{C_i}(u), \quad \forall u \in \mathbb{B}(x, \delta).$$

□

Theorem 4.4.2 *Let* X *be a smooth reflexive Banach space. Let* C *be a Chebyshev subset of* X *with* $C = \bigcup_{i=1}^{m} C_i$ *where* C_i *are closed convex sets. Then* C *is convex.*

Proof. By Theorem 4.2.1 it is sufficient to show that C is sun. Let us prove the sun property of C. Suppose that $x \notin C$ and $P_C(x) = y$. Put

$$\sigma = \sup\{t \geq 0 : y = P_C(q_t)\},$$

where $q_t = y + t(x - y)$. We want to show that $\sigma = +\infty$. Suppose that $\sigma < +\infty$. Then we have

$$d_C(q_\sigma) = \lim_{t \nearrow \sigma} d_C(y + t(x - y)) = \lim_{t \nearrow \sigma} \|y + t(x - y) - y\| = \|q_\sigma - y\|,$$

that is $y \in P_C(q_\sigma)$ and therefore $P_C(q_\sigma) = y$. Let J and J^c denote as in Proposition 4.4.1, $J = \{i : 1 \leq i \leq m,\ d_C(q_\sigma) = d_{C_i}(q_\sigma)\}$ and $J^c = \{1, 2, ..., m\} \backslash J$. Then, by Proposition 4.4.1, we have

$$P_C(q_\sigma) = \bigcup_{i \in J} P_{C_i}(q_\sigma) \tag{4.4.2}$$

and there is $\delta > 0$ such that for all $u \in \mathbb{B}(q_\sigma, \delta)$

$$\max_{i \in J} d_{C_i}(u) < \min_{i \in J^c} d_{C_i}(u). \tag{4.4.3}$$

By the non-vacuity of $P_{C_i}(x)$ we get from (4.4.2) that

$$P_{C_i}(q_\sigma) = y \quad \text{for all} \quad i \in J. \tag{4.4.4}$$

Let $\sigma' > \sigma$ such that $y + \sigma'(x - y) = q_{\sigma'} \in \mathbb{B}(q_\sigma, \delta)$, (4.4.3) provides

$$d_C(q_{\sigma'}) = \min_{1 \leq i \leq m} d_{C_i}(q_{\sigma'}) = \min_{i \in J} d_{C_i}(q_{\sigma'}). \tag{4.4.5}$$

As C_i is a convex and hence a sun, (4.4.4) ensures

$$d_{C_i}(q_{\sigma'}) = \|q_{\sigma'} - y\| \qquad \text{for all} \quad i \in J.$$

Finally we get that

$$d_C(q_{\sigma'}) = \min_{i \in J} d_{C_i}(q_{\sigma'}) = \|q_{\sigma'} - y\|,$$

or equivalently $y = P_C(q_{\sigma'})$. This contradicts the definition of σ and the proof is completed. □

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Index