THÈSE DE DOCTORAT DE L'ÉTABLISSEMENT UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ PREPARÉE À L'INSTITUT DE MATHÉMATIQUES DE BOURGOGNE

École doctorale n° 553 : Carnot-Pasteur

Doctorat de Mathématiques

par

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Constructing group actions on non-positively curved spaces for Dyer groups and Garside groups

Thèse présentée et soutenue à Dijon, le 22 mai 2023

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Suite aux rapports de Frédéric PAULIN et Piotr PRZYTYCKI

Résumé

Cette thèse est consacrée à la construction d'actions géométriques sur des espaces à courbure négative ou nulle. Une première partie étudie les groupes de Dyer, qui généralisent les groupes de Coxeter et les groupes d'Artin à angles droits. Nous démontrons que ces groupes sont des sous-groupes distingués d'indice fini de groupes de Coxeter. Nous construisons ensuite des actions géométriques de groupes de Dyer sur des complexes euclidiens par morceaux, qui étendent les actions de groupes de Coxeter sur les complexes de Davis-Moussong et les actions de groupes d'Artin à angles droits sur les complexes de Salvetti. Les complexes euclidiens par morceaux construits sont CAT(0). La seconde partie de cette thèse est consacrée aux complexes simpliciaux systoliques. Nous donnons une réponse à la question suivante : soit G un groupe avec présentation finie $\langle S \mid R \rangle$. Quelles sont des conditions nécessaires et suffisantes sur S pour que le complexe de drapeaux du graphe de Cayley de G soit systolique? Nous appliquons notre résultat aux groupes de Garside et aux groupes d'Artin.

Abstract

This thesis addresses the construction of geometric group actions on spaces of non-positive curvature. In a first part we study Dyer groups, which generalize Coxeter groups and right-angled Artin groups. We show that Dyer groups are finite index subgroups of Coxeter groups. We then construct geometric actions of Dyer groups on piecewise Euclidian cell complexes, which extend actions of Coxeter groups on Davis-Moussong complexes and of right-angled Artin groups on Salvetti complexes. The constructed complexes are CAT(0). The second part of this thesis is devoted to systolic simplicial complexes. We answer the following question: let G be a group with finite presentation $\langle S \mid R \rangle$. What are necessary and sufficient conditions on S that ensure systolicity of the flag complex of the Cayley graph? We apply our result to Garside groups and Artin groups.

An meine Großeltern
“Denkst du oder weißt du?”

Remerciements

Mes premiers remerciements vous reviennent Luis et Thomas. Si cette thèse est une expérience que j'ai beaucoup appreciée, c'est en partie grâce à vous, votre soutien, votre écoute et vos encouragements tout au long de cette thèse. Je vous remercie aussi pour votre bienveillance, votre patience et vos relectures attentives.

Je remercie Frédéric Paulin et Piotr Przytycki d'avoir accepté d'être rapporteurs ainsi que Anthony Genevois, Gilbert Levitt et Michele Triestino d'avoir bien voulu être membres du jury. Je remercie Indira Chatterji d'avoir fait partie de mon comité de suivi de thèse et de m'avoir soutenue au cours de ces dernières années.

Je remercie aussi Marc Burger et Alessandra Iozzi de m'avoir accueillie à Zürich pour cette dernière année de thèse. Je n'aurais pu espérer finir en de meilleures conditions.

Merci à Anissa Bellaassali, Nadia Bader, Magali Crochot, Francis Léger et Noémie Perrin d'avoir facilité toutes les démarches administratives. Merci à tous les autres membres du labo croisés au détour d'un couloir ou après le déjeuner pour le sacro-saint café. Merci à Peggy Cénac d'avoir fait partie de mon comité de suivi. J'ai été bien entourée à l'IMB, c'est aussi grâce à vous.

Merci à l'ensemble des doctorants et post-docs de Dijon. Merci à Ajinkhia, Quentin, Pablo et Joao pour le tableau et le radiateur partagé, Nicolas pour ton écoute, Inti and his back, Polina et Cindy pour le café, Jacques et Ioannis pour la musique, Oscar and his hat, Eddy, Hashem, Lamis, Pierre-Alexandre, Clémentine, Helal, Sion, Meriem, Federica, Gabriella, Alan et Vladimiro. Merci pour les déjeuners dans l'herbe et les stocks de café dans les bureaux.

Merci aussi aux doctorants de Montpellier, je n'aurais pu espérer un meilleur accueil. Merci à Marien, Paul et Camille pour la place dans le bureau, Tanguy, Juliette pour le gouter et la belote, Aurelio pour les pâtes, Tanguy, Pablo, Victor, Thibault, Morgane, Thiziri, Tom, Julien, Hermès, Emmanuel pour ta bonne humeur, Nathan pour les dîners, la musique et la didactique. Merci à vous pour les déjeuners au soleil, la belote qui s'éternise et les meilleurs goûters.

Thank you to everyone who welcomed me in Zürich. Thanks to Paula and Xenia for your laugh, Matt, Francesco and Benjamin for welcoming me into your office, Martina, Lauro, Luca, Raphael, and Victor. Thank you for the hot chocolate, the discussions in the common room and the difficult lunch decisions.

Finally many thanks to all the people I met at various conferences. Thank you for sharing your enthusiasm about mathematics with me. All of you opened my world in more ways than I could have ever imagined.

Cette thèse n'aurait pas vu le jour sans le soutien sans faille de ma famille. Je chéri ces trois mois passé auprès de vous pendant le premier confinement. Malgré les difficultés, je suis profondément reconnaissante du temps partagé que cela nous a accordé. Merci pour les cafés sur la terrasse, le porto à l'apéro, la musique, les promenades. Merci à Dorian d'être le premier de cordée, à Christophe pour la musique, à Pascal pour les promenades philosophiques et à Cyril pour le fond sonore. Merci à Marion pour les discussions dans la cuisine. Merci à Nicholas pour les relectures et les trop rares visites. Merci à mes parents pour leur bienveillance, leurs encouragements et leur accueil. J'ai la chance de faire partie de ceux qui ont encore les clés du paradis.

Und schliesslich, Matthias, danke.

Contents

Introduction

One approach to the study of discrete groups is the construction of properly discontinuous and cocompact actions by isometries on locally compact spaces with interesting geometric properties. We will be focusing on proper metric spaces with non-positive curvature. A classical notion of curvature is the sectional curvature of a Riemannian manifold. Alexandrov generalized the Riemannian curvature notion to metric spaces. Gromov popularized this notion unter the name CAT(κ)-inequality, in honour of Cartan, Alexandrov and Topogonov. A presentation of the properties of CAT(κ) spaces and of groups acting on them can be found in [BH99]. CAT(0) cube complexes have been a subject of a particular interest in this context. In [JŚ06], Januszkiewicz and Świątkowski introduced systolicity as a non-positive curvature criterion for simplicial complexes. Other notions of non-positive curvature have attracted attention recently such as weakly modular graphs [CCHO20], Helly graphs [HO21] or injective metric spaces [Lan13]. However the work in this thesis focuses in a first part on the construction of geometric actions of Dyer groups on CAT(0) spaces and in a second part on the construction of geometric actions of Garside groups on systolic complexes.

Let X be a geodesic metric space. A geodesic triangle T in X consists of the union of three points $x, y, z \in X$, its vertices, and three geodesic segments $[x, y]$, $[y, z]$ and $[x, z]$ joining them. A geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with vertices $\overline{x}, \overline{y}, \overline{z} \in \mathbb{R}^2$ satisfying $d(\overline{x}, \overline{y}) = d(x, y)$, $d(\overline{y}, \overline{z}) = d(y, z)$ and $d(\overline{x}, \overline{z}) = d(x, z)$, is called a comparison triangle for T. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is a comparison point for $p \in [x, y]$ if $d(\overline{p}, \overline{x}) = d(p, x)$. Comparison points on $[\overline{y}, \overline{z}]$ and $[\overline{x}, \overline{z}]$ are defined similarly. The triangle T satisfies the CAT(0) inequality if for all $p, q \in T$ and all comparison points $\overline{p}, \overline{q} \in \overline{T}$ we have $d(p, q) \leq d(\overline{p}, \overline{q})$. The geodesic metric space X is said to be CAT(0) if all geodesic triangles satisfy the CAT(0) inequality. In Chapter 1 we will also define CAT(κ) spaces for any real number κ. The other curvature related notion of interest in this thesis are systolic complexes. A simplicial complex X is *systolic* if it is simply connected and the link of every vertex is 6-large, where 6-large means it is a flag complex and all cycles of length 4 or 5 have diagonals. More generally one can consider k-systolic complexes for any $k \in \mathbb{N}$. A group G is said to be CAT(0), resp. systolic, if it acts geometrically on a proper CAT(0) space, resp. a proper systolic complex. However the two notions are not equivalent, as we will see in Section 1.3.

The first class of groups studied in this thesis are Dyer groups. In his study of reflection subgroups of Coxeter groups [Dye90], Dyer introduces a family of groups which contains both Coxeter groups and graph products of cyclic groups. A close study of [Dye90] also implies that this class of groups, which we call Dyer groups, has the same solution to the word problem as Coxeter groups, given by Tits [Tit69], and graph products of cyclic groups, given by Green [Gre90]. Similarly to Coxeter groups and right-angled Artin groups, the presentation of a Dyer group can be encoded in a graph. Consider a simplicial graph Γ with set of vertices $V = V(\Gamma)$ and set of edges $E = E(\Gamma)$, a vertex labeling $f : V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ and an edge labeling $m : E \to \mathbb{N}_{\geq 2}$. We say that the triple (Γ, f, m) is a Dyer graph if for every edge $e = \{v, w\}$ with $f(v) \geq 3$ we have $m(e) = 2$. The associated Dyer group $D = D(\Gamma, f, m)$ is given by the following presentation

$$D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E \rangle,$$

where $[a, b]_k = \underbrace{aba\ldots}_{k}$ for any $a, b \in D$, $k \in \mathbb{N}$ and we denote the identity with $\mathbf{e}$. So one can see that in particular Coxeter groups and right-angled Artin groups are examples of Dyer groups. It is natural to ask the following question. Consider a property $\mathcal{P}$ satisfied by both Coxeter groups and graph products of cyclic groups. Do Dyer groups also satisfy $\mathcal{P}$? There is extensive literature on Coxeter groups as well as on right-angled Artin groups and graph products of cyclic groups. In [DJ00] Davis and Januszkiewicz show that right-angled Artin groups are finite index subgroups of right-angled Coxeter groups. For a right-angled Artin group A, they give right-angled Coxeter groups W and W' where W' and A are both finite index subgroups of W and moreover the cubical complexes corresponding to A and W' are identical. Inspired by this work, we consider the following question: Are Dyer groups finite index subgroups of Coxeter groups? Out of a Dyer graph (Γ, f, m), we build a Dyer graph (Λ, g, n). By construction, we have $g(v) = 2$ for all $v \in V(\Lambda)$, so $D(\Lambda, g, n)$ is a Coxeter group, which we denote by $W(\Lambda)$. We prove the following statement.

Theorem (Theorem 3.1.8)**.** *We have* $W(\Lambda) \cong D(\Gamma, f, m) \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^k$ *for some determined* $k \in \mathbb{N}$ *and action* $\xi : (\mathbb{Z}/2\mathbb{Z})^k \times D(\Gamma, f, m) \to D(\Gamma, f, m)$.

The next corollary is a direct consequence.

Corollary (Corollary 3.1.9)**.** *Every Dyer group is a finite index subgroup of some Coxeter group.*

This has many interesting consequences, among others it implies that Dyer groups are CAT(0) [Dav08, Theorem 12.3.3], linear [Bou81, Corollary 2], and biautomatic [OP22]. This is the starting point for a more precise study of their geometry. Coxeter groups are known to act geometrically by isometries on the Davis-Moussong complex, right-angled Artin groups are known to act geometrically by isometries on the Salvetti complex. Moreover graph products of cyclic groups are known to be CAT(0) by [Gen17, Theorem 8.20]. The aim is to construct an analog of the Davis-Moussong and Salvetti complexes for Dyer groups and by way of the construction give a unified description of them. The piecewise Euclidean cell complex Σ associated with a Dyer group D is constructed as follows. One considers an appropriate simple category without loops $\mathcal{X}$ and a complex of groups $\mathfrak{D}(\mathcal{X})$. The development $\mathcal{C}$ of $\mathfrak{D}(\mathcal{X})$ will then encode the necessary information in order to build Σ.

Theorem (Theorem 3.2.22)**.** *The cell complex* Σ *is* CAT(0)*.*

The second class of groups studied here are Garside groups. They were introduced by Dehornoy and Paris in [DP99] as a generalization of spherical Artin groups. The Garside structure of a Garside group naturally gives a presentation leading to a simplicial Cayley graph, we call this the Garside presentation of a Garside group. So it is natural to ask when the flag complex of the Cayley graph with respect to the Garside presentation is systolic. More generally, any group endowed with a finite generating set acts geometrically on its Cayley graph. Can we give conditions on a group presentation which ensure that the flag complex of its Cayley graph is well-defined and systolic? We introduce the notion of a restricted triangular presentation. We say that a presentation $\langle S \mid R \rangle$ of some group G is a restricted triangular presentation if $S \cap S^{-1} = \emptyset$, $R = \{a \cdot b \cdot c^{-1} \mid a, b, c \in S \text{ and } abc^{-1} = \mathbf{e} \text{ in } G\}$ and for $a, b, c \in S$ $abc \in S$ implies $ab, bc \in S$. Note that the Garside presentation of a Garside group is a restricted triangular presentation. We have the following result.

Theorem. *(Theorem 4.2.5) Consider a group G endowed with a finite generating set S, where G has a finite restricted triangular presentation with respect to S. Then the complex* $\mathrm{Flag}(G, S)$ *is a simply connected simplicial complex. It is systolic if and only if the generating set S satisfies the following conditions:*

1) *If there exists $u, w, a, b, c, d \in S$, $u \neq w$, $a \neq d$, with $ua = wb \in S$ and $ud = wc \in S$, then there exists $k \in S$ such that $w = uk$ or $ua = udk$ or $u = wk$ or $ud = uak$.*

2) If there exist $v, x, a, b, c, d \in S$, $v \neq x$, $a \neq b$, with $bv = cx \in S$ and $av = dx \in S$, then there exists $k \in S$ such that $v = kx$ or $av = kbv$ or $x = kv$ or $kav = bv$.

3) If there exist $u, v, x, b, c \in S$, $v \neq x$, with $ux \in S$, $uv \in S$ and $vb = xc \in S$, then there exists $k \in S$ such that $k = uvb$ or $v = xk$ or $x = vk$.

4) If there exist $v, w, x, a, d \in S$, $v \neq x$, with $vw \in S$, $xw \in S$ and $dx = av \in S$, then there exists $k \in S$ such that $k = avw$ or $x = kv$ or $v = kx$.

5) If there exist $u, v, w, x \in S$, $v \neq x$, $u \neq w$, with $wv \in S$, $wx \in S$, $uv \in S$ and $ux \in S$, then there exists $k \in S$ such that $w = ku$ or $x = vk$ or $u = kw$ or $v = xk$.

Moreover, this implies that G is a systolic group.

An important class of examples are the following Garside groups. Consider the following definitions. Let $x_1, \dots, x_n$ be n letters and let m be a positive integer. We define

$$\operatorname{prod}(x_1, \dots, x_n; m) = \underbrace{x_1 x_2 \dots x_n x_1 x_2 \dots}_{m}$$

and $\operatorname{prod}(x_1, \dots, x_n; 0) = \mathbf{e}$. Consider the group

$$G_{n,m} = \langle x_1, \dots, x_n \mid \operatorname{prod}(x_1, \dots, x_n; m) = \operatorname{prod}(x_2, \dots, x_n, x_1; m) = \dots \\ = \operatorname{prod}(x_n, x_1, \dots, x_{n-1}; m) \rangle.$$

Consider the following set of generators

$$S_{n,m} = \{\operatorname{prod}(x_i, \dots, x_{i+n}; k) \mid 1 \leq i \leq n, 1 \leq k \leq m\} \subset G,$$

where we consider all indices modulo n. Moreover let

$$\Delta_{n,m} = \operatorname{prod}(x_1, \dots, x_n; m) \in G_{n,m}.$$

By Theorem 4.2.5, $\operatorname{Flag}(G_{n,m}, S_{n,m})$ is systolic. Applying Theorem 4.2.5 to the Garside presentation of a Garside group gives the following characterization.

Theorem. *(Theorem 4.3.10) Let G be a Garside group of finite type. Then G has a systolic Garside presentation if and only if $G \cong (*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$ for some positive integers $p, n_1, \dots, n_p$ and $m_1, \dots, m_p$.*

We also apply Theorem 4.2.5 to Artin groups. In order to state the next result, we require a few definitions. An *orientation* on a simplicial graph Γ is an

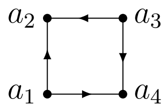


Figure 1: a misdirected 4-cycle

assignment $o(e)$ for each edge $e \in E(\Gamma)$ where $o(e)$ is a set of one or two endpoints of e. An edge with both endpoints assigned is *bioriented.* The startpoint $i(e)$ is an assignment of one or two startpoints of e. If $o(e)$ consists of one point, $i(e)$ consists of one point such that $e = (i(e), o(e))$, if $o(e)$ consists of two points then so does $i(e)$. So the startpoint $i(e)$ is consistent with the choice of $o(e)$. We say that a cycle γ is *directed* if for each $v \in \gamma$ there is exactly one edge $e \in \gamma$ with $v \in o(e)$. A cycle is *undirected* if it is not directed. We say that a 4-cycle $\gamma = (a_1, a_2, a_3, a_4)$ is *misdirected* if $a_2 \in o(a_1, a_2), a_2 \in o(a_2, a_3), a_4 \in o(a_3, a_4)$ and $a_4 \in o(a_4, a_1)$, see Figure 1.

Given a finite labeled simplicial graph Γ, the *Artin group associated with* Γ is given by

$$\begin{aligned} A_\Gamma = \langle s_v, v \in V \mid [s_v s_w s_v \dots]_{m_e} = [s_w s_v s_w \dots]_{m_e} \\ \text{for all edges } e = (v, w) \text{ with label } m_e \rangle, \end{aligned}$$

where as above $[xyx\dots]_k = \underbrace{xyx\dots}_{k}$ for $k \in \mathbb{N}$.

Theorem. *(Theorem 4.3.21) Let Γ be a simplicial graph, with edges labeled by numbers ≥ 2 and with an orientation o such that an edge is bioriented if and only if it has label 2. Assume that every 3-cycle is directed and no 4-cycle is misdirected. Let A_Γ be the Artin group associated with Γ. Then there exists a presentation of A_Γ which is systolic.*

This manuscript is structured as follows. Chapter 1 is dedicated to the two notions of non-positive curvature we will consider. We define the more general notions of $\mathrm{CAT}(\kappa)$ spaces for $\kappa \in \mathbb{R}$ and k-systolic simplicial complexes for $k \in \mathbb{N}$. We then concentrate on $\mathrm{CAT}(0)$ spaces and systolic complexes and mention some important results as well as examples. We finally compare the two notions. Indeed the geometric realization of a two-dimensional equilateral triangle complex is known to be $\mathrm{CAT}(0)$ if and only if the simplicial complex is systolic. We present known counter-examples to such a statement in higher dimensions. In Chapter 2, we provide necessary background on simple categories without loop (scwols) and complexes of groups. We also present the contruction of actions of Coxeter groups on their Davis-Moussong complexes, as well as actions of right-angled Artin groups on their Salvetti complexes. Chapter 3 is devoted to Dyer groups. We first recall the definition and show Theorem 3.1.8 mentioned above. Then we construct a cell complex Σ associated with any given Dyer group, generalizing the Davis-Moussong complex. In Chapter 4, we go back to simplicial non-positive curvature and show Theorems 4.2.5, 4.3.10 and 4.3.21.

Chapter 1

Non-positive curvature

In this chapter we recall definitions and properties of CAT(0) spaces and systolic complexes, the two notions of non-positive curvature considered in this thesis. In Section 1.1, we define CAT(0) spaces and give some of their properties. We also present Moussong's Lemma 1.1.16, which is an essential tool to show that a piecewise Euclidean cell complex is CAT(0). In Section 1.2, we define systolic complexes, mention important properties and give some examples. In Section 1.3, we compare those two notions of non-positive curvature. In particular we present examples given by Januszkiewicz and Świątkowski showing that for a simplicial complex endowed with the standard metric these two notions are not equivalent.

1.1 CAT(0) spaces

Definition 1.1.1. Let (X, d) be a metric space. A geodesic segment, or simply a geodesic, from $x \in X$ to $y \in X$ is the image of an isometric embedding c from an interval $[0, \ell] \subset \mathbb{R}$ to X such that $c(0) = x$ and $c(\ell) = y$. We also call ℓ the length of the geodesic. A metric space (X, d) is *geodesic* if any two points $x, y \in X$ can be joined by a geodesic.

Let $\kappa \in \mathbb{R}$. In order to define CAT(κ) spaces, we first need to introduce the model space M^2_κ, which denotes the (unique up to isometry) simply connected, complete, Riemannian 2-manifold of constant curvature κ. More concretely, if $\kappa = 0$, we define M^2_κ to be the Euclidean plane $\mathbb{R}^2$. If $\kappa > 0$, we define M^2_κ to be the 2-sphere $\mathbb{S}^2$ with its metric rescaled so that its curvature is κ, i.e. it is the sphere of radius $1/\sqrt{\kappa}$. If $\kappa < 0$, we define M^2_κ to be the real hyperbolic plane $\mathbb{H}^2$ with its metric rescaled, i.e. distances are multiplied by $1/\sqrt{-\kappa}$. Let D_κ be the diameter of M_κ, so it is equal to $\pi/\sqrt{\kappa}$ if $\kappa > 0$ and $+\infty$ otherwise.

Definition 1.1.2. A *geodesic triangle* T in X consists of the union of three points $x, y, z \in X$, its vertices, and three geodesic segments $[x, y]$, $[y, z]$ and $[x, z]$ joining them. The perimeter of T is $d(x, y) + d(y, z) + d(x, z)$. A geodesic triangle $\overline{T}$ in M_κ^2 with vertices $\overline{x}, \overline{y}, \overline{z} \in M_\kappa^2$ satisfying $d(\overline{x}, \overline{y}) = d(x, y)$, $d(\overline{y}, \overline{z}) = d(y, z)$ and $d(\overline{x}, \overline{z}) = d(x, z)$, is called a *comparison triangle* for T. Such a triangle always exists, if $d(x, y) + d(y, z) + d(x, z) < 2D_\kappa$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is a *comparison point* for $p \in [x, y]$ if $d(\overline{p}, \overline{x}) = d(p, x)$. Comparison points on $[\overline{y}, \overline{z}]$ and $[\overline{x}, \overline{z}]$ are defined similarly.

Definition 1.1.3. Let X be a metric space and $\kappa \in \mathbb{R}$. Let T be a geodesic triangle in X with perimeter less than $2D_\kappa$. Let $\overline{T}$ be a comparison triangle in M_κ^2. The triangle T satisfies the $\mathrm{CAT}(\kappa)$ *inequality* if for all $p, q \in T$ and all comparison points $\overline{p}, \overline{q} \in \overline{T}$, we have $d(p, q) \leq d(\overline{p}, \overline{q})$.

Definition 1.1.4. For $\kappa \leq 0$, a geodesic metric space X is said to be $\mathrm{CAT}(\kappa)$ if every geodesic triangle satisfies the $\mathrm{CAT}(\kappa)$ inequality. If $\kappa > 0$, a D_κ-geodesic space X is said to be $\mathrm{CAT}(\kappa)$ if every geodesic triangle of perimeter less than $2D_\kappa$ satisfies the $\mathrm{CAT}(\kappa)$ inequality. (Here D_κ-geodesic means that all pairs of points $x, y \in X$ with $d(x, y) < D_\kappa$ are joined by a geodesic.) A metric space X is said to be *of curvature* $\leq \kappa$ if it is locally a $\mathrm{CAT}(\kappa)$ space, i.e. if for every $x \in X$ there exists $r_x > 0$ such that the ball $B_{r_x}(x)$, endowed with the induced metric, is a $\mathrm{CAT}(\kappa)$ space. A discrete group G is $\mathrm{CAT}(\kappa)$ if it acts properly discontinuously and cocompactly on a proper $\mathrm{CAT}(\kappa)$ space.

Example 1.1.5. Convex subsets of $\mathrm{CAT}(0)$ spaces are $\mathrm{CAT}(0)$. If X and Y are $\mathrm{CAT}(0)$ spaces, then the product space $X \times Y$ endowed with the ℓ_2 metric is $\mathrm{CAT}(0)$.

Example 1.1.6. In Chapter 2 Sections 2.2 and 2.3, we will present the Davis-Moussong complex associated to a Coxeter group and the Salvetti complex associated to a right-angled Artin group, which are both examples of $\mathrm{CAT}(0)$ spaces.

Let us state some fundamental results about $\mathrm{CAT}(\kappa)$ spaces.

Proposition 1.1.7. *([BH99, Proposition II.1.4]) Let X be a $\mathrm{CAT}(\kappa)$ space. Then there is a unique geodesic segment joining each pair of points $x, y \in X$ provided that $d(x, y) < D_\kappa$, and this geodesic segment varies continuously with its endpoints. Moreover the balls of radius less than D_κ are contractible.*

Corollary 1.1.8. *([BH99, Corollary II.1.5]) For $\kappa \leq 0$, any $\mathrm{CAT}(\kappa)$ space is contractible, in particular it is simply connected and all of its homotopy groups are trivial.*

Theorem 1.1.9. *([BH99, Cartan-Hadamard Theorem II.4.1]) Let X be a complete connected metric space and $\kappa \leq 0$. If X is of curvature $\leq \kappa$, then any universal cover $\widetilde{X}$ of X is a* CAT(κ) *space.*

Some notable properties of CAT(0) groups are mentioned in the next two theorems.

Theorem 1.1.10. *([Dav08, Theorem I.4.1.]) Suppose G is a* CAT(0) *group. Then*

1. *There is a model for $\underline{E}G$ (the universal space for proper G-actions) with $\underline{E}G/G$ compact.*
2. *The group G is finitely presented.*
3. *There are only finitely many conjugacy classes of finite subgroups in G.*
4. *We have $\mathrm{cd}_{\mathbb{Q}}(G) < \infty$.*
5. *The cohomology space $H^*(G;\mathbb{Q})$ is finite dimensional.*
6. *The Word and Conjugacy Problems for G are solvable.*
7. *Any abelian subgroup of G is finitely generated.*
8. *Any virtually solvable subgroup of G is virtually abelian.*

Theorem 1.1.11. *([Dav08, Bruhat-Tits Fixed Point Theorem I.2.11.]) Let G be a group of isometries of a* CAT(0) *space X. If G is compact (or more generally if G has a bounded orbit on X), then G has a fixed point on X.*

As we will be constructing a Euclidean polyhedral complex in Chapter 3, we give some criteria for recognizing when such a complex is CAT(0).

Definition 1.1.12. Let K be a Euclidean polyhedral complex. Let v be a vertex in K. The link complex $\mathrm{Lk}(v, K)$ of v in K is the set of directions at v. It is naturally endowed with a piecewise spherical simplicial cell structure.

Definition 1.1.13. An M_κ-polyhedral complex K satisfies the link condition if for every vertex $v \in K$ the link complex $\mathrm{Lk}(v, K)$ is a CAT(1) space.

Theorem 1.1.14. *([BH99, Theorem II.5.4]) Let K be an M_κ-polyhedral complex with only finitely many isometry types of cells. If $\kappa \leq 0$, the following conditions are equivalent:*

1. *K is a* CAT(κ) *space.*

2. K is uniquely geodesic.

3. K satisfies the link condition and contains no isometrically embedded circles.

4. K is simply connected and satisfies the link condition.

$If \kappa > 0$ then the following conditions are equivalent:

1. K is a $\mathrm{CAT}(\kappa)$ space.

2. K is $(\pi/\sqrt{\kappa})$-uniquely geodesic.

3. K satisfies the link condition and contains no isometrically embedded circles of length less than $2\pi/\sqrt{\kappa}$.

The main tool used to check the link condition for Euclidean cell complexes is Moussong's Lemma.

Definition 1.1.15. A simplicial complex L with a piecewise spherical structure has simplices of size $\geq \pi/2$ if each of its edges has length $\geq \pi/2$. Such a simplicial complex is a metric flag complex if the following condition holds. Suppose the set $\{v_0, \ldots, v_k\}$ is a set of pairwise adjacent vertices of L. Put $c_{ij} = \cos(d(v_i, v_j))$. Then $\{v_0, \ldots, v_k\}$ spans a simplex if and only if the matrix $(c_{ij})_{0 \leq i,j \leq k}$ is positive definite.

Lemma 1.1.16 (Moussong's Lemma [Dav08] Lemma I.7.4.)**.** *Suppose L is a piecewise spherical simplicial cell complex in which all cells are simplices of size $\geq \pi/2$. Then L is $\mathrm{CAT}(1)$ if and only if it is a metric flag complex.*

Remark 1.1.17. Let K be a simplicial complex endowed with a piecewise Euclidean metric. Then the link $\mathrm{Lk}(v, K)$ of any vertex v is a piecewise spherical simplicial complex but not all simplices in $\mathrm{Lk}(v, K)$ are of size $\geq \pi/2$. So we cannot apply Moussong's Lemma directly to such a setting.

For cube complexes, this characterization was given earlier by Gromov.

Corollary 1.1.18. *([Dav08, Corollary I.6.3.]) A piecewise Euclidean cubical complex X is nonpositively curved if and only if the link of each of its vertices is a metric flag complex.*

In Chapter 3, we will be studying the link complex in the future piecewise Euclidean cell complex Σ. We will need the notion of spherical joins, which we define here.

Definition 1.1.19. [Dav08][Appendix I] The spherical join of two metric spaces (X, d_x) and (Y, d_y) is $X \star Y = (X \times Y \times [0, pi/2])/\sim$, where $\sim$ is defined by: $(x, y, \theta) \sim (x', y', \theta')$ whenever $\theta = \theta' = 0$ and $x = x'$ or $\theta = \theta' = 1$ and $y = y'$. We define a metric d_π^X on X by $d_\pi^X(x, x') = \min\{\pi, d_X(x, x')\}$ and a metric d_π^Y on Y by $d_\pi^Y(y, y') = \min\{\pi, d_Y(y, y')\}$.We denote the image of (x, y, θ) in $X \star Y$ by $\cos(\theta)x + \sin(\theta)y$. The distance d between points $z = \cos(\theta)x + \sin(\theta)y$ and $z' = \cos(\theta')x' + \sin(\theta')y'$ is defined by requiring it to be at most π and satisfying $\cos(d(z, z')) = \cos(\theta)\cos(\theta')\cos(d_\pi^X(x, x')) + \sin(\theta)\sin(\theta')\cos(d_\pi^Y(y, y'))$.

The following result is important to understand iterated joins.

Lemma 1.1.20. *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces. Then the product of Euclidean cones* $\mathrm{Cone}_0(X) \times \mathrm{Cone}_0(Y)$ *endowed with the product metric is isometric with* $\mathrm{Cone}_0(X \star Y)$.

1.2 Systolic complexes

For simplicial complexes, Januszkiewicz and Świątkowski introduced a notion of non-positive curvature inspired by the characterizations of CAT(0) cube complexes.

Definition 1.2.1. Let X, Y be simplicial complexes. We denote its k-skeleton by $X^{(k)}$. Then $X^{(0)}$ is the set of vertices of X. The subcomplex spanned by $A \subset X^{(0)}$ is the largest subcomplex of X which has A as its set of vertices. For a simplex $\sigma \in X$ we can define its *link in* X, $\mathrm{Lk}(\sigma, X) = \{\tau \in X \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in X\}$. The *join* $X \star Y$ of X and Y is the simplicial complex, whose simplicies are the disjoint union of simplices in X and Y so $X \star Y = \{\sigma \sqcup \tau \mid \sigma \in X, \tau \in Y\}$.

Remark 1.2.2. Let X, Y be simplicial complexes. If X and Y are each endowed with a piecewise spherical structure, their simplicial join $X \star Y$ will carry the piecewise spherical struture of the spherical join given in Definition 1.1.19. In particular the distance between two vertices $v \in X \subset X \star Y$ and $y \in Y \subset X \star Y$ is $\pi/2$. Moreover using Lemma 1.1.20, one can see that the iterated spherical join $\star_{i=1}^n X_i$ of piecewise spherical simplicial complexes X_i, $i \in \{1, \ldots, n\}$ is a well defined piecewise spherical simplicial complex. Its combinatorial structure is given by the (combinatorial) join $\star_{i=1}^n X_i$ defined in 1.2.1. Denote with d_i the metric on X_i. The distance d between two vertices u, v in $\star_{i=1}^n X_i$ is $d(u, v) = \max\{\pi, d_i(u, v)\}$ if $u, v \in X_i$ for some $i \in \{1, \ldots, n\}$ and $d(u, v) = \pi/2$ otherwise.

Definition 1.2.3. The simplicial complex X is *a flag complex* if any set of pairwise adjacent vertices spans a simplex in X.

Remark 1.2.4. A flag complex X is uniquely determined by its 1-skeleton. Moreover the link of a vertex $v \in X$ is the subcomplex spanned the set of vertices adjacent to v.

Definition 1.2.5. Let X be a simplicial complex. A *cycle* in X is the image of a simplicial map $f : S^1 \to X$ from a triangulation of the 1-sphere to X. If f is injective, the cycle is said to be embedded. Let γ be an embedded cycle in X. The length $|\gamma|$ of γ is the number of edges of γ. We say that γ is a $|\gamma|$-cycle. A *diagonal* of γ is an edge that connects two nonconsecutive vertices of γ. An embedded cycle is *diagonal-free* if there are no edges between nonconsecutive vertices. We say that two vertices v and w are adjacent if there exists an edge between v and w, we then write $v \sim w$. For every $k \in \mathbb{N}$, a simplicial complex is k*-large*, if it is flag and every embedded cycle γ with $4 \leq |\gamma| < k$ has a diagonal.

Definition 1.2.6. A simplicial complex X is systolic if it is connected, simply connected and if the links of all vertices in X are 6–large. A group G is called systolic if it acts cocompactly and properly by simplicial automorphisms on a systolic complex X. (Properly means that X is locally finite and for each compact subcomplex $K \subset X$ the set of $g \in G$ such that $g(K) \cap K \neq \emptyset$ is finite.) If the links of all vertices of X are additionally k–large with $k \geq 6$, we call it (and the group) k–systolic.

Remark 1.2.7. Januszkiewicz's and Świątkowski's original definition given in [JŚ06] of a k-systolic complex required that the links of all simplices be k-large. Moreover by [JŚ06, Proposition 1.4] a k-systolic complex with $k \geq 6$ is k-large.

Let us consider some examples of systolic complexes and groups.

Example 1.2.8. ([JŚ06, Example 1.8.]) The triangulation of the Euclidean plane by equilateral triangles is systolic. The triangulation of the hyperbolic plane by equilateral triangles with angles $2\pi/m$, $m \geq 7$ is m-systolic. The triangulation of the 2-sphere is never k-large, for any $k \geq 6$.

Remark 1.2.9. By [JŚ07, Corollary 6.4], there is no systolic simplicial subdivision of $\mathbb{R}^n$ for $n > 2$. In particular there are no systolic flats of dimension strictly greater than 2.

Remark 1.2.10. Note that the Cayley 2-complex of $\langle a, b, \Delta \mid \Delta = ab, \Delta = ba\rangle \cong \mathbb{Z}^2$ is simplicially isomorphic to the triangulation of the Euclidean plane by equilateral triangles. In particular it is systolic. In Chapter 4 we will investigate the following question: When is the flag complex of a Cayley graph systolic?

Example 1.2.11. By [EP13], the product of two free groups $\mathbb{F}_n \times \mathbb{F}_m$, $n, m \geq 2$ is systolic. More generally Elsner and Przytycki show that the fundamental group of a compact nonpositively curved $\mathcal{VH}$-complex is systolic. In particular any right-angled Artin group with bipartite defining graph is systolic.

Let us state some results about systolic complexes and groups.

Theorem 1.2.12. *[JŚ06, Theorem 4.1.1] Any finite dimensional systolic complex is contractible.*

Theorem 1.2.13. *[JŚ06, Theorem 13.1] Let G be a systolic group. Then G is biautomatic. In particular, this implies the solvability of the Word Problem and of the Conjugacy Problem and a quadratic Dehn function.*

Remark 1.2.14. In [HO20], Huang and Osajda show that Artin groups of almost large type are systolic and hence biautomatic by Theorem 1.2.13. In [HR21], Holt and Rees describe a slightly simpler variant of the biautomatic structure.

Lemma 1.2.15. *[Zad14] Finitely presented subgroups of systolic groups are systolic.*

Theorem 1.2.16. *[OP18, Proposition 5.6] The centraliser of an infinite order element in a systolic group is commensurable with $\mathbb{F}_n \times \mathbb{Z}$, where $\mathbb{F}_n$ is the free group on n generators for some $n \geq 0$.*

Theorem 1.2.17. *([HO20, Theorem 2.2]) Solvable subgroups of systolic groups are either virtually cyclic or virtually $\mathbb{Z}^2$.*

Remark 1.2.18. Lemma 1.2.15 together with Remark 1.2.9 implies that if G is a group with $\mathbb{Z}^3 < G$, then G is not systolic.

1.3 CAT(0) versus Systolic

We discuss the relationship between the CAT(0) and the k-systolic conditions. For a two-dimensional simplicial complex the connection between the CAT(0) and the systolic conditions is clear.

Lemma 1.3.1. *The geometric realization of a two-dimensional equilateral triangle complex X is* CAT(0) *if and only if the simplicial complex is systolic.*

Proof. This follows from [BH99, Lemma II.5.6] and [BH99, Proposition II.5.25]. □

In higher dimensions Januszkiewicz und Świątkowski showed that the CAT(0) and the systolic conditions are not equivalent. One can build simplicial complexes of dimension $n > 2$ that are either systolic but not CAT(0) or CAT(0) but not systolic.

Example 1.3.2 (CAT(0) but not systolic). [JŚ06, Section 14] Let $n \in \mathbb{N}$. Let Y_n be the simplicial join of an $(n-2)$-dimensional simplex σ and a 1-dimensional cycle consisting of 5 edges. The link $\mathrm{Lk}(\sigma, Y_n)$ is not 6-large so Y_n is not systolic. But the dihedral angle β_n between two codimension 1 faces in the regular n-simplex grows to $\pi/2$ as n goes to ∞. Moreover $\beta_n > 2\pi/5$ for all $n \geq 4$. So Y_n is CAT(0) but not systolic for all $n \geq 4$.

Example 1.3.3 (systolic but not CAT(0)). [JŚ06, Section 14] Let v_1, v_2, v_3 be vertices and e_1, e_2, e_3 be 1-simplices. Let X be the simplicial cone over

$$(v_1 \star e_1) \cup (e_1 \star v_2) \cup (v_2 \star e_2) \cup (e_2 \star v_3) \cup (v_3 \star e_3) \cup (e_3 \star v_1).$$

Then X is 6-systolic but the link of the cone vertex contains a closed geodesic of length $< 2\pi$. Indeed in a regular Euclidean 3-simplex the angle α between a 2-face and a 1-face is less than $\pi/3$. The construction can be extended to higher dimensions. Let $k \geq 6$, $n \in \mathbb{N}$. Let X be the simplicial cone over $\bigcup_{i \in \mathbb{Z}/k\mathbb{Z}} \sigma_i \star \sigma_{i+1}$ where each σ_j, $1 \leq j \leq k$ is an n-dimensional simplex. Then X is k-systolic but not CAT(0).

Remark 1.3.4. In [JŚ06, Theorem 16.1] Januszkiewicz and Świątkowski show that for $k \geq \frac{7\pi\sqrt{2}}{2} \cdot n + 2$, any k-systolic simplicial complex with $\dim(X) \leq n$ is CAT(0) with respect to the standard piecewise Euclidean metric. By considering $n = 2$, we obtain $k \geq 34$, hence this bound is clearly not optimal (by Lemma 1.3.1 we only need $k \geq 6$). The question whether a 6-systolic complex admits a piecewise Euclidean metric for which it is CAT(0) remains open.

Remark 1.3.5. Besides the relationship between systolic complexes and CAT(0) spaces, one can wonder about the relationship between systolic groups and CAT(0) groups. Clearly CAT(0) does not imply systolic, as evidenced by the group $\mathbb{Z}^3$ which is CAT(0) but not systolic. The question whether systolic groups are CAT(0) groups remains open.

Remark 1.3.6. Even though these two notions of non-positive curvature are not equivalent, they share many properties. For example in [Els09] Elsner gives a Systolic Flat Torus Theorem. In [Prz08] Przytycki gives a coarse fixed point result for systolic complexes.

Chapter 2

Complexes of groups

As mentioned in the introduction, Chapter 3 will be devoted to the construction of actions of Dyer groups on a Euclidean cell complex Σ and the proof of its non-positive curvature. We present the necessary background here. The construction of Σ uses simple categories without loops and complexes of groups, introduced by Haefliger in [Hae91], [Hae92]. We also develop some specific examples, used later on. These are presented in Section 2.1. The construction in Chapter 3 aims to generalize actions of Coxeter groups on the Davis-Moussong complex, presented in Section 2.2, and actions of right-angled Artin groups on the Salvetti complex, presented in Section 2.3. The material presented here also appeared in [Soe22].

2.1 Complexes of groups

Small categories without loops (scwols) and complexes of groups were introduced by Haefliger in [Hae91], [Hae92]. Based on [BH99], we would like to recall some notions about scwols and complexes of groups, as we do not assume the reader to be familiar with these constructions. We hope to give all necessary definitions and results, details can be found in [BH99].

A *small category without loops* (scwol) $\mathcal{X}$ consists of a set $V(\mathcal{X})$, called the vertex set of $\mathcal{X}$ and a set $E(\mathcal{X})$, called the set of edges of $\mathcal{X}$. Additionally two maps $i : E(\mathcal{X}) \to V(\mathcal{X})$ and $t : E(\mathcal{X}) \to V(\mathcal{X})$ are given. We call $i(\alpha)$ the initial vertex of $\alpha \in E(\mathcal{X})$ and $t(\alpha)$ the terminal vertex of $\alpha \in E(\mathcal{X})$. The set $E^{(2)}(\mathcal{X})$ denotes the pairs $(\alpha, \beta) \in E(\mathcal{X}) \times E(\mathcal{X})$ with $i(\alpha) = t(\beta)$. A third map $\circ : E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$ associates to each pair (α, β) an edge $\alpha\beta$, called their composition. These sets and maps need to satisfy the following conditions:

1. for all $(\alpha, \beta) \in E^{(2)}$, we have $i(\alpha\beta) = i(\beta)$ and $t(\alpha\beta) = t(\alpha)$,

2. for all $\alpha, \beta, \gamma \in E(\mathcal{X})$, if $i(\alpha) = t(\beta)$ and $i(\beta) = t(\gamma)$, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,

3. for each $\alpha \in E(\mathcal{X})$ we have $i(\alpha) \neq t(\alpha)$.

A subscwol $\mathcal{X}'$ of $\mathcal{X}$ is given by subsets $V(\mathcal{X}') \subset V(\mathcal{X})$ and $E(\mathcal{X}') \subset E(\mathcal{X})$ such that if $\alpha \in E(\mathcal{X}')$, then $i(\alpha), t(\alpha) \in V(\mathcal{X}')$, and if $\alpha, \beta \in E(\mathcal{X}')$ with $i(\alpha) = t(\beta)$ then $\alpha\beta \in E(\mathcal{X}')$.

Example 2.1.1. To any poset $(\mathcal{P}, <)$ we can associate a scwol $\mathcal{X}$ where the set of vertices is $\mathcal{P}$ and the set of edges are pairs $(a, b) \in \mathcal{P} \times \mathcal{P}$ such that $b < a$, $t((a, b)) = a$ and $i((a, b)) = b$.

Definition 2.1.2 (Product of scwols)**.** Given two scwols $\mathcal{X}$ and $\mathcal{Y}$, their product $\mathcal{X} \times \mathcal{Y}$ is the scwol defined as follows: $V(\mathcal{X} \times \mathcal{Y}) = V(\mathcal{X}) \times V(\mathcal{Y})$ and

$$E(\mathcal{X} \times \mathcal{Y}) = (E(\mathcal{X}) \times V(\mathcal{Y})) \bigsqcup (E(\mathcal{X}) \times E(\mathcal{Y})) \bigsqcup (V(\mathcal{X}) \times E(\mathcal{Y})) \,.$$

The maps $i, t : E(\mathcal{X} \times \mathcal{Y}) \to V(\mathcal{X} \times \mathcal{Y})$ are defined by $i(\alpha, \beta) = (i(\alpha), i(\beta))$ and $t(\alpha, \beta) = (t(\alpha), t(\beta))$ (we consider $i(v) = t(v) = v$ for any $v \in V(\mathcal{X}) \bigsqcup V(\mathcal{Y})$) the composition $(\alpha, \alpha')(\beta, \beta') = (\alpha\beta, \alpha'\beta')$ whenever $t(\beta, \beta') = i(\alpha, \alpha')$ (we consider $\alpha\beta = \alpha$ whenever $\alpha \in E(\mathcal{X}) \sqcup E(\mathcal{Y})$ and $\beta = i(\alpha) \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$, and $\alpha\beta = \beta$ whenever $\beta \in E(\mathcal{X}) \sqcup E(\mathcal{Y})$ and $\alpha = t(\beta) \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$).

Remark 2.1.3. Let $[n] = \{1, \ldots, n\}$. One can inductively define the product of n scwols $\mathcal{X}_1, \ldots, \mathcal{X}_n$. Then the product $\mathcal{X} = \Pi_{i\in[n]} \mathcal{X}_i$ is the scwol with vertices $V(\mathcal{X}) = \Pi_{i\in[n]} V(\mathcal{X}_i)$ and edges $E(\mathcal{X}) = \bigsqcup_{S\subset[n], S\neq\emptyset} (\Pi_{i\in S^c} V(\mathcal{X}_i)) \times (\Pi_{i\in S} E(\mathcal{X}_i))\,.$ The maps $i, t : E(\mathcal{X}) \to V(\mathcal{X})$ are defined by $i(\alpha) = i((\alpha_j)_{j\in[n]}) = (i(\alpha_j))_{j\in[n]}$ and $t(\alpha) = t((\alpha_j)_{j\in[n]}) = (t(\alpha_j))_{j\in[n]}$ (we consider $i(v) = t(v) = v$ for any vertex $v \in \bigsqcup_{i\in[n]} V(\mathcal{X}_i)$) and the composition $\alpha\beta = (\alpha_j)_{j\in[n]}(\beta_j)_{j\in[n]} = (\alpha_j\beta_j)_{j\in[n]}$ whenever defined.

Example 2.1.4. Consider a finite set S. Let $\mathcal{P}(S)$ be the set of all subsets of S. Consider the poset $(\mathcal{P}(S), \subset)$ and its associated scwol $\mathcal{Y}_S$. Then $\mathcal{Y}_S = \Pi_{v\in S} \mathcal{Y}_{\{v\}}$. Moreover for any $v \in S$ the scwol $\mathcal{Y}_{\{v\}}$, also denoted by $\mathcal{Y}_v$, has two vertices $\emptyset$ and $\{v\}$ and a single edge e_v with $i(e_v) = \emptyset$ and $t(e_v) = \{v\}$.

Example 2.1.5. Consider a finite set S. For $v \in S$ let $\mathcal{Z}_{\{v\}} = \mathcal{Z}_v$ be the scwol consisting of two vertices $\emptyset$ and $\{v\}$ and of two edges denoted $(\emptyset, \{v\}, \emptyset)$ and $(\emptyset, \{v\}, \{v\})$ with initial vertex $i(\emptyset, \{v\}, \emptyset) = i(\emptyset, \{v\}, \{v\}) = \emptyset$ and terminal vertex $t(\emptyset, \{v\}, \emptyset) = t(\emptyset, \{v\}, \{v\}) = \{v\}$. Let $\mathcal{Z}_S = \Pi_{v\in S} \mathcal{Z}_{\{v\}}$. Note that $V(\mathcal{Z}_S) = \mathcal{P}(S)$ and the edges can be described as $E(\mathcal{Z}_S) = \{(A, B, \lambda) \mid A \subsetneq B \subset S, \lambda \subset B \setminus A\}$, where $i(A, B, \lambda) = A$ and $t(A, B, \lambda) = B$. This example seems artificial at this point but will be quite useful later on, as the geometric realization of $\mathcal{Z}_S$ is a torus

$\mathbb{T}^S$ and its fundamental group is $\mathbb{Z}^S$. Indeed in Chapter 3 we will be particularly interested in the case where S is the vertex set of a complete Dyer graph Γ for which all vertices are labeled by ∞.

A simple complex of groups $\mathcal{G}(\mathcal{X}) = (G_v, \psi_\alpha)$ over a scwol $\mathcal{X}$ is given by the following data:

1. for each $v \in V(\mathcal{X})$, a group G_v called the local group at v,

2. for each $\alpha \in E(\mathcal{X})$ an injective homomorphism $\psi_\alpha : G_{i(\alpha)} \to G_{t(\alpha)}$, with the following compatibility condition: $\psi_{\alpha\beta} = \psi_\alpha \psi_\beta$ whenever defined.

A simple complex of groups $\mathcal{G}(\mathcal{X}) = (G_v, \psi_\alpha)$ over a scwol $\mathcal{X}$ is called *inclusive* if it additionally satisfies the following condition: for each $\alpha \in E(\mathcal{X})$ we have $G_{i(\alpha)} < G_{t(\alpha)}$ and $\psi_\alpha(g) = g$ for all $g \in G_{i(\alpha)}$. We will only be considering simple inclusive complexes of groups. These are restrictions on the more general definition of complexes of groups which can be found in [BH99].

Definition 2.1.6. The product $\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y})$ of two simple complexes of groups $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$ is the simple complex of groups over the scwol $\mathcal{X} \times \mathcal{Y}$ given by the following data:

1. for each $v = (v_1, v_2) \in V(\mathcal{X} \times \mathcal{Y})$, the local group $G_v = G_{v_1} \times G_{v_2}$ is the direct product of the corresponding local groups in $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$,

2. for each $\alpha = (\alpha_1, \alpha_2) \in E(\mathcal{X} \times \mathcal{Y})$, the injective homomorphism is $\psi_\alpha = \psi_{\alpha_1} \times \psi_{\alpha_2}$ (if one index α_i is a vertex, we set ψ_{α_i} to be the identity on G_{α_i}).

As $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$ are simple complexes of groups so is $\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y})$.

Similarly to the definition of products of scwols, this definition can be extended to finite products of simple complexes of groups. The product $\Pi_{i\in[n]}\, \mathcal{G}(\mathcal{X}_i)$ of simple complexes of groups $\mathcal{G}(\mathcal{X}_i)$, $i \in [n]$, is the simple complex of groups over the scwol $\Pi_{i\in[n]}\, \mathcal{X}_i$ given by the following data:

1. for each $v = (v_i)_{i\in[n]} \in V(\Pi_{i\in[n]}\, \mathcal{X}_i)$, the local group $G_v = \Pi_{i\in[n]} G_{v_i}$ is the direct product of the corresponding local groups in $\mathcal{G}(\mathcal{X}_i)$,

2. for each $\alpha = (\alpha_i)_{i\in[n]} \in E(\Pi_{i\in[n]}\, \mathcal{X}_i)$, the injective homomorphism is $\psi_\alpha = \Pi_{i\in[n]} \psi_{\alpha_i}$ (if an index α_i is a vertex, we set ψ_{α_i} to be the identity on G_{α_i}).

We will now give fundamental examples of complexes of groups and products of complexes of groups over the scwols introduced in Examples 2.1.4 and 2.1.5.

Example 2.1.7. We consider the scwols defined in Example 2.1.4. For every $v \in S$ we choose a positive integer p_v. Let C_v be the finite cyclic group of order p_v. For $v \in S$, let $\mathfrak{D}(\mathcal{Y}_v)$ be the simple complex of groups over $\mathcal{Y}_v$ defined by choosing $G_\emptyset = \langle \mathbf{e} \rangle$ the trivial group, $G_{\{v\}} = C_v$ and ψ_{e_v} the trivial map. We define a simple complex of groups $\mathfrak{D}(\mathcal{Y}_S)$ over $\mathcal{Y}_S$ as follows:

1. for $A \in V(\mathcal{Y}_S)$ we set $G_A = \Pi_{v \in A} C_v$,
2. for $e \in E(\mathcal{Y}_S)$ with $i(e) = A$ and $t(e) = B$ we have $A \subset B$ so $G_A < G_B$ and so there is a canonical inclusion $\psi_e : G_A \to G_B$. These inclusions satisfy the compatibility condition.

We indeed have $\mathfrak{D}(\mathcal{Y}_S) = \Pi_{v \in S} \mathfrak{D}(\mathcal{Y}_{\{v\}})$.

Example 2.1.8. For a finite Coxeter system (W, S), we can define a simple complex of groups over $\mathcal{Y}_S$ denoted by $\mathfrak{W}(\mathcal{Y}_S)$ as follows:

1. for $A \in V(\mathcal{Y}_S)$, we choose the local group to be W_A, the subgroup of W generated by A,
2. for $e \in E(\mathcal{Y}_S)$ with $i(e) = A$ and $t(e) = B$, we have $A \subset B$ so there is a canonical inclusion $\psi_e : W_A \to W_B$. These inclusions satisfy the compatibility condition.

In general in this case we have $\mathfrak{W}(\mathcal{Y}_S) \neq \Pi_{v \in S} \mathfrak{W}(\mathcal{Y}_{\{v\}})$ even though the scwols satisfy $\mathcal{Y}_S = \Pi_{v \in S} \mathcal{Y}_{\{v\}}$.

Example 2.1.9. We consider the scwols defined in Example 2.1.5. We can always define the trivial complex of groups over a scwol. The product of trivial complexes of groups will again be trivial. This leads to the following notation. For every $v \in S$, we define a simple complex of groups $\mathfrak{D}(\mathcal{Z}_v)$ over each scwol $\mathcal{Z}_v$ by choosing $G_\emptyset = G_{\{v\}} = \langle \mathbf{e} \rangle$ the trivial group and $\psi_{(\emptyset, \{v\}, \emptyset)} = \psi_{(\emptyset, \{v\}, \{v\})}$ the trivial map. Similarly we define a simple complex of groups $\mathfrak{D}(\mathcal{Z}_S)$ by choosing $G_A = \langle \mathbf{e} \rangle$ the trivial group for every $A \in V(\mathcal{Z}_S)$ and $\psi_{(A,B,\lambda)}$ the trivial map for every $(A, B, \lambda) \in E(\mathcal{Z}_S)$. We have $\mathfrak{D}(\mathcal{Z}_S) = \Pi_{v \in S} \mathfrak{D}(\mathcal{Z}_v)$.

Assume that the scwol $\mathcal{X}$ is connected, i.e. there is only one equivalence class on $V(\mathcal{X})$ for the equivalence relation generated by $i(\alpha) \sim t(\alpha)$ for every edge $\alpha \in E(\mathcal{X})$. One can define the fundamental group of a complex of groups $\mathcal{G}(\mathcal{X})$ over a scwol $\mathcal{X}$. For simplicity and as it suffices for the cases we consider, we give the following characterization.

Definition 2.1.10. Consider a simple complex of groups $\mathcal{G}(\mathcal{X})$ over a connected scwol $\mathcal{X}$. Assume that each group G_v is finitely presented with $G_v = \langle S_v \mid R_v\rangle$. Choose a maximal tree T in the graph with vertex set $V(\mathcal{X})$ and set of edges $E(\mathcal{X})$. Let $E(\mathcal{X})^{\pm} = \{\alpha^{\epsilon} \mid \alpha \in E(\mathcal{X}), \epsilon \in \{+,-\}\}$. Then the fundamental group $\pi_1(\mathcal{G}(\mathcal{X}), T)$ is generated by the set

$$\left(\bigsqcup_{v\in V(\mathcal{X})} S_v\right) \sqcup E(\mathcal{X})^{\pm}$$

subject to the relations:

$$\begin{aligned}
&\text{all the relations } R_v \text{ in the groups } G_v,\\
&(\alpha^+)^{-1} = \alpha^- \text{ for all edges } \alpha \in E(\mathcal{X}),\\
&\alpha^+\beta^+ = (\alpha\beta)^+ \text{ for all } \alpha,\beta \in E(\mathcal{X}), \text{ such that } \alpha\beta \in E(\mathcal{X}) \text{ is defined},\\
&\psi_\alpha(s) = \alpha^+ s\alpha^-, \text{ for all } \alpha \in E(\mathcal{X}) \text{ and every } s \in S_{i(\alpha)},\\
&\alpha^+ = \mathbf{e}, \text{ for every } \alpha \in T.
\end{aligned}$$

Different choices of T and of presentation $G_v = \langle S_v \mid R_v\rangle$ will give isomorphic fundamental groups. So we can consider $\pi_1(\mathcal{G}(\mathcal{X})) = \pi_1(\mathcal{G}(\mathcal{X}), T)$ for any choice of maximal tree T. Moreover the subgroup of $\pi_1(\mathcal{G}(\mathcal{X}), T)$ generated by $\{\alpha^+, \alpha \in E(\mathcal{X})\}$ defines the fundamental group of the scwol $\mathcal{X}$.

Remark 2.1.11. This definition is not the original definition given in III.$\mathcal{C}$.3.5 [BH99] but it is equivalent to it by Theorem III.$\mathcal{C}$.3.7 in [BH99] and better suited to our use.

Lemma 2.1.12. *For two simple inclusive complexes of groups $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$, we have*

$$\pi_1(\mathcal{G}(\mathcal{X})) \times \pi_1(\mathcal{G}(\mathcal{Y})) = \pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y})).$$

Proof. We start by choosing maximal trees $T_{\mathcal{X}}$ in $\mathcal{X}$ and $T_{\mathcal{Y}}$ in $\mathcal{Y}$. In $\mathcal{X}\times\mathcal{Y}$ we fix a vertex $v_0 = (v_{\mathcal{X},0}, v_{\mathcal{Y},0})$ and consider the tree $T_{\mathcal{X}\times\mathcal{Y}}$ with vertices $V(\mathcal{X}) \times V(\mathcal{Y})$ and edges

$$\begin{aligned}\{(v,\alpha) \in E(\mathcal{X}\times\mathcal{Y}) \mid v \in V(\mathcal{X}), \alpha \in E(T_{\mathcal{Y}})\}&\\
\cup \{(\alpha, v_{\mathcal{Y},0}) \in E(\mathcal{X}\times\mathcal{Y}) \mid \alpha \in E(T_{\mathcal{X}})\}&.\end{aligned}$$

Since $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ are maximal trees, so is $T_{\mathcal{X}\times\mathcal{Y}}$. In order to prove the statement, we give explicit homomorphisms

$$\phi : \pi_1(\mathcal{G}(\mathcal{X}), T_{\mathcal{X}}) \times \pi_1(\mathcal{G}(\mathcal{Y}), T_{\mathcal{Y}}) \to \pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}}),$$

and

$$\xi : \pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}}) \to \pi_1(\mathcal{G}(\mathcal{X}), T_{\mathcal{X}}) \times \pi_1(\mathcal{G}(\mathcal{Y}), T_{\mathcal{Y}})$$

and show that their composition is the identity. Recall that the generating set of $\pi_1(\mathcal{G}(\mathcal{X}), T_{\mathcal{X}})$ is $S_{\mathcal{X}} = \left(\bigsqcup_{v\in V(\mathcal{X})} S_v\right) \sqcup E(\mathcal{X})^{\pm}$ and that the generating set of $\pi_1(\mathcal{G}(\mathcal{Y}), T_{\mathcal{Y}})$ is $S_{\mathcal{Y}} = \left(\bigsqcup_{v\in V(\mathcal{Y})} S_v\right) \sqcup E(\mathcal{Y})^{\pm}$. So the generating set of $\pi_1(\mathcal{G}(\mathcal{X})) \times \pi_1(\mathcal{G}(\mathcal{Y}))$ is $S_{\mathcal{X}} \sqcup S_{\mathcal{Y}}$. This generating set is subject to the relations in $\pi_1(\mathcal{G}(\mathcal{X}))$ and $\pi_1(\mathcal{G}(\mathcal{Y}))$ and to the commutation relations $\{st = ts \mid s \in S_{\mathcal{X}}, t \in S_{\mathcal{Y}}\}$. The generating set of $\pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}})$ is $S_{\mathcal{X}\times\mathcal{Y}} = \left(\bigsqcup_{v\in V(\mathcal{X}\times\mathcal{Y})} S_v\right) \sqcup E(\mathcal{X}\times\mathcal{Y})^{\pm}$. For a vertex $v = (v_{\mathcal{X}}, v_{\mathcal{Y}}) \in V(\mathcal{X}\times\mathcal{Y})$ we have $G_v = G_{v_{\mathcal{X}}} \times G_{v_{\mathcal{Y}}}$ so we may assume that $S_v = S_{v_{\mathcal{X}}} \sqcup S_{v_{\mathcal{Y}}}$ and that the relations in the groups G_v are $R_v = R_{v_{\mathcal{X}}} \sqcup R_{v_{\mathcal{Y}}} \sqcup \{xy = yx \mid x \in S_{v_{\mathcal{X}}}, y \in S_{v_{\mathcal{Y}}}\}$. There is a lot of redundancy in the generators of the group $\pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}})$. In particular:

1. For all $\alpha \in E(\mathcal{X})$, $v, w \in V(\mathcal{Y})$ we claim that $(\alpha, v)^+ = (\alpha, w)^+$. First assume that v and w are adjacent in $T_{\mathcal{Y}}$. So there is an edge $\gamma \in E(T_{\mathcal{Y}})$ with $i(\gamma) = v$ and $t(\gamma) = w$. By Definition 2.1.2 we have $(t(\alpha), \gamma)(\alpha, v) = (\alpha, \gamma) = (\alpha, w)(i(\alpha), \gamma)$ in $\mathcal{X}\times\mathcal{Y}$. As $(t(\alpha), \gamma), (i(\alpha), \gamma) \in E(T_{\mathcal{X}\times\mathcal{Y}})$, we get $(t(\alpha), \gamma)^+ = (i(\alpha), \gamma)^+ = \mathbf{e}$, which implies $(\alpha, v)^+ = (\alpha, \gamma)^+ = (\alpha, w)^+$. If there is no such edge in $T_{\mathcal{Y}}$, there is a sequence of vertices $v_1, \ldots, v_k$ with $v_i = v$ and $v_k = w$ and such that for $1 \le i \le k-1$ the vertices v_i, v_{i+1} are adjacent in $T_{\mathcal{Y}}$. So for $1 \le i \le k-1$ we have $(\alpha, v_i)^+ = (\alpha, v_{i+1})^+$ and hence $(\alpha, v)^+ = (\alpha, w)^+$.

2. The previous statement implies that for every $\gamma \in E(T_{\mathcal{X}})$ and every vertex $v \in V(\mathcal{Y})$ we have $(\gamma, v)^+ = \mathbf{e}$. So we can do a similar construction as for the previous statement to show that for all $\alpha \in E(\mathcal{Y})$, $v, w \in V(\mathcal{X})$ we have $(v, \alpha)^+ = (w, \alpha)^+$.

3. For $\alpha \in E(\mathcal{X})$, $\beta \in E(\mathcal{Y})$, $v \in V(\mathcal{X}), w \in V(\mathcal{Y})$ we have on one hand $(\alpha, \beta)^+ = (\alpha, t(\beta))^+(i(\alpha), \beta)^+ = (\alpha, w)^+(v, \beta)^+$ and on the other hand we have $(\alpha, \beta)^+ = (t(\alpha), \beta)^+(\alpha, i(\beta))^+ = (v, \beta)^+(\alpha, w)^+$. So we get $(\alpha, \beta)^+ = (\alpha, w)^+(v, \beta)^+ = (v, \beta)^+(\alpha, w)^+$.

4. Let $v \in V(\mathcal{X})$, $\alpha \in E(T_{\mathcal{Y}})$ and $s \in S_v$. For $w \in V(\mathcal{Y})$, write $s_{(v,w)}$ for $s \in S_{(v,w)} = S_v \sqcup S_w \subset S_{\mathcal{X}\times\mathcal{Y}}$. Then $s_{(v,i(\alpha))} = s_{(v,t(\alpha))}$ in $\pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}})$. Indeed $(v, \alpha)^+ = (v, \alpha)^- = \mathbf{e}$ and $\psi_{(v,\alpha)}(s_{(v,i(\alpha))}) = (v, \alpha)^+ s_{(v,i(\alpha))}(v, \alpha)^-$, hence $\psi_{(v,\alpha)}(s_{(v,i(\alpha))}) = s_{(v,i(\alpha))}$. As $s_{(v,i(\alpha))} = s \in S_v \subset S_{(v,i(\alpha))}$, $\psi_{(v,\alpha)}(s_{(v,i(\alpha))}) = s_{(v,t(\alpha))} = s \in S_v \subset S_{(v,t(\alpha))}$. So $s_{(v,i(\alpha))} = s_{(v,t(\alpha))}$. Moreover this implies that for all $w, w' \in V(\mathcal{Y})$, $s_{(v,w)} = s_{(v,w')}$.

5. Similarly let $v, v' \in V(\mathcal{X})$, $w \in V(\mathcal{Y})$, $s \in S_w$. We can write $s_{(v,w)}$ for $s \in S_{(v,w)} = S_v \sqcup S_w \subset S_{\mathcal{X}\times\mathcal{Y}}$. So we have $s_{(v,w)} = s_{(v',w)}$.

6. Let $v \in V(\mathcal{X}), w \in V(\mathcal{Y})$ and $s \in S_v$, $t \in S_w$. The local group at (v, w) is $G_{(v,w)} = G_v \times G_w$. So for $s_{(v,w)} \in S_{\mathcal{X}\times\mathcal{Y}}$, $t_{(v,w)} \in S_{\mathcal{X}\times\mathcal{Y}}$, we have $s_{(v,w)}t_{(v,w)} = t_{(v,w)}s_{(v,w)}$. By the two previous statements, this implies that for $v' \in V(\mathcal{X})$ and $w' \in V(\mathcal{Y})$ we have $s_{(v,w')}t_{(v',w)} = t_{(v',w)}s_{(v,w')}$.

7. Let $v, v' \in V(\mathcal{X})$, $w, w' \in V(\mathcal{Y})$, $\alpha \in E(\mathcal{X})$, $\beta \in E(\mathcal{Y})$, $t \in S_w$, $s \in S_v$. Then $t_{(v,w)} = t_{(i(\alpha),w)} = t_{(t(\alpha),w)} = \psi_{(\alpha,w)}(t_{(i(\alpha),w)})$ and $(\alpha, w')^+ = (\alpha, w)^+$. So $(\alpha, w')^+t_{(v,w)}(\alpha, w')^- = t_{(v,w)}$ and hence $(\alpha, w')^+t_{(v,w)} = t_{(v,w)}(\alpha, w')^+$. Similarly $(v', \beta)^+s_{(v,w)}(v', \beta)^- = s_{(v,w)}$ and hence $(v', \beta)^+s_{(v,w)} = s_{(v,w)}(v', \beta)^+$.

We define the map $\phi : \pi_1(\mathcal{G}(\mathcal{X}), T_{\mathcal{X}}) \times \pi_1(\mathcal{G}(\mathcal{Y}), T_{\mathcal{Y}}) \to \pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}})$ on the generators. For $v \in V(\mathcal{X})$, $s \in S_v$, set $\phi(s) = s_{(v,v_{\mathcal{Y},0})}$ and for $w \in V(\mathcal{Y})$, $s \in S_w$ set $\phi(s) = s_{(v_{\mathcal{X},0},w)}$. For $\alpha \in E(\mathcal{X})$, set $\phi(\alpha^\pm) = (\alpha, v_{\mathcal{Y},0})^\pm$ and for $\beta \in E(\mathcal{Y})$ set $\phi(\beta^\pm) = (v_{\mathcal{X},0}, \beta)^\pm$. The map ϕ restricts to a map $E(\mathcal{X})^+ \sqcup E(\mathcal{Y})^+ \to E(\mathcal{X} \times \mathcal{Y})^+$, which induces a map $E(\mathcal{X}) \sqcup E(\mathcal{Y}) \to E(\mathcal{X} \times \mathcal{Y})$. This map from $E(\mathcal{X}) \sqcup E(\mathcal{Y}) \to E(\mathcal{X} \times \mathcal{Y})$ respects the composition and sends an edge in $T_{\mathcal{X}} \sqcup T_{\mathcal{Y}}$ to an edge in $T_{\mathcal{X}\times\mathcal{Y}}$. For $\alpha \in E(\mathcal{X})$, $s \in S_{i(\alpha)}$ we have by inclusiveness $\phi(\psi_\alpha(s)) = s_{(t(\alpha),v_{\mathcal{Y},0})} = (\alpha, v_{\mathcal{Y},0})^+s_{(i(\alpha),v_{\mathcal{Y},0})}(\alpha, v_{\mathcal{Y},0})^- = \phi(\alpha^+)\phi(s)\phi(\alpha^-)$. Also for $v \in V(\mathcal{X})$, we have $R_v \subset R_{(v,v_{\mathcal{Y},0})}$. Similar statements hold when choosing $w \in V(\mathcal{Y})$ and $\beta \in E(\mathcal{Y})$. So the relations in $\pi_1(\mathcal{G}(\mathcal{X}))$ and $\pi_1(\mathcal{G}(\mathcal{Y}))$ are respected under ϕ. By the consequences listed above, the commutation relations are also satisfied. Indeed for $\alpha \in E(\mathcal{X})$, $\beta \in E(\mathcal{Y})$, by the claim 3 above

$$\phi(\alpha^+\beta^+) = (\alpha, v_{\mathcal{Y},0})^+(v_{\mathcal{X},0}, \beta)^+ = (v_{\mathcal{X},0}, \beta)^+(\alpha, v_{\mathcal{Y},0})^+ = \phi(\beta^+\alpha^+).$$

For $v \in V(\mathcal{X})$, $w \in V(\mathcal{Y})$, $s \in S_v$, $t \in S_w$ we have $\phi(s)\phi(t) = \phi(t)\phi(s)$ by the claim 6 above. Finally for $\alpha \in E(\mathcal{X})$, $w \in V(\mathcal{Y})$, $t \in S_w$ we have $\phi(t)\phi(\alpha^+) = \phi(\alpha^+)\phi(t)$ by using the claim 7. There is a corresponding equality for $\beta \in E(\mathcal{Y})$, $v \in V(\mathcal{X})$, $s \in S_v$. So ϕ is a homomorphism.

We define the map $\xi : \pi_1(\mathcal{G}(\mathcal{X}) \times \mathcal{G}(\mathcal{Y}), T_{\mathcal{X}\times\mathcal{Y}}) \to \pi_1(\mathcal{G}(\mathcal{X}), T_{\mathcal{X}}) \times \pi_1(\mathcal{G}(\mathcal{Y}), T_{\mathcal{Y}})$ on the generators. For $s = s_{(v,w)} \in S_{\mathcal{X}\times\mathcal{Y}}$, set $\xi(s_{(v,w)}) = s \in S_v \sqcup S_w \subset S_{\mathcal{X}} \sqcup S_{\mathcal{Y}}$. For $\alpha \in E(\mathcal{X})$, $w \in V(\mathcal{Y})$, let $\xi((\alpha, w)^\pm) = \alpha^\pm$. For $\beta \in E(\mathcal{Y})$, $v \in V(\mathcal{X})$, let $\xi((v, \beta)^\pm) = \beta^\pm$. For $(\alpha, \beta) \in E(\mathcal{X}) \times E(\mathcal{Y})$, let $\xi((\alpha, \beta)^+) = \alpha^+\beta^+$ and $\xi((\alpha, \beta)^-) = \beta^-\alpha^-$. Similarly to ϕ one can check that ξ is a homomorphism.

Finally for $v \in V(\mathcal{X}) \sqcup V(\mathcal{Y})$, $s \in S_v$ we have $\xi(\phi(s)) = s$. For $\alpha \in E(\mathcal{X}) \sqcup E(\mathcal{Y})$, $\xi(\phi(\alpha^\pm)) = \alpha^\pm$. For $v \in V(\mathcal{X})$, $w \in V(\mathcal{Y})$, $s \in S_v$, $t \in S_w$ we have $\phi(\xi(s_{(v,w)})) = \phi(s) = s_{(v,v_{\mathcal{Y},0})} = s_{(v,w)}$ and $\phi(\xi(t_{(v,w)})) = t_{(v,w)}$. For $v \in V(\mathcal{X})$, $w \in V(\mathcal{Y})$,

$\alpha \in E(\mathcal{X})$, $\beta \in E(\mathcal{Y})$ we have $\phi(\xi((v,\beta)^+)) = (v,\beta)^+$ and $\phi(\xi((\alpha,\beta)^+)) = \phi(\alpha^+\beta^+) = (\alpha, v_{\mathcal{Y},0})^+(v_{\mathcal{X},0},\beta)^+ = (\alpha,\beta)^+$ as well as $\phi(\xi((\alpha,w)^+)) = \phi(\alpha^+) = (\alpha, v_{\mathcal{Y},0})^+ = (\alpha,w)^+$. So ϕ and ξ are isomorphisms.

□

Example 2.1.13. In Example 2.1.7, the fundamental group of $\mathfrak{D}(\mathcal{Y}_S)$ is $\times_{v\in S}C_v$. In Example 2.1.9, the fundamental group of $\mathfrak{D}(\mathcal{Z}_S)$ is $\mathbb{Z}^S$. In Example 2.1.8, the fundamental group of $\mathfrak{W}(\mathcal{Y}_S)$ is the Coxeter group W.

We will now consider morphisms. Consider two scwols $\mathcal{X}$ and $\mathcal{Y}$. A *morphism of scwols* $f : \mathcal{X} \to \mathcal{Y}$ is a map that sends $V(\mathcal{X})$ to $V(\mathcal{Y})$, sends $E(\mathcal{X})$ to $E(\mathcal{Y})$ and such that:

1. for every $\alpha \in E(\mathcal{X})$, $f(i(\alpha)) = i(f(\alpha))$ and $f(t(\alpha)) = t(f(\alpha))$,

2. for composable edges $\alpha, \beta \in E(\mathcal{X})$, $f(\alpha\beta) = f(\alpha)f(\beta)$.

Let $\mathcal{G}(\mathcal{X}) = (G_v, \psi_\alpha)$ and $\mathcal{H}(\mathcal{Y}) = (H_v, \xi_\alpha)$ be two simple complexes of groups over scwols $\mathcal{X}$ and $\mathcal{Y}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of scwols. A *morphism of complexes of groups* $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to \mathcal{H}(\mathcal{Y})$ over f consists of:

1. a homomorphism $\phi_v : G_v \to H_{f(v)}$ for every $v \in V(\mathcal{X})$ and

2. an element $\phi(\alpha) \in H_{t(f(\alpha))}$ for every edge $\alpha \in E(\mathcal{X})$ such that $\mathrm{Ad}(\phi(\alpha)) \circ \xi_{f(\alpha)} \circ \phi_{i(\alpha)} = \phi_{t(\alpha)} \circ \psi_\alpha$ and $\phi(\alpha\beta) = \phi(\alpha)\xi_{f(\alpha)}(\phi(\beta))$ for all composable edges $\alpha, \beta \in E(\mathcal{X})$,

where $\mathrm{Ad}(\phi(\alpha))$ is the conjugation by $\phi(\alpha)$ (where $\mathrm{Ad}(\phi(\alpha))(g) = \phi(\alpha)g\phi(\alpha^{-1})$ for $g \in H_{t(f(\alpha))}$). This is illustrated in the following diagram:

$$\begin{tikzcd}
G_{i(\alpha)} \arrow[r, "\phi_{i(\alpha)}"] \arrow[d, "\mathrm{Id}"] & H_{i(f(\alpha))} \arrow[r, "\xi_{f(\alpha)}"] & H_{t(f(\alpha))} \arrow[d, "\mathrm{Ad}(\phi(\alpha))"] \\
G_{i(\alpha)} \arrow[r, "\psi_\alpha"] & G_{t(\alpha)} \arrow[r, "\phi_{t(\alpha)}"] & H_{t(f(\alpha))}.
\end{tikzcd}$$

Finally let G be a group. A *morphism* $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to G$ consists of a homomorphism $\phi_v : G_v \to G$ for each $v \in V(\mathcal{X})$ and an element $\phi(\alpha) \in G$ for each $\alpha \in E(\mathcal{X})$ such that $\mathrm{Ad}(\phi(\alpha)) \circ \phi_{i(\alpha)} = \phi_{t(\alpha)} \circ \psi_\alpha$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ whenever composable. There is always a morphism from the complex of groups to the fundamental group of the complex of groups $\phi = (\phi_v, \phi(\alpha)) : \mathcal{G}(\mathcal{X}) \to \pi_1(\mathcal{G}(\mathcal{X}))$ with $\phi(\alpha) = \alpha^+ \in \pi_1(\mathcal{G}(\mathcal{X}))$ for every edge $\alpha \in E(\mathcal{X})$.

Example 2.1.14. Consider the complex of groups $\mathfrak{D}(\mathcal{Y}_S)$ given in Example 2.1.7. Its fundamental group is $\pi_1(\mathfrak{D}(\mathcal{Y}_S)) = \times_{v\in S} C_v$. For $A \in V(\mathcal{Y}_S)$, let $\phi^S_A : \times_{v\in A} C_v \to \times_{v\in S} C_v$ be the inclusion $\phi^S_A(g) = g$ and for $\alpha \in E(\mathcal{Y}_S)$ let $\phi^S(\alpha) = \mathbf{e} \in \times_{v\in S} C_v$. The morphism $\phi^S = (\phi^S_A, \phi^S(\alpha)) : \mathfrak{D}(\mathcal{Y}_S) \to \times_{v\in S} C_v$ is injective on the local groups.

Example 2.1.15. Consider the complex of groups $\mathfrak{W}(\mathcal{Y}_S)$ given in Example 2.1.8. Its fundamental group is $\pi_1(\mathfrak{W}(\mathcal{Y}_S)) = W$. For $A \in V(\mathcal{Y}_S)$, let $\phi_A : W_A \to W$ be the inclusion $\phi_A(g) = g$ and for $\alpha \in E(\mathcal{Y}_S)$ let $\phi(\alpha) = \mathbf{e} \in W$. The morphism $\phi = (\phi_A, \phi(\alpha)) : \mathfrak{W}(\mathcal{Y}_S) \to W$ is injective on the local groups.

Example 2.1.16. Consider the complex of groups $\mathfrak{D}(\mathcal{Z}_S)$ given in Example 2.1.9. Its fundamental group is $\pi_1(\mathfrak{D}(\mathcal{Z}_S)) = \mathbb{Z}^S$. For the notation: let $\mathbf{e}$ be the trivial element in $\mathbb{Z}^S$ and for $s \in S$, let x_s be the standard generators of $\mathbb{Z}^S$. For $A \in V(\mathcal{Z}_S)$, let $\phi^S_A : \langle \mathbf{e} \rangle \to \mathbb{Z}^S$ with $\phi^S_A(\mathbf{e}) = \mathbf{e}$ and for $(A, B, \lambda) \in E(\mathcal{Z}_S)$ let $\phi^S((A, B, \lambda)) = \Pi_{s\in\lambda} x_s$. The morphism $\phi^S = (\phi^S_A, \phi^S(\alpha)) : \mathfrak{D}(\mathcal{Z}_S) \to \mathbb{Z}^S$ is injective on the local groups (which are actually trivial).

Definition 2.1.17. A complex of groups $\mathcal{G}(\mathcal{X})$ over a scwol $\mathcal{X}$ is *developable* if there exists a morphism ϕ from $\mathcal{G}(\mathcal{X})$ to some group G which is injective on the local groups.

Remark 2.1.18. This definition is not the original definition given in III.$\mathcal{C}$.2.11 [BH99] but it is equivalent to it by Corollary III.$\mathcal{C}$.2.15 in [BH99] and better suited to our use.

Let $\mathcal{G}(\mathcal{X})$ be a complex of groups over a scwol $\mathcal{X}$, let G be a group and let $\phi : \mathcal{G}(\mathcal{X}) \to G$ be a morphism. The development of $\mathcal{X}$ with respect to ϕ is the scwol $\mathcal{C}(\mathcal{X}, \phi)$ given as follows:

1. its vertices are
$$V(\mathcal{C}(\mathcal{X}, \phi)) = \{(g\phi_v(G_v), v) \mid v \in V(\mathcal{X}), g\phi_v(G_v) \in G/\phi_v(G_v)\},$$
2. its edges are
$$E(\mathcal{C}(\mathcal{X}, \phi)) = \{(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) \mid \alpha \in E(\mathcal{X}), g\phi_{i(\alpha)}(G_{i(\alpha)}) \in G/\phi_{i(\alpha)}(G_{i(\alpha)})\},$$
3. the maps $i, t : E(\mathcal{C}(\mathcal{X}, \phi)) \to V(\mathcal{C}(\mathcal{X}, \phi))$ are
$$i(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) = (g\phi_{i(\alpha)}(G_{i(\alpha)}), i(\alpha))$$
and
$$t(g\phi_{i(\alpha)}(G_{i(\alpha)}), \alpha) = (g\phi(\alpha)^{-1}\phi_{t(\alpha)}(G_{t(\alpha)}), t(\alpha)),$$

4. the composition is $(g\phi_{i(\alpha)}(G_{i(\alpha)}),\alpha)(h\phi_{i(\beta)}(G_{i(\beta)}),\beta) = (h\phi_{i(\beta)}(G_{i(\beta)}),\alpha\beta)$, where α,β are composable and $g\phi_{i(\alpha)}(G_{i(\alpha)}) = h\phi(\beta)^{-1}\phi_{i(\alpha)}(G_{i(\alpha)})$.

Note that by Theorem III.$\mathcal{C}$.2.13 in [BH99], $\mathcal{C}(\mathcal{X},\phi)$ is indeed well-defined. Moreover there is an action of G on $\mathcal{C}(\mathcal{X},\phi)$ where $G\backslash\,\mathcal{C}(\mathcal{X},\phi) = \mathcal{X}$.

As for simplicial complexes, we can define geometric realizations of scwols. For a scwol $\mathcal{X}$, denote its geometric realization by $|\,\mathcal{X}\,|$. If a scwol does not have multiple edges, this construction coincides with the geometric realization of simplicial complexes. Indeed if a scwol does not have multiple edges, the graph with vertices $V(\mathcal{X})$ and edges $E(\mathcal{X})$ is simplicial and hence determines a unique flag simplicial complex. This is the only case we will need in this article and details on the general construction can be found in Chapter III.$\mathcal{C}$.1 in [BH99]. The action of G on $\mathcal{C}(\mathcal{X},\phi)$ induces an action of G on $|\,\mathcal{C}(\mathcal{X},\phi)|$. If we require the action of G on $|\,\mathcal{C}(\mathcal{X},\phi)|$ to be by isometries, putting a metric on $|\,\mathcal{C}(\mathcal{X},\phi)|$ corresponds to putting a metric on $|\,\mathcal{X}\,|$ as $G\backslash|\,\mathcal{C}(\mathcal{X},\phi)| = |\,\mathcal{X}\,|$.

Example 2.1.19. Consider the complex $\mathfrak{D}(\mathcal{Y}_S)$ and the morphism $\phi^S : \mathfrak{D}(\mathcal{Y}_S) \to \Pi_{v\in S}C_v$ from Example 2.1.14. One can check that the development $\mathcal{C}(\mathcal{Y}_S,\phi^S)$ is the product $\Pi_{v\in S}\,\mathcal{C}(\mathcal{Y}_{\{v\}},\phi^{\{v\}})$. Each $\mathcal{C}(\mathcal{Y}_{\{v\}},\phi^{\{v\}})$ is a scwol with set of vertices $\{(g,\emptyset) \mid g\in C_v\} \cup \{(C_v,\{v\})\}$ and set of edges $\{(g,e_v) \mid g\in C_v\}$ with $i(g,e_v) = (g,\emptyset)$ and $t(g,e_v) = (C_v,\{v\})$. So it is an oriented star on $|C_v|$ branches, the tips correspond to the vertices $\{(g,\emptyset) \mid g\in C_v\}$, the central vertex is $(C_v,\{v\})$ and the edges are oriented from the tips to the center. The group C_v acts by rotation and stabilizes the central vertex. For each $v\in S$, choose $\ell_v > 0$. Let $\mathrm{Stern}(v)$ be the geometric realization of $\mathcal{C}(\mathcal{Y}_v,\phi^{\{v\}})$ as follows: for $g\in C_v$ consider the interval $I_g = [0,\ell_v]$ then $\mathrm{Stern}(v) = \bigcup_{g\in C_v} I_g/\sim$ where $\sim$ is the equivalence relation generated by $0\in I_g \sim 0\in I_{\mathbf{e}}$. Note that C_v acts by isometries on $\mathrm{Stern}(v)$. The space $\mathrm{Stern}(S) = \Pi_{v\in S}\,\mathrm{Stern}(v)$ with the product metric is a geometric realization of $\mathcal{C}(\mathcal{Y}_S,\phi^S)$, due to the product structure of $\mathcal{C}(\mathcal{Y}_S,\phi^S)$. Moreover $\Pi_{v\in S}C_v$ acts by isometries on $\mathrm{Stern}(S)$.

Example 2.1.20. Consider the complex $\mathfrak{W}(\mathcal{Y}_S)$ and the morphism $\phi : \mathfrak{W}(\mathcal{Y}_S) \to W$ from Example 2.1.15. The development $\mathcal{C}(\mathcal{Y}_S,\phi)$ is a scwol with set of vertices $\{(gW_A,A) \mid A\subset S, gW_A\in W/W_A\}$ and $\{(gW_A,(A,B)) \mid A\subsetneq B,\ gW_A\in W/W_A\}$ as set of edges where $i(gW_A,(A,B)) = (gW_A,A)$ and $t(gW_A,(A,B)) = (gW_B,B)$. It is the scwol associated to the poset $W\mathcal{P}(S) = \bigcup_{T\subset S} W/W_T$, the poset of parabolic cosets ordered by inclusion. In Section 2.2, we will introduce the Coxeter polytope C_W of W, which is a geometric realization of $\mathcal{C}(\mathcal{Y}_S,\phi)$.

Example 2.1.21. Consider the complex $\mathfrak{D}(\mathcal{Z}_S)$ and the morphism $\phi^S : \mathfrak{D}(\mathcal{Z}_S) \to \mathbb{Z}^S$ from Example 2.1.16. One can check that the development

$\mathcal{C}(\mathcal{Z}_S, \phi^S)$ of $\mathcal{Z}_S$ with respect to ϕ^S is the product $\times_{v\in S}\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$. Each $\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$ is a scwol with vertices

$$V(\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})) = \{(g, \emptyset) \mid g \in \langle x_v\rangle\} \cup \{(g, \{v\}) \mid g \in \langle x_v\rangle\}$$

and edges

$$E(\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})) = \{(g, (\emptyset, \{v\}, \emptyset)) \mid g \in \langle x_v\rangle\} \cup \{(g, (\emptyset, \{v\}, \{v\})) \mid g \in \langle x_v\rangle\}$$

where $i(g, (\emptyset, \{v\}, \emptyset)) = (g, \emptyset)$, and $t(g, (\emptyset, \{v\}, \emptyset)) = (g, \{v\})$, and $i(g, (\emptyset, \{v\}, \{v\})) = (g, \emptyset)$, and $t(g, (\emptyset, \{v\}, \{v\})) = (gx_v^{-1}, \{v\})$. There are two vertex orbits for the action of $\mathbb{Z} = \langle x_v\rangle$ on $V(\mathcal{X})$, one corresponds to the vertices with incoming edges, one to the vertices with outgoing edges. A geometric realization of $\mathcal{C}(\mathcal{Z}_{\{v\}}, \phi^{\{v\}})$ is the real line $\mathbb{R}$, where we identify $(\mathbf{e}, \emptyset) \in \mathcal{C}(\mathbb{Z}_{\{v\}}, \phi^{\{v\}})$ with $0 \in \mathbb{R}$, and $(\mathbf{e}, \{v\})$ with 0.5 and $(x_v^{-1}, \{v\})$ with -0.5. Since we want $\mathbb{Z}$ to act by isometries, this means that for every $x_v^k \in \mathbb{Z}$ we identify the vertex $(x_v^k, \emptyset)$ with $k \in \mathbb{R}$ and $(x_v^k, \{v\})$ with $k + 0.5 \in \mathbb{R}$. Using the product structure with the ℓ_2-metric, we get that $\mathbb{R}^S$ with the Euclidean metric is a geometric realization of $\mathcal{C}(\mathcal{Z}_S, \phi^S)$ on which $\mathbb{Z}^S$ acts by translation.

2.2 The Davis-Moussong complex

The discussion of the Davis-Moussong complex is based primarily on [Dav08]. We will omit most proofs as they can be found in the literature, in particular in [Dav08] and [Bou81].

Let S be a finite set. Let $M = (m(s,t))_{s,t\in S}$ be a symmetric matrix with $m(s,t) \in \mathbb{N}\cup\{\infty\}$, $m(s,s) = 1$ and $m(s,t) = m(t,s) \geq 2$ if $s \neq t$. Such a matrix is called a Coxeter matrix. The Coxeter group associated to M is given by the following presentation

$$W = \langle s \in S \mid (st)^{m(s,t)} = \mathbf{e} \text{ for all } s,t \in S\rangle,$$

where $m(s,t) = \infty$ means there is no relation given between s and t. The pair (W, S) is called a Coxeter system. Consider the $S \times S$ matrix c defined by $c_{st} = \cos(\pi - \pi/m(s,t))$, the matrix c is called the cosine matrix of the Coxeter matrix M. When $m(s,t) = \infty$, we intepret π/∞ to be 0 and $\cos(\pi - \pi/\infty) = -1$. The following fact states a classical result giving a necessary and sufficient condition for a Coxeter group to be finite.

Fact 2.2.1 (Theorem 6.12.9 [Dav08]). *A Coxeter group W is finite if and only if the cosine matrix c is positive definite.*

For $T \subseteq S$, let W_T be the subgroup of W generated by T. Consider the poset of spherical subsets $\mathcal{S} = \{T \subseteq S \mid W_T \text{ is finite }\}$ ordered by inclusion. In an abuse of notation, let us also write $\mathcal{S}$ for the scwol associated to the poset $\mathcal{S}$. Similarly to examples 2.1.8, 2.1.15 and 2.1.20, let $\mathfrak{W}(\mathcal{S})$ be the complex of groups over $\mathcal{S}$ where the local group at $T \in \mathcal{S}$ is W_T and for an edge (R,T) the associated map $\psi_{(R,T)} : W_R \to W_T$ is the inclusion $\psi_{(R,T)}(r) = r$ for every $r \in R$. The fundamental group of $\mathfrak{W}(\mathcal{S})$ is W and there is an injective morphism $\phi = (\phi_T, \phi((R,T)))$ where $\phi_T : W_T \to W$ is the inclusion and $\phi((R,T)) = \mathbf{e}$ for every edge (R,T). Let $\mathcal{C}(\mathcal{S},\phi)$ be the development of $\mathfrak{W}(\mathcal{S})$ with respect to ϕ. Let us also consider the poset $W\mathcal{S} = \bigsqcup_{T\in\mathcal{S}} W/W_T$ ordered by inclusion, called the poset of spherical cosets. In a similar abuse of notation, let us also write $W\mathcal{S}$ for the scwol associated to the poset $W\mathcal{S}$.

Remark 2.2.2. The set of vertices of $\mathcal{C}(\mathcal{S},\phi)$ is $\{(wW_T, T) \mid T \in \mathcal{S}, wW_T \in W/W_T\}$. The set of edges of $\mathcal{C}(\mathcal{S},\phi)$ is $\{(wW_R, (R,T)) \mid R, T \in \mathcal{S},\ R \subsetneq T,\ wW_R \in W/W_R\}$, where $(wW_R, (R,T))$ is an edge from the vertex (wW_R, R) to the vertex (wW_T, T). In particular there is an edge from a vertex (wW_R, R) to a vertex $(w'W_T, T)$ if and only if $R \subseteq T$ and $wW_T = w'W_T$ (i.e. $w'^{-1}w \in W_T$).

Lemma 2.2.3. *The scwols $\mathcal{C}(\mathcal{S},\phi)$ and $W\mathcal{S}$ are equal.*

Proof. It follows from Remark 2.2.2 that the two scwols have the same set of vertices. For the edges, note that for $wW_R, w'W_T \in W\mathcal{S}$, we have $wW_R \subset w'W_T$ if and only if $R \subset T$ and $w'^{-1}w \in W_T$. So using Remark 2.2.2, the sets of edges also coincide. □

Coxeter polytope For now, assume W is finite. Let us recall the canonical representation of W. Consider $\Pi = \{\alpha_s \mid s \in S\}$ and $V = \oplus_{s\in S}\mathbb{R}\,\alpha_s$. Let $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ be the scalar product on V given by $\langle\alpha_s, \alpha_t\rangle = -\cos(\frac{\pi}{m(s,t)})$. The canonical representation of W on $GL(V)$ is given by $\rho : W \to GL(V)$ with $\rho(s)(x) = x - 2\langle\alpha_s, x\rangle\alpha_s$, for $s \in S$, $x \in V$. The scalar product on V is $\rho(W)$-invariant. There is a dual basis $\Pi^* = \{\alpha_s^* \mid s \in S\}$, satisfying $\langle\alpha_s^*, \alpha_t\rangle = 0$ if $s \neq t$ and $\langle\alpha_s^*, \alpha_s\rangle = 1$. We choose $x_0 = \sum_{s\in S}\alpha_s^* \in V$. The Coxeter polytope of (W,S), denoted $\mathrm{Cox}(W)$, is the convex hull of $\{\rho(w)(x_0) \in V \mid w \in W\}$. It is endowed with the Euclidean metric. Note that its interior is non-empty. For a subset $T \subset S$, let $\Pi_T = \{\alpha_s \mid s \in T\}$ and V_T be the subvector space of V spanned by Π_T. Let $\Pi_T^* = \{\alpha_{s,T}^* \mid s \in T\}$ be the dual basis of Π_T in V_T. Fix $x_{0,T} = \sum_{s\in T}\alpha_{s,T}^*$. Let $\mathrm{Cox}(W_T)$ be the convex hull of $\{\rho(w)(x_{0,T}) \in V_T \mid w \in W_T\}$. Moreover let

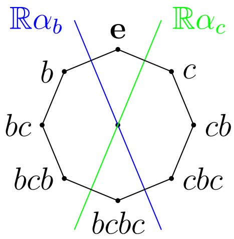


Figure 2.1: The Coxeter polytope of $(\langle b, c \mid b^2 = c^2 = (bc)^4 = \mathbf{e}\rangle, \{b, c\})$

$\mathrm{Cox}_T(W)$ be the convex hull of $\{\rho(w)(x_0) \in V \mid w \in W_T\}$. Let $u = x_0 - x_{0,T}$ and $t_u : V \to V$ be the translation by the vector u. This translation sends $\rho(w)(x_{0,T})$ to $\rho(w)(x_0)$ for every $w \in W_T$. Specifically it is an isometry from $\mathrm{Cox}(W_T)$ to $\mathrm{Cox}_T(W)$.

Lemma 2.2.4 (Lemma 7.3.3 [Dav08])**.** *The poset $W\mathcal{S}$ and the face poset $\mathcal{F}(\mathrm{Cox}(W))$ of $\mathrm{Cox}(W)$ are isomorphic. Specifically the correspondance $wW_T \to w\,\mathrm{Cox}_T(W)$ induces an isomorphism of posets.*

So we can identify $W\mathcal{S}$ and hence $\mathcal{C}(\mathcal{S}, \phi)$ with the barycentric subdivision of the Coxeter polytope $\mathrm{Cox}(W)$, thus identifying $|\mathcal{C}(\mathcal{S}, \phi)|$ isometrically with $\mathrm{Cox}(W)$. The metric on $|\mathcal{C}(\mathcal{S}, \phi)|$ induced by the identification with $\mathrm{Cox}(W)$ is called the Moussong metric. In particular for $wW_T \in W\mathcal{S}$ the geometric realisation $|W\mathcal{S}_{\leq wW_T}| \subseteq |W\mathcal{S}|$ is identified with the face $w\,\mathrm{Cox}_T(W)$.

The General Case We now consider any Coxeter group W, so W need not necessarily be finite. We put a coarser cell structure on $W\mathcal{S}$ (or equivalently on $\mathcal{C}(\mathcal{S}, \phi)$) in order to build the Davis-Moussong complex Σ by identifying each subposet $(W\mathcal{S})_{\leq wW_T}$, $T \in \mathcal{S}$, which is isomorphic to the poset $W_T(\mathcal{S}_{\leq T})$, with a Coxeter polytope $\mathrm{Cox}(W_T)$. We can give the following description of Σ.

Theorem 2.2.5 ([Dav08] Proposition 7.3.4.)**.** *There is a natural cell structure on Σ so that*

1. *its vertex set is W, its 1-skeleton is the Cayley graph $\mathrm{Cay}(W, S)$, and its 2-skeleton is the Cayley 2-complex over $\mathrm{Cay}(W, S)$ with the relations $(st)^{m(s,t)} = \mathbf{e}$ for all $s, t \in S$, $s \neq t$.*

2. *each cell is a Coxeter polytope,*

3. *the link $\mathrm{Lk}(v, \Sigma)$ of each vertex is isomorphic to the abstract simplicial complex $\mathcal{S}_{>\emptyset}$,*

4. *a subset of W is the vertex set of a cell if and only if it is a spherical coset,*
5. *the poset of cells in Σ is $W\mathcal{S}$.*

Note that here the Cayley graph $\mathrm{Cay}(W, S)$ is considered to be undirected, hence there are no double edges between vertices, even though all elements of S have order 2 in W. Furthermore all edges in $\mathrm{Cay}(W, S)$ are labeled, hence the edges of Σ are labeled. This labeling coincides with the labeling of vertices in $\mathrm{Lk}(v, \Sigma)$. By [Dav08][Lemma 12.1.1.], the piecewise Euclidean structure on Σ induces a piecewise spherical structure on the link $\mathrm{Lk}(v, \Sigma)$ of a vertex and as such on the abstract simplicial complex $\mathcal{S}_{>\emptyset}$ with edge length $d(u, v) = \pi - \pi/m(u, v)$ for two adajcent vertices $u, v \in S$.

Now that we have an appropriate description of Σ, let us state the following geometric property of Σ.

Theorem 2.2.6 (Moussong's Theorem [Mou88])**.** *For any Coxeter system, the associated cell complex Σ, equipped with its natural piecewise Euclidean metric, is* $\mathrm{CAT}(0)$.

Moussong's Theorem is the consequence of the following lemma and Moussong's Lemma 1.1.16.

Lemma 2.2.7 ([Dav08] Lemma 12.3.1.)**.** *Let* Lk *be the link of a vertex in Σ with its natural piecewise spherical structure inherited from Σ. Then* Lk *is a simplicial complex and has simplices of size $\geq \pi/2$. Moreover, it is a metric flag complex.*

Note that using Fact 2.2.1 and Theorem 2.2.5, the set of vertices of Lk is S and $T \subseteq S$ spans a simplex in Lk if and only if W_T is finite. Moreover the distance between two vertices in Lk is given by $d(v, w) = \pi - \pi/m(v, w)$.

2.3 The Salvetti complex

Every Coxeter group has an associated Artin group. We will concentrate on the class of right-angled Artin groups and present their analog to the Davis-Moussong complex, the Salvetti complex S_Γ. An extensive discussion of right-angled Artin groups can be found in Charney's survey [Cha07].

Given a simplicial graph Γ, with vertex set V and edge set E, the associated right-angled Artin group A_Γ is given by the following presentation

$$A_\Gamma = \langle x_v, v \in V \mid \text{ for every } e = \{v, w\} \in E,\ x_v x_w = x_w x_v\rangle.$$

If Γ has no edges, then A_Γ is the free group of rank $|V|$. If Γ is a complete graph, then A_Γ is the free abelian group of rank $|V|$.

Salvetti complex S_Γ Let Γ be a simplicial graph with vertex set V. For any set of pairwise adjacent vertices $V' = \{v_1, \ldots, v_n\}$ consider the corresponding generators $x_i = x_{v_i}$ and let

$$C(V') = \{x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in A_\Gamma \mid \epsilon_i \in \{0,1\} \text{ for every } i \in \{1, \ldots, n\}\}.$$

Note that for two sets $V', V'' \subset V$ of pairwise adjacent vertices, we have $V' \neq V''$ if and only if $C(V') \neq C(V'')$. The Salvetti complex S_Γ is the cube complex with vertex set A_Γ and for every $a \in A_\Gamma$ and every set of pairwise adjacent vertices $V' \subset V$ there is a cube $aC(V')$. The Salvetti complex S_Γ is known to be a CAT(0) cube complex by [CD95] and A_Γ acts properly and cocompactly on S_Γ.

Chapter 3

A generalization of the Davis-Moussong complex for Dyer groups

There is extensive literature on Coxeter groups as well as on graph products of cyclic groups. One common feature of these two families of groups is their solution to the word problem. It was given by Tits for Coxeter groups [Tit69] and by Green for graph products of cyclic groups [Gre90]. The algorithm does not only give a solution to the word problem but also allows to detect whether a word is reduced or not. In his study of reflection subgroups of Coxeter groups, Dyer introduces a family of groups which contains both Coxeter groups and graph products of cyclic groups. A close study of [Dye90] also implies that this class of groups, which we call Dyer groups, has the same solution to the word problem as Coxeter groups and graph products of cyclic groups. A complete and explicit proof is given in [PS22]. The isomorphism problem is solved for right-angled Artin groups but it is not solved for Coxeter groups, so its solution for Dyer groups seems currently out of reach. It is therefore natural to ask the following questions. Which of the properties shared by Coxeter groups and right-angled Artin groups are also satisfied by Dyer groups?

In Section 3.1, we define Dyer groups and Dyer systems, and show that every Dyer group is a finite index subgroup of a Coxeter group. In Section 3.2, actions of Dyer groups on CAT(0) spaces are constructed that extend those of Coxeter groups on Davis–Moussong complexes (presented in Section 2.2) and those of right-angled Artin groups on Salvetti complexes (presented in Section 2.3). This chapter also appeared in [Soe22].

3.1 Dyer groups

Similarly to Coxeter groups and right-angled Artin groups, the presentation of a Dyer group can be encoded in a graph.

Definition 3.1.1. Let Γ be a simplicial graph with set of vertices $V = V(\Gamma)$ and set of edges $E = E(\Gamma)$. Consider maps $f : V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $m : E \to \mathbb{N}_{\geq 2}$ such that for every edge $e = \{v, w\}$ with $f(v) \geq 3$ we have $m(e) = 2$. We call the triple (Γ, f, m) a *Dyer graph.*

Definition 3.1.2. Let (Γ, f, m) be a Dyer graph. The *Dyer group* $D = D(\Gamma, f, m)$ associated with the Dyer graph (Γ, f, m) is given by the following presentation

$$D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E\rangle,$$

where $[a, b]_k = \underbrace{aba\ldots}_{k}$ for any $a, b \in D$, $k \in \mathbb{N}$ and we denote the identity with $\mathbf{e}$. The pair $(D(\Gamma, f, m), \{x_v, v \in V(\Gamma)\})$ is called a *Dyer system.*

Example 3.1.3. As mentioned in the introduction, Coxeter groups, right-angled Artin groups and graph products of cyclic groups are examples of Dyer groups.

Remark 3.1.4. For a subset $W \subset V$, we can consider Γ_W the full subgraph of Γ spanned by W and the restrictions $f_W = f_{|W}$ and $m_W = m_{|E(\Gamma_W)}$. The triple (Γ_W, f_W, m_W) is again a Dyer graph. We denote the associated Dyer group by D_W. From [Dye90], we know that that the homomorphism $D_W \to D$ induced by the inclusion $W \hookrightarrow V$ is injective, hence D_W can be regarded as a subgroup of D.

Definition 3.1.5. Let $V_2 = \{v \in V \mid f(v) = 2\}$, $V_\infty = \{v \in V \mid f(v) = \infty\}$ and $V_p = V \setminus \{V_2 \cup V_\infty\}$. For $i \in \{2, p, \infty\}$, let Γ_i be the full subgraph spanned by V_i and D_i be the Dyer group associated to the triple $(\Gamma_i, f_{V_i}, m_{V_i})$. Note that D_2 is a Coxeter group, D_∞ a right angled Artin group and D_p a graph product of finite cyclic groups.

Example 3.1.6. Let $m, q \in \mathbb{N}_{\geq 2}$. Consider the Dyer graph $\Gamma_{m,q}$ given in Figure 3.1. The associated Dyer group is

$$D_{m,q} = \langle a, b, c, d \mid b^2 = c^2 = d^q = \mathbf{e}, ab = ba, (bc)^m = \mathbf{e}, cd = dc\rangle.$$

We recall the definition of Coxeter groups given by a labeled graph, instead of a matrix as in Section 2.2. One can easily go from one to the other by interpreting the labeling of the graph as a Coxeter matrix.

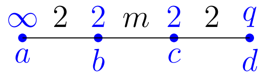


Figure 3.1: Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$

Definition 3.1.7. Let Λ be a simplicial graph with set of vertices $V = V(\Lambda)$ and set of edges $E = E(\Lambda)$. Let $m : E \to \mathbb{N}_{\geq 2}$ be an edge labeling of Λ. The *Coxeter group* $W = W(\Lambda)$ associated to the graph Λ is given by the following presentation

$$W = \langle x_v, v \in V \mid x_v^2 = \mathbf{e} \text{ for all } v \in V, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E\rangle,$$

where $[a, b]_k = \underbrace{aba\dots}_{k}$ for any $a, b \in W$, $k \in \mathbb{N}$ and we denote the identity with $\mathbf{e}$. Note that for an edge $e = \{u, v\} \in E$ the relation $[x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)}$ is equivalent to the relation $(x_v x_u)^{m(e)} = \mathbf{e}$, since $x_u^2 = x_v^2 = \mathbf{e}$.

Dyer groups are finite index subgroups of Coxeter groups The aim is now to show that every Dyer group is a finite index subgroup of a Coxeter group. From a given Dyer graph (Γ, f, m) we build a graph Λ with edge labeling m'. We then show that $D(\Gamma, f, m)$ is a finite index subgroup of $W(\Lambda)$. See Example 3.1.11 for a simple case. We define the undirected labeled simplicial graph Λ. Its set of vertices is $V(\Lambda) = V \coprod (V_p \cup V_\infty)$. We will refer to the elements of the disjoint copy of $V_p \cup V_\infty$ as v' for $v \in V_p \cup V_\infty$. Two vertices $u, v \in V \subset V(\Lambda)$ span an edge in Λ if and only if they span an edge $e = \{u, v\}$ in Γ, and we set the label of the edge $e = \{u, v\} \in E(\Lambda)$ to be $m'(e) = m(e)$. For all $u \in V_p \cup V_\infty$ and $v \in V(\Lambda) \setminus \{u, u'\}$, there is an edge $e = \{u', v\} \in E(\Lambda)$ labeled by $m'(e) = 2$. Finally for all $u \in V_p$ there is an edge $e = \{u, u'\} \in E(\Lambda)$ labeled by $m'(e) = f(u)$. So $V \subset V(\Lambda)$ spans a copy of Γ in Λ and the disjoint copy $V_p \cup V_\infty \subset V(\Lambda)$ spans a complete graph in Λ. Let $W = W(\Lambda)$ be the Coxeter group associated to the graph Λ. We give an action of $(\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$ on D. For $v \in V_p \cup V_\infty$ let $\xi_v : \{x_u, u \in V\} \to D$ with $\xi_v(x_u) = x_u$ for any $u \in V \setminus \{v\}$ and $\xi_v(x_v) = x_v^{-1}$. For all $v \in V_p \cup V_\infty$, the map ξ_v extends to an involutive homomorphism $\xi_v : D \to D$, as all relations involving x_v are commutation relations and ξ_v restricts to the inclusion on $\{x_u, u \in V \setminus \{v\}\}$. Moreover for all $u, v \in V_p \cup V_\infty$, $\xi_v \circ \xi_u = \xi_u \circ \xi_v$ and $(\xi_v)^2 = \mathbf{e}$. Hence we have an action $\xi : (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty} \times D \to D$.

Theorem 3.1.8. *We have* $W \cong D \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

Proof. Let us first recall the presentations of W, D and $U = D \rtimes_\xi (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

$$W = \langle y_v, v \in V(\Lambda) \mid \forall v \in V(\Lambda), (y_v)^2 = \mathbf{e} \\ \text{and } \forall e = \{u, v\} \in E(\Lambda), (y_u y_v)^{m'(e)} = \mathbf{e}\rangle,$$

$$D = \langle x_v, v \in V \mid x_v^{f(v)} = \mathbf{e} \text{ if } f(v) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E\rangle,$$

$$U = \langle \{x_u, u \in V\} \cup \{\xi_v, v \in V_p \cup V_\infty\} \mid x_u^{f(u)} = \mathbf{e} \text{ for all } u \in V \text{ with } f(u) \neq \infty, \\ [x_v, x_u]_{m(e)} = [x_u, x_v]_{m(e)} \text{ for all } e = \{u, v\} \in E, \\ \xi_v^2 = \mathbf{e} \text{ for all } v \in V_p \cup V_\infty,\ \xi_v \xi_u = \xi_u \xi_v \text{ for all } u, v \in V_p \cup V_\infty, \\ \xi_u x_v = x_v \xi_u \text{ for all } u \in V_p \cup V_\infty, v \in V \setminus \{u\},\ \xi_u x_u \xi_u = x_u^{-1} \text{ for all } u \in V_p \cup V_\infty\rangle.$$

We show that U is isomorphic to W by giving explicit homomorphisms $\phi : W \to U$ and $\psi : U \to W$ satisfying $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$.

First consider the map $\phi : \{y_v, v \in V(\Lambda)\} \to U$ defined as follows: for $u \in V_2$, $\phi(y_u) = x_u$ and for $u \in V_p \cup V_\infty$, $\phi(y_u) = \xi_u x_u$ and $\phi(y_{u'}) = \xi_u$. We show that ϕ extends to a homomorphism $\phi : W \to U$.

1. For $u \in V_2$, $\phi(y_u)^2 = x_u^2 = \mathbf{e}$. For $u \in V_p \cup V_\infty$, $\phi(y_{u'})^2 = \xi_u^2 = \mathbf{e}$ and $\phi(y_u)^2 = \xi_u x_u \xi_u x_u = x_u^{-1} x_u = \mathbf{e}$. So $\phi(y_u)^2 = \mathbf{e}$ for all $u \in V(\Lambda)$.

2. Let $u, v \in V \subset V(\Lambda)$ with $e = \{u, v\} \in E(\Lambda)$ so $e \in E$ and $m'(e) = m(e)$. If $u, v \in V_2$, we have $(\phi(y_u)\phi(y_v))^{m'(e)} = (x_u x_v)^{m(e)} = \mathbf{e}$ since $x_u^2 = x_v^2 = \mathbf{e}$ and hence $[x_u, x_v]_{m(e)} = [x_v, x_u]_{m(e)}$ is equivalent to $(x_u x_v)^{m(e)} = \mathbf{e}$. If $u \in V_2$ and $v \in V_p \cup V_\infty$, we have $m'(e) = 2$ and so the relations in U give the equality $\phi(y_u)\phi(y_v) = x_u \xi_v x_v = \xi_v x_u x_v = \xi_v x_v x_u = \phi(y_v)\phi(y_u)$. If $u, v \in V_p \cup V_\infty$, $m'(e) = 2$ and we have $\phi(y_u)\phi(y_v) = \xi_u x_u \xi_v x_v = \xi_u \xi_v x_u x_v = \xi_v \xi_u x_v x_u = \xi_v x_v \xi_u x_u = \phi(y_v)\phi(y_u)$.

3. Let $u \in V_p \cup V_\infty$ and $v \in V \setminus \{u\}$. Then there is an edge $\{u', v\} \in E(\Lambda)$ with $m'(\{u', v\}) = 2$. If $v \in V_2$, $\phi(y_{u'})\phi(y_v) = \xi_u x_v = x_v \xi_u = \phi(y_v)\phi(y_{u'})$. If $v \in V_p \cup V_\infty \setminus \{u\}$, we have $\phi(y_{u'})\phi(y_v) = \xi_u \xi_v x_v = \xi_v \xi_u x_v = \xi_v x_v \xi_u = \phi(y_v)\phi(y_{u'})$.

4. Let $u \in V_p \cup V_\infty$ and $v \in (V_p \cup V_\infty) \setminus \{u\}$, then there is an edge $\{u', v'\} \in E(\Lambda)$ with $m'(\{u', v'\}) = 2$ and we have $\phi(y_{u'})\phi(y_{v'}) = \xi_u \xi_v = \xi_v \xi_u = \phi(y_{v'})\phi(y_{u'})$.

5. Finally for every $u \in V_p$ there is an edge $\{u', u\} \in E(\Lambda)$ with $m'(\{u', u\}) = f(u)$ and $(\phi(y_{u'})\phi(y_u))^{f(u)} = (\xi_u \xi_u x_u)^{f(u)} = x_u^{f(u)} = \mathbf{e}$.

So the map ϕ extends to a homomorphism $\phi : W \to U$.

Now consider the map $\psi : \{x_u, u \in V\} \cup \{\xi_v, v \in V_p \cup V_\infty\} \to W$ defined as follows: for $u \in V_2$, $\psi(x_u) = y_u$ and for $u \in V_p \cup V_\infty$, $\psi(x_u) = y_{u'} y_u$ and $\psi(\xi_u) = y_{u'}$. We show that ψ extends to a homomorphism from U to W.

1. For all $v \in V_2$, $f(v) = 2$ and so $\psi(x_v)^{f(v)} = y_v^2 = \mathbf{e}$ and for all $v \in V_p$ there is an edge $e = \{v, v'\} \in E(\Lambda)$ with $m'(e) = f(v)$ so $\psi(x_v)^{f(v)} = (y_{v'} y_v)^{f(v)} = \mathbf{e}$.

2. For all $e = \{u, v\} \in E$, there is an edge $e = \{u, v\} \in E(\Lambda)$ with $m'(e) = m(e)$. If $u, v \in V_2$ we have

$$[\psi(x_u), \psi(x_v)]_{m(e)} = [y_u, y_v]_{m(e)} = [y_v, y_u]_{m(e)} = [\psi(x_v), \psi(x_u)]_{m(e)}.$$

If $u \in V_2$ and $v \in V_p \cup V_\infty$, we have $m(e) = 2$ and

$$\psi(x_u)\psi(x_v) = y_u y_{v'} y_v = y_{v'} y_u y_v = y_{v'} y_v y_u = \psi(x_v)\psi(x_u).$$

If $u, v \in V_p \cup V_\infty$, $m(e) = 2$ and

$$\psi(x_u)\psi(x_v) = y_{u'} y_u y_{v'} y_v = y_{u'} y_{v'} y_u y_v = y_{v'} y_{u'} y_v y_u = y_{v'} y_v y_{u'} y_u = \psi(x_v)\psi(x_u),$$

as $y_u y_{v'} = y_{v'} y_u$, $y_v y_{u'} = y_{u'} y_v$ and $y_{u'} y_{v'} = y_{v'} y_{u'}$.

3. For all $v \in V_p \cup V_\infty$ we have $\psi(\xi_v)^2 = y_{v'}^2 = \mathbf{e}$.

4. For all $u, v \in V_p \cup V_\infty$ distinct, we have $e = \{u', v'\} \in E(\Lambda)$ with $m'(e) = 2$, so $\psi(\xi_u)\psi(\xi_v) = y_{u'} y_{v'} = y_{v'} y_{u'} = \psi(\xi_v)\psi(\xi_u)$.

5. For all $u \in V_p \cup V_\infty$ and $v \in V \setminus \{u\}$, we have $\{u', v\}, \{u', v'\} \in E(\Lambda)$ with labels $m'(\{u', v\}) = 2$ and $m'(\{u', v'\}) = 2$. If $v \in V_2$, we have $\psi(\xi_u)\psi(x_v) = y_{u'} y_v = y_v y_{u'} = \psi(x_v)\psi(\xi_u)$. If $v \in V_p \cup V_\infty$, we have $\psi(\xi_u)\psi(x_v) = y_{u'} y_{v'} y_v = y_{v'} y_{u'} y_v = y_{v'} y_v y_{u'} = \psi(x_v)\psi(\xi_u)$.

6. For all $u \in V_p \cup V_\infty$, $\psi(\xi_u)\psi(x_u)\psi(\xi_u) = y_{u'} y_{u'} y_u y_{u'} = y_u y_{u'} = (y_{u'} y_u)^{-1} = \psi(x_u)^{-1}$.

So the map ψ extends to a homomorphism $\psi : U \to W$.

We now check that $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$ by showing that these maps are the identity on the generators. For $v \in V_2$, we have $\phi(\psi(x_v)) = \phi(y_v) = x_v$ and $\psi(\phi(y_v)) = \psi(x_v) = y_v$. For $v \in V_p \cup V_\infty$, $\phi(\psi(x_v)) = \phi(y_{v'} y_v) = \xi_v \xi_v x_v = x_v$

and $\phi(\psi(\xi_v)) = \phi(y_{v'}) = \xi_v$. For $v \in V_p \cup V_\infty$, $\psi(\phi(y_v)) = \psi(\xi_v x_v) = y_{v'} y_{v'} y_v = y_v$ and $\psi(\phi(y_{v'})) = \psi(\xi_v) = y_{v'}$. □

Corollary 3.1.9. *Every Dyer group is a finite index subgroup of some Coxeter group.*

Remark 3.1.10. As mentioned in the introduction, Corollary 3.1.9 has many interesting consequences. It implies that Dyer groups are CAT(0) [Dav08][Theorem 12.3.3.], linear [Bou81], and biautomatic [OP22], that they satisfy the Baum-Connes conjecture, the Farrell-Jones conjecture, the Haagerup property and the strong Tits alternative [Nos02]. They also admit a proper and virtually special action on a CAT(0) cube complex.

Example 3.1.11. We apply the previous theorem to Example 3.1.6. The corresponding graph Λ is given in Figure 3.2. So by Theorem 3.1.8, the Dyer group $D_{m,q}$ is an index 4 subgroup of the Coxeter group

$$\begin{aligned} W = \langle a, b, c, d, a', d' \mid\ & a^2 = b^2 = c^2 = d^2 = a'^2 = d'^2 = \mathbf{e}, \\ & (ab)^2 = (bc)^m = (cd)^2 = \mathbf{e}, (a'b)^2 = (a'c)^2 = (a'd)^2 = (a'd')^2 = \mathbf{e}, \\ & (d'a)^2 = (d'b)^2 = (d'c)^2 = \mathbf{e}, (d'd)^q = \mathbf{e} \rangle. \end{aligned}$$

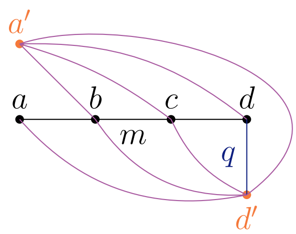


Figure 3.2: The graph $\Lambda_{m,q}$ built out of the Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$. We color coded the vertices $V \subset V(\Lambda)$ and $\{v' \mid v \in V_p \cup V_\infty\}$. For the edges: for edges of the form $e = \{u, u'\}$, $u \in V_p$ and for edges of the form $e = \{u', v\}$ where $v \in V(\Lambda) \setminus \{u, u'\}$ and $u' \in \{v' \mid v \in V_p \cup V_\infty\}$. Every edge is labeled by 2 if not specified otherwise.

The Dyer group D is not the only Dyer group, up to isomorphism, which is a finite index subgroup of W. We describe such another Dyer group $D' = D(\Omega, g, n)$ by giving the Dyer graph (Ω, g, n). The vertices of Ω are $V(\Omega) = V \coprod V_\infty$. We will refer to the elements of the disjoint copy of V_∞ as v' for $v \in V_\infty$. The labeling of the vertices is defined as follows $g_{|(V_2 \cup V_\infty) \coprod V_\infty} = 2$ and $g_{|V_p} = f_{|V_p}$. The subsets

$V_2 \cup V_p \cup V_\infty$ and $(V_2 \cup V_p) \coprod V_\infty$ both span copies of Γ, with same labeling of edges, and for $u, v \in V_\infty$ the vertices v, u' span an edge labeled by 2 in Ω if and only if v and u span an edge in Γ. Let D' be the Dyer group associated to Ω. Note that every generator x_v, $v \in V(\Omega)$ of D' has finite order. We now give an action of $(\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$ on D'. For $v \in V_p$, let $\kappa_v : \{x_u, u \in V(\Omega)\} \to D'$ with $\kappa_v(x_u) = x_u$ for any $u \in V(\Omega) \setminus \{v\}$ and $\kappa_v(x_v) = x_v^{-1}$. As all relations involving x_v are commutation relations and ξ_v restricts to the inclusion on $\{x_u, u \in V \setminus \{v\}$, the map κ_v extends to a homomorphism $\kappa_v : D' \to D'$. For $v \in V_\infty$, consider the map $\kappa_v : \{x_u, u \in V(\Omega)\} \to D'$ with $\kappa_v(x_u) = x_u$ for any $u \in V(\Omega) \setminus \{v, v'\}$ and $\kappa_v(x_v) = x_{v'}$ and $\kappa_v(x_{v'}) = x_v$. The only relations involving x_v or $x_{v'}$ are commutation relations and x_v commutes with some other generator if and only if $x_{v'}$ does, the map κ_v extends to a homomorphism $\kappa_v : D' \to D'$. Moreover for all $u, v \in V_p \cup V_\infty$, $\kappa_v \circ \kappa_u = \kappa_u \circ \kappa_v$ and $(\kappa_v)^2 = \mathbf{e}$. Hence we have an action $\kappa : (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty} \times D' \to D'$.

Theorem 3.1.12. *We have* $W \cong D' \rtimes_\kappa (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

Proof. Let us first recall the presentations of W, D' and $U = D' \rtimes_\kappa (\mathbb{Z}/2\mathbb{Z})^{V_p \cup V_\infty}$.

$$\begin{aligned} W = \langle y_v, v \in V(\Lambda) \mid \forall v \in V(\Lambda), (y_v)^2 = \mathbf{e} \\ \text{and } \forall e = \{u, v\} \in E(\Lambda), (y_u y_v)^{m'(e)} = \mathbf{e} \rangle, \end{aligned}$$

$$\begin{aligned} D' = \langle x_v, v \in V(\Omega) \mid x_v^{g(v)} = \mathbf{e} \text{ for all } v \in V(\Omega), \\ [x_v, x_u]_{n(e)} = [x_u, x_v]_{n(e)} \text{ for all } e = \{u, v\} \in E(\Omega) \rangle, \end{aligned}$$

$$\begin{gathered} U = \langle \{x_u, u \in V(\Omega)\} \cup \{\kappa_v, v \in V_p \cup V_\infty\} \mid x_u^{g(u)} = \mathbf{e} \text{ for all } u \in V(\Omega), \\ [x_v, x_u]_{n(e)} = [x_u, x_v]_{n(e)} \text{ for all } e = \{u, v\} \in E(\Omega), \\ \kappa_v^2 = \mathbf{e} \text{ for all } v \in V_p \cup V_\infty,\ \kappa_v \kappa_u = \kappa_u \kappa_v \text{ for all } u, v \in V_p \cup V_\infty, \\ \kappa_u x_v = x_v \kappa_u \text{ for all } u \in V_p, v \in V(\Omega) \setminus \{u\}, \\ \kappa_u x_v = x_v \kappa_u \text{ for all } u \in V_\infty, v \in V(\Omega) \setminus \{u, u'\}, \\ \kappa_u x_u \kappa_u = x_u^{-1} \text{ for all } u \in V_p,\ \kappa_u x_u \kappa_u = x_{u'} \text{ for all } u \in V_\infty \rangle. \end{gathered}$$

As in Theorem 3.1.8, we can check that U is isomorphic to W by considering explicit homomorphisms $\phi : W \to U$ and $\psi : U \to W$ satisfying $\phi \circ \psi = \mathrm{Id}_U$ and $\psi \circ \phi = \mathrm{Id}_W$.

The map $\phi : \{y_v, v \in V(\Lambda)\} \to U$ is given as follows: for $u \in V_2$, $\phi(y_u) = x_u$, for $u \in V_p$, $\phi(y_u) = \kappa_u x_u$ and $\phi(y_{u'}) = \kappa_u$ and for $u \in V_\infty$, $\phi(y_u) = x_u$ and $\phi(y_{u'}) = \kappa_u$.

Using similar methods to those used in the proof of Theorem 3.1.8, we show that the map ϕ induces a homomorphism $\phi : W \to U$.

1. For every $u \in V_2$, we have $g(u) = 2$ and so $\phi(y_u)^2 = x_u^2 = \mathbf{e}$. For every $u \in V_p$, we have $(\phi(y_u))^2 = (\kappa_u x_u)^2 = \kappa_u x_u \kappa_u x_u = x_u^{-1} x_u = \mathbf{e}$ and $(\phi(y_{u'}))^2 = \kappa_u^2 = \mathbf{e}$. For every $u \in V_\infty$, we have $g(u) = 2$ and so $(\phi(y_u))^2 = x_u^2 = \mathbf{e}$ and $(\phi(y_{u'}))^2 = \kappa_u^2 = \mathbf{e}$.

2. For every edge $e = \{u, v\} \in E(\Lambda)$ with $u, v \in V_2$, we have $m'(e) = n(e)$ and so $(\phi(y_u)\phi(y_v))^{m'(e)} = (x_u x_v)^{n(e)} = \mathbf{e}$, as $x_u^2 = x_v^2 = \mathbf{e}$.

3. For every edge $e = \{u, v\} \in E(\Lambda)$ with $u \in V_2$ and $v \in V_p$, we have $m'(e) = 2$ and $e = \{u, v\} \in E(\Omega)$ with $n(e) = 2$. So $(\phi(y_u)\phi(y_v))^{m'(e)} = (x_u \kappa_v x_v)^{m'(e)} = x_u \kappa_v x_v x_u \kappa_v x_v = x_u^2 (\kappa_v x_v)^2 = \mathbf{e}$.

   For every edge $e = \{u, v\} \in E(\Lambda)$ with $u \in V_2$ and $v \in V_\infty$, we have $m'(e) = 2$ and $e = \{u, v\} \in E(\Omega)$ with $n(e) = 2$. So $(\phi(y_u)\phi(y_v))^{m'(e)} = (x_u x_v)^{m'(e)} = x_u x_v x_u x_v = x_u x_u x_v x_v = \mathbf{e}$, as $g(u) = g(v) = 2$.

4. For every $u \in V_2$ and every $v \in V_p \cup V_\infty$, there is an edge $e = \{u, v'\} \in E(\Lambda)$ with $m'(e) = 2$. Then $(\phi(y_u)\phi(y_{v'}))^{m'(e)} = (x_u \kappa_v)^2 = x_u \kappa_v x_u \kappa_v = x_u^2 \kappa_v^2 = \mathbf{e}$ as $g(u) = 2$.

5. For every edge $e = \{u, v\} \in E(\Lambda)$ with $u, v \in V_p$, there is an edge $e = \{u, v\} \in E(\Omega)$ and we have $m'(e) = n(e) = 2$. So $(\phi(y_u)\phi(y_v))^2 = (\kappa_u x_u \kappa_v x_v)^2 = (\kappa_u x_u)^2 (\kappa_v x_v)^2 = \mathbf{e}$.

   For every edge $e = \{u, v\} \in E(\Lambda)$ with $u \in V_p$, $v \in V_\infty$, there is an edge $e = \{u, v\} \in E(\Omega)$ and we have $m'(e) = n(e) = 2$. So $(\phi(y_u)\phi(y_v))^2 = (\kappa_u x_u x_v)^2 = (\kappa_u x_u)^2 (x_v)^2 = \mathbf{e}$.

   For every edge $e = \{u, v\} \in E(\Lambda)$ with $u, v \in V_\infty$, there is an edge $e = \{u, v\} \in E(\Omega)$ and we have $m'(e) = n(e) = 2$ as well as $g(u) = g(v) = 2$. So $(\phi(y_u)\phi(y_v))^2 = (x_u x_v)^2 = x_u^2 x_v^2 = \mathbf{e}$.

6. For every $u \in V_p$, $v \in V_p \cup V_\infty \setminus \{u\}$, there is an edge $e = \{u, v'\} \in E(\Lambda)$ with $m'(e) = 2$. So $(\phi(y_u)\phi(y_{v'}))^2 = (\kappa_u x_u \kappa_v)^2 = (\kappa_u x_u)^2 \kappa_v^2 = \mathbf{e}$.

   For every $u \in V_\infty$, $v \in V_p \cup V_\infty \setminus \{u\}$, there is an edge $e = \{u, v'\} \in E(\Lambda)$ with $m'(e) = 2$. So $(\phi(y_u)\phi(y_{v'}))^2 = (x_u \kappa_v)^2 = x_u^2 \kappa_v^2 = \mathbf{e}$, as $g(u) = 2$.

7. For every $u \in V_p$, we have $f(u) = g(u)$ and there is an edge $e = \{u, u'\} \in E(\Lambda)$ with $m'(e) = f(u)$. Then $(\phi(y_u)\phi(y_{u'}))^{f(u)} = (\kappa_u x_u \kappa_u)^{f(u)}$ $= (x_u)^{-f(u)} = x_u^{-g(u)} = \mathbf{e}$.

8. For every $u, v \in V_p \cup V_\infty$, there is an edge $e = \{u', v'\} \in E(\Lambda)$ with $m'(e) = 2$. We have $(\phi(y_{u'})\phi(y_{v'}))^2 = (\kappa_u \kappa_v)^2 = \kappa_u^2 \kappa_v^2 = \mathbf{e}$.

So the map ϕ induces a homomorphism $\phi : W \to U$.

The map $\psi : \{x_u, u \in V(\Omega)\} \cup \{\kappa_v, v \in V_p \cup V_\infty\} \to W$ is given as follows: for $u \in V_2$, $\psi(x_u) = y_u$ and for $u \in V_p$, $\psi(x_u) = y_{u'} y_u$ and $\psi(\kappa_u) = y_{u'}$, finally for $u \in V_\infty$, $\psi(x_u) = y_u$, $\psi(x_{u'}) = y_{u'} y_u y_{u'}$ and $\phi(\kappa_u) = y_{u'}$. Using similar methods to those used in the proof of Theorem 3.1.8, we show that the map ψ induces a homomorphism $\psi : U \to W$.

1. For every $u \in V_2$, we have $g(u) = 2$ and $\psi(x_2)^2 = y_u^2 = \mathbf{e}$. For every $u \in V_\infty$, we have $g(u) = g(u') = 2$. So $\psi(x_u)^2 = y_u^2 = \mathbf{e}$ and $\psi(x_{u'})^2 = (y_{u'} y_u y_{u'})^2 = y_u^2 = \mathbf{e}$. For every $u \in V_p$, we have $g(u) = f(u)$ and there is an edge $e = \{u, u'\} \in E(\Lambda)$ with $m'(e) = f(u)$ wo $\psi(x_u)^{f(u)} = (y_{u'} y_u)^{f(u)} = \mathbf{e}$.

2. For every edge $e = \{u, v\} \in E(\Omega)$ with $u, v \in V_2$, there is an edge $e = \{u, v\} \in E(\Lambda)$ with $m'(e) = n(e)$. So we have $[\psi(x_v), \psi(x_u)]_{n(e)} = [y_v, y_u]_{m'(e)} = [y_u, y_v]_{m'(e)} = [\psi(x_u), \psi(x_v)]_{n(e)}$.

   For every edge $e = \{u, v\} \in E(\Omega)$ $u \in V_2, v \in V_p$, there is are edges $e = \{u, v\}, e' = \{u, v'\} \in E(\Lambda)$ with $m'(e') = m'(e) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_u y_{v'} y_v = y_{v'} y_v y_u = \psi(x_v)\psi(x_u)$.

   For every edge $e = \{u, v\} \in E(\Omega)$ $u \in V_2, v \in V_\infty$, there is an edge $e = \{u, v\} \in E(\Lambda)$ with $m'(e) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_u y_v = y_v y_u = \psi(x_v)\psi(x_u)$.

   For every edge $e = \{u, v'\} \in E(\Omega)$ $u \in V_2, v \in V_\infty$, there is are edges $e = \{u, v\}, e' = \{u, v'\} \in E(\Lambda)$ with $m'(e') = m'(e) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_u y_{v'} y_v y_{v'} = y_{v'} y_v y_{v'} y_u = \psi(x_v)\psi(x_u)$.

3. For every edge $e = \{u, v\} \in E(\Omega)$ with $u \in V_p$, $v \in V_p$, there are edges $e_1 = \{u, v\}, e_2 = \{u', v\}, e_3 = \{u, v'\}, e_4 = \{u', v'\} \in E(\Lambda)$ with $m'(e_1) = m'(e_2) = m'(e_3) = m'(e_4) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_{u'} y_u y_{v'} y_v = y_{'} y_v y_{u'} y_u = \psi(x_v)\psi(x_u)$.

   For every edge $e = \{u, v\} \in E(\Lambda)$ with $u \in V_p$, $v \in V_\infty$, there are edges $e = \{u, v\}, e' = \{u', v\} \in E(\Lambda)$ with $m'(e) = m'(e') = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_{u'} y_u y_v = y_v y_{u'} y_u = \psi(x_v)\psi(x_u)$.

   For every edge $e = \{u, v'\} \in E(\Omega)$ with $u \in V_p$, $v \in V_\infty$, there are edges $e_1 = \{u, v\}, e_2 = \{u', v\}, e_3 = \{u, v'\}, e_4 = \{u', v'\} \in E(\Lambda)$ with $m'(e_1) = m'(e_2) = m'(e_3) = m'(e_4) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_{u'} y_u y_{v'} y_v y_{v'} = y_{v'} y_v y_{v'} y_{u'} y_u = \psi(x_v)\psi(x_u)$.

4. For every edge $e = \{u,v\} \in E(\Omega)$ with $u,v \in V_\infty$, there is an edge $e = \{u,v\} \in E(\Lambda)$ with $m'(e) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_u y_v = y_v y_u = \psi(x_v)\psi(x_u)$.

   For every edge $e = \{u,v'\} \in E(\Lambda)$ with $u,v \in V_\infty$, there are edges $e = \{u,v\}, e' = \{u,v'\} \in E(\Lambda)$ with $m'(e) = m'(e') = n(e) = 2$. So $\psi(x_u)\psi(x_{v'}) = y_u y_{v'} y_v y_{v'} = y_{v'} y_v y_{v'} y_u = \psi(x_{v'})\psi(x_u)$.

5. For every edge $e = \{u',v'\} \in E(\Omega)$ with $u,v \in V_\infty$, there are edges $e_1 = \{u,v\}, e_2 = \{u',v\}, e_3 = \{u,v'\}, e_4 = \{u',v'\} \in E(\Lambda)$ with $m'(e_1) = m'(e_2) = m'(e_3) = m'(e_4) = n(e) = 2$. So $\psi(x_u)\psi(x_v) = y_{u'} y_u y_{u'} y_{v'} y_v y_{v'} = y_{v'} y_v y_{v'} y_{u'} y_u y_{u'} = \psi(x_v)\psi(x_u)$.

6. For every $u \in V_p \cup V_\infty$, we have $\psi(\kappa_u)^2 = y_{u'}^2 = \mathbf{e}$. For all distinct $u,v \in V_p \cup V_\infty$, there is an edge $e = \{u',v'\} \in E(\Lambda)$ with $m'(e) = 2$. So $\psi(\kappa_u)\psi(\kappa_v) = y_{u'} y_{v'} = y_{v'} y_{u'} = \psi(\kappa_v)\psi(\kappa_u)$.

7. For every $u \in V_p \cup V_\infty$ and every $v \in V_p \setminus \{u\}$, there are edges $e = \{u',v\}, e' = \{u',v'\} \in E(\Lambda)$ with $m'(e) = m'(e') = 2$. So $\psi(\kappa_u)\psi(x_v) = y_{u'} y_{v'} y_v = y_{v'} y_v y_{u'} = \psi(x_v)\psi(\kappa_u)$.

   For every $u \in V_p \cup V_\infty$ and every $v \in V_\infty \setminus \{u\}$, there are edges $e = \{u',v\}, e' = \{u',v'\} \in E(\Lambda)$ with $m'(e) = m'(e') = 2$. So $\psi(\kappa_u)\psi(x_{v'}) = y_{u'} y_{v'} y_v y_{v'} = y_{v'} y_v y_{v'} y_{u'} = \psi(x_{v'})\psi(\kappa_u)$ and $\psi(\kappa_u)\psi(x_v) = y_{u'} y_v = y_v y_{u'} = \psi(x_v)\psi(\kappa_u)$.

   For every $u \in V_p \cup V_\infty$ and every $v \in V_2$, there is an edge $e = \{u',v\} \in E(\Lambda)$ with $m'(e) = 2$. So $\psi(\kappa_u)\psi(x_v) = y_{u'} y_v = y_v y_{u'} = \psi(x_v)\psi(\kappa_u)$.

8. For every $v \in V_p$, we have $\psi(\kappa_u)\psi(x_u)\psi(\kappa_u) = y_{u'} y_{u'} y_u y_{u'} = y_u y_{u'} = (y_{u'} y_u)^{-1}$ $= \psi(x_u)^{-1}$. For every $u \in V_\infty$, we have $\psi(\kappa_u)\psi(x_u)\psi(\kappa_u) = y_{u'} y_u y_{u'} = \psi(x_{u'})$.

So the map ψ induces a homomorphism $\psi : U \to W$.

We now check that $\phi \circ \psi = \mathrm{Id}_U$ and that $\psi \circ \phi = \mathrm{Id}_W$. For $v \in V_2$, we have $\psi(\phi(y_u)) = y_u$ and $\phi(\psi(x_u)) = x_u$. For $v \in V_p$, we have $\psi(\phi(y_u)) = \psi(\kappa_u x_u) = y_{u'} y_{u'} y_u = y_u$ and $\psi(\phi(y_{u'})) = \psi(\kappa_u) = y_{u'}$, as well as $\phi(\psi(x_u)) = \phi(y_{u'} y_u) = \kappa_u \kappa_u x_u = x_u$ and $\phi(\psi(\kappa_u)) = \phi(y_{u'}) = \kappa_u$. For $v \in V_\infty$, we have $\psi(\phi(y_u)) = c\psi(x_u) = y_u$ and $\psi(\phi(y_{u'})) = \psi(\kappa_u) = y_{u'}$, as well as $\phi(\psi(x_u)) = \phi(y_u) = x_u$ and $\phi(\psi(x_{u'})) = \phi(y_{u'} y_u y_{u'}) = \kappa_u x_u \kappa_u = x_{u'}$ and $\phi(\psi(\kappa_u)) = \phi(y_{u'}) = \kappa_u$. $\square$

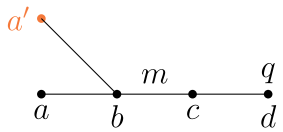


Figure 3.3: The graph $\Omega_{m,q}$ build out of the Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$. There are two types of vertices: $V \subset V(\Lambda_{m,q})$ and $\{v' \mid v \in V_\infty\}$. Every vertex and every edge is labeled by 2 if not specified otherwise.

Example 3.1.13. We apply the previous theorem to Example 3.1.6. The corresponding graph $\Omega_{m,q}$ is given in Figure 3.3. The associated Dyer group is

$$D'_{m,q} = \langle a, b, c, d, a' \mid a^2 = a'^2 = b^2 = c^2 = d^q = \mathbf{e}, \\ ab = ba, a'b = ba', (bc)^m = \mathbf{e}, cd = dc\rangle.$$

It is an index 4 subgroup of the Coxeter group W associated to the graph $\Lambda_{m,q}$ given in Figure 3.2.

Remark 3.1.14. If the Dyer group D is a right-angled Artin group, i.e. $V = V_\infty$, the constructions described here are those given in [DJ00]. In particular if D is a right-angled Artin group, the groups W and D' are right-angled Coxeter groups. So there is a decomposition of W as a semi direct product of a right-angled Artin group and the right-angled Coxeter group $(\mathbb{Z}/2\mathbb{Z})^V$. On the other hand if $V_\infty = \emptyset$, we have $(\Omega, g, n) = (\Gamma, f, m)$ and so $D = D'$.

Remark 3.1.15. There is a Coxeter group W' associated to the Dyer group D' such that $W' = D' \rtimes (\mathbb{Z}/2\mathbb{Z})^{V_p}$. It follows from Remark 3.1.14, that the Dyer groups D and D' are not necessarily isomorphic. Question: do W and W' relate in any (meaningful) way? What can we say about their Davis-Moussong complexes? How do D and D' relate to each other? What are all the Dyer subgroups of a given Coxeter group?

3.2 The piecewise Euclidean cell complex Σ

The goal of this section is to show intrinsically that Dyer groups are CAT(0) by constructing an appropriate Euclidean cell complex Σ. The first step is to construct a scwol $\mathcal{C}$ associated to a Dyer group. The scwol $\mathcal{C}$ encodes the necessary information in order to build Σ. The vertices of $\mathcal{C}$ will correspond to subcomplexes of Σ and the edges of $\mathcal{C}$ will encode identifications between subcomplexes of Σ. Finally we will also be able to interpret $\mathcal{C}$ as a simplicial subdivision of the complex

Σ. We will first focus on spherical Dyer groups D which factor as a direct product of a finite Coxeter group and cyclic groups. We start with the construction of a scwol $\mathcal{X}$ associated to a spherical Dyer group D and then define a complex of groups $\mathfrak{D}(\mathcal{X})$. The scwol $\mathcal{C}$ will be the development of the complex of groups $\mathfrak{D}(\mathcal{X})$. The second subsection will discuss this for general Dyer groups. The third subsection will be devoted to the Euclidean cell complex Σ.

3.2.1 A combinatorial structure for spherical Dyer groups

We say that a Dyer graph (Γ, f, m) is *spherical*, if Γ is comple and the group D_2 associated with the subgraph $(\Gamma_2, f_{V_2}, m_{V_2})$ is a finite group. A Dyer group $D = D(\Gamma, f, m)$ is *spherical* if it the Dyer graph (Γ, f, m) is spherical. In this section, we will assume that D is a spherical Dyer group. In particular with Remark 3.1.4, we then have $D = D_2 \times D_p \times D_\infty$, where D_2 is a finite Coxeter group, D_p is a direct product of finite cyclic groups and $D_\infty = \mathbb{Z}^{V_\infty}$.

As with Coxeter groups, we can characterize spherical Dyer groups through the cosine matrix. Let (Γ, f, m) be a Dyer graph and let $V = V(\Gamma)$ and $E = E(\Gamma)$. We extend the map $m : E \to \mathbb{N}_{\geq 2}$ to a map $m : V \times V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ by setting $m(u, v) = m(\{u, v\})$ if $\{u, v\} \in E$, and $m(u, v) = \infty$ if $u \neq v$ and $\{u, v\} \notin E$, and $m(u, u) = 1$. We interpret π/∞ to be 0 and $\cos(\pi - \pi/\infty) = \cos(\pi) = -1$. The cosine matrix associated with a Dyer graph (Γ, f, m) is the $V \times V$ matrix $c = (c_{uv})_{u,v \in V}$ defined by $c_{uv} = \cos(\pi - \pi/m(u, v))$. Note $c_{uu} = 1$ for all $u \in V$ and that $c_{uv} = 0$ whenever u, v are in disjoint components of $V = V_2 \sqcup V_p \sqcup V_\infty$. This induces a decomposition of the matrix c into three blocks $\begin{pmatrix} c_2 & 0 & 0 \\ 0 & c_p & 0 \\ 0 & 0 & c_\infty \end{pmatrix}$, where $c_2 = (c_{uv})_{u,v \in V_2}$, $c_p = (c_{uv})_{u,v \in V_p}$ and $c_\infty = (c_{uv})_{u,v \in V_\infty}$. The blocks c_p and c_∞ are the identity matrix, as $m(u, v) = 2$ and so $c_{uv} = 0$ for all distinct $u, v \in V_p \cup V_\infty$.The following characterization of spherical Dyer groups follows from Fact 2.2.1.

Lemma 3.2.1. *A Dyer group $D(\Gamma, f, m)$ is spherical if and only if the cosine matrix c associated to (Γ, f, m) is positive definite.*

Proof. Assume D is a spherical Dyer group. Then the restriction of c to $V_2 \times V_2$ is positive definite. Since additionally Γ is a complete Dyer graph, the matrix c admits a block decomposition as above, where all three blocks are positive definite matrices. This implies that the matrix c is positive definite. Now assume the cosine matrix c associated to (Γ, f, m) is positive definite. Consider the matrix $M = (m(u, v))_{u,v \in V}$. Then the cosine matrix c associated to (Γ, f, m) is equal to the cosine matrix of the Coxeter matrix M as defined in Section 2.2. So by

Fact 2.2.1 the cosine matrix c is positive definite if and only if the Coxeter group associated to M is finite. So we have $m(u,v) \neq \infty$ for all $u, v \in V$. Moreover since Γ is a Dyer graph this also implies that the restriction of c to $V_2 \times V_2$ is positive definite. So the graph Γ is complete and D_2 is a finite Coxeter group by Fact 2.2.1. Hence D is a spherical Dyer group.

□

Let $\mathcal{X} = \mathcal{X}(\Gamma)$ be the scwol with set of vertices $V(\mathcal{X}) = \{X \subseteq V\}$ and set of edges $E(\mathcal{X}) = \{(X, Y, \omega) \mid X \subsetneq Y \subseteq V(\Gamma),\ \omega \subseteq (Y \setminus X)_\infty\}$ with $i(X, Y, \omega) = X$ and $t(X, Y, \omega) = Y$ and $(Y, Z, \omega')(X, Y, \omega) = (X, Z, \omega \cup \omega')$. We call $\mathcal{X}$ the scwol associated with the spherical Dyer graph Γ. Similarly to the group D, we can also describe $\mathcal{X}$ as a direct product of scwols.

Lemma 3.2.2. *Let $\mathcal{X}_2 = \mathcal{X}(\Gamma_2)$, $\mathcal{X}_p = \mathcal{X}(\Gamma_p)$ and $\mathcal{X}_\infty = \mathcal{X}(\Gamma_\infty)$. Then we have the product decomposition $\mathcal{X} = \mathcal{X}_2 \times \mathcal{X}_p \times \mathcal{X}_\infty$. Moreover $\mathcal{X}_p = \mathcal{Y}_{V_p} = \times_{v \in V_p} \mathcal{Y}_v$ and $\mathcal{X}_\infty = \mathcal{Z}_{V_\infty} = \times_{v \in V_\infty} \mathcal{Z}_v$ as in Examples 2.1.4 and 2.1.5.*

Proof. Since $V = V_2 \sqcup V_p \sqcup V_\infty$, every $X \in V(\mathcal{X})$ can be decomposed as a disjoint union $X = X_2 \sqcup X_p \sqcup X_\infty$. Identifying X with (X_2, X_p, X_∞) gives $V(\mathcal{X}) = V(\mathcal{X}_2) \times V(\mathcal{X}_p) \times V(\mathcal{X}_\infty)$. For the edges, note that $(X, Y, \omega) \in E(\mathcal{X})$ if and only if $X_i \subset Y_i$ for every $i \in \{2, p, \infty\}$ and at least one of those inclusions is strict and $\omega \subset Y_\infty \setminus X_\infty$. The edge $(X, Y, \omega) \in E(\mathcal{X})$ is identified with $(E_2, E_p, E_\infty) \in E(\mathcal{X}_2 \times \mathcal{X}_p \times \mathcal{X}_\infty)$, where $E_i = X_i \in V(\mathcal{X}_i)$ whenever $X_i = Y_i$, and $E_i = (X_i, Y_i, \omega \cap V_i) \in E(\mathcal{X}_i)$ whenever $X_i \subsetneq Y_i$ for $i \in \{2, p, \infty\}$. □

We now define a simple complex of groups $\mathfrak{D}(\mathcal{X})$ over the scwol $\mathcal{X}$. For each $X \in V(\mathcal{X})$, let the local group be $D_X^f = D_{X_2 \cup X_p}$. As mentioned in Remark 3.1.4, we know by [Dye90] that if $X \subset Y$, then $D_X^f < D_Y^f < D$. For each edge $(X, Y, \omega) \in E(\mathcal{X})$, let $\psi_{(X,Y,\omega)} : D_X^f \to D_Y^f$ be the map induced by $\psi(x_v) = x_v$ for every $v \in X_2 \cup X_p$. These maps are all injective. Note that they do not depend on ω. We also introduce the morphism $\phi = \phi^\Gamma : \mathfrak{D}(\mathcal{X}) \to D$ where $\phi_X = \phi_X^\Gamma : D_X^f \to D$ is the natural inclusion and $\phi(X, Y, \omega) = \phi^\Gamma(X, Y, \omega) = \Pi_{v \in \omega} x_v$. Note that $\phi(X, Y, \omega)$ is well-defined since the subgraph spanned by $\omega \subset V_\infty$ is complete. On can see that $\mathrm{Ad}(\phi(X, Y, \omega)) \circ \phi_X = \phi_X = \phi_Y \circ \psi_{(X,Y,\omega)}$, as $x_v x_u x_v^{-1} = x_u$ for every $u \in X \subset Y$ and $v \in \omega \subset Y$ holds, and that for edges (Y, Z, λ), (X, Y, ω) and their composition $(Y, Z, \lambda)(X, Y, \omega) = (X, Z, \omega \cup \lambda)$ we have $\phi(Y, Z, \lambda)\phi(X, Y, \omega) = \Pi_{v \in \lambda} x_v \Pi_{v \in \omega} x_v = \Pi_{v \in \omega \cup \lambda} x_v = \phi(X, Z, \lambda \cup \omega)$. Moreover $\phi(X, Y, \omega)$ only depends on ω, so we will write $\phi(X, Y, \omega) = \phi(\omega) = \Pi_{v \in \omega} x_v$. Also note that each local group D_X^f is finite.

Remark 3.2.3. For every $X \in V(\mathcal{X})$, the local group D_X^f can be decomposed as $D_X^f = D_{X_2}^f \times D_{X_p}^f$. As $D_{X_\infty}^f$ is the trivial group, we have $D_X^f \cong D_{X_2}^f \times D_{X_p}^f \times D_{X_\infty}^f$. So using Lemma 3.2.2, we have that the complex $\mathfrak{D}(\mathcal{X})$ is isomorphic to the product $\mathfrak{D}(\mathcal{X}_2) \times \mathfrak{D}(\mathcal{X}_p) \times \mathfrak{D}(\mathcal{X}_\infty)$. Moreover $\mathfrak{D}(\mathcal{X}_p)$ is isomorphic to $\Pi_{v \in V_p} \mathfrak{D}(\mathcal{Y}_v)$ and $\mathfrak{D}(\mathcal{X}_\infty)$ is isomorphic to $\Pi_{v \in V_\infty} \mathfrak{D}(\mathcal{Z}_v)$. The morphism $\phi = \phi^\Gamma$ also decomposes as a product $\phi = \phi_2 \times \phi_p \times \phi_\infty$ where $\phi_2 = \phi^{\Gamma_2}$, $\phi_p = \phi^{\Gamma_p}$ and $\phi_\infty = \phi^{\Gamma_\infty}$.

Lemma 3.2.4. *The fundamental group of $\mathfrak{D}(\mathcal{X})$ is D and the complex of groups $\mathfrak{D}(\mathcal{X})$ is developable.*

Proof. We use the product decomposition given in Remark 3.2.3. The scwol $\mathcal{X}_2$ is associated to the poset $\mathcal{P}(V_2)$ so it is simply connected. Moreover it contains a unique maximal element V_2 so the fundamental group of $\mathfrak{D}(\mathcal{X}_2)$ is $D_{V_2}^f = D_2$. The same argument implies that the fundamental group of $\mathfrak{D}(\mathcal{X}_p)$ is D_p. Recall that $\mathfrak{D}(\mathcal{X}_\infty)$ is isomorphic to $\Pi_{v \in V_\infty} \mathfrak{D}(\mathcal{Z}_v)$. The fundamental group of each $\mathfrak{D}(\mathcal{Z}_v)$ is $\mathbb{Z}$. So the fundamental group of $\mathfrak{D}(\mathcal{X}_\infty)$ is $\mathbb{Z}^{|V_\infty|} = D_\infty$. So the fundamental group of $\mathfrak{D}(\mathcal{X})$ is $D_2 \times D_p \times D_\infty = D$. Since the maps ϕ_X are injective for all $X \in V(\mathcal{X})$, the complex $\mathfrak{D}(\mathcal{X})$ is developable. □

Since the complex $\mathfrak{D}(\mathcal{X})$ is developable, we can define its development

$$\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi).$$

Remark 3.2.5. Since $D_X^f < D$ and the maps ϕ_X are canonical inclusions, we will identify the image $\phi_X(D_X^f)$ with $D_X^f < D$. The set of vertices of $\mathcal{C}$ is

$$V(\mathcal{C}) = \{(gD_X^f, X) \mid X \in V(\mathcal{X}),\ gD_X^f \in D/D_X^f\}.$$

The set of edges of $\mathcal{C}$ is

$$E(\mathcal{C}) = \{(gD_X^f, (X, Y, \omega)) \mid (X, Y, \omega) \in E(\mathcal{X}),\ gD_X^f \in D/D_X^f\}$$

with initial vertex given by $i(gD_X^f, (X, Y, \omega)) = (gD_X^f, X)$ and terminal vertex given by $t(gD_X^f, (X, Y, \omega)) = (g\phi(\omega)^{-1}D_Y^f, Y)$. For a simpler notation, we write gX for a vertex (gD_X^f, X) and $g(X, Y, \omega)$ for an edge $(gD_X^f, (X, Y, \omega))$. Note that $gX = hY$ if and only if $X = Y$ and $g^{-1}h \in D_X^f$. Similarly $g(X, Y, \omega) = h(X', Y', \omega')$ if and only if $X' = X$, $Y' = Y$, $\omega' = \omega$ and $g^{-1}h \in D_X^f$. In particular, the scwol $\mathcal{X}$ is the quotient of $\mathcal{C}$ by the action of the group D.

Lemma 3.2.6. *The development $\mathcal{C}(\mathcal{X}, \phi)$ has a product decomposition*

$$\mathcal{C}(\mathcal{X}_2, \phi_2) \times \mathcal{C}(\mathcal{X}_p, \phi_p) \times \mathcal{C}(\mathcal{X}_\infty, \phi_\infty).$$

Proof. This follows from the product decomposition of $\mathcal{X}$, $\mathfrak{D}(\mathcal{X})$, D and ϕ. $\square$

Remark 3.2.7. For $i \in \{2, p, \infty\}$, Lemma 3.2.6 implies that we can consider each scwol $\mathcal{C}(\mathcal{X}_i, \phi_i)$ to be a subscwol of $\mathcal{C}(\mathcal{X}, \phi)$. There is a canonical inclusion by identifying a vertex $gX \in \mathcal{C}(\mathcal{X}_i, \phi_i)$ with $gX \in \mathcal{C}(\mathcal{X}, \phi)$. The subscwol $\mathcal{C}(\mathcal{X}_i, \phi_i)$ is then stable under the action of D_i.

Links and stars in $\mathcal{C}$ For a well-defined construction of the future cell complex Σ we need to understand the local combinatorial structure of the scowl $\mathcal{C}$, when (Γ, f, m) is spherical. As edges in $\mathcal{C}(\mathcal{X}, \phi)$ are oriented, we will distinguish the incoming and the outgoing link and star of a vertex. Let $gY \in V(\mathcal{C})$. The incoming link $\mathrm{Lk}_{in}(gY, \mathcal{C})$ is the full subscwol of $\mathcal{C}$ spanned by the vertices $\{hZ \mid \exists e \in E(\mathcal{C}) : t(e) = gY \text{ and } i(e) = hZ\}$. Similarly the outgoing link $\mathrm{Lk}_{out}(gY, \mathcal{C})$ is the full subscwol of $\mathcal{C}$ spanned by the vertices $\{hZ \mid \exists e \in E(\mathcal{C}) : i(e) = gY \text{ and } t(e) = hZ\}$. The incoming star is the subscwol spanned by gY and its incoming link so it is the oriented combinatorial join

$$\mathrm{St}_{in}(gY, \mathcal{C}) = \mathrm{Lk}_{in}(gY, \mathcal{C}) \star \{gY\}$$

and the outgoing star is defined similarly

$$\mathrm{St}_{out}(gY, \mathcal{C}) = \{gY\} \star \mathrm{Lk}_{out}(gY, \mathcal{C}).$$

Remark 3.2.8. The incoming star $\mathrm{St}_{in}(gY, \mathcal{C})$ is isomorphic to the incoming star $\mathrm{St}_{in}(\mathbf{e}\,Y, \mathcal{C}(\mathcal{X}(\Gamma_Y), \phi^{\Gamma_Y}))$. Moreover the product decomposition of $\mathcal{C}$ induces a product decomposition of the incoming star

$$\mathrm{St}_{in}(\mathbf{e}\,Y, \mathcal{C}) = \mathrm{St}_{in}(\mathbf{e}\,Y_2, \mathcal{C}(\mathcal{X}_2, \phi_2)) \times \mathrm{St}_{in}(\mathbf{e}\,Y_p, \mathcal{C}(\mathcal{X}_p, \phi_p)) \times \mathrm{St}_{in}(\mathbf{e}\,Y_\infty, \mathcal{C}(\mathcal{X}_\infty, \phi_\infty))$$

and as such also a product decomposition for every $\mathrm{St}_{in}(gY, \mathcal{C})$. Moreover for a vertex $hZ \in \mathrm{St}_{in}(gY, \mathcal{C})$, the star $\mathrm{St}_{in}(hZ, \mathcal{C})$ is a subscwol of $\mathrm{St}_{in}(gY, \mathcal{C})$.

3.2.2 A combinatorial structure for general Dyer groups

Let us now give a similar construction with analogous results for general Dyer groups. Let (Γ, f, m) be a Dyer graph and $D = D(\Gamma)$ be the associated Dyer group. We set $V = V(\Gamma)$. Let $\mathcal{X} = \mathcal{X}(\Gamma)$ be the scwol with set of vertices $V(\mathcal{X}) = \{X \subset V \mid D(\Gamma_X) \text{ is a spherical Dyer group}\}$ and set of edges $E(\mathcal{X}) = \{(X, Y, \omega) \mid X, Y \in V(\mathcal{X}),\ X \subsetneq Y,\ \omega \subset (Y \backslash X)_\infty\}$ with $i(X, Y, \omega) = X$ and $t(X, Y, \omega) = Y$ and $(Y, Z, \omega')(X, Y, \omega) = (X, Z, \omega \cup \omega')$. The main difference with the spherical case, is the set of vertices of $\mathcal{X}$. Indeed we do not consider all subsets $X \subset V$ but only those for which Γ_X is complete and the group $D_X^f = D_{X_2 \cup X_p}$ is finite. We also define a complex of groups $\mathfrak{D}(\mathcal{X})$ over $\mathcal{X}$. For each $X \in V(\mathcal{X})$, let the local group be $D_X^f = D_{X_2 \cup X_p}$ and for each edge $(X, Y, \omega) \in E(\mathcal{X})$, let $\psi_{(X,Y,\omega)} : D_X^f \to D_Y^f$ be the natural inclusion map. By [Dye90], these maps are all injective. The local groups are all finite. We also introduce the morphism $\phi : \mathfrak{D}(\mathcal{X}) \to D$ where $\phi_X : D_X^f \to D$ is the natural inclusion and $\phi(X, Y, \omega) = \phi(\omega) = \Pi_{v \in \omega} x_v$ (this element is well defined since $\omega \subset V_\infty$ and Γ_ω is complete). We verify the compatibility conditions. First $\mathrm{Ad}(\phi(X, Y, \omega)) \circ \phi_X = \phi_X = \phi_Y \circ \psi_{(X,Y,\omega)}$, as D_Y is a spherical Dyer group and so $x_v x_u x_v^{-1} = x_u$ for every $u \in X \subset Y$ and $v \in \omega \subset Y$. Secondly for edges (Y, Z, λ), (X, Y, ω) and their composition $(Y, Z, \lambda)(X, Y, \omega) = (X, Z, \omega \cup \lambda)$ we have $\phi(Y, Z, \lambda)\phi(X, Y, \omega) = \Pi_{v \in \lambda} x_v \Pi_{v \in \omega} x_v = \Pi_{v \in \omega \cup \lambda} x_v = \phi(X, Z, \lambda \cup \omega)$, as $\omega \cup \lambda \subset Z$ and D_Z is a spherical Dyer group. As in the spherical case, we can write $\mathfrak{D}(\mathcal{X}(\Gamma))$ and ϕ^Γ when also considering the same construction on a subgraph. As before, we are interested in the development of the complex of groups $\mathfrak{D}(\mathcal{X})$, so we first show that $\mathfrak{D}(\mathcal{X})$ is developable.

Example 3.2.9. Consider the Dyer graph $\Gamma_{m,q}$, given again in Figure 3.4, and the Dyer group $D_{m,q}$ from Example 3.1.6. The associated scwol $\mathcal{X}_{m,q} = \mathcal{X}(\Gamma_{m,q})$ is drawn in Figure 3.5. Its vertex set is

$$V(\mathcal{X}_{m,q}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}\}.$$

Its set of edges is $E(\mathcal{X}_{m,q}) = \{(X, Y, \omega) \mid X \subsetneq Y \subset V(\Gamma_{m,q}), \omega \subseteq Y_\infty \setminus X_\infty\}$. So in particular as $a \in V(\Gamma_{m,q})_\infty$, there are two edges between $\emptyset$ and $\{a\}$, namely $(\emptyset, \{a\}, \emptyset)$ and $(\emptyset, \{a\}, \{a\})$, between $\{b\}$ and $\{a, b\}$, namely $(\{b\}, \{a, b\}, \emptyset)$ and $(\{b\}, \{a, b\}, \{a\})$, and between $\emptyset$ and $\{a, b\}$, namely $(\emptyset, \{a, b\}, \emptyset)$ and $(\emptyset, \{a, b\}, \{a\})$.

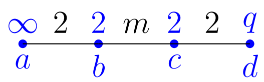


Figure 3.4: Dyer graph $\Gamma_{m,q}$ for some $m, q \in \mathbb{N}_{\geq 2}$ as given in Figure 3.1

Lemma 3.2.10. *The scwol $\mathcal{X}$ is isomorphic to the union of scwols $\mathcal{Y} = \bigcup_{Y \in V(\mathcal{X})} \mathcal{X}_Y$, where $\mathcal{X}_Y$ is the scwol associated with the spherical Dyer group D_Y. The fundamental group of $\mathfrak{D}(\mathcal{X})$ is D. In particular, the complex of groups $\mathfrak{D}(\mathcal{X})$ is developable.*

Proof. First we compare the sets of vertices. If $Y \in V(\mathcal{X})$ then $Y \in V(\mathcal{X}_Y)$ hence $Y \in V(\mathcal{Y})$. On the other hand if $Y \in V(\mathcal{Y})$, we have $Y \in V(\mathcal{X}_Z)$ for some $Z \in V(\mathcal{X})$ hence $Y \subset Z$ so that D_Y is spherical. This implies $V(\mathcal{X}) = V(\mathcal{Y})$. Now

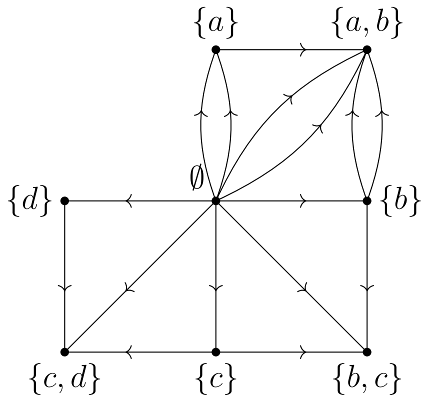


Figure 3.5: The scwol $\mathcal{X}_{m,q}$ associated with the graph $\Gamma_{m,q}$ given in Figure 3.4.

we compare the sets of edges. If $e = (X, Y, \omega) \in E(\mathcal{X})$ then $e \in E(\mathcal{X}_Y)$ hence $e \in E(\mathcal{Y})$. Conversely if $e \in E(\mathcal{Y})$, then $e \in E(\mathcal{X}_Z)$ for some $Z \in V(\mathcal{X})$ hence $e = (X, Y, \omega)$ with $X \subsetneq Y \subset Z$ and $\omega \subset (Y \setminus X)_\infty$ so that $e \in E(\mathcal{X})$. This implies that $E(\mathcal{X}) = E(\mathcal{Y})$. We can now apply the Seifert-van Kampen theorem for the fundamental group of a complex of groups [BH99][III.C Example 3.11(5)] to $\mathcal{Y}$. The set $V(\mathcal{X})$ is finite and each scwol $\mathcal{X}_Y$ is connected. We have $\emptyset \in V(\mathcal{X}_Y)$ for all $Y \in V(\mathcal{X})$ and $\emptyset$ is adjacent to any vertex in any $\mathcal{X}_Y$. So $\bigcap_{Y\in V(\mathcal{X})} \mathcal{X}_Y$ is nonempty and connected. We can then use the presentations to see that the fundamental group of $\mathfrak{D}(\mathcal{X})$ is D. Finally, by [Dye90], the maps $\phi_X : D^f_X \to D$ are all injective. Therefore $\mathfrak{D}(\mathcal{X})$ is developable. □

Since the complex $\mathfrak{D}(\mathcal{X})$ is developable, we can define its development

$$\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi).$$

Remark 3.2.11. Since $D^f_X < D$ and the maps ϕ_X are canonical inclusions, we will identify the image $\phi_X(D^f_X)$ with D^f_X. The set of vertices of $\mathcal{C}$ is

$$V(\mathcal{C}) = \{(gD^f_X, X) \mid X \in V(\mathcal{X}),\ gD^f_X \in D/D^f_X\}.$$

The set of edges of $\mathcal{C}(\mathcal{X}, \phi)$ is

$$E(\mathcal{C}) = \{(gD^f_X, (X, Y, \omega)) \mid (X, Y, \omega) \in E(\mathcal{X}),\ gD^f_X \in D/D^f_X\}$$

where initial vertices are given by $i(gD^f_X, (X, Y, \omega)) = (gD^f_X, X)$ and terminal vertices are given by $t(gD^f_X, (X, Y, \omega)) = (g\phi(\omega)^{-1}D^f_Y, Y)$. For a simpler notation, we write gX for a vertex (gD^f_X, X) and $g(X, Y, \omega)$ for an edge $(gD^f_X, (X, Y, \omega))$.

Note that $gX = hY$ if and only if $X = Y$ and $g^{-1}h \in D_X^f$. Similarly $g(X, Y, \omega) = h(X', Y', \omega')$ if and only if $X' = X$, $Y' = Y$, $\omega' = \omega$ and $g^{-1}h \in D_X^f$. As in the spherical case, the scwol $\mathcal{C}$ does not have multiple edges between two vertices.

Links and stars in $\mathcal{C}$ As before, we need to understand the local combinatorial strucuture of the scwol $\mathcal{C}$. As edges in $\mathcal{C}$ are oriented, we will distinguish the incoming and the outgoing link and star of a vertex. We recall the definitions here. Let $gY \in V(\mathcal{C})$. The incoming link $\mathrm{Lk}_{in}(gY, \mathcal{C})$ is the full subscwol of $\mathcal{C}$ spanned by the set of vertices $\{hZ \mid \exists e \in E(\mathcal{C}) : t(e) = gY \text{ and } i(e) = hZ\}$. Similarly the outgoing link $\mathrm{Lk}_{out}(gY, \mathcal{C})$ is the full subscwol of $\mathcal{C}$ spanned by the set of vertices $\{hZ \mid \exists e \in E(\mathcal{C}) : i(e) = gY \text{ and } t(e) = hZ\}$. The incoming star is the subscwol spanned by gY and its incoming link so it is the oriented combinatorial join

$$\mathrm{St}_{in}(gY, \mathcal{C}) = \mathrm{Lk}_{in}(gY, \mathcal{C}) \star \{gY\}$$

and the outgoing star is defined similarly

$$\mathrm{St}_{out}(gY, \mathcal{C}) = \{gY\} \star \mathrm{Lk}_{out}(gY, \mathcal{C}).$$

Lemma 3.2.12. *For every vertex $gY \in V(\mathcal{C})$, the scwols $\mathrm{St}_{in}(gY, \mathcal{C}(\mathcal{X}, \phi))$ and $\mathrm{St}_{in}(\mathbf{e}\,Y, \mathcal{C}(\mathcal{X}_Y, \phi^{\Gamma_Y}))$ are isomorphic.*

Proof. It suffices to show this for $g = \mathbf{e}$. Then the statement is clear as it follows directly from the definitions. □

3.2.3 The piecewise Euclidean cell complex Σ

The scwol $\mathcal{C}$, which is also a simplicial complex, described in the previous section is a combinatorial object. In order to build the cell complex Σ, we could try to endow the geometric realization of $\mathcal{C}$ with a CAT(0) metric. This would give a simplicial complex with a non-standard piecewise Euclidean metric. The problem is that Moussong's Lemma 1.1.16 does not apply directly to simplicial complexes with a piecewise Euclidean metric since dihedral angles should be at least $\pi/2$. The idea is to interpret $\mathcal{C}$ as some generalized face poset of Σ. Indeed $\mathcal{C}$ does not give us the face structure of Σ but some form of subcomplex structure. Each vertex in $\mathcal{C}$ corresponds to a subcomplex of Σ and edges give identifications between these subcomplexes. Nevertheless we will be able to interpret $\mathcal{C}$ as a simplicial subdivision of Σ. We start with the description and study of the subcomplexes associated to vertices, then build Σ and finally show that Σ is CAT(0) using Moussong's Lemma 1.1.16.

Let (Γ, f, m) be a Dyer graph, $D = D(\Gamma, f, m)$ the associated Dyer group, $\mathcal{X} = \mathcal{X}(\Gamma)$ the associated scwol and $\mathfrak{D}(\mathcal{X})$ the associated complex of groups. Consider the injective morphism $\phi : \mathfrak{D}(\mathcal{X}) \to D$ given by the natural inclusion maps $\phi_X : D_X^f \to D$ and $\phi(X, Y, \omega) = \phi(\omega) = \Pi_{v\in\omega} x_v$. As in the previous section, we construct the development $\mathcal{C} = \mathcal{C}(\mathcal{X}, \phi)$.

Elementary building blocks Let $Y \in V(\mathcal{X})$. First we consider elementary building blocks in the cases $Y = Y_2$, $Y = Y_\infty$ and $Y = Y_p$. For $Y \in V(\mathcal{X})$ with $Y = Y_2$, let $\mathrm{Cox}(Y)$ be the Coxeter polytope associated to the Coxeter group D_Y endowed with its natural Euclidean metric as described in Section 2.2. Its set of vertices is D_Y. For $Y \in V(\mathcal{X})$ with $Y = Y_\infty$, consider $[0,1]^Y \subset \mathbb{R}^Y$ with its standard cubical structure. Its set of vertices is $\mathcal{P}(Y)$, where $0 \in \mathbb{R}^Y$ corresponds to $\emptyset \in \mathcal{P}(Y)$. For $v \in V_p$, let $\mathrm{Stern}(v)$ be the $f(v)$-branched star where each edge of the star is identified with $[0,1]$. Its center is denoted by c_v and its leaves are identified with the elements of the finite cyclic group $C_{f(v)}$ of order $f(v)$. For $Y \in V(\mathcal{X})$ with $Y = Y_p$, let $\mathrm{Stern}(Y)$ be the product of stars $\Pi_{v\in Y} \mathrm{Stern}(v)$ endowed with the ℓ_2 metric. So its vertex set is $\Pi_{v\in Y}(\{c_v\} \cup C_{f(v)})$. Note that $V(\mathrm{Stern}(Y)) = \Pi_{v\in Y}(\{c_v\} \cup C_{f(v)})$ can be identified with $\coprod_{Z\subset Y} D_Y/D_Z$. We identify a vertex $(g_v)_{v\in Y} \in \Pi_{v\in Y}(\{c_v\} \cup C_{f(v)})$ with $gD_Y/D_Z \in \coprod_{Z\subset Y} D_Y/D_Z$ where $Z = \{v \in Y \mid g_v = c_v\}$ and $g = \underset{v\in Y\setminus Z}{\Pi} g_v$. Since Γ_Y is a complete graph and $Y = Y_p$ the element $g \in D_Y$ is well-defined.Let us denote a vertex $gD_Z \in D_Y/D_Z$ in $\mathrm{Stern}(Y)$ with gZ.

The cell complex $\mathrm{Cc}(Y)$ To every $Y \in V(\mathcal{X})$, we associate a Euclidean cell complex $\mathrm{Cc}(Y)$ as follows. Let $\mathrm{Cc}(Y)$ be the product $\mathrm{Cox}(Y_2) \times [0,1]^{Y_\infty} \times \mathrm{Stern}(Y_p)$ endowed with the ℓ_2 metric. Each of its factors is a piecewise Euclidean cell complex, so $\mathrm{Cc}(Y)$ is a piecewise Euclidean cell complex. In particular, $\mathrm{Cc}(Y) = \mathrm{Cc}(Y_2) \times \mathrm{Cc}(Y_\infty) \times \mathrm{Cc}(Y_p)$. The set of vertices of $\mathrm{Cc}(Y)$ is $D_{Y_2} \times \mathcal{P}(Y_\infty) \times \Pi_{v\in Y_p}(\{c_v\} \cup C_{f(v)})$. The group D_Y^f acts by isometries on $\mathrm{Cc}(Y)$. Indeed $D_Y^f = D_{Y_2} \times \Pi_{v\in Y_p} C_{f(v)}$. So D_Y^f acts through D_{Y_2} on $\mathrm{Cox}(Y_2)$ and through $C_{f(v)}$ on $\mathrm{Stern}(v)$ for $v \in Y_p$. These actions are all isometries.

Remark 3.2.13 (Links of vertices). As we will need to understand links of vertices in subcomplexes, we start by studying links of vertices in $\mathrm{Cc}(Y)$. Recall 1.2.2 for the iterated spherical join of piecewise spherical simplicial complexes. Consider a vertex $l = (w, \lambda, gZ) \in \mathrm{Cc}(Y)$. Its link is the spherical join

$$\mathrm{Lk}(w, \mathrm{Cox}(Y_2)) \star \mathrm{Lk}(\lambda, [0,1]^{|Y_\infty|}) \star \mathrm{Lk}(gZ, \mathrm{Stern}(Y_p)).$$

As mentioned in 2.2, the term $\mathrm{Lk}(w, \mathrm{Cox}(Y_2))$ is identified with the piecewise spherical flag complex with 1-skeletton Γ_{Y_2} and edge length $d(u, v) = \pi - \pi/m(u, v)$ for two vertices $u, v \in Y_2$. The link $\mathrm{Lk}(gZ, \mathrm{Stern}(Y_p))$ is isometric to $\mathrm{Lk}(Z, \mathrm{Stern}(Y_p))$ which is isometric to the spherical join

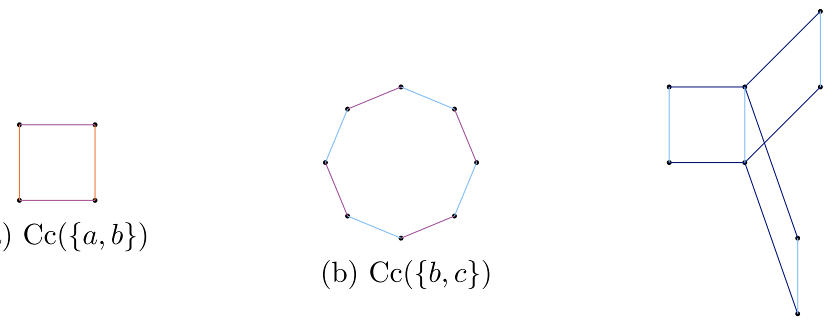
$$\star_{v\in Z} \mathrm{Lk}(c_v, \mathrm{Stern}(v)) \star_{v\in Y_p\setminus Z} \mathrm{Lk}(\mathbf{e}, \mathrm{Stern}(v)).$$

Each term $\mathrm{Lk}(c_v, \mathrm{Stern}(v))$ consists of $|C_{f(v)}|$ disjoint vertices and each term $\mathrm{Lk}(\mathbf{e}, \mathrm{Stern}(v))$ consists of a single vertex. For every $\lambda \subset Y_\infty$, the term $\mathrm{Lk}(\lambda, [0, 1]^{|Y_\infty|})$ is isometric to the spherical join $(\star_{v\in\lambda}\lambda \setminus \{v\}) \star \left(\star_{v\in Y_\infty\setminus\lambda}\lambda \cup \{v\}\right)$. So the link $\mathrm{Lk}(l, \mathrm{Cc}(Y))$ is isometric to the spherical join

$$\mathrm{Lk}(w, \mathrm{Cox}(Y_2)) \star \left((\star_{v\in\lambda}\lambda \setminus \{v\}) \star \left(\star_{v\in Y_\infty\setminus\lambda}\lambda \cup \{v\}\right)\right) \star (\star_{v\in Z} \mathrm{Lk}(c_v, \mathrm{Stern}(v))) \star (\star_{v\in Y_p\setminus Z} \mathrm{Lk}(e, \mathrm{Stern}(v))).$$

Note that in particular for two vertices $u, v \in V(\mathrm{Lk}(l, \mathrm{Cc}(Y)))$ in two different terms of the decomposition, we have $d(u, v) = \pi/2$.

Example 3.2.14. Let $m, p \in \mathbb{N}_{\geq 2}$. We go back to the example of the Dyer graph $\Gamma_{m,p}$ with associated Dyer group $D_{m,p}$ and scwol $\mathcal{X}_{m,p}$ given in Figure 3.1, Example 3.1.6 and Figure 3.5. Figure 3.6 shows the cell complexes $\mathrm{Cc}(\{a, b\})$, $\mathrm{Cc}(\{b, c\})$, $\mathrm{Cc}(\{c, d\})$ in the case $m = 4$ and $p = 3$.



(a) $\mathrm{Cc}(\{a, b\})$ (b) $\mathrm{Cc}(\{b, c\})$ (c) $\mathrm{Cc}(\{c, d\})$

Figure 3.6: The cell complexes associated to some vertices of $\mathcal{X}_{m,p}$.

The cell complex $\Sigma(gY)$ Let $Y \in V(\mathcal{X})$ and $gD^f_Y \in D/D^f_Y$ so that $gY \in V(\mathcal{C})$. We now describe the subcomplexes of Σ associated to vertices of $\mathcal{C}$. We start by identifying the vertex set of $\mathrm{Cc}(Y)$ with a subset of $V(\mathrm{St}_{in}(Y, \mathcal{C}))$ and more generally with a subset of $V(\mathrm{St}_{in}(gY, \mathcal{C}))$. Let $V_p(gY)$ be the following subset of

$V(\mathrm{St}_{in}(gY, \mathcal{C}))$:

$$V_p(gY) = \{kX \in V(\mathcal{C}) \mid X \subseteq V_p \text{ and } kX \in V(\mathrm{St}_{in}(gY, \mathcal{C}))\}.$$

By definition $kX \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$ if and only if $kX = gY$ or there exists an unique edge $h(X, Y, \omega)$ in $\mathcal{C}$ with initial vertex $kX = hX$ and terminal vertex $gY = k\phi(\omega)^{-1}Y$. So $kX \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$ if and only if $X \subseteq Y$ and there exists unique $\omega \subseteq Y_\infty \setminus X_\infty$ with $k(\Pi_{v\in\omega}x_v)^{-1}D_Y^f = gD_Y^f$. So

$$V_p(gY) = \{kX \in V(\mathcal{C}) \mid X \subseteq V_p \cap Y \text{ and } k(\Pi_{v\in\omega}x_v)^{-1}D_Y^f = gD_Y^f \text{ with } \omega \subseteq Y_\infty\}.$$

Lemma 3.2.15. *The map $j : V(\mathrm{Cc}(Y)) \to V_p(\mathbf{e}Y)$ given by $j(w, \lambda, hZ) = w\phi(\lambda)hZ$ is bijective. Moreover it induces a bijective map $j_g : V(\mathrm{Cc}(Y)) \to V_p(gY)$ with $j_g(w, \lambda, hZ) = g \cdot j(w, \lambda, hZ) = gw\phi(\lambda)hZ$.*

Proof. Let $Z \in V(\mathcal{X})$ and $kD_Z^f \in D/D_Z^f$ so that $kZ \in V_p(\mathbf{e}Y)$. So we have $Z \subseteq V_p \cap Y$ and $k(\Pi_{v\in\lambda}x_v)^{-1}D_Y^f = D_Y^f$ for some $\lambda \subseteq Y_\infty$. As $D_Y^f = D_{Y_2} \times D_{Y_p}$ the representative $k(\Pi_{v\in\lambda}x_v)^{-1}$ has a unique decomposition $k(\Pi_{v\in\lambda}x_v)^{-1} = k_2\Pi_{v\in Y_p}k_v$, with $k_2 \in D_{Y_2}$ and $k_v \in C_{f(v)}$ for every $v \in Y_p$. This gives a unique decomposition $k = k_2(\Pi_{v\in\lambda}x_v)(\Pi_{v\in Y_p}k_v)$. In particular $k \in D_Y$. As Γ_Y is complete and the coset $kD_Z^f \in D/D_Z^f$, we can assume $k_v = \mathbf{e}$ for every $v \in Z_p$. As D_{Z_2} is a parabolic subgroup of the Coxeter group D_{Y_2}, we can also assume k_2 to be the unique element of minimal length in $k_2D_{Z_2}$. So $j(k_2, \lambda, \Pi_{v\in Y_p\setminus Z}k_vZ) = kZ$. Hence the map j is surjective. Such a choice of k_2 and k_v, $v \in Y_p \setminus Z$ is independent of the representative k. Indeed let k' be another representative, so $k'D_Z^f = kD_Z^f$. Then again $k' = k_2'(\Pi_{v\in\lambda}x_v)(\Pi_{v\in Y_p}k_v')$. As $k^{-1}k' \in D_Z^f$, we have $k_2^{-1}k_2' \in D_{Z_2}$ so $k_2D_{Z_2} = k_2D_{Z_2}$ and so by uniqueness of the minimal representative $k_2 = k_2'$. Similarly $k_v = k_v'$ for every $v \in Y_p$. As there is a unique edge from kZ to $\mathbf{e}Y$, the subset $\lambda \subseteq Y_\infty$ is uniquely determined. Hence the map j is also injective so it is bijective.

Finally $kZ \in V_p(gY)$ if and only if $Z \subseteq V_p \cap Y$ and $k(\Pi_{v\in\lambda}x_v)^{-1}D_Y^f = gD_Y^f$ for some $^\lambda \subseteq Y_\infty$. So $kZ \in V_p(gY)$ if and only if $Z \subseteq V_p \cap Y$ and $g^{-1}k(\Pi_{v\in\lambda}x_v)^{-1}D_Y^f = D_Y^f$ for some $\lambda \subseteq Y_\infty$. So $kZ \in V_p(gY)$ if and only if $g^{-1}kZ \in V_p(\mathbf{e}Y)$. So the map j_g is bijective. □

For $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$, let $\Sigma(gY)$ be the piecewise Euclidean cell complex given as follows:

1. The set of vertices (or 0-cells) is $V_p(gY)$.

2. Every cell in $\Sigma(gY)$ is isometric to a cell in $\mathrm{Cc}(Y)$.

3. The map $j_g : V(\mathrm{Cc}(Y)) \to V_p(gY)$ extends to a cellular isometry $j_g : \mathrm{Cc}(Y) \to \Sigma(gY)$.

Let $hD_Y^f \in D/D_Y^f$ with $hD_Y^f = gD_Y^f$ then $j_g \circ j_h^{-1}$ is a cellular isometry from $\Sigma(hY)$ to $\Sigma(gY)$. So the cell structure on $\Sigma(gY)$ is well-defined.

We now discuss identifications of subcomplexes. Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. Let $Z \in V(\mathcal{X})$ and $hD_Z^f \in D/D_Z^f$ so that $hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$. Then $\mathrm{St}_{in}(hZ, \mathcal{C})$ is a subscwol of $\mathrm{St}_{in}(gY, \mathcal{C})$ and hence $V_p(hZ) \subset V_p(gY)$. The following lemma shows that this inclusion induces an isometric embedding of the cell complex $\Sigma(hZ)$ into $\Sigma(gY)$.

Lemma 3.2.16. *Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. Let $Z \in V(\mathcal{X})$ and $hD_Z^f \in D/D_Z^f$ so that $hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$. The cellular map $\iota : \Sigma(hZ) \to \Sigma(gY)$ satisfying $\iota(v) = v$ for every vertex $v \in V(\Sigma(hZ))$ is an isometric embedding. In particular we can identify $\Sigma(hZ)$ with $\iota(\Sigma(hZ))$.*

Proof. Since $hZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$ if and only if $g^{-1}hZ \in \mathrm{St}_{in}(Y, \mathcal{C})$, it suffices to consider the case $g = \mathbf{e}$. For $hZ \in V(\mathrm{St}_{in}(\mathbf{e}\,Y, \mathcal{C}))$ we can write $h = h_2 h_\infty h_p$ with $h_2 \in D_{Y_2}$, $h_\infty = \phi(\kappa)$ for a unique $\kappa \subset (Y \setminus Z)_\infty$ and $h_p \in D_{Y_p \setminus Z_p}$. We claim that the cellular map $\iota_h : \mathrm{Cc}(Z) \to \mathrm{Cc}(Y)$ given by $\iota(w, \lambda, mM) = (h_2 w, \lambda \cup \kappa, h_p mM)$ for $(w, \lambda, mM) \in V(\mathrm{Cc}(Z))$ (so $w \in D_{Z_2}$, $\lambda \subseteq Z_\infty$, $M \subseteq Z_p$ and $m \in D_{Z_p \setminus M}$) is an isometric embedding. Both $\mathrm{Cc}(Z) = \mathrm{Cox}(Z_2) \times [0,1]^{Z_\infty} \times \mathrm{Stern}(Z_p)$ and $\mathrm{Cc}(Y) = \mathrm{Cox}(Y_2) \times [0,1]^{Y_\infty} \times \mathrm{Stern}(Y_p)$ are endowed with the ℓ_2 metric. By Section 2.2, the cellular map $\iota_2 : \mathrm{Cox}(Z_2) \to \mathrm{Cox}(Y_2)$ with $\iota(w) = h_2 w$ for $w \in V(\mathrm{Cox}(Z_2))$ is an isometric embedding indentifying $\mathrm{Cox}(D_{Z_2}))$ with $h_2 \cdot \mathrm{Cox}_{Z_2}(D_{Y_2})$. The cellular map $\iota_\infty : [0,1]^{Z_\infty} \to [0,1]^{Y_\infty}$ with $\iota_\infty(\lambda) = \lambda \cup \kappa$ for $\lambda \in \mathcal{P}(Z_\infty)$ is also an isometric embedding identifying $\Pi_{v \in Z_\infty}[0,1]$ with $\Pi_{v \in Z_\infty}[0,1] \times \Pi_{v \in \kappa}\{1\} \times \Pi_{v \in Y_\infty \setminus (\kappa \cup Z_\infty)}\{0\} \subset \Pi_{v \in Y_\infty}[0,1]$. The cellular map $\iota_p : \mathrm{Stern}(Z_p) \to \mathrm{Stern}(Y_p)$ with $\iota_p(mM) = h_p mM$ for $mM \in V(\mathrm{Stern}(Z_p))$ is an isometric embedding identifying $\mathrm{Stern}(Z_p)$ with $\mathrm{Stern}(Z_p) \times \{h_p\} \subset \mathrm{Stern}(Y_p)$. So map ι_h decomposes as $\iota_h = (\iota_2, \iota_\infty, \iota_p) : \mathrm{Cc}(Z) \to \mathrm{Cc}(Y)$, hence it is a cellular isometric embedding. Then the map $\iota : \Sigma(hZ) \to \Sigma(\mathbf{e}\,Y)$ given by $j_{\mathbf{e}} \circ \iota_h \circ j_h^{-1}$ is a cellular isometric embedding. $\square$

Simplicial subdivision Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. We describe a simplicial subdivision of $\mathrm{Cc}(Y)$ and $\Sigma(gY)$. Let $Bar(\mathrm{Cox}(Y_2))$ be the barycentric subdivision of $\mathrm{Cox}(Y_2)$. Let $Bar([0,1])$ be the barycentric subdivision of $[0,1]$. Then $Bar(\mathrm{Cox}(Y_2))$, $Bar[0,1]$ and $\mathrm{Stern}(v)$, for $v \in Y_p$ are piecewise Euclidean simplicial complexes. Moreover the simplicial complex $Bar(\mathrm{Cox}(Y_2))$ is isomorphic to $\mathrm{St}_{in}(Y_2, \mathcal{C})$ by Lemma 2.2.3. For $v \in Y_\infty$, the simplicial complex $Bar([0,1])$ is isomorphic to $\mathrm{St}_{in}(\{v\}, \mathcal{C})$. For $v \in Y_p$ the simplicial complex $\mathrm{Stern}(v)$

is isomorphic to $\mathrm{St}_{in}(\{v\}, \mathcal{C})$. The isomorphisms induce a scwol structure on the barycentric subdivisions and on $\mathrm{Stern}(v)$. So the scwol

$$Bar(\mathrm{Cc}(Y)) = Bar(\mathrm{Cc}(Y_2)) \times \Pi_{v \in Y_\infty} Bar(\mathrm{Cc}(\{v\})) \times \Pi_{v \in Y_p} \mathrm{Cc}(\{v\})$$

is well-defined and isomorphic to

$$\mathrm{St}_{in}(Y, \mathcal{C}) = \mathrm{St}_{in}(Y_2, \mathcal{C}) \times \Pi_{v \in Y_\infty} \mathrm{St}_{in}(\{v\}, \mathcal{C}) \times \Pi_{v \in Y_p} \mathrm{St}_{in}(\{v\}, \mathcal{C}).$$

We endow $Bar(\mathrm{Cc}(Y))$ with the ℓ_2 metric, so it is piecewise Euclidean simplicial complex isometric to $\mathrm{Cc}(Y)$. We call $Bar(\mathrm{Cc}(Y))$ the *nice simplicial subdivision* of $\mathrm{Cc}(Y)$. We call the simplicial subdivision of $\Sigma(gY)$ induced by the isometry j_g the nice simplicial subdivision of $\Sigma(gY)$.

The next lemma discusses how $\mathrm{St}_{in}(gY, \mathcal{C})$ can be interpreted as a simplicial subdivision of $\Sigma(gY)$.

Lemma 3.2.17. *Let $Y \in V(\mathcal{X})$ and $gD_Y^f \in D/D_Y^f$ so that $gY \in V(\mathcal{C})$. The nice simplicial subdivision of $\Sigma(gY)$ is simplicially isomorphic to the scwol $\mathrm{St}_{in}(gY, \mathcal{C})$. For $Z \in V(\mathcal{X})$ and $kD_Z^f \in D/D_Z^f$ so that $kZ \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$, the isometric embedding $\iota : \Sigma(hZ) \to \Sigma(gY)$ given in Lemma 3.2.16 preserves the nice simplicial subdivision.*

Proof. The nice simplicial subdivision is isomorphic to $Bar(\mathrm{Cc}(Y))$ which is isomorphic to $\mathrm{St}_{in}(Y, \mathcal{C})$ which is isomorphic to $\mathrm{St}_{in}(gY, \mathcal{C})$.

The second statement follows from the product decomposition of the nice simplicial subdivision, the map ι and the complexes $\Sigma(hZ)$ and $\Sigma(gY)$. $\square$

Example 3.2.18. Let $m, q \in \mathbb{N}_{\geq 2}$. We go back to the example of the Dyer graph $\Gamma_{m,q}$ with associated Dyer group $D_{m,q}$ and scwol $\mathcal{X}_{m,q}$ given in Figure 3.1, Example 3.1.6 and Figure 3.5. Figure 3.7 shows the subcomplexes $\Sigma(\mathbf{e}\{a,b\})$, $\Sigma(\mathbf{e}\{b,c\})$, $\Sigma(\mathbf{e}\{c,d\})$ and their simplicial subdivision in the case $m = 4$ and $q = 3$.

The cell complex Σ We now have the tools needed to build the cell complex Σ. Consider

$$\Sigma = \bigcup_{gY \in V(\mathcal{C})} \Sigma(gY),$$

where we identify $\Sigma(hZ)$ with $\iota(\Sigma(hZ)) \subset \Sigma(gY)$ whenever $hZ \in \mathrm{St}_{in}(gY, \mathcal{C})$. So by Lemma 3.2.16, Σ has a well-defined piecewise Euclidean metric. We endow Σ with the associated length metric. The set of vertices of Σ is

$$V_p(\mathcal{C}) = \{gY \in V(\mathcal{C}) \mid Y \in V(\mathcal{X}),\ Y \subset V_p,\ gD_Y^f \in D/D_Y^f\}.$$

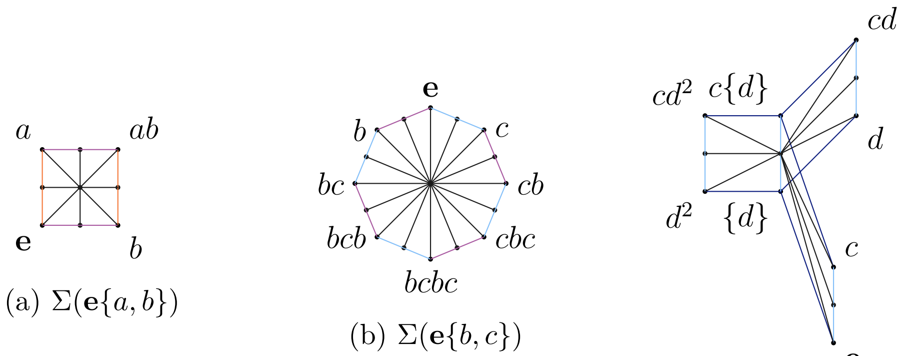


Figure 3.7: The subcomplexes associated to some vertices of the development of $\mathcal{X}_{m,q}$ and their simplicial subdivision.

The action of D on $V_p(\mathcal{C})$ induces an action by isometries of D on Σ, in particular for $d \in D$ we have $d \cdot \Sigma(gY) = \Sigma(dgY)$. By Lemma 3.2.17, the nice simplicial subdivision of each $\Sigma(gY)$ induces a simplicial subdivision of Σ, which we call the *nice simplicial subdivision* of Σ.

Lemma 3.2.19. *The scwol $\mathcal{C}$ is isomorphic to the nice simplicial subdivision of Σ. In particular this implies that Σ is a simply connected metric space.*

Proof. Since $\mathcal{C} = \bigcup_{gY \in V(\mathcal{C})} \mathrm{St}_{in}(gY, \mathcal{C})$ and by Lemma 3.2.17 every $\mathrm{St}_{in}(gY, \mathcal{C})$ is isomorphic to the nice simplicial subdivision of $\Sigma(gY)$ preserved by ι, the complex $\mathcal{C}$ is isomorphic to the nice simplicial subdivion of Σ. This induces a metric on $\mathcal{C}$ with respect to which the geometric realization $|\,\mathcal{C}\,|$ is isometric to Σ. By [BH99][Theorem III.C.3.14], the scwol $\mathcal{C}$ is simply connected. So Σ is a well-defined simply connected metric space. □

We are finally in a position to show that Σ is CAT(0). Since Σ is simply connected, we only need to understand its local structure, so we are back to studying links of vertices. In order to have a precise description of the links of vertices, we introduce an edge labeling of Σ by $V(\Gamma)$.

Edge labeling Let $Y \in V(\mathcal{X})$ and $hD_Y^f \in D/D_Y^f$ so that $hY \in V(\mathcal{C})$. We start by labeling the edges of $\Sigma(hY)$ by elements of Y. To define this edge labeling, we study when two vertices of $\Sigma(hY)$ are adjacent and then give the corresponding label. Let $X, Z \in V(\mathcal{X})$ and $kD_X^f \in D/D_X^f$, $lD_Z^f \in D/D_Z^f$ so that $kX, lZ \in V_p(hY)$, i.e. they are vertices of $\Sigma(hY)$. Then kX and lZ are adjacent in

$\Sigma(hY)$ if and only if their pre-images $j_h^{-1}(kX), j_h^{-1}(lZ) \in V(\mathrm{Cc}(Y))$ are adjacent in $\mathrm{Cc}(Y)$. Let $j_h^{-1}(kX) = (k_2, \lambda_k, k_pX), j_h^{-1}(lZ) = (l_2, \lambda_l, l_pZ) \in V(\mathrm{Cc}(Y))$, hence $hk_2\phi(\lambda_k)k_pX = kX$ and $hl_2\phi(\lambda_l)l_pZ = lZ$ in $V_p(hY)$. Remember that $\mathrm{Cc}(Y) = \mathrm{Cc}(Y)\times\mathrm{Cc}(Y)\times\mathrm{Cc}(Y)$. Then $j_h^{-1}(kX), j_h^{-1}(lZ) \in V(\mathrm{Cc}(Y))$ are adjacent in $\mathrm{Cc}(Y)$ if and only if one of the following holds

1. k_2, l_2 are adjacent in $V(\mathrm{Cc}(Y_2))$ and $\lambda_k = \lambda_l$ and $k_pX = l_pZ$. Equivalently $k_2^{-1}l_2 = x_v$ for some $v \in Y_2$ and $\lambda_k = \lambda_l$ and $X = Z$ and $k_pl_p \in D_X^f$.

2. $k_2 = l_2$ and λ_k, λ_l are adjacent in $V(\mathrm{Cc}_\infty)$ and $k_pX = l_pZ$. This is equivalent to one of the following:

   (a) $k_2 = l_2$ and $\lambda_k \subset \lambda_l$ and $\lambda_l\backslash\lambda_k = \{v\} \subset Y_\infty$ and $X = Z$ and $k_p^{-1}l_p \in D_Z^f$.

   (b) $k_2 = l_2$ and $\lambda_l \subset \lambda_k$ and $\lambda_k\backslash\lambda_l = \{v\} \subset Y_\infty$ and $X = Z$ and $k_p^{-1}l_p \in D_Z^f$.

3. $k_2 = l_2$ and $\lambda_k = \lambda_l$ and k_pX, l_pZ are adjacent in $V(\mathrm{Cc}(Y_p))$. This is equivalent to one of the following:

   (a) $k_2 = l_2$ and $\lambda_k = \lambda_l$ and $X \subset Z$ and $Z \setminus X = \{x_v\}$ for some $v \in Y_p$ and $k_p^{-1}l_p \in D_Z^f$.

   (b) $k_2 = l_2$ and $\lambda_k = \lambda_l$ and $Z \subset X$ and $X \setminus Z = \{x_v\}$ for some $v \in Y_p$ and $k_p^{-1}l_p \in D_X^f$.

Using that $Y \in V(\mathcal{X})$ so D_Y is a spherical Dyer group, this leads to the following characterization and labeling of edges by $Y \subset V(\Gamma)$. The vertices $kX, lZ \in V_p(hY)$ are adjacent in $\Sigma(hY)$ if and only if one of the following holds

1. $X = Z$ and $k^{-1}l \in x_vD_X^f$ for some $v \in Y_2$. In this case, we label the edge by $v \in Y_2 \subset V(\Gamma)$.

2. $X = Z$ and $k^{-1}l = x_v^{\pm 1}D_X^f$ for some $v \in Y_\infty$. In this case, we label the edge by $v \in Y_\infty \subset V(\Gamma)$.

3. (a) $X \subset Z$ and $Z \setminus X = \{x_v\}$ for some $v \in Y_p$ and $k^{-1}l \in k_p^{-1}l_p \in D_Z^f$. In this case, we label the edge by $v \in Y_p \subset V(\Gamma)$.

   (b) $Z \subset X$ and $X \setminus Z = \{x_v\}$ for some $v \in Y_p$ and $k^{-1}l \in k_p^{-1}l_p \in D_X^f$. In this case, we label the edge by $v \in Y_p \subset V(\Gamma)$.

Note that for $h'Y' \in V(\mathrm{St}_{in}(gY, \mathcal{C}))$, the labeling of an edge in $\Sigma(h'Y')$ is invariant under the inclusion $\iota : \Sigma(h'Y') \to \Sigma(gY)$. Moreover the labeling of edges in $\Sigma(\mathbf{e}Y)$ is invariant under the left action of D_Y^f. So this defines an equivariant labeling by $V(\Gamma)$ of the edges of Σ.

Remark 3.2.20 (Links of vertices). As our goal is to apply Moussong's Lemma to Σ, we need to understand links of vertices in Σ. We start with links of vertices in $\Sigma(gY)$. This is crucial to prove later on that Σ is CAT(0). Let $Y \in V(\mathcal{X})$ and $hD_Y^f \in D/D_Y^f$ so that $hY \in V(\mathcal{C})$. Let $X \in V(\mathcal{X})$ and $kD_X^f \in D/D_X^f$ so that $kX \in V_p(hY)$. The edge labeling on Σ and $\Sigma(gY)$ induce a vertex labeling $l : V(\mathrm{Lk}(kX, \Sigma)) \to V$, which restricts to $l : V(\mathrm{Lk}(kX, \Sigma(hY))) \to Y$. Using the map j_h in Lemma 3.2.15, the link $\mathrm{Lk}(kX, \Sigma(hY))$ is isometric to the link $\mathrm{Lk}(j_h^{-1}(kX), \mathrm{Cc}(Y))$. With Remark 3.2.13, this implies that $\mathrm{Lk}(kX, \Sigma(hY))$ can be identified with the spherical flag complex $\Gamma_{Y_2} \star \Gamma_{Y_\infty} \star \Gamma_{Y_p \setminus X} \star (\star_{v \in X}\{v^i \mid 1 \leq i \leq f(v)\})$. The vertex labeling is given by $l(v) = v$ for every $v \in Y_2 \cup Y_\infty \cup Y_p \setminus X$ and $l(v^i) = v$ for every $v^i \in \{v^i \mid v \in X, 1 \leq i \leq f(v)\}$. By Remark 3.2.13, the edge length in $\mathrm{Lk}(kX, \Sigma(hY))$ is given by $d(v, w) = \pi - \pi/m(l(v), l(w))$. As $Y \in V(\mathcal{X})$, the matrix $(\cos(d(v, w)))_{v,w \in Y}$ is positive definite by Lemma 3.2.1. So $\mathrm{Lk}(kX, \Sigma(hY))$ is a metric flag complex. Additionally we have that v, w are adjacent vertices in $\mathrm{Lk}(kX, \Sigma(hY))$ if and only if $l(v) \neq l(w)$. As this holds for every $gY \in V(\mathcal{C})$, it implies that if v, w are adjacent vertices in $\mathrm{Lk}(kX, \Sigma)$, we have $l(v) \neq l(w)$. So for pairwise adjacent vertices $v_1, \ldots, v_n \in \mathrm{Lk}(kX, \Sigma)$, we have $l(v_i) \neq l(v_j)$ for every $i \neq j$. To simplify the notation we will write $\hat{v_i} = l(v_i) \in V$ when considering pairwise adjacent vertices $v_1, \ldots, v_n \in \mathrm{Lk}(kX, \Sigma)$.

Let us now give some more details on the link of vertices in Σ.

Lemma 3.2.21. *Let $Y \in V(\mathcal{X})$ with $Y \subseteq V_p$ so that $Y \in V_p(\mathcal{C})$. Let the vertices $v_1, \ldots, v_k \in V(\mathrm{Lk}(Y, \Sigma))$ be pairwise adjacent. There exist $Z \in V(\mathcal{X})$ and $g \in D$ such that $Y \in V(\Sigma(gZ))$ and $v_1, \ldots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$ if and only if $Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \in V(\mathcal{X})$.*

Proof. As $v_1, \ldots, v_k \in V(\mathrm{Lk}(Y, \Sigma))$ are pairwise adjacent, we have $\hat{v_i} \neq \hat{v_j}$. Assume that there exists $Z \in V(\mathcal{X})$ and $g \in D$ such that $Y \in V(\Sigma(gZ))$ and the vertices $v_1, \ldots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$. Then $Y \in V_p(gZ)$ so $Y \subset Z$ and $\{\hat{v_1}, \ldots, \hat{v_k}\} \subset Z$. Hence $Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \subset Z$ which implies that $Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \in V(\mathcal{X})$.

Now assume that $Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \in V(\mathcal{X})$. Each vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge in Σ between $\mathbf{e}Y$ and some vertex $h_v Z_v \in \Sigma$. Let us define an element $g_v \in D$.

(i) If $\hat{v} \in V_2$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $x_{\hat{v}}Y$. In this case let $g_v = \mathbf{e}$.

(ii) If $\hat{v} \in V_\infty$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $\phi(\hat{v})Y$ or between Y and $\phi(\hat{v})^{-1}Y$. In the first case let $g_v = \mathbf{e}$. In the second case let $g_v = \phi(\hat{v})^{-1} = x_{\hat{v}}^{-1}$. Note that only one of these cases can occur as $v_1, \ldots, v_k$ are pairwise adjacent.

(iii) For $\hat{v} \in V_p \backslash Y$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $Y \cup \{\hat{v}\}$. In this case we fix $g_v = \mathbf{e}$.

(iv) For $\hat{v} \in Y$, the vertex $v \in V(\mathrm{Lk}(Y, \Sigma))$ is an edge between Y and $x^t_{\hat{v}}(Y \setminus \{\hat{v}\})$, for some $1 \leq t \leq f(\hat{v})$. In this case fix $g_v = \mathbf{e}$.

We claim that for $Z = Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\}$ and $g = \Pi^k_{i=1} g_{v_i}$ we have $Y \in V(\Sigma(gZ))$ and $v_1, \ldots, v_k \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$. Since $Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \in V(\mathcal{X})$, we have $g_v g_w = g_w g_v$ for all $v, w \in \{v_1, \ldots, v_k\}$. In fact $g = \phi(\omega)^{-1}$ for $\omega = \{\hat{v} \in Z \mid g_v = x^{-1}_{\hat{v}}\} \subset (Z \setminus Y)_\infty$. Hence $Y \in V_p(gZ)$. Let $v \in \{v_1, \ldots, v_k\}$. Now we need to show that the element $h_v Z_v \in V_p(gZ)$. We use the case by case analysis above to fix the following notation.

(i) If $\hat{v} \in V_2$, we have $h_v Z_v = x_{\hat{v}} Y$ and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

(ii) If $\hat{v} \in V_\infty$ and $h_v Z_v = \phi(\hat{v}) Y$, let $\lambda_v = \omega \cup \{v\} \subset (Z \setminus Z_v)_\infty$. If $\hat{v} \in V_\infty$ and $h_v Z_v = \phi(\hat{v})^{-1} Y$, let $\lambda_v = \omega \setminus \{\hat{v}\} \subset (Z \setminus Z_v)_\infty$.

(iii) If $\hat{v} \in V_p \setminus Y$, we have $h_v Z_v = Y \cup \{\hat{v}\}$ and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

(iv) If $\hat{v} \in Y$, we have $h_v Z_v = x^t_{\hat{v}}(Y \setminus \{\hat{v}\})$ for some $1 \leq t \leq f(\hat{v})$ and we set $\lambda_v = \omega \subset (Z \setminus Z_v)_\infty$.

As $Z = Y \cup \{\hat{v_1}, \ldots, \hat{v_k}\} \in V(\mathcal{X})$, we have $gZ = h_v \phi(\lambda_v)^{-1} Z$ and $Z_v \subset Z$, so $h_v Z_v \in V(\mathrm{St}_{in}(gZ, \mathcal{C}))$. As additionally $Z_v \subset Z \cap V_p$, we have $h_v Z_v \in V_p(gZ)$. $\square$

We now have the necessary tools to show the following statement.

Theorem 3.2.22. *The cell complex Σ is* CAT(0).

Proof. By [BH99] Theorem II.5.4, Σ is CAT(0) if and only if it is simply connected and the link of every vertex is CAT(1). By Lemma 3.2.19, the cell complex Σ is simply connected. Let us now prove that the link of every vertex is CAT(1) by using Moussong's Lemma 1.1.16. Let $Y \in V(\mathcal{X})$ with $Y \subset V_p$, and $gD^f_Y \in D/D^f_Y$ so that $gY \in V(\Sigma)$. Assume $gD^f_Y = D^f_Y$ so $gY = \mathbf{e}\, Y = Y \in V(\Sigma)$.

Claim 3.2.23. Every edge in the link $\mathrm{Lk}(Y, \Sigma)$ of Y in Σ has length $\geq \pi/2$.

Proof. Since the vertex $Y \in V(\Sigma)$ is contained in $\Sigma(gZ)$ if and only if the vertex $gZ \in \mathrm{St}_{out}(Y, \mathcal{C})$ we can describe $\mathrm{Lk}(Y, \Sigma)$ as the union $\bigcup_{gZ \in \mathrm{St}_{out}(Y,\mathcal{C})} \mathrm{Lk}(Y, \Sigma(gZ))$, where $\mathrm{Lk}(Y, \Sigma(gZ))$ is the link of Y in the subcomplex $\Sigma(gZ)$. By Remark 3.2.20, for two adjacent vertices $u, v \in V(\mathrm{Lk}(Y, \Sigma(gZ)))$ the length of the edge is $d(u, v) = \pi - \pi/m(\hat{u}, \hat{v}) \geq \pi/2$ as $m(\hat{u}, \hat{v}) \geq 2$. So each edge in $\mathrm{Lk}(Y, \Sigma)$ has length $\geq \pi/2$. $\square$

Claim 3.2.24. The link $\mathrm{Lk}(Y,\Sigma)$ of Y in Σ is metrically flag.

Proof. Consider a set of pairwise adjacent vertices $v_1,\dots,v_k \in \mathrm{Lk}(Y,\Sigma)$. As $\mathrm{Lk}(Y,\Sigma)$ is a piecewise spherical simplicial complex, the vertices $v_1,\dots,v_k$ are then pairwise distinct. As mentioned in Remark 3.2.20, then $\hat{v_1},\dots,\hat{v_k}$ are pairwise distinct. So the map $\{v_1,\dots,v_k\} \to \{\hat{v_1},\dots,\hat{v_k}\}$, $(v \mapsto \hat{v})$ is a bijection. In particular $Y \cup \{\hat{v_1},\dots,\hat{v_k}\}$ spans a complete subgraph of Γ. So $v_1,\dots,v_k$ span a simplex in $\mathrm{Lk}(Y,\Sigma)$ if and only $v_1,\dots,v_k$ span a simplex in $\mathrm{Lk}(Y,\Sigma(gZ))$ for some $gZ \in V(\mathcal{C})$. By Remark 3.2.20, the link $\mathrm{Lk}(Y,\Sigma(gZ))$ is a piecewise spherical flag complex. So the vertices $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma))$ span a simplex in $\mathrm{Lk}(Y,\Sigma)$ if and only if there exists some $gZ \in V(\mathcal{C})$ with $Y \in V(\Sigma(gZ))$ and the vertices $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma(gZ)))$. By Lemma 3.2.21, this is the case if and only if $Y' = Y \cup \{\hat{v_1},\dots,\hat{v_k}\} \in V(\mathcal{X})$. By Fact 3.2.1, $Y' \in V(\mathcal{X})$ if and only if the matrix $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'}$ is positive definite. As $\pi - \pi/m(u,v) = \pi/2$ for all $u \in Y' \setminus V_2$, $v \in Y' \setminus \{u\}$ and $\pi - \pi/m(u,u) = 0$ for all $u \in Y'$, the matrix $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'}$ is positive definite if and only if its restriction $(\cos(\pi - \pi/m(u,v)))_{u,v\in Y'\cap V_2}$ is positive definite. As $Y' \cap V_2 = \{\hat{v_1},\dots,\hat{v_k}\} \cap V_2$ and $\pi - \pi/m(\hat{u},\hat{v}) = d(u,v)$ for all $\hat{u},\hat{v} \in Y'\cap V_2$, the matrix $(\cos(\pi-\pi/m(\hat{u},\hat{v})))_{\hat{u},\hat{v}\in Y'\cap V_2}$ is positive definite if and only if the matrix $(\cos(d(u,v))_{\hat{u},\hat{v}\in Y'\cap V_2}$ is positive definite. Finally $d(u,u) = 0$ for all $\hat{u} \in \{\hat{v_1},\dots,\hat{v_k}\}$, and $d(u,v) = \pi/2$ for all $\hat{u} \in \{\hat{v_1},\dots,\hat{v_k}\}\setminus V_2$ and $\hat{v} \in \{\hat{v_1},\dots,\hat{v_k}\}\setminus\{u\}$, so the matrix $(\cos(d(u,v))_{\hat{u},\hat{v}\in\{\hat{v_1},\dots,\hat{v_k}\}\cap V_2}$ is positive definite if and only if $(\cos(d(u,v))_{\hat{u},\hat{v}\in\{\hat{v_1},\dots,\hat{v_k}\}}$ is positive definite. So we conclude that $v_1,\dots,v_k \in V(\mathrm{Lk}(Y,\Sigma))$ span a simplex if and only if the matrix $(\cos(d(u,v))_{u,v\in\{v_1,\dots,v_k\}}$ is positive definite. So $\mathrm{Lk}(Y,\Sigma)$ is metrically flag. □

So by Moussong's Lemma, $\mathrm{Lk}(Y,\Sigma)$ is CAT(1). Since D acts by isometries on Σ, the link $\mathrm{Lk}(gY,\Sigma)$ is CAT(1) for every $g \in D$. We conclude that Σ is CAT(0). □

Remark 3.2.25. If D is a spherical Dyer group the scwol $\mathcal{C}$ decomposes as a product $\mathcal{C}_2 \times \mathcal{C}_p \times \mathcal{C}_\infty$. For every $i \in \{2,p,\infty\}$, let Σ_i be the cell complex associated to D_i. So $\Sigma_2 = \mathrm{Cox}(V_2)$, $\Sigma_\infty = \mathbb{R}^{|V_\infty|}$ and $\Sigma_p = \mathrm{Stern}(V_p)$. Then $\Sigma = \Sigma_2 \times \Sigma_\infty \times \Sigma_p$ where each factor is known to be CAT(0). So Σ is CAT(0).

Corollary 3.2.26. *The Dyer group D is* CAT(0)*.*

Proof. The group D acts properly discontinuously and cocompactly by isometries on Σ. □

Remark 3.2.27. If the Dyer group D is a Coxeter group, Σ is the Davis-Moussong complex described in Theorem 2.2.5. If the Dyer group D is a right-angled Artin group, Σ is the Salvetti complex described in Section 2.3. The dimension of Σ is $\dim(\Sigma) = \max\{|Y| \mid Y \in V(\mathcal{X})\}$. Consider the Coxeter group W from Theorem 3.1.8 and its associated Davis-Moussong complex $\Sigma(W)$. The dimension of $\Sigma(W)$ is $\dim(\Sigma(W)) = \max\{|S| \mid S \subset V(\Lambda), W_S \text{ is finite}\}$. Looking at the construction of the graph Λ we can see that $\dim(\Sigma(W)) = \max\{|Y|+|V_p|+|V_\infty \setminus Y| \mid Y \in V(\mathcal{X})\}$. So $\dim(\Sigma) \leq \dim(\Sigma(W))$.

Chapter 4

Systolic complexes and group presentations

In Chapter 1, we introduced systolic complexes as a form of non-positive curvature for simplicial complexes. As a group endowed with a generating set naturally acts on its Cayley graph, it is natural to wonder when such a Cayley graph is systolic. The goal is to give conditions on a group presentation $\langle S \mid R \rangle$, that ensure systolicity of the flag complex of the Cayley graph $\mathrm{Cay}(G, S)$. Recall that a simplicial complex X is *systolic* if it is connected, simply connected and if $\mathrm{Lk}(v, X)$ is 6-large for all vertices $v \in X$. A group is *systolic* if it acts properly discontinuously and cocompactly on a systolic complex.

In section 4.1, we investigate when the flag complex $\mathrm{Flag}(G, S)$ of the Cayley graph $\mathrm{Cay}(G, S)$ is simply connected. The goal of Section 4.2 is the proof of Theorem 4.2.5, which gives condition on a presentation ensuring that $\mathrm{Flag}(G, S)$ is systolic. In order to do so, we investigate the cycles of length 4 and 5 in $\mathrm{Lk}(v[\mathbf{e}], \mathrm{Flag}(G, S))$ in Lemmas 4.2.2, 4.2.3 and 4.2.4. In Section 4.3, we apply Theorem 4.2.5 to Garside groups and Artin groups. The material presented here also appeared in [Soe23].

4.1 Simply connected flag complexes

Let G be a group and $S \subset G$ a finite generating set. Suppose additionally that $S \cap S^{-1} = \emptyset$, this especially implies $\mathbf{e} \notin S$ and $s^2 \neq \mathbf{e}$ for all $s \in S$. This convention is common in the context of Garside groups and monoids, which will be one of our main examples. Let $\mathrm{Cay}(G, S)$ be the Cayley graph of G relative to S. Its vertices and edges are $V(\mathrm{Cay}(G, S)) = \{v[g] \mid g \in G\}$ and $E(\mathrm{Cay}(G, S)) = \{e[g, s] \mid g \in G, s \in S\}$ where the edge $e[g, s]$ goes from $v[g]$ to $v[gs]$. We also write

$e[g,s] = (v[g], v[gs])$. As $S \cap S^{-1} = \emptyset$, the graph $\mathrm{Cay}(G,S)$ is simplicial. So we can define $\mathrm{Flag}(G,S)$ as the flag complex of $\mathrm{Cay}(G,S)$. As $\mathrm{Flag}(G,S)$ is the flag complex of $\mathrm{Cay}(G,S)$, the group G naturally acts properly discontinuously and cocompactly by isometries on $\mathrm{Flag}(G,S)$. Note that even though G acts freely on $\mathrm{Cay}(G,S)$, it does not necessarily act freely on $\mathrm{Flag}(G,S)$.

Proposition 4.1.1. *Let G be a group and $S \subset G$ a finite generating set. Suppose additionally that $S \cap S^{-1} = \emptyset$ and that the action of G on $\mathrm{Flag}(G,S)$ is free. Then $\mathrm{Flag}(G,S)$ is a simply connected simplicial complex and $\pi_1(\mathrm{Flag}(G,S)/G) = G$ if and only if G admits the presentation $G = \langle S \mid R \rangle$ where*

$$\begin{aligned} R = \{a \cdot b \cdot c \mid a,b,c \in S \text{ with } abc = \mathbf{e} \text{ in } G\} \\ \cup \{a \cdot b \cdot c^{-1} \mid a,b,c \in S \text{ with } abc^{-1} = \mathbf{e} \text{ in } G\}. \end{aligned}$$

Proof. To see when $\mathrm{Flag}(G,S)$ is simply connected, it is enough to take a look at its 2-skeleton. There is a 2-simplex in $\mathrm{Flag}(G,S)$ for every set of 3 pairwise adjacent vertices. As in $\mathrm{Cay}(G,S)$ edges are labeled by elements in S and vertices correspond to elements of G, so are edges and vertices in $\mathrm{Flag}(G,S)$. Hence we can interpret the existence of a 2-simplex in terms of relations on the generators. At each vertex $v[g]$ in $\mathrm{Flag}(G,S)$ and for every triple $a,b,c \in S$ with $a \cdot b \cdot c = \mathbf{e}$ or $a \cdot b \cdot c^{-1} = \mathbf{e}$ in G there is a 2-simplex with vertices $v[g]$, $v[ga]$, $v[gab]$ and edges $e[g,a]$, $e[ga,b]$, $e[gab,c]$ or $e[g,a]$, $e[ga,b]$, $e[g,c]$. On the other hand each 2-simplex has vertices and edges which correspond to such a triple of generators. This implies

$$\begin{aligned} \pi_1(\mathrm{Flag}(G,S)/G) = \langle S \mid a \cdot b \cdot c = \mathbf{e} \text{ or } a \cdot b \cdot c^{-1} = \mathbf{e} \text{ for all } a,b,c \\ \text{for which one of these equalities holds in } G \rangle. \end{aligned}$$

Finally $\mathrm{Flag}(G,S)$ is simply connected if and only if it is the universal cover of the quotient, so if and only if $\pi_1(\mathrm{Flag}(G,S)/G) = G$. ☐

We call a presentation satisfying the conditions of Proposition 4.1.1 a *triangular* presentation.

Proposition 4.1.2. *Assume a group G has a triangular presentation $\langle S \mid R \rangle$. Assume additionally $R = \{a \cdot b \cdot c^{-1} \mid a,b,c \in S \text{ with } abc^{-1} = \mathbf{e} \text{ in } G\}$, so in particular $s^3 \neq \mathbf{e}$ for all $s \in S$. Then the action of G on $\mathrm{Flag}(G,S)$ is free.*

Proof. Let $g \in G$ and $x \in \mathrm{Flag}(G,S)$ such that $g \cdot x = x$. Let V be the set of vertices of the smallest simplex containing x. As the action of G is simplicial,

$g \cdot V = V$. The restriction of possible relations in R imposes an orientation on triangles in the graph, which in turn implies that in the subcomplex spanned by V, there exists a unique vertex $v_0 \in V$ with only incoming edges i.e. there exists a unique vertex $v_0 \in V$ such that for all $w \in V \setminus \{v_0\}$, there exists $k \in S$ such that $v_0 = wk$. Since $S \cap S^{-1} = \emptyset$, the action of G on $\mathrm{Cay}(G, S)$ preserves the orientation of the edges and is free on the vertices. So $g \cdot x = x$ implies $g \cdot v_0 = v_0$ hence $g = \mathbf{e}$. □

4.2 Systolic Cayley Complexes

Consider a group G with a finite triangular presentation $G = \langle S \mid R \rangle$. Then we know that $\mathrm{Flag}(G, S)$ is a simply connected simplicial complex. We now want to know when $\mathrm{Flag}(G, S)$ is systolic. We already know that $\mathrm{Flag}(G, S)$ is simply connected. So we need to check whether $\mathrm{Lk}(v, \mathrm{Flag}(G, S))$ is 6-large for all vertices of $\mathrm{Flag}(G, S)$. As the action of G on $\mathrm{Flag}(G, S)$ is transitive and by isometries on the vertices, we only need to check if $\mathrm{Lk}(v[\mathbf{e}], \mathrm{Flag}(G, S))$ is 6-large. Moreover as $\mathrm{Flag}(G, S)$ is flag, $\mathrm{Lk}(v[\mathbf{e}], \mathrm{Flag}(G, S))$ is flag. So we are left with checking whether in $\mathrm{Lk}(v[\mathbf{e}], \mathrm{Flag}(G, S))$ every embedded cycle γ with $4 \leq |\gamma| < 6$ has a diagonal.

For simpler notation we set $L = \mathrm{Lk}(v[\mathbf{e}], \mathrm{Flag}(G, S))$. Then the vertices of L are $V(L) = \{v[s] \mid s \in S \cup S^{-1}\}$. As we differentiate between elements in S and elements in S^{-1}, we will call vertices $v[s]$ with $s \in S$ *positive* and vertices $v[s]$ with $s \in S^{-1}$ *negative*. Edges between these vertices are oriented and labeled by elements in S. We have $E(L) = \{e[g, a] \mid g \in S \cup S^{-1} \text{ and } a \in S \text{ and } ga \in S \cup S^{-1}\}$, where $e[g, a]$ is the edge labeled by $a \in S$ going from $v[g]$ to $v[ga]$. Note that $e[g, a] \in E(L)$ implies $ga \in S \cup S^{-1}$ so in particular $ga \neq \mathbf{e}$ and hence $a \neq g^{-1}$. So $E(L) = \{e[g, a] \mid g \in S \cup S^{-1} \text{ and } a \in S \setminus \{g^{-1}\} \text{ and } ga \in S \cup S^{-1}\}$. We write $v \sim w$ for two adjacent vertices v and w and $e = (v, w)$ for the edge e between v and w. If there is an edge from v to w, we might also use the notation $v \to w$. To simplify the notation, we will also denote the vertex $v[s]$ with s and say the vertex $s \in S$ is positive, $s \in S^{-1}$ is negative.

We put some additional conditions on S and R:

(1) For all $a, b, c \in S$, if $abc \in S$ then $ab \in S$ and $bc \in S$.

(2) $R = \{a \cdot b \cdot c^{-1} \mid abc^{-1} = \mathbf{e} \text{ in } G\}$, so we do not have relations of the form $a \cdot b \cdot c$ in R.

We call a triangular presentation satisfying these additional conditions a *restricted triangular presentation*. These conditions are mostly technical. They limit the possible diagonal-free cycles in L and by Proposition 4.1.2 ensure that the action of G on $\text{Flag}(G, S)$ is free. We don't know how to decide whether L is 6-large without them. So what can we say about diagonal free cycles in L of length 4 or 5 under these conditions?

Remark 4.2.1. The additional condition on R implies that there are no edges from vertices in S to vertices in S^{-1}, so from positive to negative vertices. The additional conditions on S imply that every cycle γ in L which contains one of the following configurations of adjacent vertices has a diagonal. So if $a, b, c \in S$ with

a) $a, b, c \in V(\gamma)$ and $a \to b \to c$ then $a \to c$.

b) $a^{-1}, b^{-1}, c^{-1} \in V(\gamma)$ and $a^{-1} \to b^{-1} \to c^{-1}$ then $a^{-1} \to c^{-1}$.

c) $a^{-1}, b, c \in V(\gamma)$ and $a^{-1} \to b \leftarrow c$ then $a^{-1} \to c$.

d) $a^{-1}, b^{-1}, c \in V(\gamma)$ and $a^{-1} \leftarrow b^{-1} \to c$ then $a^{-1} \to c$.

So all of these configurations of vertices cannot occur in diagonal-free cycles. This leads us to the following statement about potential cycles of length 4 or 5.

Lemma 4.2.2. *Every cycle of length 5 in L contains a diagonal.*

Proof. Let γ be a cycle of length 5 in L.

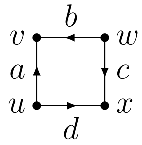
As 5 is odd, if γ has 5 positive or 5 negative vertices, it has a diagonal (situation a) or b) always occurs).

Assume γ contains one negative and four positive vertices. As there are only edges from negative to positive vertices, the direction of two of the edges in γ is already determined. Every possible direction of the three other edges leads to one of the situations above (avoiding situation c) necessarily leads to situation a)). The same argument holds if γ has one positive and four negative vertices.

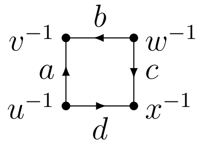
Assume γ has two negative and 3 positive vertices. As each negative vertex is adjacent to at least one positive vertex, the direction of at least two edges of γ is already determined. Every possible direction of the three other edges leads to one of the situations above (if the two negative vertices are adjacent we have situation d) otherwise we have situation c)). The same argument holds if γ has two positive and three negative vertices. $\square$

Lemma 4.2.3. *Let $u, v, w, x, a, b, c, d \in S$. The only cycles of length 4 in L that do not contain one of the situations mentioned in Remark 4.2.1 are:*

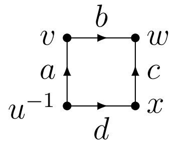
1)



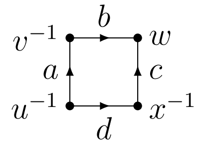
2)



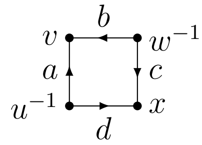
3)



4)



5)



Proof. Let γ be a cycle of length 4.

Assume γ has 4 positive vertices. In order not to be in the situation of Remark 4.2.1 a), each vertex has either two incoming or two outgoing edges. This corresponds to the first cycle. The same argument holds if γ has 4 negative vertices. Then we have the second cycle.

Assume γ has 1 negative and 3 positive vertices. Then the negative vertex is adjacent to two positive vertices and hence the direction of two edges is already determined. As we do not allow the configuration of Remark 4.2.1 c), the only possible cycle is the third cycle. The same argument holds for 1 positive and 3 negative vertices, which gives the fourth cycle.

Assume γ has 2 positive and 2 negative vertices. Then if the two negative vertices are adjacent, we are necessarily in the situation of Remark 4.2.1 d). So the negative vertices are not adjacent. Hence they are both adjacent to the two positive vertices and the direction of these edges is determined. This gives the fifth cycle. $\square$

So to see if L is 6-large we need to concentrate on the cycles of length 4 presented in Lemma 4.2.3. When do they exist? Under which conditions do they have a diagonal? The next lemma aims to answer those questions.

Lemma 4.2.4. *The link L is 6-large if and only if the following additional conditions on S are satisfied:*

1) *If there exist $u, w, a, b, c, d \in S$, $u \neq w$, $a \neq d$, with $ua = wb \in S$ and $ud = wc \in S$, then there exists $k \in S$ such that $w = uk$ or $ua = udk$ or $u = wk$ or $ud = uak$.*

2) *If there exist $v, x, a, b, c, d \in S$, $v \neq x$, $a \neq b$, with $bv = cx \in S$ and $av = dx \in S$, then there exists $k \in S$ such that $v = kx$ or $av = kbv$ or $x = kv$ or $kav = bv$.*

3) *If there exist $u, v, x, b, c \in S$, $v \neq x$, with $ux \in S$, $uv \in S$ and $vb = xc \in S$, then there exists $k \in S$ such that $k = uvb$ or $v = xk$ or $x = vk$.*

4) *If there exist $v, w, x, a, d \in S$, $v \neq x$, with $vw \in S$, $xw \in S$ and $dx = av \in S$, then there exists $k \in S$ such that $k = avw$ or $x = kv$ or $v = kx$.*

5) *If there exist $u, v, w, x \in S$, $v \neq x$, $u \neq w$, with $wv \in S$, $wx \in S$, $uv \in S$ and $ux \in S$, then there exists $k \in S$ such that $w = ku$ or $x = vk$ or $u = kw$ or $v = xk$.*

Note that $u, v, w, x \in S$ correspond to vertices in 4-cycles in L and $a, b, c, d \in S$ to edges. Also note that these conditions are all necessary as one can see in Lemma 4.2.6.

Proof. As mentioned above, L is a flag complex as $\mathrm{Flag}(G, S)$ is a flag complex. We know by Lemma 4.2.2 that there are no diagonal-free cycles of length 5 in L. By Lemma 4.2.3, we know that there are only five problematic cycles of length 4. We show here under which conditions on S such cycles exists and which conditions are necessary for the existence of a diagonal. If those five tyoes of 4-cycles have a diagonal, all cycles of length 4 have a diagonal. So all cycles of length < 6 have a diagonal, so L is 6-large. The existence of the 4-cycle relies on two elements: we need four distinct vertices u, v, w, x and we need the appropriate edges a, b, c, d between these vertices.

i) In the first cycle of length 4: the existence of the cycle is equivalent to the following statement about elements of S:

$$\exists\ a, b, c, d, u, v, w, x \in S \text{ with } u, v, w, x \text{ pairwise distinct}$$
$$\text{such that } v = ua = wb \text{ and } x = ud = wc$$

where $u, v, w, x \in S$ are the labels on the vertices and a, b, c, d are the labels on the edges. There is a diagonal if $v \sim x$ or $u \sim w$ which is equivalent to

$$\exists\ k \in S : w = uk, u = wk, v = xk \text{ or } x = vk$$

where $k \in S$ is the label on the diagonal. As $v = ua = wb$ and $x = ud = wc$, this corresponds to condition 1).

ii) In the second cycle of length 4: the existence of the cycle is equivalent to the following statement about elements of S:

$$\exists\ a, b, c, d, u, v, w, x \in S, u^{-1}, v^{-1}, w^{-1}, x^{-1} \text{ pairwise distinct}$$
$$\text{such that } v^{-1} = u^{-1}a = w^{-1}b \text{ and } x^{-1} = w^{-1}c = u^{-1}d$$

where $a, b, c, d \in S$ are the labels on the edges. There is a diagonal if $v^{-1} \sim x^{-1}$ or $u^{-1} \sim w^{-1}$ which is equivalent to

$$\exists\ k \in S : w^{-1} = u^{-1}k, u^{-1} = w^{-1}k, v^{-1} = x^{-1}k \text{ or } x^{-1} = v^{-1}k$$

where $k \in S$ is the label on the diagonal. As $v^{-1} = u^{-1}a = w^{-1}b$ and $x^{-1} = w^{-1}c = u^{-1}d$, this corresponds to condition 2).

iii) In the third cycle of length 4: the existence of the cycle is equivalent to the following statement about elements of S:

$$\exists\ a, b, c, d, u, v, w, x \in S \text{ with } u^{-1}, v, w, x \text{ pairwise distinct}$$
$$\text{such that } v = u^{-1}a, x = u^{-1}d, w = vb \text{ and } w = xc$$

where $a, b, c, d \in S$ are the labels on the edges. There is a diagonal if $v \sim x$ or $u^{-1} \sim w$ which is equivalent to

$$\exists k \in S : uw = k, v = xk \text{ or } x = vk$$

where $k \in S$ is the label on the diagonal. As u^{-1} is a negative vertex and w is a positive one, there is only one possible direction for the diagonal from u^{-1} to w. As $v = u^{-1}a, x = u^{-1}d, w = vb$ and $w = xc$, this corresponds to condition 3).

iv) In the fourth cycle of length 4: the existence of the cycle is equivalent to the following statement about elements of S:

$$\exists\ a, b, c, d, u, v, w, x \in S \text{ with } u^{-1}, v^{-1}, w, x^{-1} \text{ pairwise distinct}$$
$$\text{such that } v^{-1} = u^{-1}a, w = v^{-1}b = x^{-1}c \text{ and } x^{-1} = u^{-1}d$$

where $a, b, c, d \in S$ are the labels on the edges. There is a diagonal if $v^{-1} \sim x^{-1}$ or $u^{-1} \sim w$ which is equivalent to

$$\exists k \in S : uw = k, v^{-1} = x^{-1}k \text{ or } x^{-1} = v^{-1}k$$

where $k \in S$ is the label on the diagonal. As u^{-1} is a negative vertex and w is a positive one, there is only one possible direction for the diagonal from u^{-1} to w. As $v^{-1} = u^{-1}a, w = v^{-1}b = x^{-1}c$ and $x^{-1} = u^{-1}d$, this corresponds to condition 4).

v) In the fifth cycle of length 4: the existence of the cycle is equivalent to the following statement about elements of S:

$$\exists\ a,b,c,d,u,v,w,x \in S \text{ with } u^{-1},v,w^{-1},x \text{ pairwise distinct}$$
$$\text{such that } v = u^{-1}a = w^{-1}b \text{ and } x = u^{-1}d = w^{-1}c$$

where $a,b,c,d \in S$ are the labels on the edges. There is a diagonal if $v \sim x$ or $u^{-1} \sim w^{-1}$ which is equivalent to

$$\exists\ k \in S : w^{-1} = u^{-1}k, u^{-1} = w^{-1}k, v = xk \text{ or } x = vk$$

where $k \in S$ is the label on the diagonal. As $v = u^{-1}a = w^{-1}b$ and $x = u^{-1}d = w^{-1}c$, this corresponds to condition 5).

□

We can now get back to the original question: when is $\mathrm{Flag}(G,S)$ systolic?

Theorem 4.2.5. *Consider a group G with generating set S, where G has a finite restricted triangular presentation with respect to S. Then the complex* $\mathrm{Flag}(G,S)$ *is a simply connected simplicial complex. It is systolic if and only if the generating set S satisfies the following conditions:*

1) *If there exists $u,w,a,b,c,d \in S$, $u \neq w$, $a \neq d$, with $ua = wb \in S$ and $ud = wc \in S$, then there exists $k \in S$ such that $w = uk$ or $ua = udk$ or $u = wk$ or $ud = uak$.*

2) *If there exist $v,x,a,b,c,d \in S$, $v \neq x$, $a \neq b$, with $bv = cx \in S$ and $av = dx \in S$, then there exists $k \in S$ such that $v = kx$ or $av = kbv$ or $x = kv$ or $kav = bv$.*

3) *If there exist $u,v,x,b,c \in S$, $v \neq x$, with $ux \in S$, $uv \in S$ and $vb = xc \in S$, then there exists $k \in S$ such that $k = uvb$ or $v = xk$ or $x = vk$.*

4) *If there exist $v,w,x,a,d \in S$, $v \neq x$, with $vw \in S$, $xw \in S$ and $dx = av \in S$, then there exists $k \in S$ such that $k = avw$ or $x = kv$ or $v = kx$.*

5) *If there exist $u,v,w,x \in S$, $v \neq x$, $u \neq w$, with $wv \in S$, $wx \in S$, $uv \in S$ and $ux \in S$, then there exists $k \in S$ such that $w = ku$ or $x = vk$ or $u = kw$ or $v = xk$.*

Moreover, this implies that G is a systolic group.

Proof. It follows from Propositions 4.1.1 and 4.1.2, and the definition of a restricted triangular presentation that $\mathrm{Cay}(G, S)$ is a connected simplicial graph and $\mathrm{Flag}(G, S)$ is a welldefined simply connected flag complex. The complex $\mathrm{Flag}(G, S)$ is systolic if and only if the link of every vertex is 6-large. The action of G on the vertices of $\mathrm{Cay}(G, S) = \mathrm{Flag}(G, S)^{(1)}$ is transitive and by isometries. So $\mathrm{Flag}(G, S)$ is systolic if and only if the link $L = \mathrm{Lk}(\mathbf{e}, \mathrm{Flag}(G, S))$ is 6-large. As every link in a flag complex is also a flag complex, this is equivalent to the conditions given by Lemma 4.2.4. Since G acts properly discontinuously and cocompactly on $\mathrm{Flag}(G, S)$, the group G is systolic if $\mathrm{Flag}(G, S)$ is systolic. $\square$

We call a presentation satisfying the conditions of Theorem 4.2.5, a *systolic presentation*. We say a group is *Cayley systolic* if it admits a systolic presentation. By Proposition 4.1.2, a Cayley systolic group G with systolic presentation $\langle S \mid R\rangle$ acts freely on $\mathrm{Flag}(G, S)$. One can also note that free products of Cayley systolic groups are also Cayley systolic. More generally we do not know under which conditions amalgamated products of Cayley systolic groups are systolic or Cayley systolic.

Lemma 4.2.6. *Consider the set $S = \{a, b, c, d, u, v, w, x\}$ and the sets*

$$\begin{aligned}
R_1 &= \{uav^{-1}, wbv^{-1}, udx^{-1}, wcx^{-1}\},\\
R_2 &= \{bvw^{-1}, cxw^{-1}, avu^{-1}, dxu^{-1}\},\\
R_3 &= \{vbw^{-1}, xcw^{-1}, uxd^{-1}, uva^{-1}\},\\
R_4 &= \{dxu^{-1}, avu^{-1}, vwb^{-1}, xwc^{-1}\},\\
R_5 &= \{vua^{-1}, vwb^{-1}, xwc^{-1}, xud^{-1}\}.
\end{aligned}$$

For all $i \in \{1, 2, 3, 4, 5\}$, the presentation $\langle S \mid R_i\rangle$ is a restricted triangular presentation. Additionally the presentation $\langle S \mid R_i\rangle$ satisfies all the conditions of Theorem 4.2.5 except condition i.

Proof. We show this for $\langle S \mid R_3\rangle$. The other cases can be checked in the same way. First note that $a = uv$, $b = v^{-1}w$, $c = x^{-1}w$ and $d = ux$. So the group $C_3 = \langle S \mid R_3\rangle$ is in fact $\mathbb{F}(u, v, w, x)$ the free group on the generators u, v, w and x. So the word problem in C_3 is solvable. We do the following calculations using SageMath. We check for all triples $(\alpha, \beta, \gamma) \in S^3$ that $\alpha\beta\gamma \neq \mathbf{e}$, $\alpha\beta\gamma \notin S$ and $\alpha\beta\gamma^{-1} = \mathbf{e} \Leftrightarrow \alpha \cdot \beta \cdot \gamma^{-1} \in R_3$. Note that $\alpha\beta\gamma \notin S$ implies $S \cap S^{-1} = \emptyset$. So the presentation $\langle S \mid R_3\rangle$ is a restricted triangular presentation. Now we need to check the different conditions of Theorem 4.2.5. Using SageMath, we see that there are no elements $s_u, s_v, s_w, s_x, s_a, s_b, s_c, s_d \in S$ satisfying the hypothesis of one of the conditions 1), 2), 4) and 5) but the tuples $(s_u, s_v, s_x, s_b, s_c) = (u, v, x, b, c)$ and

$(s_u, s_v, s_x, s_b, s_c) = (u, x, v, c, b)$ satisfy $s_v s_b = s_x s_c \in S$, $s_u s_x \in S$ and $s_u s_v \in S$. We check that condition 3) fails in at least one of those cases so we check that for all $s \in S$ we have $s \neq s_u s_v s_b$ and $s_v \neq s_x s$ for at least one of those tuples. □

Example 4.2.7. The conditions in Theorem 4.2.5 are all necessary as shown in Lemma 4.2.6. One can note that the given presentations are not systolic but the underlying groups are, since free groups are known to be systolic and even Cayley systolic with respect to the standard generating system. Also note that the following presentation

$$\mathbb{F}_2 \times \mathbb{F}_2 = \langle a, b, c, d, \Delta_1, \Delta_2, \Delta_3, \Delta_4 \mid \Delta_1 = ab = ba,\ \Delta_2 = bc = cb,\ \Delta_3 = cd = dc,\ \Delta_4 = da = ad \rangle$$

satisfies conditions 1)–4) but not condition 5) (as $u = a$, $v = b$, $w = c$ and $x = d$ do not satisfy condition 5)). Again $\mathbb{F}_2 \times \mathbb{F}_2$ is known to be systolic, using a construction by Elsner and Przytycki [EP13]. We do not know whether it is Cayley systolic.

Remark 4.2.8. We can also interpret the conditions in Lemma 4.2.4 as conditions using orders on S. We first introduce left and right orders on $S \cup \mathbf{e}$ by defining for $a, b \in S \cup \{\mathbf{e}\}$:

a) $a \leq_L b$ if $\exists c \in S \cup \{\mathbf{e}\}$ such that $ac = b$.

b) $a \leq_R b$ if $\exists c \in S \cup \{\mathbf{e}\}$ such that $ca = b$.

These are indeed orders on $S \cup \{\mathbf{e}\}$: as $\mathbf{e} \in S \cup \{\mathbf{e}\}$ they are reflexive, as there are no inverses in S they are antisymmetric and by the additional condition (1) on S they are transitive. The existence of an edge $v \to w$ in L is equivalent to

1. $v \leq_L w$ if $v, w \in S$
2. $w^{-1} \leq_R v^{-1}$ if $v, w \in S^{-1}$
3. $v^{-1}w \in S$ if $w \in S$ and $v \in S^{-1}$.

Then the conditions on the existence of diagonals in given 4-cycles could be reformulated in the following way:

1) If there exist $u, v, w, x \in S$ pairwise distinct with $u \leq_L v$ and $w \leq_L v$ and $u \leq_L x$ and $w \leq_L x$ then $u \leq_L w$ or $x \leq_L v$ or $w \leq_L u$ or $v \leq_L v$.

2) If there exist $u, v, w, x \in S$ pairwise distinct with $v \leq_R u$ and $x \leq_R u$ and $v \leq_R w$ and $x \leq_R w$ then $u \leq_R w$ or $x \leq_R v$ or $w \leq_R u$ or $v \leq_R x$.

3) If there exist $u, v, w, x \in S$ pairwise distinct with $uv \in S$ and $ux \in S$ and $v \leq_L w$ and $x \leq_L w$ then $uw \in S$ or $x \leq_L v$ or $v \leq_L x$.

4) If there exist $u, v, w, x \in S$ pairwise distinct with $vw \in S$ and $xw \in S$ and $x \leq_R u$ and $v \leq_R u$ then $uw \in S$ or $v \leq_R x$ or $x \leq_R v$.

5) If there exist $u, v, w, x \in S$ pairwise distinct with $uv \in S$ and $ux \in S$ and $wv \in S$ and $wx \in S$ then $v \leq_L x$ or $u \leq_R w$ or $x \leq_L v$ or $w \leq_R u$.

Here the conditions are only given in terms of elements of S associated to the vertices of the 4-cycles.

4.3 Applications

4.3.1 Garside groups

Garside groups were introduced by Dehornoy and Paris in [DP99] as a generalization of spherical Artin groups.

Definition 4.3.1. A group G is said to be a *Garside group* with *Garside structure* (G, P, Δ) if it admits a submonoid P with $P \cap P^{-1} = \{\mathbf{e}\}$, called the *monoid of positive elements* and a special element $\Delta \in P$ called the *Garside element* such that the following properties are satisfied:

1. The partial order $\leq_L$ defined by $a \leq_L b \Leftrightarrow a^{-1}b \in P$ is a lattice order, i.e. for every $a, b \in G$ there exists an lcm $a \vee_L b$ and a gcd $a \wedge_L b$ with respect to $\leq_L$ i.e. for all $a, b \in G$ there exists an element $(a \vee_L b)$ such that $a \leq_L (a \vee_L b)$, $b \leq_L (a \vee_L b)$ and for every $c \in G$, if $a \leq_L c$ and $b \leq_L c$ then $(a \vee_L b) \leq_L c$. Similarly for all $a, b \in G$ there exists an element $(a \wedge_L b)$ such that $(a \wedge_L b) \leq_L a$, $(a \wedge_L b) \leq_L b$ and for every $c \in G$, if $c \leq_L a$ and $c \leq_L b$ imply $c \leq_L (a \wedge_L b)$.

2. The set $[\mathbf{e}, \Delta] = \{a \in G \mid \mathbf{e} \leq_L a \leq_L \Delta\}$, called the set of *simple elements*, generates the monoid P.

3. Conjugation by Δ preserves P i.e. $\Delta^{-1} P \Delta = P$.

4. For all $x \in P \setminus \{\mathbf{e}\}$ we have

$$\|x\| = \sup\{k \in \mathbb{N} \mid \exists a_1, \ldots, a_k \in P \setminus \{\mathbf{e}\} \text{ such that } x = a_1 \cdots a_k\} < \infty.$$

Definition 4.3.2. A Garside structure (G, P, Δ) is said to be of *finite type* if the set of simple elements $[\mathbf{e}, \Delta]$ is finite. A group G is said to be a *Garside group of finite type* if it admits a Garside structure of finite type. Elements $x \in P \setminus \{\mathbf{e}\}$ with $\|x\| = 1$ are called *atoms*. The set of atoms also generates P.

Remark 4.3.3. Note that the lcm and gcd of two elements $a, b \in G$ is unique. So $a \vee_L b$ is the unique lcm and $a \wedge b$ is the unique gcd of the elements a and b.

Remark 4.3.4. The monoid P also induces a partial order $\leq_R$ which is invariant under right multiplication. We define $a \leq_R b \Leftrightarrow ba^{-1} \in P$. It follows from the properties of G that $\leq_R$ is also a lattice order, that P is the set of elements a such that $\mathbf{e} \leq_R a$ and that the simple elements are the elements a such that $\mathbf{e} \leq_R a \leq_R \Delta$. We denote by $a \vee_R b$ the lcm and by $a \wedge_R b$ the gcd with respect to $\leq_R$. We say that an element $g \in G$ is *balanced* if $\{a \in G \mid \mathbf{e} \leq_L a \leq_L g\} = \{a \in G \mid \mathbf{e} \leq_R a \leq_R g\}$. So $\{a \in G \mid \mathbf{e} \leq_L a \leq_L \Delta\} = \{a \in G \mid \mathbf{e} \leq_R a \leq_R \Delta\}$, hence Δ is balanced.

Example 4.3.5. Spherical Artin groups are Garside groups, in particular braid groups are Garside. Torus knot groups $\langle x, y \mid x^p = y^q \rangle$ with $p, q > 1$ are Garside groups with Garside element $\Delta = x^p = y^q$ and monoid of positive elements $P = \langle x, y \mid x^p = y^q \rangle^+$. The fundamental group of the complement of n lines through the origin in $\mathbb{C}^2$, with presentation

$$\langle x_1, \ldots, x_n \mid x_1 x_2 \ldots x_n = x_2 \ldots x_n x_1 = \cdots = x_n x_1 \ldots x_{n-1} \rangle$$

is also a Garside group with Garside element $\Delta = x_1 x_2 \ldots x_n$ and monoid of positive words $P = \langle x_1, \ldots, x_n \mid x_1 x_2 \ldots x_n = x_2 \ldots x_n x_1 = \cdots = x_n x_1 \ldots x_{n-1} \rangle^+$.

Lemma 4.3.6. *Let G be a Garside group with Garside structure (G, P, Δ) and set of non-trivial simple elements S. Then $\langle S \mid s \cdot t = st \ \forall s, t \in S$ such that $st \in S \rangle$ is a restricted triangular presentation of G.*

Proof. This presentation is a direct consequence of Theorem 6.1 in [DP99]. It can also be found in [DDG+15, Proposition IV.3.4] in the more general context of Garside categories and groupoids, alternatively it is also mentioned for Garside groups in [Pic00]. It is a restricted triangular presentation as all relations are of the form $a \cdot b = c$ for some $a,\ b,\ c \in S$ and S is the set of non-trivial simple elements of a Garside group. $\square$

We call this presentation the *Garside presentation* of G associated with the Garside structure (G, P, Δ). We can now state one of our main results.

Theorem 4.3.7. *Let G be a Garside group of finite type with Garside structure (G, P, Δ) and set of non trivial simple elements S. Then* $\mathrm{Flag}(G, S)$ *is systolic if and only if for all $a, b \in S$, $a \wedge_L b \in \{\mathbf{e}, a, b\}$ and $a \wedge_R b \in \{\mathbf{e}, a, b\}$. In particular if* $\mathrm{Flag}(G, S)$ *is systolic then so is G.*

Proof. By Lemma 4.3.6, the Garside presentation of G associated with the Garside structure (G, P, Δ) is a restricted triangular presentation. By Proposition 4.1.1, $\mathrm{Flag}(G, S)$ is well-defined. By Theorem 4.2.5, $\mathrm{Flag}(G, S)$ is systolic if and only if the conditions 1)–5) are satisfied.

Assume first that for all $a, b \in S$, we have $a \wedge_L b, a \wedge_R b \in \{\mathbf{e}, a, b\}$. Then

1) If there exist $u, w, a, b, c, d \in S, u \neq w, a \neq b$ with $ua = wb \in S$ and $ud = wc \in S$, then $u \leq_L ua$ and $u \leq_L ud$ so $ua \wedge_L ud \neq \mathbf{e}$, so either $ua \wedge_L ud = ua$ or $ua \wedge_L ud = ud$, say $ua \wedge_L ud = ud$ so $ua = udk$ for some $k \in P$. As $k \leq_R ua$ and $ua \in S$, we have $k \in S$.

2) If there exist $v, x, a, b, c, d \in S, v \neq x, a \neq b$ with $bv = cx \in S$ and $av = dx \in S$ then $v \leq_R bv$ and $v \leq_R av$ then $av \wedge_R bv \neq \mathbf{e}$, so either $av \wedge_R bv = av$ or $av \wedge_R bv = bv$, say $av \wedge_R bv = bv$ so $av = kbv$ for some $k \in P$. As $k \leq_L av$ and $av \in S$, we have $k \in S$.

3) If there exist $u, v, x, b, c \in S, v \neq x$ with $ux \in S, uv \in S$ and $vb = xc \in S$ then $u \leq_L ux$ and $u \leq_L uv$ so $uv \wedge_L ux \neq \mathbf{e}$, so either $uv \wedge_L ux = uv$ or $uv \wedge_L ux = ux$, say $uv \wedge_L ux = ux$ so $uv = uxk$ for some $k \in P$ and then $v = xk$. As $k \leq_R v$ and $v \in S$, we have $k \in S$.

4) If there exist $v, w, x, a, d \in S$, $v \neq x$ with $vw \in S, xw \in S$ and $dx = av \in S$ then $w \leq_R vw$ and $w \leq_R xw$ so $vw \wedge_R xw \neq \mathbf{e}$, so either $vw \wedge_R xw = vw$ or $vw \wedge_R xw = xw$, say $xw \wedge_R vw = vw$ so $xw = kvw$ for some $k \in P$ and then $x = kv$. As $k \leq_L u$ and $u \in S$, we have $k \in S$.

5) If there exist $u, v, w, x \in S, v \neq x, u \neq w$ with $wv \in S, wx \in S, uv \in S$ and $ux \in S$ then $u \leq_L uv$ and $u \leq_L ux$ so $uv \wedge_L ux \neq \mathbf{e}$, so either $uv \wedge_L ux = uv$ or $uv \wedge_L ux = ux$, say $uv \wedge_L ux = uv$ so $uv = uxk$ for some $k \in P$ and then $v = xk$. As $k \leq_R v$ and $v \in S$, we have $k \in S$.

So if for all $a, b \in S$, $a \wedge_R b, a \wedge_L b \in \{\mathbf{e}, a, b\}$, the conditions 1)–5) of Theorem 4.2.5 are satisfied and so $\mathrm{Flag}(G, S)$ is systolic, which directly implies that G is sytolic.

We now show the other implication. Note that for every $a, b \in S$, we have $a \wedge_L b \leq_L a$ and so $a \wedge_L b \in S \cup \{\mathbf{e}\}$. First assume there exist $a, b \in S$, $a \neq b$, with $a \wedge_L b = c$ for some $c \in S \setminus \{a, b\}$, i.e $c \neq \mathbf{e}, a, b$. Then there exist $k_a, k_b, r_a, r_b \in S$ with $a = ck_a$, $b = ck_b$ and $\Delta = k_a r_a = k_b r_b$. Then $c, k_a, k_b, r_a, r_b \in S$ and $k_a \neq k_b$, $ck_a = a \in S$, $ck_b = b \in S$ and $\Delta = k_a r_a = k_b r_b \in S$. But $ck_a r_a = c\Delta \notin S$ and $\nexists k \in S$ with $k_a = k_b k$ since $c = a \wedge_L b \neq b$, similarly there is no $k \in S$ with $k_b = k_a k$

since $c = a \wedge_L b \neq a$. So condition 3) of Theorem 4.2.5 fails. So $\mathrm{Flag}(G, S)$ is not systolic. Finally assume there exist $a, b \in S$, $a \neq b$, with $a \wedge_R b = c$ for some $c \in S \setminus \{a, b\}$, i.e. $c \neq \mathbf{e}, a, b$. Then there exist $k_a, k_b, r_a, r_b \in S$ with $a = k_a c$, $b = k_b c$ and $\Delta = r_a k_a = r_b k_b$. Then $k_a, c, k_b, r_a, r_b \in S$ and $k_a \neq k_b$, $k_a c = a \in S$, $k_b c = b \in S$ and $\Delta = r_a k_a = r_b k_b \in S$. But $r_a k_a c = \Delta c \notin S$ and there is no $k \in S$ with $k_a = k k_b$ or $k_b = k k_a$ since $c = a \wedge_R b \neq a, b$. So condition 4) of Theorem 4.2.5 fails. So $\mathrm{Flag}(G, S)$ is not systolic. So if there exist $a, b \in S$ with $a \wedge_L b \notin \{\mathbf{e}, a, b\}$ or $a \wedge_R b \notin \{\mathbf{e}, a, b\}$, the complex $\mathrm{Flag}(G, S)$ is not systolic. □

Remark 4.3.8. Note that the statement of Theorem 4.3.7 is equivalent to the following statement. Let G be a Garside group of finite type with Garside structure (G, P, Δ) and set of simple elements $\hat{S}$. Then $\mathrm{Flag}(G, \hat{S} \setminus \{\mathbf{e}\})$ is systolic if and only if for all $a, b \in \hat{S}$, $a \wedge_L b \in \{\mathbf{e}, a, b\}$ and $a \wedge_R b \in \{\mathbf{e}, a, b\}$. In particular if $\mathrm{Flag}(G, \hat{S} \setminus \{\mathbf{e}\})$ is systolic then so is G.

Example 4.3.9. Let $x_1, \ldots, x_n$ be n letters and let m be a positive integer. We define

$$\mathrm{prod}(x_1, \ldots, x_p; m) = \underbrace{x_1 x_2 \ldots x_p x_1 x_2 \ldots}_{m}.$$

and $\mathrm{prod}(x_1, \ldots, x_p; 0) = \mathbf{e}$. Consider the group

$$\begin{aligned} G_{n,m} = \langle x_1, \ldots, x_n \mid & \mathrm{prod}(x_1, \ldots, x_n; m) = \mathrm{prod}(x_2, \ldots, x_n, x_1; m) = \ldots \\ & = \mathrm{prod}(x_n, x_1, \ldots, x_{n-1}; m) \rangle. \end{aligned}$$

By Proposition 5.2 in [DP99], this is a Garside group with Garside element $\Delta_{n,m} = \mathrm{prod}(x_1, \ldots, x_n; m)$ and monoid of positive elements

$$\begin{aligned} P_{n,m} = \langle x_1, \ldots, x_n \mid & \mathrm{prod}(x_1, \ldots, x_n; m) = \mathrm{prod}(x_2, \ldots, x_n, x_1; m) = \ldots \\ & = \mathrm{prod}(x_n, x_1, \ldots, x_{n-1}; m) \rangle^+. \end{aligned}$$

When considering all indices modulo n, we can write the set of non trivial simple elements as

$$S = \{\mathrm{prod}(x_i, \ldots, x_{i+n}; k) \mid 1 \leq k \leq m \text{ and } 1 \leq i \leq n\}.$$

In particular, for $n = 1$, we have $G_{1,m} = \langle x_1 \rangle$ with Garside element x_1^m and the simple elements are $S = \{x_1^i \mid 1 \leq i \leq m\}$. Note that if $m = n = 1$ we have $x_1 = \Delta_{1,1}$ and $S = \{x_1\}$. For $n \in \mathbb{N}$, we have $G_{n,1} = \langle x_1, \ldots, x_n \mid x_1 = x_2 = \cdots = x_n \rangle$, with Garside element $\Delta_{n,1} = x_1$ and set of non trivial simple elements $S = \{x_1, \ldots, x_n\} = \{x_1\}$. So for $n > 1$, we can assume $m > 1$.

More generally by Proposition 5.3 in [DP99], for some positive integers p, $n_1, \dots, n_p$, $m_1, \dots, m_p$, the product

$$G = (*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$$

is a Garside group with Garside element $\Delta = \Delta_{n_1,m_1} = \cdots = \Delta_{n_p,m_p}$. The monoid of positive elements is the monoid $P = (*_{i=1}^{p} P_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$. We would like to remark that if $p > 1$ and $n_i = m_i = 1$ for some i, say $i = p$, we have $(*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j) \cong (*_{i=1}^{p-1} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$. So we can assume that if $p > 1$, we have $m_i \geq 2$ for all $i \in \{1, \dots, p\}$. The next theorem shows that these are the only Garside groups with systolic Garside presentation.

Theorem 4.3.10. *Let G be a Garside group of finite type with Garside structure (G, P, Δ). Then the Garside presentation of G associated with the Garside structure (G, P, Δ) is systolic if and only if G is isomorphic to the group*

$$(*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$$

for some positive integers $p, n_1, \dots, n_p$ and $m_1, \dots, m_p$ and the isomorphism maps Δ to Δ_{n_1,m_1} and P to the monoid $(*_{i=1}^{p} P_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$.

Proof. Let $p, n_1, \dots, n_p, m_1, \dots, m_p \mathbb{N}$.

We start by showing that if $G = (*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$ and $P = (*_{i=1}^{p} P_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$ and $\Delta = \Delta_{n_1,m_1}$, the Garside presentation of G associated with the Garside structure (G, P, Δ) is systolic. Let S be the set of non trivial simple elements. Assume that $p = 1$. If $n = m = 1$, we have $S = \{x_1\}$ so the Garside presentation is systolic. Otherwise $G = G_{n,m}$ for some positive integers n and m, $2 \leq m$ and the Garside element is $\Delta = \Delta_{n,m} = \mathrm{prod}(x_1, \dots, x_n; m)$. For simpler notation we always consider the index i modulo n. The set of non trivial simple elements is

$$S = \{\mathrm{prod}(x_i, \dots, x_{i+n}; k) \mid 1 \leq i \leq n \text{ and } 1 \leq k \leq m\}.$$

Then for $0 \leq k \leq l \leq m$, we have

$$\mathrm{prod}(x_i, \dots, x_{i+n}; k) \wedge_L \mathrm{prod}(x_j, \dots, x_{j+n}; l) \\ = \begin{cases} \mathbf{e} & \text{if } i \neq j \text{ and } l < m \\ \mathrm{prod}(x_i, \dots, x_{i+n}; k) & \text{if } i = j \text{ or } l = m. \end{cases}$$

Similarly for $0 \leq k \leq l \leq m$, we have

$$\operatorname{prod}(x_i, \ldots, x_{i+n}; k) \wedge_R \operatorname{prod}(x_j, \ldots, x_{j+n}; l) = \begin{cases} \mathbf{e} & \text{if } i+k \not\equiv j+l \text{ and } l < m \\ \operatorname{prod}(x_i, \ldots, x_{i+n}; k) & \text{if } i+k \equiv j+l \text{ or } l = m. \end{cases}$$

So by Theorem 4.3.7 the Garside presentation of $G_{n,m}$ associated with the Garside structure $(G_{n,m}, P_{n,m}, \Delta_{n,m})$ is systolic.

Now assume that $p > 1$. Then the element $\Delta = \Delta_{n_1,m_1} = \cdots = \Delta_{n_p,m_p}$ is the Garside element of G. Let $\hat{S}$ be the set of simple elements of G and $\hat{S}_i$ the set of simple elements of G_{n_i,m_i} for $i = 1, \ldots, p$. Then $\hat{S} = \sqcup_{i=1}^{p}(\hat{S}_i \setminus \{\Delta_{n_i,m_i}, \mathbf{e}\}) \sqcup \{\Delta, \mathbf{e}\}$ is a partition of the set of simple elements. For every $i, j \in \{1, \ldots, p\}$, S_i satisfies $s \wedge_L t, s \wedge_R t \in \{\mathbf{e}, s, t\}$ for all $s, t \in \hat{S}_i$ and we have $s \wedge_R t = s \wedge_L t = \mathbf{e}$ if $s \in \hat{S}_i \setminus \{\Delta\}, t \in \hat{S}_j \setminus \{\Delta\}$, $i \neq j$. So for all $s, t \in \hat{S}$ we have $s \wedge_R t, s \wedge_L t \in \{s, t, \mathbf{e}\}$. Hence by Theorem 4.3.7, the Garside presentation of G associated with the Garside structure (G, P, Δ) is systolic.

Now let G be a Garside group with Garside structure (G, P, Δ). Assume the Garside presentation of G associated with the Garside structure (G, P, Δ) is systolic. Let $\hat{S}$ the set of simple elements and A the set of atoms of P. So by Theorem 4.3.7, we have for all $s, t \in \hat{S}$, $s \wedge_L t$, $s \wedge_R t \in \{\mathbf{e}, s, t\}$. First note that if $\Delta \in A$ we have $\hat{S} = \{\mathbf{e}, \Delta\}$, $A = \{\Delta\}$, $G = \mathbb{Z}$ and $P = \mathbb{N}$. Hence we can write G as $(*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j)$ with $p = 1$ and $n_1 = m_1 = 1$. So we can now assume that $\Delta \notin A$.

We start with showing that for all $a \in A$ there exists a unique $\xi(a) \in A$ such that $a\xi(a) \in \hat{S}$. As $\Delta \notin A$, such a $\xi(a) \in A$ exists. Assume it is not unique, so let $a_1, a_2 \in A$ with $aa_1, aa_2 \in \hat{S}$. Since $aa_1 \wedge_L aa_2 \in \{\mathbf{e}, aa_1, aa_2\}$ and a, a_1, a_2 are atoms, $aa_1 = aa_2$ and hence $a_1 = a_2$. Similarly for all $a \in A$ there exists a unique $\rho(a) \in A$ such that $\rho(a)a \in \hat{S}$. In particular for all $a \in A$, $\rho(\xi(a)) = a = \xi(\rho(a))$. So the map $\xi : A \to A$ is bijective with inverse map $\rho : A \to A$. As A is finite, the map ξ is a permutation of A. So the orbits of ξ form a partition of A and ξ can be written as a product of cycles of disjoint support,

$$\xi = (a_{1,1}, a_{1,2}, \ldots, a_{1,n_1})(a_{2,1}, \ldots, a_{2,n_2}) \cdots (a_{p,1}, \ldots, a_{p,n_p}).$$

Then for any $1 \leq i \leq p$, as A generates P, there exists $m_i \geq 2$ such that $\Delta = \operatorname{prod}(a_{i,1}, \ldots, a_{i,n_i}; m_i)$. Since Δ is balanced, every subproduct of

$\mathrm{prod}(a_{i,1}, \dots, a_{i,n_i}; m_i)$ is a simple element and so

$$\begin{aligned}\mathrm{prod}(a_{i,1}, \dots, a_{i,n_i}; m_i) &= \mathrm{prod}(a_{i,2}, a_{i,3}, \dots, a_{i,n_i}, a_{i,1}; m_i) \\ &= \dots = \mathrm{prod}(a_{i,n_i}, a_{i,1}, \dots, a_{i,n_i-1}; m_i).\end{aligned}$$

The set $A_i = \{a_{i,1}, \dots, a_{i,n_i}\}$ corresponds to the atoms of G_{n_i,m_i}. As for $1 \le i < j \le p$, $a \in A_i$ and $b \in A_j$ we have $ab \notin S$, using Lemma 4.3.6, we get that

$$G \cong (*_{i=1}^{p} G_{n_i,m_i})/(\Delta_{n_i,m_i} = \Delta_{n_j,m_j} \forall i, j).$$

□

Corollary 4.3.11. *1. The group*

$$G_{n,n} = \langle x_1, \dots, x_n \mid x_1x_2 \dots x_n = x_2x_3 \dots x_nx_1 = \dots = x_nx_1 \dots x_{n-1} \rangle,$$

which is the fundamental group of the complement of n lines through the origin in $\mathbb{C}^2$, has a systolic Garside presentation with respect to the Garside element $\Delta = x_1x_2 \dots x_n$.

2. Consider n positive integers $p_1, \dots, p_n$, $p_i \ge 2$. The group

$$G = \langle x_1, \dots, x_n \mid x_1^{p_1} = x_2^{p_2} = \dots = x_n^{p_n} \rangle$$

has a systolic Garside presentation with respect to the Garside element $\Delta = x_1^{p_1}$. In particular torus knot groups $\langle x, y \mid x^p = y^q \rangle$ have a systolic Garside presentation with respect to the Garside element $\Delta = x^p$.

Remark 4.3.12. In [DP99], Example 5 mentions a generalization of the groups $G_{n,m}$. Let $p, n, m \in \mathbb{N}$, $2 \le m$, $2 \le p \le n$. It claims that

$$\begin{aligned}K_{n,p,m} = \langle x_1, x_2, \dots, x_n \mid \mathrm{prod}(x_1, \dots, x_p; m) &= \mathrm{prod}(x_2, \dots, x_{p+1}; m) = \dots \\ &= \mathrm{prod}(x_{n-p+1}, \dots, x_n; m) \\ &= \mathrm{prod}(x_{n-p+2}, \dots, x_n, x_1; m) = \dots \\ &= \mathrm{prod}(x_n, x_1, \dots, x_{p-1}; m) \rangle\end{aligned}$$

is a Garside group by [DP99] Proposition 5.2. But this is a wrong application of [DP99] Proposition 5.2, as one can see by considering for example the case $n = 5$, $p = 3$ and $m = 4$ or more generally $m = p + 1$. So the question of whether $K_{n,p,m}$ is a Garside group when $p \neq n$ remains open.

Remark 4.3.13. For $k \geq 2$, the group $G_k = \langle a, b \mid b^k = aba \rangle$ is a Garside group with Garside element $\Delta = b^{k+1} = abab = baba$. The elements b^k and bab are both simple elements, but $b^k \wedge_L bab = b \notin \{\mathbf{e}, b^k, bab\}$. So it does not satisfy the conditions of Theorem 4.3.7. Yet it is systolic as it is isomorphic to the group $\langle x, b \mid x^2 = b^{k+1} \rangle$. This leads to the following question: Do all systolic Garside groups have a Garside structure with respect to which they have a systolic Garside presentation?

Remark 4.3.14 (Restrictions on systolicity in Garside groups). Consider a Garside group of finite type G with Garside element Δ. Then Δ^k is in the center of G for some positive integer k. Let S be the set of simple elements. Suppose there is some balanced element $\delta \in S \setminus \{\Delta, \mathbf{e}\}$. Let $T = \{a \in G \mid \mathbf{e} \leq_L a \leq_L \delta\} = \{a \in G \mid \mathbf{e} \leq_R a \leq_R \delta\}$. Let $a \in T$ be an atom and suppose $\delta \notin \langle a \rangle$. Then δ^l is in the center of the subgroup of G generated by T for some positive integer l. If $\langle T \rangle \neq G$, we have $\langle a, \delta^l, \Delta^k \rangle \cong \mathbb{Z}^3$. By Theorem 1.2.17, this implies in particular that G is not systolic.

Remark 4.3.15 (Cohomological dimension). The cohomological dimension of Garside groups is known to be bounded [DL03], [CMW04]. For $n \in \mathbb{N}$, we know that $\mathrm{cd}(G_{n,n}) = 2$ as $G_{n,n}$ is the direct product of $\mathbb{Z}$ with a free group of rank $n-1$. For $n, m \in \mathbb{N}$ we expect that $\mathrm{cd}(G_{n,m}) = 2$. Moreover for $p, q \in \mathbb{N}$ the torus knot groups $T_{p,q} = \langle x, y \mid x^p = y^q \rangle$ also satisfies $\mathrm{cd}(T_{p,q}) = 2$. So we do not expect these groups to produce knew examples of groups with higher cohomological dimension.

4.3.2 Artin groups

Recall the notation $[xyx\ldots]_k = \underbrace{xyx\ldots}_{k}$ and $[\ldots xyx]_k = \underbrace{\ldots xyx}_{k}$ for some $k \in \mathbb{N}$.
Given a finite labeled simplicial graph Γ, the *Artin group associated to* Γ is given by

$$A_\Gamma = \langle s_v, v \in V \mid [s_v s_w s_v \ldots]_{m_e} = [s_w s_v s_w \ldots]_{m_e} \quad \text{for all edges } e = (v, w) \text{ with label } m_e \rangle.$$

For $n \in \mathbb{N}_{\geq 2}$, the *dihedral Artin group* DA_n is the Artin group defined by the graph $a \bullet\!\overset{n}{\text{———}}\!\bullet b$. So $DA_n = \langle a, b \mid [aba\ldots]_n = [bab\ldots]_n \rangle$.

The *Artin monoid associated to* Γ is given by

$$A_\Gamma^+ = \langle s_v, v \in V \mid [s_v s_w s_v \dots]_{m_e} = [s_w s_v s_w \dots]_{m_e} \text{ for all edges } e = (v,w) \text{ with label } m_e \rangle^+.$$

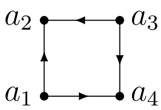
By [Par02] Theorem 1.1, the canonical homomorphism $\iota : A_\Gamma^+ \hookrightarrow A_\Gamma$ is an injection.

Corollary 4.3.16. *The dihedral Artin group DA_n is Cayley systolic for all $n \in \mathbb{N}_{\geq 2}$.*

Proof. The dihedral Artin group DA_n corresponds to the Garside group $G_{2,n}$ with Garside element $\Delta = [aba\dots]_n$. So by Theorem 4.3.10 it is Cayley systolic. □

Consider the following definitions.

Definition 4.3.17. An *orientation* on a simplicial graph Γ is an assignement $o(e)$ for each edge $e = \{v,w\} \in E(\Gamma)$ where $o(e) \subseteq \{v,w\}$ is a set of one or two endpoints of e. An edge with both endpoints assigned is *bioriented.* The startpoint $i(e) \subseteq \{v,w\}$ is an assignement of one or two startpoints of $e = \{v,w\}$. If $o(e)$ consists of one point, $i(e)$ consists of one point such that $e = (i(e), o(e))$, if $o(e)$ consists of two points then so does $i(e)$ and we have $i(e) = o(e)$. So the startpoint $i(e)$ is consistent with the choice of $o(e)$. We say that a cycle γ is *directed* if for each vertex $v \in \gamma$ there is exactly one edge $e \in \gamma$ with $v \in o(e)$. A cycle is *undirected* if it is not directed. We say that a 4-cycle $\gamma = (a_1, a_2, a_3, a_4)$ is *misdirected* if $a_2 \in o(a_1,a_2), a_2 \in o(a_2,a_3), a_4 \in o(a_3,a_4)$ and $a_4 \in o(a_4,a_1)$.



Lemma 4.3.18. *Let Γ be a labeled simplicial graph with an orientation o such that an edge is bioriented if and only if it has label 2. Assume that every 3-cycle in Γ is directed. Let $V(\Gamma) = \{v_1, \dots, v_n\}$. Consider the set*

$$S = \{x_1, x_2, \dots, x_n\} \cup \{\Delta_e, t_1^e, t_2^e, \dots, t_{m_e-2}^e \mid e \in E(\Gamma) \text{ with label } m_e\}.$$

In particular there is non $t_i^e \in S$ if $m_e = 2$. For each $e \in E(\Gamma)$ with label $m_e \geq 3$ and with $i(e) = v_i$ and $o(e) = v_j$, we consider the set

$$R_e = \{x_i x_j \Delta_e^{-1}, x_j t_1^e \Delta_e^{-1}, t_1^e t_2^e \Delta_e^{-1}, t_2^e t_3^e \Delta_e^{-1}, \dots, t_{m_e-3}^e t_{m_e-2}^e \Delta_e^{-1}, t_{m_e-2}^e x_i \Delta_e^{-1}\}.$$

For each $e \in E(\Gamma)$ with label $m_e = 2$ and with $o(e) = i(e) = \{v_i, v_j\}$, we consider the set $R_e = \{x_i x_j \Delta_e^{-1}, x_j x_i \Delta_e^{-1}\}$. Let $R = \bigcup_{e \in E(\Gamma)} R_e$. Then the presentation

$\langle S \mid R\rangle$ is a restricted triangular presentation of the Artin group A_Γ. We call this the dual presentation of A_Γ with orientation o.

Proof. The standard presentation of A_Γ is

$$A_\Gamma = \langle x_1, \ldots, x_n \mid [x_i x_j x_i \ldots]_{m_e} = [x_j x_i x_j \ldots]_{m_e} \quad \text{for all edges } e = (v_i, v_j) \text{ with label } m_e\rangle.$$

We first see that the dual presentation is indeed a presentation of A_Γ. Let $e \in E(\Gamma)$ with $m_e = 2$, $i(e) = o(e) = \{v_i, v_j\}$. Then the standard presentation states $x_i x_j = x_j x_i$. In the dual presentation the relations R_e imply $\Delta_e = x_i x_j$ and $\Delta_e = x_j x_i$ and hence $x_i x_j = x_j x_i$. Let $e \in E(\Gamma)$ with $m_e \geq 3$, $i(e) = v_i$ and $o(e) = v_j$. For $k \in \{1, \ldots, m_e - 2\}$, the relations R_e imply $t^e_1 = x_j^{-1} x_i x_j$ and $t^e_2 = (t^e_1)^{-1} x_j t^e_1$ and $t^e_k = (t^e_{k-1})^{-1} t^e_{k-2} t^e_{k-1}$ for $k \in \{3, \ldots, m_e - 2\}$. So for $k \in \{1, \ldots, m_e - 2\}$ we have

$$t^e_k = ([\ldots x_j x_i x_j]_k)^{-1} [\ldots x_j x_i x_j]_{k+1} . \tag{4.1}$$

For $k \in \{1, \ldots, m_e - 2\}$, the relations R_e imply $t^e_{m_e-2} = x_i x_j x_i^{-1}$ and $t^e_{m_e-3} = (t^e_{m_e-2}) x_i (t^e_{m_e-2})^{-1}$ and $t^e_k = (t^e_{k+1}) t^e_{k+2} (t^e_{k+1})^{-1}$ for $k \in \{1, \ldots, m_e - 4\}$. So for $k \in \{1, \ldots, m_e - 2\}$ we have

$$t^e_k = [x_i x_j x_i \ldots]_{m_e - k} ([x_i x_j x_i \ldots]_{m_e-k-1})^{-1}. \tag{4.2}$$

This implies the relation $[x_i x_j x_i \ldots]_{m_e} = [x_j x_i x_j \ldots]_{m_e}$. Conversely the relation $[x_i x_j x_i \ldots]_{m_e} = [x_j x_i x_j \ldots]_{m_e}$ implies $x_i x_j = [x_j x_i x_j \ldots]_{m_e} ([x_i x_j x_i \ldots]_{m_e-2})^{-1} = x_j t^e_1$. For $k \in \{1, \ldots, m_e - 3\}$ we also have

$$\begin{aligned} t^e_k t^e_{k+1} &= ([\ldots x_j x_i x_j]_k)^{-1} [\ldots x_j x_i x_j]_{k+1} ([\ldots x_j x_i x_j]_{k+1})^{-1} [\ldots x_j x_i x_j]_{k+2} \\ &= ([\ldots x_j x_i x_j]_k)^{-1} [\ldots x_j x_i x_j]_{k+2} \\ &= x_i x_j. \end{aligned}$$

Finally when $k = m_e - 2$ we have

$$\begin{aligned} t^e_{m_e-2} x_i &= ([\ldots x_j x_i x_j]_{m_e-2})^{-1} [\ldots x_j x_i x_j]_{m_e-1} x_i \\ &= ([\ldots x_j x_i x_j]_{m_e-2})^{-1} [\ldots x_i x_j x_i]_{m_e} \\ &= ([\ldots x_j x_i x_j]_{m_e-2})^{-1} [\ldots x_j x_i x_j]_{m_e} \\ &= x_i x_j. \end{aligned}$$

Let $a, b, c \in S$. In order to see that the dual presentation is a restricted triangular presentation we check that $abc \neq \mathbf{e}$, $abc \notin S$ and that $abc^{-1} = \mathbf{e}$ implies $abc^{-1} \in R$. Note that $abc \notin S$ implies that $S \cap S^{-1} = \emptyset$. We consider the following map $\xi : S \to \mathbb{Z}$ defined by $\xi(\Delta_e) = 2$ and $\xi(t_i^e) = 1$ for all $e \in E(\Gamma)$, $i \in \{1, \dots, m_e - 2\}$ and $\xi(x_i) = 1$ for $i \in \{1, \dots, n\}$. As $\xi(a) + \xi(b) - \xi(c) = 0$ for all $abc^{-1} \in R$, the map extends to a homomorphism $\xi : A_\Gamma \to \mathbb{Z}$. For any $a, b, c \in S$, we have $\xi(a) + \xi(b) \geq 2$ and $\xi(abc) = \xi(a) + \xi(b) + \xi(c) \geq 3$ so it follows that $ab \neq \mathbf{e}$, $abc \neq \mathbf{e}$ and $abc \notin S$. Also for any $a, b, c \in S$ such that $abc^{-1} = \mathbf{e}$, we have $\xi(abc^{-1}) = 0$, which implies $\xi(c) = \xi(a) + \xi(b)$ so we necessarily have $c = \Delta_e$ for some $e \in E(\Gamma)$ and $a, b \in S \setminus \{\Delta_e, e \in E(\Gamma)\}$. So fix $e \in E(\Gamma)$ with v_i, v_j the vertices of e and $c = \Delta_e = x_i x_j$. Let $S_0 = S \setminus \{\Delta_e, e \in E(\Gamma)\}$. We need to verify for all $(a, b) \in S_0 \times S_0$ that if $abc^{-1} = \mathbf{e}$ we have $abc^{-1} \in R$.

Our proof relies on the following property of A_Γ^+: Let $\alpha = x_{i_1} x_{i_2} \dots x_{i_l}$ and $\beta = x_{j_1} x_{j_2} \dots x_{j_k}$ be two words on $x_1, \dots, x_n$. They represent the same element in A_Γ^+ if and only if we can transform α into β using a finite number of transformations of the form $u \cdot [x_p x_q x_p \dots]_{m_f} \cdot u' = u \cdot [x_q x_p x_q \dots]_{m_f} \cdot u'$ for some $f = (v_p, v_q) \in E(\Gamma)$. If α does not contain a subword of this form, no transformation is possible and the expression is unique.

Let $a = x_k$ and $b = x_l$ for some $k, l \in \{1, \dots, n\}$. Then $ab = c$ implies $x_k x_l = x_i x_j$ in A_Γ hence the equality also holds in A_Γ^+. If $m_e \geq 3$ this necessarily implies $k = i$ and $l = j$. If $m_e = 2$, this implies either $k = i$ and $l = j$ or $k = j$ and $l = i$. In all of these cases the corresponding, relation is in R.

Let $a = x_k$ and $b = t_l^f$ with $k \in \{1, \dots, n\}$, $f = (v_p, v_q) \in E(\Gamma)$, $m_f \geq 3$ and $1 \leq l \leq m_f - 2$. Set $l' = m_f - l - 1$, so $1 \leq l' \leq m_f - 2$. Then $ab = c$ implies that $x_k t_l^f = x_i x_j$ so by equation (4.2), we have that $x_k \left[x_p x_q x_p \dots\right]_{m_f - l} \left(\left[x_p x_q x_p \dots\right]_{m_f - l - 1}\right)^{-1} = x_i x_j$ and hence $x_k \left[x_p x_q x_p \dots\right]_{l'+1} = x_i x_j \left[x_p x_q x_p \dots\right]_{l'}$ in A_Γ^+. As all 3-cycles are directed, neither x_i, nor x_j, nor x_k can commute with both x_p and x_q. So the last letter on the left hand side and on the right hand side is different and is either x_p or x_q. The only way to change this last letter is to apply the relation $\left[x_p x_q x_p \dots\right]_{m_f} = \left[x_q x_p x_q \dots\right]_{m_f}$. So we need $k = q$ and $l' = m_f - 2$ so $l = 1$. Since $t_1^f = x_q^{-1} x_p x_q$ by equation (4.1), this implies the equality $x_p x_q = x_i x_j$ so $p = i$ and $j = q$. Hence $f = e$, $a = x_j$, $b = t_1^e$ and the corresponding relation is in R.

Let $a = t_l^f$ and $b = x_k$ with $k \in \{1, \dots, n\}$, $f = (v_p, v_q) \in E(\Gamma)$, $m_f \geq 3$ and $1 \leq l \leq m_f - 2$. Then $ab = c$ implies that $t_l^f x_k = x_i x_j$ so by equation (4.1) $\left(\left[\dots x_q x_p x_q\right]_l\right)^{-1} \left[\dots x_q x_p x_q\right]_{l+1} x_k = x_i x_j$ so $\left[\dots x_q x_p x_q\right]_{l+1} x_k = \left[\dots x_q x_p x_q\right]_l x_i x_j$ in A_Γ^+. As all 3-cycles are directed, neither x_i, nor x_j, nor x_k can commute with both x_p and x_q. The first letter on the left hand side and on the right hand side is different and is either x_p or x_q. The only way to change this first letter is to use the relation $[x_p x_q x_p \dots]_{m_f} = [x_q x_p x_q \dots]_{m_f}$. So we need $k = p$ and $l = m_f - 2$. This implies $x_p x_q = x_i x_j$ hence $i = p$ and $j = q$. So $f = e$, $a = t^e_{m_e-2}$ and $b = x_i$ and the corresponding relation is in R.

Finally let $a = t^f_k$ and $b = t^g_l$ with $f = (v_p, v_q), g = (v_r, v_s) \in E(\Gamma)$, $m_f, m_g \geq 3$, $1 \leq k \leq m_f - 2$ and $1 \leq l \leq m_g - 2$. Set $l' = m_g - l - 1$. Then $ab = c$ implies $t^f_k t^g_l = x_i x_j$ so by equations (4.1) and (4.2), we have $[\dots x_q x_p x_q]_{k+1} [x_r x_s x_r \dots]_{l'+1} = [\dots x_q x_p x_q]_k \, x_i x_j \, [x_r x_s x_r \dots]_{l'}$ in A^+_Γ. As all 3-cycles are directed, x_r and x_i cannot commute with both x_p and x_q. Similarly x_q and x_j cannot commute with both x_r and x_s. So the first letter on the left and on the right hand side is different and is either x_p or x_q. Similarly the last letter on the left and on the right hand side is different and is either x_r or x_s. So we need to apply the relations $[x_p x_q x_p \dots]_{m_f} = [x_q x_p x_q \dots]_{m_f}$ and $[x_r x_s x_r \dots]_{m_g} = [x_s x_r x_s \dots]_{m_g}$. This requires $r = p$ and $s = q$ and $m_f \leq k + l' + 2$. If $k + l' + 2 > m_f$, we can apply the relation $[x_p x_q x_p \dots]_{m_f} = [x_q x_p x_q \dots]_{m_f}$ on a piece of the left hand side of length m_f. But since $k+l'+2 < 2m_f$, this only allows us to change either the first or the last letter but not both. So we necessarily have $k + l' + 2 = m_f$, so $k = m_f - l' - 2 = l - 1$. Applying the relation leads to $x_i x_j = x_p x_q$ and $i = p$ and $j = q$. So $f = g = e$, $a = t^e_k$ and $b = t^e_{k+1}$ for some $k \in \{1, \dots, m_e - 3\}$ and the corresponding relation is in R.

So we have indeed that $abc \neq \mathbf{e}$, $abc \notin S$ and that $abc^{-1} = \mathbf{e}$ implies $abc^{-1} \in R$ for $a, b, c \in S$ so this is a restricted triangular presentation. □

Remark 4.3.19. Let Γ be a simplicial graph, with edges labeled by numbers ≥ 2 and with an orientation o such that an edge is bioriented if and only if it has label 2. Then it follows from the proof of Lemma 4.3.18 that the dual presentation of A_Γ with orientation o is always a presentation of A_Γ but it is not always a restricted triangular presentation. For example if Γ is a 3-cycle with one edge labeled by 2 and the two others labeled by $m, n \geq 3$, the presentation is not a restricted triangular presentation.

Remark 4.3.20. In Corollary 4.3.16, we used the Garside structure on DA_n induced by $G_{2,n}$. Lemma 4.3.18 implies that $DA_n \cong G_{n,2}$, this presentation corresponds to another Garside structure. So in particular $G_{2,n} \cong G_{n,2}$ as groups but with different Garside structures.

Theorem 4.3.21. *Let Γ be a simplicial graph, with edges labeled by numbers ≥ 2 and with an orientation o such that an edge is bioriented if and only if it has label 2. Assume that every 3-cycle is directed and no 4-cycle is misdirected. Let A_Γ be the Artin group associated to Γ. Then A_Γ is Cayley systolic.*

Proof. We prove using Theorem 4.2.5 that $\mathrm{Flag}(A_\Gamma, S)$ with respect to the dual presentation with orientation o is systolic. Note that with respect to the partial order defined in Remark 4.2.8, since the presentation $\langle S \mid R\rangle$ is a restricted triangular presentation by Lemma 4.3.18 and by a careful inspection of the relations R, we have that:

- If $a, b \in S$ with $ab \in S$, then $ab = \Delta_e$ for some $e \in E(\Gamma)$.
- For $s \in S$, $e \in E(\Gamma)$, we have $s \leq_L \Delta_e \Leftrightarrow s \leq_R \Delta_e$.
- If $s \leq_L \Delta_e$ and $s \leq_L \Delta_f$ for some $s \in S$, $e, f \in E(\Gamma)$, $e \neq f$, then $s \in \{x_1, x_2, \dots, x_n\}$.
- If $s \leq_R \Delta_e$ and $s \leq_R \Delta_f$ for some $s \in S$, $e, f \in E(\Gamma)$, $e \neq f$, then $s \in \{x_1, x_2, \dots, x_n\}$.
- If $s, t \leq_L \Delta_e$ and $s, t \leq_L \Delta_f$ for some $s, t \in S$, $s \neq t$, then $e = f$.
- If $s, t \leq_R \Delta_e$ and $s, t \leq_R \Delta_f$ for some $s, t \in S$, $s \neq t$, then $e = f$.
- If $x_i \leq_L \Delta_e$, then $v_i \in o(e) \cup i(e)$.
- If $x_i \leq_R \Delta_e$, then $v_i \in o(e) \cup i(e)$.
- If $\Delta_e = x_i x_j$, then $i(e) \cup o(e) = \{v_i, v_j\}$.

We check the different conditions of Theorem 4.2.5:

1) Assume there exist $u, w, a, b, c, d \in S$, $u \neq w$, $a \neq d$ with $ua = wb \in S$ and $ud = wc \in S$. Then $ua = wb = \Delta_e$ and $ud = wc = \Delta_f$ for some $e, f \in E(\Gamma)$. As $u \neq w$ we have $e = f$. But then $a = d$ which is a contradiction.

2) Assume there exist $v, x, a, b, c, d \in S$, $v \neq x$, $a \neq b$ with $bv = cx \in S$ and $av = dx \in S$. Then $bv = cx = \Delta_e$ and $av = dx = \Delta_f$ for some $e, f \in E(\Gamma)$. As $v \neq x$ we have $e = f$. And so $a = b$ which is a contradiction.

3) Assume there exist $u, v, x, b, c \in S$, $v \neq x$ with $ux, uv, vb, xc \in S$ and $vb = xc$. Then $uv = \Delta_e$, $ux = \Delta_f$ and $vb = xc = \Delta_g$ for some $e, f, g \in E(\Gamma)$.

- If $e = f$ we have $x = v$ which is a contradiction.
- If $e = g$ and $e \neq f$ then $u, x \in \{x_1, \dots, x_n\}$. But then $o(e) \cup i(e) = \{u, x\} = o(f) \cup i(f)$. So $e = f$, which is a contradiction.
- If $e \neq g$, $f \neq g$, $e \neq f$ then $u, v, x \in \{x_1, \dots, x_n\}$ and $u \in i(e)$, $u \in i(f)$, $v \in o(e)$, $x \in o(f)$, and $\{x, v\} = i(g) \cup o(g)$. But this corresponds to an undirected triangle in the defining graph Γ.

4) Assume there exist $v, w, x, a, b \in S$, $v \neq x$, with $vw, xw, dx, av \in S$ and $dx = av$. Then $vw = \Delta_e$, $xw = \Delta_f$ and $dx = av = \Delta_g$ for some $e, f, g \in E(\Gamma)$.

- If $e = f$ we have $x = v$ which is a contradiction.
- If $e = g$ and $e \neq f$ then $w, x \in \{x_1, \dots, x_n\}$. But then $o(e) \cup i(e) = \{w, x\} = o(f) \cup i(f)$. So $e = f$, which is a contradiction.
- If $e \neq g$, $f \neq g$, $e \neq f$ then $v, w, x \in \{x_1, \dots, x_n\}$ and $w \in o(e)$, $w \in o(f)$ and $v \in i(e)$, $x \in i(f)$ and $o(g) \cup i(g) = \{v, x\}$. But this corresponds to an undirected triangle in the defining graph Γ.

5) Assume there exist $u, v, w, x \in S$, $v \neq x, u \neq w$ with $wv, wx, uv, ux \in S$. Then $wv = \Delta_e$, $wx = \Delta_f$, $uv = \Delta_g$ and $ux = \Delta_h$ for some $e, f, g, h \in E(\Gamma)$. Then $v \neq x$ implies $e \neq f$ and $g \neq h$, and $u \neq w$ implies $e \neq g$ and $f \neq h$. So $u, v, w, x \in \{x_1, \dots, x_n\}$ and $v \in o(e) \cap o(g)$, $x \in o(f) \cap o(h)$, $w \in i(e) \cap i(f)$ and $u \in i(g) \cap i(h)$. Furthermore $i(e) \cup o(e) = \{w, v\}$ which implies $v \neq w$ and $i(h) \cup o(h) = \{u, x\}$ which implies $u \neq x$. So the 4-cycle (u, v, w, x) is misdirected. This is a contradiction to the orientation assumption on Γ.

□

Remark 4.3.22. The Artin groups in Theorem 4.3.21 are all of almost large type. As such they were known to be systolic by [HO20]. The complex given here is of independent interest as it is two-dimensional. One can note that for any Artin group A_Γ satisfying the conditions of Theorem 4.3.21 we have $\mathrm{cd}(A_\Gamma) = 2$. For a general Artin group A_Γ, its cohomological dimension is conjectured to be $\mathrm{cd}(A_\Gamma) = \max\{|X| \mid X \subset S,\ A_X \text{ is spherical }\}$.

Remark 4.3.23. The Cayley complex of A_Γ with respect to the dual presentation was known to be CAT(0) by [BM00]. Since it is a two dimensional simplicial complex, this already implies that it is systolic. One may note that [BM00] predates the introduction of systolic complexes in [JŚ06].

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