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A NEW INVARIANT OF QUADRATIC LIE ALGEBRAS AND QUADRATIC LIE SUPERALGEBRAS

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Contents

Introduction

In this thesis, we study Lie algebras, Lie superalgebras, Jordan algebras and Novikov algebras equipped with a non-degenerate associative bilinear form. Such algebras are considered over the field of complex numbers and finite-dimensional. We add the condition that the bilinear form is symmetric or even, supersymmetric in the graded case. We call them respectively **quadratic Lie algebras**, **quadratic Lie superalgebras**, **pseudo-Euclidean Jordan algebras** and **symmetric Novikov algebras**.

Let $\mathfrak{g}$ be a finite-dimensional algebra over $\mathbb{C}$ and $(X,Y) \mapsto XY$ be its product. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called **associative** (or **invariant**) if it satisfies:

$$B(XY,Z) = B(X,YZ)$$

for all X,Y,Z in $\mathfrak{g}$ and **non-degenerate** if $B(X,\mathfrak{g}) = 0$ implies $X = 0$. Such a bilinear form has arisen in several areas of Mathematics and Physics. It can be seen as a generalization of the Killing form on a semisimple Lie algebra, the inner product of an Euclidean Jordan algebra or simply, as the Frobenius form of a Frobenious algebra. The associativity of a bilinear form also can be found in the conditions of an admissible trace function defined on a power-associative algebra. For details, the reader can refer to a paper by M. Bordemann [Bor97].

We begin with a quadratic Lie algebra $\mathfrak{g}$ and its product, the bracket $[\ ,\ ]$. A result in the work of G. Pinczon and R. Ushirobira [PU07] leads to our first problem: define the 3-form I on $\mathfrak{g}$ by $I(X,Y,Z) = B([X,Y],Z)$ for all X,Y,Z in $\mathfrak{g}$. Then I satisfies $\{I,I\} = 0$ where $\{\ ,\ \}$ is the super-Poisson bracket defined on $\mathscr{A}(\mathfrak{g})$, the Grassmann algebra of skew-symmetric multilinear forms on $\mathfrak{g}$ by:

$$\{\Omega,\Omega'\} = (-1)^{k+1}\sum_{j=1}^{n} \iota_{X_j}(\Omega)\wedge\iota_{X_j}(\Omega'),\ \forall\, \Omega \in \mathscr{A}^k(\mathfrak{g}), \Omega' \in \mathscr{A}(\mathfrak{g})$$

in a fixed orthonormal basis $\{X_1,\ \dots,\ X_n\}$ of $\mathfrak{g}$.

In this case, the element I is called the **3-form associated to** $\mathfrak{g}$. Conversely, given a quadratic vector space $(\mathfrak{g},B)$ and a non-zero 3-form I on $\mathfrak{g}$ such that $\{I,I\} = 0$, then there is a non-Abelian quadratic Lie algebra structure on $\mathfrak{g}$ such that I is the 3-form associated to $\mathfrak{g}$. By a classical result in a N. Bourbaki's book [Bou58] that is also recalled in Proposition 2.2.3, we set the following vector space:

$$\mathcal{V}_I = \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\}$$

The **dup-number** $\mathrm{dup}(\mathfrak{g})$ of a non-Abelian quadratic Lie algebra $\mathfrak{g}$ is defined by $\mathrm{dup}(\mathfrak{g}) = \dim(\mathcal{V}_I)$. It measures the decomposability of the 3-form I and its range is $\{0,1,3\}$. For instance, I is decomposable if and only if $\mathrm{dup}(\mathfrak{g}) = 3$ and then the corresponding quadratic Lie algebra structures can be determined completely up to isometrical isomorphisms (or i-isomorphisms, for short) [PU07]. Such Lie algebras appear also in the classification of Lie algebras whose coadjoint orbits of dimension at most 2 (done by D. Arnal, M. Cahen and J. Ludwig [ACL95]). A remarkable point is that the dup-number is invariant by i-isomorphism, that is, two i-isomorphic quadratic Lie algebras have the same dup-number.

The first goal of our study is to determine quadratic Lie algebra structures in the case $\mathrm{dup}(\mathfrak{g}) = 1$. The classification of such structures is one of the aims of this thesis. Here, we want to emphasize that our classification is considered in two senses: up to i-isomorphisms and more strongly, up to isomorphisms. This study is interesting by itself. It allows us to regard two distinguished kinds of classes: quadratic Lie algebras whose dup-number is non-zero and those whose dup-number is zero.

We say that a non-Abelian quadratic Lie algebra $\mathfrak{g}$ is **ordinary** if $\mathrm{dup}(\mathfrak{g}) = 0$. Otherwise, $\mathfrak{g}$ is called **singular**. By a technical requirement, we separate singular quadratic Lie algebras into two classes: those of **type** S_1 if their dup-number is 1 and of **type** S_3 if their dup-number is 3.

For $n \geq 1$, let $\mathcal{O}(n)$ be the set of ordinary, $\mathcal{S}(n)$ be the set of singular and $\mathcal{Q}(n)$ be the set of non-Abelian quadratic Lie algebra structures on $\mathbb{C}^n$. The distinction of two sets $\mathcal{O}(n)$ and $\mathcal{S}(n)$ is shown in Theorem 2.2.13 as follows:

THEOREM 1:

(1) $\mathcal{O}(n)$ *is Zariski-open and* $\mathcal{S}(n)$ *is Zariski-closed in* $\mathcal{Q}(n)$.

(2) $\mathcal{Q}(n) \neq \emptyset$ *if and only if* $n \geq 3$.

(3) $\mathcal{O}(n) \neq \emptyset$ *if and only if* $n \geq 6$.

Next, we shall give **a complete classification** of singular quadratic Lie algebras, up to i-isomorphisms and up to isomorphisms. It is done mainly on a **solvable** framework by the reason below. There are four main steps to reach this goal:

(1) Using the identity $\{I, I\} = 0$, we determine the Lie bracket on a solvable singular quadratic Lie algebra (Proposition 2.2.22).

(2) We describe a solvable singular quadratic Lie algebra as a **double extension** of a quadratic vector space by a skew-symmetric map (or double extension, for short) (this notion is initiated by V. Kac [Kac85] and generally developed by A. Medina and P. Revoy [MR85]). As a consequence of (1), a quadratic Lie algebra is singular and solvable if and only if it is a double extension (Proposition 2.2.28 and Proposition 2.2.29).

(3) We find the i-isomorphic and isomorphic conditions for two solvable singular quadratic Lie algebras (Theorem 2.2.30 and Corollary 2.2.31). These conditions allow us to establish a one-to-one correspondence between the set of i-isomorphic class of solvable singular quadratic Lie algebras and the set $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ of $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$, where $\mathbb{P}^1(\mathfrak{o}(n))$ denotes the projective space of the Lie algebra $\mathfrak{o}(n)$.

(4) Finally, we prove that the i-isomorphic and isomorphic notions coincide for solvable singular quadratic Lie algebras.

What about **non-solvable** singular quadratic Lie algebras? Such a Lie algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathfrak{s} \overset{\perp}{\oplus} \mathfrak{z}$$

where $\mathfrak{z}$ is a central ideal of $\mathfrak{g}$ and $\mathfrak{s} \simeq \mathfrak{o}(3)$ equipped with a bilinear form $\lambda\kappa$ for some non-zero $\lambda \in \mathbb{C}$ where κ is the Killing form of $\mathfrak{o}(3)$. Note that different from the solvable case, the notions of i-isomophism and isomorphism are not equivalent in this case.

We denote by $\mathcal{S}_s(n+2)$ the set of solvable singular quadratic Lie algebra structures on $\mathbb{C}^{n+2}$, by $\widehat{\mathcal{S}_s}(n+2)$ the set of isomorphism classes of elements in $\mathcal{S}_s(n+2)$ and by $\widehat{\mathcal{S}_s}^{\mathrm{i}}(n+2)$ the set of i-isomorphism classes. Given $\overline{C} \in \mathfrak{o}(n)$, there is an associated double extension $\mathfrak{g}_{\overline{C}} \in \mathcal{S}_s(n+2)$ (Definition 2.2.26) and then (Theorem 2.2.35):

THEOREM 2:
The map $\overline{C} \to \mathfrak{g}_{\overline{C}}$ induces a bijection from $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ onto $\widehat{\mathcal{S}_s}^{\mathrm{i}}(n+2)$.

A weak form of Theorem 2 was stated in the paper by G. Favre and L. J. Santharoubane [FS87], in the case of i-isomorphisms satisfying some (dispensable) conditions. A strong improvement to Theorem 2 will be given in Theorem 5 where the i-isomorphic notion is replaced by the isomorphic notion.

We detail Theorem 2 in some particular cases. Let $\mathcal{N}(n+2)$ be the set of **nilpotent singular structures** on $\mathbb{C}^{n+2}$, $\widehat{\mathcal{N}}^{\mathrm{i}}(n+2)$ be the set of i-isomorphism classes and $\widehat{\mathcal{N}}(n+2)$ be the set of isomorphism classes of elements in $\mathcal{N}(n+2)$. We denote $\mathfrak{g}$ and $\mathfrak{g}'$ i-isomorphic by $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$. Using the Jacobson-Morozov theorem, we prove that (Theorem 2.2.37):

THEOREM 3:

(1) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be in $\mathcal{N}(n+2)$. Then $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$ if and only if $\mathfrak{g} \simeq \mathfrak{g}'$. Thus $\widehat{\mathcal{N}}^{\mathrm{i}}(n+2) = \widehat{\mathcal{N}}(n+2)$.

(2) Let $\widetilde{\mathscr{N}}(n)$ be the set of nilpotent $\mathrm{O}(n)$-adjoint orbits in $\mathfrak{o}(n)$. Then the map $\overline{C} \mapsto \mathfrak{g}_{\overline{C}}$ induces a bijection from $\widetilde{\mathscr{N}}(n)$ onto $\widehat{\mathcal{N}}(n+2)$.

In Chapter 1, we recall the well-known classification of nilpotent $\mathrm{O}(n)$-adjoint orbits in $\mathfrak{o}(n)$. An important ingredient is the Jacobson-Morosov and Kostant theorems on $\mathfrak{sl}(2)$-triples in semisimple Lie algebras (see the book by D. H. Collingwood and W. M. McGovern [CM93] for more details). Using this classification, we obtain a classification of $\widehat{\mathcal{N}}(n+2)$ in term of *special* partitions of n, that is, there is a one-to-one correspondence between $\widehat{\mathcal{N}}(n+2)$ and the set $\mathscr{P}_1(n)$ of partitions of n in which even parts occur with even multiplicity (Theorem 2.2.38). In other words, we can parametrize the set $\widehat{\mathcal{N}}(n+2)$ by the set of indices $\mathscr{P}_1(n)$. This parametrization is detailed by means of amalgamated products of **nilpotent Jordan-type Lie algebras**.

Let $\mathcal{D}(n+2)$ be the set of **diagonalizable singular structures** on $\mathbb{C}^{n+2}$ (i.e. $\overline{C}$ is a semisimple element of $\mathfrak{o}(n)$) and $\mathcal{D}_{\mathrm{red}}(n+2)$ be the set of reduced ones (see Definition 2.1.7 for the definition of a **reduced** quadratic Lie algebra). Denote by $\widehat{\mathcal{D}}(n+2)$, $\widehat{\mathcal{D}}^{\mathrm{i}}(n+2)$, $\widehat{\mathcal{D}}_{\mathrm{red}}(n+2)$

and $\widehat{\mathcal{D}}^{\mathrm{i}}_{\mathrm{red}}(n+2)$ the corresponding sets of isomorphism and i-isomorphism classes of elements in $\mathcal{D}(n+2)$ and $\mathcal{D}_{\mathrm{red}}(n+2)$. It is clear by Theorem 2 that $\widehat{\mathcal{D}}^{\mathrm{i}}(n+2)$ is in bijection with the well-known set of semisimple $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. More precisely, we have the following result (Proposition 2.2.40, Corollary 2.2.43 and Proposition 2.2.44):

THEOREM 4:

(1) There is a bijection between $\widehat{\mathcal{D}}^{\mathrm{i}}(n+2)$ and the set of semisimple $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. The same result holds for $\widehat{\mathcal{D}}^{\mathrm{i}}_{\mathrm{red}}(n+2)$ and semisimple invertible orbits in $\mathbb{P}^1(\mathfrak{o}(n))$.

(2) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be in $\mathcal{D}_{\mathrm{red}}(n+2)$. Then n must be even and $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$ if and only if $\mathfrak{g} \simeq \mathfrak{g}'$. Thus, $\widehat{\mathcal{D}}_{\mathrm{red}}(2p+2) = \widehat{\mathcal{D}}^{\mathrm{i}}_{\mathrm{red}}(2p+2)$, for all $p \geq 1$.

(3) Let $(\mathfrak{g}, B)$ be a diagonalizable reduced singular quadratic Lie algebra. Consider $\mathfrak{g}_4$ the double extension of $\mathbb{C}^2$ by $\overline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathfrak{g}$ is an amalgamated product of singular quadratic Lie algebras all i-isomorphic to $\mathfrak{g}_4$.

Combined with the classification of semisimple $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ in Chapter 1, the sets $\widehat{\mathcal{D}}(2p+2)$ and $\widehat{\mathcal{D}}_{\mathrm{red}}(2p+2)$ can be parametrized as in Theorem 2.2.41 where the set $\widehat{\mathcal{D}}(2p+2)$ is in bijection with Λ_p/H_p and the set $\widehat{\mathcal{D}}_{\mathrm{red}}(2p+2)$ is in bijection with Λ_p^+/H_p (see Section 1.3 and Subsection 2.2.5 for the respective definitions).

Clearly, the parametrization of i-isomorphic classes of nilpotent or diagonalizable singular quadratic Lie algebras can be regarded as a direct corollary of the classification of nilpotent or semisimple $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$. However, we do not find any reference that shows how to parametrize $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ in the general case. Therefore, our next objective is to determine such a parametrization. This solution allows us to go further in the classification of singular quadratic Lie algebras.

We continue with the notion of an **invertible singular quadratic Lie algebra** (i.e. $\overline{C}$ is invertible). Let $\mathcal{S}_{\mathrm{inv}}(2p+2)$ be the set of such structures on $\mathbb{C}^{2p+2}$ and $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ be the set of isomorphism classes of elements in $\mathcal{S}_{\mathrm{inv}}(2p+2)$. The isomorphic and i-isomorphic notions coincide in the invertible case as we show in Lemma 2.2.42. Moreover, a description of invertible singular quadratic Lie algebras in term of amalgamated product can be found in Proposition 2.2.49. The classification of the set $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ is deduced from Theorem 6 below.

We turn our attention to the general case. Given a solvable singular quadratic Lie algebra $\mathfrak{g}$, realized as the double extension of $\mathbb{C}^n$ by $\overline{C} \in \mathfrak{o}(n)$, we consider the Fitting components $\overline{C}_I$ and $\overline{C}_N$ of $\overline{C}$ and the corresponding double extensions $\mathfrak{g}_I = \mathfrak{g}_{\overline{C}_I}$ and $\mathfrak{g}_N = \mathfrak{g}_{\overline{C}_N}$ that we call the **Fitting components** of $\mathfrak{g}$. We have $\mathfrak{g}_I$ invertible, $\mathfrak{g}_N$ nilpotent and $\mathfrak{g}$ is the amalgamated product of $\mathfrak{g}_I$ and $\mathfrak{g}_N$. We prove that (Theorem 2.2.52):

THEOREM 5:

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be solvable singular quadratic Lie algebras and let $\mathfrak{g}_N$, $\mathfrak{g}_I$, $\mathfrak{g}'_N$, $\mathfrak{g}'_I$ be their Fitting components, respectively. Then:

(1) $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$ *if and only if* $\begin{cases} \mathfrak{g}_N \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_N \\ \mathfrak{g}_I \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_I \end{cases}$.

The result remains valid if we replace $\overset{\mathrm{i}}{\simeq}$ *by* $\simeq$.

(2) $\mathfrak{g} \simeq \mathfrak{g}'$ *if and only if* $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$. *Therefore* $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2) = \widehat{\mathcal{S}_{\mathrm{s}}}^{\mathrm{i}}(n+2)$.

Theorem 5 is a really interesting and unexpected property of solvable singular quadratic Lie algebras.

From the above facts, in order to describe more precisely the set $\mathcal{S}_{\mathrm{s}}(n+2)$, we turn to the problem of classification of $\mathrm{O}(n)$-adjoint orbits in $\mathfrak{o}(n)$. Since the study of the nilpotent orbits is complete, we begin with the invertible case. Let $\mathscr{I}(n)$ be the set of invertible elements in $\mathfrak{o}(n)$ and $\widetilde{\mathscr{I}}(n)$ be the set of $\mathrm{O}(n)$-adjoint orbits of elements in $\mathscr{I}(n)$. Notice that $\mathscr{I}(2p+1) = \emptyset$ (Appendix A) then we consider $n = 2p$. Define the set

$$\mathscr{D} = \bigcup_{r \in \mathbb{N}^*} \{(d_1, \dots, d_r) \in \mathbb{N}^r \mid d_1 \geq d_2 \geq \dots \geq d_r \geq 1\}$$

and the map $\Phi : \mathscr{D} \to \mathbb{N}$ defined by $\Phi(d_1, \dots, d_r) = \sum_{i=1}^r d_i$. We introduce the set $\mathscr{J}_p$ of all triples (Λ, m, d) such that:

(1) Λ is a subset of $\mathbb{C} \setminus \{0\}$ with $\sharp\Lambda \leq 2p$ and $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$.

(2) $m : \Lambda \to \mathbb{N}^*$ satisfies $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ and $\sum_{\lambda \in \Lambda} m(\lambda) = 2p$.

(3) $d : \Lambda \to \mathscr{D}$ satisfies $d(\lambda) = d(-\lambda)$, for all $\lambda \in \Lambda$ and $\Phi \circ d = m$.

To every $\overline{C} \in \mathscr{I}(2p)$, we can associate an element (Λ, m, d) of $\mathscr{J}_p$ as follows: write $\overline{C} = S + N$ as a sum of its semisimple and nilpotent parts. Then Λ is the spectrum of S, m is the multiplicity map on Λ and d gives the size of the Jordan blocks of N. Therefore, we obtain a map $i : \mathscr{I}(2p) \to \mathscr{J}_p$ and we prove:

THEOREM 6:
The map $i : \mathscr{I}(2p) \to \mathscr{J}_p$ *induces a bijection from* $\widetilde{\mathscr{I}}(2p)$ *onto* $\mathscr{J}_p$.

As a corollary, we deduce a bijection from $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ onto $\mathscr{J}_p/\mathbb{C}^*$ (Theorem 2.2.50) where the action of $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathscr{J}_p$ is defined by

$$\mu \cdot (\Lambda, m, d) = (\mu\Lambda, m', d'), \text{ with } m'(\mu\lambda) = m(\lambda) \text{ and } d'(\mu\lambda) = d(\lambda), \forall\, \lambda \in \Lambda.$$

Combined with the previous theorems, we obtain a complete classification of $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2)$ as follows. Let $\mathscr{D}(n)$ be the set of all pairs $([d], T)$ such that $[d] \in \mathscr{P}_1(m)$, the set of partitions of m in which even parts occur with even multiplicity, and $T \in \mathscr{J}_\ell$ satisfying $m + 2\ell = n$. We set an action of the multiplicative group $\mathbb{C}^*$ on $\mathscr{D}(n)$ by:

$$\mu \cdot ([d], T) = ([d], \mu \cdot T)\ ,\ \forall\, \mu \in \mathbb{C}^*,\ ([d], T) \in \mathscr{D}(n).$$

and obtain the following result (Theorem 2.2.54):

THEOREM 7:
The set $\widehat{\mathcal{S}}_{\mathrm{s}}(n+2)$ *is in bijection with* $\mathscr{D}(n)/\mathbb{C}^*$.

By this process, we also obtain a complete classification of $\mathrm{O}(n)$-adjoint orbits in $\mathfrak{o}(n)$, a result which is certainly known, but for which we have no available reference.

We close the first problem with the result as follows (Theorem 2.3.7):

THEOREM 8:
The dup-*number is invariant under isomorphisms, i.e. if*

$$\mathfrak{g} \simeq \mathfrak{g}' \ \textit{then} \ \mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{g}').$$

Its proof is not really obvious as in the case of i-isomorphisms. It is obtained through a computation of **centromorphisms** in the reduced singular case. Here, we use a result of I. Bajo and S. Benayadi given in [BB97]. We also have the quadratic dimension $d_q(\mathfrak{g})$ of $\mathfrak{g}$ in this case (Proposition 2.3.6):

$$d_q(\mathfrak{g}) = 1 + \frac{\dim(\mathcal{Z}(\mathfrak{g}))(1+\dim(\mathcal{Z}(\mathfrak{g}))}{2},$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

As we will see in Chapter 4, we are also interested in 2-step nilpotent quadratic Lie algebras. Thanks to double extensions, the simplest case of a quadratic Lie algebra is a solvable singular quadratic Lie algebra. We have a similar situation for 2-step nilpotent quadratic Lie algebras in term of T^*-extensions, a notion given by M. Bordemann [Bor97]. Such algebras with a characterization of i-isomorphic classes and isomorphic classes were introduced in a paper of G. Ovando [Ova07]. By studying the set of linear transformations in $\mathfrak{o}(\mathfrak{h})$ where $\mathfrak{h}$ is a vector space with a fixed inner product, the author shows that if the dimension of the vector space $\mathfrak{h}$ is three or greater than four, there exists a reduced 2-step nilpotent quadratic Lie algebra. Moreover, there is only one six-dimensional reduced 2-step nilpotent quadratic Lie algebra (up to i-isomorphisms). In this thesis, once again, we want to approach 2-step nilpotent quadratic Lie algebras through the method of double extensions and the associated 3-form I. In term of double extensions, we have a rather obvious result: every 2-step nilpotent quadratic Lie algebra can be obtained from an Abelian algebra by a sequence of double extensions by one-dimensional algebra (Proposition 2.4.12).

In order to observe the appearance of the element I, we recall the notion of T^***-extension** of a Lie algebra in [Bor97] but with some restricted conditions as follows. Let $\mathfrak{h}$ be a complex vector space and $\theta : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*$ be a non-degenerate skew-symmetric bilinear map. We assume that θ is *cyclic* (that means $\theta(x,y)z = \theta(y,z)x$ for all $x,y,z \in \mathfrak{g}$). Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ be the vector space equipped with the bracket

$$[x+f, y+g] = \theta(x,y)$$

and the bilinear form

$$B(x+f, y+g) = f(y) + g(x),$$

for all $x,y \in \mathfrak{h}, f,g \in \mathfrak{h}^*$. Then $(\mathfrak{g},B)$ is a 2-step nilpotent quadratic Lie algebra and called the **T^*-extension of $\mathfrak{h}$ by θ** (or T^*-extension, simply).

The set of T^*-extensions is enough to represent all 2-step nilpotent quadratic Lie algebras by a result in Proposition 2.4.14 that every reduced 2-step nilpotent quadratic Lie algebra is i-isomorphic to a T^*-extension. Thus, we focus on the isomorphic classes and i-isomorphic classes of T^*-extensions. We prove that (Theorem 2.4.16):

THEOREM 9:
Let $\mathfrak{g}$ and $\mathfrak{g}'$ be T^*-extensions of $\mathfrak{h}$ by θ_1 and θ_2 respectively. Then:

(1) there exists a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist an isomorphism A_1 of $\mathfrak{h}$ and an isomorphism A_2 of $\mathfrak{h}^*$ such that

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)),\ \forall\, x,y \in \mathfrak{h}.$$

(2) there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isomorphism A_1 of $\mathfrak{h}$ such that

$$\theta_1(x,y) = \theta_2(A_1(x),A_1(y)) \circ A_1,\ \forall\, x,y \in \mathfrak{h}.$$

Now, if we define $I(x,y,z) = \theta(x,y)z$, for all $x,y,z \in \mathfrak{h}$ then the element I is the 3-form associated to the T^*-extension of $\mathfrak{h}$ by θ. Moreover, there is a one-to-one correspondence between the set of T^*-extensions of $\mathfrak{h}$ and the set of 3-forms $\{I \in \mathscr{A}^3(\mathfrak{h}) \mid \iota_x(I) \neq 0,\ \forall\, x \in \mathfrak{h} \setminus \{0\}\}$. Remark that by Theorem 9 the i-isomorphic classification of T^*-extensions of $\mathfrak{h}$ can be reduced to the isomorphic classification of such 3-forms on $\mathfrak{h}$. As a consequence, we obtain the same result as in [Ova07] and further that there exists only one reduced 2-step nilpotent quadratic Lie algebra of dimension 10 (Appendix C and Remark 2.4.21).

In Chapter 3, we give a graded version of the main results in Chapter 2: **singular quadratic Lie superalgebras**. We begin with a quadratic $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a non-degenerate bilinear form B. Recall that the bilinear form B is symmetric on $V_{\bar{0}}$, skew-symmetric on $V_{\bar{1}}$ and $B(V_{\bar{0}},V_{\bar{1}}) = 0$.

Consider the **super-exterior algebra** of V^* defined by a $\mathbb{Z} \times \mathbb{Z}_2$-gradation

$$\mathcal{E}(V) = \mathscr{A}(V_{\bar{0}}) \underset{\mathbb{Z}\times\mathbb{Z}_2}{\otimes} \mathscr{S}(V_{\bar{1}})$$

with the natural **super-exterior product**

$$(\Omega \otimes F) \wedge (\Omega' \otimes F') = (-1)^{f\omega'}(\Omega \wedge \Omega') \otimes FF',$$

for all Ω, Ω' in the algebra $\mathscr{A}(V_{\bar{0}})$ of alternating multilinear forms on $V_{\bar{0}}$ and F,F' in the algebra $\mathscr{S}(V_{\bar{1}})$ of symmetric multilinear forms on $V_{\bar{1}}$. It is clear that this algebra is commutative and associative. In [MPU09], I. A. Musson, G. Pinczon and R. Ushirobira presented the **super $\mathbb{Z} \times \mathbb{Z}_2$-Poisson bracket** on $\mathcal{E}(V)$ as follows:

$$\{\Omega \otimes F, \Omega' \otimes F'\} = (-1)^{f\omega'}\left(\{\Omega,\Omega'\} \otimes FF' + (\Omega \wedge \Omega') \otimes \{F,F'\}\right),$$

for all $\Omega \in \mathscr{A}(V_{\bar{0}})$, $\Omega' \in \mathscr{A}^{\omega'}(V_{\bar{0}})$, $F \in \mathscr{S}^f(V_{\bar{1}})$, $F' \in \mathscr{S}(V_{\bar{1}})$.

We realize that with a quadratic Lie superalgebra $(\mathfrak{g},B)$, if we define a trilinear form I on $\mathfrak{g}$ by

$$I(X,Y,Z) = B([X,Y],Z),\ \forall\, X,Y,Z \in \mathfrak{g}$$

then I is a super-antisymmetric trilinear form in $\mathcal{E}^{(3,\bar{0})}(\mathfrak{g})$ and therefore, it seems to be natural to ask the question: does it happen $\{I,I\} = 0$? We give an affirmative answer this in the first part of Chapter 3 (Theorem 3.1.17). Moreover, we obtain that quadratic Lie superalgebra structures with bilinear form B are in one-to-one correspondence with elements I in $\mathcal{E}^{(3,\bar{0})}(\mathfrak{g})$ satisfying $\{I,I\} = 0$ (Proposition 3.1.18)

As in Chapter 2, we give the notion of dup-number of a non-Abelian quadratic Lie superalgebra $\mathfrak{g}$ defined by

$$\mathrm{dup}(\mathfrak{g}) = \dim(\{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\})$$

and suggest considering the set of quadratic Lie superalgebras where the dup-number is non-zero. Thanks to Lemma 3.2.1, we focus on a singular quadratic Lie superalgebra $\mathfrak{g}$ with $\mathrm{dup}(\mathfrak{g}) = 1$. It is called a **singular quadratic Lie superalgebra of type** S_1. Remark that in this case, the element I may be decomposable.

We detail some particular cases. When the element I is decomposable, we obtain a classification of reduced corresponding Lie superalgebras as in Proposition 3.3.3 where the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is a singular quadratic Lie algebra or 2-dimensional. Actually, we prove in Proposition 3.4.1 that if $\mathfrak{g}$ is a non-Abelian quadratic Lie superalgebra with 2-dimensional even part then $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_1.

Note that if we replace the quadratic vector space $\mathfrak{q}$ in the definition of double extension by a symplectic vector space then we obtain a quadratic Lie superalgebra with 2-dimensional even part (Definitions 2.2.26 and 3.4.6). Thus, we have the following result (Theorem 3.4.8):

THEOREM 10:
A quadratic Lie superalgebra has the 2-dimensional even part if and only if it is a double extension.

By a completely similar process as in Chapter 2, a classification of quadratic Lie superalgebras with 2-dimensional even part is given as follows. Let $\mathcal{S}(2+2n)$ be the set of such structures on $\mathbb{C}^{2+n}$. We call an algebra $\mathfrak{g} \in \mathcal{S}(2+2n)$ **diagonalizable** (resp. **invertible**) if it is the double extension by a diagonalizable (resp. invertible) map. Denote the subsets of nilpotent elements, diagonalizable elements and invertible elements in $\mathcal{S}(2+2n)$, respectively by $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$ and by $\mathcal{S}_{\mathrm{inv}}(2+2n)$. Denote by $\widehat{\mathcal{N}}(2+2n)$, $\widehat{\mathcal{D}}(2+2n)$, $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ the sets of isomorphic classes in $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$, $\mathcal{S}_{\mathrm{inv}}(2+2n)$, respectively and $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ the subset of $\widehat{\mathcal{D}}(2+2n)$ including reduced ones. Keeping the other notations then we have the classification result (Theorems 3.4.13 and 3.4.14):

THEOREM 11:

(1) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be elements in $\mathcal{S}(2+2n)$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if they are isomorphic.

(2) There is a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of nilpotent $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ that induces a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of partitions $\mathscr{P}_{-1}(2n)$ of $2n$ in which odd parts occur with even multiplicity.

(3) *There is a bijection between* $\widehat{\mathcal{D}}(2+2n)$ *and the set of semisimple* $\mathrm{Sp}(2n)$- *orbits of* $\mathbb{P}^1(\mathfrak{sp}(2n))$ *that induces a bijection between* $\widehat{\mathcal{D}}(2+2n)$ *and* Λ_n/H_n. *In the reduced case,* $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ *is bijective to* Λ_n^+/H_n.

(4) *There is a bijection between* $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ *and the set of invertible* $\mathrm{Sp}(2n)$- *orbits of* $\mathbb{P}^1(\mathfrak{sp}(2n))$ *that induces a bijection between* $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ *and* $\mathscr{J}_n/\mathbb{C}^*$.

(5) *There is a bijection between* $\widehat{\mathcal{S}}(2+2n)$ *and the set of* $\mathrm{Sp}(2n)$- *orbits of* $\mathbb{P}^1(\mathfrak{sp}(2n))$ *that induces a bijection between* $\widehat{\mathcal{S}}(2+2n)$ *and* $\mathscr{D}(2n)/\mathbb{C}^*$.

As for quadratic Lie algebras, we have the notion of quadratic dimension for quadratic Lie superalgebras. In the case $\mathfrak{g}$ having a 2-dimensional even part, we can compute its quadratic dimension as follows:

$$d_q(\mathfrak{g}) = 2 + \frac{(\dim(\mathscr{Z}(\mathfrak{g})-1))(\dim(\mathscr{Z}(\mathfrak{g})-2)}{2}.$$

We turn now to $(\mathfrak{g},B)$ a singular quadratic Lie superalgebra of type S_1. By Definition 3.5.3 and Lemma 3.5.5, the Lie superalgebra $\mathfrak{g}$ can be realized as the double extension of a quadratic $\mathbb{Z}_2$-graded vector space $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ by a map $\overline{C} = \overline{C}_0 + \overline{C}_1 \in \mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$. Denote by $\mathscr{L}(\mathfrak{q}_{\bar{0}})$ (resp. $\mathscr{L}(\mathfrak{q}_{\bar{1}})$) the set of endomorphisms of $\mathfrak{q}_{\bar{0}}$ (resp. $\mathfrak{q}_{\bar{1}}$). We give a characterization as follows (Theorem 3.5.7).

THEOREM 12:
Let $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be two double extensions of* $\mathfrak{q}$ *by* $\overline{C} = \overline{C}_0 + \overline{C}_1$ *and* $\overline{C}' = \overline{C}_0' + \overline{C}_1'$, *respectively. Assume that* $\overline{C}_1$ *is non-zero. Then*

(1) *there exists a Lie superalgebra isomorphism between* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if there exist invertible maps* $P \in \mathscr{L}(\mathfrak{q}_{\bar{0}})$, $Q \in \mathscr{L}(\mathfrak{q}_{\bar{1}})$ *and a non-zero* $\lambda \in \mathbb{C}$ *such that*

(i) $\overline{C}_0' = \lambda P \overline{C}_0 P^{-1}$ *and* $P^* P \overline{C}_0 = \overline{C}_0$.

(ii) $\overline{C}_1' = \lambda Q \overline{C}_1 Q^{-1}$ *and* $Q^* Q \overline{C}_1 = \overline{C}_1$.

where P^* *and* Q^* *are the adjoint maps of* P *and* Q *with respect to* $B|_{\mathfrak{q}_{\bar{0}}\times\mathfrak{q}_{\bar{0}}}$ *and* $B|_{\mathfrak{q}_{\bar{1}}\times\mathfrak{q}_{\bar{1}}}$.

(2) *there exists an i-isomorphism between* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if there is a non-zero* $\lambda \in \mathbb{C}$ *such that* $\overline{C}_0'$ *is in the* $\mathrm{O}(\mathfrak{q}_{\bar{0}})$*-adjoint orbit through* $\lambda\overline{C}_0$ *and* $\overline{C}_1'$ *is in the* $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$*-adjoint orbit through* $\lambda\overline{C}_1$.

We close the problem on singular quadratic Lie superalgebras by an assertion that the dup-number is invariant under Lie superalgebra isomorphisms (Theorem 3.5.9).

In the last section of Chapter 3, we study the structure of a quadratic Lie superalgebra $\mathfrak{g}$ such that its element I has the form:

$$I = J \wedge p$$

where $p \in \mathfrak{g}_{\bar{1}}^*$ is non-zero and $J \in \mathscr{A}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^1(\mathfrak{g}_{\bar{1}})$ is indecomposable. We call $\mathfrak{g}$ a **quasi-singular quadratic Lie superalgebra**. With the notion of **generalized double extension** given by I. Bajo, S. Benayadi and M. Bordemann in [BBB], we prove that (Corollary 3.6.5 and Theorem 3.6.8)

THEOREM 13:
A quasi-singular quadratic Lie superalgebra is a generalized double extension of a quadratic $\mathbb{Z}_2$-graded vector space. This superalgebra is 2-nilpotent.

The algebras obtained in Chapter 2 and Chapter 3 lead us to the general framework as follows: let $\mathfrak{q}$ be a complex vector space equipped with a non-degenerate bilinear form $B_{\mathfrak{q}}$ and $C : \mathfrak{q} \to \mathfrak{q}$ be a linear map. We associate a vector space :

$$\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

to the triple $(\mathfrak{q}, B_{\mathfrak{q}}, C)$ where $(\mathfrak{t} = \mathrm{span}\{x_1, y_1\}, B_{\mathfrak{t}})$ is a 2-dimensional vector space and $B_{\mathfrak{t}} : \mathfrak{t} \times \mathfrak{t} \to \mathbb{C}$ is the bilinear form defined by

$$B_{\mathfrak{t}}(x_1, x_1) = B_{\mathfrak{t}}(y_1, y_1) = 0,\ B_{\mathfrak{t}}(x_1, y_1) = 1.$$

Define a product $\star$ on the vector space $\mathfrak{J}$ such that $\mathfrak{t}$ is a subalgebra of $\mathfrak{J}$,

$$y_1 \star x = C(x),\ x_1 \star x = 0,\ x \star y = B_{\mathfrak{q}}(C(x), y)x_1$$

for all $x, y \in \mathfrak{q}$ and such that the bilinear form $B_{\mathfrak{J}} = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ is associative. We call $\mathfrak{J}$ the **double extension of $\mathfrak{q}$ by C**. It can be completely characterized by the pair $(B_{\mathfrak{q}}, C)$. Solvable singular quadratic Lie algebras and singular quadratic Lie superalgebras are only particular cases of this notion. Therefore, it is natural to consider similar algebras corresponding to the remaining different cases of the pair $(B_{\mathfrak{q}}, C)$. In Chapter 4 we give a condition that $\mathfrak{J}$ is a pseudo-Euclidean (commutative) Jordan algebra (i.e a Jordan algebra endowed with a non-degenerate associative symmetric bilinear form). Consequently, the bilinear forms $B_{\mathfrak{q}}$, $B_{\mathfrak{t}}$ are symmetric, C must be also symmetric (with respect to $B_{\mathfrak{q}}$) and the product $\star$ is defined by:

$$\begin{aligned}(x + \lambda x_1 + \mu y_1) \star (y + \lambda' x_1 + \mu' y_1) \;&=\; \mu C(y) + \mu' C(x) + B_{\mathfrak{q}}(C(x), y)x_1 \\ &\quad + \varepsilon\left((\lambda\mu' + \lambda'\mu)\, x_1 + \mu\mu' y_1\right),\end{aligned}$$

$\varepsilon \in \{0, 1\}$, for all $x, y \in \mathfrak{q}$, $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$.

Since there exist only two one-dimensional Jordan algebras, one Abelian and one simple, then we have two types of extensions called respectively **nilpotent double extension** and **diagonalizable double extension**. The first result (Proposition 4.2.1, Corollary 4.2.2, Lemma 4.2.7 and Appendix D) is the following:

THEOREM 14:

(1) If $\mathfrak{J}$ is the nilpotent double extension of $\mathfrak{q}$ by C then $C^3 = 0$, $\mathfrak{J}$ is k-step nilpotent, $k \leq 3$, and $\mathfrak{t}$ is an Abelian subalgebra of $\mathfrak{J}$.

(2) If $\mathfrak{J}$ is the diagonalizable double extension of $\mathfrak{q}$ by C then $3C^2 = 2C^3 + C$, $\mathfrak{J}$ is not solvable and $\mathfrak{t} \star \mathfrak{t} = \mathfrak{t}$. In the reduced case, y_1 acts diagonally on $\mathfrak{J}$ with eigenvalues 1 and $\frac{1}{2}$.

This result can be obtained by checking **Jordan identity** for the algebra $\mathfrak{J}$. However, it can be seen as a particular case of the general theory of double extension on pseudo-Euclidean (commutative) Jordan algebras given by A. Baklouti and S. Benayadi in [BB], that is the double extension of an Abelian algebra by one dimensional Jordan algebra. By the similar method as in Chapter 2 and Chapter 3, we obtain the classification result (Theorem 4.2.5, Theorem 4.2.8 and Corollary 4.2.9):

THEOREM 15:

(1) Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be nilpotent double extensions of $\mathfrak{q}$ by symmetric maps C and C', respectively. Then there exists a Jordan algebra isomorphism $A : \mathfrak{J} \to \mathfrak{J}'$ such that $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$ if and only if there exist an invertible map $P \in \mathrm{End}(\mathfrak{q})$ and a non-zero $\lambda \in \mathbb{C}$ such that $\lambda C' = PCP^{-1}$ and $P^*PC = C$ where P^* is the adjoint map of P with respect to B. In this case A i-isomorphic then $P \in \mathrm{O}(\mathfrak{q})$.

(2) Let $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ be diagonalizable double extensions of $\mathfrak{q}$ by symmetric maps C and C', respectively. Then $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic if and only if they are i-isomorphic. In this case, C and C' have the same spectrum.

The next part of Chapter 4 can be regarded as the symmetric version of 2-step nilpotent quadratic Lie algebras, that is the class of 2-step nilpotent pseudo-Euclidean Jordan algebras. We introduce the notion of generalized double extension but with a restricting condition for 2-step nilpotent pseudo-Euclidean Jordan algebras. As a consequence, we obtain in this way the inductive characterization of those algebras (Proposition 4.3.11): a non-Abelian 2-step nilpotent pseudo-Euclidean Jordan algebra is obtained from an Abelian algebra by a sequence of generalized double extensions.

To characterize (up to isomorphisms and i-isomorphisms) 2-step nilpotent pseudo-Euclidean Jordan algebras we need to use again the concept of a T^*-extension as above with a little change. Given a complex vector space $\mathfrak{a}$ and a non-degenerate cyclic *symmetric* bilinear map $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$. On the vector space $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{a}^*$ we define the product

$$(x+f)(y+g) = \theta(x,y).$$

Then $\mathfrak{J}$ is a 2-step nilpotent pseudo-Euclidean Jordan algebra and it is called the **T^*-extension of $\mathfrak{a}$ by θ** (or T^*-extension, simply). Moreover, every reduced 2-step nilpotent pseudo-Euclidean Jordan algebra is i-isomorphic to some T^*-extension (Proposition 4.3.14). An i-isomorphic and isomorphic characterization of T^*-extensions is given in Theorem 4.3.15 as follows:

THEOREM 16:
Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be T^*-extensions of $\mathfrak{a}$ by θ_1 and θ_2 respectively. Then:

(1) there exists a Jordan algebra isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exist an isomorphism A_1 of $\mathfrak{a}$ and an isomorphism A_2 of $\mathfrak{a}^*$ satisfying:

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)),\ \forall\, x,y \in \mathfrak{a}.$$

(2) there exists an i-isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exists an isomorphism A_1 of $\mathfrak{a}$ satisfying

$$\theta_1(x,y) = \theta_2(A_1(x),A_1(y)) \circ A_1, \ \forall\, x,y \in \mathfrak{a}.$$

As a consequence, the classification of i-isomorphic T^*-extensions of $\mathfrak{a}$ is equivalent to the classification of symmetric 3-forms on $\mathfrak{a}$. We will detail it in the cases of $\dim(\mathfrak{a}) = 1$ and 2 (Examples 4.3.18 and 4.3.19).

In the last of Chapter 4, we study Novikov algebras. These objects appeared in the study of the Hamiltonian condition of an operator in the formal calculus of variations (see the paper by I. M. Gel'fand and I. Y. Dorfman [GD79]) as well as in the classification of Poisson brackets of hydrodynamic type (done by A. A. Balinskii and S. P. Novikov in [BN85]). A detailed classification of Novikov algebras up to dimension 3 is given by C. Bai and D. Meng in [BM01].

It is known that an associative algebra is both Lie-admissible and Jordan-admissible. This is not true for Novikov algebras although they are Lie-admissible. Therefore, it is natural to search a condition for a Novikov algebra to become Jordan-admissible. The condition we give here (weaker than associativity) is the following (Theorem 4.4.17):

THEOREM 17:
A Novikov algebra $\mathfrak{N}$ is Jordan-admissible if it satisfies the condition

$$(x,x,x) = 0, \ \forall\, x \in \mathfrak{N}.$$

A corollary of Theorem 17 is that Novikov algebras are not power-associative since there exist Novikov algebras not Jordan-admissible.

Next, we consider symmetric Novikov algebras. In this case, $\mathfrak{N}$ will be associative, its sub-adjacent Lie algebra $\mathfrak{g}(\mathfrak{N})$ is a 2-step nilpotent quadratic Lie algebra (shown in the paper [AB10] of I. Ayadi and S. Benayadi) and the associated Jordan algebra $\mathfrak{J}(\mathfrak{N})$ is pseudo-Euclidean. Therefore, the study of 2-step nilpotent quadratic Lie algebras and pseudo-Euclidean Jordan algebras is closely related to symmetric Novikov algebras.

It is known that every symmetric Novikov algebra up to dimension 5 is commutative [AB10] and a non-commutative example of dimension 6 is given by F. Zhu and Z. Chen in [ZC07]. This algebra is 2-step nilpotent. We will show in Proposition 4.4.28 that every non-commutative symmetric Novikov algebra of dimension 6 is 2-step nilpotent.

As for quadratic Lie algebras and pseudo-Euclidean Jordan algebras, we define the notion of a **reduced** symmetric Novikov algebra. Using this notion, we obtain the result (Proposition 4.4.29): if $\mathfrak{N}$ is a non-commutative symmetric Novikov algebra such that it is reduced then

$$3 \leq \dim(\mathrm{Ann}(\mathfrak{N})) \leq \dim(\mathfrak{N}^2) \leq \dim(\mathfrak{N}) - 3.$$

In other words, we do not have $\mathfrak{N}^2 = \mathfrak{N}$ in the non-commutative case . Note that this may happen in the commutative case (see Example 4.4.13). As a consequence, we have a characterization for the non-commutative case of dimension 7 (Proposition 4.4.31). Finally, we give an indecomposable non-commutative example for 3-step nilpotent symmetric Novikov algebras of dimension 7.

✠

The thesis has been divided into four parts. The classifications of O(m)-adjoint orbits of $\mathfrak{o}(m)$ and Sp$(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ are presented in Chapter 1. Singular quadratic Lie algebras and 2-step nilpotent quadratic Lie algebras are studied in Chapter 2. We will prove the equality $\{I,I\} = 0$ and introduce the class of singular quadratic Lie superalgebras in Chapter 3. However, since the classifying method is not new, we only focus on two cases: elementary and 2-dimensional even part. The classification of singular quadratic Lie algebras and singular quadratic Lie superalgebras having 2-dimensional even part can be regarded as an application of the problem of orbits classification in Chapter 1. We present quasi-singular quadratic Lie superalgebras without classification in Chapter 3. Such algebras can be found in [BBB] where the generalized double extension notion is reduced into the one-dimensional extension of an Abelian superalgebra. Pseudo-Euclidean Jordan algebras that are the one-dimensional double extension of an Abelian algebra and 2-step nilpotent pseudo-Euclidean Jordan algebras are given in Chapter 4 with a classifying characterization. The structure of symmetric Novikov algebras is studied in the last section of Chapter 4 with a more detail than in [AB10].

There are four appendices containing rather obvious and lengthy results but yet useful for our problems. Appendix A supplies a source about skew-symmetric maps for Chapter 1 and Chapter 2. Appendix B gives a non trivial proof of a fact that every non-Abelian 5-dimensional quadratic Lie algebra is singular. Another proof can be found in Appendix C where we classify (up to isomorphisms) 3-forms on a vector space V with $1 \leq \dim(V) \leq 5$. Appendix D is a small result used in Chapter 4 for pseudo-Euclidean Jordan algebras.

Notations

We use the notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the set of natural numbers, the set of integers, the set of real numbers and the set of complex numbers, respectively. The ring of residue classes modulo 2 of integers is denoted by $\mathbb{Z}_2$ which contains two elements $\overline{0}$ and $\overline{1}$. If $p \in \mathbb{Z}$, the notation $\overline{p}$ indicates its residue classes modulo 2. For a set of numbers $\mathbb{K}$, we denote by $\mathbb{K}^*$ the set of non-zero numbers in $\mathbb{K}$. Let Λ be a finite set then we use the notation $\sharp\Lambda$ for the number of elements of Λ.

If V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, the notation $\mathrm{End}(V)$ signs the set of endomorphisms of V. The space $\mathrm{End}(V)$ is also an algebra over $\mathbb{K}$ and it is denoted by $\mathscr{L}(V)$. We denote by V^* the dual vector space of V, that is the set of linear maps from V into $\mathbb{K}$. For each $f \in V^*$ and each $X \in V$, there is a natural bilinear form $\langle\ ,\ \rangle$ is defined by

$$\langle f, X\rangle = f(X).$$

Let W be a subset of V, we denote by $W^{\perp_*}$ an orthogonal complement of W in V^* by the bilinear form $\langle\ ,\ \rangle$. In addition, if V has a non-degenerate bilinear form $B : V \times V \to \mathbb{K}$ then $W^{\perp_B}$ (or $W^{\perp}$, for short) also denotes the set $\{X \in V \mid B(X,W) = 0\}$.

The Grassmann algebra of V, that is the algebra of alternating multilinear forms on V, with the wedge product is denoted by $\mathscr{A}(V)$. We have $\mathscr{A}(V) = \bigwedge(V^*)$, where $\bigwedge(V^*)$ denotes the exterior algebra of the dual space V^*. We also use the notation $\mathscr{S}(V)$ for the algebra of symmetric multilinear forms on V, i.e. $\mathscr{S}(V) = \mathsf{S}(V^*)$ where $\mathsf{S}(V^*)$ denotes the symmetric algebra of V^*. The algebras $\mathscr{A}(V)$ and $\mathscr{S}(V)$ are $\mathbb{Z}$-graded, we denote their homogeneous subspaces of degee n by $\mathscr{A}^n(V)$ and $\mathscr{S}^n(V)$, respectively. Thus one has

$$\mathscr{A}(V) = \bigoplus_{n\in\mathbb{Z}} \mathscr{A}^n(V) \text{ and } \mathscr{S}(V) = \bigoplus_{n\in\mathbb{Z}} \mathscr{S}^n(V)$$

where $\mathscr{A}^n(V) = \{0\}$ if $n \notin \{0, 1, \ldots, \dim(V)\}$ and $\mathscr{S}^n(V) = \{0\}$ if n is negative.

For $n \in \mathbb{N}^*$, the Lie algebra of complex square matrices of size n is denoted by $\mathfrak{gl}(n,\mathbb{C})$ or $\mathfrak{gl}(n)$ for short. The subalgebras $\mathfrak{sl}(n,\mathbb{C})$ of zero trace matrices and $\mathfrak{o}(n,\mathbb{C})$ of skew-symmetric matrices of $\mathfrak{gl}(n)$ are defined as follows:

$$\mathfrak{sl}(n,\mathbb{C}) = \{M \in \mathfrak{gl}(n) \mid \ tr(M) = 0\},$$

$$\mathfrak{o}(n,\mathbb{C}) = \{M \in \mathfrak{gl}(n) \mid {}^tM = -M\}$$

where ${}^tM$ denotes the transpose matrix of the matrix M. If $n = 2k$ then the subalgebra $\mathfrak{sp}(2k,\mathbb{C})$ of symplectic matrices of $\mathfrak{gl}(n)$ is defined by:

$$\mathfrak{sp}(2k,\mathbb{C}) = \{M \in \mathfrak{gl}(2k) \mid {}^tMJ + JM = 0\}$$

where $J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \in \mathfrak{gl}(2k)$, I_k denotes the unit matrix of size k. From here, we use the notations $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$ instead of $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{o}(n,\mathbb{C})$, $\mathfrak{sp}(2k,\mathbb{C})$, respectively. We also denote by $\mathbb{P}^1(\mathfrak{o}(n))$ and $\mathbb{P}^1(\mathfrak{sp}(2k))$ the projective spaces associated to $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$.

For complex numbers $\lambda_1,\ldots,\lambda_n$, the notation $\mathrm{diag}_n(\lambda_1,\ldots,\lambda_n)$ indicates the diagonal matrix whose diagonal entries are $\lambda_1,\ldots,\lambda_n$, respectively.

Finally, if $\mathfrak{g}$ is a finite dimensional algebra over $\mathbb{K}$ then we denote the algebra of derivations of $\mathfrak{g}$ by $\mathrm{Der}(\mathfrak{g})$. Recall that an endomorphism $D:\mathfrak{g}\to\mathfrak{g}$ is called a *derivation* of $\mathfrak{g}$ if it satisfies $D(XY) = D(X)Y + XD(Y)$ for all X,Y in $\mathfrak{g}$.

Chapter 1

Adjoint orbits of $\mathfrak{sp}(2n)$ and $\mathfrak{o}(m)$

In the first chapter, we will turn to a fundamental and really interesting problem in Lie theory: the classification of orbits of classical Lie algebras $\mathfrak{sl}(m)$, $\mathfrak{sp}(2n)$ and $\mathfrak{o}(m)$ where $m,n \in \mathbb{N}^*$. However, we only emphasize two special cases $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$ that are necessary for next chapters. The Jordan decomposition allows us to consider two kinds of orbits: nilpotent and semisimple (represented respectively by diagonal matrices and strictly upper triangular matrices). The classfication of nilpotent orbits that we present here follows the work of Gerstenhaber and an important point is the Jacobson-Morozov theorem. In the semisimple case, a well-known result is that the orbits are parametrized by a Cartan subalgebra under an action of the associated Weyl group. A brief overview can be found in [Hum95] with interesting discussions. Many results with detailed proofs can be found in [CM93] and [BBCM02].

A different point here is to use the Fitting decomposition to review this problem. In particular, we parametrize the *invertible* component in the Fitting decomposition of a skew-symmetric map and from this, we give an explicit classification for $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$ in the general case. In other words, we establish a one-to-one correspondence between the set of orbits and some set of indices. This is an rather obvious and classical result but in our knowledge there is not a reference for that mentioned before.

1.1 Definitions

Let V be a m-dimensional complex vector space endowed with a non-degenerate bilinear form B_ε where $\varepsilon = \pm 1$ such that $B_\varepsilon(X,Y) = \varepsilon B_\varepsilon(Y,X)$, for all $X,Y \in V$. If $\varepsilon = 1$ then the form B_1 is symmetric and we say V a *quadratic vector space*. If $\varepsilon = -1$ then m must be even and we say V a *symplectic vector space* with symplectic form B_{-1}. We denote by $\mathscr{L}(V)$ the *algebra of linear operators* of V and $\mathrm{GL}(V)$ the *group of invertible operators* in $\mathscr{L}(V)$. A map $C \in \mathscr{L}(V)$ is called *skew-symmetric* (with respect to B_ε) if it satisfies the following condition:

$$B_\varepsilon(C(X),Y) = -B_\varepsilon(X,C(Y)),\ \forall\, X,Y \in V.$$

We define

$$I_\varepsilon(V) = \{A \in \mathrm{GL}(V) \mid\ B_\varepsilon(A(X),A(Y)) = B_\varepsilon(X,Y),\ \forall\, X,Y \in V\}$$

$$\text{and } \mathfrak{g}_\varepsilon(V) = \{C \in \mathscr{L}(V) \mid C \text{ is skew-symmetric}\}.$$

Then $I_\varepsilon(V)$ is the *isometry group* of the bilinear form B_ε and $\mathfrak{g}_\varepsilon(V)$ is its Lie algebra. Denote by $A^* \in \mathscr{L}(V)$ the *adjoint map* of an element $A \in \mathscr{L}(V)$ with respect to B_ε, then $A \in I_\varepsilon(V)$ if and only if $A^{-1} = A^*$ and $C \in \mathfrak{g}_\varepsilon(V)$ if and only if $C^* = -C$. If $\varepsilon = 1$ then $I_\varepsilon(V)$ is denoted by $\mathrm{O}(V)$ and $\mathfrak{g}_\varepsilon(V)$ is denoted by $\mathfrak{o}(V)$. If $\varepsilon = -1$ then $\mathrm{Sp}(V)$ stands for $I_\varepsilon(V)$ and $\mathfrak{sp}(V)$ stands for $\mathfrak{g}_\varepsilon(V)$.

Recall that the *adjoint action* Ad of $I_\varepsilon(V)$ on $\mathfrak{g}_\varepsilon(V)$ is given by

$$\mathrm{Ad}_U(C) = UCU^{-1}, \ \forall \ U \in I_\varepsilon(V), \ C \in \mathfrak{g}_\varepsilon(V).$$

We denote by $\mathscr{O}_C = \mathrm{Ad}_{I_\varepsilon(V)}(C)$, the *adjoint orbit* of an element $C \in \mathfrak{g}_\varepsilon(V)$ by this action.

If $V = \mathbb{C}^n$, we call B_ε a *canonical bilinear form* of $\mathbb{C}^n$. And with respect to B_ε, we define a *canonical basis* $\mathscr{B} = \{E_1, \dots, E_m\}$ of $\mathbb{C}^m$ as follows. If m even, $m = 2n$, write $\mathscr{B} = \{E_1, \dots, E_n, F_1, \dots, F_n\}$, if m is odd, $m = 2n+1$, write $\mathscr{B} = \{E_1, \dots, E_n, G, F_1, \dots, F_n\}$ and one has:

- if $m = 2n$ then

$$B_1(E_i, F_j) = B_1(F_j, E_i) = \delta_{ij}, \ B_1(E_i, E_j) = B_1(F_i, F_j) = 0,$$

$$B_{-1}(E_i, F_j) = -B_{-1}(F_j, E_i) = \delta_{ij}, \ B_{-1}(E_i, E_j) = B_{-1}(F_i, F_j) = 0,$$

  where $1 \le i, j \le n$.

- if $m = 2n+1$ then $\varepsilon = 1$ and

$$\begin{cases} B_1(E_i, F_j) = \delta_{ij}, \ B_1(E_i, E_j) = B_1(F_i, F_j) = 0, \\ B_1(E_i, G) = B_1(F_j, G) = 0, \\ B_1(G, G) = 1 \end{cases}$$

  where $1 \le i, j \le n$.

Also, in the case $V = \mathbb{C}^m$, we denote by $\mathrm{GL}(m)$ instead of $\mathrm{GL}(V)$, $\mathrm{O}(m)$ stands for $\mathrm{O}(V)$ and $\mathfrak{o}(m)$ stands for $\mathfrak{o}(V)$. If $m = 2n$ then $\mathrm{Sp}(2n)$ stands for $\mathrm{Sp}(V)$ and $\mathfrak{sp}(2n)$ stands for $\mathfrak{sp}(V)$. We will also write $I_\varepsilon = I_\varepsilon(\mathbb{C}^m)$ and $\mathfrak{g}_\varepsilon = \mathfrak{g}_\varepsilon(\mathbb{C}^m)$. The goal of this chapter is classifying all of I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$.

Finally, let V is an m-dimensional vector space. If V is quadratic then V is isometrically isomorphic to the quadratic space $(\mathbb{C}^m, B_1)$ and if V is symplectic then V is isometrically isomorphic to the symplectic space $(\mathbb{C}^m, B_{-1})$ [Bou59].

1.2 Nilpotent orbits

Let $n \in \mathbb{N}^*$, a *partition* $[d]$ of n is a tuple $[d_1, ..., d_k]$ of positive integers satisfying

$$d_1 \geq ... \geq d_k \text{ and } d_1 + ... + d_k = n.$$

Occasionally, we use the notation $[t_1^{i_1}, \ldots, t_r^{i_r}]$ to replace the partition $[d_1, ..., d_k]$ where

$$d_j = \begin{cases} t_1 & 1 \leq j \leq i_1 \\ t_2 & i_1 + 1 \leq j \leq i_1 + i_2 \\ t_3 & i_1 + i_2 + 1 \leq j \leq i_1 + i_2 + i_3 \\ \ldots \end{cases}$$

Each i_j is called the *multiplicity* of t_j. Denote by $\mathscr{P}(n)$ the set of partitions of n. For example, $\mathscr{P}(3) = \{[3],\ [2,1],\ [1^3]\}$ and $\mathscr{P}(4) = \{[4],\ [3,1],\ [2^2],\ [2,1^2],\ [1^4]\}$.

Let $p \in \mathbb{N}^*$. We denote the *Jordan block of size* p by $J_1 = (0)$ and for $p \geq 2$,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then J_p is a nilpotent endomorphism of $\mathbb{C}^p$. Given a partition $[d] = [d_1, ..., d_k] \in \mathscr{P}(n)$ there is a nilpotent endomorphism of $\mathbb{C}^n$ defined by

$$X_{[d]} := \text{diag}_k(J_{d_1}, ..., J_{d_k}).$$

Moreover, $X_{[d]}$ is also a nilpotent element of $\mathfrak{sl}(n)$ since its trace is zero. Conversely, if C is a nilpotent element in $\mathfrak{sl}(n)$ then C is $\mathrm{GL}(n)$-conjugate to its *Jordan normal form* $X_{[d]}$ for some partition $[d] \in \mathscr{P}(n)$.

Given two different partitions $[d] = [d_1, ..., d_k]$ and $[d'] = [d'_1, ..., d'_l]$ of n then the $\mathrm{GL}(n)$-adjoint orbits through $X_{[d]}$ and $X_{[d']}$ respectively are disjoint by the unicity of Jordan normal form. Therefore, one has the following proposition:

Proposition 1.2.1. *There is a one-to-one correspondence between the set of nilpotent* $\mathrm{GL}(n)$*-adjoint orbits of* $\mathfrak{sl}(n)$ *and the set* $\mathscr{P}(n)$.

It results that $\mathfrak{sl}(n)$ has only finitely many nilpotent $\mathrm{GL}(n)$-adjoint orbits, exactly $\sharp\mathscr{P}(n)$. However, this does not assure the same for its semisimple subalgebras and the classification of nilpotent adjoint orbits of $\mathfrak{g}_\varepsilon$ is rather more difficult since the action of the subgroup I_ε does not coincide with the action of $\mathrm{GL}(n)$. However, by many works of Dynkin, Kostant and Mal'cev (see [CM93]), there is an important bijection between nilpotent adjoint orbits of a semisimple Lie algebra $\mathfrak{g}$ and a subset of $3^{rank(\mathfrak{g})}$ possible weight Dynkin diagrams where $rank(\mathfrak{g})$ is the dimension of a Cartan subalgebra of $\mathfrak{g}$, and thus $\mathfrak{g}_\varepsilon$ has only finitely many nilpotent adjoint orbits.

The main tool in the classical work on nilpotent adjoint orbits is the representation theory of the Lie algebra $\mathfrak{sl}(2)$ (or $\mathfrak{sl}(2)$-*theory*, for short) applied to the adjoint action on a semisimple

Lie algebra $\mathfrak{g}$. We start with a review of the basic results give in [Hum72]. Recall that $\mathfrak{sl}(2)$ is spanned by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and these satisfy the following relations:

$$[H,X] = 2X, \ \ [H,Y] = -2Y \text{ and } [X,Y] = H.$$

Let V be a finite-dimensional $\mathfrak{sl}(2)$-module. Since H is semisimple then H acts diagonally on V. Therefore, we can decompose V as a direct sum of eigenspaces $V_\lambda = \{v \in V \mid H.v = \lambda v\}$, $\lambda \in \mathbb{C}$ where the notation $H.v$ denotes H acting on v by the representation.

Definition 1.2.2. If $V_\lambda \neq \{0\}$ then we call λ a *weight* of H in V and we call V_λ a *weight space*.

Lemma 1.2.3. *If $v \in V_\lambda$ then $X.v \in V_{\lambda+2}$ and $Y.v \in V_{\lambda-2}$.*

Proof. Since $[H,X] = 2X$ one has $H.(X.v) = X.(H.v) + 2X.v = (\lambda+2)X.v$. So $X.v \in V_{\lambda+2}$. And this is done similarly for Y. □

Since $V = \bigoplus_\lambda V_\lambda$ and $\dim(V)$ is finite then there must exist $V_\lambda \neq \{0\}$ such that $V_{\lambda+2} = \{0\}$. In this case, each non-zero $x \in V_\lambda$ is called a *maximal vector* of weight λ (note that $X.v = 0$ if v is a maximal vector).

Now, we assume that V is an irreducible $\mathfrak{sl}(2)$-module. Choose a maximal vector, say $v_0 \in V_\lambda$. Set $v_{-1} = 0$, $v_i = \frac{1}{i!}Y^i.v_0$ $(i \geq 0)$. Then one has the following lemma.

Lemma 1.2.4.

(1) $H.v_i = (\lambda - 2i)v_i$,

(2) $Y.v_i = (i+1)v_{i+1}$,

(3) $X.v_i = (\lambda - i + 1)v_{i-1}$, $(i \geq 0)$.

Proof.

(1) One has $H.v_i = \frac{1}{i!}HY^i.v_0 = \frac{1}{i!}(YH - 2Y)Y^{i-1}.v_0$. Since $H.v_0. = \lambda v_0$, $Y.v_{i-1} = iv_i$ and by induction on i, we get $H.v_i = (\lambda - 2i)v_i$.

(2) It follows from the definition of v_i.

(3) We prove (3) by induction on i. If $i = 0$, it is clear since $v_{i-1} = 0$ and $X.v_0 = 0$. If $i > 0$, one has

$$iX.v_i = XY.v_{i-1} = [X,Y].v_{i-1} + YX.v_{i-1} = H.v_{i-1} + YX.v_{i-1}.$$

By (1), (2) and induction, we obtain

$$iX.v_i = (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)Y.v_{i-2} = i(\lambda - i + 1)v_{i-1}.$$

□

By induction, it is easy to show that the non-zero v_i are all linearly independent. Since $\dim(V)$ is finite then there must exist the smallest m such that $v_m \neq 0$ and $v_{m+1} = 0$, obviously $v_{m+i} = 0$ for all $i > 0$. Therefore, the subspace of V spanned by vectors $v_0, \dots, v_m$ is a $\mathfrak{sl}(2)$-module. Since V is irreducible then $V = \mathrm{span}\{v_0, \dots, v_m\}$. Moreover, the formula (3) shows $\lambda = m$ by checking with $i = m+1$. It means that the weight λ of a maximal vector is a nonnegative integer (equal to $\dim(V) - 1$) and we call it the *highest weight* of V. Conversely, for arbitrary $m \geq 0$, formulas (1)- (3) of Lemma 1.2.4 can be used to define a representation of $\mathfrak{sl}(2)$ on an $m+1$-dimensional vector space with a basis $\{v_0, \dots, v_m\}$. Moreover, it is easy to check that this representation is irreducible and then we have the following corollary:

Corollary 1.2.5.

(1) Let V be an irreducible $\mathfrak{sl}(2)$-module then V is the direct sum of its weight spaces V_μ, $\mu = m, m-2, \dots, -(m-2), -m$ where $m = \dim(V) - 1$ and $\dim(V_\mu) = 1$ for each μ.

(2) For each integer $m \geq 0$, there is (up to isomorphisms) one irreducible $\mathfrak{sl}(2)$-module of dimension $m+1$.

Next, let $\mathfrak{g}$ be a complex semisimple Lie algebra. If there is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$ and spanned by $\{H, X, Y\}$ then we called $\{H, X, Y\}$ a $\mathfrak{sl}(2)$-*triple* of $\mathfrak{g}$. In this case, the triple $\{H, X, Y\}$ satisfies the bracket relations:

$$[H,X] = 2X, \ \ [H,Y] = -2Y \text{ and } [X,Y] = H.$$

We call H (resp. X, Y) the *neutral* (resp. *nilpositive, nilnegative*) element of the triple $\{H, X, Y\}$. Since $\mathrm{ad}(H)$ is semisimple in the subalgebra $\mathfrak{a} = \mathrm{span}\{H, X, Y\}$ of $\mathfrak{g}$ then it is known that H is also semisimple in $\mathfrak{g}$. Similarly, X, Y are nilpotent in $\mathfrak{g}$.

Fix an integer $r \geq 0$ and define a linear map $\rho_r : \mathfrak{sl}(2) \to \mathfrak{sl}(r+1)$ by

$$\rho_r(H) = \begin{pmatrix} r & 0 & 0 & \dots & 0 & 0 \\ 0 & r-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -r+2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -r \end{pmatrix},$$

$$\rho_r(X) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$\rho_r(Y) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \mu_r & 0 \end{pmatrix},$$

where $\mu_i = i(r+1-i)$ for $1 \le i \le r$.

By Corollary 1.2.5, ρ_r defines an irreducible representation of $\mathfrak{sl}(2)$ of highest weight r and dimension $r+1$. Moreover, every irreducible finite-dimensional representation of $\mathfrak{sl}(2)$ arises in this way.

Conveniently, for a partition $[d] = [d_1, ..., d_k]$ of n, denote by $\mathscr{O}_{[d]}$ the orbit through $X_{[d]}$. Now, let $\mathscr{O}$ be a non-zero nilpotent orbit in $\mathfrak{sl}(n)$ then there exists a partition $[d_1, ..., d_k]$ of n such that $\mathscr{O} = \mathscr{O}_{[d_1,...,d_k]}$. We define the homomorphism $\phi_{\mathscr{O}} : \mathfrak{sl}(2) \to \mathfrak{sl}(n)$ by

$$\phi_{\mathscr{O}} = \bigoplus_{1\le i\le k} \rho_{d_i-1}.$$

Then $\phi_{\mathscr{O}}(X) = X_{[d_1,...,d_k]}$ where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2)$. Note that if $\mathscr{O}$ is a non-zero nilpotent orbit then the partition $[d_1, ..., d_k]$ does not coincide with $[1^n]$, the image $\phi_{\mathscr{O}}$ is not trivial and it is isomorphic to $\mathfrak{sl}(2)$. Therefore, to each non-zero nilpotent orbit $\mathscr{O} = \mathscr{O}_X$ in $\mathfrak{sl}(n)$, we can attach a $\mathfrak{sl}(2)$-triple $\{H, X, Y\}$ such that the nilpositive element is X. More precisely, choose X exactly having the Jordan form and set

$$H := \phi_{\mathscr{O}}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),\ X := \phi_{\mathscr{O}}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \text{ and } Y := \phi_{\mathscr{O}}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right).$$

This can be done for an arbitrary complex semisimple algebra $\mathfrak{g}$, not necessarily $\mathfrak{sl}(n)$, by the theorem of Jacobson-Morozov as follows.

Proposition 1.2.6 (Jacobson-Morozov)**.**
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If X is a non-zero nilpotent element of $\mathfrak{g}$ then it is the nilpositive element of a $\mathfrak{sl}(2)$-triple. Equivalently, for any nilpotent element X, there exists a homomorphism $\phi : \mathfrak{sl}(2) \to \mathfrak{g}$ such that: $\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X$.

Proof. We follow the proof given in [CM93]. First, we prove the following lemma.

Lemma 1.2.7. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and X be a nilpotent element in $\mathfrak{g}$. Then one has:*

(1) $\kappa(X, \mathfrak{g}^X) = 0$ where κ is the Killing form and $\mathfrak{g}^X$ is the centralizer of X in $\mathfrak{g}$ defined by $\mathfrak{g}^X = \{Y \in \mathfrak{g} \mid [X,Y] = 0\}$.

(2) $[\mathfrak{g}, X] = (\mathfrak{g}^X)^\perp$ where the notation $(\mathfrak{g}^X)^\perp$ denotes the orthogonal subspace of $\mathfrak{g}^X$ by κ.

Proof.

(1) Let Z be an element of $\mathfrak{g}^X$ then by the Jacobi identity, one has $\mathrm{ad}(X) \circ \mathrm{ad}(Z) = \mathrm{ad}(Z) \circ \mathrm{ad}(X)$. As a consequence, $(\mathrm{ad}(X) \circ \mathrm{ad}(Z))^k = \mathrm{ad}^k(X) \circ \mathrm{ad}^k(Z)$ for any k in $\mathbb{N}$. Since X is nilpotent then $\mathrm{ad}(X)$ is nilpotent. It implies that $\mathrm{ad}^k(X) = 0$ for some k and therefore $\mathrm{ad}(X) \circ \mathrm{ad}(Z)$ is nilpotent. That means $trace(\mathrm{ad}(X) \circ \mathrm{ad}(Z)) = \kappa(X, Z) = 0$.

(2) By the invariance of κ, one has $\kappa([\mathfrak{g}, X], \mathfrak{g}^X) = \kappa(\mathfrak{g}, [X, \mathfrak{g}^X]) = 0$. Hence, $[\mathfrak{g}, X] \subset (\mathfrak{g}^X)^\perp$. Since $\dim(\mathfrak{g}) = \dim(\ker(\mathrm{ad}(X))) + \dim(\mathrm{Im}(\mathrm{ad}(X)))$ then $\dim(\mathfrak{g}) = \dim(\mathfrak{g}^X) + \dim([\mathfrak{g}, X])$. By the non-degeneracy of κ, we obtain $[\mathfrak{g}, X] = (\mathfrak{g}^X)^\perp$.

□

We prove the Jacobson-Morozov theorem by induction on the dimension of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 3$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2)$ and the result follows. Assume that $\dim(\mathfrak{g}) > 3$. If X is in a proper semisimple subalgebra of $\mathfrak{g}$ then by induction, there is a $\mathfrak{sl}(2)$-triple with X its nilpositive element. Hence we may assume that X is not in any proper semisimple subalgebra of $\mathfrak{g}$. By Lemma 1.2.7 and $[X,X] = 0$, one has $X \in (\mathfrak{g}^X)^{\perp} = [\mathfrak{g},X]$. Therefore, there exists $H' \in \mathfrak{g}$ such that $[H',X] = 2X$.

Now, let $H' = H'_s + H'_n$ be the Jordan decomposition of H' in $\mathfrak{g}$ where H'_s is semisimple and H'_n is nilpotent. Remark that any subspace which is stable by $\mathrm{ad}(H')$ is also stable by $\mathrm{ad}(H'_s)$ and $\mathrm{ad}(H'_n)$. The nilpotency of the $\mathrm{ad}(H'_n)$ action on the stable subspace $\mathbb{C}X$ gives $[H'_n,X] = 0$ and therefore $[H'_s,X] = 2X$. Set $H = H'_s$.

If $H \in [\mathfrak{g},X]$ then there exists $Y \in \mathfrak{g}$ such that $H = [X,Y]$. Since $\mathrm{ad}(H)$ acts semisimply on $\mathfrak{g}$ then $\mathfrak{g}$ is decomposed by

$$\mathfrak{g} = \mathfrak{g}_{\lambda_1} \oplus \cdots \oplus \mathfrak{g}_{\lambda_k},$$

where $\mathfrak{g}_{\lambda_1}, \ldots, \mathfrak{g}_{\lambda_k}$ are $\mathrm{ad}(H)$-eigenspaces. Write $Y = Y_{\lambda_1} + \cdots + Y_{\lambda_k}$ with $Y_i \in \mathfrak{g}_{\lambda_i}$. Let $Z \in \mathfrak{g}_{\lambda_i}$ then $[H,Z] = \lambda_i Z$. Hence, $[X,[H,Z]] = \lambda_i [X,Z]$. By the Jacobi identity, one has

$$[Z,[H,X]] + [H,[X,Z]] = \lambda_i [X,Z].$$

Then we get $\mathrm{ad}(H)([X,Z]) = (\lambda_i + 2)[X,Z]$. It shows that $[X,\mathfrak{g}_{\lambda_i}] \subset \mathfrak{g}_{\lambda_i+2}, 1 \leq i \leq k$. Since $\mathrm{ad}(H)(H) = 0$, one has $H \in \mathfrak{g}_0$. Moreover, $H = [X,Y] = \sum_k^{i=1}[X,Y_i]$. Therefore, there is some $Y' \in \mathfrak{g}_{-2}$ such that $H = [X,Y']$. If we replace Y by Y' then $[H,X] = 2X$, $[H,Y] = -2Y$ and $[X,Y] = H$. That means $\{H,X,Y\}$ is a $\mathfrak{sl}(2)$-triple with X its nilpositive element.

From the above reason, it remains to prove $H \in [\mathfrak{g},X]$. By contradiction, assume that $H \notin [\mathfrak{g},X]$ then $\kappa(H,\mathfrak{g}^X) \neq 0$. According to the Jacobi identity

$$[X,[H,\mathfrak{g}^X]] + [\mathfrak{g}^X,[X,H]] + [H,[\mathfrak{g}^X,X]] = 0,$$

one get $[X,[H,\mathfrak{g}^X]] = 0$. It implies that $\mathrm{ad}(H)(\mathfrak{g}^X) \subset \mathfrak{g}^X$, i.e. $\mathfrak{g}^X$ is $\mathrm{ad}(H)$-invariant. By acting semisimply of $\mathrm{ad}(H)$ on $\mathfrak{g}^X$, $\mathfrak{g}^X$ is decomposed into $\mathrm{ad}(H)$-eigenspaces:

$$\mathfrak{g}^X = \mathfrak{g}^X_{\tau_1} \oplus \cdots \oplus \mathfrak{g}^X_{\tau_i} = \mathfrak{g}^X_0 \oplus \sum_{\tau_i \neq 0} \mathfrak{g}^X_{\tau_i}.$$

From the invariance of the Killing form, one has $\kappa(H,[H,\mathfrak{g}^X]) = \kappa([H,H],\mathfrak{g}^X) = 0$. Therefore if Z is a non-zero element of $\mathfrak{g}^X_{\tau_i}$ with $\tau_i \neq 0$ then

$$0 = \kappa(H,[H,Z]) = \kappa(H,\tau_i Z) = \tau_i \kappa(H,Z).$$

This shows that $H \in (\mathfrak{g}^X_{\tau_i})^{\perp}$. Since $\kappa(H,\mathfrak{g}^X) \neq 0$ there must exist $Z \in \mathfrak{g}^X_0 = \{Y \in \mathfrak{g}^X \mid [H,Y] = 0\} = (\mathfrak{g}^X)^H$ such that $\kappa(H,Z) \neq 0$. If Z is nilpotent then $\kappa(H,Z) = 0$ as in the proof of Lemma 1.2.7. Therefore, Z is non-nilpotent. That means its semisimple component $Z_s \neq 0$ and hence $\mathfrak{g}^{Z_s}$ is reductive. As a consequence, $[\mathfrak{g}^{Z_s},\mathfrak{g}^{Z_s}]$ is a semisimple subalgebra of $\mathfrak{g}$ and it is a proper subalgebra since $\mathfrak{g}^{Z_s} = \mathfrak{g}$ only if $Z_s = 0$.

On the other hand, since $[X,Z] = [H,Z] = 0$, apply the property of Jordan decomposition one has $[X,Z_s] = [H,Z_s] = 0$. That means $Z_s \in (\mathfrak{g}^X)^H$ and then $X \in \mathfrak{g}^{Z_s}$. Also, $H \in \mathfrak{g}^{Z_s}$ so we get $2X = [H,X] \in [\mathfrak{g}^{Z_s},\mathfrak{g}^{Z_s}]$. This is a contradiction then we obtain the result. □

The point of the above proof is that the Killing form of $\mathfrak{g}$ is non-degenerate and invariant. However, the existence of a $\mathfrak{sl}(2)$-triple $\{H,X,Y\}$ having the nilpositive element X is not unique, that is, it may exist another $\mathfrak{sl}(2)$-triple $\{H',X,Y'\}$ with the same nilpositive element. The following theorem shows that our choice is is unique up to an element in G_{ad}, the identity component of the automorphism group $\mathrm{Aut}(\mathfrak{g}) = \{\phi \in \mathrm{GL}(\mathfrak{g}) \mid [\phi(X),\phi(Y)] = \phi([X,Y])\}$.

Proposition 1.2.8 (Kostant)**.**
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two $\mathfrak{sl}(2)$-triples $\{H,X,Y\}$ and $\{H',X,Y'\}$ with the same nilpositive element are conjugate by an element of G_{ad}.

Denote by A_{triple} the set of $\mathfrak{sl}(2)$-triples of $\mathfrak{g}$, $\widetilde{A}_{triple}$ the set of G_{ad}-conjugacy classes of $\mathfrak{sl}(2)$-triples in A_{triple} and $\mathscr{N}(\mathfrak{g})$ the set of non-zero nilpotent orbits in $\mathfrak{g}$ then we obtain the corollary:

Corollary 1.2.9. *The map $\omega : A_{triple} \to \mathscr{N}(\mathfrak{g})$ defined by $\omega(\{H,X,Y\}) = \mathscr{O}_X$ induces a bijection $\Omega : \widetilde{A}_{triple} \to \mathscr{N}(\mathfrak{g})$.*

Proof. Let $\mathscr{O}$ be a non-zero nilpotent orbit in $\mathfrak{g}$. Fix $X \neq 0$ in $\mathscr{O}$. By Proposition 1.2.6, there exists a $\mathfrak{sl}(2)$-triple $\{H,X,Y\}$ such that X is its nilpositive element so Ω is onto. If there exists another $\mathfrak{sl}(2)$-triple $\{H',X,Y'\}$ such that X is also nilpositive then by Proposition 1.2.8 $\{H',X,Y'\}$ must lie in G_{ad}-conjugacy class of $\{H,X,Y\}$. Therefore, Ω is one-to-one. □

Now we turn to our problem of classification nilpotent I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$. Define the set

$$\mathscr{P}_\varepsilon(m) = \{[d_1,...,d_m] \in \mathscr{P}(m) | \sharp\{j \mid d_j = i\} \text{ is even for all i such that } (-1)^i = \varepsilon\}.$$

In particular, $\mathscr{P}_1(m)$ is the set of partitions of m in which even parts occur with even multiplicity and $\mathscr{P}_{-1}(m)$ is the set of partitions of m in which odd parts occur with even multiplicity.

Proposition 1.2.10 (Gerstenhaber)**.**
Nilpotent I_ε-adjoint orbits in $\mathfrak{g}_\varepsilon$ are in one-to-one correspondence with the set of partitions in $\mathscr{P}_\varepsilon(m)$.

Proof. A proof of the proposition can be found in [CM93], Theorem 5.1.6. □

Here, we give the construction of a nilpotent element in $\mathfrak{g}_\varepsilon$ from a partition $[d]$ of m that is useful for next two chapters. Define maps in $\mathfrak{g}_\varepsilon$ as follows:

- For $p \geq 2$, we equip the vector space $\mathbb{C}^{2p}$ with its canonical bilinear form B_ε and the map C^J_{2p} having the matrix

$$C^J_{2p} = \begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$$

  in a canonical basis where ${}^tJ_p$ denotes the *transpose* matrix of the Jordan block J_p. Then $C^J_{2p} \in \mathfrak{g}_\varepsilon(\mathbb{C}^{2p})$.

- For $p \geq 1$ we equip the vector space $\mathbb{C}^{2p+1}$ with its canonical bilinear form B_1 and the map C^J_{2p+1} having the matrix

$$C^J_{2p+1} = \begin{pmatrix} J_{p+1} & M \\ 0 & -{}^tJ_p \end{pmatrix}$$

  in a canonical basis where $M = (m_{ij})$ denotes the $(p+1) \times p$-matrix with $m_{p+1,p} = -1$ and $m_{ij} = 0$ otherwise. Then $C^J_{2p+1} \in \mathfrak{o}(2p+1)$

- For $p \geq 1$, we consider the vector space $\mathbb{C}^{2p}$ equipped with its canonical bilinear form B_{-1} and the map C^J_{p+p} with matrix

$$\begin{pmatrix} J_p & M \\ 0 & -{}^tJ_p \end{pmatrix}$$

  in a canonical basis where $M = (m_{ij})$ denotes the $p \times p$-matrix with $m_{p,p} = 1$ and $m_{ij} = 0$ otherwise. Then $C^J_{p+p} \in \mathfrak{sp}(2p)$.

For each partition $[d] \in \mathscr{P}_{-1}(2n)$, $[d]$ can be written as

$$(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1, \dots, 2q_\ell)$$

with all p_i odd, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$. We associate a map $C_{[d]}$ with the matrix:

$$\mathrm{diag}_{k+\ell}(C^J_{2p_1}, C^J_{2p_2}, \dots, C^J_{2p_k}, C^J_{q_1+q_1}, \dots, C^J_{q_\ell+q_\ell})$$

in a canonical basis of $\mathbb{C}^{2n}$ then $C_{[d]} \in \mathfrak{sp}(2n)$.

Similarly, let $[d] \in \mathscr{P}_1(m)$, $[d]$ can be written as

$$(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1+1, \dots, 2q_\ell+1)$$

with all p_i even, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$. We associate a map $C_{[d]}$ with the matrix:

$$\mathrm{diag}_{k+\ell}(C^J_{2p_1}, C^J_{2p_2}, \dots, C^J_{2p_k}, C^J_{2q_1+1}, \dots, C^J_{2q_\ell+1}).$$

in a canonical basis of $\mathbb{C}^m$ then $C_{[d]} \in \mathfrak{o}(m)$.

By Proposition 1.2.10, it is sure that our construction is a bijection between the set $\mathscr{P}_\varepsilon(m)$ and the set of nilpotent I_ε-adjoint orbits in $\mathfrak{g}_\varepsilon$.

1.3 Semisimple orbits

We recall the well-known result [CM93]:

Proposition 1.3.1. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra,* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *and* W *be the associated Weyl group. Then there is a bijection between the set of semisimple orbits of* $\mathfrak{g}$ *and* $\mathfrak{h}/W$.

For each $\mathfrak{g}_\varepsilon$, we choose the Cartan subalgebra $\mathfrak{h}$ given by the vector space of diagonal matrices of type

$$\mathrm{diag}_{2n}(\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n)$$

if $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n)$ or $\mathfrak{g}_\varepsilon = \mathfrak{sp}(2n)$ and of type

$$\mathrm{diag}_{2n+1}(\lambda_1,\ldots,\lambda_n,0,-\lambda_1,\ldots,-\lambda_n)$$

if $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n+1)$.

Any diagonalizable (equivalently semisimple) $C \in \mathfrak{g}_\varepsilon$ is conjugate to an element of $\mathfrak{h}$ (see Appendix A for a direct proof).

If $\mathfrak{g}_\varepsilon = \mathfrak{sp}(2n)$ then any two eigenvectors $v,w \in \mathbb{C}^{2n}$ of $X \in \mathfrak{g}_\varepsilon$ with eigenvalues $\lambda,\lambda' \in \mathbb{C}$ such that $\lambda+\lambda' \neq 0$ are orthorgonal. Moreover, each eigenvalue pair $\lambda,-\lambda$ is corresponding to an eigenvector pair (v,w) satisfying $B_\varepsilon(v,w)=1$ and we can easily arrange for vectors v,v' lying in a distinct pair $(v,w),(v',w')$ to be orthogonal, regardless of the eigenvalues involved. That means the associated Weyl group is of all coordinate permutations and sign changes of $(\lambda_1,\ldots,\lambda_n)$. We denote it by G_n.

If $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n)$, the associated Weyl group, when considered in the action of the group $\mathrm{SO}(2n)$, consists all coordinate permutations and even sign changes of $(\lambda_1,\ldots,\lambda_n)$. However, we only focus on $\mathrm{O}(2n)$-adjoint orbits of $\mathfrak{o}(2n)$ obtained by the action of the full orthogonal group, then similarly to preceding analysis any sign change effects. The corresponding group is still G_n. If $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n+1)$, the Weyl group is G_n and there is nothing to add.

Now, let $\Lambda_n = \{(\lambda_1,\ldots,\lambda_n) \mid \lambda_1,\ldots,\lambda_n \in \mathbb{C},\ \lambda_i \neq 0 \text{ for some } i\}$.

Corollary 1.3.2. *There is a bijection between non-zero semisimple* I_ε*-adjoint orbits of* $\mathfrak{g}_\varepsilon$ *and* Λ_n/G_n.

1.4 Invertible orbits

Definition 1.4.1. We say that the I_ε-orbit $\mathscr{O}_X$ is *invertible* if X is an invertible element in $\mathfrak{g}_\varepsilon$.

Keep the above notations. We say an element $X \in V$ *isotropic* if $B(X,X) = 0$ and a subset $W \subset V$ *totally isotropic* if $B(X,Y) = 0$ for all $X,Y \in W$. To classify invertible adjoint orbits in $\mathfrak{g}_\varepsilon$, we need the following lemma:

Lemma 1.4.2. *Let V be an even-dimensional vector space with a non-degenerate bilinear form B_ε. Assume that $V = V_+ \oplus V_-$ where $V_\pm$ are totally isotropic vector subspaces.*

(1) Let $N \in \mathscr{L}(V)$ such that $N(V_\pm) \subset V_\pm$. We define maps $N_\pm$ by $N_+|_{V_+} = N|_{V_+}$, $N_+|_{V_-} = 0$, $N_-|_{V_-} = N|_{V_-}$ and $N_-|_{V_+} = 0$. Then $N \in \mathfrak{g}_\varepsilon(V)$ if and only if $N_- = -N_+^*$ and, in this case, $N = N_+ - N_+^*$.

(2) Let $U_+ \in \mathscr{L}(V)$ such that U_+ is invertible, $U_+(V_+) = V_+$ and $U_+|_{V_-} = \mathrm{Id}_{V_-}$. We define $U \in \mathscr{L}(V)$ by $U|_{V_+} = U_+|_{V_+}$ and $U|_{V_-} = \left(U_+^{-1}\right)^*|_{V_-}$. Then $U \in I_\varepsilon(V)$.

(3) Let $N' \in \mathfrak{g}_\varepsilon(V)$ such that N' satisfies the assumptions of (1). Define $N_\pm$ as in (1). Moreover, we assume that there exists $U_+ \in \mathscr{L}(V_+)$, U_+ invertible such that

$$N'_+|_{V_+} = \left(U_+ N_+ U_+^{-1}\right)|_{V_+}.$$

We extend U_+ to V by $U_+|_{V_-} = \mathrm{Id}_{V_-}$ and define $U \in I_\varepsilon(V)$ as in (2). Then

$$N' = U\, N\, U^{-1}.$$

Proof.

(1) It is obvious that $N = N_+ + N_-$. Recall that $N \in \mathfrak{g}_\varepsilon(V)$ if and only if $N^* = -N$ so $N_+^* + N_-^* = -N_+ - N_-$. Since $B_\varepsilon(N_+^*(V_+), V) = B_\varepsilon(V_+, N_+(V)) = 0$ then $N_+^*(V_+) = 0$. Similarly, $N_-^*(V_-) = 0$. Hence, $N_- = -N_+^*$.

(2) We shows that $B_\varepsilon(U(X), U(Y)) = B_\varepsilon(X,Y)$, for all $X,Y \in V$. Indeed, let $X = X_+ + X_-, Y = Y_+ + Y_- \in V_+ \oplus V_-$, one has

$$B_\varepsilon(U(X_+ + X_-), U(Y_+ + Y_-)) = B_\varepsilon(U_+(X_+) + \left(U_+^{-1}\right)^*(X_-), U_+(Y_+) + \left(U_+^{-1}\right)^*(Y_-))$$

$$= B_\varepsilon(U_+(X_+), \left(U_+^{-1}\right)^*(Y_-)) + B_\varepsilon(\left(U_+^{-1}\right)^*(X_-), U_+(Y_+))$$
$$= B_\varepsilon(X_+, Y_-) + B_\varepsilon(X_-, Y_+) = B_\varepsilon(X,Y).$$

(3) Since $B_\varepsilon(U^{-1}(V_+), V_+) = B_\varepsilon(V_+, U(V_+)) = 0$, one has $U^{-1}(V_+) = V_+$ and $U^{-1}(V_-) = V_-$. Consequently, $(U\,N\,U^{-1})(V_+) \subset V_+$ and $(U\,N\,U^{-1})(V_-) \subset V_-$. Clearly, $U\,N\,U^{-1} \in \mathfrak{g}_\varepsilon(V)$. By (1), we only show that

$$(U\,N\,U^{-1})|_{V_+} = N'_+$$

This is obvious since $U^{-1}|_{V_+} = U_+^{-1}$.

□

Let us now consider $C \in \mathfrak{g}_\varepsilon$, C invertible. Then, m must be even (obviously, it happened if $\varepsilon = -1$), $m = 2n$ (see Appendix A). We decompose $C = S + N$ into semisimple and nilpotent parts, S, $N \in \mathfrak{g}_\varepsilon$ by its Jordan decomposition. It is clear that S is invertible. We have $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$ (Appendix A) where Λ is the spectrum of S. Also, $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ with the multiplicity $m(\lambda)$. Since N and S commute, we have $N(V_{\pm\lambda}) \subset V_{\pm\lambda}$ where V_λ is the eigenspace of S corresponding to $\lambda \in \Lambda$. Denote by $W(\lambda)$ the direct sum

$$W(\lambda) = V_\lambda \oplus V_{-\lambda}.$$

Define the equivalence relation $\mathcal{R}$ on Λ by:

$$\lambda \mathcal{R} \mu \text{ if and only if } \lambda = \pm\mu.$$

Then

$$\mathbb{C}^{2n} = \overset{\perp}{\bigoplus_{\lambda \in \Lambda/\mathcal{R}}} W(\lambda),$$

and each $(W(\lambda), B_\lambda)$ is a vector space with the non-degenerate form $B_\lambda = B_\varepsilon|_{W(\lambda)\times W(\lambda)}$.

Fix $\lambda \in \Lambda$. We write $W(\lambda) = V_+ \oplus V_-$ with $V_\pm = V_{\pm\lambda}$. Then, according to the notation in Lemma 1.4.2, define $N_{\pm\lambda} = N_\pm$. Since $N|_{V_-} = -N_\lambda^*$, it is easy to verify that the matrices of $N|_{V_+}$ and $N|_{V_-}$ have the same Jordan form. Let $(d_1(\lambda), \dots, d_{r_\lambda}(\lambda))$ be the size of the Jordan blocks in the Jordan decomposition of $N|_{V_+}$. This does not depend on a possible choice between $N|_{V_+}$ or $N|_{V_-}$ since both maps have the same Jordan type.

Next, we consider

$$\mathscr{D} = \bigcup_{r \in \mathbb{N}^*} \{(d_1, \dots, d_r) \in \mathbb{N}^r \mid d_1 \geq d_2 \geq \dots \geq d_r \geq 1\}.$$

Define $d : \Lambda \to \mathscr{D}$ by $d(\lambda) = (d_1(\lambda), \dots, d_{r_\lambda}(\lambda))$. It is clear that $\Phi \circ d = m$ where $\Phi : \mathscr{D} \to \mathbb{N}$ is the map defined by $\Phi(d_1, \dots, d_r) = \sum_{i=1}^r d_i$.

Finally, we can associate to $C \in \mathfrak{g}_\varepsilon$ a triple (Λ, m, d) defined as above.

Definition 1.4.3. Let $\mathscr{J}_n$ be the set of all triples (Λ, m, d) such that:

(1) Λ is a subset of $\mathbb{C} \setminus \{0\}$ with $\sharp\Lambda \leq 2n$ and $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$.

(2) $m : \Lambda \to \mathbb{N}^*$ satisfies $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ and $\sum_{\lambda \in \Lambda} m(\lambda) = 2n$.

(3) $d : \Lambda \to \mathscr{D}$ satisfies $d(\lambda) = d(-\lambda)$, for all $\lambda \in \Lambda$ and $\Phi \circ d = m$.

Let $\mathscr{I}(2n)$ be the set of invertible elements in $\mathfrak{g}_\varepsilon$ and $\widetilde{\mathscr{I}}(2n)$ be the set of I_ε-adjoint orbits of elements in $\mathscr{I}(2n)$. By the preceding remarks, there is a map $i : \mathscr{I}(2n) \to \mathscr{J}_n$. Then we have a parametrization of the set $\widetilde{\mathscr{I}}(2n)$ as follows:

Proposition 1.4.4.
The map $i : \mathscr{I}(2n) \to \mathscr{J}_n$ induces a bijection $\widetilde{i} : \widetilde{\mathscr{I}}(2n) \to \mathscr{J}_n$.

Proof. Let C and $C' \in \mathscr{I}(2n)$ such that $C' = U\ C\ U^{-1}$ with $U \in I_\varepsilon$. Let S, S', N, N' be respectively the semisimple and nilpotent parts of C and C'. Write $i(C) = (\Lambda, m, d)$ and $i(C') = (\Lambda', m', d')$. One has

$$S' + N' = U\ (S+N)\ U^{-1} = U\ S\ U^{-1} + U\ N\ U^{-1}.$$

By the unicity of Jordan decomposition, $S' = U\ S\ U^{-1}$ and $N' = U\ N\ U^{-1}$. So $\Lambda' = \Lambda$ and $m' = m$. Also, since $U\ S = S'\ U$ one has $U\ S(V_\lambda) = S'\ U(V_\lambda)$. It implies that

$$S'\ (U(V_\lambda)) = \lambda U(V_\lambda).$$

That means $U(V_\lambda) = V'_\lambda$, for all $\lambda \in \Lambda$. Since $N' = U\ N\ U^{-1}$ then $N|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan decomposition, so $d = d'$ and $\widetilde{i}$ is well defined.

To prove that $\widetilde{i}$ is onto, we start with $\Lambda = \{\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k\}$, m and d as in Definition 1.4.3. Define on the canonical basis:

$$S = \operatorname{diag}_{2n}(\overbrace{\lambda_1, \ldots, \lambda_1}^{m(\lambda_1)}, \ldots, \overbrace{\lambda_k, \ldots, \lambda_k}^{m(\lambda_k)}, \overbrace{-\lambda_1, \ldots, -\lambda_1}^{m(\lambda_1)}, \ldots, \overbrace{-\lambda_k, \ldots, -\lambda_k}^{m(\lambda_k)}).$$

For all $1 \le i \le k$, let $d(\lambda_i) = (d_1(\lambda_i) \ge \cdots \ge d_{r_{\lambda_i}}(\lambda_i) \ge 1)$ and define

$$N_+(\lambda_i) = \operatorname{diag}_{d(\lambda_i)}\left(J_{d_1(\lambda_i)}, J_{d_2(\lambda_i)}, \ldots, J_{d_{r_{\lambda_i}}(\lambda_i)}\right)$$

on the eigenspace V_{λ_i} and 0 on the eigenspace $V_{-\lambda_i}$ where J_d is the Jordan block of size d.

By Lemma 1.4.2, $N(\lambda_i) = N_+(\lambda_i) - N_+^*(\lambda_i)$ is skew-symmetric on $V_{\lambda_i} \oplus V_{-\lambda_i}$. Finally,

$$\mathbb{C}^{2n} = \overset{\perp}{\bigoplus_{1\le i\le k}} \left(V_{\lambda_i} \oplus V_{-\lambda_i}\right).$$

Define $N \in \mathfrak{g}_\varepsilon$ by $N\left(\sum_{i=1}^k v_i\right) = \sum_{i=1}^k N(\lambda_i)(v_i)$, $v_i \in V_{\lambda_i} \oplus V_{-\lambda_i}$ and $C = S + N \in \mathfrak{g}_\varepsilon$. By construction, $i(C) = (\Lambda, m, d)$, so $\widetilde{i}$ is onto.

To prove that $\widetilde{i}$ is one-to-one, assume that C, $C' \in \mathscr{I}(2n)$ and that $i(C) = i(C') = (\Lambda, m, d)$. Using the previous notation, since their respective semisimple parts S and S' have the same spectrum and same multiplicities, there exist $U \in I_\varepsilon$ such that $S' = USU^{-1}$. For $\lambda \in \Lambda$, we have $U(V_\lambda) = V'_\lambda$ for eigenspaces V_λ and V'_λ of S and S' respectively.

Also, for $\lambda \in \Lambda$, if N and N' are the nilpotent parts of C and C', then $N''(V_\lambda) \subset V_\lambda$, with $N'' = U^{-1}N'U$. Since $i(C) = i(C')$, then $N|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan type. Since $N'' = U^{-1}N'U$, then $N''|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan type. So $N|_{V_\lambda}$ and $N''|_{V_\lambda}$ have the same Jordan type. Therefore, there exists $D_+ \in \mathscr{L}(V_\lambda)$ such that $N''|_{V_\lambda} = D_+ N|_{V_\lambda} D_+^{-1}$. By Lemma 1.4.2, there exists $D(\lambda) \in I_\varepsilon(V_\lambda \oplus V_{-\lambda})$ such that

$$N''|_{V_\lambda \oplus V_{-\lambda}} = D(\lambda) N|_{V_\lambda \oplus V_{-\lambda}} D(\lambda)^{-1}.$$

We define $D \in I_\varepsilon$ by $D|_{V_\lambda \oplus V_{-\lambda}} = D(\lambda)$, for all $\lambda \in \Lambda$. Then $N'' = DND^{-1}$ and D commutes with S since $S|_{V_{\pm\lambda}}$ is scalar. Then $S' = (UD)S(UD)^{-1}$ and $N' = (UD)N(UD)^{-1}$ and we conclude

$$C' = (UD)C(UD)^{-1}.$$

□

1.5 Adjoint orbits in the general case

Let us now classify I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$ in the general case as follows. Let C be an element in $\mathfrak{g}_\varepsilon$ and consider the Fitting decomposition of C

$$\mathbb{C}^m = V_N \oplus V_I,$$

where V_N and V_I are stable by C, $C_N = C|_{V_N}$ is nilpotent and $C_I = C|_{V_I}$ is invertible. Since C is skew-symmetric, $B_\varepsilon(C^k(V_N), V_I) = (-1)^k B_\varepsilon(V_N, C^k(V_I))$ for any k then one has $V_I = (V_N)^\perp$. Also, the restrictions $B_\varepsilon^N = B_\varepsilon|_{V_N\times V_N}$ and $B_\varepsilon^I = B_\varepsilon|_{V_I\times V_I}$ are non-degenerate. Clearly, $C_N \in \mathfrak{g}_\varepsilon(V_N)$ and $C_I \in \mathfrak{g}_\varepsilon(V_I)$. By Section 1.2 and Section 1.4, C_N is attached with a partition $[d] \in \mathscr{P}_\varepsilon(n)$ and C_I corresponds to a triple $T \in \mathscr{J}_\ell$ where $n = \dim(V_N)$, $2\ell = \dim(V_I)$. Let $\mathscr{D}(m)$ be the set of all pairs $([d], T)$ such that $[d] \in \mathscr{P}_\varepsilon(n)$ and $T \in \mathscr{J}_\ell$ satisfying $n + 2\ell = m$. By the preceding remarks, there exists a map $p : \mathfrak{g}_\varepsilon \to \mathscr{D}(m)$. Denote by $\mathscr{O}(\mathfrak{g}_\varepsilon)$ the set of I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$ then we obtain the classification of $\mathscr{O}(\mathfrak{g}_\varepsilon)$ as follows:

Proposition 1.5.1. *The map $p : \mathfrak{g}_\varepsilon \to \mathscr{D}(m)$ induces a bijection $\widetilde{p} : \mathscr{O}(\mathfrak{g}_\varepsilon) \to \mathscr{D}(m)$.*

Proof. Let C and C' be two elements in $\mathfrak{g}_\varepsilon$. Assume that C and C' lie in the same I_ε-adjoint orbit. It means that there exists an isometry P such that $C' = PCP^{-1}$. So $C'^k\ P = P\ C^k$ for any k in $\mathbb{N}$. As a consequence, $P(V_N) \subset V'_N$ and $P(V_I) \subset V'_I$. However, P is an isometry then $V'_N = P(V_N)$ and $V'_I = P(V_I)$. Therefore, one has

$$C'_N = P_N\, C_N P_N^{-1} \text{ and } C'_I = P_I\, C_I P_I^{-1},$$

where $P_N = P : V_N \to V'_N$ and $P_I = P : V_I \to V'_I$ are isometries. It implies that C_N, C'_N have the same partition and C_I, C'_I have the same triple. Hence, the map $\widetilde{p}$ is well defined.

For a pair $([d], T) \in \mathscr{D}(m)$ with $[d] \in \mathscr{P}_\varepsilon(n)$ and $T \in \mathscr{J}_\ell$, we set a nilpotent map $C_N \in \mathfrak{g}_\varepsilon(V_N)$ corresponding to $[d]$ as in Section 1.2 and an invertible map $C_I \in \mathfrak{g}_\varepsilon(V_I)$ as in Proposition 1.4.4 where $\dim(V_N) = n$ and $\dim(V_I) = 2\ell$. Define $C \in \mathfrak{g}_\varepsilon$ by $C(X_N + X_I) = C_N(X_N) + C_I(X_I)$, for all $X_N \in V_N$, $X_I \in V_I$. By construction, $p(C) = ([d], T)$ and $\widetilde{p}$ is onto.

To prove $\widetilde{p}$ is one-to-one, let $C, C' \in \mathfrak{g}_\varepsilon$ such that $p(C) = p(C') = ([d], T)$. Keep the above notations, since C_N and C'_N have the same partition then there exists an isometry $P_N : V_N \to V'_N$ such that $C'_N = P_N\ C_N\ P_N^{-1}$. Similarly C_I and C'_I have the same triple and then there exists an isometry $P_I : V_I \to V'_I$ such that $C'_I = P_I\ C_I\ P_I^{-1}$. Define $P : V \to V$ by $P(X_N + X_I) = P_N(X_N) + P_I(X_I)$, for all $X_N \in V_N, X_I \in V_I$ then P is an isometry and $C' = P\ C\ P^{-1}$. Therefore, $\widetilde{p}$ is one-to-one.

□

Chapter 2

Quadratic Lie algebras

In the first part of this chapter, we recall some preliminary definitions and results of quadratic Lie algebras. Next, we study a new invariant under isomorphisms of quadratic Lie algebras that we call dup-*number*. Moreover, we give a classification of singular quadratic Lie algebras, i.e. those for which the invariant does not vanish. The classification is closely related to $O(m)$-adjoint orbits of $\mathfrak{o}(m)$ mentioned in the Chapter 1. To prove these results, we need to fully describe the structure of singular quadratic Lie algebras by properties of super-Poisson bracket defined on the ($\mathbb{Z}$-graded) Grassmann algebra of alternating multilinear forms of an n-dimensional quadratic vector space [PU07] and in terms of *double extensions* ([Kac85], [FS87] and [MR85]). The invariance of dup-number is a consequence of calculating the quadratic dimension of singular quadratic Lie algebras in the reduced case.

Another effective method called T^**-extension* is introduced by M. Bordemann to descibe solvable quadratic Lie algebras [Bor97] and we use it to study the 2-step nilpotent case. Finally, we also obtain a familiar result: the classification of 2-step nilpotent quadratic Lie algebras up to isometrical isomorphisms is equivalent to the classification all of associated 3-forms.

2.1 Preliminaries

Definition 2.1.1. A *quadratic Lie algebra* $(\mathfrak{g}, B)$ is a vector space $\mathfrak{g}$ equipped with a non-degenerate symmetric bilinear form B and a Lie algebra structure on $\mathfrak{g}$ such that B is invariant (that means, $B([X,Y],Z) = B(X,[Y,Z])$, for all X, Y, $Z \in \mathfrak{g}$).

Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Since B is non-degenerate and invariant, we have some simple properties of $\mathfrak{g}$ as follows:

Proposition 2.1.2.

(1) If I is an ideal of $\mathfrak{g}$ then $I^\perp$ is also an ideal of $\mathfrak{g}$. Moreover, if I is non-degenerate then so is $I^\perp$ and $\mathfrak{g} = I \oplus I^\perp$. Conveniently, in this case we use the notation $\mathfrak{g} = I \overset{\perp}{\oplus} I^\perp$.

(2) $\mathscr{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$ where $\mathscr{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. And then

$$\dim(\mathscr{Z}(\mathfrak{g})) + \dim([\mathfrak{g}, \mathfrak{g}]) = \dim(\mathfrak{g}).$$

(3) *Set the map* $\phi : \mathfrak{g} \to \mathfrak{g}^*$ *defined by* $\phi(X) = B(X,.)$, *for all* $X \in \mathfrak{g}$ *then* ϕ *is an isomorphism. Moreover, the adjoint representation and coadjoint representation of* $\mathfrak{g}$ *are equivalent by* ϕ.

Definition 2.1.3. Let $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ be two quadratic Lie algebras. We say that $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ are *isometrically isomorphic* (or *i-isomorphic*) if there exists a Lie algebra isomorphism A from $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying

$$B'(A(X),A(Y)) = B(X,Y),\ \forall\, X,Y \in \mathfrak{g}.$$

In this case, A is called an *i-isomorphism*. In other words, A is an i-isomorphism if it is a Lie algebra isomorphism and an isometry. We write $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$.

Consider two quadratic Lie algebras $(\mathfrak{g},B)$ and $(\mathfrak{g},B')$ (same Lie algebra) with $B' = \lambda B$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. They are not necessarily i-isomorphic, as shown by the example below:

Example 2.1.4. Let $\mathfrak{g} = \mathfrak{o}(3)$ and κ its Killing form. Then A is a Lie algebra automorphism of $\mathfrak{g}$ if and only if $A \in \mathrm{O}(\mathfrak{g})$. So $(\mathfrak{g},\kappa)$ and $(\mathfrak{g},\lambda\kappa)$ cannot be i-isomorphic if $\lambda \neq 1$.

We have a characteristic of quadratic Lie algebras as follows:

Proposition 2.1.5. [PU07]
Let $(\mathfrak{g},B)$ *be a non-Abelian quadratic Lie algebra. Then there exists a central ideal* $\mathfrak{z}$ *and an ideal* $\mathfrak{l} \neq \{0\}$ *such that:*

(1) $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ *where* $(\mathfrak{z},B|_{\mathfrak{z}\times\mathfrak{z}})$ *and* $(\mathfrak{l},B|_{\mathfrak{l}\times\mathfrak{l}})$ *are quadratic Lie algebras. Moreover,* $\mathfrak{l}$ *is non-Abelian.*

(2) *The center* $\mathscr{Z}(\mathfrak{l})$ *is totally isotropic, equivalently* $\mathscr{Z}(\mathfrak{l}) \subset [\mathfrak{l},\mathfrak{l}]$, *and*

$$\dim(\mathscr{Z}(\mathfrak{l})) \leq \frac{1}{2}\dim(\mathfrak{l}) \leq \dim([\mathfrak{l},\mathfrak{l}]).$$

(3) *Let* $\mathfrak{g}'$ *be a quadratic Lie algebra and* $A : \mathfrak{g} \to \mathfrak{g}'$ *be a Lie algebra isomorphism. Then*

$$\mathfrak{g}' = \mathfrak{z}' \overset{\perp}{\oplus} \mathfrak{l}'$$

where $\mathfrak{z}' = A(\mathfrak{z})$ *is central,* $\mathfrak{l}' = A(\mathfrak{z})^\perp$, $\mathscr{Z}(\mathfrak{l}')$ *is totally isotropic and* $\mathfrak{l}$ *and* $\mathfrak{l}'$ *are isomorphic. Moreover if* A *is an i-isomorphism, then* $\mathfrak{l}$ *and* $\mathfrak{l}'$ *are i-isomorphic.*

Proof. Let $\mathfrak{z}_0 = \mathscr{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$. Define $\mathfrak{z}$ a complementary subspace of $\mathfrak{z}_0$ in $\mathscr{Z}(\mathfrak{g})$. Since $\mathscr{Z}(\mathfrak{g})^\perp = [\mathfrak{g},\mathfrak{g}]$, one has $B(\mathfrak{z}_0,\mathfrak{z}) = \{0\}$ and $\mathfrak{z} \cap \mathfrak{z}^\perp = \{0\}$. Therefore $\mathfrak{z}$ and $\mathfrak{z}^\perp$ are non-degenerate and $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$, where $\mathfrak{l} = \mathfrak{z}^\perp$. It is obvious that $\mathfrak{l}$ is non-Abelian since $\mathfrak{z}$ is central in $\mathfrak{g}$.

Since $B([\mathfrak{g},\mathfrak{g}],\mathfrak{z}) = \{0\}$, one has $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{l}$. It is easy to check that $\mathscr{Z}(\mathfrak{l}) = \mathfrak{z}_0$ and $[\mathfrak{l},\mathfrak{l}] = [\mathfrak{g},\mathfrak{g}] = \mathscr{Z}(\mathfrak{g})^\perp$ so $\mathscr{Z}(\mathfrak{l})$ is totally isotropic. Moreover, $\mathscr{Z}(\mathfrak{l}) \subset [\mathfrak{l},\mathfrak{l}] = \mathscr{Z}(\mathfrak{l})^\perp$ implies $\dim(\mathfrak{l}) - \dim([\mathfrak{l},\mathfrak{l}]) \leq \dim([\mathfrak{l},\mathfrak{l}])$ and (2) is finished.

For (3), one has $A(\mathscr{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]) = \mathscr{Z}(\mathfrak{g}') \cap [\mathfrak{g}',\mathfrak{g}']$ and $\mathscr{Z}(\mathfrak{g}') = \mathfrak{z}' \oplus (\mathscr{Z}(\mathfrak{g}') \cap [\mathfrak{g}',\mathfrak{g}'])$. Therefore $\mathfrak{l}'$ satisfies $\mathfrak{g}' = \mathfrak{z}' \overset{\perp}{\oplus} \mathfrak{l}'$ and $\mathscr{Z}(\mathfrak{l}')$ is totally isotropic. Since A is an isomorphism from $\mathfrak{z}$ onto $\mathfrak{z}'$, A induces an isomorphism from $\mathfrak{g}/\mathfrak{z}$ onto $\mathfrak{g}'/\mathfrak{z}'$, and it results that $\mathfrak{l}$ and $\mathfrak{l}'$ are isomorphic Lie algebras. Same reasoning works for A i-isomorphism. □

Corollary 2.1.6. *Let* $(\mathfrak{g}, B)$ *be a non-Abelian solvable quadratic Lie algebra. Then there exists a central element* X *of* $\mathfrak{g}$ *such that* X *is isotropic.*

Proof. By the above proposition, $\mathfrak{g}$ can be decomposed by $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ where $\mathfrak{l}$ is non-Abelian and $\mathcal{Z}(\mathfrak{l})$ is totally isotropic. Since $\mathfrak{g}$ is solvable then $\mathfrak{l}$ is solvable. Moreover, $\mathfrak{l}$ is a quadratic Lie algebra then $\mathcal{Z}(\mathfrak{l}) \neq 0$ and the result follows. □

Definition 2.1.7. A quadratic Lie algebra $\mathfrak{g}$ is *reduced* if:

(1) $\mathfrak{g} \neq \{0\}$

(2) $\mathcal{Z}(\mathfrak{g})$ is totally isotropic.

Notice that a reduced quadratic Lie algebra is necessarily non-Abelian.

Definition 2.1.8. Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and C be an endomorphism of $\mathfrak{g}$. We say that C is *skew-symmetric* (or *B-antisymmetric*) if $B(C(X), Y) = -B(X, C(Y))$, for all $X, Y \in \mathfrak{g}$. Denote by $\mathrm{End}_a(\mathfrak{g})$ (resp. $\mathrm{Der}_a(\mathfrak{g})$) space of skew-symmetric endomorphisms (resp. derivations) of $\mathfrak{g}$.

Next, we recall two effective methods to construct quadratic Lie algebras: double extensions and T^*-extensions. The former method is initiated by V. Kac for the solvable case ([Kac85] and [FS87]), after that developed generally by A. Medina and Ph. Revoy [MR85]; the later is given by M. Bordemann [Bor97].

Definition 2.1.9. Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra, $\mathfrak{h}$ be another Lie algebra and $\pi : \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g})$ be a representation of $\mathfrak{h}$ by means of skew-symmetric derivations of $\mathfrak{g}$. Define the map $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*$ by $\varphi(X,Y)Z = B(\pi(Z)X, Y)$, for all $X, Y \in \mathfrak{g}$, $Z \in \mathfrak{h}$. Denote by ad^* the coadjoint representation of $\mathfrak{h}$. Then the vector space $\overline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{h}^*$ with the product:

$$[X+Y+f, X'+Y'+f'] = [X,X']_{\mathfrak{h}} + [Y,Y']_{\mathfrak{g}} + \pi(X)Y' - \pi(X')Y + \mathrm{ad}^*(X)f' - \mathrm{ad}^*(X')f + \varphi(Y,Y')$$

for all $X, X' \in \mathfrak{h}$, $Y, Y' \in \mathfrak{g}$, $f, f' \in \mathfrak{h}^*$ is a Lie algebra and it is called the *double extension of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *by means of* π. It is easy to show that $\overline{\mathfrak{g}}$ is also a quadratic Lie algebra with the bilinear form $\overline{B}$ defined by:

$$\overline{B}(X+Y+f, X'+Y'+f') = B(Y,Y') + f(X') + f'(X)$$

for all $X, X' \in \mathfrak{h}$, $Y, Y' \in \mathfrak{g}$, $f, f' \in \mathfrak{h}^*$.

If there is an invariant symmetric bilinear form γ on $\mathfrak{h}$ (not necessarily non-degenerate) then $\overline{\mathfrak{g}}$ is also a quadratic Lie algebra with the bilinear form $\overline{B}_\gamma$ as follows:

$$\overline{B}_\gamma(X+Y+f, X'+Y'+f') = B(Y,Y') + \gamma(X,X') + f(X') + f'(X)$$

for all $X, X' \in \mathfrak{h}$, $Y, Y' \in \mathfrak{g}$, $f, f' \in \mathfrak{h}^*$.

Proposition 2.1.10. *(*[Kac85]*, 2.11,* [MR85]*, Theorem I)*

Let $(\mathfrak{g}, B)$ *be an indecomposable quadratic Lie algebra (see Definition 2.2.17) such that it is not simple nor one-dimensional. Then* $\mathfrak{g}$ *is the double extension of a quadratic Lie algebra by a simple or one-dimensional algebra.*

Proof. Let I be a minimal ideal of $\mathfrak{g}$. Since $I \cap I^{\perp}$ is also an ideal of $\mathfrak{g}$ then we must have $I \cap I^{\perp} = I$ or $I \cap I^{\perp} = \{0\}$. But $\mathfrak{g}$ is indecomposable so the second case does not happen. It means that $I \subset I^{\perp}$, i.e. I is totally isotropic. So $B([I, I^{\perp}], \mathfrak{g}) = B(I, [I^{\perp}, \mathfrak{g}]) = 0$. Therefore, $[I, I^{\perp}] = 0$ by the non-degeneracy of B.

Consider two exact sequences of Lie algebras

$$0 \to I \to I^{\perp} \to I^{\perp}/I \to 0,$$
$$0 \to I^{\perp} \to \mathfrak{g} \to \mathfrak{g}/I^{\perp} \to 0.$$

Denote by $\mathfrak{h} = \mathfrak{g}/I^{\perp}$ then $\mathfrak{h}$ is simple or one-dimensional since $I^{\perp}$ is a maximal ideal of $\mathfrak{g}$. We can identify $\mathfrak{h}$ with a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ is the semi-direct product of $I^{\perp}$ by $\mathfrak{h}$. Let $W = I^{\perp}/I$, $p : I^{\perp} \to W$ be the canonical projection and define on W the bilinear form T that is the restriction of B on W. Then (W, T) is a quadratic Lie algebra. Since the subspace $H = I \oplus \mathfrak{h}$ is non-degenerate, we can identify W (regarded as a subspace of $\mathfrak{g}$) with $H^{\perp}$, i.e. $\mathfrak{g} = I \oplus W \oplus \mathfrak{h}$.

Now, we will define an action π of $\mathfrak{h}$ on W as follows. Let $x \in W$, regarded as an element of $H^{\perp}$, and take $h \in \mathfrak{h}$. We will show that $[h, x] \in H^{\perp}$. Indeed, let $h' \in \mathfrak{h}$ then $B(h', [h, x]) = B([h', h], x) = 0$ since $[h', h] \in \mathfrak{h}$. Therefore, $[h, x] \in \mathfrak{h}^{\perp}$. Let $y \in I$ then $B([x, h], y) = B(x, [h, y]) = 0$. Hence, $[h, x]$ must be in $H^{\perp}$. We set $\pi : \mathfrak{h} \to \mathrm{Der}_a(W)$ by $\pi(h)(x) = [h, x]$, for all $h \in \mathfrak{h}$, $x \in W$.

For all $x, y \in W$ one has:

$$[x, y]_{\mathfrak{g}} = [x, y]_W + \varphi(x, y),$$

where $\varphi : W \times W \to I$ satisfies $B(\varphi(x, y), z) = B(\pi(z)x, y)$, for all $x, y \in W$, $z \in \mathfrak{h}$.

Finally, since $I \oplus \mathfrak{h}$ is non-degenerate and I is a totally isotropic subspace of $I \oplus \mathfrak{h}$, we can identify I with $\mathfrak{h}^*$ and the adjoint action of $\mathfrak{h}$ into I becomes the coadjoint action ad^* of $\mathfrak{h}$ into $\mathfrak{h}^*$. Then $\mathfrak{g}$ is the double extension of W by $\mathfrak{h}$ by means of π. □

Corollary 2.1.11. [FS87]

Let $(\mathfrak{g}, B)$ be a non-Abelian solvable quadratic Lie algebra then $\mathfrak{g}$ is the double extension of a quadratic Lie algebra of dimension $\dim(\mathfrak{g}) - 2$ by a one-dimensional algebra.

Proof. By Corollary 2.1.6, there is a totally isotropic central ideal $\mathbb{C}X$ of $\mathfrak{g}$. Let $Y \in \mathfrak{g}$ isotropic such that $B(X, Y) = 1$ then $\mathfrak{g}$ is the double extension of $(\mathbb{C}X \oplus \mathbb{C}Y)^{\perp}$ by the one-dimensional algebra $\mathbb{C}Y$. □

Let us present now the second construction of quadratic Lie algebras which is given by M. Bordemann in [Bor97] as follows:

Definition 2.1.12. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, V be a complex vector space and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ in V. That means

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X), \ \forall\, X, Y \in \mathfrak{g}.$$

In this case, V is also called a $\mathfrak{g}$-*module*. For an integer $k \geq 0$, denote by $C^k(\mathfrak{g}, V)$ the space of alternating k-linear maps from $\mathfrak{g} \times ... \times \mathfrak{g}$ into V if $k \geq 1$ and $C^0(\mathfrak{g}, V) = V$. Let $C(\mathfrak{g}, V) =$

$\sum_{k=0}^{\infty} C^k(\mathfrak{g},V)$. The coboundary operator $\delta : C(\mathfrak{g},V) \to C(\mathfrak{g},V)$ is defined by

$$\begin{aligned}\delta f(X_0,...,X_k) &= \sum_{i=0}^{k}(-1)^i\rho(X_i)(f(X_0,...,\widehat{X_i},...,X_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}f([X_i,X_j],X_0,...,\widehat{X_i},...,\widehat{X_j},...,X_k)\end{aligned}$$

for all $f \in C^k(\mathfrak{g},V)$, $X_0,\dots,X_k \in \mathfrak{g}$. It is known that $\delta^2 = 0$. We say that $f \in C^k(\mathfrak{g},V)$ is a *k-cocycle* if $\delta f = 0$ and f is a *k-coboundary* if there is $g \in C^{k-1}(\mathfrak{g},V)$ such that $f = \delta g$.

In particular, the dual $\mathfrak{g}^*$ is a $\mathfrak{g}$-module with respect to the coadjoint representation of $\mathfrak{g}$. Consider a bilinear map $\theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}^*$ and define on the vector space $T^*_\theta(\mathfrak{g}) = \mathfrak{g}\oplus\mathfrak{g}^*$ the product as follows:

$$[X+f,Y+g] = [X,Y]_\mathfrak{g} + f\circ \mathrm{ad}_\mathfrak{g}(Y) - g\circ\mathrm{ad}_\mathfrak{g}(X) + \theta(X,Y),\ \forall\, X,Y\in\mathfrak{g},\ f,g\in\mathfrak{g}^*.$$

It is easy to check that $T^*_\theta(\mathfrak{g})$ is a Lie algebra if and only if θ is a 2-cocycle. In this case, $T^*_\theta(\mathfrak{g})$ is called the *T^*-extension of* $\mathfrak{g}$ *by means of* θ. Moreover, if θ satisfies $\theta(X,Y)Z = \theta(Y,Z)X$, for all $X,Y,Z\in\mathfrak{g}$ (*cyclic* condition) then $T^*_\theta(\mathfrak{g})$ becomes a quadratic Lie algebra with the bilinear form B defined by

$$B(X+f,Y+g) = f(Y)+g(X),\ \forall\, X,Y\in\mathfrak{g},\ f,g\in\mathfrak{g}^*.$$

Proposition 2.1.13. [Bor97]

Let $(\mathfrak{g},B)$ be an even-dimensional quadratic Lie algebra over $\mathbb{C}$. If $\mathfrak{g}$ is solvable then $\mathfrak{g}$ is i-isomorphic to a T^*-extension $T^*_\theta(\mathfrak{h})$ of $\mathfrak{h}$ where $\mathfrak{h}$ is the quotient algebra of $\mathfrak{g}$ by a totally isotropic ideal.

2.2 Singular quadratic Lie algebras

2.2.1 Super-Poisson bracket and quadratic Lie algebras

Let (V,B) be a quadratic vector space. Denote by $\mathscr{A}(V)$ the ($\mathbb{Z}$-graded) Grassmann algebra of alternating multilinear forms on V. For $X \in V$, we recall the derivation ι_X of $\mathscr{A}(V)$ defined by:

$$\iota_X(\Omega)(Y_1,\dots,Y_k) = \Omega(X,Y_1,\dots,Y_k),\ \forall\, \Omega \in \mathscr{A}^{k+1}(V),\ X,Y_1,\dots,Y_k \in V\ (k \geq 0),$$

and $\iota_X(1) = 0$. Then the super-Poisson bracket on $\mathscr{A}(V)$ is defined as follows (see [PU07] for details): fix an orthonormal basis $\{v_1,\dots,v_n\}$ of V, then one has

$$\{\Omega,\Omega'\} = (-1)^{k+1}\sum_{j=1}^{n} \iota_{v_j}(\Omega)\wedge\iota_{v_j}(\Omega'),\ \forall\, \Omega \in \mathscr{A}^k(V),\ \Omega' \in \mathscr{A}(V). \tag{I}$$

For instance, if $\alpha \in V^*$, one has

$$\{\alpha,\Omega\} = \iota_{\phi^{-1}(\alpha)}(\Omega),\ \forall\, \Omega \in \mathscr{A}(V),$$

and if $\alpha' \in V^*$, $\{\alpha,\alpha'\} = B(\phi^{-1}(\alpha),\phi^{-1}(\alpha'))$. This definition does not depend on the choice of the basis.

For any $\Omega \in \mathscr{A}^k(V)$, define $\mathrm{ad}_{\mathrm{P}}(\Omega)$ by

$$\mathrm{ad}_{\mathrm{P}}(\Omega)\left(\Omega'\right) = \{\Omega,\Omega'\},\ \forall\, \Omega' \in \mathscr{A}(V).$$

Then $\mathrm{ad}_{\mathrm{P}}(\Omega)$ is a super-derivation of degree $k-2$ of the algebra $\mathscr{A}(V)$. One has:

$$\mathrm{ad}_{\mathrm{P}}(\Omega)\left(\{\Omega',\Omega''\}\right) = \{\mathrm{ad}_{\mathrm{P}}(\Omega)(\Omega'),\Omega''\} + (-1)^{kk'}\{\Omega',\mathrm{ad}_{\mathrm{P}}(\Omega)(\Omega'')\},$$

for all $\Omega' \in \mathscr{A}^{k'}(V)$, $\Omega'' \in \mathscr{A}(V)$. That implies that $\mathscr{A}(V)$ is a graded Lie algebra for the super-Poisson bracket.

Proposition 2.2.1. [PU07]
Let $(\mathfrak{g},B)$ be a quadratic Lie algebra. We define a 3-form I on $\mathfrak{g}$ as follows:

$$I(X,Y,Z) = B([X,Y],Z),\ \forall\, X,Y,Z \in \mathfrak{g}.$$

Then one has:

(1) $I \in \mathscr{A}^3(\mathfrak{g})$.

(2) $\{I,I\} = 0$.

Conversely, assume that $\mathfrak{g}$ is a finite-dimensional quadratic vector space. Let $I \in \mathscr{A}^3(\mathfrak{g})$ and define

$$[X,Y] = \phi^{-1}\left(\iota_{X\wedge Y}(I)\right),\ \forall\, X,Y \in \mathfrak{g}.$$

This bracket satisfies the Jacobi identity if and only if $\{I,I\} = 0$ [PU07]. In this case, $\mathfrak{g}$ becomes a quadratic Lie algebra with the 3-form I.

Definition 2.2.2. The 3-form I in the previous proposition is called the *3-form associated* to $\mathfrak{g}$.

2.2.2 The dup number of a quadratic Lie algebra

Let V be a vector space and $I \in \mathscr{A}^k(V)$, for $k \geq 1$. We introduce two subspaces of V^*:

$$\begin{aligned} \mathcal{V}_I &= \{\alpha \in V^* \mid \alpha \wedge I = 0\}, \\ \mathcal{W}_I &= \{v \in V \mid \iota_v(I) = 0\}^{\perp_*}. \end{aligned}$$

Proposition 2.2.3. [Bou58]
Let $I \in \mathscr{A}^k(V)$, $I \neq 0$. Then:

(1) $\mathcal{V}_I \subset \mathcal{W}_I$, $\dim(\mathcal{V}_I) \leq k$ *and* $\dim(\mathcal{W}_I) \geq k$.

(2) If $\{\alpha_1, \dots, \alpha_r\}$ is a basis of $\mathcal{V}_I$, then $\alpha_1 \wedge \dots \wedge \alpha_r$ divides I. Moreover, I belongs to the k-th exterior power of $\mathcal{W}_I$, also denoted by $\bigwedge^k(\mathcal{W}_I)$.

(3) I is decomposable if and only if $\dim(\mathcal{V}_I) = k$ *or* $\dim(\mathcal{W}_I) = k$. *In this case, $\mathcal{V}_I = \mathcal{W}_I$ and if $\{\alpha_1, \dots, \alpha_k\}$ is a basis of $\mathcal{V}_I$, there is some non-zero $\lambda \in \mathbb{C}$ such that:*

$$I = \lambda \alpha_1 \wedge \dots \wedge \alpha_k.$$

Proof. First, we need the following lemmas:

Lemma 2.2.4. *Let $l \leq k$ and $\alpha_1, ..., \alpha_l$ be linear forms independent in V^* and satisfying $\alpha_i \wedge I = 0$, for all $i = 1, \dots, l$. One has:*

(1) if $l \leq k-1$ then there exists a multilinear form $\beta \in \mathscr{A}^{k-l}(V)$ such that $I = \alpha_1 \wedge \dots \wedge \alpha_l \wedge \beta$,

(2) if $l = k$ then there exists a non-zero complex ξ such that $I = \xi \alpha_1 \wedge \dots \wedge \alpha_k$.

Proof. Since $\alpha_1, ..., \alpha_l$ are linearly independent in V^* then we can complete this system by vectors to get a basis $\{\alpha_1, ..., \alpha_n\}$ of V^*. In this basis, assume that I is as follows:

$$I = \sum_{J \subset [[1,n]], |J| = k} \xi_J \alpha_{j_1} \wedge \dots \wedge \alpha_{j_k},$$

where $\xi_J \in \mathbb{C}$ and the indices meant $J = (j_1, \dots, j_k) \in \mathbb{N}^k$ with $1 \leq j_1 < \dots < j_k \leq n$.

One has $\alpha_1 \wedge I = 0$ then $\xi_J = 0$ if $j_1 \neq 1$. Therefore, we obtain

$$I = \alpha_1 \wedge \sum_{2 \leq j_2 < \dots < j_k \leq n} \xi_{1, j_2, \dots, j_k} \alpha_{j_2} \wedge \dots \wedge \alpha_{j_k}.$$

Similarly, one has $\alpha_2 \wedge I = 0$ then $\xi_J = 0$ if $j_2 \neq 2$ and we get

$$I = \alpha_1 \wedge \alpha_2 \wedge \sum_{3 \leq j_3 < \dots < j_k \leq n} \xi_{1,2,j_3, \dots, j_k} \alpha_{j_3} \wedge \dots \wedge \alpha_{j_k}.$$

This continues until α_l and finally we obtain if $l \leq k-1$:

$$I = \alpha_1 \wedge \dots \wedge \alpha_l \wedge \left(\sum_{l+1 \leq j_{l+1} < \dots < j_k \leq n} \xi_{1, \dots, l, j_{l+1}, \dots, j_k} \alpha_{j_{l+1}} \wedge \dots \wedge \alpha_{j_k} \right).$$

If $l = k$ then we stand at $(k-1)$-step and

$$I = \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \sum_{k \le j_k \le n} \xi_{1,\ldots,k-1,j_k} \alpha_{j_k}.$$

Since $\alpha_k \wedge I = 0$ then $\xi_J = 0$ if $j_k \neq k$. Therefore, we obtain

$$I = \xi \alpha_1 \wedge \cdots \wedge \alpha_k.$$

□

Lemma 2.2.5. $\mathcal{W}_I = \mathcal{G}_I = \{\iota_A(I) \mid A \in \bigwedge^{k-1}(V)\}$.

Proof. Let $X \in V$ such that $\iota_X(I) = 0$. If $A \in \bigwedge^{k-1}(V)$ then one has

$$\iota_A(I)(X) = I(A \wedge X) = (-1)^{k-1} I(X \wedge A) = (-1)^{k-1} \iota_X(I)(A) = 0.$$

So $\mathcal{G}_I \subset \mathcal{W}_I$. Let $X \in \mathcal{G}_I^\perp$. If $A \in \bigwedge^{k-1}(V)$ then $\iota_A(I)(X) = 0$. That means $\iota_X(I)(A) = 0$, for all $A \in \bigwedge^{k-1}(V)$. Therefore $\iota_X(I) = 0$ and then $X \in \mathcal{W}_I^\perp$. It implies that $\mathcal{W}_I = \mathcal{G}_I$. □

We turn now to the proof of the proposition.

(1) Let $\alpha \in \mathcal{V}_I$, we show that $\alpha \in \mathcal{W}_I$. It means that if $X \in V$ such that $\iota_X(I) = 0$ then $\alpha(X) = 0$. Indeed, let $X \in V$ such that $\iota_X(I) = 0$. Since $\alpha \in \mathcal{V}_I$ one has $\alpha \wedge I = 0$. This implies that
$$0 = \iota_X(\alpha \wedge I) = \iota_X(\alpha) \wedge I - \alpha \wedge \iota_X(I) = \alpha(X) I.$$
Therefore, $\alpha(X) = 0$ since $I \neq 0$.

To prove $\dim(\mathcal{V}_I) \le k$, we assume to the contrary, i.e. $\dim(\mathcal{V}_I) \ge k+1$. Let $\{\alpha_1, \ldots, \alpha_{k+1}\}$ be an independent system of vectors of $\mathcal{V}_I$. Apply Lemma 2.2.4 one has a non-zero complex ξ such that
$$I = \xi \alpha_1 \cdots \wedge \alpha_k.$$
Since $\alpha_{k+1} \wedge I = 0$ we get $\xi \alpha_1 \cdots \wedge \alpha_k = 0$, i.e. $\xi = 0$. This is a contradiction. Therefore, $\dim(\mathcal{V}_I) \le k$.

To prove $\dim(\mathcal{W}_I) \ge k$, let $\{\alpha_1, \ldots, \alpha_r\}$ be a basis of $\mathcal{V}_I$ and complete it to get a basis $\{\alpha_1, \ldots, \alpha_n\}$ of V^*. By Lemma 2.2.4, there exists a $(k-r)$-form β such that

$$I = \alpha_1 \cdots \wedge \alpha_r \wedge \beta.$$

We can write β as follows:

$$\beta = \sum_{r+1 \le j_{r+1} < \cdots < j_k \le n} \xi_{j_{r+1},\ldots j_k} \alpha_{j_{r+1}} \wedge \cdots \wedge \alpha_{j_k} = \sum_J \xi_J \alpha_J.$$

Let $\{X_1, \ldots, X_n\}$ be the dual basis of $\{\alpha_1, \ldots, \alpha_n\}$ in V and choose J such that $\xi_J \neq 0$. By Lemma 2.2.5, one has:

- if $A_s = X_1 \wedge \cdots \wedge \widehat{X_s} \wedge \cdots \wedge X_r \wedge X_{j_{r+1}} \wedge \cdots \wedge X_{j_k}$ then $\iota_{A_s} I = \pm \xi_J \alpha_s \in \mathcal{W}_I (1 \le s \le r)$.

- if $A_s = X_1 \wedge \cdots \wedge X_r \wedge X_{j_{r+1}} \wedge \cdots \wedge \widehat{X_{j_{r+s}}} \wedge \cdots \wedge X_{j_k}$ then $\iota_{A_s} I = \pm \xi_J \alpha_{j_{r+s}} \in \mathcal{W}_I (1 \leq s \leq k-r)$.

Therefore, $\dim(\mathcal{W}_I) \geq k$.

(2) The first statement of (2) follows Lemma 2.2.4. By the proof of (1), $\alpha_1, ..., \alpha_r, \alpha_{j_{r+1}}, ..., \alpha_{j_{r+s}} \in \mathcal{W}_I$ then $I \in \bigwedge^k(\mathcal{W}_I)$.

(3) Lemma 2.2.4 shows that I is decomposable if and only if $\dim(\mathcal{V}_I) = k$. We only prove that I is decomposable if and only if $\dim(\mathcal{W}_I) = k$. The last assertion follows. Indeed, if I is decomposable, then $I = \alpha_1 \wedge \cdots \wedge \alpha_k$. Therefore, $\mathcal{W}_I = \operatorname{span}\{\alpha_1, \ldots, \alpha_k\}$, i.e. $\dim(\mathcal{W}_I) = k$.

Conversely, if $\dim(\mathcal{W}_I) = k$, we assume that I is not decomposable. Let $\{\alpha_1, ..., \alpha_r\}$ be a basis of $\mathcal{V}_I$ and complete it to get a basis $\{\alpha_1, ..., \alpha_n\}$ of V^*. By Lemma 2.2.4, there exists a $(k-r)$-form β such that

$$I = \alpha_1 \cdots \wedge \alpha_r \wedge \beta.$$

We can write β as follows:

$$\beta = \sum_{r+1 \leq j_{r+1} < \cdots < j_k \leq n} \xi_{j_{r+1}, \ldots j_k} \alpha_{j_{r+1}} \wedge \cdots \wedge \alpha_{j_k} = \sum_J \xi_J \alpha_J.$$

If there exist two index families J and J' distinct such that $\xi_J \neq 0$ and $\xi_{J'} \neq 0$ then

$$\sharp(\{j_{r+1}, \ldots, j_k\} \cup \{j'_{r+1}, \ldots, j'_k\}) \geq k - r + 1.$$

So one has $r + (k-r+1) = k+1$ vectors independent in $\mathcal{W}_I$ which is of dimension k. This is a contradiction. Therefore, I is decomposable.

□

Corollary 2.2.6. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie algebra and I be its associated 3-form. Then one has:*

(1) $\dim(\mathcal{V}_I) \in \{0, 1, 3\}$.

(2) $\mathcal{W}_I = \phi([\mathfrak{g}, \mathfrak{g}])$ *and* $\dim([\mathfrak{g}, \mathfrak{g}]) \geq 3$.

(3) I is decomposable if and only if $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$.

Corollary 2.2.7. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie algebra with its associated 3-form I. If $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ is a decomposition of $\mathfrak{g}$ as in Proposition 2.1.5, where $\mathfrak{z}$ is a central ideal and $\mathfrak{l}$ has a totally isotropic center then $\mathcal{W}_I = \phi([\mathfrak{l}, \mathfrak{l}])$.*

Recall that if $A : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism then it induces an isomorphism ${}^tA : \mathfrak{g}'^* \to \mathfrak{g}^*$ defined by ${}^tA(f) = f \circ A$, for all $f \in \mathfrak{g}'^*$, that extends to an algebra isomorphism from $\mathscr{A}(\mathfrak{g}')$ onto $\mathscr{A}(\mathfrak{g})$. We also denote this isomorphism by ${}^tA$.

Lemma 2.2.8. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be quadratic Lie algebras with associated 3-forms I and I' respectively. Let A be an i-isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}'$. Then $I = {}^tA(I')$, $\mathcal{V}_I = {}^tA(\mathcal{V}_{I'})$ and $\mathcal{W}_I = {}^tA(\mathcal{W}_{I'})$.*

Proof. Assume that A is an i-isomorphism from $(\mathfrak{g},B)$ onto $(\mathfrak{g}',B')$ then

$$B'([A(X),A(Y)],A(Z)) = B'(A[X,Y],A(Z)) = B([X,Y],Z),\ \forall\, X,Y,Z \in \mathfrak{g}.$$

That means $I = {}^tA(I')$.

Let $\alpha' \in \mathcal{V}_{I'}$ then $\alpha' \wedge I' = 0$. So ${}^tA(\alpha' \wedge I') = {}^tA(\alpha') \wedge {}^tA(I') = 0$. It means that ${}^tA(\mathcal{V}_{I'}) \subset \mathcal{V}_I$. Similarly, ${}^tA^{-1}(\mathcal{V}_I) \subset \mathcal{V}_{I'}$. Therefore, $\mathcal{V}_I = {}^tA(\mathcal{V}_{I'})$.

For all $X \in \mathfrak{g},\ Y,Z \in \mathfrak{g}'$ one has $\iota_{A(X)}(I')(Y,Z) = \iota_X(I)(A^{-1}(Y),A^{-1}(Z))$. Therefore, the restriction of A to the subspace $\{v \in \mathfrak{g} \mid \iota_v(I) = 0\}$ is an i-isomorphism from $\{v \in \mathfrak{g} \mid \iota_v(I) = 0\}$ onto $\{v \in \mathfrak{g}' \mid \iota_v(I') = 0\}$ then $\mathcal{W}_I = {}^tA(\mathcal{W}_{I'})$. □

It results from the previous lemma that $\dim(\mathcal{V}_I)$ and $\dim(\mathcal{W}_I)$ are invariant under i-isomorphisms. This is not new for $\dim(\mathcal{W}_I)$ since $\dim(\mathcal{W}_I) = \dim([\mathfrak{g},\mathfrak{g}])$. Actually, $\dim(\mathcal{W}_I)$ is invariant under isomorphisms.

For $\dim(\mathcal{V}_I)$, to our knowledge this fact was not remarked up to now, so we introduce the following definition:

Definition 2.2.9. Let $\mathfrak{g}$ be a quadratic Lie algebra. The dup *number* $\mathrm{dup}(\mathfrak{g})$ is defined by

$$\mathrm{dup}(\mathfrak{g}) = \dim(\mathcal{V}_I).$$

Remark 2.2.10. By Corollary 2.2.6, when $\mathfrak{g}$ is non-Abelian, one has $\mathrm{dup}(\mathfrak{g}) \in \{0,1,3\}$ and $\dim([\mathfrak{g},\mathfrak{g}]) \geq 3$. Moreover, I is decomposable if and only if $\mathrm{dup}(\mathfrak{g}) = \dim([\mathfrak{g},\mathfrak{g}]) = 3$, a simple but rather interesting remark. Finally, if $\mathfrak{g}$ is decomposed by $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ as in Proposition 2.1.5 then $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{l})$ since $I \in \bigwedge^3(\mathcal{W}_I)$ and $\mathcal{W}_I = \phi([\mathfrak{g},\mathfrak{g}]) = \phi([\mathfrak{l},\mathfrak{l}])$.

We separate non-Abelian quadratic Lie algebras as follows:

Definition 2.2.11. Let $\mathfrak{g}$ be a non-Abelian quadratic Lie algebra.

(1) $\mathfrak{g}$ is an *ordinary* quadratic Lie algebra if $\mathrm{dup}(\mathfrak{g}) = 0$.

(2) $\mathfrak{g}$ is a *singular* quadratic Lie algebra if $\mathrm{dup}(\mathfrak{g}) \geq 1$.

   (i) $\mathfrak{g}$ is a *singular* quadratic Lie algebra of *type* S_1 if $\mathrm{dup}(\mathfrak{g}) = 1$.

   (ii) $\mathfrak{g}$ is a *singular* quadratic Lie algebra of *type* S_3 if $\mathrm{dup}(\mathfrak{g}) = 3$.

Now, given a non-Abelian n-dimensional quadratic Lie algebra $\mathfrak{g}$, we can assume, up to i-isomorphisms, that $\mathfrak{g}$ is regarded as the quadratic vector space $\mathbb{C}^n$ equipped with its canonical bilinear form B. Our problem is considering Lie algebra structures on $\mathfrak{g}$ such that B is invariant. So we introduce the following sets:

Definition 2.2.12. For $n \geq 1$:

(1) $\mathcal{Q}(n)$ is the set of non-Abelian quadratic Lie algebra structures on $\mathbb{C}^n$.

(2) $\mathcal{O}(n)$ is the set of *ordinary* quadratic Lie algebra structures on $\mathbb{C}^n$.

(3) $\mathcal{S}(n)$ is the set of *singular* quadratic Lie algebra structures on $\mathbb{C}^n$.

By [PU07], there is a one-to-one map from $\mathcal{Q}(n)$ onto the subset

$$\left\{I \in \mathscr{A}^3(\mathbb{C}^n) \mid I \neq 0, \{I,I\} = 0\right\} \subset \mathscr{A}^3(\mathbb{C}^n).$$

In the sequel, we identify these two sets, so that $\mathcal{Q}(n) \subset \mathscr{A}^3(\mathbb{C}^n)$.

Theorem 2.2.13. *One has:*

(1) $\mathcal{Q}(n)$ is an affine variety in $\mathscr{A}^3(\mathbb{C}^n)$.

(2) $\mathcal{O}(n)$ is a Zariski-open subset of $\mathcal{Q}(n)$.

(3) $\mathcal{S}(n)$ is a Zariski-closed subset of $\mathcal{Q}(n)$.

Proof. The map $I \mapsto \{I,I\}$ is a polynomial map from $\mathscr{A}^3(\mathbb{C}^n)$ into $\mathscr{A}^4(\mathbb{C}^n)$, so the first claim follows.

Fix $I \in \mathscr{A}^3(\mathbb{C}^n)$ such that $\{I,I\} = 0$. Consider the map $\mathsf{m} : (\mathbb{C}^n)^* \to \mathscr{A}^4(\mathbb{C}^n)$ defined by $\mathsf{m}(\alpha) = \alpha \wedge I$, for all $\alpha \in (\mathbb{C}^n)^*$. Then, if $\mathfrak{g}$ is the quadratic Lie algebra associated to I, one has $\mathrm{dup}(\mathfrak{g}) = 0$ if and only if $\mathrm{rank}(\mathsf{m}) = n$. This can never happen for $n \leq 4$. Assume that $n \geq 5$. Let M be a matrix of m and Δ_i be the minors of order n, for $1 \leq i \leq \binom{n}{4}$. Then $\mathfrak{g} \in \mathcal{O}(n)$ if and only if there exists i such that $\Delta_i \neq 0$. But Δ_i is a polynomial function and from that the second and the third claims follow. □

Lemma 2.2.14. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be non-Abelian quadratic Lie algebras. Then $\mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$ is an ordinary quadratic Lie algebra.*

Proof. Set $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$. Denote by I, I_1 and I_2 the non-trivial 3-forms associated to $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

One has $\mathscr{A}(\mathfrak{g}) = \mathscr{A}(\mathfrak{g}_1) \otimes \mathscr{A}(\mathfrak{g}_2)$, $\mathscr{A}^k(\mathfrak{g}) = \bigoplus_{r+s=k} \mathscr{A}^r(\mathfrak{g}_1) \otimes \mathscr{A}^s(\mathfrak{g}_2)$ and $I = I_1 + I_2$, with $I_1 \in \mathscr{A}^3(\mathfrak{g}_1)$ and $I_2 \in \mathscr{A}^3(\mathfrak{g}_2)$. It immediately results that for $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$, one has $\alpha \wedge I = 0$ if and only if $\alpha_1 = \alpha_2 = 0$. □

Proposition 2.2.15. *One has:*

(1) $\mathcal{Q}(n) \neq \emptyset$ if and only if $n \geq 3$.

(2) $\mathcal{O}(3) = \mathcal{O}(4) = \emptyset$ and $\mathcal{O}(n) \neq \emptyset$ if $n \geq 6$.

Proof. If $\mathfrak{g}$ is a non-Abelian quadratic Lie algebra, using Remark 2.2.10, one has $\dim([\mathfrak{g},\mathfrak{g}]) \geq 3$, so $\mathcal{Q}(n) = \emptyset$ if $n < 3$.

We shall now use some elementary quadratic Lie algebras given in Section 6 of [PU07] (see also in Proposition 2.2.29), i.e. those are quadratic Lie algebras such that their associated 3-form is decomposable. We denote these algebras by $\mathfrak{g}_i$, according to their dimension, i.e. $\dim(\mathfrak{g}_i) = i$, for $3 \leq i \leq 6$. Note that $\mathfrak{g}_3 = \mathfrak{o}(3)$, $\mathfrak{g}_4$, $\mathfrak{g}_5$ and $\mathfrak{g}_6$ are examples of elements of $\mathcal{Q}(3)$, $\mathcal{Q}(4)$, $\mathcal{Q}(5)$ and $\mathcal{Q}(6)$, respectively.

Consider

$$\mathfrak{g} = \overset{\perp}{\bigoplus_{3\leq i\leq 6}} (\overbrace{\mathfrak{g}_i \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} \mathfrak{g}_i}^{k_i \text{ times}}).$$

Then $\dim(\mathfrak{g}) = \sum_{i=3}^{6} ik_i$ and by Lemma 2.2.14, $\mathrm{dup}(\mathfrak{g}) = 0$, so we obtain $\mathcal{O}(n) \neq \emptyset$ if $n \geq 6$.

Finally, let $\mathfrak{g}$ be a non-Abelian quadratic Lie algebra of dimension 3 or 4 with associated 3-form I. Then I is decomposable, so $\mathfrak{g}$ is singular. Therefore $\mathcal{O}(3) = \mathcal{O}(4) = \emptyset$.

□

Remark 2.2.16. We shall prove in Appendix B and Appendix C by two different ways that $\mathcal{O}(5) = \emptyset$. So, generically a non-Abelian quadratic Lie algebra is ordinary if $n \geq 6$.

Definition 2.2.17. A quadratic Lie algebra $\mathfrak{g}$ is *indecomposable* if $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, implies $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$.

The proposition below gives another characterization of reduced singular quadratic Lie algebras.

Proposition 2.2.18. *Let* $\mathfrak{g}$ *be a singular quadratic Lie algebra. Then* $\mathfrak{g}$ *is reduced if and only if* $\mathfrak{g}$ *is indecomposable.*

Proof. If $\mathfrak{g}$ is indecomposable, by Proposition 2.1.5, $\mathfrak{g}$ is reduced. If $\mathfrak{g}$ is reduced and $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, then $\mathcal{Z}(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$ for $i = 1, 2$. So $\mathfrak{g}_i$ is reduced or $\mathfrak{g}_i = \{0\}$. But if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are both reduced, by Lemma 2.2.14, one has $\mathrm{dup}(\mathfrak{g}) = 0$. Hence $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$. □

2.2.3 Quadratic Lie algebras of type S_1

Let $(\mathfrak{g}, B)$ be a quadratic vector space and I be a non-zero 3-form in $\mathscr{A}^3(\mathfrak{g})$. As in Subsection 2.2.1, we define a Lie bracket on $\mathfrak{g}$ by:

$$[X,Y] = \phi^{-1}(\iota_{X\wedge Y}(I)), \ \forall\, X, Y \in \mathfrak{g}.$$

Then $\mathfrak{g}$ becomes a quadratic Lie algebra with an invariant bilinear form B if and only if $\{I, I\} = 0$ [PU07].

In the sequel, we assume that $\dim(\mathcal{V}_I) = 1$. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega \in \mathscr{A}^2(\mathfrak{g})$ such that $I = \alpha \wedge \Omega$ as follows: let $\{\alpha, \alpha_1, \dots, \alpha_r\}$ be a basis of $\mathcal{W}_I$. Then, $I \in \bigwedge^3(\mathcal{W}_I)$ by Proposition 2.2.3. We set:

$$X_0 = \phi^{-1}(\alpha) \ \text{ and } X_i = \phi^{-1}(\alpha_i), \ 1 \leq i \leq r.$$

So, we can choose $\Omega \in \mathscr{A}^2(V)$ where $V = \mathrm{span}\{X_1, \dots, X_r\}$. Note that Ω is an indecomposable bilinear form, so $\dim(V) > 3$.

We define $C : \mathfrak{g} \to \mathfrak{g}$ by

$$B(C(X), Y) = \Omega(X, Y).$$

Therefore C is skew-symmetric with respect to B.

Lemma 2.2.19. *The following assertions are equivalent:*

(1) $\{I,I\} = 0$

(2) $\{\alpha,\alpha\} = 0$ *and* $\{\alpha,\Omega\} = 0$

(3) $B(X_0,X_0) = 0$ *and* $C(X_0) = 0$

In this case, one has $\dim([\mathfrak{g},\mathfrak{g}]) > 4$, $\mathcal{Z}(\mathfrak{g}) \subset \ker(C)$, $\mathrm{Im}(C) \subset [\mathfrak{g},\mathfrak{g}]$ *and* $X_0 \in \mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$.

Proof. It is easy to see that:

$$\{I,I\} = 0 \Leftrightarrow \{\alpha,\alpha\} \wedge \Omega \wedge \Omega = 2I \wedge \{\alpha,\Omega\}.$$

If $\Omega \wedge \Omega = 0$, then Ω is decomposable and that is a contradiction since $\dim(\mathcal{V}_I) = 1$. So $\Omega \wedge \Omega \neq 0$.

If $\{\alpha,\alpha\} \neq 0$, then α divides $\Omega \wedge \Omega \in \mathscr{A}^4(V)$, another contradiction. That implies $\{\alpha,\alpha\} = 0 = B(X_0,X_0)$. It results that $\{\alpha,\Omega\} \in \mathcal{V}_I = \mathbb{C}\alpha$, hence $\{\alpha,\Omega\} = \lambda\alpha$ for some $\lambda \in \mathbb{C}$. But $\{\alpha,\Omega\}$ is an element of $\mathscr{A}^1(V)$, so λ must be zero and by Subsection 2.2.1, $\iota_{X_0}(\Omega) = 0$, therefore $C(X_0) = 0$. Moreover, since $\{\alpha,\alpha\} = \{\alpha,\Omega\} = 0$, using $I = \alpha \wedge \Omega$, we deduce that $\{\alpha,I\} = 0$. Again by Subsection 2.2.1, it results that $B(X_0,[X,Y]) = \{\alpha,I\}(X \wedge Y) = 0$, for all X, $Y \in \mathfrak{g}$. So $X_0 \in [\mathfrak{g},\mathfrak{g}]^\perp = \mathcal{Z}(\mathfrak{g})$. Also, $\mathcal{V}_I \subset \mathcal{W}_I$, so $X_0 = \phi^{-1}(\alpha) \in \phi^{-1}(\mathcal{W}_I) = [\mathfrak{g},\mathfrak{g}]$.

Write $\Omega = \sum_{i<j} a_{ij}\alpha_i \wedge \alpha_j$, with $a_{ij} \in \mathbb{C}$. Since $\mathcal{W}_I = \phi([\mathfrak{g},\mathfrak{g}])$ and $X_1,\dots,X_r \in [\mathfrak{g},\mathfrak{g}]$, we deduce that

$$C = \sum_{i<j} a_{ij}(\alpha_i \otimes X_j - \alpha_j \otimes X_i)$$

Hence $\mathrm{Im}(C) \subset [\mathfrak{g},\mathfrak{g}]$. Since C is skew-symmetric, one has $\ker(C) = \mathrm{Im}(C)^\perp$ and it follows $\mathcal{Z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp \subset \ker(C)$.

Finally, $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus V$ and since $\dim(V) > 3$, we conclude that $\dim([\mathfrak{g},\mathfrak{g}]) > 4$. □

Remark 2.2.20. It is important to notice that our choice of Ω such that $I = \alpha \wedge \Omega$ is not unique, it depends on the choice of V, so C is not uniquely defined. If we consider another vector space V' and $I = \alpha \wedge \Omega'$. Then $\Omega' = \Omega + \alpha \wedge \beta$ for some $\beta \in \mathfrak{g}^*$. Let $X_1 = \phi^{-1}(\beta)$ and let C' be the map associated to Ω'. By a straightforward computation, $C' = C + \alpha \otimes X_1 - \beta \otimes X_0$. Since $C'(X_0) = 0$, we must have $B(X_0,X_1) = 0$.

Lemma 2.2.21. *There exists* $Y_0 \in V^\perp$ *such that*

$$V^\perp = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0,\ B(Y_0,Y_0) = 0 \ \text{and} \ B(X_0,Y_0) = 1.$$

Moreover

$$C(Y_0) = 0.$$

Proof. One has $\phi^{-1}(\mathcal{W}_I) = [\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus V$, therefore $\mathcal{Z}(\mathfrak{g}) \subset V^\perp$ and $\dim(\mathcal{Z}(\mathfrak{g})) = \dim(\mathfrak{g}) - \dim([\mathfrak{g},\mathfrak{g}]) = \dim(V^\perp) - 1$. So there exists $Y \in V^\perp$ such that $V^\perp = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y$. Now, Y cannot be orthogonal to X_0, since it would be orthogonal to $[\mathfrak{g},\mathfrak{g}]$ and therefore an element of $\mathcal{Z}(\mathfrak{g})$. So we can assume that $B(X_0,Y) = 1$. Replace Y by $Y_0 = Y - \frac{1}{2}B(Y,Y)X_0$ to obtain $B(Y_0,Y_0) = 0$ (recall $B(X_0,X_0) = 0$).

By Lemma 2.2.19, $\mathrm{Im}(C) \subset V$ and that implies $B(Y_0,C(X)) = -B(C(Y_0),X) = 0$, for all $X \in \mathfrak{g}$. Then $C(Y_0) = 0$. □

Proposition 2.2.22. *We keep the previous notation and assumptions. Then:*

(1) $[X,Y] = B(X_0,X)C(Y) - B(X_0,Y)C(X) + B(C(X),Y)X_0$, *for all* X, $Y \in \mathfrak{g}$.

(2) $C = \mathrm{ad}(Y_0)$ *and* $\mathrm{rank}(C)$ *is even.*

(3) $\ker(C) = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$, $\mathrm{Im}(C) = V$ *and* $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus \mathrm{Im}(C)$.

(4) the Lie algebra $\mathfrak{g}$ *is solvable. Moreover,* $\mathfrak{g}$ *is nilpotent if and only if* C *is nilpotent.*

(5) the dimension of $[\mathfrak{g},\mathfrak{g}]$ *is greater than or equal to 5 and it is odd.*

Proof.

(1) For all $X,Y,Z \in \mathfrak{g}$ one has

$$B([X,Y],Z) = (\alpha \wedge \Omega)(X,Y,Z) = \alpha(X)\Omega(Y,Z) - \alpha(Y)\Omega(X,Z) + \alpha(Z)\Omega(X,Y)$$
$$= B(B(X_0,X)C(Y) - B(X_0,Y)C(X) + B(C(X),Y)X_0, Z).$$

Since B is non-degenerate then

$$[X,Y] = B(X_0,X)C(Y) - B(X_0,Y)C(X) + B(C(X),Y)X_0,\ \forall\, X,Y \in \mathfrak{g}.$$

(2) Set $X = Y_0$ in (1) and use Lemma 2.2.21 to show $C = \mathrm{ad}(Y_0)$. Since $C(\mathfrak{g}) = \mathrm{ad}(Y_0)(\mathfrak{g}) = \phi^{-1}(\mathrm{ad}^*(\mathfrak{g})(\phi(Y_0)))$, the rank of C is the dimension of the coadjoint orbit through $\phi(Y_0)$, so it is even (see also Appendix A).

(3) We may assume that $\mathfrak{g}$ is reduced. Then $\mathscr{Z}(\mathfrak{g})$ is totally isotropic and $\mathscr{Z}(\mathfrak{g}) \subset X_0^\perp$. Write $X_0^\perp = \mathscr{Z}(\mathfrak{g}) \oplus \mathfrak{h}$ with $\mathfrak{h}$ a complementary subspace of $\mathscr{Z}(\mathfrak{g})$. Therefore $\mathfrak{g} = \mathscr{Z}(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathbb{C}Y_0$ and for an element $X = Z + H + \lambda Y_0 \in \ker(C)$, we deduce $H \in \ker(C)$ by Lemmas 2.2.19 and 2.2.21.

But $B(X_0,H) = 0$, so using (1), $H \in \mathscr{Z}(\mathfrak{g})$. It results that $H = 0$. Then $\ker(C) = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$. In addition,

$$\dim(\mathrm{Im}(C)) = \dim(\mathfrak{h}) = \dim(X_0^\perp) - \dim(\mathscr{Z}(\mathfrak{g})) = \dim([\mathfrak{g},\mathfrak{g}]) - 1.$$

Our choice of V implies that $[\mathfrak{g},\mathfrak{g}] = \phi^{-1}(\mathscr{W}_I) = \mathbb{C}X_0 \oplus V$ and $\mathrm{Im}(C) \subset V$ (see the proof of Lemma 2.2.19). Therefore $\mathrm{Im}(C) = V$ and $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus \mathrm{Im}(C)$.

(4) Since $B(X_0, \mathrm{Im}(C)) = 0$, then $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = [\mathrm{Im}(C), \mathrm{Im}(C)] \subset \mathbb{C}X_0$. We conclude that $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is nilpotent, then $C = \mathrm{ad}(Y_0)$ is nilpotent. If C is nilpotent, using $\mathrm{Im}(C) \subset X_0^\perp$, we obtain by induction that $(\mathrm{ad}(X))^k(\mathfrak{g}) \subset \mathbb{C}X_0 \oplus \mathrm{Im}(C^k)$ for any $k \in \mathbb{N}$. So $\mathrm{ad}(X)$ is nilpotent, for all $X \in \mathfrak{g}$ and that implies $\mathfrak{g}$ nilpotent.

(5) Notice that $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus \mathrm{Im}(C)$ and $\mathrm{rank}(C)$ is even, so $\dim([\mathfrak{g},\mathfrak{g}])$ is odd. By Lemma 2.2.19, $\dim([\mathfrak{g},\mathfrak{g}]) \geq 5$.

□

Recall that C is not unique (see Remark 2.2.20) and it depends on the choice of V. Let

$$\mathfrak{a} = X_0^{\perp}/\mathbb{C}X_0.$$

We denote by $\widehat{X}$ the class of an element $X \in X_0^{\perp}$.

Proposition 2.2.23.
Keep the notation above. One has:

(1) the Lie algebra $\mathfrak{a}$ is Abelian.

(2) define

$$\widehat{B}(\widehat{X},\widehat{Y}) = B(X,Y),\ \forall\, X,Y \in X_0^{\perp}.$$

Then $\widehat{B}$ is a non-degenerate symmetric bilinear form on $\mathfrak{a}$.

(3) define

$$\widehat{C}(\widehat{X}) = C(X),\ \forall\, X \in X_0^{\perp}.$$

Then $\widehat{C} \in \mathscr{L}(\mathfrak{a})$ is a skew-symmetric map with $\operatorname{rank}(\widehat{C}) = \operatorname{rank}(C)$ even and $\operatorname{rank}(\widehat{C}) \geq 4$.

(4) $\widehat{C}$ does not depend on the choice of V. More precisely, if $\mathcal{W}_I = \mathbb{C}\alpha \oplus \phi(V')$ and C' is the associated map to V' (see Remark 2.2.20), then $\widehat{C'} = \widehat{C}$.

(5) the Lie algebra $\mathfrak{g}$ is reduced if and only if $\ker(\widehat{C}) \subset \operatorname{Im}(\widehat{C})$.

Proof.

(1) It follows from Proposition 2.2.22 (1).

(2) It is clear that $\widehat{B}$ is well-defined. Now, since $B(X_0,Y_0) = 1$, $B(X_0,X_0) = B(Y_0,Y_0) = 0$, the restriction of B to $\operatorname{span}\{X_0,Y_0\}$ is non-degenerate. So

$$\mathfrak{g} = \operatorname{span}\{X_0,Y_0\} \overset{\perp}{\oplus} \operatorname{span}\{X_0,Y_0\}^{\perp},$$

$X_0^{\perp} = \mathbb{C}X_0 \oplus \operatorname{span}\{X_0,Y_0\}^{\perp}$ and $X_0^{\perp\perp} = X_0^{\perp} \cap \operatorname{span}\{X_0,Y_0\} = \mathbb{C}X_0$. We conclude that $\widehat{B}$ is non degenerate.

(3) We have $C(X_0^{\perp}) = \operatorname{ad}(Y_0)(X_0^{\perp}) \subset X_0^{\perp}$ since $X_0^{\perp}$ is an ideal of $\mathfrak{g}$. Moreover, $C(X_0) = 0$, so $\widehat{C}$ is well-defined. The image of C is contained in $X_0^{\perp}$ and $\operatorname{Im}(C) \cap \mathbb{C}X_0 = \{0\}$, therefore $\dim(\operatorname{Im}(C)/\mathbb{C}X_0) = \dim(\operatorname{Im}(\widehat{C})) = \dim(\operatorname{Im}(C))$. Now it is enough to apply Proposition 2.2.22.

(4) By Remark 2.2.20, we have $C' = C + \alpha \otimes X_1 - \beta \otimes X_0$. But $\alpha(X_0) = 0$, so $\widehat{C'} = \widehat{C}$.

(5) By Proposition 2.2.22, we have $\ker(C) = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$ and by Lemma 2.2.19, we have $\mathscr{Z}(\mathfrak{g}) \subset X_0^{\perp}$. Again by Proposition 2.2.22, we conclude that $\ker(\widehat{C}) = \mathscr{Z}(\mathfrak{g})/\mathbb{C}X_0$. Applying Proposition 2.2.22 once more, we have $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_0 \oplus \mathrm{Im}(C)$, so $\mathrm{Im}(\widehat{C}) = [\mathfrak{g},\mathfrak{g}]/\mathbb{C}X_0$. Then $\ker(\widehat{C}) \subset \mathrm{Im}(\widehat{C})$ if and only if $\mathscr{Z}(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}] + \mathbb{C}X_0$. But $X_0 \in [\mathfrak{g},\mathfrak{g}]$ (see Lemma 2.2.19), so the result follows.

□

We should notice that $\widehat{C}$ still depends on the choice of α (see Remark 2.2.20): if we replace α by $\lambda\alpha$, for a non-zero $\lambda \in \mathbb{C}$, that will change $\widehat{C}$ into $\frac{1}{\lambda}\widehat{C}$. So there is not a *unique* map $\widehat{C}$ associated to $\mathfrak{g}$ but rather a *family* $\{\lambda\widehat{C} \mid \lambda \in \mathbb{C}\setminus\{0\}\}$ of associated maps. In other words, there is a line

$$[\widehat{C}] = \{\lambda\widehat{C} \mid \lambda \in \mathbb{C}\} \in \mathbb{P}^1(\mathfrak{o}(\mathfrak{a}))$$

where $\mathbb{P}^1(\mathfrak{o}(\mathfrak{a}))$ is the projective space associated to the space $\mathfrak{o}(\mathfrak{a})$.

Definition 2.2.24. We call $[\widehat{C}]$ the *line of skew-symmetric maps* associated to the quadratic Lie algebra $\mathfrak{g}$ of type S_1.

Remark 2.2.25. The unicity of $[\widehat{C}]$ is valuable, but the fact that $\widehat{C}$ acts on a quotient space and not on a subspace of $\mathfrak{g}$ could be a problem. Hence it is convenient to use the following decomposition of $\mathfrak{g}$: the restriction of B to $\mathbb{C}X_0 \oplus \mathbb{C}Y_0$ is non-degenerate, so we can write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ where $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^{\perp}$. Since $C(X_0) = C(Y_0) = 0$ and $C \in \mathfrak{o}(\mathfrak{g})$, C maps $\mathfrak{q}$ into $\mathfrak{q}$. Let $\pi : X_0^{\perp} \to X_0^{\perp}/\mathbb{C}X_0$ be the canonical surjection and $\overline{C} = C|_{\mathfrak{q}}$. Then the restriction $\pi_{\mathfrak{q}} : \mathfrak{q} \to X_0^{\perp}/\mathbb{C}X_0$ is an isometry and $\widehat{C} = \pi_{\mathfrak{q}}\, \overline{C}\, \pi_{\mathfrak{q}}^{-1}$.

Remark that Y_0 is not unique, but if Y_0' satisfies Lemma 2.2.21, consider $C' = \mathrm{ad}(Y_0')$ and $\mathfrak{q}'$ such that $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0') \overset{\perp}{\oplus} \mathfrak{q}'$, therefore $\widehat{C} = \pi_{\mathfrak{q}}'\, \overline{C'}\, \pi_{\mathfrak{q}}'^{-1}$ with the obvious notation. It results that $\pi_{\mathfrak{q}}'^{-1}\pi_{\mathfrak{q}}$ is an isometry from $\mathfrak{q}$ to $\mathfrak{q}'$ and that

$$\overline{C'} = \left(\pi_{\mathfrak{q}}'^{-1}\pi_{\mathfrak{q}}\right)\overline{C}\left(\pi_{\mathfrak{q}}'^{-1}\pi_{\mathfrak{q}}\right)^{-1}.$$

We shall develop this aspect in the next Section.

2.2.4 Solvable singular quadratic Lie algebras and double extensions

In this subsection, we will apply Definition 2.1.9 for a particular case that is the double extension of a quadratic vector space by a skew-symmetric map as follows:

Definition 2.2.26.

(1) Let $(\mathfrak{q}, B_{\mathfrak{q}})$ be a quadratic vector space and $\overline{C} : \mathfrak{q} \to \mathfrak{q}$ be a skew-symmetric map. Let $(\mathfrak{t} = \mathrm{span}\{X_1, Y_1\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vector space with $B_{\mathfrak{t}}$ defined by

$$B_{\mathfrak{t}}(X_1,X_1) = B_{\mathfrak{t}}(Y_1,Y_1) = 0,\ B_{\mathfrak{t}}(X_1,Y_1) = 1.$$

Consider

$$\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with a bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ and define a bracket on $\mathfrak{g}$ by

$$[X + \lambda X_1 + \mu Y_1, Y + \lambda' X_1 + \mu' Y_1] = \mu \overline{C}(Y) - \mu' \overline{C}(X) + B(\overline{C}(X), Y) X_1,$$

for all $X, Y \in \mathfrak{q}$, $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$. Then $(\mathfrak{g}, B)$ is a quadratic solvable Lie algebra. We say that $\mathfrak{g}$ is the *double extension* of $\mathfrak{q}$ by $\overline{C}$.

(2) Let $\mathfrak{g}_i$ be double extensions of quadratic vector spaces $(\mathfrak{q}_i, B_i)$ by skew-symmetric maps $\overline{C}_i \in \mathscr{L}(\mathfrak{q}_i)$, for $1 \leq i \leq k$. The *amalgamated product*

$$\mathfrak{g} = \mathfrak{g}_1 \underset{\mathrm{a}}{\times} \mathfrak{g}_2 \underset{\mathrm{a}}{\times} \dots \underset{\mathrm{a}}{\times} \mathfrak{g}_k$$

is defined as follows:

- consider $(\mathfrak{q}, B)$ be the quadratic vector space with $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \cdots \oplus \mathfrak{q}_k$ and the bilinear form B such that $B(\sum_{i=I}^k X_i, \sum_{i=I}^k Y_i) = \sum_{i=I}^k B_i(X_i, Y_i)$, for $X_i, Y_i \in \mathfrak{q}_i$, $1 \leq i \leq k$.
- the skew-symmetric map $\overline{C} \in \mathscr{L}(\mathfrak{q})$ is defined by $\overline{C}(\sum_{i=I}^k X_i) = \sum_{i=I}^k \overline{C}_i(X_i)$, for $X_i \in \mathfrak{q}_i$, $1 \leq i \leq k$.

Then $\mathfrak{g}$ is the double extension of $\mathfrak{q}$ by $\overline{C}$.

Next, we will show that double extensions are highly related to solvable singular quadratic Lie algebras and amalgamated products can be used to *decompose* double extensions that are useful in the nilpotent case. However, we notice here that generally this decomposition is a bad behavior with respect to i-isomorphisms as follows: if $\mathfrak{g}_1 \overset{\mathrm{i}}{\simeq} \mathfrak{g}_1'$ and $\mathfrak{g}_2 \overset{\mathrm{i}}{\simeq} \mathfrak{g}_2'$, it may happen that $\mathfrak{g}_1 \underset{\mathrm{a}}{\times} \mathfrak{g}_2$ and $\mathfrak{g}_1' \underset{\mathrm{a}}{\times} \mathfrak{g}_2'$ are not even isomorphic. An example will be given in Remark 2.2.45.

Lemma 2.2.27. *We keep the notation above.*

(1) Let $\mathfrak{g}$ be the double extension of $\mathfrak{q}$ by $\overline{C}$. Then

$$[X, Y] = B(X_1, X) C(Y) - B(X_1, Y) C(X) + B(C(X), Y) X_1, \ \forall\, X, Y \in \mathfrak{g},$$

where $C = \mathrm{ad}(Y_1)$. Moreover, $X_1 \in \mathscr{Z}(\mathfrak{g})$ and $C|_{\mathfrak{q}} = \overline{C}$.

(2) Let $\mathfrak{g}'$ be the double extension of $\mathfrak{q}$ by $\overline{C'} = \lambda \overline{C}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic.

Proof.

(1) This is a straightforward computation from the previous definition.

(2) Write $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t} = \mathfrak{g}'$. Denote by $[\cdot,\cdot]'$ the Lie bracket on $\mathfrak{g}'$. Define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(X_1) = \lambda X_1$, $A(Y_1) = \dfrac{1}{\lambda} Y_1$ and $A|_{\mathfrak{q}} = \mathrm{Id}_{\mathfrak{q}}$. Then $A([Y_1, X]) = C(X) = [A(Y_1), A(X)]'$ and $A([X, Y]) = [A(X), A(Y)]'$, for all $X, Y \in \mathfrak{q}$. So A is an i-isomorphism.

□

A natural consequence of formulas in Proposition 2.2.22 (1) and Lemma 2.2.27 is given by the proposition below:

Proposition 2.2.28.

(1) Consider the notation in Subection 2.2.3, Remark 2.2.25. Let $\mathfrak{g}$ be a singular quadratic Lie algebra of type S_1 (that is, $\mathrm{dup}(\mathfrak{g}) = 1$). Then $\mathfrak{g}$ is the double extension of $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^\perp$ by $\overline{C} = \mathrm{ad}(Y_0)|_{\mathfrak{q}}$.

(2) Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Let $\mathfrak{g}'$ be the double extension of a quadratic vector space $(\mathfrak{q}', B')$ by a map $\overline{C'}$. Let A be an i-isomorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$ and write $\mathfrak{q} = A(\mathfrak{q}')$. Then $\mathfrak{g}$ is the double extension of $(\mathfrak{q}, B|_{\mathfrak{q}\times\mathfrak{q}})$ by the map $\overline{C} = \overline{A}\,\overline{C'}\,\overline{A}^{-1}$ where $\overline{A} = A|_{\mathfrak{q}'}$.

(3) Let $\mathfrak{g}$ be the double extension of a quadratic vector space $\mathfrak{q}$ by a map $\overline{C} \neq 0$. Then $\mathfrak{g}$ is a solvable singular quadratic Lie algebra. Moreover:

- *(i) $\mathfrak{g}$ is of type S_3 if and only if $\mathrm{rank}(\overline{C}) = 2$.*
- *(ii) $\mathfrak{g}$ is of type S_1 if and only if $\mathrm{rank}(\overline{C}) \geq 4$.*
- *(iii) $\mathfrak{g}$ is reduced if and only if $\ker(\overline{C}) \subset \mathrm{Im}(\overline{C})$.*
- *(iv) $\mathfrak{g}$ is nilpotent if and only if $\overline{C}$ is nilpotent.*

Proof.

(1) Let $\mathfrak{b} = \mathbb{C}X_0 \oplus \mathbb{C}Y_0$. Then $B|_{\mathfrak{b}\times\mathfrak{b}}$ is non-degenerate and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{q}$. Since $\mathrm{ad}(Y_0)(\mathfrak{b}) \subset \mathfrak{b}$ and $\mathrm{ad}(Y_0)$ is skew-symmetric, we have $\mathrm{ad}(Y_0)(\mathfrak{q}) \subset \mathfrak{q}$. By Proposition 2.2.22 (1), we have

$$[X, X'] = B(\overline{C}(X), X')X_0, \ \forall\, X, X' \in \mathfrak{q}.$$

Set $X_1 = X_0$ and $Y_1 = Y_0$ to obtain the result.

(2) Write $\mathfrak{g}' = (\mathbb{C}X_1' \oplus \mathbb{C}Y_1') \overset{\perp}{\oplus} \mathfrak{q}'$. Let $X_1 = A(X_1')$ and $Y_1 = A(Y_1')$. Then $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ since A is i-isomorphic. One has:

$$[Y_1, X] = A[Y_1', A^{-1}(X)] = (A\overline{C'}A^{-1})(X), \ \forall\, X \in \mathfrak{q}, \text{ and}$$

$$[X, Y] = A[A^{-1}(X), A^{-1}(Y)] = B((A\overline{C'}A^{-1})(X), Y)X_1, \ \forall\, X, Y \in \mathfrak{q}.$$

Hence, this proves the result.

(3) Let $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$, $C = \mathrm{ad}(Y_1)$, $\alpha = \phi(X_1)$, $\Omega(X, Y) = B(C(X), Y)$, for all $X, Y \in \mathfrak{g}$ and I be the 3-form associated to $\mathfrak{g}$. Then the formula for the Lie bracket in Lemma 2.2.27 (1) can be translated as $I = \alpha \wedge \Omega$, hence $\mathrm{dup}(\mathfrak{g}) \geq 1$ and $\mathfrak{g}$ is singular.

Let W_Ω be the set $W_\Omega = \{\iota_X(\Omega) = \phi(\overline{C}(X)), X \in \mathfrak{g}\}$ then $W_\Omega = \phi(\mathrm{Im}(\overline{C}))$. Therefore $\mathrm{rank}(\overline{C}) \geq 2$ by Proposition 2.2.3 and Ω is decomposable if and only if $\mathrm{rank}(\overline{C}) = 2$.

If $\operatorname{rank}(\overline{C}) > 2$, then $\mathfrak{g}$ is of type S_1 and by Proposition 2.2.23, we have $\operatorname{rank}(\overline{C}) \geq 4$.

Finally, $\mathscr{Z}(\mathfrak{g}) = \mathbb{C}X_1 \oplus \ker(\overline{C})$ and $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_1 \oplus \operatorname{Im}(\overline{C})$, so $\mathfrak{g}$ is reduced if and only if $\ker(\overline{C}) \subset \operatorname{Im}(\overline{C})$.

The proof of the last claim is exactly the same as in Proposition 2.2.22 (4).

□

A complete classification (up to i-isomorphisms) of quadratic Lie algebras of type S_3 is given in [PU07] by applying the formula $\{I,I\} = 0$ for the case I decomposable. We shall recall the characterization of these algebras here and describe them in terms of double extensions:

Proposition 2.2.29. *Let $\mathfrak{g}$ be a quadratic Lie algebra of type S_3. Then $\mathfrak{g}$ is i-isomorphic to an algebra $\mathfrak{l} \overset{\perp}{\oplus} \mathfrak{z}$ where $\mathfrak{z}$ is a central ideal of $\mathfrak{g}$ and $\mathfrak{l}$ is one of the following algebras:*

(1) $\mathfrak{g}_3(\lambda) = \mathfrak{o}(3)$ *equipped with the bilinear form $B = \lambda\kappa$ where κ is the Killing form and $\lambda \in \mathbb{C}$, $\lambda \neq 0$.*

(2) $\mathfrak{g}_4$*, a 4-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^2$, $\{E_1,E_2\}$ its canonical basis and the bilinear form B defined by $B(E_1,E_1) = B(E_2,E_2) = 0$ and $B(E_1,E_2) = 1$. Then $\mathfrak{g}_4$ is the double extension of $\mathfrak{q}$ by the skew-symmetric map*

$$\overline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, $\mathfrak{g}_4$ is solvable, but it is not nilpotent. The Lie algebra $\mathfrak{g}_4$ is known in the literature as the diamond algebra (see for instance [Dix74]*).*

(3) $\mathfrak{g}_5$*, a 5-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^3$, $\{E_1,E_2,E_3\}$ its canonical basis and the bilinear form B defined by $B(E_1,E_1) = B(E_3,E_3) = B(E_1,E_2) = B(E_2,E_3) = 0$ and $B(E_1,E_3) = B(E_2,E_2) = 1$. Then $\mathfrak{g}_5$ is the double extension of $\mathfrak{q}$ by the skew-symmetric map*

$$\overline{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, $\mathfrak{g}_5$ is nilpotent.

(4) $\mathfrak{g}_6$*, a 6-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^4$, $\{E_1,E_2,E_3,E_4\}$ its canonical basis and the bilinear form B defined by $B(E_1,E_3) = B(E_2,E_4) = 1$ and $B(E_i,E_j) = 0$ otherwise. Then $\mathfrak{g}_6$ is the double extension of $\mathfrak{q}$ by the skew-symmetric map*

$$\overline{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Moreover, $\mathfrak{g}_6$ is nilpotent.

Therefore, we can conclude that ***all solvable quadratic Lie algebras of type*** S_3 ***are double extensions of a quadratic vector space by a skew-symmetric map.***

We remark that in the nilpotent Lie algebras classification, the Lie algebras $\mathfrak{g}_5$ and $\mathfrak{g}_6$ can be identified respectively as $\mathfrak{g}_{5,4}$ and $\mathfrak{g}_{6,4}$, for more details the reader should refer to [Oom09] and [Mag10].

2.2.5 Classification singular quadratic Lie algebras

Let $(\mathfrak{q}, B)$ be a quadratic vector space. We recall the notations in Chapter 1 that $\mathrm{O}(\mathfrak{q})$ is the group of orthogonal maps and $\mathfrak{o}(\mathfrak{q})$ is its Lie algebra, i.e. the Lie algebra of skew-symmetric maps. Also, recall that the *adjoint action* is the action of $\mathrm{O}(\mathfrak{q})$ on $\mathfrak{o}(\mathfrak{q})$ by conjugation.

Theorem 2.2.30. *Let* $(\mathfrak{q}, B)$ *be a quadratic vector space. Let* $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ *and* $\mathfrak{g}' = (\mathbb{C}X_1' \oplus \mathbb{C}Y_1') \overset{\perp}{\oplus} \mathfrak{q}$ *be double extensions of* $\mathfrak{q}$*, by nonzero skew-symmetric maps* $\overline{C}$ *and* $\overline{C'}$ *respectively. Then:*

(1) there exists a Lie algebra isomorphism between $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if there exist an invertible map* $P \in \mathscr{L}(\mathfrak{q})$ *and a non-zero* $\lambda \in \mathbb{C}$ *such that* $\overline{C'} = \lambda\, P\overline{C}P^{-1}$ *and* $P^*P\overline{C} = \overline{C}$ *where* P^* *is the adjoint map of* P *with respect to* B*.*

(2) there exists an i-isomorphism between $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if* $\overline{C'}$ *is in the* $\mathrm{O}(\mathfrak{q})$*-adjoint orbit through* $\lambda\overline{C}$ *for some non-zero* $\lambda \in \mathbb{C}$*.*

Proof.

(1) Let $A : \mathfrak{g} \to \mathfrak{g}'$ be a Lie algebra isomorphism. We know by Proposition 2.2.28 that $\mathfrak{g}$ and $\mathfrak{g}'$ are singular. Assume that $\mathfrak{g}$ is of type S_3. Then $3 = \dim([\mathfrak{g},\mathfrak{g}]) = \dim([\mathfrak{g}',\mathfrak{g}'])$. So $\mathfrak{g}'$ is also of type S_3 ([PU07]). Therefore, $\mathfrak{g}$ and $\mathfrak{g}'$ are either both of type S_1 or both of type S_3. Let us study these two cases.

(i) First, assume that $\mathfrak{g}$ and $\mathfrak{g}'$ are both of type S_1. We start by proving that $A(\mathbb{C}X_1 \oplus \mathfrak{q}) = \mathbb{C}X_1' \oplus \mathfrak{q}$. If this is not the case, there is $X \in \mathfrak{q}$ such that $A(X) = \beta X_1' + \gamma Y_1' + Y$ with $Y \in \mathfrak{q}$ and $\gamma \neq 0$. Then

$$[A(X), \mathbb{C}X_1' \oplus \mathfrak{q}]' = \gamma\overline{C'}(\mathfrak{q}) + [Y, \mathfrak{q}]'.$$

Since $\mathfrak{g}'$ is of type S_1, we have $\operatorname{rank}(\overline{C}') \geq 4$ (see Proposition 2.2.28) and it follows that $\dim([A(X), \mathbb{C}X_1' \oplus \mathfrak{q}]') \geq 4$. On the other hand, $[A(X), \mathbb{C}X_1' \oplus \mathfrak{q}]'$ is contained in $A([X,\mathfrak{g}])$ and $\dim([X,\mathfrak{g}]) \leq 2$, so we obtain a contradiction.
Next, we prove that $A(X_1) \in \mathbb{C}X_1'$. By the definition of a double extension and B non-degenerate, then there exist X, $Y \in \mathfrak{q}$ such that $X_1 = [X,Y]$. Then $A(X_1) = [A(X), A(Y)]' \in [\mathbb{C}X_1' \oplus \mathfrak{q}, \mathbb{C}X_1' \oplus \mathfrak{q}]' = \mathbb{C}X_1'$. Hence $A(X_1) = \mu X_1'$ for some non-zero $\mu \in \mathbb{C}$.
Now, write $A|_{\mathfrak{q}} = P + \beta \otimes X_1'$ with $P : \mathfrak{q} \to \mathfrak{q}$ and $\beta \in \mathfrak{q}^*$. If $X \in \ker(P)$, then $A\left(X - \dfrac{1}{\mu}\beta(X)X_1\right) = 0$, so $X = 0$ and therefore, P is invertible.

For all X, $Y \in \mathfrak{q}$, we have $A([X,Y]) = \mu B(\overline{C}(X),Y)X_1'$. Also,

$$\begin{aligned} A([X,Y]) &= [P(X)+\beta(X)X_1', P(Y)+\beta(Y)X_1']' \\ &= B(\overline{C'}P(X),P(Y))X_1'. \end{aligned}$$

So it results that $P^*\overline{C'}P = \mu\overline{C}$.

Moreover, $A([Y_1,X]) = P(C(X)+\beta(C(X))X_1'$, for all $X \in \mathfrak{q}$. Let $A(Y_1) = \gamma Y_1' + Y + \delta X_1'$, with $Y \in \mathfrak{q}$. Therefore

$$A([Y_1,X]) = \gamma\overline{C'}P(X) + B(\overline{C'}(Y),P(X))X_1'$$

and we conclude that $P\,\overline{C}\,P^{-1} = \gamma\overline{C'}$ and since $P^*\overline{C'}P = \mu\overline{C}$, then $P^*P\overline{C} = \gamma\mu\overline{C}$.

Set $Q = \dfrac{1}{(\mu\gamma)^{\frac{1}{2}}}P$. It follows that $Q\overline{C}Q^{-1} = \gamma\overline{C'}$ and $Q^*Q\overline{C} = \overline{C}$. This finishes the proof in the case $\mathfrak{g}$ and $\mathfrak{g}'$ of type S_1.

(ii) We proceed to the case when $\mathfrak{g}$ and $\mathfrak{g}'$ of type S_3: the proof is a straightforward case-by-case verification. By Proposition 2.2.5, we can assume that $\mathfrak{g}$ and $\mathfrak{g}'$ are reduced. Then $\dim(\mathfrak{q}) = 2,3$ or 4 by Proposition 2.2.29.

Recall that $\mathfrak{g}$ is nilpotent if and only if $\overline{C}$ is nilpotent (see Proposition 2.2.28 (3)). The same is valid for $\mathfrak{g}'$.

If $\dim(\mathfrak{q}) = 2$, then $\mathfrak{g}$ is not nilpotent, so $\overline{C}$ is not nilpotent, $\mathrm{Tr}(\overline{C}) = 0$ and $\overline{C}$ must be semisimple. Therefore we can find a basis $\{e_1,e_2\}$ of $\mathfrak{q}$ such that $B(e_1,e_2) = 1$, $B(e_1,e_1) = B(e_2,e_2) = 0$ and the matrix of $\overline{C}$ is $\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$. The same holds for $\overline{C'}$: there exists a basis $\{e_1',e_2'\}$ of $\mathfrak{q}$ such that $B(e_1',e_2') = 1$ and $B(e_1',e_1)' = B(e_2',e_2') = 0$ such that the matrix of $\overline{C'}$ is $\begin{pmatrix} \mu' & 0 \\ 0 & -\mu' \end{pmatrix}$. It results that $\overline{C'}$ and $\dfrac{\mu'}{\mu}\overline{C}$ are $\mathrm{O}(\mathfrak{q})$-conjugate and we are done.

If $\dim(\mathfrak{q}) = 3$ or 4, then $\mathfrak{g}$ and $\mathfrak{g}'$ are nilpotent. We use the classification of nilpotent orbits given for instance in Chapter 1: there is only one non-zero orbit in dimension 3 or 4 (corresponding to only one partition different from $[1^3]$ or $[1^4]$), so $\overline{C}$ and $\overline{C'}$ are conjugate by $\mathrm{O}(\mathfrak{q})$.

This finishes the proof of the necessary condition. To prove the sufficiency, we replace $\overline{C'}$ by $\lambda P\overline{C}P^{-1}$ to obtain $P^*\overline{C'}P = \lambda\overline{C}$. Then we define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(X_1) = \lambda X_1'$, $A(Y_1) = \dfrac{1}{\lambda}Y_1'$ and $A(X) = P(X)$, for all $X \in \mathfrak{q}$. By a direct computation, we have for all X and $Y \in \mathfrak{q}$:

$$A([X,Y]) = [A(X),A(Y)]' \text{ and } A([Y_1,X]) = [A(Y_1),A(X)]',$$

so A is a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$.

(2) If $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic, then the isomorphism A in the proof of (1) is an isometry. Hence $P \in \mathrm{O}(\mathfrak{q})$ and $P\overline{C'}P^{-1} = \mu\overline{C}$ gives the result.

Conversely, define A as above (sufficiency of (1)). Then A is an isometry and it is easy to check that A is an i-isomorphism.

□

Corollary 2.2.31. *Let $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ be double extensions of $(\mathfrak{q},\overline{B})$ and $(\mathfrak{q}',\overline{B'})$ respectively, where $\overline{B}=B|_{\mathfrak{q}\times\mathfrak{q}}$ and $\overline{B'}=B'|_{\mathfrak{q}'\times\mathfrak{q}'}$. Write $\mathfrak{g}=(\mathbb{C}X_1\oplus\mathbb{C}Y_1)\overset{\perp}{\oplus}\mathfrak{q}$ and $\mathfrak{g}'=(\mathbb{C}X_1'\oplus\mathbb{C}Y_1')\overset{\perp}{\oplus}\mathfrak{q}'$. Then:*

(1) there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isometry $\overline{A}:\mathfrak{q}\to\mathfrak{q}'$ such that $\overline{C'}=\lambda\,\overline{A}\,\overline{C}\,\overline{A}^{-1}$, for some non-zero $\lambda\in\mathbb{C}$.

(2) there exists a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist invertible maps $\overline{Q}:\mathfrak{q}\to\mathfrak{q}'$ and $\overline{P}\in\mathscr{L}(\mathfrak{q})$ such that

(i) $\overline{C'}=\lambda\,\overline{Q}\,\overline{C}\,\overline{Q}^{-1}$ for some non-zero $\lambda\in\mathbb{C}$,

(ii) $\overline{P}^*\,\overline{P}\,\overline{C}=\overline{C}$ and

(iii) $\overline{Q}\,\overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ onto $\mathfrak{q}'$.

Proof.

(1) We can assume that $\dim(\mathfrak{g})=\dim(\mathfrak{g}')$. Define a map $F:\mathfrak{g}'\to\mathfrak{g}$ by $F(X_1')=X_1$, $F(Y_1')=Y_1$ and $\overline{F}=F|_{\mathfrak{q}'}$ is an isometry from $\mathfrak{q}'$ onto $\mathfrak{q}$. Then define a new Lie bracket on $\mathfrak{g}$ by

$$[X,Y]''=F\left([F^{-1}(X),F^{-1}(Y)]'\right),\ \forall\,X,Y\in\mathfrak{g}.$$

Denote by $(\mathfrak{g}'',[\cdot,\cdot]'')$ this new Lie algebra. So F is an i-isomorphism from $\mathfrak{g}'$ onto $\mathfrak{g}''$.

Moreover $\mathfrak{g}''=(\mathbb{C}X_1\oplus\mathbb{C}Y_1)\overset{\perp}{\oplus}\mathfrak{q}$ is the double extension of $\mathfrak{q}$ by $\overline{C}''$ with $\overline{C}''=\overline{F}\,\overline{C}'\,\overline{F}^{-1}$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{g}''$ are i-isomorphic. Applying Theorem 2.2.30, this is the case if and only if there exists $\overline{A}\in\mathrm{O}(\mathfrak{q})$ such that $\overline{C}''=\lambda\,\overline{A}\,\overline{C}\,\overline{A}^{-1}$ for some non-zero complex λ. That implies

$$\overline{C}'=\lambda\ (\overline{F}^{-1}\,\overline{A})\,\overline{C}\ (\overline{F}^{-1}\overline{A})^{-1}$$

and proves (1).

(2) We keep the notation in (1). We have that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic. Applying Theorem 2.2.30, $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic if and only if there exist an invertible map $\overline{P}\in\mathscr{L}(\mathfrak{q})$ and a non-zero $\lambda\in\mathbb{C}$ such that $\overline{C}''=\lambda\,\overline{P}\,\overline{C}\,\overline{P}^{-1}$ and $\overline{P}^*\,\overline{P}\,\overline{C}=\overline{C}$ and we conclude that $\overline{C}'=\lambda\,\overline{Q}\,\overline{C}\,\overline{Q}^{-1}$ with $\overline{Q}=\overline{F}^{-1}\,\overline{P}$. Finally, $\overline{F}^{-1}=\overline{Q}\,\overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ to $\mathfrak{q}'$.

On the other hand, if $\overline{C}'=\lambda\,\overline{Q}\,\overline{C}\,\overline{Q}^{-1}$ and $\overline{P}^*\,\overline{P}\,\overline{C}=\overline{C}$ with $\overline{P}=\overline{F}\,\overline{Q}$ for some isometry $\overline{F}:\mathfrak{q}'\to\mathfrak{q}$, then construct $\mathfrak{g}''$ as in (1). We deduce $\overline{C}''=\lambda\,\overline{P}\,\overline{C}\,\overline{P}^{-1}$ and $\overline{P}^*\,\overline{P}\,\overline{C}=\overline{C}$. So, by Theorem 2.2.30, $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic and therefore, $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

□

Remark 2.2.32. Let $\mathfrak{g}$ be a solvable singular quadratic Lie algebra. Consider $\mathfrak{g}$ as the double extension of two quadratic vectors spaces $\mathfrak{q}$ and $\mathfrak{q}'$:

$$\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q} \text{ and } \mathfrak{g} = (\mathbb{C}X_1' \oplus \mathbb{C}Y_1') \overset{\perp}{\oplus} \mathfrak{q}'.$$

Let $\overline{C} = \mathrm{ad}(Y_1)|_{\mathfrak{q}}$ and $\overline{C}' = \mathrm{ad}(Y_1')|_{\mathfrak{q}'}$ Since $\mathrm{Id}_{\mathfrak{g}}$ is obviously an i-isomorphism, there exist an isometry $\overline{A} : \mathfrak{q} \to \mathfrak{q}'$ and a non-zero $\lambda \in \mathbb{C}$ such that

$$\overline{C}' = \lambda\, \overline{A}\, \overline{C}\, \overline{A}^{-1}.$$

Remark 2.2.33. A weak form of Corollary 2.2.31 (1) was stated in [FS87], in the case of i-isomorphisms satisfying some (dispensable) conditions. So (1) is an improvement. To our knowledge, (2) is completely new. Corollary 2.2.31 and Remark 2.2.32 can be applied directly to solvable singular quadratic Lie algebras: by Proposition 2.2.28 and Proposition 2.2.29, they are double extensions of quadratic vector spaces by skew-symmetric maps.

Apply results in Chapter 1, we shall now classify solvable singular quadratic Lie algebra structures on $\mathbb{C}^{n+2}$ up to i-isomorphisms in terms of $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. We need the lemma below.

Lemma 2.2.34. *Let V be a quadratic vector space such that $V = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}'$ with X_1, Y_1 isotropic elements and $B(X_1,Y_1) = 1$. Let $\mathfrak{g}$ be a solvable singular quadratic Lie algebra with* $\dim(\mathfrak{g}) = \dim(V)$. *Then, there exists a skew-symmetric map $\overline{C'} : \mathfrak{q}' \to \mathfrak{q}'$ such that V considered as the the double extension of $\mathfrak{q}'$ by $\overline{C'}$ is i-isomorphic to $\mathfrak{g}$.*

Proof. By Proposition 2.2.28 and Proposition 2.2.29, $\mathfrak{g}$ is a double extension. Let us write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ and $\overline{C} = \mathrm{ad}(Y_0)|_{\mathfrak{q}}$. Define $A : \mathfrak{g} \to V$ by $A(X_0) = X_1$, $A(Y_0) = Y_1$ and $\overline{A} = A|_{\mathfrak{q}}$ any isometry from $\mathfrak{q} \to \mathfrak{q}'$. It is clear that A is an isometry from $\mathfrak{g}$ to V. Now, define the Lie bracket on V by:

$$[X,Y] = A\left([A^{-1}(X), A^{-1}(Y)]\right),\ \forall\, X,Y \in V.$$

Then V is a quadratic Lie algebra, that is i-isomorphic to $\mathfrak{g}$, by definition. Moreover, V is obviously the double extension of $\mathfrak{q}'$ by $\overline{C'} = \overline{A}\,\overline{C}\,\overline{A}^{-1}$. □

We denote by $\mathcal{S}_{\mathrm{s}}(n+2)$ the set of solvable elements of $\mathcal{S}(n+2)$ (the set of singular Lie algebras structures on $\mathbb{C}^{n+2}$), for $n \geq 2$. Given $\mathfrak{g} \in \mathcal{S}(n+2)$, we denote by $[\mathfrak{g}]_{\mathrm{i}}$ its i-isomorphism class and by $\widehat{\mathcal{S}_{\mathrm{s}}}^{\mathrm{i}}(n+2)$ the set of classes in $\mathcal{S}_{\mathrm{s}}(n+2)$. For $[\overline{C}] \in \mathbb{P}^1(\mathfrak{o}(n))$, we denote by $\mathcal{O}_{[\overline{C}]}$ its $\mathrm{O}(n)$-adjoint orbit and by $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ the set of orbits.

Theorem 2.2.35. *There exists a bijection $\theta : \widetilde{\mathbb{P}^1(\mathfrak{o}(n))} \to \widehat{\mathcal{S}_{\mathrm{s}}}^{\mathrm{i}}(n+2)$.*

Proof. We consider $\mathcal{O}_{[\overline{C}]} \in \widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$. There is a double extension $\mathfrak{g}$ of $\mathfrak{q} = \mathrm{span}\{E_2, \dots, E_{n+1}\}$ by $\overline{C}$ realized on $\mathbb{C}^{n+2} = (\mathbb{C}E_1 \oplus \mathbb{C}E_{n+2}) \overset{\perp}{\oplus} \mathfrak{q}$. Then, by Proposition 2.2.28, $\mathfrak{g} \in \mathcal{S}_{\mathrm{s}}(n+2)$ and $[\mathfrak{g}]_{\mathrm{i}}$ does not depend on the choice of $\overline{C}$. We define $\theta(\mathcal{O}_{[\overline{C}]}) = [\mathfrak{g}]_{\mathrm{i}}$. If $\mathfrak{g}' \in \mathcal{S}_{\mathrm{s}}(n+2)$ then by Lemma 2.2.34, $\mathfrak{g}'$ can be realized (up to i-isomorphism) as a double extension on $\mathbb{C}^{n+2} = (\mathbb{C}E_1 \oplus \mathbb{C}E_{n+2}) \overset{\perp}{\oplus} \mathfrak{q}$. So θ is onto. Finally, θ is one-to-one by Corollary 2.2.31. □

It results from the previous theorem that the classification (up to i-isomorphisms) of the solvable singular quadratic Lie algebras of dimension $n+2$ can be reduced to the classification of adjoint orbits of $\mathfrak{o}(n)$ that details in Chapter 1. However, here we are interesting in classifying up to isomorphisms these algebras. According to Chapter 1, we consider case-by-case particular subsets of $\mathcal{S}_s(n+2)$: the set of nilpotent elements $\mathcal{N}(n+2)$, the set of *diagonalizable* elements $\mathcal{D}(n+2)$ and the set of *invertible* elements $\mathcal{S}_{\text{inv}}(2p+2)$.

Let $\mathfrak{g} \in \mathcal{N}(n+2)$. Recall that $\mathfrak{g}$ will be a double extension by a nilpotent map $\overline{C}$. Let $\mathscr{N}(n)$ be the set of non-zero nilpotent elements of $\mathfrak{o}(n)$ then $\overline{C} \in \mathscr{N}(n)$, we denote by $\mathscr{O}_{\overline{C}}$ its $\mathrm{O}(n)$-adjoint orbit. The set of nilpotent orbits is denoted by $\widetilde{\mathscr{N}}(n)$. We need the following lemma:

Lemma 2.2.36. *Let $\overline{C}$ and $\overline{C}' \in \mathscr{N}(n)$. Then $\overline{C}$ is conjugate to $\lambda\overline{C}'$ modulo $\mathrm{O}(n)$ for some non-zero $\lambda \in \mathbb{C}$ if and only if $\overline{C}$ is conjugate to $\overline{C}'$.*

Proof. It is enough to show that $\overline{C}$ and $\lambda\overline{C}$ are conjugate, for any non-zero $\lambda \in \mathbb{C}$. By the Jacobson-Morozov theorem, there exists a $\mathfrak{sl}(2)$-triple $\{X, H, \overline{C}\}$ in $\mathfrak{o}(n)$ such that $[H, \overline{C}] = 2\overline{C}$, so $\mathrm{e}^{t\,\mathrm{ad}(H)}(\overline{C}) = \mathrm{e}^{2t}\overline{C}$, for all $t \in \mathbb{C}$. We choose t such that $\mathrm{e}^{2t} = \lambda$, then $\mathrm{e}^{tH}\overline{C}\mathrm{e}^{-tH} = \lambda\overline{C}$ and $\mathrm{e}^{tH} \in \mathrm{O}(n)$. □

Theorem 2.2.37. *One has:*

(1) Let $\mathfrak{g}$ and $\mathfrak{g}' \in \mathcal{N}(n+2)$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if and only if they are i-isomorphic, so $[\mathfrak{g}]_i = [\mathfrak{g}]$, where $[\mathfrak{g}]$ denotes the isomorphism class of $\mathfrak{g}$, and $\widehat{\mathcal{N}}^i(n+2) = \widehat{\mathcal{N}}(n+2)$.

(2) There is a bijection $\tau : \widetilde{\mathscr{N}}(n) \to \widehat{\mathcal{N}}(n+2)$.

(3) $\widehat{\mathcal{N}}(n+2)$ is finite.

Proof.

(1) Assum that $\mathfrak{g}$ and $\mathfrak{g}' \in \mathcal{N}(n+2)$ are double extensions by $\overline{C}$ and $\overline{C}'$ respectively. Using Lemma 2.2.34, Proposition 2.2.28 (3) and Corollary 2.2.31, if $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic then there exists $P \in \mathrm{GL}(n)$ such that $\overline{C}' = \lambda P\overline{C}P^{-1}$, for some non-zero $\lambda \in \mathbb{C}$. Then $\lambda\overline{C}$ and $\overline{C}'$ are conjugate under $\mathrm{O}(n)$ by Lemma 2.2.36. Therfore, $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic.

(2) As in the proof of Theorem 2.2.35, for a given $\mathscr{O}_{\overline{C}} \in \widetilde{\mathscr{N}}(n)$, we construct the double extension $\mathfrak{g}$ of $\mathfrak{q} = \mathrm{span}\{E_2, \dots, E_{n+1}\}$ by $\overline{C}$ realized on $\mathbb{C}^{n+2}$. Then, by Proposition 2.2.28 (3), $\mathfrak{g} \in \mathcal{N}(n+2)$ and $[\mathfrak{g}]$ does not depend on the choice of $\overline{C}$. We define $\tau(\mathcal{O}_{\overline{C}}) = [\mathfrak{g}]$. Then by (1) and Corollary 2.2.31, τ is one-to-one and onto.

(3) $\widehat{\mathcal{N}}(n+2)$ is finite since the set of nilpotent orbits $\widetilde{\mathscr{N}}(n)$ is finite.

□

Next, we describe more explicitly the set $\widehat{\mathcal{N}}(n+2)$ by nilpotent Jordan-type maps mentioned in Chapter 1 and by the amalgamated product defined in the previous subsection.

In a canonical basis of the quadratic vector space $\mathfrak{q} = \mathbb{C}^{2p}$ consider the map $\overline{C}^J_{2p}$ with matrix $\begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$ and in a canonical basis of the quadratic vector space $\mathfrak{q} = \mathbb{C}^{2p+1}$ consider the map $\overline{C}^J_{2p+1}$ with matrix $\begin{pmatrix} J_{p+1} & M \\ 0 & -{}^tJ_p \end{pmatrix}$ where J_p is the Jordan block of size p, $M = (m_{ij})$ denotes the $(p+1)\times p$-matrix with $m_{p+1,p} = -1$ and $m_{ij} = 0$ otherwise. Then $\overline{C}^J_{2p} \in \mathfrak{o}(2p)$ and $\overline{C}^J_{2p+1} \in \mathfrak{o}(2p+1)$. Denote by $\mathfrak{j}_{2p}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{2p}$ and by $\mathfrak{j}_{2p+1}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{2p+1}$. So $\mathfrak{j}_{2p} \in \mathcal{N}(2p+2)$ and $\mathfrak{j}_{2p+1} \in \mathcal{N}(2p+3)$. Lie algebras $\mathfrak{j}_{2p}$ or $\mathfrak{j}_{2p+1}$ will be called *nilpotent Jordan-type Lie algebras*.

Keep the notations in Chapter 1, each $[d] \in \mathscr{P}_1(n)$ can be written as $(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1+1, \dots 2q_\ell+1)$ with all p_i even, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$. We associate a map $\overline{C}_{[d]} \in \mathfrak{o}(n)$ with the matrix

$$\mathrm{diag}_{k+\ell}(\overline{C}^J_{2p_1}, \overline{C}^J_{2p_2}, \dots, \overline{C}^J_{2p_k}, \overline{C}^J_{2q_1+1}, \dots, \overline{C}^J_{2q_\ell+1})$$

in a canonical basis of $\mathbb{C}^n$ and denote by $\mathfrak{g}_{[d]}$ the double extension of $\mathbb{C}^n$ by $\overline{C}_{[d]}$. Then $\mathfrak{g}_{[d]} \in \mathcal{N}(n+2)$ and $\mathfrak{g}_{[d]}$ is an amalgamated product of nilpotent Jordan-type Lie algebras. More precisely,

$$\mathfrak{g}_{[d]} = \mathfrak{j}_{2p_1} \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_2} \underset{\mathrm{a}}{\times} \cdots \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_k} \underset{\mathrm{a}}{\times} \mathfrak{j}_{2q_1+1} \underset{\mathrm{a}}{\times} \cdots \underset{\mathrm{a}}{\times} \mathfrak{j}_{2q_\ell+1}.$$

By Proposition 1.2.10, the map $[d] \mapsto \overline{C}_{[d]}$ from $\mathscr{P}_1(n)$ to $\mathfrak{o}(n)$ induces a bijection from $\mathscr{P}_1(n)$ onto $\widetilde{\mathscr{N}}(n)$. Therefore, combined with Theorem 2.2.37, we deduce:

Theorem 2.2.38.

(1) The map $[d] \mapsto \mathfrak{g}_{[d]}$ from $\mathscr{P}_1(n)$ to $\mathcal{N}(n+2)$ induces a bijection from $\mathscr{P}_1(n)$ onto $\widehat{\mathcal{N}}(n+2)$.

(2) Each nilpotent singular $n+2$-dimensional Lie algebra is i-isomorphic to a unique amalgamated product $\mathfrak{g}_{[d]}$, $[d] \in \mathscr{P}_1(n)$ of nilpotent Jordan-type Lie algebras.

Definition 2.2.39. Let $\mathfrak{g}$ be a solvable singular quadratic Lie algebra and write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ a decomposition of $\mathfrak{g}$ as a double extension (Proposition 2.2.28). Let $\overline{C} = \mathrm{ad}(Y_0)|_{\mathfrak{q}}$. We say that $\mathfrak{g}$ is *diagonalizable* if $\overline{C}$ is diagonalizable.

We denote by $\mathcal{D}(n+2)$ the set of such structures on the quadratic vector space $\mathbb{C}^{n+2}$, by $\mathcal{D}_{\mathrm{red}}(n+2)$ the reduced ones, by $\widehat{\mathcal{D}}(n+2)$, $\widehat{\mathcal{D}}^{\mathrm{i}}(n+2)$, $\widehat{\mathcal{D}}_{\mathrm{red}}(n+2)$, $\widehat{\mathcal{D}}^{\mathrm{i}}_{\mathrm{red}}(n+2)$ the corresponding sets of isomorphic and i-isomorphic classes of elements in $\mathcal{D}(n+2)$ and $\mathcal{D}_{\mathrm{red}}(n+2)$.

Remark that the property of being diagonalizable does not depend on the chosen decomposition of $\mathfrak{g}$ (see Remark 2.2.32) and a diagonalizable $\overline{C}$ satisfies $\ker(\overline{C}) \subset \mathrm{Im}(\overline{C})$ if and only if $\ker(\overline{C}) = \{0\}$. By Corollary 2.2.31 and using a proof completely similar to Theorem 2.2.35 or Theorem 2.2.37, we conclude:

Proposition 2.2.40. *There is a bijection between $\widehat{\mathcal{D}}^{\mathrm{i}}(n+2)$ and the set of semisimple $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. The same result holds for $\widehat{\mathcal{D}}^{\mathrm{i}}_{\mathrm{red}}(n+2)$ and semisimple invertible orbits in $\mathbb{P}^1(\mathfrak{o}(n))$.*

Keep the notations as in Chapter 1. To describe the set of semisimple $\mathrm{O}(n)$-orbits in $\mathbb{P}^1(\mathfrak{o}(n))$, we need to add maps $(\lambda_1,\dots,\lambda_p)\mapsto\lambda(\lambda_1,\dots,\lambda_p)$, for all $\lambda\in\mathbb{C}$, $\lambda\neq 0$ to the group G_p. We obtain a group denoted by H_p. As a consequence of Section 1.3, we have the classification result for the diagonalizable case:

Theorem 2.2.41. *There is a bijection between $\widehat{\mathfrak{D}}^{\mathrm{i}}(n+2)$ and Λ_p/H_p with $n=2p$ or $n=2p+1$. Moreover, if $n=2p+1$, $\widehat{\mathfrak{D}}^{\mathrm{i}}_{\mathrm{red}}(n+2)=\emptyset$ and if $n=2p$, then $\widehat{\mathfrak{D}}^{\mathrm{i}}_{\mathrm{red}}(2p+2)$ is in bijection with Λ_p^+/H_p where $\Lambda_p^+=\{(\lambda_1,\dots,\lambda_p)\mid \lambda_i\in\mathbb{C},\lambda_i\neq 0,\ \forall\, i\}$.*

To go further in the study of diagonalizable reduced case, we need the following Lemma:

Lemma 2.2.42.

Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be solvable singular quadratic Lie algebras, $\mathfrak{g}'=(\mathbb{C}X_1'\oplus\mathbb{C}Y_1')\overset{\perp}{\oplus}\mathfrak{q}'$ a decomposition of $\mathfrak{g}'$ as a double extension and $\overline{C}'=\mathrm{ad}(Y_1')|_{\mathfrak{q}'}$. We assume that $\overline{C}'$ is invertible. Then $\mathfrak{g}'$ and $\mathfrak{g}''$ are isomorphic if and only if they are i-isomorphic.

Proof. Write $\mathfrak{g}''=(\mathbb{C}X_1''\oplus\mathbb{C}Y_1'')\overset{\perp}{\oplus}\mathfrak{q}''$ a decomposition of $\mathfrak{g}''$ as a double extension and $\overline{C}''=\mathrm{ad}(Y_1'')|_{\mathfrak{q}''}$.

Assume that $\mathfrak{g}'$ and $\mathfrak{g}''$ are isomorphic. By Corollary 2.2.31, there exist $\overline{Q}:\mathfrak{q}'\to\mathfrak{q}''$ and $\overline{P}\in\mathscr{L}(\mathfrak{q}')$ such that $\overline{Q}\,\overline{P}^{-1}$ is an isometry, $\overline{P}^*\,\overline{P}\,\overline{C}'=\overline{C}'$ and $\overline{C}''=\lambda\,\overline{Q}\,\overline{C}'\,\overline{Q}^{-1}$ for some non-zero $\lambda\in\mathbb{C}$. But $\overline{C}'$ is invertible, so $\overline{P}^*\,\overline{P}=\mathrm{Id}_{\mathfrak{q}'}$. Therefore, $\overline{P}$ is an isometry of $\mathfrak{q}'$ and then $\overline{Q}$ is an isometry from $\mathfrak{q}'$ to $\mathfrak{q}''$. The conditions of Corollary 2.2.31 (1) are satisfied, so $\mathfrak{g}'$ and $\mathfrak{g}''$ are i-isomorphic. □

Corollary 2.2.43. *One has:*

$$\widehat{\mathfrak{D}}_{\mathrm{red}}(2p+2)=\widehat{\mathfrak{D}}^{\mathrm{i}}_{\mathrm{red}}(2p+2),\ \forall\, p\geq 1.$$

Next, we describe diagonalizable reduced singular quadratic Lie algebras using the amalgamated products. First, let $\mathfrak{g}_4(\lambda)$ be the double extension of $\mathfrak{q}=\mathbb{C}^2$ by $\overline{C}=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}$, $\lambda\neq 0$. By Lemma 2.2.27, $\mathfrak{g}_4(\lambda)$ is i-isomorphic to $\mathfrak{g}_4(1)$, call it $\mathfrak{g}_4$.

Proposition 2.2.44. *Let $(\mathfrak{g},B)$ be a diagonalizable reduced singular quadratic Lie algebra. Then $\mathfrak{g}$ is an amalgamated product of singular quadratic Lie algebras all i-isomorphic to $\mathfrak{g}_4$.*

Proof. We write $\mathfrak{g}=(\mathbb{C}X_0\oplus\mathbb{C}Y_0)\overset{\perp}{\oplus}\mathfrak{q}$, $C=\mathrm{ad}(Y_0)$, $\overline{C}=C|_{\mathfrak{q}}$ and $\overline{B}=B_{\mathfrak{q}\times\mathfrak{q}}$. Then $\overline{C}$ is a diagonalizable invertible element of $\mathfrak{o}(\mathfrak{q},\overline{B})$. Apply Appendix A to obtain a basis $\{e_1,\dots,e_p,f_1,\dots,f_p\}$ of $\mathfrak{q}$ and $\lambda_1,\dots,\lambda_p\in\mathbb{C}$, all non-zero, such that $B(e_i,e_j)=B(f_i,f_j)=0$, $B(e_i,f_j)=\delta_{ij}$ and $\overline{C}(e_i)=\lambda_i e_i$, $\overline{C}(f_i)=-\lambda_i f_i$, for all $1\leq i,j\leq p$. Let $\mathfrak{q}_i=\mathrm{span}\{e_i,f_i\}$, $1\leq i\leq p$. Then

$$\mathfrak{q}=\overset{\perp}{\underset{1\leq i\leq p}{\oplus}}\mathfrak{q}_i.$$

Furthermore, $\mathfrak{h}_i=(\mathbb{C}X_0\oplus\mathbb{C}Y_0)\overset{\perp}{\oplus}\mathfrak{q}_i$ is a Lie subalgebra of $\mathfrak{g}$ for all $1\leq i\leq p$ and

$$\mathfrak{g}=\mathfrak{h}_1\underset{\mathrm{a}}{\times}\mathfrak{h}_2\underset{\mathrm{a}}{\times}\dots\underset{\mathrm{a}}{\times}\mathfrak{h}_p\ \text{ with }\ \mathfrak{h}_i\overset{\mathrm{i}}{\simeq}\mathfrak{g}_4(\lambda_i)\overset{\mathrm{i}}{\simeq}\mathfrak{g}_4.$$

□

Remark 2.2.45. For non-zero λ, $\mu \in \mathbb{C}$, consider the amalgamated product:

$$\mathfrak{g}(\lambda,\mu) = \mathfrak{g}_4(\lambda) \underset{\mathrm{a}}{\times} \mathfrak{g}_4(\mu).$$

Then $\mathfrak{g}(\lambda,\mu)$ is the double extension of $\mathbb{C}^4$ by

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}.$$

Therefore $\mathfrak{g}(\lambda,\mu)$ is isomorphic to $\mathfrak{g}(1,1)$ if and only if $\mu = \pm\lambda$ (Lemma 2.2.42 and Section 1.3). So, though $\mathfrak{g}_4(\lambda)$ and $\mathfrak{g}_4(\mu)$ are i-isomorphic to $\mathfrak{g}_4$, the amalgamated product $\mathfrak{g}(\lambda,\mu)$ is not even isomorphic to $\mathfrak{g}(1,1) = \mathfrak{g}_4 \underset{\mathrm{a}}{\times} \mathfrak{g}_4$ if $\mu \neq \pm\lambda$. This illustrates that amalgamated products may have a rather bad behavior with respect to isomorphisms.

Definition 2.2.46. A double extension is called an *invertible quadratic Lie algebra* if the corresponding skew-symmetric map is invertible.

Remark 2.2.47.

- By Remark 2.2.32, the property of being an invertible quadratic Lie algebra does not depend on the chosen decomposition.
- By Appendix A, the dimension of an invertible quadratic Lie algebra is even.
- By Lemma 2.2.42, two invertible quadratic Lie algebras are isomorphic if and only if they are i-isomorphic.

For $p \geq 1$ and $\lambda \in \mathbb{C}$, let $J_p(\lambda) = \mathrm{diag}_p(\lambda,\dots,\lambda) + J_p$ and

$$\overline{C}^J_{2p}(\lambda) = \begin{pmatrix} J_p(\lambda) & 0 \\ 0 & -{}^tJ_p(\lambda) \end{pmatrix}$$

in a canonical basis of quadratic vector space $\mathbb{C}^{2p}$. Then $\overline{C}^J_{2p}(\lambda) \in \mathfrak{o}(2p)$.

Definition 2.2.48. For $\lambda \in \mathbb{C}$, let $\mathrm{j}_{2p}(\lambda)$ be the double extension of $\mathbb{C}^{2p}$ by $\overline{C}^J_{2p}(\lambda)$. We say that $\mathrm{j}_{2p}(\lambda)$ is a *Jordan-type quadratic Lie algebra*.

When $\lambda = 0$ and $p \geq 2$, we recover the nilpotent Jordan-type Lie algebras j_{2p} from nilpotent case.

When $\lambda \neq 0$, $\mathrm{j}_{2p}(\lambda)$ is an invertible singular quadratic Lie algebra and

$$\mathrm{j}_{2p}(-\lambda) \simeq \mathrm{j}_{2p}(\lambda).$$

Proposition 2.2.49. *Let $\mathfrak{g}$ be a solvable singular quadratic Lie algebra. Then $\mathfrak{g}$ is an invertible quadratic Lie algebra if and only if $\mathfrak{g}$ is an amalgamated product of Lie algebras all i-isomorphic to Jordan-type Lie algebras $\mathrm{j}_{2p}(\lambda)$, with $\lambda \neq 0$.*

Proof. Let $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$, B be the bilinear form of $\mathfrak{g}$, $\overline{B} = B|_{\mathfrak{q}\times\mathfrak{q}}$, $C = \mathrm{ad}(Y_0)$ and $\overline{C} = C|_{\mathfrak{q}} \in \mathfrak{o}(\mathfrak{q},\overline{B})$. We decompose $\overline{C}$ into its semisimple and nilpotent parts, $\overline{C} = \overline{S} + \overline{N}$. It is well known that $\overline{S}$ and $\overline{N} \in \mathfrak{o}(\mathfrak{q},\overline{B})$.

Let $\Lambda \subset \mathbb{C}\setminus\{0\}$ be the spectrum of $\overline{S}$. We have that $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$ (see Appendix A). Let V_λ be the eigenspace corresponding to the eigenvalue λ. We have $\dim(V_\lambda) = \dim(V_{-\lambda})$. Denote by $\mathfrak{q}(\lambda)$ the direct sum $\mathfrak{q}(\lambda) = V_\lambda \oplus V_{-\lambda}$. If $\mu \in \Lambda$, $\mu \neq \pm\lambda$, then $\mathfrak{q}(\lambda)$ and $\mathfrak{q}(\mu)$ are orthogonal (Appendix A). Choose Λ_+ such that $\Lambda = \Lambda_+ \cup (-\Lambda_+)$ and $\Lambda_+ \cap (-\Lambda_+) = \emptyset$. We have (see Appendix A):

$$\mathfrak{q} = \overset{\perp}{\underset{\lambda\in\Lambda_+}{\oplus}} \mathfrak{q}(\lambda).$$

So the restriction $B_\lambda = B|_{\mathfrak{q}(\lambda)\times\mathfrak{q}(\lambda)}$ is non-degenerate. Moreover, V_λ and $V_{-\lambda}$ are maximal isotropic subspaces in $\mathfrak{q}(\lambda)$.

Now, consider the map $\Psi : V_{-\lambda} \to V_\lambda^*$ defined by $\Psi(u)(v) = B_\lambda(u,v)$, for all $u \in V_{-\lambda}$, $v \in V_\lambda$. Then Ψ is an isomorphism. Given any basis $\mathscr{B}(\lambda) = \{e_1(\lambda),\dots,e_{n_\lambda}(\lambda)\}$ of V_λ, there is a basis $\mathscr{B}(-\lambda) = \{e_1(-\lambda),\dots,e_{n_\lambda}(-\lambda)\}$ of $V_{-\lambda}$ such that $B_\lambda(e_i(\lambda),e_j(-\lambda)) = \delta_{ij}$, for all $1 \le i,j \le n_\lambda$: simply define $e_i(-\lambda) = \psi^{-1}(e_i(\lambda)^*)$, for all $1 \le i \le n_\lambda$.

Remark that $\overline{N}$ and $\overline{S}$ commute, so $\overline{N}(V_\lambda) \subset V_\lambda$, for all $\lambda \in \Lambda$. Define $\overline{N}_\lambda = \overline{N}|_{\mathfrak{q}(\lambda)}$, then $\overline{N}_\lambda \in \mathfrak{o}(\mathfrak{q}(\lambda),B_\lambda)$. Hence, if $\overline{N}_\lambda|_{V_\lambda}$ has a matrix M_λ with respect to $\mathscr{B}(\lambda)$, then $\overline{N}_\lambda|_{V_{-\lambda}}$ has a matrix $-{}^tM_\lambda$ with respect to $\mathscr{B}(-\lambda)$. We choose the basis $\mathscr{B}(\lambda)$ such that M_λ is of Jordan type, i.e.

$$\mathscr{B}(\lambda) = \mathscr{B}(\lambda,1) \cup \dots \cup \mathscr{B}(\lambda,r_\lambda),$$

the multiplicity m_λ of λ is $m_\lambda = \sum_{i=1}^{r_\lambda} d_\lambda(i)$ where $d_\lambda(i) = \sharp\mathscr{B}(\lambda,i)$ and

$$M_\lambda = \mathrm{diag}_{r_\lambda}\left(J_{d_\lambda(1)},\dots,J_{d_\lambda(r_\lambda)}\right).$$

The matrix of $C|_{\mathfrak{q}(\lambda)}$ written on the basis $\mathscr{B}(\lambda)\cup\mathscr{B}(-\lambda)$ is:

$$\mathrm{diag}_{n_\lambda}\left(J_{d_\lambda(1)}(\lambda),\dots,J_{d_\lambda(r_\lambda)}(\lambda),-{}^tJ_{d_\lambda(1)}(\lambda),\dots,-{}^tJ_{d_\lambda(r_\lambda)}(\lambda)\right).$$

Let $\mathfrak{q}(\lambda,i)$ be the subspace generated by $\mathscr{B}(\lambda,i)\cup\mathscr{B}(-\lambda,i)$, for all $1 \le i \le r_\lambda$ and let $C(\lambda,i) = C|_{\mathfrak{q}(\lambda,i)}$. We have

$$\mathfrak{q}(\lambda) = \overset{\perp}{\underset{1\le i\le r_\lambda}{\oplus}} \mathfrak{q}(\lambda,i).$$

The matrix of $C(\lambda,i)$ written on the basis of $\mathfrak{q}(\lambda,i)$ is $C^J_{2d_\lambda(i)}(\lambda)$. Let $\mathfrak{g}(\lambda,i)$, $\lambda\in\Lambda_+$, $1\le i\le r_\lambda$ be the double extension of $\mathfrak{q}(\lambda,i)$ by $C(\lambda,i)$. Then $\mathfrak{g}(\lambda,i)$ is i-isomorphic to $\mathfrak{j}_{2d_\lambda(i)}(\lambda)$. But

$$\mathfrak{q} = \overset{\perp}{\underset{\substack{\lambda\in\Lambda_+\\ 1\le i\le r_\lambda}}{\oplus}} \mathfrak{q}(\lambda,i) \ \text{ and } \ C|_{\mathfrak{q}(\lambda,i)} = C(\lambda,i).$$

Therefore, $\mathfrak{g}$ is the amalgamated product

$$\mathfrak{g} = \underset{\substack{\lambda\in\Lambda_+\\ 1\le i\le r_\lambda}}{\underset{\mathrm{a}}{\times}} \mathfrak{g}(\lambda,i).$$

□

Denote by $\mathcal{S}_{\mathrm{inv}}(2p+2)$ the set of invertible singular quadratic Lie algebra structures on $\mathbb{C}^{2p+2}$, by $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ the set of isomorphism (or i-isomorphism) classes of $\mathcal{S}_{\mathrm{inv}}(2p+2)$. The classification of $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ can be deduced from the classification of the set of orbits $\widetilde{\mathscr{I}}(2p)$ by $\mathscr{J}_p$ as follows (see Chapter 1): introduce an action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on $\mathscr{J}_p$ by

$$\text{for all } \mu \in \mathbb{C}^*,\ \mu\cdot(\Lambda,m,d) = (\mu\Lambda,m',d'),\ \forall\ (\Lambda,m,d)\in\mathscr{J}_p, \lambda\in\Lambda,$$

where $m'(\mu\lambda) = m(\lambda), d'(\mu\lambda) = d(\lambda)$, for all $\lambda\in\Lambda$. Since $i(\mu C) = \mu i(C)$, for all $C\in\mathscr{I}(2p)$ and $\mu\in\mathbb{C}^*$ where $i:\mathscr{I}(2p)\to\mathscr{J}_p$ the bijection is defined as in Chapter 1, then there is a bijection $\widehat{i}:\mathbb{P}^1(\widetilde{\mathscr{I}}(2p))\to\mathscr{J}_p/\mathbb{C}^*$ given by $\widehat{i}([C]) = [i(C)]$, if $[C]$ is the class of $C\in\mathscr{I}(2p)$ and $[(\Lambda,m,d)]$ is the class of $(\Lambda,m,d)\in\mathscr{J}_p$.

Theorem 2.2.50. *The set $\widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$ is in bijection with $\mathscr{J}_p/\mathbb{C}^*$.*

Proof. By Theorem 2.2.35, there is a bijection between $\widehat{\mathcal{S}_{\mathrm{s}}}^{\mathrm{i}}(2p+2)$ and $\mathbb{P}^1(\widetilde{\mathfrak{o}(2p)})$. By restriction, that induces a bijection between $\widehat{\mathcal{S}_{\mathrm{inv}}}^{\mathrm{i}}(2p+2)$ and $\mathbb{P}^1(\widetilde{\mathscr{I}(2p)})$. By Lemma 2.2.42, we have $\widehat{\mathcal{S}_{\mathrm{inv}}}^{\mathrm{i}}(2p+2) = \widehat{\mathcal{S}_{\mathrm{inv}}}(2p+2)$. Then, the result follows: given $\mathfrak{g}\in\mathcal{S}_{\mathrm{inv}}(2p+2)$ and an associated $\overline{C}\in\mathscr{I}(2p)$, the bijection maps $\widetilde{\mathfrak{g}}$ to $[i(\overline{C})]$ where $\widetilde{\mathfrak{g}}$ is the isomorphism class of $\mathfrak{g}$. □

Let $\mathfrak{g}$ be a solvable singular quadratic Lie algebra. We fix a realization of $\mathfrak{g}$ as a double extension, $\mathfrak{g} = (\mathbb{C}X_0\oplus\mathbb{C}Y_0)\overset{\perp}{\oplus}\mathfrak{q}$ (Proposition 2.2.28 and Lemma 2.2.34). Let $C = \mathrm{ad}(Y_0)$, $\overline{C} = C|_{\mathfrak{q}}$ and $\overline{B} = B|_{\mathfrak{q}\times\mathfrak{q}}$. We consider the Fitting decomposition of $\overline{C}$:

$$\mathfrak{q} = \mathfrak{q}_N\oplus\mathfrak{q}_I,$$

where $\mathfrak{q}_N$ and $\mathfrak{q}_I$ are $\overline{C}$-stable, $\overline{C}_N = \overline{C}|_{\mathfrak{q}_N}$ is nilpotent and $\overline{C}_I = \overline{C}|_{\mathfrak{q}_I}$ is invertible.

We recall the facts in Section 1.5, one has $\mathfrak{q}_I = \mathfrak{q}_N^{\perp}$, the restrictions $\overline{B}_N = \overline{B}|_{\mathfrak{q}_N\times\mathfrak{q}_N}$ and $\overline{B}_I = \overline{B}|_{\mathfrak{q}_I\times\mathfrak{q}_I}$ are non-degenerate, $\overline{C}_N$ and $\overline{C}_I$ are skew-symmetric and $[\mathfrak{q}_I,\mathfrak{q}_N] = 0$. Let $\mathfrak{g}_N = (\mathbb{C}X_0\oplus\mathbb{C}Y_0)\overset{\perp}{\oplus}\mathfrak{q}_N$ and $\mathfrak{g}_I = (\mathbb{C}X_0\oplus\mathbb{C}Y_0)\overset{\perp}{\oplus}\mathfrak{q}_I$. Then $\mathfrak{g}_N$ and $\mathfrak{g}_I$ are Lie subalgebras of $\mathfrak{g}$, $\mathfrak{g}_N$ is the double extension of $\mathfrak{q}_N$ by $\overline{C}_N$, $\mathfrak{g}_I$ is the double extension of $\mathfrak{q}_I$ by $\overline{C}_I$ and $\mathfrak{g}_N$ is a nilpotent singular quadratic Lie algebra. Moreover, we have

$$\mathfrak{g} = \mathfrak{g}_N\underset{\mathrm{a}}{\times}\mathfrak{g}_I.$$

Definition 2.2.51. The Lie subalgebras $\mathfrak{g}_N$ and $\mathfrak{g}_I$ are respectively the *nilpotent* and *invertible Fitting components* of $\mathfrak{g}$.

This definition is justified by:

Theorem 2.2.52. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be solvable singular quadratic Lie algebras and $\mathfrak{g}_N$, $\mathfrak{g}_I$, $\mathfrak{g}'_N$, $\mathfrak{g}'_I$ be their Fitting components. Then*

(1) $\mathfrak{g}\overset{\mathrm{i}}{\simeq}\mathfrak{g}'$ if and only if $\mathfrak{g}_N\overset{\mathrm{i}}{\simeq}\mathfrak{g}'_N$ and $\mathfrak{g}_I\overset{\mathrm{i}}{\simeq}\mathfrak{g}'_I$. The result remains valid if we replace $\overset{\mathrm{i}}{\simeq}$ by $\simeq$.

(2) $\mathfrak{g} \simeq \mathfrak{g}'$ *if and only if* $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$.

Proof. We assume that $\mathfrak{g} \simeq \mathfrak{g}'$. Then by Corollary 2.2.31, there exists an invertible $\overline{P} : \mathfrak{q} \to \mathfrak{q}'$ and a non-zero $\lambda \in \mathbb{C}$ such that $\overline{C}' = \lambda\, \overline{P}\, \overline{C}\, \overline{P}^{-1}$, so $\mathfrak{q}'_N = \overline{P}(\mathfrak{q}_N)$ and $\mathfrak{q}'_I = \overline{P}(\mathfrak{q}_I)$, then $\dim(\mathfrak{q}'_N) = \dim(\mathfrak{q}_N)$ and $\dim(\mathfrak{q}'_I) = \dim(\mathfrak{q}_I)$. Thus, there exist isometries $F_N : \mathfrak{q}'_N \to \mathfrak{q}_N$ and $F_I : \mathfrak{q}'_I \to \mathfrak{q}_I$ and we can define an isometry $\overline{F} : \mathfrak{q}' \to \mathfrak{q}$ by $\overline{F}(X'_N + X'_I) = F_N(X'_N) + F_I(X'_I)$, for all $X'_N \in \mathfrak{q}'_N$ and $X'_I \in \mathfrak{q}'_I$. We now define $F : \mathfrak{g}' \to \mathfrak{g}$ by $F(X'_1) = X_1$, $F(Y'_1) = Y_1$, $F|_{\mathfrak{q}'} = \overline{F}$ and a new Lie bracket on $\mathfrak{g}$:

$$[X,Y]'' = F\left([F^{-1}(X), F^{-1}(Y)]'\right),\ \forall\, X,Y \in \mathfrak{g}.$$

Call $\mathfrak{g}''$ this new quadratic Lie algebra. We have $\mathfrak{g}'' = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$, i.e., $\mathfrak{q}'' = \mathfrak{q}$ and $\overline{C}'' = \overline{F}\ \overline{C}'\ \overline{F}^{-1}$. So $\mathfrak{q}''_N = F(\mathfrak{q}'_N) = \mathfrak{q}_N$ and $\mathfrak{q}''_I = F(\mathfrak{q}'_I) = \mathfrak{q}_I$. But $\mathfrak{g} \simeq \mathfrak{g}''$, so there exists an invertible $Q : \mathfrak{q} \to \mathfrak{q}$ such that $\overline{C}'' = \lambda\ \overline{Q}\ \overline{C}\ \overline{Q}^{-1}$ for some non-zero $\lambda \in \mathbb{C}$ (Corollary 2.2.31). It follows that $\mathfrak{q}''_N = Q(\mathfrak{q}_N)$ and $\mathfrak{q}''_I = Q(\mathfrak{q}_I)$, so $Q(\mathfrak{q}_N) = \mathfrak{q}_N$ and $Q(\mathfrak{q}_I) = \mathfrak{q}_I$.

Moreover, we have $Q^*\, Q\, \overline{C} = \overline{C}$ (Corollary 2.2.31), so $Q^*\, Q\, \overline{C}^k = \overline{C}^k$ for all k. There exists k such that $\mathfrak{q}_I = \mathrm{Im}(C^k)$ and $(Q^*\, Q\, \overline{C}^k)(X) = \overline{C}^k(X)$, for all $X \in \mathfrak{g}$. So $Q^*Q|_{\mathfrak{q}_I} = \mathrm{Id}_{\mathfrak{q}_I}$ and $Q_I = Q|_{\mathfrak{q}_I}$ is an isometry. Since $\overline{C}''_I = \lambda\ \overline{Q}_I\ \overline{C}_I\ \overline{Q}_I^{-1}$, then $\mathfrak{g}_I \overset{\mathrm{i}}{\simeq} \mathfrak{g}''_I$ (Corollary 2.2.31).

Let $Q_N = Q|_{\mathfrak{q}_N}$. Then $\overline{C}''_N = \lambda\ \overline{Q}_N\ \overline{C}_N\ \overline{Q}_N^{-1}$ and $Q^*_N\ Q_N\ \overline{C}_N = \overline{C}_N$, so by Corollary 2.2.31, $\mathfrak{g}_N \simeq \mathfrak{g}''_N$. Since $\mathfrak{g}_N$ and $\mathfrak{g}''_N$ are nilpotent, then $\mathfrak{g}''_N \overset{\mathrm{i}}{\simeq} \mathfrak{g}_N$ by Theorem 2.2.37.

Conversely, assume that $\mathfrak{g}_N \simeq \mathfrak{g}'_N$ and $\mathfrak{g}_I \simeq \mathfrak{g}'_I$. Then $\mathfrak{g}_N \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_N$ and $\mathfrak{g}_I \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_I$ by Theorem 2.2.37 and Lemma 2.2.42.

So, there exist isometries $P_N : \mathfrak{g}_N \to \mathfrak{g}'_N$, $P_I : \mathfrak{g}_I \to \mathfrak{g}'_I$ and non-zero λ_N and $\lambda_I \in \mathbb{C}$ such that $\overline{C}'_N = \lambda_N\ \overline{P}_N\ \overline{C}_N\ \overline{P}_N^{-1}$ and $\overline{C}'_I = \lambda_I\ \overline{P}_I\ \overline{C}_I\ \overline{P}_I^{-1}$. By Lemma 2.2.36, since $\mathfrak{g}_N$ and $\mathfrak{g}'_N$ are nilpotent, we can assume that $\lambda_N = \lambda_I = \lambda$. Now we define $P : \mathfrak{q} \to \mathfrak{q}'$ by $P(X_N + X_I) = P_N(X_N) + P_I(X_I)$, for all $X_N \in \mathfrak{q}_N$, $X_I \in \mathfrak{q}_I$, so P is an isometry. Moreover, since $\overline{C}(X_N + X_I) = \overline{C}_N(X_N) + \overline{C}_I(X_I)$, for all $X_N \in \mathfrak{q}_N$, $X_I \in \mathfrak{q}_I$ and $\overline{C}'(X'_N + X'_I) = \overline{C}'_N(X'_N) + \overline{C}'_I(X'_I)$, for all $X'_N \in \mathfrak{q}_N$, $X'_I \in \mathfrak{q}_I$, we conclude $\overline{C}' = \lambda\ P\ \overline{C}P^{-1}$ and finally, $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$, by Corollary 2.2.31. □

Remark 2.2.53. The class of solvable singular quadratic Lie algebras has the remarkable property that two Lie algebras in this class are isomorphic if and only if they are i-isomorphic. In addition, the Fitting components do not depend on the realizations of the Lie algebra as a double extension and they completely characterize the Lie algebra (up to isomorphisms).

It results that the classification of $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2)$ can be reduced from the classification of $\mathrm{O}(n)$-orbits of $\mathfrak{o}(n)$ in Chapter 1. We set an action of the group $\mathbb{C}^*$ on $\mathscr{D}(n)$ by:

$$\mu \cdot ([d],T) = ([d], \mu \cdot T)\ ,\ \forall\, \mu \in \mathbb{C}^*,\ ([d],T) \in \mathscr{D}(n).$$

Then, we have the classification result of $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2)$ as follows:

Theorem 2.2.54. *The set* $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2)$ *is in bijection with* $\mathscr{D}(n)/\mathbb{C}^*$.

Proof. By Theorems 2.2.35 and 2.2.52, there is a bijection between $\widehat{\mathcal{S}_{\mathrm{s}}}(n+2)$ and $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$. It needs only to show that there is a bijection between $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ and $\mathscr{D}(n)/\mathbb{C}^*$. Let $C \in \mathfrak{o}(n)$ then there is the bijection p mapping C onto a pair $([d],T)$ in $\mathscr{D}(n)$ by Proposition 1.5.1. Moreover, C_N and μC_N have the same partition so $p(\mu C) = ([d], \mu \cdot T) = \mu \cdot p(C)$. Therefore, the map p induces a bijection $\widehat{p} : \mathbb{P}^1(\widetilde{\mathfrak{o}(n)}) \to \mathscr{D}(n)/\mathbb{C}^*$ given by $\widehat{p}([C]) = [([d],T)]$, if $[C]$ is the class of $C \in \mathfrak{o}(n)$ and $[([d],T)]$ is the class of $([d],T) \in \mathscr{D}(n)$. The result follows. □

2.3 Quadratic dimension of quadratic Lie algebras

In this section, we will study an interesting characteristic of a quadratic Lie algebra $\mathfrak{g}$ called the *quadratic dimension*. It is defined by the dimension of the space of invariant symmetric bilinear forms on $\mathfrak{g}$ and denoted by $d_q(\mathfrak{g})$. This notion involves special maps which commute with inner derivations of $\mathfrak{g}$. We call such maps *centromorphisms*. Some simple properties of a centromorphism are given in the first subsection. We calculate the formula $d_q(\mathfrak{g})$ for reduced singular quadratic Lie algebras in the second subsection and use it to show that dup-number is invariant under isomorphisms. Finally, we study centromorphisms with respect to some extensions of a quadratic Lie algebra $\mathfrak{g}$.

2.3.1 Centromorphisms of a quadratic Lie algebra

Let $(\mathfrak{g},B)$ be a quadratic Lie algebra. To any symmetric bilinear form B' on $\mathfrak{g}$, there is an associated map $D:\mathfrak{g}\to\mathfrak{g}$ satisfying

$$B'(X,Y)=B(D(X),Y),\ \forall\, X,Y\in\mathfrak{g}.$$

Since B and B' are symmetric, one has D *symmetric* (with respect to B), i.e. $B(D(X),Y)=B(X,D(X))$ for all $X,\ Y\in\mathfrak{g}$.

Lemma 2.3.1.

(1) B' is invariant if and only if D satisfies

$$D([X,Y])=[D(X),Y]=[X,D(Y)],\ \forall\, X,Y\in\mathfrak{g}. \tag{II}$$

(2) B' is non-degenerate if and only if D is invertible.

Proof.

(1) Let $X,Y,Z\in\mathfrak{g}$ then $B'([X,Y],Z)=B(D([X,Y]),Z)$. Since B is invariant one has:

$$B'(X,[Y,Z])=B(D(X),[Y,Z])=B([D(X),Y],Z).$$

Therefore B' is invariant if and only if $D([X,Y])=[D(X),Y]$ since B is non-degenerate. The Lie bracket anticommutative implies that $D([X,Y])=[X,D(Y)]$.

(2) Assume that B' is non-degenerate. If X is an element in $\mathfrak{g}$ such that $D(X)=0$ then $B(D(X),\mathfrak{g})=0$. Then one has $B'(X,\mathfrak{g})=0$. It implies that $X=0$ and therefore D is invertible. Conversely, if D is invertible then $B'(X,\mathfrak{g})=0$ reduces to $B(D(X),\mathfrak{g})=0$. Since B is non-degenerate one has $D(X)=0$. Thus $X=0$ and B' is non-degenerate.

□

A symmetric map D satisfying (II) is called a *centromorphism* of $\mathfrak{g}$. The equality (II) is equivalent to

$$D \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ D = \mathrm{ad}(D(X)),\ \forall\, X \in \mathfrak{g}.$$

Denote by $\mathcal{C}(\mathfrak{g})$ the space of centromorphisms of $\mathfrak{g}$ and by $\mathcal{C}_I(\mathfrak{g})$ the subspace spanned by invertible centromorphisms in $\mathcal{C}(\mathfrak{g})$. We recall Lemma 2.1 in [BB97] as follows.

Lemma 2.3.2. *One has* $\mathcal{C}(\mathfrak{g}) = \mathcal{C}_I(\mathfrak{g})$.

Proof. Let D be an invertible centromorphism and $\varphi \in \mathcal{C}(\mathfrak{g})$. Fix $\mathscr{B}$ a basis of $\mathfrak{g}$. Denote by $M(D)$ and $M(\varphi)$ respectively the associated matrices of D and φ in $\mathscr{B}$. Consider the polynomial $P(x) = det(M(\varphi) - xM(D))$. Since $P(x)$ is a non-zero polynomial so there exists $\lambda \in \mathbb{C}$ such that $P(\lambda) \neq 0$. It means that $\varphi - \lambda D$ is invertible and thus $\varphi = (\varphi - \lambda D) + \lambda D \in \mathcal{C}_I(\mathfrak{g})$. It shows that $\mathcal{C}(\mathfrak{g}) = \mathcal{C}_I(\mathfrak{g})$. □

Therefore the space of invariant symmetric bilinear forms on $\mathfrak{g}$ and the subspace generated by non-degenerated ones are the same. Let us denote it by $\mathcal{B}(\mathfrak{g})$. The dimension of $\mathcal{B}(\mathfrak{g})$ is called the *quadratic dimension* of $\mathfrak{g}$ and denoted by $d_q(\mathfrak{g})$. As a consequence of the previous lemmas, one has $d_q(\mathfrak{g}) = \dim(\mathcal{C}(\mathfrak{g}))$. Moreover, one has other properties of $\mathcal{C}(\mathfrak{g})$ as follows:

Proposition 2.3.3. *Let* $D \in \mathcal{C}(\mathfrak{g})$ *then*

(1) $D^n \in \mathcal{C}(\mathfrak{g})$ *for all* $n \geq 1$. *Furthermore, if D is invertible then* $D^{-1} \in \mathcal{C}(\mathfrak{g})$.

(2) $\mathcal{Z}(\mathfrak{g})$ *and* $[\mathfrak{g},\mathfrak{g}]$ *are stable subspaces under* D.

Proposition 2.3.4. *Let* $\delta \in \mathrm{Der}_a(\mathfrak{g})$ *be a skew-symmetric derivation of* $\mathfrak{g}$. *Assume* $D \in \mathcal{C}(\mathfrak{g})$ *such that D and δ commute. Then* $D \circ \delta$ *is also a skew-symmetric derivation of* $\mathfrak{g}$.

Proof. Since $(D \circ \delta)[X,Y] = D[\delta(X),Y] + D[X,\delta(Y)] = [(D\circ\delta)(X),Y] + [X,(D\circ\delta)(Y)]$ and $B((D\circ\delta)(X),Y) = -B(X,(\delta\circ D)(Y)) = -B(X,(D\circ\delta)(Y))$ for all $X,Y \in \mathfrak{g}$, one has $D\circ\delta \in \mathrm{Der}_a(\mathfrak{g})$. □

Corollary 2.3.5. *For all* $X \in \mathfrak{g}$ *and* $D \in \mathcal{C}(\mathfrak{g})$, $D \circ \mathrm{ad}(X) \in \mathrm{Der}_a(\mathfrak{g})$.

It is known that $d_q(\mathfrak{g}) = 1$ if $\mathfrak{g}$ is simple or one-dimensional Lie algebra. If $\mathfrak{g}$ is reductive, but neither simple, nor one-dimensional, then

$$d_q(\mathfrak{g}) = s(\mathfrak{g}) + \frac{\dim(\mathcal{Z}(\mathfrak{g}))(1+\dim(\mathcal{Z}(\mathfrak{g})))}{2}$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $s(\mathfrak{g})$ is the number of simple ideals of a Levi factor of $\mathfrak{g}$ (Corollary 2.1 in [Ben03], see also in [BB07]). A general formula for $d_q(\mathfrak{g})$ is not known. Next, we give a formula of $d_q(\mathfrak{g})$ for reduced singular quadratic Lie algebras.

2.3.2 Quadratic dimension of reduced singular quadratic Lie algebras and the invariance of dup number

Proposition 2.3.6. *Let $\mathfrak{g}$ be a reduced singular quadratic Lie algebra and $D \in \mathscr{L}(\mathfrak{g})$ be a symmetric map. Then:*

(1) D is a centromorphism if and only if there exist $\mu \in \mathbb{C}$ and a symmetric map $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ such that $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and $D = \mu\,\mathrm{Id} + \mathsf{Z}$. Moreover D is invertible if and only if $\mu \neq 0$.

(2)

$$d_q(\mathfrak{g}) = 1 + \frac{\dim(\mathscr{Z}(\mathfrak{g}))(1+\dim(\mathscr{Z}(\mathfrak{g}))}{2}.$$

Proof.

(1) If $\mathfrak{g} = \mathfrak{o}(3)$, with $B = \lambda\kappa$ and κ the Killing form, the two results are obvious. So, we examine the case where $\mathfrak{g}$ is solvable, and then $\mathfrak{g}$ can be realized as a double extension: $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$, with corresponding bilinear form $\overline{B}$ on $\mathfrak{q}$, $C = \mathrm{ad}(Y_1)$, $\overline{C} = C|_{\mathfrak{q}} \in \mathfrak{o}(\mathfrak{q})$.

Let D be an invertible centromorphism. One has $D \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ D$, for all $X \in \mathfrak{g}$ and that implies $DC = CD$. Using formula (1) of Lemma 2.2.27 and $CD = DC$, from $[D(X), Y_1] = [X, D(Y_1)]$, we find $D(C(X)) = B(D(X_1), Y_1)C(X)$. Let $\mu = B(D(X_1), Y_1)$. Since D is invertible, one has $\mu \neq 0$ and $C(D - \mu\,\mathrm{Id}) = 0$. Since $\ker(C) = \mathbb{C}X_1 \oplus \ker(\overline{C}) \oplus \mathbb{C}Y_1 = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_1$, there exists a map $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ and $\varphi \in \mathfrak{g}^*$ such that $D - \mu\,\mathrm{Id} = \mathsf{Z} + \varphi \otimes Y_1$. But D maps $[\mathfrak{g},\mathfrak{g}]$ into itself, so $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$. One has $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_1 \oplus \mathrm{Im}(\overline{C})$. If $X \in \mathrm{Im}(\overline{C})$, let $X = C(Y)$. Then $D(X) = D(C(Y)) = \mu C(Y)$, so $D(X) = \mu X$. For Y_1, $D([Y_1, X]) = DC(X) = \mu C(X)$ for all $X \in \mathfrak{g}$. But also, $D([Y_1, X]) = [D(Y_1), X] = \mu C(X) + \varphi(Y_1)C(X)$, hence $\varphi(Y_1) = 0$.

Assume we have shown that $D(X_1) = \mu X_1$. Then if $X \in \mathfrak{q}$, $B(D(X_1), X) = \mu B(X_1, X) = 0$. Moreover, $B(D(X_1), X) = B(X_1, D(X))$, so $\varphi(X) = 0$. Thus, to prove (1), we must prove that $D(X_1) = \mu X_1$. We decompose $\mathfrak{q}$ respectively to $\overline{C}$ as in Appendix A. Let $\mathfrak{l} = \ker(\overline{C})$. Then:

$$\mathfrak{q} = (\mathfrak{l} \oplus \mathfrak{l}') \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$$

and C is an isomorphism from $\mathfrak{l}' \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ onto $\mathfrak{l} \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$. Or

$$\mathfrak{q} = (\mathfrak{l} + \mathfrak{l}') \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$$

and C is an isomorphism from $\mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ onto $\mathfrak{l} \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$.

If $\mathfrak{u} \oplus \mathfrak{u}' \neq \{0\}$, there exist X', $Y' \in \mathfrak{u} \oplus \mathfrak{u}'$ such that $B(X', Y') = -1$ and $X, Y \in \mathfrak{l}' \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ (resp. $\mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$) such that $X' = C(X)$, $Y' = C(Y)$. It follows that $[C(X), Y] = X_1$ and then $D(X_1) = [DC(X), Y] = \mu[C(X), Y] = \mu X_1$.

If $\mathfrak{u} \oplus \mathfrak{u}' = \{0\}$, then either $\mathfrak{q} = (\mathfrak{l} + \mathfrak{l}') \overset{\perp}{\oplus} \mathbb{C}T$ or $\mathfrak{q} = \mathfrak{l} + \mathfrak{l}'$. The first case is similar to the situation above, setting $X' = Y' = \dfrac{T}{i}$ and $X, Y \in \mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T$. In the second case, $\mathfrak{l} = \mathrm{Im}(\overline{C})$ is totally isotropic and C is an isomorphism from $\mathfrak{l}'$ onto $\mathfrak{l}$. For any non-zero $X \in \mathfrak{l}'$, choose a non-zero $Y \in \mathfrak{l}'$ such that $B(C(X),Y) = 0$. Then $D([X,Y]) = D(B(C(X),Y)X_1) = 0$. But this is also equal to $[D(X),Y] = \mu[X,Y] + \varphi(X)C(Y)$. Since D is invertible, $[X,Y] = 0$ and we conclude that $\varphi(X) = 0$. Therefore $\varphi|_{\mathfrak{l}'} = 0$. There exist $L, L' \in \mathfrak{l}'$ such that $X_1 = [L,L']$ and then $D(X_1) = \mu X_1$.

Finally, $\mathcal{C}(\mathfrak{g})$ is generated by invertible centromorphisms, so the necessary condition of (1) follows. The sufficiency is a simple verification.

(2) As in (1), we can restrict ourselves to a double extension and follow the same notation. By (1), D is a centromorphism if and only if $D(X) = \mu X + \mathsf{Z}(X)$, for all $X \in \mathfrak{g}$ with $\mu \in \mathbb{C}$ and Z is a symmetric map from $\mathfrak{g}$ into $\mathcal{Z}(\mathfrak{g})$ satisfying $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$. To compute $d_q(\mathfrak{g})$, we use Appendix A. Assume $\dim(\mathfrak{q})$ is even and write $\mathfrak{q} = (\mathfrak{l} \oplus \mathfrak{l}') \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ with $\mathfrak{l} = \ker(\overline{C})$, $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_1 \oplus \mathfrak{l}$, $\mathrm{Im}(\overline{C}) = \mathfrak{l} \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ and $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_1 \oplus \mathrm{Im}(\overline{C})$. Let us define $\mathsf{Z} : \mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}Y_1 \to \mathfrak{l} \overset{\perp}{\oplus} \mathbb{C}X_1$: set basis $\{X_1, X_2, \dots, X_r\}$ of $\mathfrak{l} \oplus \mathbb{C}X_1$ and $\{Y_1' = Y_1, Y_2', \dots, Y_r'\}$ of $\mathfrak{l}' \oplus \mathbb{C}Y_1$ such that $B(Y_i', X_j) = \delta_{ij}$. Then Z is completely defined by

$$\mathsf{Z}\left(\sum_{j=1}^{r} \mu_j Y_j'\right) = \sum_{i=1}^{r}\left(\sum_{j=1}^{r} \nu_{ij}\mu_j\right) X_i$$

with $\nu_{ij} = \nu_{ji} = B(Y_i', \mathsf{Z}(Y_j'))$ and the formula follows. The case of $\dim(\mathfrak{q})$ odd is completely similar.

□

As a consequence of Proposition 2.3.6, we have:

Theorem 2.3.7. *The* dup*-number is invariant under isomorphisms, i.e. if* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are quadratic Lie algebras with* $\mathfrak{g} \simeq \mathfrak{g}'$*, then* $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{g}')$.

Proof. Assume that $\mathfrak{g} \simeq \mathfrak{g}'$. Since an i-isomorphism does not change $\mathrm{dup}(\mathfrak{g}')$, we can assume that $\mathfrak{g} = \mathfrak{g}'$ as Lie algebras equipped with invariant bilinear forms B and B'. Thus, we have two dup-numbers, $\mathrm{dup}_B(\mathfrak{g})$ and $\mathrm{dup}_{B'}(\mathfrak{g})$.

We choose $\mathfrak{z}$ such that $\mathcal{Z}(\mathfrak{g}) = (\mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]) \oplus \mathfrak{z}$. Then $\mathfrak{z} \cap \mathfrak{z}^{\perp_B} = \{0\}$, $\mathfrak{z}$ is a central ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{l} \overset{\perp_B}{\oplus} \mathfrak{z}$ with $\mathfrak{l}$ a reduced quadratic Lie algebra. Then $\mathrm{dup}_B(\mathfrak{g}) = \mathrm{dup}_B(\mathfrak{l})$ (see Remark 2.2.10). Similarly, $\mathfrak{z} \cap \mathfrak{z}^{\perp_{B'}} = \{0\}$, $\mathfrak{g} = \mathfrak{l}' \overset{\perp_{B'}}{\oplus} \mathfrak{z}$ with $\mathfrak{l}$ a reduced quadratic Lie algebra and $\mathrm{dup}_{B'}(\mathfrak{g}) = \mathrm{dup}_{B'}(\mathfrak{l}')$. Now, $\mathfrak{l}$ and $\mathfrak{l}'$ are isomorphic to $\mathfrak{g}/\mathfrak{z}$, so $\mathfrak{l} \simeq \mathfrak{l}'$. Therefore, it is enough to prove the result for reduced quadratic Lie algebras to conclude that $\mathrm{dup}_B(\mathfrak{l}) = \mathrm{dup}_{B'}(\mathfrak{l})$ and then that $\mathrm{dup}_B(\mathfrak{g}) = \mathrm{dup}_{B'}(\mathfrak{g})$.

Consider a reduced quadratic Lie algebra $\mathfrak{g}$ equipped with bilinear forms B and B' and associated 3-forms I and I'. We have $\mathrm{dup}_B(\mathfrak{g}) = \dim(\mathcal{V}_I)$ and $\mathrm{dup}_{B'}(\mathfrak{g}) = \dim(\mathcal{V}_{I'})$ with $\mathcal{V}_I = \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\}$ and $\mathcal{V}_{I'} = \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I' = 0\}$.

We start with the case $\mathrm{dup}_B(\mathfrak{g}) = 3$. This is true if and only if $\dim([\mathfrak{g},\mathfrak{g}]) = 3$ by Remark 2.2.10. Then $\mathrm{dup}_{B'}(\mathfrak{g}) = 3$.

If $\mathrm{dup}_B(\mathfrak{g}) = 1$, then $\mathfrak{g}$ is of type S_1 with respect to B. We apply Proposition 2.3.6 to obtain an invertible centromorphism $D = \mu\,\mathrm{Id} + \mathsf{Z}$ for a non-zero $\mu \in \mathbb{C}$, $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ satisfying $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and such that $B'(X,Y) = B(D(X),Y)$, for all $X,Y \in \mathfrak{g}$. Then $I'(X,Y,Z) = B'([X,Y],Z) = B([D(X),Y],Z) = \mu B([X,Y],Z) = \mu I(X,Y,Z)$, for all X, Y, $Z \in \mathfrak{g}$. So $I' = \mu I$ and $\mathrm{dup}_{B'}(\mathfrak{g}) = \mathrm{dup}_B(\mathfrak{g})$.

Finally, if $\mathrm{dup}_B(\mathfrak{g}) = 0$, then from the previous cases, $\mathfrak{g}$ cannot be of type S_3 or S_1 with respect to B', so $\mathrm{dup}_{B'}(\mathfrak{g}) = 0$.

□

2.3.3 Centromorphisms and extensions of a quadratic Lie algebra

First we recall the definition of double extension of a quadratic Lie algebra by a one-dimensional algebra as follows:

Definition 2.3.8. Let $(\mathfrak{g},B)$ be a quadratic Lie algebra and $\delta \in \mathrm{Der}_a(\mathfrak{g})$ the space of skew-symmetric derivations of $\mathfrak{g}$. Denote by $\overline{\mathfrak{g}}$ the Lie algebra defined by $\overline{\mathfrak{g}} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{g}$ with the bracket:

$$[\alpha X_1 + \beta Y_1 + X, \alpha' X_1 + \beta' Y_1 + Y] = [X,Y]_{\mathfrak{g}} + \beta\delta(Y) - \beta'\delta(X) + B(\delta(X),Y)X_1$$

and the non-degenerate invariant symmetric bilinear form B on $\mathfrak{g}$ is extended on $\overline{\mathfrak{g}}$ by:

$$\overline{B}(\alpha X_1 + \beta Y_1 + X, \alpha' X_1 + \beta' Y_1 + Y) = B(X,Y) + \alpha\beta' + \alpha'\beta,$$

for all $X,Y \in \mathfrak{g}$, $\alpha,\alpha',\beta,\beta' \in \mathbb{C}$. Then the quadratic Lie algebra $(\overline{\mathfrak{g}},\overline{B})$ is called the *double extension of* $(\mathfrak{g},B)$ *by means of* δ.

Proposition 2.3.9. *Let* $(\mathfrak{g},B)$ *be a quadratic Lie algebra and* $D \in \mathscr{C}_I(\mathfrak{g})$. *Assume that there exist a derivation* $\delta \in \mathrm{Der}_a(\mathfrak{g})$, *a non-zero* $x \in \mathbb{C}$ *and an element* $U \in \ker(\delta)$ *such that* $\delta D = D\delta = x\delta + \mathrm{ad}_{\mathfrak{g}}(U)$. *Let* $\overline{\mathfrak{g}}$ *be the double extension of* $\mathfrak{g}$ *by means of* δ. *Then the endomorphism* $\overline{D}$ *of* $\overline{\mathfrak{g}}$ *defined by:*

$$\overline{D}|_{\mathfrak{g}} = D + \phi(U) \otimes X_1,\ \overline{D}(X_1) = xX_1,\ \overline{D}(Y_1) = xY_1 + U + yX_1$$

with $y \in \mathbb{C}$ *is an invertible centromorphism of* $\overline{\mathfrak{g}}$.

Proof. It is obvious that $\overline{D}$ is symmetric and invertible. Let $\alpha X_1 + \beta Y_1 + X$ and $\alpha' X_1 + \beta' Y_1 + Y$ be elements in $\overline{\mathfrak{g}}$, one has:

$$\begin{aligned}\overline{D}[\alpha X_1 + \beta Y_1 + X, \alpha' X_1 + \beta' Y_1 + Y] &= D[X,Y]_{\mathfrak{g}} + B(U,[X,Y]_{\mathfrak{g}})X_1 + \beta D\delta(Y) \\ &\quad - \beta' D\delta(X) + xB(\delta(X),Y)X_1.\end{aligned}$$

Furthermore, we get:

$$\begin{aligned}\overline{D}(\alpha X_1 + \beta Y_1 + X), \alpha' X_1 + \beta' Y_1 + Y] &= [D(X),Y]_{\mathfrak{g}} + \beta[U,Y]_{\mathfrak{g}} + \beta x\delta(Y) \\ &\quad - \beta'\delta D(X) + B(\delta D(X),Y)X_1.\end{aligned}$$

Therefore, since $\delta D = D\delta = x\delta + \mathrm{ad}_{\mathfrak{g}}(U)$ we obtain $\overline{D}$ a centromorphism of $\overline{\mathfrak{g}}$. □

Corollary 2.3.10. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and $\delta \in \mathrm{Der}_a(\mathfrak{g})$. Let $\overline{\mathfrak{g}}$ be the double extension of $\mathfrak{g}$ by means of δ. Then the endomorphism $\overline{D}$ of $\overline{\mathfrak{g}}$ defined by:*

$$\overline{D}|_{\mathfrak{g}} = x\,\mathrm{Id},\ \overline{D}(X_1) = xX_1,\ \overline{D}(Y_1) = xY_1 + yX_1$$

with $x \in \mathbb{C}^*, y \in \mathbb{C}$ is an invertible endomorphism of $\overline{\mathfrak{g}}$. Consequently, $d_q(\overline{\mathfrak{g}}) \geq 2$.

Proof. The result can be obtained from the previous proposition by setting $D = x\,\mathrm{Id}, x \in \mathbb{C}^*$ and $U = 0$. □

Keep the notations as in Proposition 2.3.9 and define the set:

$$E(\mathfrak{g}, B, \delta) = \{(x, y, U, D) \in \mathbb{C} \times \mathbb{C} \times \ker\delta \times \mathcal{C}(\mathfrak{g}) \mid \ \delta D = D\delta = x\delta + \mathrm{ad}_{\mathfrak{g}}(U)\}.$$

Therefore, if $d_q(\overline{\mathfrak{g}}) = 2$ then $E(\mathfrak{g}, B, \delta) = \{(x, y, 0, x\,\mathrm{Id}) \mid \ x, y \in \mathbb{C}\}$ [BB07].

Proposition 2.3.11. *Let $\mathfrak{g}$ be a Lie algebra and $D : \mathfrak{g} \to \mathfrak{g}$ be an invertible linear map satisfying $D[X,Y] = [D(X), Y]$, for all $X, Y \in \mathfrak{g}$. Assume that there exists a cyclic 2-cocycle $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ such that $\theta(D(X), Y) = \theta(X, D(Y))$, for all $X, Y \in \mathfrak{g}$. Denote by $T^*_\theta(\mathfrak{g})$ the T^*-extension of $\mathfrak{g}$ by means of θ then the endomorphism $\overline{D}$ of $T^*_\theta(\mathfrak{g})$ defined by:*

$$\overline{D}(X + f) = D(X) + f \circ D,\ \forall\, X \in \mathfrak{g}, f \in \mathfrak{g}^*$$

is an invertible centromorphism of $T^*_\theta(\mathfrak{g})$.

Proof. Since D is invertible, so is $\overline{D}$. Let $X + f, Y + g \in T^*_\theta(\mathfrak{g})$, one has:

$$B(\overline{D}(X + f), Y + g) = f \circ D(Y) + g \circ D(X) = B(X + f, \overline{D}(Y + g)),$$

$$\overline{D}[X + f, Y + g] = D[X,Y]_{\mathfrak{g}} + \theta(X,Y) \circ D + f \circ \mathrm{ad}_{\mathfrak{g}}(Y) \circ D - g \circ \mathrm{ad}_{\mathfrak{g}}(X) \circ D$$

and $[\overline{D}(X + f), Y + g] = [D(X), Y]_{\mathfrak{g}} + \theta(D(X), Y) + f \circ D \circ \mathrm{ad}_{\mathfrak{g}}(Y) - g \circ D \circ \mathrm{ad}_{\mathfrak{g}}(X)$.

Remark that the condition $D[X,Y] = [D(X), Y]$, for all $X, Y \in \mathfrak{g}$ is equivalent to $D \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X) \circ D$, for all $X \in \mathfrak{g}$. Since θ is cyclic then $\theta(X,Y) \circ D = \theta(D(X), Y)$. Therefore $\overline{D}[X + f, Y + g] = [\overline{D}(X + f), Y + g]$, $\forall\, X + f, Y + g \in T^*_\theta(\mathfrak{g})$ and so $\overline{D}$ is a centromorphism of $T^*_\theta(\mathfrak{g})$. □

A more general result is given in the proposition below:

Proposition 2.3.12. *Let $\mathfrak{g}$ be a Lie algebra endowed with an invariant symmetric bilinear form ω (not necessarily non-degenerate) and $D : \mathfrak{g} \to \mathfrak{g}$ be the invertible linear map satisfying $D[X,Y] = [D(X), Y]$, for all $X, Y \in \mathfrak{g}$. Assume that there exists a cyclic 2-cocycle $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ such that $\theta(D(X), Y) = \theta(X, D(Y))$, for all $X, Y \in \mathfrak{g}$ then the endomorphism $\overline{D}$ of $T^*_\theta(\mathfrak{g})$ defined by:*

$$\overline{D}(X + f) = D(X) + \varphi(X) + f \circ D,\ \forall\, X \in \mathfrak{g}, f \in \mathfrak{g}^*$$

is an invertible centromorphism of $T^*_\theta(\mathfrak{g})$ where $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ is defined by $\varphi(X) = \omega(X, .)$, for all $X \in \mathfrak{g}$.

Proof. It is easy to see that $\overline{D}$ is invertible. Since ω is symmetric then $\overline{D}$ is also symmetric. Prove similarly to Proposition 2.3.11 and note that the condition invariance of ω is equivalent to $\varphi([X,Y]_{\mathfrak{g}}) = \varphi(X) \circ \mathrm{ad}_{\mathfrak{g}}(Y)$, for all $X, Y \in \mathfrak{g}$, we get the result. □

2.4 2-step nilpotent quadratic Lie algebras

Conveniently, we redefine a 2-step nilpotent Lie algebra in another way as follows:

Definition 2.4.1. An algebra $\mathfrak{g}$ over $\mathbb{C}$ with a bilinear product $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}, (x,y)\mapsto[x,y]$ is called a *2-step nilpotent Lie algebra* if it satisfies $[x,y]=-[y,x]$ and $[[x,y],z]=0$ for all $x,y,z\in\mathfrak{g}$. Sometimes, we use the notion *2SN-Lie algebra* as an abbreviation.

According to this definition, a commutative Lie algebra is a trivial case of 2SN-Lie algebras.

2.4.1 Some extensions of 2-step nilpotent Lie algebras

Definition 2.4.2. Let $\mathfrak{g}$ be a 2SN-Lie algebra, V be a vector space and $\varphi:\mathfrak{g}\times\mathfrak{g}\to V$ be a bilinear map. On the space $\overline{\mathfrak{g}}=\mathfrak{g}\oplus V$ we define the following product:

$$[x+u,y+v]=[x,y]+\varphi(x,y),\ \forall\, x,y\in\mathfrak{g},u,v\in V.$$

Then it is easy to see that $\overline{\mathfrak{g}}$ is a 2SN-Lie algebra if and only if φ is skew-symmetric and $\varphi([x,y],z)=0$, for all $x,y,z\in\mathfrak{g}$. In this case V is contained in the center $\mathcal{Z}(\overline{\mathfrak{g}})$ of $\overline{\mathfrak{g}}$ so the Lie algebra $\overline{\mathfrak{g}}$ is called the *2SN-central extension* of $\mathfrak{g}$ by V by means of φ.

Proposition 2.4.3. *Let $\mathfrak{g}$ be a 2SN-Lie algebra then $\mathfrak{g}$ is the 2SN-central extension of an Abelian algebra $\mathfrak{h}$ by some vector space V.*

Proof. Denote by $V=[\mathfrak{g},\mathfrak{g}]$ and let $\mathfrak{h}=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Then $\mathfrak{h}$ is Abelian. Set the map $\varphi:\mathfrak{h}\times\mathfrak{h}\to V$ by

$$\varphi(p(x),p(y))=[x,y],\ \forall\, x,y\in\mathfrak{g},$$

where $p:\mathfrak{g}\to\mathfrak{h}$ is the canonical projection. This map is well defined since $\mathfrak{g}$ is 2-step nilpotent. So $\mathfrak{g}$ is the 2SN-central extension of $\mathfrak{h}$ by V by means of φ. □

Let $\mathfrak{g}$ be a 2SN-Lie algebra, V be a vector space and $\pi:\mathfrak{g}\to\mathrm{End}(V)$ be a linear map. On the space $\overline{\mathfrak{g}}=\mathfrak{g}\oplus V$ we define the following product:

$$[x+u,y+v]=[x,y]+\pi(x)v-\pi(y)u,\ \forall\, x,y\in\mathfrak{g},u,v\in V.$$

Proposition 2.4.4. *The vector space $\overline{\mathfrak{g}}$ is a 2SN-Lie algebra if and only if π satisfies the condition:*

$$\pi([x,y])=\pi(x)\pi(y)=0,\ \forall\, x,y\in\mathfrak{g}.$$

In this case, π is called a **2SN-representation** *of $\mathfrak{g}$ in V.*

Proof. For all $x,y,z\in\mathfrak{g}$, $u,v,w\in V$ the condition $[[x+u,y+v],z+w]=0$ is equivalent to $\pi([x,y])w-\pi(z)\pi(x)v+\pi(z)\pi(y)u=0$ for all $u,v,w\in V$. This happens if and only if $\pi([x,y])=\pi(x)\pi(y)=0$, for all $x,y\in\mathfrak{g}$. □

Remark 2.4.5. The adjoint representation and the coadjoint representation are 2SN-representations of a 2SN-Lie algebra $\mathfrak{g}$. Therefore, the extensions of $\mathfrak{g}$ by itself or its dual space with respect to these representations are 2-step nilpotent.

Definition 2.4.6. Let $\mathfrak{g}$ be a 2SN-Lie algebra, V and W be vector spaces. If $\pi : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho : \mathfrak{g} \to \mathrm{End}(W)$ are two 2SN-representations of $\mathfrak{g}$ then the map $\pi \oplus \rho : \mathfrak{g} \to \mathrm{End}(V \oplus W)$ defined by

$$(\pi \oplus \rho)(x)(v+w) = \pi(x)v + \rho(x)w,\ \forall\, x \in \mathfrak{g}, v \in V, w \in W$$

is also a 2SN-representation of $\mathfrak{g}$ and it is called the *direct sum* of the two representations π and ρ.

Proposition 2.4.7. *Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be 2SN-Lie algebras and $\pi : \mathfrak{g}_1 \to \mathrm{End}(\mathfrak{g}_2)$ be a linear map. We define on the vector space $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ the following product:*

$$[x+y, x'+y'] = [x,x']_{\mathfrak{g}_1} + \pi(x)y' - \pi(x')y + [y,y']_{\mathfrak{g}_2},\ \forall\, x,x' \in \mathfrak{g}_1, y,y' \in \mathfrak{g}_2.$$

Then $\mathfrak{g}$ with this product is a 2SN-Lie algebra if and only if π satisfies the following conditions:

(1) $\pi([x,x']_{\mathfrak{g}_1}) = \pi(x)\pi(x') = 0.$

(2) $\pi(x)([y,y']_{\mathfrak{g}_2}) = [\pi(x)y,y']_{\mathfrak{g}_2} = 0.$

for all $x,x' \in \mathfrak{g}_1, y,y' \in \mathfrak{g}_2$.

Proof. Assume that $\mathfrak{g}$ is a 2SN-Lie algebra. For all $x,x',x'' \in \mathfrak{g}_1$, $y,y',y'' \in \mathfrak{g}_2$, one has:

$$0 = [[x+y,x'+y'],x''+y''] = \pi([x,x']_{\mathfrak{g}_1})y'' - \pi(x'')\pi(x)y' +$$

$$+\pi(x'')\pi(x')y + [\pi(x)y',y'']_{\mathfrak{g}_2} - [\pi(x')y,y'']_{\mathfrak{g}_2} - \pi(x'')([y,y']_{\mathfrak{g}_2}).$$

Let $y = y' = x'' = 0$ we obtain $\pi([x,x']_{\mathfrak{g}_1})y'' = 0$, for all $y'' \in \mathfrak{g}$. It means that $\pi([x,x']_{\mathfrak{g}_1}) = 0$. Similarly, $\pi(x)\pi(x')$, $\pi(x)([y,y']_{\mathfrak{g}_2})$ and $[\pi(x)y,y']_{\mathfrak{g}_2}$ are zero for all $x,x' \in \mathfrak{g}_1$, $y,y' \in \mathfrak{g}_2$.

Conversely, if π satisfies (1) and (2) then it is easy to check $\mathfrak{g}$ is a 2SN-Lie algebra by Definition 2.4.1. □

Clearly, the map π in Proposition 2.4.7 is a 2SN-representation of $\mathfrak{g}_1$ in $\mathfrak{g}_2$. Hence,, we obtain the following definition:

Definition 2.4.8. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be 2SN-Lie algebras and $\pi : \mathfrak{g}_1 \to \mathrm{End}(\mathfrak{g}_2)$ be a 2SN-representation of $\mathfrak{g}_1$ in $\mathfrak{g}_2$. Then the vector $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the product:

$$[x+y, x'+y'] = [x,x']_{\mathfrak{g}_1} + \pi(x)y' - \pi(x')y + [y,y']_{\mathfrak{g}_2},\ \forall\, x,x' \in \mathfrak{g}_1, y,y' \in \mathfrak{g}_2.$$

become a 2SN-Lie algebra if and only if

$$\pi(x)([y,y']_{\mathfrak{g}_2}) = [\pi(x)y,y']_{\mathfrak{g}_2} = 0,\ \forall\, x \in \mathfrak{g}_1, y,y' \in \mathfrak{g}_2.$$

In this case, π is called a *2SN-admissible representation* of $\mathfrak{g}_1$ in $\mathfrak{g}_2$ and we say that $\mathfrak{g}$ is the *semi-direct product of* $\mathfrak{g}_2$ *by* $\mathfrak{g}_1$ *by means of* π.

Remark 2.4.9.

(1) The condition in the above definition ensures $\pi(x) \in \mathrm{Der}(\mathfrak{g}_2)$, for all $x \in \mathfrak{g}_1$.

(2) The adjoint representation of a 2SN-Lie algebra is a 2SN-admissible representation.

2.4.2 2-step nilpotent quadratic Lie algebras

Let $(\mathfrak{g},B)$ be a quadratic Lie algebra and $\overline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{h}^*$ be the double extension of $\mathfrak{g}$ by $\mathfrak{h}$ by means of π as in Definition 2.1.9. If $\overline{\mathfrak{g}}$ is a 2SN-Lie algebra then $\mathfrak{g}$ and $\mathfrak{h}$ should be also 2-step nilpotent.

Proposition 2.4.10. *Let $(\mathfrak{g},B)$ be a 2-step nilpotent quadratic Lie algebra (or* **2SNQ-Lie algebra** *for short), $\mathfrak{h}$ be another 2SN-Lie algebra and $\pi : \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g})$ be a representation of $\mathfrak{h}$ by means of skew-symmetric derivations of $\mathfrak{g}$. Then the double extension of $\mathfrak{g}$ by $\mathfrak{h}$ by means of π is 2-step nilpotent if and only if π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{g}$.*

Proof. We can prove directly by checking the conditions of Definition 2.4.1 for the Lie algebra $\overline{\mathfrak{g}}$. However, it is easy to see that $\overline{\mathfrak{g}}$ is the semi-direct product of $\mathfrak{h}$ by $\mathfrak{g} \oplus \mathfrak{h}^*$ by means of $\pi \oplus \mathrm{ad}^*$ where $\mathfrak{g} \oplus \mathfrak{h}^*$ is the central extension of $\mathfrak{g}$ by $\mathfrak{h}^*$ by means of ϕ. Therefore, the result follows. □

Combined with Proposition 2.4.7 and Definition 2.4.8 we obtain the following result:

Corollary 2.4.11. *Let $(\mathfrak{g},B)$ be a 2SNQ-Lie algebra and $D \in \mathrm{Der}_a(\mathfrak{g})$ be a skew-symmetric derivation of $\mathfrak{g}$. Then the double extension of $\mathfrak{g}$ by means of D is a 2SNQ-Lie algebra if and only if $D^2 = 0$ and $[D(x),y] = 0$, for all $x,y \in \mathfrak{g}$.*

Proposition 2.4.12. *Let $(\mathfrak{g},B)$ be a 2SNQ-Lie algebra of dimension $n+2$, $n \geq 0$. Then $\mathfrak{g}$ is the double extension of a 2SNQ-Lie algebra of dimension n. Consequently, every 2SNQ-Lie algebra can be obtained from an Abelian algebra by a sequence of double extensions by one-dimensional algebra.*

Proof. If $\mathfrak{g}$ is Abelian then $\mathfrak{g}$ is the double extension of an Abelian algebra by means of the zero map. If $\mathfrak{g}$ is non-Abelian. By Corollary 2.1.6, there exists a central element x such that x is isotropic. Then there is an isotropic element y such that $B(x,y) = 1$ and $\mathfrak{g}$ is the double extension of $(\mathfrak{h} = (\mathbb{C}x \oplus \mathbb{C}y)^\perp, B')$ where $B' = B|_{\mathfrak{h}\times\mathfrak{h}}$. Certainly, $\mathfrak{h}$ is still 2-step nilpotent. □

Proposition 2.4.13. *Let $\mathfrak{g}$ be a Lie algebra, θ be a cyclic 2-cocycle of $\mathfrak{g}$ with value in $\mathfrak{g}^*$ and $T^*_\theta(\mathfrak{g})$ be the T^*-extension of $\mathfrak{g}$ by means of θ (see Definition 2.1.12). Then $T^*_\theta(\mathfrak{g})$ is a 2SNQ-Lie algebra if and only if $\mathfrak{g}$ is 2-step nilpotent and θ satisfies*

$$\theta(x,y) \circ \mathrm{ad}_{\mathfrak{g}}(z) + \theta([x,y]_{\mathfrak{g}},z) = 0, \ \forall\, x,y,z \in \mathfrak{g}.$$

Proof. For all $x,y,z \in \mathfrak{g}$, $f,g,h \in \mathfrak{g}^*$ one has

$$[[x+f,y+g],z+h] = [[x,y],z]_{\mathfrak{g}} + (f \circ \mathrm{ad}_{\mathfrak{g}}(y) - g \circ \mathrm{ad}_{\mathfrak{g}}(x) + \theta(x,y)) \circ \mathrm{ad}_{\mathfrak{g}}(z)$$
$$-h \circ \mathrm{ad}_{\mathfrak{g}}([x,y]_{\mathfrak{g}}) + \theta([x,y]_{\mathfrak{g}},z).$$

Therefore, $T^*_\theta(\mathfrak{g})$ is 2-step nilpotent if and only if $\mathfrak{g}$ is 2-step nilpotent and θ satisfies

$$\theta(x,y) \circ \mathrm{ad}_{\mathfrak{g}}(z) + \theta([x,y],z) = 0, \ \forall\, x,y,z \in \mathfrak{g}.$$

□

By the above proposition, if $T^*_\theta(\mathfrak{g})$ is a 2SNQ-Lie algebra then $\mathfrak{g}$ should be 2-step nilpotent. However, we can only consider T^*_θ-extensions of an Abelian algebra by the following proposition.

Proposition 2.4.14. *Let $(\mathfrak{g}, B)$ be a reduced quadratic Lie algebra. Then $\mathfrak{g}$ is 2-step nilpotent if and only if it is i-isomorphic to a T^*_θ-extension of an Abelian algebra by means of a non-degenerate cyclic 2-cocycle θ.*

Proof. Assume that $\mathfrak{g}$ is 2-nilpotent then $[\mathfrak{g},\mathfrak{g}] \subset \mathscr{Z}(\mathfrak{g})$. Since $\mathfrak{g}$ is reduced one has $[\mathfrak{g},\mathfrak{g}] = \mathscr{Z}(\mathfrak{g})$ and $\dim(\mathfrak{g})$ even. By Proposition 2.1.13 (given in [Bor97]) and $\mathscr{Z}(\mathfrak{g})$ a totally isotropic ideal, we write $\mathfrak{g} = V \oplus \mathscr{Z}(\mathfrak{g})$ with V totally isotropic. We can identify V with the quotient algebra $\mathfrak{h} \simeq \mathfrak{g}/\mathscr{Z}(\mathfrak{g})$ and $\mathscr{Z}(\mathfrak{g})$ with $\mathfrak{h}^*$. Then $\mathfrak{g}$ is i-isomorphic to the T^*_θ-extension of $\mathfrak{h}$ by θ defined by

$$\theta(p_0(x), p_0(y)) = \phi(p_1([x,y])),$$

where p_0, p_1 are respectively the projections from $\mathfrak{g}$ into V and $\mathscr{Z}(\mathfrak{g})$. Certainly, $\mathfrak{h}$ is Abelian since $[\mathfrak{g},\mathfrak{g}] = \mathscr{Z}(\mathfrak{g})$. We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ and the bracket on $\mathfrak{g}$ becomes

$$[x+f, y+g] = \theta(x,y), \ \forall\, x,y \in \mathfrak{h}, f,g \in \mathfrak{h}^*.$$

Since $\mathscr{Z}(\mathfrak{g}) = \mathfrak{h}^*$ then θ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$.

Conversely, if $\mathfrak{g}$ is i-isomorphic to the T^*_θ-extension $T^*_\theta(\mathfrak{h})$ of Abelian algebra $\mathfrak{h}$ by means of a non-degenerate cyclic 2-cocycle θ, it is obvious that $\mathfrak{g}$ is 2-step nilpotent and $\mathscr{Z}(\mathfrak{g}) \simeq \mathscr{Z}(T^*_\theta(\mathfrak{h})) = \mathfrak{h}^*$. Since $\mathfrak{h}^*$ is totally isotropic then $\mathscr{Z}(\mathfrak{g})$ is also totally isotropic. Therefore $\mathfrak{g}$ is reduced. □

Consequently, we have a restricted definition for the reduced 2-step nilpotent case as follows:

Definition 2.4.15. Let $\mathfrak{h}$ be a complex vector space and $\theta : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*$ be a non-degenerate cyclic skew-symmetric bilinear map. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ be the vector space equipped with the bracket

$$[x+f, y+g] = \theta(x,y)$$

and the bilinear form

$$B(x+f, y+g) = f(y) + g(x),$$

for all $x,y \in \mathfrak{h}, f,g \in \mathfrak{h}^*$. Then $(\mathfrak{g}, B)$ is a 2SNQ-Lie algebra. We say that $\mathfrak{g}$ is the *T^*-extension* of $\mathfrak{h}$ by θ.

Theorem 2.4.16. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be T^*-extensions of $\mathfrak{h}$ by θ_1 and θ_2 respectively. Then:*

(1) there exists a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist an isomorphism A_1 of $\mathfrak{h}$ and an isomorphism A_2 of $\mathfrak{h}^*$ such that

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y)), \ \forall\, x,y \in \mathfrak{h}.$$

(2) there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isomorphism A_1 of $\mathfrak{h}$ such that

$$\theta_1(x,y) = \theta_2(A_1(x), A_1(y)) \circ A_1, \ \forall\, x,y \in \mathfrak{h}.$$

Proof.

(1) Let $A : \mathfrak{g} \to \mathfrak{g}'$ be a Lie algebra isomorphism. Since $\mathfrak{h}^* = \mathscr{Z}(\mathfrak{g}) = \mathscr{Z}(\mathfrak{g}')$ is stable by A then there exist linear maps $A_1 : \mathfrak{h} \to \mathfrak{h}$, $A_1' : \mathfrak{h} \to \mathfrak{h}^*$ and $A_2 : \mathfrak{h}^* \to \mathfrak{h}^*$ such that

$$A(x+f) = A_1(x) + A_1'(x) + A_2(f),\ \forall\, x \in \mathfrak{h}, f \in \mathfrak{h}^*.$$

It is obvious that A_2 is an isomorphism of $\mathfrak{h}^*$. We show that A_1 is also an isomorphism of $\mathfrak{h}$. Indeed, if there exists $x_0 \in \mathfrak{h}$ such that $A_1(x_0) = 0$ then $0 = [A(x_0), \mathfrak{g}']' = A([x_0, A^{-1}(\mathfrak{g}')]) = A([x_0, \mathfrak{g}])$. It means that $[x_0, \mathfrak{g}] = 0$. Since $\mathfrak{h}^* = \mathscr{Z}(\mathfrak{g})$ then $x_0 = 0$. Therefore A_1 is an isomorphism of $\mathfrak{h}$.

For all $x, y \in \mathfrak{h}, f, g \in \mathfrak{h}^*$, one has

$$A([x+f, y+g]) = A(\theta_1(x,y)) = A_2(\theta_1(x,y)).$$

and $[A(x+f), A(y+g)]' = \theta_2(A_1(x), A_1(y))$.

Therefore, $A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y))$ for all $x, y \in \mathfrak{h}$.

Conversely, if there exist an isomorphism A_1 of $\mathfrak{h}$ and an isomorphism A_2 of $\mathfrak{h}^*$ such that $A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y))$ for all $x, y \in \mathfrak{h}$, we define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(x+f) = A_1(x) + A_2(f)$ for all $x + f \in \mathfrak{g}$. Then it is easy to check that A is a Lie algebra isomorphism.

(2) Asumme $A : \mathfrak{g} \to \mathfrak{g}'$ is an i-isomorphism then there exist A_1 and A_2 defined as in (1). Let $x \in \mathfrak{h}, f \in \mathfrak{h}^*$, one has

$$B'(A(x), A(f)) = B(x, f) \Rightarrow A_2(f)(A_1(x)) = f(x).$$

Therefore, $A_2(f) = f \circ A_1^{-1}$, for all $f \in \mathfrak{h}^*$.

On the other hand, since $A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y))$ we obtain

$$\theta_1(x,y) = \theta_2(A_1(x), A_1(y)) \circ A_1,\ \forall\, x, y \in \mathfrak{h}.$$

Conversely, define $A(x+f) = A_1(x) + f \circ A_1^{-1}$, for all $x \in \mathfrak{h}, f \in \mathfrak{h}^*$ then A is an i-isomorphism.

□

Example 2.4.17. We keep the notations as above. Let $\mathfrak{g}'$ be the T^*-extension of $\mathfrak{h}$ by $\theta' = \lambda\theta$ where $\lambda \neq 0$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic by $A : \mathfrak{g} \to \mathfrak{g}'$ defined by $A(x+f) = \frac{1}{\sqrt[3]{\lambda}}x + \sqrt[3]{\lambda} f$, for all $x + f \in \mathfrak{g}$.

For a non-degenerate cyclic skew-symmetric bilinear map θ of $\mathfrak{h}$, define the 3-form

$$I(x,y,z) = \theta(x,y)z,\ \forall\, x, y, z \in \mathfrak{h}.$$

Then $I \in \mathscr{A}^3(\mathfrak{h})$. The non-degenerate condition of θ is equivalent to $\iota_x(I) \neq 0$ for all $x \in \mathfrak{h} \setminus \{0\}$. Conversely, let $\mathfrak{h}$ be a vector space and $I \in \mathscr{A}^3(\mathfrak{h})$ such that $\iota_x(I) \neq 0$ for every non-zero vector $x \in \mathfrak{h}$. Define $\theta : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*$ by $\theta(x,y) = I(x,y,.)$, for all $x, y \in \mathfrak{h}$ then θ is skew-symmetric and non-degenerate. Moreover, since I is alternating one has θ is cyclic and then we obtain a reduced 2SNQ-Lie algebra $T^*_\theta(\mathfrak{h})$ defined by θ. Therefore, there is a one-to-one map from the set of all T^*-extension of $\mathfrak{h}$ onto the subset $\{I \in \mathscr{A}^3(\mathfrak{h}) \mid \iota_x(I) \neq 0,\ \forall\, x \in \mathfrak{h} \setminus \{0\}\}$. We has a corollary of the above theorem as follows:

Corollary 2.4.18. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be T^*-extensions of $\mathfrak{h}$ with respect to I_1 and I_2. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if there exists an isomorphism A of $\mathfrak{h}$ such that $I_1(x,y,z) = I_2(A(x),A(y),A(z))$, for all $x,y,z \in \mathfrak{h}$.*

Remark 2.4.19. The element I in this case is exactly the 3-form associated to $\mathfrak{g}$ in Definition 2.2.2 and the above corollary is only a particular case of Lemma 2.2.8.

Lemma 2.4.20. *Let $\mathfrak{h}$ be a vector space and $I \in \mathscr{A}^3(\mathfrak{h})$ satisfying $\iota_x(I) \neq 0,\ \forall\, x \in \mathfrak{h} \setminus \{0\}\}$. If there are nontrivial subspaces $\mathfrak{h}_1,\ \mathfrak{h}_2$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and I is decomposed by $I = I_1 + I_2$ where $I_1 \in \mathscr{A}^3(\mathfrak{h}_1),\ I_2 \in \mathscr{A}^3(\mathfrak{h}_2)$. Then the T^*-extension $\mathfrak{g}$ of $\mathfrak{h}$ with respect to I is decomposable.*

Proof. Let $\mathfrak{a} = \mathfrak{h}_1 \oplus \mathfrak{h}_1^*$ be the T^*-extension of $\mathfrak{h}_1$ with respect to I_1 then $\mathfrak{a}$ is non-degenerate. We will show that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Indeed, one has:

$$[\mathfrak{h}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_2] = I(\mathfrak{h}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_2, .) = I(\mathfrak{h}_1, \mathfrak{h}_1, .) + I(\mathfrak{h}_1, \mathfrak{h}_2, .).$$

Since $I(\mathfrak{h}_1, \mathfrak{h}_2, .) = 0$ then $[\mathfrak{h}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_2] = I(\mathfrak{h}_1, \mathfrak{h}_1, .) = I_1(\mathfrak{h}_1, \mathfrak{h}_1, .) \subset \mathfrak{h}_1^*$. Therefore $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and then $\mathfrak{g}$ is decomposable. □

Remark 2.4.21. Denote by $N(2n)$ the set of i-isomorphism classes of $2n$-dimensional reduced 2SNQ-Lie algebras. It is obvious that $N(2) = N(4) = \emptyset$ and $N(6)$ has only an element (see Appendix C and Example 2.4.17, also in [PU07] or [Ova07]). By Appendix C and Lemma 2.4.20, $N(8) = \emptyset$, $N(10)$ contains only an element and $N(2n) \neq \emptyset$ if $n \geq 6$.

Chapter 3

Singular quadratic Lie superalgebras

This chapter is a natural adaptation of Chapter 2 for the quadratic Lie superalgebras: Lie superalgebras endowed with an even invariant non-degenerate bilinear form. In this context, we also have a trilinear form I. We will recall the construction of the super-exterior algebra ([Sch79], [Gié04]) and the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket $\{\ ,\ \}$ on it [MPU09] to get the same formula $\{I,I\}=0$ for quadratic Lie superalgebras. These guide us to define a dup-number and a subclass of quadratic Lie superalgebras having dup-number non-zero which can be characterized up to isomorphisms. Finally, we show that the dup-number is also an invariant of quadratic Lie superalgebras.

3.1 Application of $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie superalgebras to quadratic Lie superalgebras

We begin from a $\mathbb{Z}_2$-graded vector space $V=V_{\bar{0}}\oplus V_{\bar{1}}$ over $\mathbb{C}$. The subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are respectively called the *even part* and the *odd part* of V. Keep the notation $\mathscr{A}=\mathscr{A}(V_{\bar{0}})$ for the Grassmann algebra of alternating multilinear forms on $V_{\bar{0}}$ as in Chapter 2 and denote by $\mathscr{S}=\mathscr{S}(V_{\bar{1}})$ the ($\mathbb{Z}$-graded) algebra of symmetric multilinear forms on $V_{\bar{1}}$, i.e. $\mathscr{S}=\mathsf{S}(V_{\bar{1}}^*)$ where $\mathsf{S}(V_{\bar{1}}^*)$ is the symmetric algebra of $V_{\bar{1}}^*$. We define a $\mathbb{Z}\times\mathbb{Z}_2$-gradation on $\mathscr{A}$ and on $\mathscr{S}$ by

$$\mathscr{A}^{(i,\bar{0})}=\mathscr{A}^i,\quad \mathscr{A}^{(i,\bar{1})}=\{0\}$$

$$\text{and}\quad \mathscr{S}^{(i,\bar{i})}=\mathscr{S}^i,\quad \mathscr{S}^{(i,\bar{j})}=\{0\}\quad\text{if}\quad \bar{i}\neq\bar{j},$$

where $i,j\in\mathbb{Z}$ and $\bar{i},\bar{j}$ are the residue classes modulo 2 of i and j, respectively.

Set a gradation:

$$\mathcal{E}(V)=\mathscr{A}\underset{\mathbb{Z}\times\mathbb{Z}_2}{\otimes}\mathscr{S}.$$

More particularly, in terms of the $\mathbb{Z}$-gradations of $\mathscr{A}$ and $\mathscr{S}$

$$\mathcal{E}^n(V)=\bigoplus_{m=0}^{n}\left(\mathscr{A}^m\otimes\mathscr{S}^{n-m}\right),$$

and in terms of the $\mathbb{Z}_2$-gradations

$$\mathcal{E}_{\bar{0}}(V)=\mathscr{A}\otimes\left(\underset{j\geq 0}{\oplus}\mathscr{S}^{2j}\right)\ \text{ and }\ \mathcal{E}_{\bar{1}}(V)=\mathscr{A}\otimes\left(\underset{j\geq 0}{\oplus}\mathscr{S}^{2j+1}\right).$$

In other words, if $A=\Omega\otimes F\in\mathscr{A}^{\omega}\otimes\mathscr{S}^{f}$ then $A\in\mathcal{E}^{(\omega+f,\overline{f})}(V)$ where $\overline{f}$ denotes the residue class modulo 2 of f.

Next, we define the *super-exterior product* on $\mathcal{E}(V)$ as follows:

$$(\Omega\otimes F)\wedge(\Omega'\otimes F')=(-1)^{f\omega'}(\Omega\wedge\Omega')\otimes FF',$$

for all $\Omega\in\mathscr{A}$, $\Omega'\in\mathscr{A}^{\omega'}$, $F\in\mathscr{S}^{f}$, $F'\in\mathscr{S}$.

Proposition 3.1.1. *The vector space $\mathcal{E}(V)$ with this product becomes a commutative and associative $\mathbb{Z}\times\mathbb{Z}_2$-graded algebra. We call $\mathcal{E}(V)$ the* **super-exterior algebra** *of V^*.*

Proof. It is easy to see that the algebra $\mathcal{E}(V)$ with this product is a $\mathbb{Z}\times\mathbb{Z}_2$-graded algebra. Let $A=\Omega\otimes F\in\mathscr{A}^{\omega}\otimes\mathscr{S}^{f}$, $A'=\Omega'\otimes F'\in\mathscr{A}^{\omega'}\otimes\mathscr{S}^{f'}$ and $A''=\Omega''\otimes F''\in\mathscr{A}^{\omega''}\otimes\mathscr{S}^{f''}$ then

$$A'\wedge A=(\Omega'\otimes F')\wedge(\Omega\otimes F)=(-1)^{f'\omega}(\Omega'\wedge\Omega)\otimes F'F=(-1)^{f'\omega+\omega\omega'}(\Omega\wedge\Omega')\otimes FF'$$
$$=(-1)^{f'\omega+\omega\omega'+f\omega'}(\Omega\otimes F)\wedge(\Omega'\otimes F')=(-1)^{(\omega+f)(\omega'+f')+ff'}A\wedge A'.$$

Similarly, $(A\wedge A')\wedge A''=A\wedge(A'\wedge A'')=(-1)^{f(\omega'+\omega'')+f'\omega''}(\Omega\wedge\Omega'\wedge\Omega'')\otimes FF'F''$. Therefore, we get the result. □

Remark 3.1.2. There is another equivalent construction in [BP89], that is, $\mathcal{E}(V)$ is the space of super-antisymmetric multilinear mappings from V into $\mathbb{C}$. The algebras $\mathscr{A}$ and $\mathscr{S}$ are regarded as subalgebras of $\mathcal{E}(V)$ by identifying $\Omega:=\Omega\otimes 1$, $F:=1\otimes F$, and the tensor product $\Omega\otimes F=(\Omega\otimes 1)\wedge(1\otimes F)$ for all $\Omega\in\mathscr{A}$, $F\in\mathscr{S}$.

Now, we assume that the vector space V is equipped with a non-degenerate even supersymmetric bilinear form B. That means B is symmetric on $V_{\bar{0}}$, skew-symmetric on $V_{\bar{1}}$, $B(V_{\bar{0}},V_{\bar{1}})=0$ and $B|_{V_{\bar{0}}\times V_{\bar{0}}}$, $B|_{V_{\bar{1}}\times V_{\bar{1}}}$ are non-degenerate. In this case, $\dim(V_{\bar{1}})=2n$ must be even and V is also called a *quadratic $\mathbb{Z}_2$-graded vector space*. We recall the definition of the Poisson bracket on $\mathscr{S}$ as follows.

Definition 3.1.3. Let $\mathscr{B}=\{X_1,...,X_n,Y_1,...,Y_n\}$ be a canonical basis of $V_{\bar{1}}$ such that $B(X_i,X_j)=B(Y_i,Y_j)=0$, $B(X_i,Y_j)=\delta_{ij}$ and $\{p_1,...,p_n,q_1,...,q_n\}$ its dual basis. Then the Poisson bracket on the algebra $\mathscr{S}$ regarded as the polynomial algebra $\mathbb{C}[p_1,...,p_n,q_1,...,q_n]$ is defined by:

$$\{F,G\}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}-\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right),\qquad\forall\ F,G\in\mathscr{S}.$$

Combined with the notion of super-Poisson bracket on $\mathscr{A}$ in Chapter 2, we have a new bracket as follows [MPU09].

Definition 3.1.4. The *super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket* on $\mathcal{E}(V)$ is defined by:

$$\{\Omega\otimes F,\Omega'\otimes F'\}=(-1)^{f\omega'}\left(\{\Omega,\Omega'\}\otimes FF'+(\Omega\wedge\Omega')\otimes\{F,F'\}\right),$$

for all $\Omega\in\mathscr{A}$, $\Omega'\in\mathscr{A}^{\omega'}$, $F\in\mathscr{S}^{f}$, $F'\in\mathscr{S}$.

By straightforward calculating, it is easy to get simple properties of this bracket:

Proposition 3.1.5. $\mathcal{E}(V)$ *is a Lie superalgebra with the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket. In other words, for $A\in\mathcal{E}^{(a,b)}(V)$, $A'\in\mathcal{E}^{(a',b')}(V)$ and $A''\in\mathcal{E}^{(a'',b'')}(V)$:*

(1) $\{A',A\} = -(-1)^{aa'+bb'}\{A,A'\}$.

(2) $(-1)^{aa''+bb''}\{A,\{A',A''\}\}+(-1)^{a''a'+b''b'}\{A'',\{A,A'\}\}+(-1)^{a'a+b'b}\{A',\{A'',A\}\}=0.$

Moreover, one has $\{A,A'\wedge A''\}=\{A,A'\}\wedge A''+(-1)^{aa'+bb'}A'\wedge\{A,A''\}$.

Remark 3.1.6. The second formula in the previous proposition is equivalent to:

$$\{\{A,A'\},A''\}=\{A,\{A',A''\}\}-(-1)^{aa'+bb'}\{A',\{A,A''\}\}.$$

Therefore, if we denote by $\mathrm{ad}_{\mathrm{P}}(A)=\{A,.\}$, $A\in\mathcal{E}^{(a,b)}(V)$ and by $\mathrm{End}(\mathcal{E}(V))$ the vector space of endomorphisms of $\mathcal{E}(V)$ then $\mathrm{ad}_{\mathrm{P}}(A)\in\mathrm{End}(\mathcal{E}(V))$, for all $A\in\mathcal{E}(V)$ and:

$$\mathrm{ad}_{\mathrm{P}}(\{A,A'\})=\mathrm{ad}_{\mathrm{P}}(A)\circ\mathrm{ad}_{\mathrm{P}}(A')-(-1)^{aa'+bb'}\mathrm{ad}_{\mathrm{P}}(A')\circ\mathrm{ad}_{\mathrm{P}}(A)$$

for all $A'\in\mathcal{E}^{(a',b')}(V)$.

We recall that $\mathrm{End}(\mathcal{E}(V))$ has a natural $\mathbb{Z}\times\mathbb{Z}_2$-gradation as follows:

$$\deg(F)=(n,d),\ n\in\mathbb{Z},\ d\in\mathbb{Z}_2 \text{ if } \deg(F(A))=(n+a,d+b), \text{ where } A\in\mathcal{E}^{(a,b)}(V).$$

Denote by $\mathrm{End}^n_f(\mathcal{E}(V))$ the subspace of endomorphisms of degree (n,f) of $\mathrm{End}(\mathcal{E}(V))$. It is clear that if $A\in\mathcal{E}^{(a,b)}(V)$ then $\mathrm{ad}_{\mathrm{P}}(A)$ has degree $(a-2,b)$. Moreover, as we known, $\mathrm{End}(\mathcal{E}(V))$ is also a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra, frequently denoted by $\mathfrak{gl}(\mathcal{E}(V))$, with the Lie super-bracket defined by:

$$[F,G]=F\circ G-(-1)^{np+fg}G\circ F,\ \forall\, F\in\mathrm{End}^n_f(\mathcal{E}(V)),\ G\in\mathrm{End}^p_g(\mathcal{E}(V)).$$

Therefore, since Remark 3.1.6, one has the corollary:

Corollary 3.1.7.

$$\mathrm{ad}_{\mathrm{P}}(\{A,A'\})=[\mathrm{ad}_{\mathrm{P}}(A),\mathrm{ad}_{\mathrm{P}}(A')],\ \forall\, A,A'\in\mathcal{E}(V).$$

Definition 3.1.8. An endomorphism $D\in\mathfrak{gl}(\mathcal{E}(V))$ of degree (n,d) is called a *super-derivation* of $\mathcal{E}(V)$ (for the super-exterior product) if

$$D(A\wedge A')=D(A)\wedge A'+(-1)^{na+db}A\wedge D(A'),\ \forall\, A\in\mathcal{E}^{(a,b)}(V),\ A'\in\mathcal{E}(V).$$

Denote by $\mathcal{D}^n_d(\mathcal{E}(V))$ the space of super-derivations degree (n,d) of $\mathcal{E}(V)$ then we obtain a $\mathbb{Z}\times\mathbb{Z}_2$-gradation of the space of super-derivations $\mathcal{D}(\mathcal{E}(V))$ of $\mathcal{E}(V)$ as follows:

$$\mathcal{D}(\mathcal{E}(V))=\bigoplus_{d\in\mathbb{Z}_2}^{n\in\mathbb{Z}}\mathcal{D}^n_d(\mathcal{E}(V))$$

and $\mathcal{D}(\mathcal{E}(V))$ becomes a subalgebra of $\mathfrak{gl}(\mathcal{E}(V))$ [NR66]. Moreover, the last formula in Proposition 3.1.5 affirms that $\mathrm{ad}_{\mathrm{P}}(A)\in\mathcal{D}(\mathcal{E}(V))$, for all $A\in\mathcal{E}(V)$.

Example 3.1.9. Let $X \in V_x$ be a homogeneous element in V of degree x and define the endomorphism ι_X of $\mathcal{E}(V)$ by

$$\iota_X(A)(X_1,...,X_{a-1}) = (-1)^{xb}A(X,X_1,...,X_{a-1}),\ \forall\, A \in \mathcal{E}^{(a,b)}(V),\ X_1,\dots,X_{a-1} \in V.$$

Then ι_X is a super-derivation of $\mathcal{E}(V)$ of degree (-1,x) [BP89]. In particular

$$\iota_X(A\wedge A') = \iota_X(A)\wedge A' + (-1)^{-a+xb}A\wedge\iota_X(A'),\ \forall\, A\in\mathcal{E}^{(a,b)}(V),\ A'\in\mathcal{E}(V).$$

Lemma 3.1.10. *Let $X_{\bar 0}\in V_{\bar 0}$ and $X_{\bar 1}\in V_{\bar 1}$ then for all $\Omega\otimes F\in\mathscr{A}^\omega\otimes\mathscr{S}^f$, one has:*

(1) $\iota_{X_{\bar 0}}(\Omega\otimes F) = \iota_{X_{\bar 0}}(\Omega)\otimes F$,

(2) $\iota_{X_{\bar 1}}(\Omega\otimes F) = (-1)^\omega\Omega\otimes\iota_{X_{\bar 1}}(F)$.

Proof. We have:

(1) $\iota_{X_{\bar 0}}(\Omega\otimes F) = \iota_{X_{\bar 0}}((\Omega\otimes 1)\wedge(1\otimes F)) = \iota_{X_{\bar 0}}(\Omega\otimes 1)\wedge(1\otimes F) + (-1)^{-\omega}(\Omega\otimes 1)\wedge\iota_{X_{\bar 0}}(1\otimes F) = \iota_{X_{\bar 0}}(\Omega)\otimes F$.

(2) $\iota_{X_{\bar 1}}(\Omega\otimes F) = \iota_{X_{\bar 1}}((\Omega\otimes 1)\wedge(1\otimes F)) = \iota_{X_{\bar 1}}(\Omega\otimes 1)\wedge(1\otimes F) + (-1)^{-\omega}(\Omega\otimes 1)\wedge\iota_{X_{\bar 1}}(1\otimes F) = (-1)^\omega\Omega\otimes\iota_{X_{\bar 1}}(F)$.

□

Remark 3.1.11.

(1) If $\Omega\in\mathscr{A}^\omega$ then $\iota_X(\Omega)(X_1,\dots,X_{\omega-1}) = \Omega(X,X_1,\dots,X_{\omega-1})$, for all $X,X_1,\dots,X_{\omega-1}\in V_{\bar 0}$.

(2) Let X be an element of the canonical basis $\mathscr{B}$ of $V_{\bar 1}$ and $p\in V_{\bar 1}^*$ be its dual form. By Corollary II.1.52 in [Gié04] one has:

$$\iota_X(p^n)(X^{n-1}) = (-1)^n p^n(X^n) = (-1)^n(-1)^{n(n-1)/2}n!.$$

Moreover, $\dfrac{\partial p^n}{\partial p}(X^{n-1}) = n(p^{n-1})(X^{n-1}) = (-1)^{(n-1)(n-2)/2}n!$. It implies that

$$\iota_X(p^n)(X^{n-1}) = -\frac{\partial p^n}{\partial p}(X^{n-1}).$$

Since each $F\in\mathscr{S}^f$ is regarded as a polynomial in the variable p and by linearizing so one has the following property: let $X\in V_{\bar 1}$ and $p\in V_{\bar 1}^*$ be its dual form then

$$\iota_X(F) = -\frac{\partial F}{\partial p},\ \forall\, F\in\mathscr{S}.$$

Proposition 3.1.12. *Fix an orthonormal basis $\{X_{\bar 0}^1,\dots,X_{\bar 0}^m\}$ of $V_{\bar 0}$ and a canonical basis $\mathscr{B} = \{X_{\bar 1}^1,\dots,X_{\bar 1}^n,Y_{\bar 1}^1,\dots,Y_{\bar 1}^n\}$ of $V_{\bar 1}$. Then the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket on $\mathcal{E}(V)$ is given by:*

$$\{A,A'\} = (-1)^{\omega+f+1}\sum_{j=1}^m \iota_{X_{\bar 0}^j}(A)\wedge\iota_{X_{\bar 0}^j}(A') + (-1)^\omega\sum_{k=1}^n\left(\iota_{X_{\bar 1}^k}(A)\wedge\iota_{Y_{\bar 1}^k}(A') - \iota_{Y_{\bar 1}^k}(A)\wedge\iota_{X_{\bar 1}^k}(A')\right)$$

for all $A\in\mathscr{A}^\omega\otimes\mathscr{S}^f$ and $A'\in\mathcal{E}(V)$.

Proof. Let $A=\Omega\otimes F\in\mathscr{A}^{\omega}\otimes\mathscr{S}^{f}$ and $A'=\Omega'\otimes F'\in\mathscr{A}^{\omega'}\otimes\mathscr{S}^{f'}$, then

$$\{A,A'\}=(-1)^{f\omega'}\left(\{\Omega,\Omega'\}\otimes FF'+(\Omega\wedge\Omega')\otimes\{F,F'\}\right).$$

By the formula (I) in Chapter 2, one has

$$\{\Omega,\Omega'\}=(-1)^{\omega+1}\sum_{j=1}^{m}\iota_{X_{\bar 0}^j}(\Omega)\wedge\iota_{X_{\bar 0}^j}(\Omega').$$

Combined with Lemma 3.1.10 (1), we obtain

$$\{\Omega,\Omega'\}\otimes FF'=(-1)^{\omega+1}\sum_{j=1}^{m}\left(\iota_{X_{\bar 0}^j}(\Omega)\wedge\iota_{X_{\bar 0}^j}(\Omega')\right)\otimes FF'$$

$$=(-1)^{f(\omega'-1)+\omega+1}\sum_{j=1}^{m}\left(\iota_{X_{\bar 0}^j}(\Omega)\otimes F\right)\wedge\left(\iota_{X_{\bar 0}^j}(\Omega')\otimes F'\right)$$

$$=(-1)^{f\omega'+\omega+f+1}\sum_{j=1}^{m}\iota_{X_{\bar 0}^j}(A)\wedge\iota_{X_{\bar 0}^j}(A').$$

Let $\{p_1,\dots,p_n,q_1,\dots,q_n\}$ be the dual basis of $\mathscr{B}$ then

$$\{F,F'\}=\sum_{k=1}^{n}\left(\frac{\partial F}{\partial p_k}\frac{\partial F'}{\partial q_k}-\frac{\partial F}{\partial q_k}\frac{\partial F'}{\partial p_k}\right).$$

By Remark 3.1.11, one has

$$\{F,F'\}=\sum_{k=1}^{n}\left(\iota_{X_{\bar 1}^k}(F)\,\iota_{Y_{\bar 1}^k}(F')-\iota_{Y_{\bar 1}^k}(F)\,\iota_{X_{\bar 1}^k}(F')\right).$$

Combined with Lemma 3.1.10 (2), we obtain

$$(\Omega\wedge\Omega')\otimes\{F,F'\}=(\Omega\wedge\Omega')\otimes\sum_{k=1}^{n}\left(\iota_{X_{\bar 1}^k}(F)\,\iota_{Y_{\bar 1}^k}(F')-\iota_{Y_{\bar 1}^k}(F)\,\iota_{X_{\bar 1}^k}(F')\right)$$

$$=(-1)^{(f-1)\omega'}\sum_{k=1}^{n}\left(\left(\Omega\otimes\iota_{X_{\bar 1}^k}(F)\right)\wedge\left(\Omega'\otimes\iota_{Y_{\bar 1}^k}(F')\right)-\left(\Omega\otimes\iota_{Y_{\bar 1}^k}(F)\right)\wedge\left(\Omega'\otimes\iota_{X_{\bar 1}^k}(F')\right)\right)$$

$$=(-1)^{f\omega'+\omega}\sum_{k=1}^{n}\left(\iota_{X_{\bar 1}^k}(A)\wedge\iota_{Y_{\bar 1}^k}(A')-\iota_{Y_{\bar 1}^k}(A)\wedge\iota_{X_{\bar 1}^k}(A')\right).$$

Therefore, the result follows. □

Corollary 3.1.13. *Define the (even) isomorphism $\phi:V\to V^*$ by $\phi(X)=B(X,.)$, for all $X\in\mathfrak{g}$ then one has*

(1) $\{\alpha,A\}=\iota_{\phi^{-1}(\alpha)}(A)$,

(2) $\{\alpha,\alpha'\}=B(\phi^{-1}(\alpha),\phi^{-1}(\alpha'))$,

for all $\alpha,\alpha'\in V^*$, $A\in\mathcal{E}(V)$.

Proof.

(1) We apply Proposition 3.1.12, respectively for $\alpha=(X_{\bar 0}^i)^*=\phi(X_{\bar 0}^i)$, for all $i=1,\dots,m$, $\alpha=(Y_{\bar 1}^l)^*=\phi(X_{\bar 1}^l)$ and $\alpha=(-X_{\bar 1}^l)^*=\phi(Y_{\bar 1}^l)$, for all $l=1,\dots,n$ to obtain the result.

(2) Let $\alpha\in\mathfrak{g}_x^*$, $\alpha'\in\mathfrak{g}_{x'}^*$ be homogeneous forms in $\mathfrak{g}^*$, one has

$$\{\alpha,\alpha'\}=\iota_{\phi^{-1}(\alpha)}(\alpha')=(-1)^{xx'}\alpha'(\phi^{-1}(\alpha))=(-1)^{xx'}B(\phi^{-1}(\alpha'),\phi^{-1}(\alpha))$$
$$=B(\phi^{-1}(\alpha),\phi^{-1}(\alpha')).$$

□

Proposition 3.1.12 and Corollary 3.1.13 are enough for our purpose. But as a consequence of Lemma 6.9 in [PU07], one has a more general result as follows:

Proposition 3.1.14. *Let* $\{X_{\bar 0}^1,\dots,X_{\bar 0}^m\}$ *be a basis of* $V_{\bar 0}$ *and* $\{\alpha_1,\dots,\alpha_m\}$ *its dual basis. Let* $\{Y_{\bar 0}^1,\dots,Y_{\bar 0}^m\}$ *be the basis of* $V_{\bar 0}$ *defined by* $Y_{\bar 0}^i=\phi^{-1}(\alpha_i)$. *Set* $\mathscr{B}=\{X_{\bar 1}^1,\dots,X_{\bar 1}^n,Y_{\bar 1}^1,\dots,Y_{\bar 1}^n\}$ *be a canonical basis of* $V_{\bar 1}$. *Then the super* $\mathbb{Z}\times\mathbb{Z}_2$*-Poisson bracket on* $\mathcal{E}(V)$ *is given by:*

$$\begin{aligned}\{A,A'\} &= (-1)^{\omega+f+1}\sum_{i,j=1}^{m}B(Y_{\bar 0}^i,Y_{\bar 0}^j)\,\iota_{X_{\bar 0}^i}(A)\wedge\iota_{X_{\bar 0}^j}(A')\\ &\quad+(-1)^{\omega}\sum_{k=1}^{n}\Big(\iota_{X_{\bar 1}^k}(A)\wedge\iota_{Y_{\bar 1}^k}(A')-\iota_{Y_{\bar 1}^k}(A)\wedge\iota_{X_{\bar 1}^k}(A')\Big)\end{aligned}$$

for all $A\in\mathscr{A}^\omega\otimes\mathscr{S}^f$ *and* $A'\in\mathcal{E}(V)$.

Now, we consider the vector space

$$\mathscr{E}=\bigoplus_{n\in\mathbb{Z}}\mathscr{E}^n,$$

where $\mathscr{E}^n=\{0\}$ if $n\leq -2$, $\mathscr{E}^{-1}=V$ and $\mathscr{E}^n$ is the space of super-antisymmetric $n+1$-linear mappings from V to V. Each of the subspaces $\mathscr{E}^n$ is $\mathbb{Z}_2$-graded then the space $\mathscr{E}$ is $\mathbb{Z}\times\mathbb{Z}_2$-graded by

$$\mathscr{E}=\bigoplus_{f\in\mathbb{Z}_2}^{n\in\mathbb{Z}}\mathscr{E}_f^n.$$

Moreover, $\mathscr{E}$ is a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra and called the *graded Lie algebra* of the $\mathbb{Z}_2$-graded vector space V [BP89]. Recall that there exists a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra isomorphism D between $\mathscr{E}$ and $\mathcal{D}(\mathcal{E}(V))$ (see a precise construction in [Gié04]) satisfying if $F=\Omega\otimes X\in\mathscr{E}_{\omega+x}^n$ then $D_F=-(-1)^{x\omega}\Omega\wedge\iota_X\in\mathcal{D}_f^n(\mathcal{E}(V))$.

Lemma 3.1.15. *(*[BP89], [Gié04]*)*
Fix $F\in\mathscr{E}_{\bar 0}^1$, *denote by* $d=D_F$ *and define the product* $[X,Y]=F(X,Y)$, *for all* $X,Y\in V$. *Then one has*

(1) $d(\phi)(X,Y) = -\phi([X,Y])$, *for all* $X,Y \in V$, $\phi \in V^*$.

(2) The product $[\ ,\ ]$ *becomes a Lie super-bracket if and only if* $d^2 = 0$. *In this case,* d *is called a* **differential super-exterior** *of* $\mathcal{E}(V)$.

Next, we will apply the above results for quadratic Lie superalgebras defined as follows:

Definition 3.1.16. A *quadratic Lie superalgebra* $(\mathfrak{g},B)$ is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g}$ equipped with a non-degenerate even supersymmetric bilinear form B and a Lie superalgebra structure such that B is invariant, i.e. $B([X,Y],Z) = B(X,[Y,Z])$, for all $X,Y,Z \in \mathfrak{g}$.

Theorem 3.1.17. *Let* $(\mathfrak{g},B)$ *be a quadratic Lie superalgebra. Define a trilinear form* I *on* $\mathfrak{g}$ *by*

$$I(X,Y,Z) = B([X,Y],Z),\ \forall\, X,Y,Z \in \mathfrak{g}.$$

Then one has

(1) $I \in \mathcal{E}^{(3,\bar{0})}(\mathfrak{g}) = \mathscr{A}^3(\mathfrak{g}_{\bar{0}}) \oplus \left(\mathscr{A}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\bar{1}})\right)$.

(2) $d = -\operatorname{ad}_{\mathrm{P}}(I)$.

(3) $\{I,I\} = 0$.

Proof. The assertion (1) follows the properties of B, note that $B([\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}],\mathfrak{g}_{\bar{1}}) = B([\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}],\mathfrak{g}_{\bar{1}}) = 0$.

For (2), fix an orthonormal basis $\{X_{\bar{0}}^1,\dots,X_{\bar{0}}^m\}$ of $V_{\bar{0}}$ and a canonical basis $\{X_{\bar{1}}^1,\dots,X_{\bar{1}}^n, Y_{\bar{1}}^1,\dots,Y_{\bar{1}}^n\}$ of $V_{\bar{1}}$. Let $\{\alpha_1,\dots,\alpha_m\}$ and $\{\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n\}$ be their dual basis respectively. Then for all $X,Y \in \mathfrak{g}$, $i = 1,...,m$, $l = 1,...,n$ one has:

$$\begin{aligned}
\operatorname{ad}_{\mathrm{P}}(I)(\alpha_i)(X,Y) &= \left(\sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I)\wedge\iota_{X_{\bar{0}}^j}(\alpha_i) - \sum_{k=1}^n \left(\iota_{X_{\bar{1}}^k}(I)\wedge\iota_{Y_{\bar{1}}^k}(\alpha_i) - \iota_{Y_{\bar{1}}^k}(I)\wedge\iota_{X_{\bar{1}}^k}(\alpha_i)\right)\right)(X,Y)\\
&= \left(\sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I)\wedge\iota_{X_{\bar{0}}^j}(\alpha_i)\right)(X,Y) = \left(\iota_{X_{\bar{0}}^i}(I)\wedge\iota_{X_{\bar{0}}^i}(\alpha_i)\right)(X,Y)\\
&= B(X_{\bar{0}}^i,[X,Y]) = \alpha_i([X,Y]) = -d(\alpha_i)(X,Y),
\end{aligned}$$

$$\begin{aligned}
\operatorname{ad}_{\mathrm{P}}(I)(\beta_l)(X,Y) &= \left(\sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I)\wedge\iota_{X_{\bar{0}}^j}(\beta_l) - \sum_{k=1}^n \left(\iota_{X_{\bar{1}}^k}(I)\wedge\iota_{Y_{\bar{1}}^k}(\beta_l) - \iota_{Y_{\bar{1}}^k}(I)\wedge\iota_{X_{\bar{1}}^k}(\beta_l)\right)\right)(X,Y)\\
&= \left(\sum_{k=1}^n \iota_{Y_{\bar{1}}^k}(I)\wedge\iota_{X_{\bar{1}}^k}(\beta_l)\right)(X,Y) = \left(\iota_{Y_{\bar{1}}^l}(I)\wedge\iota_{X_{\bar{1}}^l}(\beta_l)\right)(X,Y)\\
&= -\iota_{Y_{\bar{1}}^l}(I)(X,Y) = -B(Y_{\bar{1}}^l,[X,Y]) = \beta_l([X,Y]) = -d(\beta_l)(X,Y),
\end{aligned}$$

$$\operatorname{ad}_{\mathrm{P}}(I)(\gamma_l)(X,Y) = \left(\sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I)\wedge\iota_{X_{\bar{0}}^j}(\gamma_l) - \sum_{k=1}^n \left(\iota_{X_{\bar{1}}^k}(I)\wedge\iota_{Y_{\bar{1}}^k}(\gamma_l) - \iota_{Y_{\bar{1}}^k}(I)\wedge\iota_{X_{\bar{1}}^k}(\gamma_l)\right)\right)(X,Y)$$

$$= -\left(\sum_{k=1}^{n} \iota_{X_{\bar 1}^k}(I)\wedge\iota_{Y_{\bar 1}^k}(\gamma_l)\right)(X,Y) = -\left(\iota_{X_{\bar 1}^l}(I)\wedge\iota_{Y_{\bar 1}^l}(\gamma_l)\right)(X,Y)$$

$$= \iota_{X_{\bar 1}^l}(I)(X,Y) = B(X_{\bar 1}^l,[X,Y]) = \gamma_l([X,Y]) = -d(\gamma_l)(X,Y).$$

Therefore, $d = -\operatorname{ad}_{\mathrm{P}}(I)$.

Moreover, $\operatorname{ad}_{\mathrm{P}}(\{I,I\}) = [\operatorname{ad}_{\mathrm{P}}(I),\operatorname{ad}_{\mathrm{P}}(I)] = [d,d] = 2d^2 = 0$. Therefore, for all $1\le i\le m$, $1\le j,k\le n$ one has $\{\alpha_i,\{I,I\}\} = \{\beta_j,\{I,I\}\} = \{\gamma_k,\{I,I\}\} = 0$. Those imply $\iota_X(\{I,I\}) = 0$ for all $X\in\mathfrak{g}$ and hence, we obtain $\{I,I\}=0$. □

Conversely, let $\mathfrak{g}$ be a quadratic $\mathbb{Z}_2$-graded vector space equipped with a bilinear form B and I be an element in $\mathcal{E}^{(3,\bar 0)}(\mathfrak{g})$. Define $d = -\operatorname{ad}_{\mathrm{P}}(I)$ then $d\in\mathcal{D}^1_{\bar 0}(\mathcal{E}(\mathfrak{g}))$. Therefore, $d^2=0$ if and only if $\{I,I\}=0$. Let F be the struture in $\mathfrak{g}$ corresponding to d by the isomorphism D in Lemma 3.1.15, one has

Proposition 3.1.18. *F becomes a Lie superalgebra structure if and only if $\{I,I\}=0$. In this case, with the notation $[X,Y] := F(X,Y)$ one has:*

$$I(X,Y,Z) = B([X,Y],Z),\ \forall\, X,Y,Z\in\mathfrak{g}.$$

Moreover, the bilinear form B is invariant.

Proof. We need to prove that if F is a Lie superalgebra structure then $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z\in\mathfrak{g}$. Indeed, let $\{X_{\bar 0}^1,\dots,X_{\bar 0}^m\}$ be an orthonormal basis of $V_{\bar 0}$ and $\{X_{\bar 1}^1,\dots,X_{\bar 1}^n, Y_{\bar 1}^1,\dots,Y_{\bar 1}^n\}$ be a canonical basis of $V_{\bar 1}$ then one has

$$d = -\operatorname{ad}_{\mathrm{P}}(I) = -\sum_{j=1}^{m}\iota_{X_{\bar 0}^j}(I)\wedge\iota_{X_{\bar 0}^j} + \sum_{k=1}^{n}\iota_{X_{\bar 1}^k}(I)\wedge\iota_{Y_{\bar 1}^k} - \sum_{k=1}^{n}\iota_{Y_{\bar 1}^k}(I)\wedge\iota_{X_{\bar 1}^k}.$$

It implies that

$$F = \sum_{j=1}^{m}\iota_{X_{\bar 0}^j}(I)\otimes X_{\bar 0}^j + \sum_{k=1}^{n}\iota_{X_{\bar 1}^k}(I)\otimes Y_{\bar 1}^k - \sum_{k=1}^{n}\iota_{Y_{\bar 1}^k}(I)\otimes X_{\bar 1}^k.$$

Therefore, for all i we obtain

$$B([X,Y],X_{\bar 0}^i) = \iota_{X_{\bar 0}^i}(I)(X,Y) = I(X_{\bar 0}^i,X,Y) = I(X,Y,X_{\bar 0}^i),$$

$$B([X,Y],X_{\bar 1}^i) = -\iota_{X_{\bar 1}^i}(I)(X,Y) = -I(X_{\bar 1}^i,X,Y) = I(X,Y,X_{\bar 1}^i),$$

$$B([X,Y],Y_{\bar 1}^i) = -\iota_{Y_{\bar 1}^i}(I)(X,Y) = -I(Y_{\bar 1}^i,X,Y) = I(X,Y,Y_{\bar 1}^i).$$

These show that $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z\in\mathfrak{g}$. □

Remark 3.1.19. The element I defined as above is also an invariant of $\mathfrak{g}$ since $\mathscr{L}_X(I)=0$, for all $X\in\mathfrak{g}$ where $\mathscr{L}_X = D(\operatorname{ad}(X))$ the Lie super-derivation of $\mathfrak{g}$. Therefore, I is called the *associated invariant* of $\mathfrak{g}$.

Lemma 3.1.20. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and I be its associated invariant. Then $\iota_X(I)=0$ if and only if $X\in\mathcal{Z}(\mathfrak{g})$.*

Proof. Since $\iota_X(I)(\mathfrak{g},\mathfrak{g}) = B(X,[\mathfrak{g},\mathfrak{g}])$ and $\mathscr{Z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp$ then one has $\iota_X(I) = 0$ if and only if $X \in \mathscr{Z}(\mathfrak{g})$. □

Definition 3.1.21. Let $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ be two quadratic Lie superalgebras. We say that $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ are *isometrically isomorphic* (or *i-isomorphic*) if there exists a Lie superalgebra isomorphism A from $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying

$$B'(A(X),A(Y)) = B(X,Y),\ \forall\, X,Y \in \mathfrak{g}.$$

In other words, A is an i-isomorphism if it is a (necessarily even) Lie superalgebra isomorphism and an isometry. We write $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$.

Note that two isomorphic quadratic Lie superalgebras $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ are not necessarily i-isomorphic by the example below:

Example 3.1.22. Let $\mathfrak{g} = \mathfrak{osp}(1,2)$ and B its Killing form. Recall that $\mathfrak{g}_{\overline{0}} = \mathfrak{o}(3)$. Consider another bilinear form $B' = \lambda B$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. In this case, $(\mathfrak{g},B)$ and $(\mathfrak{g},\lambda B)$ cannot be i-isomorphic if $\lambda \neq 1$ since $(\mathfrak{g}_{\overline{0}},B)$ and $(\mathfrak{g}_{\overline{0}},\lambda B)$ are not i-isomorphic (see Example 2.1.4).

3.2 The dup-number of a quadratic Lie superalgebra

Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and I be its associated invariant, then by Theorem 3.1.17

$$I = I_0 + I_1$$

where $I_0 \in \mathscr{A}^3(\mathfrak{g}_{\bar{0}})$ and $I_1 \in \mathscr{A}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\bar{1}})$. Since $\{I,I\} = 0$ one has $\{I_0,I_0\} = 0$. It means that $\mathfrak{g}_{\bar{0}}$ is a quadratic Lie algebra with the associated 3-form I_0. Remark that $\mathfrak{g}_{\bar{0}}$ is Abelian (resp. $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \{0\}$) if and only if $I_0 = 0$ (resp. $I_1 = 0$). Define the subspaces of $\mathfrak{g}^*$ as follows:

$$\begin{aligned}
\mathcal{V}_I &= \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\},\\
\mathcal{V}_{I_0} &= \{\alpha \in \mathfrak{g}_{\bar{0}}^* \mid \alpha \wedge I_0 = 0\},\\
\mathcal{V}_{I_1} &= \{\alpha \in \mathfrak{g}_{\bar{0}}^* \mid \alpha \wedge I_1 = 0\}.
\end{aligned}$$

The following lemma alows us to consider the notion dup-*number* for quadratic Lie superalgebras in a similar way we did for quadratic Lie algebras.

Lemma 3.2.1. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra then one has*

(1) $\dim(\mathcal{V}_I) \in \{0,1,3\}$,

(2) $\dim(\mathcal{V}_I) = 3$ *if and only if* $I_1 = 0$, $\mathfrak{g}_{\bar{0}}$ *is non-Abelian and* I_0 *is decomposable in* $\mathscr{A}^3(\mathfrak{g}_{\bar{0}})$.

Proof. Let $\alpha = \alpha_0 + \alpha_1 \in \mathfrak{g}_{\bar{0}}^* \oplus \mathfrak{g}_{\bar{1}}^*$ then one has

$$\alpha \wedge I = \alpha_0 \wedge I_0 + \alpha_0 \wedge I_1 + \alpha_1 \wedge I_0 + \alpha_1 \wedge I_1,$$

where $\alpha_0 \wedge I_0 \in \mathscr{A}^4(\mathfrak{g}_{\bar{0}})$, $\alpha_0 \wedge I_1 \in \mathscr{A}^2(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\bar{1}})$, $\alpha_1 \wedge I_0 \in \mathscr{A}^3(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^1(\mathfrak{g}_{\bar{1}})$ and $\alpha_1 \wedge I_1 \in \mathscr{A}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathscr{S}^3(\mathfrak{g}_{\bar{1}})$.

Hence, $\alpha \wedge I = 0$ if and only if $\alpha_1 = 0$ and $\alpha_0 \wedge I_0 = \alpha_0 \wedge I_1 = 0$. It means that $\mathcal{V}_I = \mathcal{V}_{I_0} \cap \mathcal{V}_{I_1}$. If $I_0 \neq 0$ then $\dim(\mathcal{V}_{I_0}) \in \{0,1,3\}$ and if $I_1 \neq 0$ then $\dim(\mathcal{V}_{I_1}) \in \{0,1\}$. Therefore, $\dim(\mathcal{V}_I) \in \{0,1,3\}$ and $\dim(\mathcal{V}_I) = 3$ if and only if $I_1 = 0$ and $\dim(\mathcal{V}_{I_0}) = 3$. □

Definition 3.2.2. Let $(\mathfrak{g},B)$ be a non-Abelian quadratic Lie superalgebra and I be its associated invariant. The dup *number* $\mathrm{dup}(\mathfrak{g})$ is defined by

$$\mathrm{dup}(\mathfrak{g}) = \dim(\mathcal{V}_I).$$

The decomposition in Proposition 2.1.5 is still right in the case of a quadratic Lie superalgebra below:

Proposition 3.2.3. *Let $(\mathfrak{g},B)$ be a non-Abelian quadratic Lie superalgebra. Keep the notations as in Chapter 2, then there exist a central ideal $\mathfrak{z}$ and an ideal $\mathfrak{l} \neq \{0\}$ such that:*

(1) $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ *where* $(\mathfrak{z}, B|_{\mathfrak{z}\times\mathfrak{z}})$ *and* $(\mathfrak{l}, B|_{\mathfrak{l}\times\mathfrak{l}})$ *are quadratic Lie superalgebras. Moreover,* $\mathfrak{l}$ *is non-Abelian.*

(2) The center $\mathcal{Z}(\mathfrak{l})$ *is totally isotropic, i.e.* $\mathcal{Z}(\mathfrak{l}) \subset [\mathfrak{l},\mathfrak{l}]$.

(3) *Let $\mathfrak{g}'$ be a quadratic Lie superalgebra and $A : \mathfrak{g} \to \mathfrak{g}'$ be a Lie superalgebra isomorphism. Then*

$$\mathfrak{g}' = \mathfrak{z}' \overset{\perp}{\oplus} \mathfrak{l}'$$

where $\mathfrak{z}' = A(\mathfrak{z})$ is central, $\mathfrak{l}' = A(\mathfrak{z})^\perp$, $\mathcal{Z}(\mathfrak{l}')$ is totally isotropic and $\mathfrak{l}$ and $\mathfrak{l}'$ are isomorphic. Moreover if A is an i-isomorphism, then $\mathfrak{l}$ and $\mathfrak{l}'$ are i-isomorphic.

Proof. The proof is exactly as Proposition 2.1.5 where $\mathfrak{z}$ is a complementary subspace of $\mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$ in $\mathcal{Z}(\mathfrak{g})$ and $\mathfrak{l} = \mathfrak{z}^\perp$. □

Clearly, if $\mathfrak{z} = \{0\}$ then $\mathcal{Z}(\mathfrak{g})$ is totally isotropic. Moreover, one has

Lemma 3.2.4. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra. Write $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ as in Proposition 3.2.3 then* $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{l})$.

Proof. Since $[\mathfrak{z},\mathfrak{g}] = \{0\}$ then $I \in \mathcal{E}^{(3,\bar{0})}(\mathfrak{l})$. Let $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge I = 0$, we show that $\alpha \in \mathfrak{l}^*$. Assume that $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \mathfrak{z}^*$ and $\alpha_2 \in \mathfrak{l}^*$. Since $\alpha \wedge I = 0$, $\alpha_1 \wedge I \in \mathcal{E}(\mathfrak{z}) \otimes \mathcal{E}(\mathfrak{l})$ and $\alpha_2 \wedge I \in \mathcal{E}(\mathfrak{l})$ then one has $\alpha_1 \wedge I = 0$. Therefore, $\alpha_1 = 0$ since I is nonzero in $\mathcal{E}^{(3,\bar{0})}(\mathfrak{l})$. That mean $\alpha \in \mathfrak{l}^*$ and then $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{l})$. □

Definition 3.2.5. A quadratic Lie superalgebra $\mathfrak{g}$ is *reduced* if:

(1) $\mathfrak{g} \neq \{0\}$

(2) $\mathcal{Z}(\mathfrak{g})$ is totally isotropic.

Notice that a reduced quadratic Lie superalgebra is necessarily non-Abelian.

Definition 3.2.6. Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra.

(1) $\mathfrak{g}$ is an *ordinary* quadratic Lie superalgebra if $\mathrm{dup}(\mathfrak{g}) = 0$.

(2) $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra if $\mathrm{dup}(\mathfrak{g}) \geq 1$.

  (i) $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra of *type* S_1 if $\mathrm{dup}(\mathfrak{g}) = 1$.

  (ii) $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra of *type* S_3 if $\mathrm{dup}(\mathfrak{g}) = 3$.

By Lemma 3.2.1, if $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_3 then $\mathfrak{g}$ is an orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \overset{\perp}{\oplus} V^{2n}$ where $\mathfrak{g}_{\bar{1}} = V^{2n}$, $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \{0\}$, $\mathfrak{g}_{\bar{0}}$ is a singular quadratic Lie algebra of type S_3 and the classification is known (more details in Proposition 3.3.3). Therefore, we are interested in singular quadratic Lie superalgebras of type S_1.

Before studying completely the structure of singular quadratic Lie superalgebras of type S_1, we begin with some simple properties as follows:

Proposition 3.2.7. *Let $(\mathfrak{g},B)$ be a singular quadratic Lie superalgebra of type S_1. If $\mathfrak{g}_{\bar{0}}$ is non-Abelian then $\mathfrak{g}_{\bar{0}}$ is a singular quadratic Lie algebra.*

Proof. By the proof of Lemma 3.2.1, one has $\mathcal{V}_I = \mathcal{V}_{I_0} \cap \mathcal{V}_{I_1}$. Therefore, $\dim(\mathcal{V}_{I_0}) \geq 1$. It means that $\mathfrak{g}_{\bar{0}}$ is a singular quadratic Lie algebra. □

Let $(\mathfrak{g}, B)$ be a singular quadratic Lie superalgebra of type S_1. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega_0 \in \mathscr{A}^2(\mathfrak{g}_{\bar{0}})$, $\Omega_1 \in \mathscr{S}^2(\mathfrak{g}_{\bar{1}})$ such that $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1$. Then one has

$$\{I, I\} = \{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} + 2\{\alpha \wedge \Omega_0, \alpha\} \otimes \Omega_1 + \{\alpha, \alpha\} \otimes \Omega_1 \Omega_1.$$

By the equality $\{I, I\} = 0$, one has $\{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} = 0$, $\{\alpha, \alpha\} = 0$ and $\{\alpha, \alpha \wedge \Omega_0\} = 0$. These imply that $\{\alpha, I\} = 0$. Hence, if we let $X_0 = \phi^{-1}(\alpha)$ then $X_0 \in \mathscr{Z}(\mathfrak{g})$ and $B(X_0, X_0) = 0$ (Corollary 3.1.13 and Lemma 3.1.20).

Proposition 3.2.8. *Let $(\mathfrak{g}, B)$ be a singular quadratic Lie superalgebra of type S_1. If $\mathfrak{g}$ is reduced then $\mathfrak{g}_{\bar{0}}$ is reduced.*

Proof. Assume that $\mathfrak{g}_{\bar{0}}$ is not reduced, i.e. $\mathfrak{g}_{\bar{0}} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ where $\mathfrak{z}$ is a non-trivial central ideal of $\mathfrak{g}_{\bar{0}}$, there is $X \in \mathfrak{z}$ such that $B(X, X) = 1$. Since $\mathfrak{g}$ is singular of type S_1 then $\mathfrak{g}_{\bar{0}}$ is also singular, the element X_0 defined as above will be in $\mathfrak{l}$ and $I_0 = \alpha \wedge \Omega_0 \in \mathscr{A}^3(\mathfrak{l})$ (see in Chapter 2). We also have $B(X, X_0) = 0$.

Let $\beta = \phi(X)$ so $\iota_X(I) = \{\beta, I\} = \{\beta, \alpha \wedge \Omega_0 + \alpha \wedge \Omega_1\} = 0$. That means $X \in \mathscr{Z}(\mathfrak{g})$. This is a contradiction since $\mathfrak{g}$ is reduced. Hence $\mathfrak{g}_{\bar{0}}$ must be reduced. □

Lemma 3.2.9. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be non-Abelian quadratic Lie superalgebras. Then $\mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$ is an ordinary quadratic Lie algebra.*

Proof. Set $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$. Denote by I, I_1 and I_2 their non-trivial associated invariants, respectively. One has $\mathcal{E}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g}_1) \otimes \mathcal{E}(\mathfrak{g}_2)$, $\mathcal{E}^k(\mathfrak{g}) = \bigoplus_{r+s=k} \mathcal{E}^r(\mathfrak{g}_1) \otimes \mathcal{E}^s(\mathfrak{g}_2)$ and $I = I_1 + I_2$ where $I_1 \in \mathcal{E}^3(\mathfrak{g}_1)$, $I_2 \in \mathcal{E}^3(\mathfrak{g}_2)$. Therefore, if $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ such that $\alpha \wedge I = 0$ then $\alpha_1 = \alpha_2 = 0$. □

Definition 3.2.10. A quadratic Lie superalgebra is *indecomposable* if $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, then $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$.

Proposition 3.2.11. *Let $\mathfrak{g}$ be a singular quadratic Lie superalgebra. Then $\mathfrak{g}$ is reduced if and only if $\mathfrak{g}$ is indecomposable.*

Proof. If $\mathfrak{g}$ is indecomposable then it is obvious that $\mathfrak{g}$ is reduced. If $\mathfrak{g}$ is reduced, assume that $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, then $\mathscr{Z}(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$ for $i = 1, 2$. Therefore, $\mathfrak{g}_i$ is reduced or $\mathfrak{g}_i = \{0\}$. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are both reduced, by Lemma 3.2.9 , then $\mathfrak{g}$ is ordinary. Hence $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$. □

To describe fully the class of singular quadratic Lie superalgebras, we shall study case-by-case its particular subclasses: elementary quadratic Lie superalgebras and quadratic Lie superalgebras with 2-dimensional even part.

3.3 Elementary quadratic Lie superalgebras

Definition 3.3.1. Let $\mathfrak{g}$ be a quadratic Lie superalgebra and I be its associated invariant. We say that $\mathfrak{g}$ is an *elementary* quadratic Lie superalgebra if I is decomposable.

It is easy to see that if I is decomposable then I_0 or $I_1 = 0$. If $\mathfrak{g}$ is a non-Abelian elementary quadratic Lie superalgebra then either $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_3 or $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_1 such that I is non-zero and decomposable in $\mathscr{A}^1(\mathfrak{g}_{\overline{0}})\otimes\mathscr{S}^2(\mathfrak{g}_{\overline{1}})$.

Assume that $I_0 = 0$ and $I = I_1$ is a non-zero decomposable element in $\mathscr{A}^1(\mathfrak{g}_{\overline{0}})\otimes\mathscr{S}^2(\mathfrak{g}_{\overline{1}})$, that means

$$I = \alpha\otimes pq$$

where $\alpha\in\mathfrak{g}_{\overline{0}}^*$ and $p,q\in\mathfrak{g}_{\overline{1}}^*$.

Lemma 3.3.2. *Let $\mathfrak{g}$ be a reduced elementary quadratic Lie superalgebra having $I = \alpha\otimes pq$. Set $X_{\overline{0}} = \phi^{-1}(\alpha)$ then one has:*

(1) $\dim(\mathfrak{g}_{\overline{0}}) = 2$ *and* $\mathfrak{g}_{\overline{0}}\cap\mathscr{Z}(\mathfrak{g}) = \mathbb{C}X_{\overline{0}}$.

(2) Let $X_{\overline{1}} = \phi^{-1}(p)$, $Y_{\overline{1}} = \phi^{-1}(q)$ and $U = \operatorname{span}\{X_{\overline{1}},Y_{\overline{1}}\}$ then

- *(i)* $\dim(\mathfrak{g}_{\overline{1}}) = 2$ *if* $\dim(U) = 1$ *or U is non-degenerate.*
- *(ii)* $\dim(\mathfrak{g}_{\overline{1}}) = 4$ *if U is totally isotropic.*

Proof.

(1) Let β be an element in $\mathfrak{g}_{\overline{0}}^*$. It is easy to see that $\{\beta,\alpha\} = 0$ if and only if $\{\beta,I\} = 0$, equivalently $\phi^{-1}(\beta)\in\mathscr{Z}(\mathfrak{g})$. Therefore, $(\phi^{-1}(\alpha))^\perp\cap\mathfrak{g}_{\overline{0}}\subset\mathscr{Z}(\mathfrak{g})$. It means that $\dim(\mathfrak{g}_{\overline{0}})\leq 2$ since $\mathfrak{g}$ is reduced (see [Bou59]). Moreover, $X_{\overline{0}} = \phi^{-1}(\alpha)$ is isotropic then $\dim(\mathfrak{g}_{\overline{0}}) = 2$. If $\dim(\mathfrak{g}_{\overline{0}}\cap\mathscr{Z}(\mathfrak{g})) = 2$ then $\mathfrak{g}_{\overline{0}}\subset\mathscr{Z}(\mathfrak{g})$. Since B is invariant one obtains $\mathfrak{g}$ Abelian (a contradiction). Therefore, $\mathfrak{g}_{\overline{0}}\cap\mathscr{Z}(\mathfrak{g}) = \mathbb{C}X_{\overline{0}}$.

(2) It is obvious that $\dim(\mathfrak{g}_{\overline{1}})\geq 2$. If $\dim(U) = 1$ then U is an isotropic subspace of $\mathfrak{g}_{\overline{1}}$ so there exists a one-dimensional subspace V of $\mathfrak{g}_{\overline{1}}$ such that B is non-degenerate on $U\oplus V$ (see [Bou59]). Let $\mathfrak{g}_{\overline{1}} = (U\oplus V)\overset{\perp}{\oplus} W$ where $W = (U\oplus V)^\perp$ then for all $f\in\phi(W)$ one has:

$$\{f,I\} = \{f,\alpha\otimes pq\} = -\alpha\otimes(\{f,p\}q + p\{f,q\}) = 0.$$

Therefore, $W\subset\mathscr{Z}(\mathfrak{g})$. Since B is non-degenerate on W and $\mathfrak{g}$ is reduced then $W = \{0\}$. If $\dim(U) = 2$ then U is non-degenerate or totally isotropic. If U is non-degenerate, let $\mathfrak{g}_{\overline{1}} = U\overset{\perp}{\oplus} W$ where $W = U^\perp$. If U is totally isotropic, let $\mathfrak{g}_{\overline{1}} = (U\oplus V)\overset{\perp}{\oplus} W$ where $W = (U\oplus V)^\perp$ and B is non-degenerate on $U\oplus V$. In the both cases, similarly as above, one has W a non-degenerate central ideal so $W = \{0\}$. Therefore, $\dim(\mathfrak{g}_{\overline{1}}) = \dim(U) = 2$ if U is non-degenerate and $\dim(\mathfrak{g}_{\overline{1}}) = \dim(U\oplus V) = 4$ if U is totally isotropic.

□

Proposition 3.3.3. *Let $\mathfrak{g}$ be a reduced elementary quadratic Lie superalgebra then $\mathfrak{g}$ is i-isomorphic to one of the following Lie superalgebras:*

(1) $\mathfrak{g}_i$ *where* $\mathfrak{g}_i$, $3 \leq i \leq 6$, *are reduced singular quadratic Lie algebras of type* S_3 *given in Proposition 2.2.29.*

(2) $\mathfrak{g}^s_{4,1} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}})$ *such that the bilinear form B is defined by*

$$B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Z_{\overline{1}}) = 1,$$

the other are zero and the Lie super-bracket is defined by $[Z_{\overline{1}}, Z_{\overline{1}}] = -2X_{\overline{0}}$, $[Y_{\overline{0}}, Z_{\overline{1}}] = -2X_{\overline{1}}$, *the other are trivial.*

(3) $\mathfrak{g}^s_{4,2} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$ *such that the bilinear form B is defined by*

$$B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Y_{\overline{1}}) = 1,$$

the other are zero and the Lie super-bracket is defined by $[X_{\overline{1}}, Y_{\overline{1}}] = X_{\overline{0}}$, $[Y_{\overline{0}}, X_{\overline{1}}] = X_{\overline{1}}$, $[Y_{\overline{0}}, Y_{\overline{1}}] = -Y_{\overline{1}}$, *the other are trivial.*

(4) $\mathfrak{g}^s_6 = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}})$ *such that the bilinear form B is defined by*

$$B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Z_{\overline{1}}) = B(Y_{\overline{1}}, T_{\overline{1}}) = 1,$$

the other are zero and the Lie super-bracket is defined by $[Z_{\overline{1}}, T_{\overline{1}}] = -X_{\overline{0}}$, $[Y_{\overline{0}}, Z_{\overline{1}}] = -Y_{\overline{1}}$, $[Y_{\overline{0}}, T_{\overline{1}}] = -X_{\overline{1}}$, *the other are trivial.*

Proof.

(1) This statement corresponds to the case where $I_1 = 0$ and $I = I_0$ a decomposable 3-form in $\mathscr{A}^3(\mathfrak{g}_{\overline{0}})$. In this case, $I(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}) = B([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}) = 0$. It implies $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$ since B is non-degenerate and then $\mathfrak{g}_{\bar{1}} \subset \mathscr{Z}(\mathfrak{g})$. On the other hand, $\mathfrak{g}$ is reduced so $\mathscr{Z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_{\bar{0}}$. Therefore, $\mathfrak{g}_{\bar{1}} = \{0\}$ and we obtain (1).

Assume that $I = \alpha \otimes pq \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\bar{1}})$. By the previous lemma, $\mathfrak{g}_{\overline{0}} = \mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}$ where $X_{\overline{0}} = \phi^{-1}(\alpha)$, $B(X_{\overline{0}}, X_{\overline{0}}) = B(Y_{\overline{0}}, Y_{\overline{0}}) = 0$, $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$. Let $X_{\bar{1}} = \phi^{-1}(p)$, $Y_{\bar{1}} = \phi^{-1}(q)$ and $U = \operatorname{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$.

(2) If $\dim(U) = 1$ then $Y_{\bar{1}} = kX_{\bar{1}}$ with some non-zero $k \in \mathbb{C}$. Therefore, $q = kp$ and $I = k\alpha \otimes p^2$. Replace with

$$X_{\overline{0}} = kX_{\overline{0}},\ Y_{\overline{0}} = \frac{Y_{\overline{0}}}{k}$$

and $Z_{\bar{1}}$ is an element in $\mathfrak{g}_{\bar{1}}$ such that $B(X_{\bar{1}}, Z_{\bar{1}}) = 1$ then $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Z_{\bar{1}})$ and $I = \alpha \otimes p^2$.

Now, let $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$. By using (1.7) and (1.8) of [BP89], one has:

$$B(X, [Y, Z]) = -2\alpha(X)p(Y)p(Z) = -2B(X_{\overline{0}}, X)B(X_{\bar{1}}, Y)B(X_{\bar{1}}, Z).$$

Since $B|_{\mathfrak{g}_{\overline{0}} \times \mathfrak{g}_{\overline{0}}}$ is non-degenerate then:

$$[Y, Z] = -2B(X_{\bar{1}}, Y)B(X_{\bar{1}}, Z)X_{\overline{0}},\ \forall\, Y, Z \in \mathfrak{g}_{\bar{1}}.$$

Similarly and by invariance of B, we also obtain:

$$[X,Y] = -2B(X_{\overline{0}},X)B(X_{\overline{1}},Y)X_{\overline{1}},\ \forall\, X \in \mathfrak{g}_{\overline{0}},\ Y \in \mathfrak{g}_{\overline{1}}$$

and (2) follows.

(3) If $\dim(U) = 2$ and U is non-degenerate then $B(X_{\overline{1}},Y_{\overline{1}}) = a \neq 0$. Replace with

$$X_{\overline{1}} := \frac{X_{\overline{1}}}{a},\ p := \frac{p}{a},\ X_{\overline{0}} := aX_{\overline{0}} \text{ and } Y_{\overline{0}} := \frac{Y_{\overline{0}}}{a}$$

then $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$, $B(X_{\overline{0}},Y_{\overline{0}}) = B(X_{\overline{1}},Y_{\overline{1}}) = 1$ and $I = \alpha \otimes pq$.
Let $X \in \mathfrak{g}_{\bar{0}},\ Y,Z \in \mathfrak{g}_{\bar{1}}$, one has:

$$B(X,[Y,Z]) = I(X,Y,Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$

Therefore, the Lie super-brackets are defined:

$$[Y,Z] = -(B(X_{\overline{1}},Y)B(Y_{\overline{1}},Z) + B(X_{\overline{1}},Z)B(Y_{\overline{1}},Y))X_{\overline{0}},\ \forall\, Y,Z \in \mathfrak{g}_{\overline{1}},$$

$$[X,Y] = -B(X_{\overline{0}},X)(B(X_{\overline{1}},Y)Y_{\overline{1}} + B(Y_{\overline{1}},Y)X_{\overline{1}}),\ \forall\, X \in \mathfrak{g}_{\overline{0}},\ Y \in \mathfrak{g}_{\overline{1}}$$

and (3) follows.

(4) If $\dim(U) = 2$ and U is totally isotropic. Let $V = \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}}$ be a totally isotropic subspace of $\mathfrak{g}_{\overline{1}}$ such that $\mathfrak{g}_{\overline{1}} = \mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}}$, $B(X_{\overline{1}},Z_{\overline{1}}) = B(Y_{\overline{1}},T_{\overline{1}}) = 1$.
If $X \in \mathfrak{g}_{\bar{0}}, Y,Z \in \mathfrak{g}_{\bar{1}}$ then:

$$B(X,[Y,Z]) = I(X,Y,Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$

We obtain the Lie super-brackets as follows:

$$[Y,Z] = -(B(X_{\overline{1}},Y)B(Y_{\overline{1}},Z) + B(X_{\overline{1}},Z)B(Y_{\overline{1}},Y))X_{\overline{0}},\ \forall\, Y,Z \in \mathfrak{g}_{\overline{1}},$$

$$[X,Y] = -B(X_{\overline{0}},X)(B(X_{\overline{1}},Y)Y_{\overline{1}} + B(Y_{\overline{1}},Y)X_{\overline{1}}),\ \forall\, X \in \mathfrak{g}_{\overline{0}},\ Y \in \mathfrak{g}_{\overline{1}}.$$

Thus, $[Z_{\overline{1}},T_{\overline{1}}] = -X_{\overline{0}}$, $[Y_{\overline{0}},Z_{\overline{1}}] = -Y_{\overline{1}}$, $[Y_{\overline{0}},T_{\overline{1}}] = -X_{\overline{1}}$.

□

3.4 Quadratic Lie superalgebras with 2-dimensional even part

Proposition 3.4.1. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. Then $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_1.*

Proof. By Remark 2.2.10, every non-Abelian quadratic Lie algebra must have the dimension more than 2 so $\mathfrak{g}_{\overline{0}}$ is Abelian and consequently, $I \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\overline{1}})$. We choose a canonical basis $\{X_{\overline{0}}, Y_{\overline{0}}\}$ of $\mathfrak{g}_{\overline{0}}$ such that $B(X_{\overline{0}}, X_{\overline{0}}) = B(Y_{\overline{0}}, Y_{\overline{0}}) = 0$ and $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$. Let $\alpha = \phi(X_{\overline{0}})$, $\beta = \phi(Y_{\overline{0}})$ and we can assume that

$$I = \alpha \otimes \Omega_1 + \beta \otimes \Omega_2$$

where $\Omega_1, \Omega_2 \in \mathscr{S}^2(\mathfrak{g}_{\overline{1}})$. Then one has:

$$\{I, I\} = 2(\Omega_1\Omega_2 + \alpha \wedge \beta \otimes \{\Omega_1, \Omega_2\}).$$

Therefore, $\{I, I\} = 0$ implies that $\Omega_1\Omega_2 = 0$. So $\Omega_1 = 0$ or $\Omega_2 = 0$. It means that $\mathfrak{g}$ is a singular quadratic Lie superalgebra of type S_1. □

Proposition 3.4.2. *Let $\mathfrak{g}$ be a singular quadratic Lie superalgebra with Abelian even part. If $\mathfrak{g}$ is reduced then* $\dim(\mathfrak{g}_{\overline{0}}) = 2$.

Proof. Let I be the associated invariant of $\mathfrak{g}$. Since $\mathfrak{g}$ has the Abelian even part one has $I \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathscr{S}^2(\mathfrak{g}_{\overline{1}})$. Moreover $\mathfrak{g}$ is singular then

$$I = \alpha \otimes \Omega$$

where $\alpha \in \mathfrak{g}_{\overline{0}}^*$, $\Omega \in \mathscr{S}^2(\mathfrak{g}_{\overline{1}})$. The proof follows exactly Lemma 3.3.2. Let $\beta \in \mathfrak{g}_{\overline{0}}^*$ then $\{\beta, \alpha\} = 0$ if and only if $\{\beta, I\} = 0$, equivalently $\phi^{-1}(\beta) \in \mathscr{Z}(\mathfrak{g})$. Therefore, $(\phi^{-1}(\alpha))^{\perp} \cap \mathfrak{g}_{\overline{0}} \subset \mathscr{Z}(\mathfrak{g})$. It means that $\dim(\mathfrak{g}_{\overline{0}}) = 2$ since $\mathfrak{g}$ is reduced and $\phi^{-1}(\alpha)$ is isotropic in $\mathscr{Z}(\mathfrak{g})$. □

Now, let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. By Proposition 3.4.1, $\mathfrak{g}$ is singular of type S_1. Fix $\alpha \in \mathscr{V}_I$ and choose $\Omega \in \mathscr{S}^2(\mathfrak{g})$ such that

$$I = \alpha \otimes \Omega.$$

We define $C : \mathfrak{g}_{\overline{1}} \to \mathfrak{g}_{\overline{1}}$ by $B(C(X), Y) = \Omega(X, Y)$, for all $X, Y \in \mathfrak{g}_{\overline{1}}$ and let $X_{\overline{0}} = \phi^{-1}(\alpha)$.

Lemma 3.4.3. *The following assertions are equivalent:*

(1) $\{I, I\} = 0$*,*

(2) $\{\alpha, \alpha\} = 0$*,*

(3) $B(X_{\overline{0}}, X_{\overline{0}}) = 0$.

In this case, one has $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$.

Proof. It is easy to see that:

$$\{I, I\} = 0 \Leftrightarrow \{\alpha, \alpha\} \otimes \Omega^2 = 0.$$

Therefore the assertions are equivalent. Moreover, since $\{\alpha, I\} = 0$ one has $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$. □

We keep the notations as in the previous sections. Then there exists an isotropic element $Y_{\overline{0}} \in \mathfrak{g}_{\overline{0}}$ such that $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$ and one has the following proposition:

Proposition 3.4.4.

(1) The map C is skew-symmetric (with respect to B), that is, $B(C(X),Y) = -B(X,C(Y))$ for all X, $Y \in \mathfrak{g}_{\overline{1}}$.

(2) $[X,Y] = B(C(X),Y)X_{\overline{0}}$, for all $X,Y \in \mathfrak{g}_{\overline{1}}$ and $C = \mathrm{ad}(Y_{\overline{0}})|_{\mathfrak{g}_{\overline{1}}}$.

(3) $\mathscr{Z}(\mathfrak{g}) = \ker(C) \oplus \mathbb{C}X_{\overline{0}}$ and $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) \oplus \mathbb{C}X_{\overline{0}}$. Therefore, $\mathfrak{g}$ is reduced if and only if $\ker(C) \subset \mathrm{Im}(C)$.

(4) The Lie superalgebra $\mathfrak{g}$ is solvable. Moreover, $\mathfrak{g}$ is nilpotent if and only if C is nilpotent.

Proof.

(1) For all X, $Y \in \mathfrak{g}_{\overline{1}}$, one has

$$B(C(X),Y) = \Omega(X,Y) = \Omega(Y,X) = B(C(Y),X) = -B(X,C(Y)).$$

(2) Let $X \in \mathfrak{g}_{\overline{0}}, Y,Z \in \mathfrak{g}_{\overline{1}}$ then

$$B(X,[Y,Z]) = (\alpha \otimes \Omega)(X,Y,Z) = \alpha(X)\Omega(Y,Z).$$

Since $\alpha(X) = B(X_{\overline{0}},X)$ and $\Omega(Y,Z) = B(C(Y),Z)$ so one has

$$B(X,[Y,Z]) = B(X_{\overline{0}},X)B(C(Y),Z).$$

The non-degeneracy of B implies $[Y,Z] = B(C(Y),Z)X_{\overline{0}}$. Set $X = Y_{\overline{0}}$ then $B(Y_{\overline{0}},[Y,Z]) = B(C(Y),Z)$. By the invariance of B, we obtain $[Y_{\overline{0}},Y] = C(Y)$.

(3) It follows from the assertion (2).

(4) $\mathfrak{g}$ is solvable since $\mathfrak{g}_{\overline{0}}$ is solvable, or since $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \subset \mathbb{C}X_{\overline{0}}$. If $\mathfrak{g}$ is nilpotent then $C = \mathrm{ad}(Y_{\overline{0}})$ is nilpotent obviously. Conversely, if C is nilpotent then it is easy to see that $\mathfrak{g}$ is nilpotent since $(\mathrm{ad}(X))^k(\mathfrak{g}) \subset \mathbb{C}X_{\overline{0}} \oplus \mathrm{Im}(C^k)$ for all $X \in \mathfrak{g}$.

□

Remark 3.4.5. The choice of C is unique up to a non-zero scalar. Indeed, assume that $I = \alpha' \otimes \Omega'$ and C' is the map associated to Ω'. Since $\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}} = \mathbb{C}X_{\overline{0}}$ and $\phi^{-1}(\alpha') \in \mathscr{Z}(\mathfrak{g})$ one has $\alpha' = \lambda\alpha$ for some non-zero $\lambda \in \mathbb{C}$. Therefore, $\alpha \otimes (\Omega - \lambda\Omega') = 0$. It means that $\Omega = \lambda\Omega'$ and then we get $C = \lambda C'$.

3.4.1 Double extension of a symplectic vector space

In Chapter 2, the double extension of a quadratic vector space by a skew-symmetric map is a solvable singular quadratic Lie algebra. Next, we have a similar definition for a symplectic vector space as follows:

Definition 3.4.6.

(1) Let $(\mathfrak{q}, B_{\mathfrak{q}})$ be a symplectic vector space equipped with the symplectic bilinear form $B_{\mathfrak{q}}$ and $\overline{C}: \mathfrak{q} \to \mathfrak{q}$ be a skew-symmetric map, that is,

$$B_{\mathfrak{q}}(\overline{C}(X), Y) = -B_{\mathfrak{q}}(X, \overline{C}(Y)),\ \forall\, X, Y \in \mathfrak{q}.$$

Let $(\mathfrak{t} = \mathrm{span}\{X_{\overline{0}}, Y_{\overline{0}}\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vector space with $B_{\mathfrak{t}}$ defined by

$$B_{\mathfrak{t}}(X_{\overline{0}}, X_{\overline{0}}) = B_{\mathfrak{t}}(Y_{\overline{0}}, Y_{\overline{0}}) = 0,\ B_{\mathfrak{t}}(X_{\overline{0}}, Y_{\overline{0}}) = 1.$$

Consider the vector space

$$\mathfrak{g} = \mathfrak{t} \overset{\perp}{\oplus} \mathfrak{q}$$

equipped with a bilinear form $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$ and define a bracket on $\mathfrak{g}$ by

$$[\lambda X_{\overline{0}} + \mu Y_{\overline{0}} + X, \lambda' X_{\overline{0}} + \mu' Y_{\overline{0}} + Y] = \mu \overline{C}(Y) - \mu' \overline{C}(X) + B(\overline{C}(X), Y) X_{\overline{0}},$$

for all $X, Y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$. Then $(\mathfrak{g}, B)$ is a quadratic solvable Lie superalgebra with $\mathfrak{g}_{\overline{0}} = \mathfrak{t}$ and $\mathfrak{g}_{\overline{1}} = \mathfrak{q}$. We say that $\mathfrak{g}$ is the *double extension* of $\mathfrak{q}$ by $\overline{C}$.

(2) Let $\mathfrak{g}_i$ be double extensions of symplectic vector spaces $(\mathfrak{q}_i, B_i)$ by skew-symmetric maps $\overline{C}_i \in \mathscr{L}(\mathfrak{q}_i)$, for $1 \leq i \leq k$. The *amalgamated product*

$$\mathfrak{g} = \mathfrak{g}_1 \underset{\mathrm{a}}{\times} \mathfrak{g}_2 \underset{\mathrm{a}}{\times} \dots \underset{\mathrm{a}}{\times} \mathfrak{g}_k$$

is defined as follows:

- consider $(\mathfrak{q}, B)$ be the symplec vector space with $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \cdots \oplus \mathfrak{q}_k$ and the bilinear form B such that $B(\sum_{i=I}^k X_i, \sum_{i=I}^k Y_i) = \sum_{i=I}^k B_i(X_i, Y_i)$, for $X_i, Y_i \in \mathfrak{q}_i$, $1 \leq i \leq k$.
- the skew-symmetric map $\overline{C} \in \mathscr{L}(\mathfrak{q})$ is defined by $\overline{C}(\sum_{i=I}^k X_i) = \sum_{i=I}^k \overline{C}_i(X_i)$, for $X_i \in \mathfrak{q}_i$, $1 \leq i \leq k$.

Then $\mathfrak{g}$ is the double extension of $\mathfrak{q}$ by $\overline{C}$.

Lemma 3.4.7. *We keep the notation above.*

(1) Let $\mathfrak{g}$ be the double extension of $\mathfrak{q}$ by $\overline{C}$. Then

$$[X, Y] = B(X_{\overline{0}}, X) C(Y) - B(X_{\overline{0}}, Y) C(X) + B(C(X), Y) X_{\overline{0}},\ \forall\, X, Y \in \mathfrak{g},$$

where $C = \mathrm{ad}(Y_{\overline{0}})$. Moreover, $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$ and $C|_{\mathfrak{q}} = \overline{C}$.

(2) *Let $\mathfrak{g}'$ be the double extension of $\mathfrak{q}$ by $\overline{C'} = \lambda\overline{C}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic.*

Proof.

(1) This is a straightforward computation by Definition 3.4.6.

(2) Write $\mathfrak{g} = \mathfrak{t} \overset{\perp}{\oplus} \mathfrak{q} = \mathfrak{g}'$. Denote by $[\cdot,\cdot]'$ the Lie super-bracket on $\mathfrak{g}'$. Define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(X_{\bar{0}}) = \lambda X_{\bar{0}}$, $A(Y_{\bar{0}}) = \frac{1}{\lambda} Y_{\bar{0}}$ and $A|_{\mathfrak{q}} = \mathrm{Id}_{\mathfrak{q}}$. Then $A([Y_{\bar{0}}, X]) = C(X) = [A(Y_{\bar{0}}), A(X)]'$ and $A([X,Y]) = [A(X), A(Y)]'$, for all $X, Y \in \mathfrak{q}$. So A is an i-isomorphism.

□

Theorem 3.4.8.

(1) *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. Keep the notations as in Proposition 3.4.4. Then $\mathfrak{g}$ is the double extension of $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp} = \mathfrak{g}_{\bar{1}}$ by $\overline{C} = \mathrm{ad}(Y_{\bar{0}})|_{\mathfrak{q}}$.*

(2) *Let $\mathfrak{g}$ be the double extension of a symplectic vector space $\mathfrak{q}$ by a map $\overline{C} \neq 0$. Then $\mathfrak{g}$ is a singular solvable quadratic Lie superalgebra with 2-dimensional even part. Moreover:*

   (i) *$\mathfrak{g}$ is reduced if and only if $\ker(\overline{C}) \subset \mathrm{Im}(\overline{C})$.*

   (ii) *$\mathfrak{g}$ is nilpotent if and only if $\overline{C}$ is nilpotent.*

(3) *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Let $\mathfrak{g}'$ be the double extension of a symplectic vector space $(\mathfrak{q}', B')$ by a map $\overline{C'}$. Let A be an i-isomorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$ and write $\mathfrak{q} = A(\mathfrak{q}')$. Then $\mathfrak{g}$ is the double extension of $(\mathfrak{q}, B|_{\mathfrak{q}\times\mathfrak{q}})$ by the map $\overline{C} = \overline{A}\ \overline{C'}\ \overline{A}^{-1}$ where $\overline{A} = A|_{\mathfrak{q}'}$.*

Proof. The assertions (1) and (2) follow Proposition 3.4.4 and Lemma 3.4.7. For (3), since A is i-isomorphic then $\mathfrak{g}$ has also 2-dimensional even part. Write $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$. Let $X_{\bar{0}} = A(X'_{\bar{0}})$ and $Y_{\bar{0}} = A(Y'_{\bar{0}})$. Then $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and one has:

$$[Y_{\bar{0}}, X] = (A\overline{C'}A^{-1})(X),\ \forall\, X \in \mathfrak{q}, \text{ and}$$

$$[X,Y] = B((A\overline{C'}A^{-1})(X), Y)X_{\bar{0}},\ \forall\, X, Y \in \mathfrak{q}.$$

This proves the result. □

Example 3.4.9. For reduced elementary quadratic Lie superalgebras with 2-dimensional even part, then

(1) $\mathfrak{g}^s_{4,1}$ is the double extension of the 2-dimensional symplectic vector space $\mathfrak{q} = \mathbb{C}^2$ by the map having matrix:

$$\overline{C} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in a canonical basis $\{E_1, E_2\}$ of $\mathfrak{q}$ where $B(E_1, E_2) = 1$.

(2) $\mathfrak{g}^s_{4,2}$ the double extension of the 2-dimensional symplectic vector space $\mathfrak{q} = \mathbb{C}^2$ by the map having matrix:

$$\overline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in a canonical basis $\{E_1, E_2\}$ of $\mathfrak{q}$ where $B(E_1, E_2) = 1$.

(3) $\mathfrak{g}^s_6$ is the double extension of the 4-dimensional symplectic vector space $\mathfrak{q} = \mathbb{C}^4$ by the map having matrix:

$$\overline{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

in a canonical basis $\{E_1, E_2, E_3, E_4\}$ of $\mathfrak{q}$ where $B(E_1, E_3) = B(E_2, E_4) = 1$, the other are zero.

Let $(\mathfrak{q}, B)$ be a symplectic vector space. We recall that $\mathrm{Sp}(\mathfrak{q})$ is the isometry group of the symplectic form B and $\mathfrak{sp}(\mathfrak{q})$ is its Lie algebra, i.e. the Lie algebra of skew-symmetric maps with respects to B. The *adjoint action* is the action of $\mathrm{Sp}(\mathfrak{q})$ on $\mathfrak{sp}(\mathfrak{q})$ by conjugation (see Chapter 1).

Proposition 3.4.10. *Let $(\mathfrak{q}, B)$ be a symplectic vector space. Let $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ be double extensions of $\mathfrak{q}$, by skew-symmetric maps $\overline{C}$ and $\overline{C'}$ respectively. Then:*

(1) there exists a Lie superalgebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist an invertible map $P \in \mathscr{L}(\mathfrak{q})$ and a non-zero $\lambda \in \mathbb{C}$ such that $\overline{C'} = \lambda\, P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$ where P^* is the adjoint map of P with respect to B.

(2) there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if $\overline{C'}$ is in the $\mathrm{Sp}(\mathfrak{q})$-adjoint orbit through $\lambda\overline{C}$ for some non-zero $\lambda \in \mathbb{C}$.

Proof. The assertions are obvious if $\overline{C} = 0$. We assume $\overline{C} \neq 0$.

(1) Let $A : \mathfrak{g} \to \mathfrak{g}'$ be a Lie superalgebra isomorphism then $A(\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) = \mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}$ and $A(\mathfrak{q}) = \mathfrak{q}$. It is obvious that $\overline{C'} \neq 0$. It is easy to see that $\mathbb{C}X_{\overline{0}} = \mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}}$ and $\mathbb{C}X'_{\overline{0}} = \mathscr{Z}(\mathfrak{g}') \cap \mathfrak{g}'_{\overline{0}}$ then one has $A(\mathbb{C}X_{\overline{0}}) = \mathbb{C}X'_{\overline{0}}$. It means $A(X_{\overline{0}}) = \mu X'_{\overline{0}}$ for some non-zero $\mu \in \mathbb{C}$. Let $A|_{\mathfrak{q}} = Q$ and assume $A(Y_{\overline{0}}) = \beta Y'_{\overline{0}} + \gamma X'_{\overline{0}}$. For all $X,\ Y \in \mathfrak{q}$, we have $A([X,Y]) = \mu B(\overline{C}(X), Y)X'_{\overline{0}}$. Also,

$$A([X,Y]) = [Q(X), Q(Y)]' = B(\overline{C'}Q(X), Q(Y))X'_{\overline{0}}.$$

It results that $Q^*\overline{C'}Q = \mu\overline{C}$.

Moreover, $A([Y_{\overline{0}},X]) = Q(\overline{C}(X)) = [\beta Y'_{\overline{0}} + \gamma X'_{\overline{0}}, Q(X)]' = \beta\overline{C'}Q(X)$, for all $X \in \mathfrak{q}$. We conclude that $Q\,\overline{C}\,Q^{-1} = \beta\overline{C'}$ and since $Q^*\overline{C'}Q = \mu\overline{C}$, then $Q^*Q\overline{C} = \beta\mu\overline{C}$.

Set $P = \dfrac{1}{(\mu\beta)^{\frac{1}{2}}}Q$ and $\lambda = \dfrac{1}{\beta}$. It follows that $\overline{C'} = \lambda P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$.

Conversely, assume that $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ be double extensions of $\mathfrak{q}$, by skew-symmetric maps $\overline{C}$ and $\overline{C'}$ respectively such that $\overline{C'} = \lambda P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$ with an invertible map $P \in \mathscr{L}(\mathfrak{q})$ and a non-zero $\lambda \in \mathbb{C}$. Define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(X_{\overline{0}}) = \dfrac{1}{\lambda}X'_{\overline{0}}$, $A(Y_{\overline{0}}) = \lambda Y'_{\overline{0}}$ and $A(X) = P(X)$, for all $X \in \mathfrak{q}$ then it is easy to check that A is isomorphic.

(2) If $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic, then the isomorphism A in the proof of (1) is an isometry. Hence $P \in \mathrm{Sp}(\mathfrak{q})$ and $\overline{C'} = \lambda P\overline{C}P^{-1}$ gives the result.

Conversely, define A as above (the sufficiency of (1)). Then A is an isometry and it is easy to check that A is an i-isomorphism.

□

Corollary 3.4.11. *Let $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ be double extensions of $(\mathfrak{q},\overline{B})$ and $(\mathfrak{q}',\overline{B'})$ respectively where $\overline{B} = B|_{\mathfrak{q}\times\mathfrak{q}}$ and $\overline{B'} = B'|_{\mathfrak{q}'\times\mathfrak{q}'}$. Write $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$. Then:*

(1) there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isometry $\overline{A} : \mathfrak{q} \to \mathfrak{q}'$ such that $\overline{C'} = \lambda\ \overline{A}\ \overline{C}\ \overline{A}^{-1}$, for some non-zero $\lambda \in \mathbb{C}$.

(2) there exists a Lie superalgebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist invertible maps $\overline{Q} : \mathfrak{q} \to \mathfrak{q}'$ and $\overline{P} \in \mathscr{L}(\mathfrak{q})$ such that

(i) $\overline{C'} = \lambda\ \overline{Q}\ \overline{C}\ \overline{Q}^{-1}$ for some non-zero $\lambda \in \mathbb{C}$,
(ii) $\overline{P}^*\ \overline{P}\ \overline{C} = \overline{C}$ and
(iii) $\overline{Q}\ \overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ onto $\mathfrak{q}'$.

Proof.
The proof is completely similar to Corollary 2.2.31 in Chapter 2. □

We shall now classify quadratic Lie superalgebra structures on the quadratic $\mathbb{Z}_2$-graded vector space $\mathbb{C}^2 \underset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$ up to i-isomorphisms in terms of $\mathrm{Sp}(2n)$-orbits in $\mathbb{P}^1(\mathfrak{sp}(2n))$. This work is like what we have done in Chapter 2. We need the following lemma:

Lemma 3.4.12. *Let V be a quadratic $\mathbb{Z}_2$-graded vector space such that its even part is 2-dimensional. We write $V = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$ with $X_{\overline{0}}$, $Y_{\overline{0}}$ isotropic elements in $V_{\overline{0}}$ and $B(X_{\overline{0}},Y_{\overline{0}}) =$ 1. Let $\mathfrak{g}$ be a quadratic Lie superalgebra with $\dim(\mathfrak{g}_{\overline{0}}) = \dim(V_{\overline{0}})$ and $\dim(\mathfrak{g}) = \dim(V)$. Then, there exists a skew-symmetric map $\overline{C'} : \mathfrak{q}' \to \mathfrak{q}'$ such that V is considered as the double extension of $\mathfrak{q}'$ by $\overline{C'}$ that is i-isomorphic to $\mathfrak{g}$.*

Proof. By Theorem 3.4.8, $\mathfrak{g}$ is a double extension. Let us write $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\overline{C} = \mathrm{ad}(Y_{\overline{0}})|_{\mathfrak{q}}$. Define $A : \mathfrak{g} \to V$ by $A(X_{\overline{0}}) = X'_{\overline{0}}$, $A(Y_{\overline{0}}) = Y'_{\overline{0}}$ and $\overline{A} = A|_{\mathfrak{q}}$ any isometry from $\mathfrak{q} \to \mathfrak{q}'$. It is clear that A is an isometry from $\mathfrak{g}$ to V. Now, define the Lie super-bracket on V by:

$$[X,Y] = A\left([A^{-1}(X), A^{-1}(Y)]\right),\ \forall\, X,Y \in V.$$

Then V is a quadratic Lie superalgebra, that is i-isomorphic to $\mathfrak{g}$. Moreover, V is obviously the double extension of $\mathfrak{q}'$ by $\overline{C'} = \overline{A}\,\overline{C}\,\overline{A}^{-1}$. □

Theorem 3.4.8, Proposition 3.4.10, Corollary 3.4.11 and Lemma 3.4.12 are enough for us to apply the method of classification in Chapter 2 for the set $\mathcal{S}(2+2n)$ of quadratic Lie superalgebra structures on the quadratic $\mathbb{Z}_2$-graded vector space $\mathbb{C}^2 \underset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$ by only replacing the isometry group $\mathrm{O}(m)$ by $\mathrm{Sp}(2n)$ and $\mathfrak{o}(m)$ by $\mathfrak{sp}(2n)$ to obtain completely similar results. One has the first characterization of the set $\mathcal{S}(2+2n)$:

Theorem 3.4.13. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be elements in $\mathcal{S}(2+2n)$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if they are isomorphic.*

By using the notion of double extension, we call the Lie superalgebra $\mathfrak{g} \in \mathcal{S}(2+2n)$ *diagonalizable* (resp. *invertible*) if it is a double extension by a diagonalizable (resp. invertible) map. Denote the subsets of nilpotent elements, diagonalizable elements and invertible elements in $\mathcal{S}(2+2n)$, respectively by $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$ and by $\mathcal{S}_{\mathrm{inv}}(2+2n)$. Denote by $\widehat{\mathcal{N}}(2+2n)$, $\widehat{\mathcal{D}}(2+2n)$, $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ the sets of isomorphism classes in $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$, $\mathcal{S}_{\mathrm{inv}}(2+2n)$, respectively and $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ the subset of $\widehat{\mathcal{D}}(2+2n)$ including reduced ones. Keep the notations as in Chapter 1 and Chapter 2. Then we have the classification result of these sets as follows:

Theorem 3.4.14.

(1) There is a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of nilpotent $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ that induces a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of partitions $\mathscr{P}_{-1}(2n)$.

(2) There is a bijection between $\widehat{\mathcal{D}}(2+2n)$ and the set of semisimple $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{D}}(2+2n)$ and Λ_n/H_n. In the reduced case, $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ is bijective to Λ_n^+/H_n.

(3) There is a bijection between $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ and the set of invertible $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ and $\mathscr{J}_n/\mathbb{C}^*$.

(4) There is a bijection between $\widehat{\mathcal{S}}(2+2n)$ and the set of $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{S}}(2+2n)$ and $\mathscr{D}(2n)/\mathbb{C}^*$.

Next, we will describe the sets $\mathcal{N}(2+2n)$, $\mathcal{D}_{\mathrm{red}}(2+2n)$ the subset of $\mathcal{D}(2+2n)$ including reduced ones, and $\mathcal{S}_{\mathrm{inv}}(2+2n)$ in term of amalgamated product in Definition 3.4.6. Remark that except for the nilpotent case, the amalgamated product may have a bad behavior with respect to isomorphisms.

Definition 3.4.15. Keep the notation J_p for the Jordan block of size p and define two types of double extension as follows:

- for $p \geq 2$, we consider the symplectic vector space $\mathfrak{q} = \mathbb{C}^{2p}$ equipped with its canonical bilinear form $\overline{B}$ and the map $\overline{C}^J_{2p}$ having matrix

$$\begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$$

  in a canonical basis. Then $\overline{C}^J_{2p} \in \mathfrak{sp}(2p)$ and we denote by $\mathfrak{j}_{2p}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{2p}$. So $\mathfrak{j}_{2p} \in \mathcal{N}(2+2p)$.

- for $p \geq 1$, we consider the symplectic vector space $\mathfrak{q} = \mathbb{C}^{2p}$ equipped with its canonical bilinear form $\overline{B}$ and the map $\overline{C}^J_{p+p}$ with matrix

$$\begin{pmatrix} J_p & M \\ 0 & -{}^tJ_p \end{pmatrix}$$

  in a canonical basis where $M = (m_{ij})$ denotes the $p \times p$-matrix with $m_{p,p} = 1$ and $m_{ij} = 0$ otherwise. Then $\overline{C}^J_{p+p} \in \mathfrak{sp}(2p)$ and we denote by $\mathfrak{j}_{p+p}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{p+p}$. So $\mathfrak{j}_{p+p} \in \mathcal{N}(2+2p)$.

Lie superalgebras $\mathfrak{j}_{2p}$ or $\mathfrak{j}_{p+p}$ will be called *nilpotent Jordan-type Lie superalgebras*.

Keep the notations as in Chapter 1. For $n \in \mathbb{N}$, $n \neq 0$, each $[d] \in \mathscr{P}_{-1}(2n)$ can be written as

$$[d] = (p_1, p_1, p_2, p_2, \ldots, p_k, p_k, 2q_1, \ldots 2q_\ell),$$

with all p_i odd, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$.

We associate the partition $[d]$ with the map $\overline{C}_{[d]} \in \mathfrak{sp}(2n)$ having matrix

$$\mathrm{diag}_{k+\ell}(\overline{C}^J_{2p_1}, \overline{C}^J_{2p_2}, \ldots, \overline{C}^J_{2p_k}, \overline{C}^J_{q_1+q_1}, \ldots, \overline{C}^J_{q_\ell+q_\ell})$$

in a canonical basis of $\mathbb{C}^{2n}$ and denote by $\mathfrak{g}_{[d]}$ the double extension of $\mathbb{C}^{2n}$ by $\overline{C}_{[d]}$. Then $\mathfrak{g}_{[d]} \in \mathcal{N}(2+2n)$ and $\mathfrak{g}_{[d]}$ is an amalgamated product of nilpotent Jordan-type Lie superalgebras, more precisely,

$$\mathfrak{g}_{[d]} = \mathfrak{j}_{2p_1} \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_2} \underset{\mathrm{a}}{\times} \ldots \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_k} \underset{\mathrm{a}}{\times} \mathfrak{j}_{q_1+q_1} \underset{\mathrm{a}}{\times} \ldots \underset{\mathrm{a}}{\times} \mathfrak{j}_{q_\ell+q_\ell}.$$

Proposition 3.4.16. *Each $\mathfrak{g} \in \mathcal{N}(2+2n)$ is i-isomorphic to a unique amalgamated product $\mathfrak{g}_{[d]}$, $[d] \in \mathscr{P}_{-1}(2n)$ of nilpotent Jordan-type Lie superalgebras.*

For reduced diagonalizable case, let $\mathfrak{g}_4^s(\lambda)$ be the double extension of $\mathfrak{q} = \mathbb{C}^2$ by $\overline{C} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $\lambda \neq 0$. By Lemma 3.4.7, $\mathfrak{g}_4^s(\lambda)$ is i-isomorphic to $\mathfrak{g}_4^s(1) = \mathfrak{g}_{4,2}^s$.

Proposition 3.4.17. *Let $\mathfrak{g} \in \mathcal{D}_{\mathrm{red}}(2+2n)$ then $\mathfrak{g}$ is an amalgamated product of quadratic Lie superalgebras all i-isomorphic to $\mathfrak{g}^s_{4,2}$.*

Finally, for invertible case, we recall the matrix $J_p(\lambda) = \mathrm{diag}_p(\lambda,\dots,\lambda)+J_p$, $p \geq 1$, $\lambda \in \mathbb{C}$ and set

$$\overline{C}^J_{2p}(\lambda) = \begin{pmatrix} J_p(\lambda) & 0 \\ 0 & -{}^tJ_p(\lambda) \end{pmatrix}$$

in a canonical basis of $\mathbb{C}^{2p}$ then $\overline{C}^J_{2p}(\lambda) \in \mathfrak{sp}(2p)$. Let $\mathfrak{j}_{2p}(\lambda)$ be the double extension of $\mathbb{C}^{2p}$ by $\overline{C}^J_{2p}(\lambda)$ then it is called a *Jordan-type quadratic Lie superalgebra*.

When $\lambda = 0$ and $p \geq 2$, we recover the nilpotent Jordan-type Lie superalgebras $\mathfrak{j}_{2p}$. If $\lambda \neq 0$, $\mathfrak{j}_{2p}(\lambda)$ becomes an invertible singular quadratic Lie superalgebra and

$$\mathfrak{j}_{2p}(-\lambda) \simeq \mathfrak{j}_{2p}(\lambda).$$

Proposition 3.4.18. *Let $\mathfrak{g} \in \mathcal{S}_{\mathrm{inv}}(2+2n)$ then $\mathfrak{g}$ is an amalgamated product of Lie superalgebras all i-isomorphic to Jordan-type quadratic Lie superalgebras $\mathfrak{j}_{2p}(\lambda)$, with $\lambda \neq 0$.*

3.4.2 Quadratic dimension of reduced quadratic Lie superalgebras with 2-dimensional even part

Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. To any bilinear form B' on $\mathfrak{g}$, there is an associated map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$B'(X,Y) = B(D(X),Y),\ \forall\, X,Y \in \mathfrak{g}.$$

Lemma 3.4.19. *If B' is even then D is even.*

Proof. Let X be an element in $\mathfrak{g}_{\bar{0}}$ and assume that $D(X) = Y + Z$ with $Y \in \mathfrak{g}_{\bar{0}}$ and $Z \in \mathfrak{g}_{\bar{1}}$. Since B' is even then $B'(X,\mathfrak{g}_{\bar{1}}) = 0$. It implies that $B(D(X),\mathfrak{g}_{\bar{1}}) = B(Z,\mathfrak{g}_{\bar{1}}) = 0$. By the non-degeneracy of B on $\mathfrak{g}_{\bar{1}}$, we obtain $Z = 0$ and then $D(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}$. Similarly to the case $X \in \mathfrak{g}_{\bar{1}}$, it concludes that $D(\mathfrak{g}_{\bar{1}}) \subset \mathfrak{g}_{\bar{1}}$. Thus, D is even. □

Lemma 3.4.20. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra, B' be an even bilinear form on $\mathfrak{g}$ and $D \in \mathscr{L}(\mathfrak{g})$ be its associated map. Then:*

(1) B' is invariant if and only if D satisfies

$$D([X,Y]) = [D(X),Y] = [X,D(Y)],\ \forall\, X,Y \in \mathfrak{g}.$$

(2) B' is supersymmetric if and only if D satisfies

$$B(D(X),Y) = B(X,D(Y)),\ \forall\, X,Y \in \mathfrak{g}.$$

In this case, D is called symmetric.

(3) B' is non-degenerate if and only if D is invertible.

Proof. Let X,Y and Z be homogeneous elements in $\mathfrak{g}$ of degrees x, y and z, respectively.

(1) If B' is invariant then
$$B'([X,Y],Z) = B'(X,[Y,Z]).$$
That means $B(D([X,Y]),Z = B(D(X),[Y,Z]) = B([D(X),Y],Z)$. Since B is non-degenerate, one has $D([X,Y]) = [D(X),Y]$. As a consequence, $D([X,Y]) = -(-1)^{xy}D([Y,X]) = -(-1)^{xy}[D(Y),X] = [X,D(Y)]$ by D even.

Conversely, if D satisfies $D([X,Y]) = [D(X),Y] = [X,D(Y)]$, for all $X,Y \in \mathfrak{g}$, it is easy to check that B' is invariant.

(2) B' is supersymmetric if and only if $B'(X,Y) = (-1)^{xy}B'(Y,X)$. Therefore, $B(D(X),Y) = (-1)^{xy}B(D(Y),X) = B(X,D(Y))$ by B supersymmetric.

(3) It is obvious since B is non-degenerate.

□

Definition 3.4.21. An even and symmetric map $D \in \mathscr{L}(\mathfrak{g})$ satisfying Lemma 3.4.20 (1) is called a *centromorphism* of $\mathfrak{g}$.

As in Subsection 2.3.1, for a quadratic Lie superalgebra $\mathfrak{g}$, the space of centromorphisms of $\mathfrak{g}$ and the space generated by invertible ones are the same, denote it by $\mathcal{C}(\mathfrak{g})$ (the proof is similar completely to Lemma 2.3.2). As a consequence, the space of even invariant supersymmetric bilinear forms on $\mathfrak{g}$ coincides with its subspace generated by non-degenerate ones. Moreover, the dimensions of all those spaces are the same and we denote it by $d_q(\mathfrak{g})$, in particular, $d_q(\mathfrak{g}) = \dim(\mathcal{C}(\mathfrak{g}))$. The following proposition gives the formula of $d_q(\mathfrak{g})$ for reduced quadratic Lie superalgebras with 2-dimensional even part.

Proposition 3.4.22. *Let $\mathfrak{g}$ be a reduced quadratic Lie superalgebra with 2-dimensional even part and $D \in \mathscr{L}(\mathfrak{g})$ be an even symmetric map. Then:*

(1) D is a centromorphism if and only if there exist $\mu \in \mathbb{C}$ and an even symmetric map $\mathsf{Z} : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ such that $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and $D = \mu \,\mathrm{Id} + \mathsf{Z}$. Moreover D is invertible if and only if $\mu \neq 0$.

(2)
$$d_q(\mathfrak{g}) = 2 + \frac{(\dim(\mathcal{Z}(\mathfrak{g}) - 1))(\dim(\mathcal{Z}(\mathfrak{g}) - 2)}{2}.$$

Proof.

(1) The proof follows exactly as Proposition 2.3.6. First, $\mathfrak{g}$ can be realized as the double extension $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ by $C = \mathrm{ad}(Y_{\bar{0}})$ where $\overline{C} = C|_{\mathfrak{q}} \in \mathfrak{sp}(\mathfrak{q})$.

Let D be an invertible centromorphism. Lemma 3.4.20 (1) implies that $D \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ D$, for all $X \in \mathfrak{g}$ and then $DC = CD$. Using Lemma 3.4.7 (1) and $CD = DC$, from $[D(X),Y_{\bar{0}}] = [X,D(Y_{\bar{0}})]$, we find
$$D(C(X)) = B(D(X_{\bar{0}}),Y_{\bar{0}})C(X),\ \forall\, X \in \mathfrak{g}.$$

Let $\mu = B(D(X_{\bar{0}}),Y_{\bar{0}})$. Since D is invertible, one has $\mu \neq 0$ and $C(D-\mu\,\mathrm{Id}) = 0$. Recall that $\ker(C) = \mathbb{C}X_{\bar{0}} \oplus \ker(\overline{C}) \oplus \mathbb{C}Y_{\bar{0}} = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar{0}}$, there exists a map $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ and $\varphi \in \mathfrak{g}^*$ such that

$$D - \mu\,\mathrm{Id} = \mathsf{Z} + \varphi \otimes Y_{\bar{0}}.$$

But D maps $[\mathfrak{g},\mathfrak{g}]$ into itself and $Y_{\bar{0}} \notin [\mathfrak{g},\mathfrak{g}]$, so $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$. One has $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_{\bar{0}} \oplus \mathrm{Im}(\overline{C})$. If $X \in \mathrm{Im}(\overline{C})$, let $X = C(Y)$. Then $D(X) = D(C(Y)) = \mu C(Y)$, so $D(X) = \mu X$. For $Y_{\bar{0}}$, $D([Y_{\bar{0}},X]) = DC(X) = \mu C(X)$ for all $X \in \mathfrak{g}$. But also, $D([Y_{\bar{0}},X]) = [D(Y_{\bar{0}}),X] = \mu C(X) + \varphi(Y_{\bar{0}})C(X)$, hence $\varphi(Y_{\bar{0}}) = 0$. As a consequence, $D(Y_{\bar{0}}) = \mu Y_{\bar{0}} + aX_{\bar{0}}$ since D is even and $X_{\bar{0}} \in \mathscr{Z}(\mathfrak{g})$.

Now, we will prove that $D(X_{\bar{0}}) = \mu X_{\bar{0}}$. Indeed, since D is even and $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}$ then one has

$$D(X_{\bar{0}}) \subset D([\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]) = [D(\mathfrak{g}_{\bar{1}}),\mathfrak{g}_{\bar{1}}] \subset [\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}.$$

It implies that, $D(X_{\bar{0}}) = bX_{\bar{0}}$. Combined with $B(D(Y_{\bar{0}}),X_{\bar{0}}) = B(Y_{\bar{0}},D(X_{\bar{0}}))$, we obtain $\mu = b$.

Let $X \in \mathfrak{q}$, $B(D(X_{\bar{0}}),X) = \mu B(X_{\bar{0}},X) = 0$. Moreover, $B(D(X_{\bar{0}}),X) = B(X_{\bar{0}},D(X))$, so $\varphi(X) = 0$.

Since $\mathscr{C}(\mathfrak{g})$ is generated by invertible centromorphisms then the necessary condition of (1) is finished. The sufficiency is obvious.

(2) By (1), D is a centromorphism if and only if $D(X) = \mu X + \mathsf{Z}(X)$, for all $X \in \mathfrak{g}$ with $\mu \in \mathbb{C}$ and Z is an even symmetric map from $\mathfrak{g}$ into $\mathscr{Z}(\mathfrak{g})$ satisfying $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$. To compute $d_q(\mathfrak{g})$, we use Appendix A. Write $\mathfrak{q} = (\mathfrak{l} \oplus \mathfrak{l}') \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ with $\mathfrak{l} = \ker(\overline{C})$, $\mathscr{Z}(\mathfrak{g}) = \mathbb{C}X_{\bar{0}} \oplus \mathfrak{l}$, $\mathrm{Im}(\overline{C}) = \mathfrak{l} \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ and $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_{\bar{0}} \oplus \mathrm{Im}(\overline{C})$. Let us define $\mathsf{Z} : \mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}Y_{\bar{0}} \to \mathfrak{l} \overset{\perp}{\oplus} \mathbb{C}X_{\bar{0}}$: set basis $\{X_1,\dots,X_r\}$ of $\mathfrak{l}$ and $\{Y_1',\dots,Y_r'\}$ of $\mathfrak{l}'$ such that $B(Y_i',X_j) = \delta_{ij}$. Note that by (1) so $\mathsf{Z}(Y_{\bar{0}}) = aX_{\bar{0}}$ and $\mathsf{Z}(\mathfrak{l}') \subset \mathfrak{l}$ since Z is even. Therefore, Z is completely defined by

$$\mathsf{Z}\left(\sum_{j=1}^{r} \mu_j Y_j'\right) = \sum_{i=1}^{r}\left(\sum_{j=1}^{r} \nu_{ij}\mu_j\right) X_i$$

with $\nu_{ij} = -\nu_{ji} = B(Y_i',\mathsf{Z}(Y_j'))$ and the formula follows.

□

3.5 Singular quadratic Lie superalgebras of type S_1 with non-Abelian even part

Let $\mathfrak{g}$ be a singular quadratic Lie superalgebra of type S_1 such that $\mathfrak{g}_{\bar{0}}$ is non-Abelian. If $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]=\{0\}$ then $\mathfrak{g}$ is an orthogonal direct sum of a singular quadratic Lie algebra of type S_1 and a vector space. There is nothing to do. Therefore, we can assume that $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]\neq\{0\}$. Fix $\alpha\in\mathcal{V}_I$ and choose $\Omega_0\in\mathscr{A}^2(\mathfrak{g}_{\bar{0}})$, $\Omega_1\in\mathscr{S}^2(\mathfrak{g}_{\bar{1}})$ such that

$$I=\alpha\wedge\Omega_0+\alpha\otimes\Omega_1.$$

Let $X_{\bar{0}}=\phi^{-1}(\alpha)$ then $X_{\bar{0}}\in\mathcal{Z}(\mathfrak{g})$ and $B(X_{\bar{0}},X_{\bar{0}})=0$. We define the maps $C_0:\mathfrak{g}_{\bar{0}}\to\mathfrak{g}_{\bar{0}}$, $C_1:\mathfrak{g}_{\bar{1}}\to\mathfrak{g}_{\bar{1}}$ by $\Omega_0(X,Y)=B(C_0(X),Y)$, for all $X,Y\in\mathfrak{g}_{\bar{0}}$ and $\Omega_1(X,Y)=B(C_1(X),Y)$, for all $X,Y\in\mathfrak{g}_{\bar{1}}$. Let $C:\mathfrak{g}\to\mathfrak{g}$ defined by $C(X+Y)=C_0(X)+C_1(Y)$, for all $X\in\mathfrak{g}_{\bar{0}}$, $Y\in\mathfrak{g}_{\bar{1}}$.

Proposition 3.5.1. *For all $X,Y\in\mathfrak{g}$, the Lie super-bracket of $\mathfrak{g}$ is defined by:*

$$[X,Y]=B(X_{\bar{0}},X)C(Y)-B(X_{\bar{0}},Y)C(X)+B(C(X),Y)X_{\bar{0}}.$$

In particular, if $X,Y\in\mathfrak{g}_{\bar{0}}$, $Z,T\in\mathfrak{g}_{\bar{1}}$ then

(1) $[X,Y]=B(X_{\bar{0}},X)C_0(Y)-B(X_{\bar{0}},Y)C_0(X)+B(C_0(X),Y)X_{\bar{0}}$,

(2) $[X,Z]=B(X_{\bar{0}},X)C_1(Z)$,

(3) $[Z,T]=B(C_1(Z),T)X_{\bar{0}}$

Proof. For all $X,Y,T\in\mathfrak{g}_{\bar{0}}$, since $B([Y,Z],T)=\alpha\wedge\Omega_0(X,Y,Z)$ then one has (1) as in Chapter 2. For all $X\in\mathfrak{g}_{\bar{0}}$, $Y,Z\in\mathfrak{g}_{\bar{1}}$

$$B([X,Y],Z)=\alpha\otimes\Omega_1(X,Y,Z)=\alpha(X)\Omega_1(Y,Z)=B(X_{\bar{0}},X)B(C_1(Y),Z).$$

Hence we obtain (2) and (3). □

Now, we show that $\mathfrak{g}_{\bar{0}}$ is solvable. Consider the quadratic Lie algebra $\mathfrak{g}_{\bar{0}}$ with 3-form $I_0=\alpha\wedge\Omega_0$. Write $\Omega_0=\sum_{i<j}a_{ij}\alpha_i\wedge\alpha_j$, with $a_{ij}\in\mathbb{C}$. Set $X_i=\phi^{-1}(\alpha_i)$ then

$$C_0=\sum_{i<j}a_{ij}(\alpha_i\otimes X_j-\alpha_j\otimes X_i).$$

Recall the space W_{I_0} in Chapter 2 as follows:

$$\mathcal{W}_{I_0}=\{\iota_{X\wedge Y}(I_0)\mid X,Y\in\mathfrak{g}_{\bar{0}}\}.$$

Since $\mathcal{W}_{I_0}=\phi([\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}])$ one has $\mathrm{Im}(C_0)\subset[\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}]$. In Section 3.2, it is known that $\{\alpha,I_0\}=0$ and then $[X_{\bar{0}},\mathfrak{g}_{\bar{0}}]=0$. As a sequence, $B(X_{\bar{0}},[\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}])$. That deduces $B(X_{\bar{0}},\mathrm{Im}(C_0))=0$. Therefore $[[\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}],[\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}]]=[\mathrm{Im}(C_0),\mathrm{Im}(C_0)]\subset\mathbb{C}X_{\bar{0}}$ and then $\mathfrak{g}_{\bar{0}}$ is solvable.

Since B is non-degenerate then we can choose $Y_{\bar{0}}\in\mathfrak{g}_{\bar{0}}$ isotropic such that $B(X_{\bar{0}},Y_{\bar{0}})=1$ as in Chapter 2 to obtain a straightforward consequence as follows:

Corollary 3.5.2.

(1) $C = \mathrm{ad}(Y_{\bar{0}})$, $\ker(C) = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar{0}}$ *and* $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) \oplus \mathbb{C}X_{\bar{0}}$.

(2) The Lie superalgebra $\mathfrak{g}$ *is solvable. Moreover,* $\mathfrak{g}$ *is nilpotent if and only if* C *is nilpotent.*

Moreover, Proposition 3.5.1 is enough for us to give the definition of double extension of a quadratic $\mathbb{Z}_2$-graded vector space as follows:

Definition 3.5.3. Let $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ be an even endomorphism of $\mathfrak{q}$. Assume that $\overline{C}$ is skew-supersymmetric, that is, $B(\overline{C}(X),Y) = -B(X,\overline{C}(Y))$, for all $X,Y \in \mathfrak{q}$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\bar{0}}, Y_{\bar{0}}\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vecor space with $B_{\mathfrak{t}}$ defined by:

$$B_{\mathfrak{t}}(X_{\bar{0}},X_{\bar{0}}) = B_{\mathfrak{t}}(Y_{\bar{0}},Y_{\bar{0}}) = 0 \text{ and } B_{\mathfrak{t}}(X_{\bar{0}},Y_{\bar{0}}) = 1.$$

Consider the vector space $\mathfrak{g} = \mathfrak{t} \overset{\perp}{\oplus} \mathfrak{q}$ equipped with the bilinear form $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$ and define on $\mathfrak{g}$ the bracket as follows:

$$[\lambda X_{\overline{0}} + \mu Y_{\overline{0}} + X, \lambda' X_{\overline{0}} + \mu' Y_{\overline{0}} + Y] = \mu\overline{C}(Y) - \mu'\overline{C}(X) + B(\overline{C}(X),Y)X_{\bar{0}},$$

for all $X,Y \in \mathfrak{q}, \lambda,\mu,\lambda',\mu' \in \mathbb{C}$. Then $(\mathfrak{g},B)$ is a quadratic solvable Lie superalgebra with $\mathfrak{g}_{\overline{0}} = \mathfrak{t} \oplus \mathfrak{q}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}}$. We say that $\mathfrak{g}$ is the *double extension* of $\mathfrak{q}$ by $\overline{C}$.

Note that an even skew-supersymmetric endomorphism $\overline{C}$ on $\mathfrak{q}$ can be written by $\overline{C} = \overline{C}_0 + \overline{C}_1$ where $\overline{C}_0 \in \mathfrak{o}(\mathfrak{q}_{\bar{0}})$ and $\overline{C}_1 \in \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$.

Corollary 3.5.4. *Let* $\mathfrak{g}$ *is the double extension of* $\mathfrak{q}$ *by* $\overline{C}$. *Denote by* $C = \mathrm{ad}(Y_{\bar{0}})$ *then one has*

(1) $[X,Y] = B(X_{\bar{0}},X)C(Y) - B(X_{\bar{0}},Y)C(X) + B(C(X),Y)X_{\bar{0}}$, *for all* $X,Y \in \mathfrak{g}$.

(2) $\mathfrak{g}$ *is a singular quadratic Lie superalgebra. If* $\overline{C}|_{\mathfrak{q}_{\bar{1}}}$ *is non-zero then* $\mathfrak{g}_{\bar{1}}$ *is of type* S_1.

Proof. The assertion (1) is direct from the above definition. Let $\alpha = \phi(X_{\bar{0}})$ and define the bilinear form $\Omega : \mathfrak{g} \to \mathfrak{g}$ by $\Omega(X,Y) = B(C(X),Y)$ for all $X,Y \in \mathfrak{g}$. By B even and supersymmetric, C even and skew-supersymmetric (with respect to B) then $\Omega = \Omega_0 + \Omega_1 \in \mathscr{A}^2(\mathfrak{g}_{\bar{0}}) \oplus \mathscr{S}^2(\mathfrak{g}_{\bar{1}})$. The formula in (1) can be replaced by $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1 = \alpha \wedge \Omega$. Therefore, $\mathrm{dup}(\mathfrak{g}) \geq 1$ and $\mathfrak{g}$ is singular. If $\overline{C}|_{\mathfrak{q}_{\bar{1}}}$ is non-zero then $\Omega_1 \neq 0$. In this case, $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] \neq \{0\}$ and thus $\mathrm{dup}(\mathfrak{g}) = 1$. □

As a consequence of Proposition 3.5.1 and Definition 3.5.3, one has

Lemma 3.5.5. *Let* $(\mathfrak{g},B)$ *be a singular quadratic Lie superalgebra of type* S_1. *Keep the notations as in Proposition 3.5.1 and Corollary 3.5.2. Then* $(\mathfrak{g},B)$ *is the double extension of* $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp}$ *by* $\overline{C} = C|_{\mathfrak{q}}$.

Remark 3.5.6. Let $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ be the double extension of $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ by $\overline{C} = \overline{C}_0 + \overline{C}_1$ then $\mathfrak{g}_{\bar{0}}$ is the double extension of $\mathfrak{q}_{\bar{0}}$ by $\overline{C}_0$ and the subalgebra $(\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}_{\bar{1}}$ is the double extension of $\mathfrak{q}_{\bar{1}}$ by $\overline{C}_1$ (see Definitions 2.2.26 and 3.4.6).

Theorem 3.5.7. *Let* $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ *and* $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ *be two double extensions of* $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ *by* $\overline{C} = \overline{C}_0 + \overline{C}_1$ *and* $\overline{C}' = \overline{C}'_0 + \overline{C}'_1$*, respectively. Assume that* $\overline{C}_1$ *is non-zero. Then*

(1) *there exists a Lie superalgebra isomorphism between* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if there exist an invertible maps* $P \in \mathscr{L}(\mathfrak{q}_{\bar{0}})$*,* $Q \in \mathscr{L}(\mathfrak{q}_{\bar{1}})$ *and a non-zero* $\lambda \in \mathbb{C}$ *such that*

   (i) $\overline{C}'_0 = \lambda P \overline{C}_0 P^{-1}$ *and* $P^* P \overline{C}_0 = \overline{C}_0$.

   (ii) $\overline{C}'_1 = \lambda Q \overline{C}_1 Q^{-1}$ *and* $Q^* Q \overline{C}_1 = \overline{C}_1$.

   where P^* *and* Q^* *are the adjoint maps of* P *and* Q *with respect to* $B|_{\mathfrak{q}_{\bar{0}} \times \mathfrak{q}_{\bar{0}}}$ *and* $B|_{\mathfrak{q}_{\bar{1}} \times \mathfrak{q}_{\bar{1}}}$.

(2) *there exists an i-isomorphism between* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *if and only if there is a non-zero* $\lambda \in \mathbb{C}$ *such that* $\overline{C}'_0$ *is in the* $\mathrm{O}(\mathfrak{q}_{\bar{0}})$*-adjoint orbit through* $\lambda \overline{C}_0$ *and* $\overline{C}'_1$ *is in the* $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$*-adjoint orbit through* $\lambda \overline{C}_1$.

Proof.

(1) Assume that there is a Lie superalgebra isomorphism $A : \mathfrak{g} \to \mathfrak{g}'$. Obviously, $A|_{\mathfrak{g}_{\bar{0}}} : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$ is a Lie algebra isomorphism. Moreover, $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}'_{\bar{0}}$ are the double extensions of $\mathfrak{q}_{\bar{0}}$ by $\overline{C}_0$ and $\overline{C}'_0$, respectively. By Theorem 2.2.30 (1), there exist an invertible map $P \in \mathscr{L}(\mathfrak{q}_{\bar{0}})$ and a non-zero $\lambda \in \mathbb{C}$ such that

$$\overline{C}'_0 = \lambda P \overline{C}_0 P^{-1},\ P^* P \overline{C}_0 = \overline{C}_0 \text{ and } A(Y_{\bar{0}}) = \frac{1}{\lambda} Y'_{\bar{0}} + Y,$$

where $Y \in \mathfrak{g}_{\bar{0}} \cap (X_{\bar{0}})^{\perp}$. Let $Q = A|_{\mathfrak{g}_{\bar{1}}}$. Since $\overline{C}_1$ is non-zero one has $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq 0$ and then $[\mathfrak{g}'_{\bar{1}}, \mathfrak{g}'_{\bar{1}}]' \neq 0$, where $[\ ,\ ]'$ denotes the Lie super-bracket on $\mathfrak{g}'$. We have:

$$A([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = A(\mathbb{C}X_{\bar{0}}) = [A(\mathfrak{g}_{\bar{1}}), A(\mathfrak{g}_{\bar{1}})]' = [\mathfrak{g}'_{\bar{1}}, \mathfrak{g}'_{\bar{1}}]' = \mathbb{C}X'_{\bar{0}}.$$

Therefore, $A(X_{\bar{0}}) = \mu X'_{\bar{0}}$ for some non-zero $\mu \in \mathbb{C}$.

Let X, Y be elements in $\mathfrak{g}_{\bar{1}}$ then

$$A([X,Y]) = \mu B(\overline{C}_1(X,Y)) X'_{\bar{0}} = [A(X), A(Y)] = B(\overline{C}'_1 Q(X), Q(Y)) X'_{\bar{0}}.$$

It implies that $Q^* \overline{C}'_1 Q = \mu \overline{C}_1$. Similarly, one has

$$Q\overline{C}_1(X) = A([Y_{\bar{0}}, X]) = [A(Y_{\bar{0}}), A(X)]' = [\frac{1}{\lambda} Y'_{\bar{0}} + Y, Q(X)]' = \frac{1}{\lambda} \overline{C}'_1 Q(X).$$

So we obtain $\overline{C}'_1 = \lambda Q \overline{C}_1 Q^{-1}$ and then $Q^* Q \overline{C}_1 = \frac{\mu}{\lambda} \overline{C}_1$. Replace with $Q := \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} Q$ then $Q^* Q \overline{C}_1 = \overline{C}_1$ and $\overline{C}'_1 = \lambda Q \overline{C}_1 Q^{-1}$.

Conversely, if there is a triple (P, Q, λ) satisfying (i) and (ii) then we set $A(X_{\bar{0}}) = \lambda X_{\bar{0}}$, $A(Y_{\bar{0}}) = \frac{1}{\lambda} Y_{\bar{0}}$ and $A(X+Y) = P(X) + Q(Y)$, for all $X \in \mathfrak{q}_{\bar{0}}$, $Y \in \mathfrak{q}_{\bar{1}}$. It is easy to check that A is a Lie superalgebra isomorphism.

(2) In the case of A i-isomorphic then the maps P and Q in the proof of (1) are isometries. Therefore, one has the necessary condition. To prove the sufficiency, we define A as in the proof the sufficiency of (1) then A is an i-isomorphism.

□

Remark 3.5.8. If let $M = P+Q$ then $M^{-1} = P^{-1} + Q^{-1}$ and $M^* = P^* + Q^*$. The formulas in Theorem 3.5.7 (1) can be written:

$$\overline{C}' = \lambda M\overline{C}M^{-1} \text{ and } M^*M\overline{C} = \overline{C}.$$

Hence, the problem of classification of singular quadratic Lie superalgebras of type S_1 (up to i-isomophisms) can be reduced to the classification of $\mathrm{O}(\mathfrak{q}_{\bar{0}}) \times \mathrm{Sp}(\mathfrak{q}_{\bar{1}})$- orbits of $\mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$, where $\mathrm{O}(\mathfrak{q}_{\bar{0}}) \times \mathrm{Sp}(\mathfrak{q}_{\bar{1}})$ denotes the direct product of two groups $\mathrm{O}(\mathfrak{q}_{\bar{0}})$ and $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$.

Theorem 3.5.9. *The* dup*-number is invariant under Lie superalgebra isomorphisms, i.e. if* $(\mathfrak{g}, B)$ *and* $(\mathfrak{g}', B')$ *are quadratic Lie superalgebras with* $\mathfrak{g} \simeq \mathfrak{g}'$*, then* $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{g}')$.

Proof. By Lemma 3.2.4 we can assume that $\mathfrak{g}$ is reduced. By Proposition 3.2.3, $\mathfrak{g}'$ is also reduced. Since $\mathfrak{g} \simeq \mathfrak{g}'$ then we can identify $\mathfrak{g} = \mathfrak{g}'$ as a Lie superalgebra equipped with the bilinear forms B, B' and we have two dup-numbers: $\mathrm{dup}_B(\mathfrak{g})$ and $\mathrm{dup}'_B(\mathfrak{g})$.

We start with the case $\mathrm{dup}_B(\mathfrak{g}) = 3$. Since $\mathfrak{g}$ is reduced then $\mathfrak{g}_{\bar{1}} = \{0\}$ and $\mathfrak{g}$ is a reduced singular quadratic Lie algebra of type S_3. By Proposition 2.3.7, $\mathrm{dup}_{B'}(\mathfrak{g}) = 3$.

If $\mathrm{dup}_B(\mathfrak{g}) = 1$, then $\mathfrak{g}$ is of type S_1 with respect to B. There are two cases: $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$. If $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$ then $\mathfrak{g}_{\bar{1}} = \{0\}$ by $\mathfrak{g}$ reduced. In this case, $\mathfrak{g}$ is a reduced singular quadratic Lie algebra of type S_1. By Proposition 2.3.7 again, $\mathfrak{g}$ is also a reduced singular quadratic Lie algebra of type S_1 with the bilinear form B', i.e. $\mathrm{dup}_{B'}(\mathfrak{g}) = 1$.

Assume that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$, we need the following lemma:

Lemma 3.5.10. *Let* $\mathfrak{g}$ *be a reduced quadratic Lie superalgebras of type* S_1 *such that* $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq 0$ *and* $D \in \mathscr{L}(\mathfrak{g})$ *be an even symmetric map. Then* D *is a centromorphism if and only if there exist* $\mu \in \mathbb{C}$ *and an even symmetric map* $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ *such that* $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ *and* $D = \mu\,\mathrm{Id} + \mathsf{Z}$. *Moreover* D *is invertible if and only if* $\mu \neq 0$.

Proof. First, $\mathfrak{g}$ can be realized as the double extension $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ by $C = \mathrm{ad}(Y_{\bar{0}})$ and let $\overline{C} = C|_{\mathfrak{q}}$.

Assume that D is an invertible centromorphism. The condition (1) of Lemma 3.4.20 implies that $D \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ D$, for all $X \in \mathfrak{g}$ and then $DC = CD$. Using formula (1) of Corollary 3.5.4 and $CD = DC$, from $[D(X), Y_{\bar{0}}] = [X, D(Y_{\bar{0}})]$ we find

$$D(C(X)) = \mu C(X),\ \forall\, X \in \mathfrak{g},\ \text{where } \mu = B(D(X_{\bar{0}}), Y_{\bar{0}}).$$

Since D is invertible, one has $\mu \neq 0$ and $C(D - \mu\,\mathrm{Id}) = 0$. Recall that $\ker(C) = \mathbb{C}X_{\bar{0}} \oplus \ker(\overline{C}) \oplus \mathbb{C}Y_{\bar{0}} = \mathscr{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar{0}}$, there exists a map $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ and $\varphi \in \mathfrak{g}^*$ such that

$$D - \mu\,\mathrm{Id} = \mathsf{Z} + \varphi \otimes Y_{\bar{0}}.$$

It needs to show that $\varphi = 0$. Indeed, D maps $[\mathfrak{g}, \mathfrak{g}]$ into itself and $Y_{\bar{0}} \notin [\mathfrak{g}, \mathfrak{g}]$, so $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$. One has $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_{\bar{0}} \oplus \mathrm{Im}(\overline{C})$. If $X \in \mathrm{Im}(\overline{C})$, let $X = C(Y)$. Then $D(X) = D(C(Y)) = \mu C(Y)$, so $D(X) = \mu X$. For $Y_{\bar{0}}$, $D([Y_{\bar{0}},X]) = DC(X) = \mu C(X)$ for all $X \in \mathfrak{g}$. But also, $D([Y_{\bar{0}},X]) = [D(Y_{\bar{0}}),X] = \mu C(X) + \varphi(Y_{\bar{0}})C(X)$, hence $\varphi(Y_{\bar{0}}) = 0$. As a consequence, $D(Y_{\bar{0}}) = \mu Y_{\bar{0}} + \mathsf{Z}(Y_{\bar{0}})$.

Now, we prove that $D(X_{\bar{0}}) = \mu X_{\bar{0}}$. Indeed, since D is even and $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}$ then one has

$$D(X_{\bar{0}}) \subset D([\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]) = [D(\mathfrak{g}_{\bar{1}}),\mathfrak{g}_{\bar{1}}] \subset [\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}.$$

It implies that, $D(X_{\bar{0}}) = aX_{\bar{0}}$. Combined with $B(D(Y_{\bar{0}}),X_{\bar{0}}) = B(Y_{\bar{0}},D(X_{\bar{0}}))$, we obtain $\mu = a$.

Let $X \in \mathfrak{q}$, $B(D(X_{\bar{0}}),X) = \mu B(X_{\bar{0}},X) = 0$. Moreover, $B(D(X_{\bar{0}}),X) = B(X_{\bar{0}},D(X))$, so $\varphi(X) = 0$.

Since $\mathcal{C}(\mathfrak{g})$ is generated by invertible centromorphisms then the necessary condition of Lemma is finished. The sufficiency is obvious. □

We turn now the proposition. By the previous lemma, the bilinear form B' defines an associated invertible centromorphism $D = \mu\,\mathrm{Id} + \mathsf{Z}$ for some non-zero $\mu \in \mathbb{C}$ and $\mathsf{Z} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{g})$ satisfying $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$. For all $X,Y,Z \in \mathfrak{g}$, one has:

$$I'(X,Y,Z) = B'([X,Y],Z) = B(D([X,Y]),Z) = B([D(X),Y],Z) = \mu B([X,Y],Z).$$

That means $I' = \mu I$ and then $\mathrm{dup}_{B'}(\mathfrak{g}) = \mathrm{dup}_B(\mathfrak{g}) = 1$.

Finally, if $\mathrm{dup}_B(\mathfrak{g}) = 0$, then $\mathfrak{g}$ cannot be of type S_3 or S_1 with respect to B', so $\mathrm{dup}_{B'}(\mathfrak{g}) = 0$. □

3.6 Quasi-singular quadratic Lie superalgebras

By Definition 3.5.3, it is natural to question: let $(\mathfrak{q} = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ be an endomorphism of $\mathfrak{q}$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\overline{1}}, Y_{\overline{1}}\}, B_{\mathfrak{t}})$ be a 2-dimensional *symplectic* vector space with $B_{\mathfrak{t}}(X_{\overline{1}}, Y_{\overline{1}}) = 1$. Is there an extension $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ such that $\mathfrak{g}$ equipped with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ becomes a quadratic Lie superalgebra such that $\mathfrak{g}_{\overline{0}} = \mathfrak{q}_{\overline{0}}$, $\mathfrak{g}_{\overline{1}} = \mathfrak{q}_{\overline{1}} \oplus \mathfrak{t}$ and the Lie super-bracket is represented by $\overline{C}$? In this last section of Chapter 3, we will give an affirmative answer to this question.

The dup-number and the form of the associated invariant I in the previous sections suggest that it would be also interesting to study a quadratic Lie superalgebra $\mathfrak{g}$ whose associated invariant I has the form

$$I = J \wedge p$$

where $p \in \mathfrak{g}_{\overline{1}}^*$ is non-zero, $J \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathscr{S}^1(\mathfrak{g}_{\overline{1}})$ is indecomposable. We obtain the first result as follows:

Proposition 3.6.1. $\{J,J\} = \{p,J\} = 0$.

Proof. Apply Proposition 3.1.5 (1) and (2) to obtain

$$\{I,I\} = \{J \wedge p, J \wedge p\} = \{J \wedge p, J\} \wedge p + J \wedge \{J \wedge p, p\}$$

$$= -\{J,J\} \wedge p \wedge p + 2J \wedge \{p,J\} \wedge p - J \wedge J \wedge \{p,p\}.$$

Since the super-exterior product is commutative then one has $J \wedge J = 0$. Moreover, $\{I,I\} = 0$ implies that:

$$\{J,J\} \wedge p \wedge p = 2J \wedge \{p,J\} \wedge p.$$

That means $\{J,J\} \wedge p = 2J \wedge \{p,J\}$.

If $\{J,J\} \neq 0$ then $\{J,J\} \wedge p \neq 0$, so $J \wedge \{p,J\} \neq 0$. Note that $\{p,J\} \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}})$ so J must contain the factor p, i.e. $J = \alpha \otimes p$ where $\alpha \in \mathfrak{g}_{\overline{0}}^*$. But $\{p,J\} = \{p, \alpha \otimes p\} = -\alpha \otimes \{p,p\} = 0$ since $\{p,p\} = 0$. This is a contradiction and therefore $\{J,J\} = 0$.

As a consequence, $J \wedge \{p,J\} = 0$. Set $\alpha = \{p,J\} \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}})$ then we have $J \wedge \alpha = 0$. If $\alpha \neq 0$ then J must have the form $J = \alpha \otimes q$ where $q \in \mathscr{S}^1(\mathfrak{g}_{\overline{1}})$. That is a contradition since J is indecomposable. ☐

Definition 3.6.2. We continue to keep the condition $I = J \wedge p$ with $p \in \mathfrak{g}_{\overline{1}}^*$ non-zero and $J \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathscr{S}^1(\mathfrak{g}_{\overline{1}})$ indecomposable. We can assume that

$$J = \sum_{i=1}^{n} \alpha_i \otimes p_i$$

where $\alpha_i \in \mathscr{A}^1(\mathfrak{g}_{\overline{0}})$, $i = 1, \ldots, n$ are linearly independent and $p_i \in \mathscr{S}^1(\mathfrak{g}_{\overline{1}})$. A quadratic Lie superalgebra having such associated invariant I is called a *quasi-singular quadratic Lie superalgebra*.

Let $U = \mathrm{span}\{\alpha_1,\dots,\alpha_n\}$ and $V = \mathrm{span}\{p_1,\dots,p_n\}$, one has $\dim(U)$ and $\dim(V) \geq 2$ by if there is a contrary then J is decomposable. Using Definition 3.1.4, we have:

$$\{J,J\} = \left\{\sum_{i=1}^{n}\alpha_i\otimes p_i, \sum_{i=1}^{n}\alpha_i\otimes p_i\right\} = -\sum_{i,j=1}^{n}\left(\{\alpha_i,\alpha_j\}\otimes p_ip_j + (\alpha_i\wedge\alpha_j)\otimes\{p_i,p_j\}\right).$$

Since $\{J,J\} = 0$ and $\alpha_i,\ i = 1,\dots,n$ are linearly independent then $\{p_i,p_j\} = 0$, for all i,j. It implies that $\{p_i,J\} = 0$, for all i.

Moreover, since $\{p,J\} = 0$ we obtain $\{p,p_i\} = 0$, consequently $\{p_i,I\} = 0$, for all i and $\{p,I\} = 0$. By Corollary 3.1.13 (2) and Lemma 3.1.20 we conclude that $\phi^{-1}(V+\mathbb{C}p)$ is a subset of $\mathscr{Z}(\mathfrak{g})$ and totally isotropic.

Now, let $\{q_1,\dots,q_m\}$ be a basis of V then J can be rewritten by

$$J = \sum_{j=1}^{m}\beta_j\otimes q_j$$

where $\beta_j\in U$, for all j. One has:

$$\{J,J\} = \left\{\sum_{j=1}^{m}\beta_j\otimes q_j, \sum_{j=1}^{m}\beta_j\otimes q_j\right\} = -\sum_{i,j=1}^{m}\left(\{\beta_i,\beta_j\}\otimes q_iq_j + (\beta_i\wedge\beta_j)\otimes\{q_i,q_j\}\right).$$

By the linear independence of the system $\{q_iq_j\}$, we obtain $\{\beta_i,\beta_j\} = 0$, for all i,j. It implies that $\{\beta_j,I\} = 0$, equivalently $\phi^{-1}(\beta_j)\in\mathscr{Z}(\mathfrak{g})$, for all j. Therefore, we always can begin with $J = \sum\limits_{i=1}^{n}\alpha_i\otimes p_i$ satisfying the following conditions:

(i) $\alpha_i,\ i = 1,\dots,n$ are linearly independent,

(ii) $\phi^{-1}(U)$ and $\phi^{-1}(V+\mathbb{C}p)$ are totally isotropic subspaces of $\mathscr{Z}(\mathfrak{g})$ where $U = \mathrm{span}\{\alpha_1,\dots,\alpha_n\}$ and $V = \mathrm{span}\{p_1,\dots,p_n\}$.

Let $X_{\bar{0}}^i = \phi^{-1}(\alpha_i)$, $X_{\bar{1}}^i = \phi^{-1}(p_i)$, for all i and $C:\mathfrak{g}\to\mathfrak{g}$ defined by

$$J(X,Y) = B(C(X),Y),\ \forall\, X,Y\in\mathfrak{g}.$$

Lemma 3.6.3. *The mapping C is a skew-supersymmetric homogeneous endomorphism of odd degree and* $\mathrm{Im}(C)\subset\mathscr{Z}(\mathfrak{g})$. *Recall that if C is a homogeneous endomorphism of degree c of $\mathfrak{g}$ satisfying*

$$B(C(X),Y) = -(-1)^{cx}B(X,C(Y)),\ \forall\, X\in\mathfrak{g}_x,\ Y\in\mathfrak{g}$$

then we say C **skew-supersymmetric** *(with respect to B).*

Proof. Since $J(\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}) = J(\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}) = 0$ and B is even then $C(\mathfrak{g}_{\bar{0}})\subset\mathfrak{g}_{\bar{1}}$ and $C(\mathfrak{g}_{\bar{1}})\subset\mathfrak{g}_{\bar{0}}$. That means C is of odd degree. For all $X\in\mathfrak{g}_{\bar{0}}$, $Y\in\mathfrak{g}_{\bar{1}}$ one has:

$$B(C(X),Y) = J(X,Y) = \sum_{i=1}^{n}\alpha_i\otimes p_i(X,Y) = \sum_{i=1}^{n}\alpha_i(X)p_i(Y) = \sum_{i=1}^{n}B(X_{\bar{0}}^i,X)B(X_{\bar{1}}^i,Y).$$

By the non-degeneracy of B and $J(X,Y) = -J(Y,X)$, we obtain:

$$C(X) = \sum_{i=1}^{n} B(X_{\bar{0}}^{i}, X)X_{\bar{1}}^{i} \text{ and } C(Y) = -\sum_{i=1}^{n} B(X_{\bar{1}}^{i}, Y)X_{\bar{0}}^{i}.$$

Combined with B supersymmetric, one has:

$$-B(Y, C(X)) = B(C(X), Y) = -B(C(Y), X) = -B(X, C(Y)).$$

It shows that C is skew-supersymmetric. Finally, $\mathrm{Im}(C) \subset \mathscr{Z}(\mathfrak{g})$ since $X_{\bar{0}}^{i}, X_{\bar{1}}^{i} \in \mathscr{Z}(\mathfrak{g})$, for all i. □

Proposition 3.6.4. *Let $X_{\bar{1}} = \phi^{-1}(p)$ then for all $X \in \mathfrak{g}_{\bar{0}}$, $Y, Z \in \mathfrak{g}_{\bar{1}}$ one has:*

(1) $[X,Y] = -B(C(X),Y)X_{\bar{1}} - B(X_{\bar{1}},Y)C(X)$,

(2) $[Y,Z] = B(X_{\bar{1}},Y)C(Z) + B(X_{\bar{1}},Z)C(Y)$,

(3) $X_{\bar{1}} \in \mathscr{Z}(\mathfrak{g})$ *and* $C(X_{\bar{1}}) = 0$.

Proof. Let $X \in \mathfrak{g}_{\bar{0}}$, $Y, Z \in \mathfrak{g}_{\bar{1}}$ then

$$\begin{aligned} B([X,Y],Z) = J \wedge p(X,Y,Z) &= -J(X,Y)p(Z) - J(X,Z)p(Y) \\ &= -B(C(X),Y)B(X_{\bar{1}},Z) - B(C(X),Z)B(X_{\bar{1}},Y). \end{aligned}$$

By the non-degeneracy of B on $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}}$, it shows that:

$$[X,Y] = -B(C(X),Y)X_{\bar{1}} - B(X_{\bar{1}},Y)C(X).$$

Combined with B invariant and C skew-supersymmetric, one has:

$$[Y,Z] = B(X_{\bar{1}},Y)C(Z) + B(X_{\bar{1}},Z)C(Y).$$

Since $\{p, I\} = 0$ then $X_{\bar{1}} \in \mathscr{Z}(\mathfrak{g})$. Moreover, $\{p, p_i\} = 0$ imply $B(X_{\bar{1}}, X_{\bar{1}}^{i}) = 0$, for all i. It means $B(X_{\bar{1}}, \mathrm{Im}(C)) = 0$. And since $B(C(X_{\bar{1}}), X) = B(X_{\bar{1}}, C(X)) = 0$, for all $X \in \mathfrak{g}$ then $C(X_{\bar{1}}) = 0$. □

Let W be a complementary subspace of $\mathrm{span}\{X_{\bar{1}}^{1}, \ldots, X_{\bar{1}}^{n}, X_{\bar{1}}\}$ in $\mathfrak{g}_{\bar{1}}$ and $Y_{\bar{1}}$ be an element in W such that $B(X_{\bar{1}}, Y_{\bar{1}}) = 1$. Let $X_{\bar{0}} = C(Y_{\bar{1}})$, $\mathfrak{q} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_{\bar{1}})^{\perp}$ and $B_{\mathfrak{q}} = B|_{\mathfrak{q}\times\mathfrak{q}}$ then we have the following corollary:

Corollary 3.6.5.

(1) $[Y_{\bar{1}}, Y_{\bar{1}}] = 2X_{\bar{0}}$, $[Y_{\bar{1}}, X] = C(X) - B(X, X_{\bar{0}})X_{\bar{1}}$ *and* $[X,Y] = -B(C(X),Y)X_{\bar{1}}$, *for all* $X, Y \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$.

(2) $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathscr{Z}(\mathfrak{g})$ *so* $\mathfrak{g}$ *is 2-step nilpotent. If* $\mathfrak{g}$ *is reduced then* $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathscr{Z}(\mathfrak{g})$.

(3) $C^2 = 0$.

Proof.

(1) The assertion (1) is obvious by Proposition 3.6.4.

(2) Note that $X_{\bar{0}} \in \mathrm{Im}(C)$ so $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}}$. By Lemma 3.6.3 and Proposition 3.6.4, $\mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathscr{Z}(\mathfrak{g})$. If $\mathfrak{g}$ is reduced then $\mathscr{Z}(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$ and therefore $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathscr{Z}(\mathfrak{g})$.

(3) Since $\mathfrak{g}$ is 2-step nilpotent then

$$0 = [Y_{\bar{1}},[Y_{\bar{1}},Y_{\bar{1}}]] = [Y_{\bar{1}},2X_{\bar{0}}] = 2C(X_{\bar{0}}) - 2B(X_{\bar{0}},X_{\bar{0}})X_{\bar{1}}.$$

Since $X_{\bar{0}} = C(Y_{\bar{1}})$ and $\mathrm{Im}(C)$ is totally isotropic then $B(X_{\bar{0}},X_{\bar{0}}) = 0$ and therefore $C(X_{\bar{0}}) = C^2(Y_{\bar{1}}) = 0$.

If $X \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ then $0 = [Y_{\bar{1}},[Y_{\bar{1}},X]] = [Y_{\bar{1}},C(X)]$. By the choice of $Y_{\bar{1}}$, it is sure that $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$. Therefore, one has:

$$0 = [Y_{\bar{1}},C(X)] = C^2(X) - B(C(X),X_{\bar{0}})X_{\bar{1}} = C^2(X) - B(C(X),C(Y_{\bar{1}}))X_{\bar{1}}.$$

By $\mathrm{Im}(C)$ totally isotropic, one has $C^2(X) = 0$.

□

Now, we consider a special case: $X_{\bar{0}} = 0$. As a consequence, $[Y_{\bar{1}},Y_{\bar{1}}] = 0$, $[Y_{\bar{1}},X] = C(X)$ and $[X,Y] = -B(C(X),Y)X_{\bar{1}}$, for all $X,Y \in \mathfrak{q}$. Let $X \in \mathfrak{q}$ and assume that $C(X) = C_1(X) + aX_1$ where $C_1(X) \in \mathfrak{q}$ then

$$0 = B([Y_{\bar{1}},Y_{\bar{1}}],X) = B(Y_{\bar{1}},[Y_{\bar{1}},X]) = B(Y_{\bar{1}},C_1(X) + aX_1) = a.$$

It shows that $C(X) \in \mathfrak{q}$, for all $X \in \mathfrak{q}$ and therefore we have the affirmative answer of the above question as follows:

Proposition 3.6.6. *Let $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ be an odd endomorphism of $\mathfrak{q}$ such that $\overline{C}$ is skew-supersymmetric and $\overline{C}^2 = 0$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\bar{1}},Y_{\bar{1}}\}, B_{\mathfrak{t}})$ be a 2-dimensional symplectic vector space with $B_{\mathfrak{t}}(X_{\bar{1}},Y_{\bar{1}}) = 1$. Consider the space $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ and define the product on $\mathfrak{g}$ by:*

$$[Y_{\bar{1}},Y_{\bar{1}}] = [X_{\bar{1}},\mathfrak{g}] = 0,\ [Y_{\bar{1}},X] = \overline{C}(X) \ \text{ and } [X,Y] = -B_{\mathfrak{q}}(\overline{C}(X),Y)X_{\bar{1}}$$

for all $X \in \mathfrak{q}$. Then $\mathfrak{g}$ becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. Moreover, one has $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$.

It remains to consider $X_{\bar{0}} \neq 0$. The fact is that C may be not stable on $\mathfrak{q}$, that is, $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ if $X \in \mathfrak{q}$ but that we need here is an action stable on $\mathfrak{q}$. Therefore, we decompose C by $C(X) = \overline{C}(X) + \varphi(X)X_{\bar{1}}$, for all $X \in \mathfrak{q}$ where $\overline{C} : \mathfrak{q} \to \mathfrak{q}$ and $\varphi : \mathfrak{q} \to \mathbb{C}$. Since $B(C(Y_{\bar{1}}),X) = B(Y_{\bar{1}},C(X)$ then $\varphi(X) = -B(X_{\bar{0}},X) = -B(X,X_{\bar{0}})$, for all $X \in \mathfrak{q}$. Moreover, C is odd degree on $\mathfrak{g}$ and skew-supersymmetric (with respect to B) implies that $\overline{C}$ is also odd on $\mathfrak{q}$ and skew-supersymmetric (with respect to $B_{\mathfrak{q}}$). It is easy to see that $\overline{C}^2 = 0$, $\overline{C}(X_{\bar{0}}) = 0$ and we have the following result:

Corollary 3.6.7. *Keep the notations as in Corollary 3.6.5 and replace $2X_{\overline{0}}$ by $X_{\overline{0}}$ then for all $X,Y \in \mathfrak{q}$, one has:*

(1) $[Y_{\overline{1}}, Y_{\overline{1}}] = X_{\overline{0}}$,

(2) $[Y_{\overline{1}}, X] = \overline{C}(X) - B(X, X_{\overline{0}})X_{\overline{1}}$,

(3) $[X,Y] = -B(\overline{C}(X),Y)X_{\overline{1}}$.

Hence, we have a more general result of Proposition 3.6.6:

Theorem 3.6.8. *Let $(\mathfrak{q} = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ an odd endomorphism of $\mathfrak{q}$ such that $\overline{C}$ is skew-supersymmetric and $\overline{C}^2 = 0$. Let $X_{\overline{0}}$ be an isotropic element $\mathfrak{q}_{\overline{0}}$, $X_{\overline{0}} \in \ker(\overline{C})$ and $(\mathfrak{t} = \mathrm{span}\{X_{\overline{1}}, Y_{\overline{1}}\}, B_{\mathfrak{t}})$ a 2-dimensional symplectic vector space with $B_{\mathfrak{t}}(X_{\overline{1}}, Y_{\overline{1}}) = 1$. Consider the space $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ and define the product on $\mathfrak{g}$ by:*

$$[Y_{\overline{1}}, Y_{\overline{1}}] = X_{\overline{0}},\ [Y_{\overline{1}}, X] = \overline{C}(X) - B_{\mathfrak{q}}(X, X_{\overline{0}})X_{\overline{1}} \text{ and } [X,Y] = -B_{\mathfrak{q}}(\overline{C}(X),Y)X_{\overline{1}}$$

for all $X \in \mathfrak{q}$. Then $\mathfrak{g}$ becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. Moreover, one has $\mathfrak{g}_{\overline{0}} = \mathfrak{q}_{\overline{0}}$, $\mathfrak{g}_{\overline{1}} = \mathfrak{q}_{\overline{1}} \oplus \mathfrak{t}$.

A quadratic Lie superalgebra obtained in the above theorem is a special case of the generalized double extensions given in [BBB] where we consider the generalized double extension of a quadratic $\mathbb{Z}_2$-graded vector space (regarded as an Abelian superalgebra) by a one-dimensional Lie superalgebra.

Chapter 4

Pseudo-Euclidean Jordan algebras

4.1 Preliminaries

Definition 4.1.1. A (non-associative) algebra $\mathfrak{J}$ over $\mathbb{C}$ is called a (commutative) *Jordan algebra* if its product is commutative and satisfies the following identity (*Jordan identity*):

$$(xy)x^2 = x(yx^2), \ \forall\, x, y, z \in \mathfrak{J}. \tag{I}$$

For instance, any commutative algebra with an associative product is a Jordan algebra. A trivial case is when the product $xy = 0$ for all $x, y \in \mathfrak{J}$. In this case, we say that $\mathfrak{J}$ is *Abelian*. A Jordan algebra $\mathfrak{J}$ is called *nilpotent* if there is an integer $k \in \mathbb{N}$ such that $\mathfrak{J}^k = \{0\}$. The smallest k for which this condition satisfied is called the *nilindex* of $\mathfrak{J}$ and we say that $\mathfrak{J}$ is $k-1$-step nilpotent. If $\mathfrak{J}$ is non-Abelian and it has only two ideals $\{0\}$ and $\mathfrak{J}$ then we say $\mathfrak{J}$ *simple*.

Given an algebra A, the *commutator* $[x, y] = xy - yx$, for all $x, y \in A$ measures the commutativity of A. Similarly the *associator* defined by

$$(x, y, z) = (xy)z - x(yz), \ \forall\, x, y, z \in A.$$

measures the associativity of A. In term of associators, the Jordan identity in a Jordan algebra $\mathfrak{J}$ becomes

$$(x, y, x^2) = 0, \ \forall\, x, y, z \in \mathfrak{J}. \tag{II}$$

An algebra A is called a *power-associative algebra* if the subalgebra generated by any element $x \in A$ is associative (see [Sch66] for more details). A Jordan algebra is an example of a power-associative algebra. A power-associative algebra A is called *trace-admissible* if there exists a bilinear form τ on A that satisfies:

(1) $\tau(x, y) = \tau(y, x)$,

(2) $\tau(xy, z) = \tau(x, yz)$,

(3) $\tau(e, e) \neq 0$ for any idempotent e of A,

(4) $\tau(x, y) = 0$ if xy is nilpotent or $xy = 0$.

It is a well-known result that simple (commutative) Jordan algebras are trace-admissible [Alb49]. A similar fact is proved for any *non-commutative* Jordan algebras of characteristic 0 [Sch55]. Recall that non-commutative Jordan algebras are algebras satisfying (I) and the *flexible* condition $(xy)x = x(yx)$ (a weaker condition than commutativity).

A bilinear form B on a Jordan algebra $\mathfrak{J}$ is *associative* if

$$B(xy,z) = B(x,yz),\ \forall\, x,y,z \in \mathfrak{J}.$$

The following definition is quite natural:

Definition 4.1.2. Let $\mathfrak{J}$ be a Jordan algebra equipped with an associative symmetric non-degenerate bilinear form B. We say that the pair $(\mathfrak{J},B)$ is a *pseudo-Euclidean Jordan algebra* and B is an *associative scalar product* on $\mathfrak{J}$.

Recall that a real finite-dimensional Jordan algebra $\mathfrak{J}$ with a unit element e (that means, $xe = ex = x$, for all $x \in \mathfrak{J}$) is called *Euclidean* if there exists an associative inner product on $\mathfrak{J}$. This is equivalent to say that the associated trace form $\mathrm{Tr}(xy)$ is positive definite where $\mathrm{Tr}(x)$ is the sum of eigenvalues in the spectral decomposition of $x \in \mathfrak{J}$. To obtain a pseudo-Euclidean Jordan algebra, we replace the base field $\mathbb{R}$ by $\mathbb{C}$ and the inner product by a non-degenerate symmetric bilinear form (considered as generalized inner product) on $\mathfrak{J}$ keeping its associativity.

Lemma 4.1.3. *Let $(\mathfrak{J},B)$ be a pseudo-Euclidean Jordan algebra and I be a* **non-degenerate ideal** *of $\mathfrak{J}$, that is, the restriction $B|_{I\times I}$ is non-degenerate. Then $I^\perp$ is also an ideal of $\mathfrak{J}$, $II^\perp = I^\perp I = \{0\}$ and $I \cap I^\perp = \{0\}$.*

Proof. Let $x \in I^\perp, y \in \mathfrak{J}$, one has $B(xy,I) = B(x,yI) = 0$ then $xy \in I^\perp$ and $I^\perp$ is an ideal.

If $x \in I^\perp$ such that $B(x,I^\perp) = 0$ then $x \in I$ and $B(x,I) = 0$. Since I is non-degenerate then $x = 0$. That implies that $I^\perp$ is non-degenerate.

Since $B(II^\perp,\mathfrak{J}) = B(I,I^\perp\mathfrak{J}) = 0$ then $II^\perp = I^\perp I = \{0\}$.

If $x \in I \cap I^\perp$ then $B(x,I) = 0$. Since I non-degenerate, then $x = 0$. □

By above Lemma, if I is a proper non-degenerate ideal of $\mathfrak{J}$ then $\mathfrak{J} = I \overset{\perp}{\oplus} I^\perp$. In this case, we say $\mathfrak{J}$ *decomposable*.

Remark 4.1.4. A pseudo-Euclidean Jordan algebra does not necessarily have a unit element. However if that is the case, this unit element is certainly unique. A Jordan algebra with unit element is called a *unital* Jordan algebra. If $\mathfrak{J}$ is not a unital Jordan algebra, we can extend $\mathfrak{J}$ to a unital Jordan algebra $\overline{\mathfrak{J}} = \mathbb{C}e \oplus \mathfrak{J}$ by the product

$$(\lambda e + x) \star (\mu e + y) = \lambda\mu e + \lambda y + \mu x + xy.$$

More particularly, $e \star e = e$, $e \star x = x \star e = x$ and $x \star y = xy$ for all $x,y \in \mathfrak{J}$. In this case, we say $\overline{\mathfrak{J}}$ the *unital extension* of $\mathfrak{J}$.

Proposition 4.1.5. *If $(\mathfrak{J},B)$ is unital then there is a decomposition:*

$$\mathfrak{J} = \mathfrak{J}_1 \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} \mathfrak{J}_k,$$

where $\mathfrak{J}_i$, $i = 1,\dots,k$ are unital and indecomposable ideals.

Proof. The assertion is obvious if $\mathfrak{J}$ is indecomposable. Assume that $\mathfrak{J}$ is decomposable, that is, $\mathfrak{J} = I \oplus I'$ with I, $I' \neq \{0\}$ proper ideals of $\mathfrak{J}$ such that I is non-degenerate. By the above Lemma, $I' = I^\perp$ and we write $\mathfrak{J} = I \overset{\perp}{\oplus} I^\perp$. Assume that $\mathfrak{J}$ has the unit element e. If $e \in I$ then for x a non-zero element in $I^\perp$, we have $ex = x \in I^\perp$. This is a contradiction. This happens similarly if $e \in I^\perp$. Therefore, $e = e_1 + e_2$ where $e_1 \in I$ and $e_2 \in I^\perp$ are non-zero vectors. For all $x \in I$, one has:

$$x = ex = (e_1 + e_2)x = e_1 x = x e_1.$$

It implies that e_1 is the unit element of I. Similarly, e_2 is also the unit element of $I^\perp$. Since the dimension of $\mathfrak{J}$ is finite then by induction, one has the result. □

Example 4.1.6. Let us recall an example in Chapter II of [FK94]: consider $\mathfrak{q}$ a vector space over $\mathbb{C}$ and $B : \mathfrak{q} \times \mathfrak{q} \to \mathbb{C}$ a symmetric bilinear form. Define the product below on the vector space $\mathfrak{J} = \mathbb{C}e \oplus \mathfrak{q}$:

$$(\lambda e + u)(\mu e + v) = (\lambda \mu + B(u,v))e + \lambda v + \mu u,$$

for all $\lambda, \mu \in \mathbb{C}, u, v \in \mathfrak{q}$. In particular, $e^2 = e$, $ue = eu = u$ and $uv = B(u,v)e$. This product makes $\mathfrak{J}$ a Jordan algebra.

Now, we add the condition that B is non-degenerate and define a bilinear form $B_{\mathfrak{J}}$ on $\mathfrak{J}$ by:

$$B_{\mathfrak{J}}(e,e) = 1,\ B_{\mathfrak{J}}(e,\mathfrak{q}) = B_{\mathfrak{J}}(\mathfrak{q},e) = 0 \text{ and } B_{\mathfrak{J}}|_{\mathfrak{q}\times\mathfrak{q}} = B.$$

Then $B_{\mathfrak{J}}$ is associative and non-degenerate and $\mathfrak{J}$ becomes a pseudo-Euclidean Jordan algebra with unit element e.

Example 4.1.7. Let us slightly change Example 4.1.6 by setting

$$\mathfrak{J}' = \mathbb{C}e \oplus \mathfrak{q} \oplus \mathbb{C}f.$$

Define the product of $\mathfrak{J}'$ as follows:

$$e^2 = e,\ ue = eu = u,\ ef = fe = f,\ uv = B(u,v)f \text{ and } uf = fu = f^2 = 0,\ \forall\, u, v \in \mathfrak{q}.$$

It is easy to see that $\mathfrak{J}'$ is the unital extension of the Jordan algebra $\mathfrak{J} = \mathfrak{q} \oplus \mathbb{C}f$ where the product on $\mathfrak{J}$ is defined by:

$$uv = B(u,v)f,\ uf = fu = 0,\ \forall\, u, v \in \mathfrak{q}.$$

Moreover, $\mathfrak{J}'$ is a pseudo-Euclidean Jordan algebra with the bilinear form $B_{\mathfrak{J}'}$ defined by:

$$B_{\mathfrak{J}'}\left(\lambda e + u + \lambda' f, \mu e + v + \mu' f\right) = \lambda \mu' + \lambda' \mu + B(u,v).$$

We will meet this algebra again in the next section.

Recall the definition of a representation of a Jordan algebra:

Definition 4.1.8. A *Jacobson representation* (or simply, a *representation*) of a Jordan algebra $\mathfrak{J}$ on a vector space V is a linear map $\mathfrak{J} \to \mathrm{End}(V)$, $x \to S_x$ satisfying for all x, y, $z \in \mathfrak{J}$,

(1) $[S_x, S_{yz}] + [S_y, S_{zx}] + [S_z, S_{xy}] = 0$,

(2) $S_xS_yS_z + S_zS_yS_x + S_{(xz)y} = S_xS_{yz} + S_yS_{zx} + S_zS_{xy}$.

Remark 4.1.9. An equivalent definition of a representation S of $\mathfrak{J}$ can be found for instance in [BB], as a necessary and sufficient condition for the vector space $\mathfrak{J}_1 = \mathfrak{J} \oplus V$ equipped with the product:

$$(x+u)(y+v) = xy + S_x(v) + S_y(u),\ \forall x,y \in \mathfrak{J}, u,v \in V$$

to be a Jordan algebra. That is:

(1) $S_{x^2}S_x - S_xS_{x^2} = 0$,

(2) $2S_{xy}S_x + S_{x^2}S_y - 2S_xS_yS_x - S_{x^2y} = 0$,

for all $x,y \in \mathfrak{J}$. In this case, Jacobson's definition is different from the usual definition of representation, that is, as a homomorphism from $\mathfrak{J}$ into the Jordan algebra of linear maps.

For $x \in \mathfrak{J}$, let $R_x \in \mathrm{End}(\mathfrak{J})$ be the endomorphism of $\mathfrak{J}$ defined by:

$$R_x(y) = xy = yx,\ \forall\, y \in \mathfrak{J}.$$

Then the Jordan identity is equivalent to $[R_x, R_{x^2}] = 0$, for all $x \in \mathfrak{J}$ where $[\cdot,\cdot]$ denotes the Lie bracket on $\mathrm{End}(\mathfrak{J})$. The linear maps

$$R : \mathfrak{J} \to \mathrm{End}(\mathfrak{J}) \ \text{ with } R(x) = R_x$$

$$\text{and } R^* : \mathfrak{J} \to \mathrm{End}(\mathfrak{J}^*) \ \text{ with } R^*(x)(f) = f \circ R_x,\ \forall\, x \in \mathfrak{J}, f \in \mathfrak{J}^*,$$

are called respectively the *adjoint representation* and the *coadjoint representation* of $\mathfrak{J}$. It is easy to check that they are indeed representations of $\mathfrak{J}$. Recall that there exists a natural non-degenerate bilinear from $\langle\cdot,\cdot\rangle$ on $\mathfrak{J} \oplus \mathfrak{J}^*$ defined by $\langle x, f\rangle = f(x)$, for all $x \in \mathfrak{J},\ f \in \mathfrak{J}^*$. For all $x, y \in \mathfrak{J}, f \in \mathfrak{J}^*$, one has:

$$f(xy) = \langle xy, f\rangle = \langle R_x(y), f\rangle = \langle y, R_x^*(f)\rangle.$$

That means that R_x^* is the adjoint map of R_x with respect to the bilinear form $\langle\cdot,\cdot\rangle$.

The following proposition gives a characterization of pseudo-Euclidean Jordan algebras ([BB], Proposition 2.1 or [Bor97], Proposition 2.4.)

Proposition 4.1.10. *Let $\mathfrak{J}$ be a Jordan algebra. Then $\mathfrak{J}$ is pseudo-Euclidean if and only if its adjoint representation and coadjoint representation are equivalent.*

Proof. Assume that $(\mathfrak{J}, B)$ is a pseudo-Euclidean Jordan algebra. We define the map $\phi : \mathfrak{J} \to \mathfrak{J}^*$ by $\phi(x) = B(x,.)$, for all $x \in \mathfrak{J}$. Since B is non-degenerate, ϕ is an isomorphism from $\mathfrak{J}$ onto $\mathfrak{J}^*$. Moreover, it satisfies

$$\phi\left(R_x(y)\right)(z) = B(xy,z) = B(y,xz) = \left(R^*(x)\phi(y)\right)(z).$$

That means $\phi \circ R_x = R^*(x) \circ \phi$, for all $x \in \mathfrak{J}$. Therefore, the representations R and R^* are equivalent.

Conversely, assume that R and R^* are equivalent then there exists an isomorphism $\phi : \mathfrak{J} \to \mathfrak{J}^*$ such that $\phi \circ R_x = R^*(x) \circ \phi$, for all $x \in \mathfrak{J}$. Set the bilinear form $T : \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}$ defined by

$T(x,y) = \phi(x)(y)$, for all $x,y \in \mathfrak{J}$ then T is non-degenerate. Moreover, it is easy to check that T is also associative since R and R^* are equivalent by ϕ. However, T is not necessarily symmetric. Then we need construct a symmetric bilinear form that is still non-degenerate and associative from T as follows. Define the symmetric (resp. skew-symmetric) part T_s (resp. T_a) by

$$T_s(x,y) = \frac{1}{2}(T(x,y)+T(y,x)) \text{ (resp. } T_a(x,y) = \frac{1}{2}(T(x,y)-T(y,x))), \ \forall\, x,y \in \mathfrak{J}.$$

By straightforward checking, T_s and T_a are also associative. Consider subspaces

$$\mathfrak{J}_s = \{x \in \mathfrak{J} \mid \ T_s(x,\mathfrak{J}) = 0\} \text{ and } \mathfrak{J}_a = \{x \in \mathfrak{J} \mid \ T_a(x,\mathfrak{J}) = 0\}.$$

If $x \in \mathfrak{J}_s \cap \mathfrak{J}_a$ then $T(x,\mathfrak{J}) = T_s(x,\mathfrak{J}) + T_a(x,\mathfrak{J}) = 0$. So $x = 0$ since T is non-degenerate. It means $\mathfrak{J}_s \cap \mathfrak{J}_a = \{0\}$. Moreover, $\mathfrak{J}_s$ and $\mathfrak{J}_a$ are also ideals of $\mathfrak{J}$ since T_s and T_a are associative.

Now, for all $x,y,z \in \mathfrak{J}$, $T_a(xy,z) = T_a(x,yz) = -T_a(xy,z)$. Therefore, $\mathfrak{J}^2 \subset \mathfrak{J}_a$ and then $\mathfrak{J}_s^2 \subset \mathfrak{J}_s \cap \mathfrak{J}_a = \{0\}$. Let $\mathfrak{J} = W \oplus \mathfrak{J}_s$ where W is a complemenary subspace of $\mathfrak{J}_s$ in $\mathfrak{J}$. It is obvious that $\mathfrak{J}_a \subset W$. Therefore, $WW \subset \mathfrak{J}^2 \subset W$ and $W\mathfrak{J}_s = \{0\}$. Consider $F : \mathfrak{J}_s \times \mathfrak{J}_s \to \mathbb{C}$ be a non-degenerate symmetric bilinear form on $\mathfrak{J}_s$. Since $\mathfrak{J}_s^2 = \{0\}$ then F is associative. Finally, we define the bilinear form $B : \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}$ by:

$$B|_{W\times W} = T_s|_{W\times W},\ B|_{\mathfrak{J}_s\times\mathfrak{J}_s} = F \text{ and } B(W,\mathfrak{J}_s) = B(\mathfrak{J}_s,W) = 0.$$

Let $x = x_w + x_s, y = y_w + y_s, z = z_w + z_s \in W \oplus \mathfrak{J}_s$. Remark that $T_s(\mathfrak{J}_s,\mathfrak{J}) = 0$ so if $T_s(x_w,W) = 0$ then $T_s(x_w,\mathfrak{J}) = 0$. It implies $x_w \in \mathfrak{J}_s$ so $x_w = 0$. One has

$$B(x_w + x_s,\mathfrak{J}) = 0 \text{ if and only if } T_s(x_w,W) = 0 \text{ and } F(x_s,\mathfrak{J}_s) = 0.$$

By preceding remark and F non-degenerate on $\mathfrak{J}_s$ then $x_w = x_s = 0$. It means B non-degenerate. It is easy to see that

$$\begin{aligned} B((x_w+x_s)(y_w+y_s),z_w+z_s) &= T_s(x_wy_w,z_w) = T_s(x_w,y_wz_w) \\ &= B(x_w+x_s,(y_w+y_s)(z_w+z_s)) \end{aligned}$$

Hence, B is associative. □

We will need some special subspaces of an arbitrary algebra $\mathfrak{J}$:

Definition 4.1.11. Let $\mathfrak{J}$ be an algebra.

(1) The subspace

$$(\mathfrak{J},\mathfrak{J},\mathfrak{J}) = \operatorname{span}\{(x,y,z) \mid \ x,y,z \in \mathfrak{J}\}$$

is the *associator* of $\mathfrak{J}$.

(2) The subspaces

$$\begin{aligned} \operatorname{LAnn}(\mathfrak{J}) &= \{x \in \mathfrak{J} \mid \ x\mathfrak{J} = 0\}, \\ \operatorname{RAnn}(\mathfrak{J}) &= \{x \in \mathfrak{J} \mid \ \mathfrak{J}x = 0\} \text{ and} \\ \operatorname{Ann}(\mathfrak{J}) &= \{x \in \mathfrak{J} \mid \ x\mathfrak{J} = \mathfrak{J}x = 0\} \end{aligned}$$

are respectively the *left-annihilator*, the *right-annihilator* and the *annihilator* of $\mathfrak{J}$. Certainly, if $\mathfrak{J}$ is commutative then these subspaces coincide.

(3) The subspace

$$N(\mathfrak{J}) = \{x \in \mathfrak{J} \mid (x,y,z) = (y,x,z) = (y,z,x) = 0, \ \forall\, y,z \in \mathfrak{J}\}$$

is the *nucleus* of $\mathfrak{J}$.

Proposition 4.1.12. *If $(\mathfrak{J},B)$ is a pseudo-Euclidean Jordan algebra then*

(1) the nucleus $N(\mathfrak{J})$ coincide with the **center** $\mathsf{Z}(\mathfrak{J})$ *of $\mathfrak{J}$ where $\mathsf{Z}(\mathfrak{J}) = \{x \in N(\mathfrak{J}) \mid xy = yx, \ \forall\, y \in \mathfrak{J}\}$, that is, the set of all elements x that commute and associate with all elements of $\mathfrak{J}$. Therefore*

$$N(\mathfrak{J}) = \mathsf{Z}(\mathfrak{J}) = \{x \in \mathfrak{J} \mid (x,y,z) = 0, \ \forall\, y,z \in \mathfrak{J}\}.$$

(2) $\mathsf{Z}(\mathfrak{J})^\perp = (\mathfrak{J},\mathfrak{J},\mathfrak{J})$.

(3) $(\mathrm{Ann}(\mathfrak{J}))^\perp = \mathfrak{J}^2$.

Proof. Since $B((x,y,z),t) = B((y,x,t),z) = B((z,t,x),y) = B((t,z,y),x)$, for all $x,y,z,t \in \mathfrak{J}$ then we get (1) and (2). The statement (3) is gained by B non-degenerate and associative. □

Definition 4.1.13. A pseudo-Euclidean Jordan algebra $\mathfrak{J}$ is *reduced* if

(1) $\mathfrak{J} \neq \{0\}$,

(2) $\mathrm{Ann}(\mathfrak{J})$ is totally isotropic.

Proposition 4.1.14. *Let $\mathfrak{J}$ be non-Abelian pseudo-Euclidean Jordan algebra. Then $\mathfrak{J} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ where $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{J})$ and $\mathfrak{l}$ is reduced.*

Proof. The proof is completely similar to Proposition 2.1.5. Let $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}^2$, $\mathfrak{z}$ be a complementary subspace of $\mathfrak{z}_0$ in $\mathrm{Ann}(\mathfrak{J})$ and $\mathfrak{l} = \mathfrak{z}^\perp$. If x is an element in $\mathfrak{z}$ such that $B(x,\mathfrak{z}) = 0$ then $B(x,\mathfrak{J}^2) = 0$ since $\mathrm{Ann}(\mathfrak{J}) = (\mathfrak{J}^2)^\perp$. As a consequence, $B(x,\mathfrak{z}_0) = 0$ and therefore $B(x,\mathrm{Ann}(\mathfrak{J})) = 0$. That implies $x \in \mathfrak{J}^2$. Hence, $x = 0$ and the restriction of B to $\mathfrak{z}$ is non-degenerate. Moreover, $\mathfrak{z}$ is an ideal, then it is easy to check that the restriction of B to $\mathfrak{l}$ is also a non-degenerate and that $\mathfrak{z} \cap \mathfrak{l} = \{0\}$.

Since $\mathfrak{J}$ is non-Abelian then $\mathfrak{l}$ is non-Abelian and $\mathfrak{l}^2 = \mathfrak{J}^2$. Moreover, $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{l})$ and the result follows. □

Next, we will define some extensions of a Jordan algebra and introduce the notion of a *double extension* of a pseudo-Euclidean Jordan algebra [BB].

Definition 4.1.15. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be Jordan algebras and $\pi : \mathfrak{J}_1 \to \mathrm{End}(\mathfrak{J}_2)$ be a representation of $\mathfrak{J}_1$ on $\mathfrak{J}_2$. We call π an *admissible representation* if it satisfies the following conditions:

(1) $\pi(x^2)(yy') + 2(\pi(x)y')(\pi(x)y) + (\pi(x)y')y^2 + 2(yy')(\pi(x)y)$
$= 2\pi(x)(y'(\pi(x)y)) + \pi(x)(y'y^2) + (\pi(x^2)y')y + 2(y'(\pi(x)y))y$,

(2) $(\pi(x)y)y^2 = (\pi(x)y^2)y$,

(3) $\pi(xx')y^2+2(\pi(x')y)(\pi(x)y)=\pi(x)\pi(x')y^2+2(\pi(x')\pi(x)y)y$,

for all $x,x'\in\mathfrak{J}_1, y,y'\in\mathfrak{J}_2$. In this case, the vector space $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_2$ with the product defined by:

$$(x+y)(x'+y')=xx'+\pi(x)y'+\pi(x')y+yy',\ \forall\, x,x'\in\mathfrak{J}_1, y,y'\in\mathfrak{J}_2$$

becomes a Jordan algebra.

Definition 4.1.16. Let $(\mathfrak{J},B)$ be a pseudo-Euclidean Jordan algebra and C be an endomorphism of $\mathfrak{J}$. We say that C is *symmetric* if

$$B(C(x),y)=B(x,C(y)),\ \forall\, x,y\in\mathfrak{J}.$$

Denote by $\mathrm{End}_s(\mathfrak{J})$ the space of symmetric endomorphisms of $\mathfrak{J}$.

The definition below was introduced in [BB], Theorem 3.8.

Definition 4.1.17. Let $(\mathfrak{J}_1,B_1)$ be a pseudo-Euclidean Jordan algebra and let $\mathfrak{J}_2$ be an arbitrary Jordan algebra. Let $\pi:\mathfrak{J}_2\to\mathrm{End}_s(\mathfrak{J}_1)$ be an admissible representation. Define a symmetric bilinear map $\varphi:\mathfrak{J}_1\times\mathfrak{J}_1\to\mathfrak{J}_2^*$ by: $\varphi(y,y')(x)=B_1(\pi(x)y,y')$, for all $x\in\mathfrak{J}_2, y,y'\in\mathfrak{J}_1$. Consider the vector space

$$\overline{\mathfrak{J}}=\mathfrak{J}_2\oplus\mathfrak{J}_1\oplus\mathfrak{J}_2^*$$

endowed with the product:

$$(x+y+f)(x'+y'+f')=xx'+yy'+\pi(x)y'+\pi(x')y+f'\circ R_x+f\circ R_{x'}+\varphi(y,y')$$

for all $x,x'\in\mathfrak{J}_2,\ y,y'\in\mathfrak{J}_1,\ f,f'\in\mathfrak{J}_2^*$. Then $\overline{\mathfrak{J}}$ is a Jordan algebra. Moreover, define a bilinear form B on $\overline{\mathfrak{J}}$ by:

$$B(x+y+f,x'+y'+f')=B_1(y,y')+f(x')+f'(x),\ \forall\, x,x'\in\mathfrak{J}_2, y,y'\in\mathfrak{J}_1, f,f'\in\mathfrak{J}_2^*.$$

Then $\overline{\mathfrak{J}}$ is a pseudo-Euclidean Jordan algebra. The Jordan algebra $(\overline{\mathfrak{J}},B)$ is called the *double extension* of $\mathfrak{J}_1$ by $\mathfrak{J}_2$ by means of π.

Remark 4.1.18. If γ is an associative bilinear form (not necessarily non-degenerate) on $\mathfrak{J}_2$ then $\overline{\mathfrak{J}}$ is again pseudo-Euclidean thanks to the bilinear form

$$B_\gamma(x+y+f,x'+y'+f')=\gamma(x,x')+B_1(y,y')+f(x')+f'(x)$$

for all x, $x'\in\mathfrak{J}_2$, y, $y'\in\mathfrak{J}_1$, f, $f'\in\mathfrak{J}_2^*$.

4.2 Jordanian double extension of a quadratic vector space

Let $\mathbb{C}c$ be a one-dimensional Jordan algebra. If $c^2 \neq 0$ then $c^2 = \lambda c$ for some non-zero $\lambda \in \mathbb{C}$. Replace c by $\frac{1}{\lambda}c$, we obtain $c^2 = c$. Therefore, there exist only two one-dimensional Jordan algebras: one Abelian and one simple. Next, we will study double extensions of a quadratic vector space by these algebras.

Let us start with $(\mathfrak{q}, B_{\mathfrak{q}})$ a quadratic vector space. We consider $(\mathfrak{t} = \operatorname{span}\{x_1, y_1\}, B_{\mathfrak{t}})$ a 2-dimensional quadratic vector space with $B_{\mathfrak{t}}$ defined by

$$B_{\mathfrak{t}}(x_1,x_1) = B_{\mathfrak{t}}(y_1,y_1) = 0,\ B_{\mathfrak{t}}(x_1,y_1) = 1.$$

Let $C : \mathfrak{q} \to \mathfrak{q}$ be a non-zero symmetric map and consider the vector space

$$\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with a product defined by

$$\begin{aligned}(x+\lambda x_1+\mu y_1)\quad &(y+\lambda' x_1+\mu' y_1) = \\ &\mu C(y) + \mu' C(x) + B_{\mathfrak{q}}(C(x),y)x_1 + \varepsilon\left(\left(\lambda\mu' + \lambda'\mu\right)x_1 + \mu\mu' y_1\right),\end{aligned}$$

$\varepsilon \in \{0,1\}$, for all $x, y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$.

Proposition 4.2.1. *Keep the notation just above.*

(1) Assume that $\varepsilon = 0$. Then $\mathfrak{J}$ is a Jordan algebra if and only if $C^3 = 0$. In this case, we call $\mathfrak{J}$ a **nilpotent double extension** *of $\mathfrak{q}$ by C.*

(2) Assume that $\varepsilon = 1$. Then $\mathfrak{J}$ is a Jordan algebra if and only if $3C^2 = 2C^3 + C$. Moreover, $\mathfrak{J}$ is pseudo-Euclidean with bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. In this case, we call $\mathfrak{J}$ a **diagonalizable double extension** *of $\mathfrak{q}$ by C.*

Proof.

(1) Let x, $y \in \mathfrak{q}$, λ, μ, λ', $\mu' \in \mathbb{C}$. One has

$$((x+\lambda x_1+\mu y_1)(y+\lambda' x_1+\mu' y_1))(x+\lambda x_1+\mu y_1)^2 = 2\mu B_{\mathfrak{q}}(C^2(\mu y+\mu' x), C(x))x_1$$

and

$$\begin{aligned}(x+\lambda x_1+\mu y_1)((y+\lambda' x_1+\mu' y_1)(x+\lambda x_1+\mu y_1)^2) \quad = \quad & 2\mu^2\mu' C^3(x) \\ & +2\mu\mu' B_{\mathfrak{q}}(C(x), C^2(x))x_1.\end{aligned}$$

Therefore, $\mathfrak{J}$ is a Jordan algebra if and only if $C^3 = 0$.

(2) The result is achieved by checking directly the equality (I) for $\mathfrak{J}$.

□

4.2.1 Nilpotent double extensions

Consider $\mathfrak{J}_1 = \mathfrak{q}$ as an Abelian algebra, $\mathfrak{J}_2 = \mathbb{C}y_1$ the nilpotent one-dimensional Jordan algebra, $\pi(y_1) = C$ and identify $\mathfrak{J}_2^*$ with $\mathbb{C}x_1$. Then by Definition 4.1.17, $\mathfrak{J} = \mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*$ is a pseudo-Euclidean Jordan algebra with a bilinear form B given by $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. In this case, C obviously satisfies the condition $C^3 = 0$.

An immediate corollary of the definition is:

Corollary 4.2.2. *If* $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ *is the nilpotent double extension of* $\mathfrak{q}$ *by* C *then*

$$y_1 x = C(x), xy = B(C(x), y)x_1 \text{ and } y_1^2 = x_1\mathfrak{J} = 0, \ \forall\, x \in \mathfrak{q}.$$

As a consequence, $\mathfrak{J}^2 = \mathrm{Im}(C) \oplus \mathbb{C}x_1$ *and* $\mathrm{Ann}(\mathfrak{J}) = \ker(C) \oplus \mathbb{C}x_1$.

Remark 4.2.3. In this case, $\mathfrak{J}$ is k-step nilpotent, $k \leq 3$ since $R_x^k(\mathfrak{J}) \subset \mathrm{Im}(C^k) \oplus \mathbb{C}x_1$.

Definition 4.2.4. Let $(\mathfrak{J}, B)$ and $(\mathfrak{J}', B')$ be pseudo-Euclidean Jordan algebras, if there exists a Jordan algebra isomorphism $A : \mathfrak{J} \to \mathfrak{J}'$ such that it is also an isometry then we say that $\mathfrak{J}$, $\mathfrak{J}'$ are *i-isomorphic* and A is an *i-isomorphism.*

Theorem 4.2.5. *Let* $(\mathfrak{q}, B)$ *be a quadratic vector space. Let* $\mathfrak{J} = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1 \oplus \mathbb{C}y_1)$ *and* $\mathfrak{J}' = \mathfrak{q} \overset{\perp}{\oplus} (\mathbb{C}x_1' \oplus \mathbb{C}y_1')$ *be nilpotent double extensions of* $\mathfrak{q}$*, by symmetric maps* C *and* C' *respectively. Then:*

(1) there exists a Jordan algebra isomorphism $A : \mathfrak{J} \to \mathfrak{J}'$ *such that* $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$ *if and only if there exist an invertible map* $P \in \mathrm{End}(\mathfrak{q})$ *and a non-zero* $\lambda \in \mathbb{C}$ *such that* $\lambda C' = PCP^{-1}$ *and* $P^*PC = C$ *where* P^* *is the adjoint map of* P *with respect to* B.

(2) there exists an i-isomorphism $A : \mathfrak{J} \to \mathfrak{J}'$ *such that* $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$ *if and only if there exists a non-zero* $\lambda \in \mathbb{C}$ *such that* C *and* $\lambda C'$ *are conjugate by an isometry* $P \in \mathrm{O}(\mathfrak{q})$.

Proof.

(1) Assume that $A : \mathfrak{J} \to \mathfrak{J}'$ is an isomorphism such that $A(\mathfrak{q} \oplus \mathbb{C}x_1) = \mathfrak{q} \oplus \mathbb{C}x_1'$. By Corollary 4.2.2 and B non-degenerate, there exist $x, y \in \mathfrak{q} \oplus \mathbb{C}x_1$ such that $xy = x_1$. Therefore $A(x_1) = A(x)A(y) \in (\mathfrak{q} \oplus \mathbb{C}x_1')(\mathfrak{q} \oplus \mathbb{C}x_1') = \mathbb{C}x_1'$. That means $A(x_1) = \mu x_1'$ for some non-zero $\mu \in \mathbb{C}$. Write $A|_{\mathfrak{q}} = P + \beta \otimes x_1'$ with $P \in \mathrm{End}(\mathfrak{q})$ and $\beta \in \mathfrak{q}^*$. If $x \in \ker(P)$ then $A\left(x - \dfrac{1}{\mu}\beta(x)x_1\right) = 0$, so $x = 0$ and therefore, P is invertible. For all $x, y \in \mathfrak{q}$, one has

$$\mu B(C(x), y)x_1' = A(xy) = A(x)A(y) = B(C'(P(x)), P(y))x_1'.$$

So we obtain $P^*C'P = \mu C$. Assume that $A(y_1) = y + \delta x_1' + \lambda y_1'$, with $y \in \mathfrak{q}$. For all $x \in \mathfrak{q}$, one has

$$P(C(x)) + \beta(C(x))x_1' = A(y_1 x) = A(y_1)A(x) = \lambda C'(P(x)) + B(C'(y), P(x))x_1'.$$

Therefore, $\lambda C' = PCP^{-1}$. Combined with $P^*C'P = \mu C$ to get $P^*PC = \lambda\mu C$. Replace P by $\dfrac{1}{(\mu\lambda)^{\frac{1}{2}}}P$ to obtain $\lambda C' = PCP^{-1}$ and $P^*PC = C$.

Conversely, define $A:\mathfrak{J}\to\mathfrak{J}'$ by $A(y_1)=\lambda y_1'$, $A(x)=P(x)$, for all $x\in\mathfrak{q}$ and $A(x_1)=\frac{1}{\lambda}x_1'$, we prove that A is a Jordan isomorphism. Indeed, for all $x,y\in\mathfrak{q}$ and $\delta,\delta',\mu,\mu'\in\mathbb{C}$ one has:

$$A((\delta y_1+x+\mu x_1)(\delta' y_1+y+\mu' x_1))=\delta PC(y)+\delta' PC(x)+\frac{1}{\lambda}B(C(x),y)x_1'$$

and $A(\delta y_1+x+\mu x_1)A(\delta' y_1+y+\mu' x_1)=\lambda\delta C'P(y)+\lambda\delta' CP(x)+B(C'P(x),P(y))x_1'$. Since $\lambda C'=PCP^{-1}$ and $P^*PC=C$, we obtain $\lambda C'P=PC$ and $P^*C'P=\frac{1}{\lambda}C$. Therefore, A is an Jordan isomorphism.

(2) If $A:\mathfrak{J}\to\mathfrak{J}'$ is an i-isomorphism then the isomorphism P in the proof of (1) is also an isometry. Hence $P\in\mathrm{O}(\mathfrak{q})$. Conversely, define A as in (1) then it is obvious that A is an i-isomorphism.

□

Proposition 4.2.6. *Let* $(\mathfrak{q},B)$ *be a quadratic vector space and let* $\mathfrak{J}=\mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1\oplus\mathbb{C}y_1)$, $\mathfrak{J}'=\mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1'\oplus\mathbb{C}y_1')$ *be nilpotent double extensions of* $\mathfrak{q}$, *by symmetric maps* C *and* C' *respectively. Assume that* $\mathrm{rank}(C')\geq 3$. *Let* $A:\mathfrak{J}\to\mathfrak{J}'$ *be an isomorphism. Then* $A(\mathfrak{q}\oplus\mathbb{C}x_1)=\mathfrak{q}\oplus\mathbb{C}x_1'$.

Proof. We assume that there is $x\in\mathfrak{q}$ such that $A(x)=y+\beta x_1'+\gamma y_1'$ where $y\in\mathfrak{q},\beta,\gamma\in\mathbb{C},\gamma\neq 0$. Then for all $q\in\mathfrak{q}$ and $\lambda\in\mathbb{C}$, we have

$$A(x)(q+\lambda x_1')=\gamma C'(q)+B(C'(y),q)x_1'.$$

Therefore, $\dim(A(x)(\mathfrak{q}\oplus\mathbb{C}x_1'))\geq 3$. But A is an isomorphism, hence

$$A(x)(\mathfrak{q}\oplus\mathbb{C}x_1')\subset A(xA^{-1}(\mathfrak{q}\oplus\mathbb{C}x_1'))\subset A(x(\mathfrak{q}\oplus\mathbb{C}x_1\oplus\mathbb{C}y_1))\subset A(\mathbb{C}C(x)\oplus\mathbb{C}x_1).$$

This is a contradiction. Hence $A(\mathfrak{q}\oplus\mathbb{C}x_1)=\mathfrak{q}\oplus\mathbb{C}x_1'$. □

4.2.2 Diagonalizable double extensions

Lemma 4.2.7. *Let* $\mathfrak{J}=\mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1\oplus\mathbb{C}y_1)$ *be a diagonalizable double extension of* $\mathfrak{q}$ *by* C. *Then*

$$y_1^2=y_1, y_1x_1=x_1, y_1x=C(x), xy=B(C(x),y)x_1 \text{ and } x_1x=x_1^2=0,\ \forall\, x\in\mathfrak{q}.$$

Note that $x_1\notin\mathrm{Ann}(\mathfrak{J})$. Let $x\in\mathfrak{q}$. Then $x\in\mathrm{Ann}(\mathfrak{J})$ if and only if $x\in\ker(C)$. Moreover, $\mathfrak{J}^2=\mathrm{Im}(C)\oplus(\mathbb{C}x_1\oplus\mathbb{C}y_1)$. Therefore $\mathfrak{J}$ is reduced if and only if $\ker(C)\subset\mathrm{Im}(C)$.

Let $x\in\mathrm{Im}(C)$. Then there exists $y\in\mathfrak{q}$ such that $x=C(y)$. Since $3C^2=2C^3+C$, one has $3C(x)-2C^2(x)=x$. Therefore, if $\mathfrak{J}$ is reduced then $\ker(C)=\{0\}$ and C is invertible. That implies that $3C-2C^2=\mathrm{Id}$ and we have the following proposition:

Theorem 4.2.8. *Let* $(\mathfrak{q},B)$ *be a quadratic vector space. Let* $\mathfrak{J}=\mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1\oplus\mathbb{C}y_1)$ *and* $\mathfrak{J}'=\mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1'\oplus\mathbb{C}y_1')$ *be diagonalizable double extensions of* $\mathfrak{q}$, *by invertible maps* C *and* C' *respectively. Then there exists a Jordan algebra isomorphism* $A:\mathfrak{J}\to\mathfrak{J}'$ *if and only if there exists an isometry* P *such that* $C'=PCP^{-1}$. *In this case,* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *are also i-isomorphic.*

Proof. Assume $\mathfrak{J}$ and $\mathfrak{J}'$ isomorphic by A. First, we will show that $A(\mathfrak{q}\oplus\mathbb{C}x_1) = \mathfrak{q}\oplus\mathbb{C}x_1'$. Indeed, if $A(x_1) = y+\beta x_1'+\gamma y_1'$ where $y\in\mathfrak{q},\beta,\gamma\in\mathbb{C}$, then

$$0 = A(x_1^2) = A(x_1)A(x_1) = 2\gamma C'(y)+(2\beta\gamma+B(C'(y),y))x_1'+\gamma^2 y_1'.$$

Therefore, $\gamma = 0$. Similarly, if there exists $x\in\mathfrak{q}$ such that $A(x) = z+\alpha x_1'+\delta y_1'$ where $z\in\mathfrak{q},\alpha,\delta\in\mathbb{C}$. Then

$$B(C(x),x)A(x_1) = A(x^2) = A(x)A(x) = 2\delta C'(z)+(2\alpha\delta+B(C'(z),z))x_1'+\delta^2 y_1'.$$

That implies $\delta = 0$ and $A(\mathfrak{q}\oplus\mathbb{C}x_1) = \mathfrak{q}\oplus\mathbb{C}x_1'$.

The rest of the proof follows exactly the proof of Theorem 4.2.5, one has $A(x_1) = \mu x_1'$ for some non-zero $\mu\in\mathbb{C}$ and there is an isomorphism of $\mathfrak{q}$ such that $A|_{\mathfrak{q}} = P+\beta\otimes x_1'$ where $\beta\in\mathfrak{q}^*$. Similarly as in the proof of Theorem 4.2.5, one also has $P^*C'P = \mu C$, where P^* is the adjoint map of P with respect to B. Assume that $A(y_1) = \lambda y_1'+y+\delta x_1'$. Since $A(y_1)A(y_1) = A(y_1)$, one has $\lambda = 1$ and therefore $C' = PCP^{-1}$. Replace P by $\dfrac{1}{(\mu)^{\frac{1}{2}}}P$ to get $P^*PC = C$. However, since C is invertible then $P^*P = I$. It means that P is an isometry of $\mathfrak{q}$.

Conversely, define $A:\mathfrak{J}\to\mathfrak{J}'$ by $A(x_1) = x_1'$, $A(y_1) = y_1'$ and $A(x) = P(x)$, for all $x\in\mathfrak{q}$ then A is an i-isomorphism. □

An invertible symmetric endomorphism of $\mathfrak{q}$ satisfying $3C-2C^2 = \mathrm{Id}$ is diagonalizable by an orthogonal basis of eigenvectors with eigenvalues 1 and $\frac{1}{2}$ (see Appendix D). Therefore, we have the following corollary:

Corollary 4.2.9. *Let $(\mathfrak{q},B)$ be a quadratic vector space and let $\mathfrak{J} = \mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1\oplus\mathbb{C}y_1)$ and $\mathfrak{J}' = \mathfrak{q}\overset{\perp}{\oplus}(\mathbb{C}x_1'\oplus\mathbb{C}y_1')$ be diagonalizable double extensions of $\mathfrak{q}$, by invertible maps C and C' respectively. Then $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic if and only if C and C' have the same spectrum.*

Example 4.2.10. Let $\mathbb{C}x$ be a one-dimensional Abelian algebra. Let $\mathfrak{J} = \mathbb{C}x\overset{\perp}{\oplus}(\mathbb{C}x_1\oplus\mathbb{C}y_1)$ and $\mathfrak{J}' = \mathbb{C}x\overset{\perp}{\oplus}(\mathbb{C}x_1'\oplus\mathbb{C}y_1')$ be diagonalizable double extensions of $\mathbb{C}x$ by $C = \mathrm{Id}$ and $C = \frac{1}{2}\mathrm{Id}$. In particular, the product on $\mathfrak{J}$ and $\mathfrak{J}'$ are defined by:

$$y_1^2 = y_1, y_1x = x, y_1x_1 = x_1, x^2 = x_1;$$

$$(y_1')^2 = y_1', y_1'x = \frac{1}{2}x, y_1x_1 = x_1, x^2 = \frac{1}{2}x_1.$$

Then $\mathfrak{J}$ and $\mathfrak{J}'$ are not isomorphic. Moreover, $\mathfrak{J}'$ has no unit element.

Remark 4.2.11. The i-isomorphic and isomorphic notions are not coinciding in general. For example, the Jordan algebras $\mathfrak{J} = \mathbb{C}e$ with $e^2 = e$, $B(e,e) = 1$ and $\mathfrak{J}' = \mathbb{C}e'$ with $(e')^2 = e'$, $B(e',e') = a\neq 1$ are isomorphic but not i-isomorphic.

4.3 Pseudo-Euclidean 2-step nilpotent Jordan algebras

4.3.1 2-step nilpotent Jordan algebras

In Chapter 2, 2-step nilpotent quadratic Lie algebras are characterized up to isometric isomorphisms and up to isomorphisms (see also [Ova07]). There is a similar natural property in the case of pseudo-Euclidean 2-step nilpotent Jordan algebras. Let us redefine 2-step nilpotent Jordan algebras in a more convenient way:

Definition 4.3.1. An algebra $\mathfrak{J}$ over $\mathbb{C}$ with a product $(x,y) \mapsto xy$ is called a *2-step nilpotent Jordan algebra* if it satisfies $xy = yx$ and $(xy)z = 0$ for all $x,y,z \in \mathfrak{J}$. Sometimes, we use *2SN-Jordan algebra* as an abbreviation.

The method of double extension is a fundamental tool used in describing algebras that are endowed with an associative non-degenerate bilinear form. This method is based on two principal notions: central extension and semi-direct product of two algebras. We have just seen it in Chapter 2 for 2-step nilpotent quadratic Lie algebras. In the next part, we apply it again but as we will see that the method of double extension is not quite effective for pseudo-Euclidean Jordan algebras, even in the 2-step nilpotent case. With our attention we will recall some definitions given in Section 3 of [BB] but with a restricting condition for pseudo-Euclidean 2-step nilpotent Jordan algebras.

Proposition 4.3.2. *Let $\mathfrak{J}$ be a 2SN-Jordan algebra, V be a vector space, $\varphi : \mathfrak{J} \times \mathfrak{J} \to V$ be a bilinear map and $\pi : \mathfrak{J} \to \mathrm{End}(V)$ be a representation. Let*

$$\overline{\mathfrak{J}} = \mathfrak{J} \oplus V$$

equipped with the following product:

$$(x+u)(y+v) = xy + \pi(x)(v) + \pi(y)(u) + \varphi(x,y), \ \forall x,y \in \mathfrak{J}, u,v \in V.$$

Then $\overline{\mathfrak{J}}$ is a 2SN-Jordan algebra if and only if for all $x,y,z \in \mathfrak{J}$ one has:

(1) φ *is symmetric and* $\varphi(xy,z) + \pi(z)(\varphi(x,y)) = 0$,

(2) $\pi(xy) = \pi(x)\pi(y) = 0$.

Definition 4.3.3. If π is the trivial representation in Proposition 4.3.2, the Jordan algebra $\overline{\mathfrak{J}}$ is called the *2SN-central extension* of $\mathfrak{J}$ by V (by means of φ).

Remark that in a 2SN-central extension $\overline{\mathfrak{J}}$, the annihilator $\mathrm{Ann}(\overline{\mathfrak{J}})$ contains the vector space V.

Proposition 4.3.4. *Let $\mathfrak{J}$ be a 2SN-Jordan algebra. Then $\mathfrak{J}$ is a 2SN-central extension of an Abelian algebra.*

Proof. Set $\mathfrak{h} = \mathfrak{J}/\mathfrak{J}^2$ and $V = \mathfrak{J}^2$. Define $\varphi : \mathfrak{h} \times \mathfrak{h} \to V$ by $\varphi(p(x), p(y)) = xy$, for all $x,y \in \mathfrak{J}$ where $p : \mathfrak{J} \to \mathfrak{h}$ is the canonical projection. Then $\mathfrak{h}$ is an Abelian algebra and $\mathfrak{J} \simeq \mathfrak{h} \oplus V$ is the 2SN-central extension of $\mathfrak{h}$ by means of φ. □

Remark 4.3.5. It is easy to see that if $\mathfrak{J}$ is a 2SN-Jordan algebra, then the coadjoint representation R^* of $\mathfrak{J}$ satisfies the condition on π in Proposition 4.3.2 (2). For a trivial φ, we conclude that $\mathfrak{J} \oplus \mathfrak{J}^*$ is also a 2SN-Jordan algebra with respect to the coadjoint representation.

Definition 4.3.6. Let $\mathfrak{J}$ be a 2SN-Jordan algebra, V and W be two vector spaces. Let $\pi : \mathfrak{J} \to \mathrm{End}(V)$ and $\rho : \mathfrak{J} \to \mathrm{End}(W)$ be representations of $\mathfrak{J}$. The *direct sum* $\pi \oplus \rho : \mathfrak{J} \to \mathrm{End}(V \oplus W)$ of π and ρ is defined by

$$(\pi \oplus \rho)(x)(v+w) = \pi(x)(v) + \rho(x)(w), \ \forall\, x \in \mathfrak{J}, v \in V, w \in W.$$

Proposition 4.3.7. *Let* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *be 2SN-Jordan algebras and let* $\pi : \mathfrak{J}_1 \to \mathrm{End}(\mathfrak{J}_2)$ *be a linear map. Let*

$$\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2.$$

Define the following product on $\mathfrak{J}$*:*

$$(x+y)(x'+y') = xx' + \pi(x)(y') + \pi(x')(y) + yy', \ \forall\, x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Then $\mathfrak{J}$ *is a 2SN-Jordan algebra if and only if* π *satisfies:*

(1) $\pi(xx') = \pi(x)\pi(x') = 0$,

(2) $\pi(x)(yy') = (\pi(x)(y))y' = 0$,

for all $x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2$.

In this case, π *satisfies the conditions of Definition 4.1.15, it is called a* **2SN-admissible representation** *of* $\mathfrak{J}_1$ *in* $\mathfrak{J}_2$ *and we say that* $\mathfrak{J}$ *is the* **semi-direct product** *of* $\mathfrak{J}_2$ *by* $\mathfrak{J}_1$ *by means of* π.

Proof. For all $x, x', x'' \in \mathfrak{J}_1, y, y', y'' \in \mathfrak{J}_2$, one has:

$$\begin{aligned}((x+y)(x'+y'))(x''+y'') = \pi(xx')(y'') + \pi(x'')(\pi(x)(y') + \pi(x')(y) + yy') \\ +(\pi(x)(y') + \pi(x')(y))y''.\end{aligned}$$

Therefore, $\mathfrak{J}$ is 2-step nilpotent if and only if $\pi(xx')$, $\pi(x)\pi(x')$, $\pi(x)(yy')$ and $(\pi(x)y)y'$ are zero, for all $x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2$. □

Remark 4.3.8.

(1) The adjoint representation of a 2SN-Jordan algebra is a 2SN-admissible representation.

(2) Consider the particular case of $\mathfrak{J}_1 = \mathbb{C}c$ a one-dimensional algebra. If $\mathfrak{J}_1$ is 2-step nilpotent then $c^2 = 0$. Let $D = \pi(c) \in \mathrm{End}(\mathfrak{J}_2)$. The vector space $\mathfrak{J} = \mathbb{C}c \oplus \mathfrak{J}_2$ with the product:

$$(\alpha c + x)(\alpha' c + x') = \alpha D(x') + \alpha' D(x) + xx', \ \forall\, x, x' \in \mathfrak{J}_2, \alpha, \alpha' \in \mathbb{C}.$$

is 2-step nilpotent if and only if $D^2 = 0$, $D(xx') = D(x)x' = 0$, for all $x, x' \in \mathfrak{J}_2$.

(3) Let us slightly change (2) by fixing $x_0 \in \mathfrak{J}_2$ and setting the product on $\mathfrak{J} = \mathbb{C}c \oplus \mathfrak{J}_2$ as follows:

$$(\alpha c + x)(\alpha' c + x') = \alpha D(x') + \alpha' D(x) + xx' + \alpha\alpha' x_0,$$

for all $x, x' \in \mathfrak{J}_2, \alpha, \alpha' \in \mathbb{C}$. Then $\mathfrak{J}$ is a 2SN-Jordan algebra if and only if

$$D^2(x) = D(xx') = D(x)x' = D(x_0) = x_0 x = 0, \ \forall\, x, x' \in \mathfrak{J}_2.$$

In this case, we say (D, x_0) a *2SN-admissible pair* of $\mathfrak{J}_2$.

Next, we see how to obtain a 2SN-Jordan algebra from a pseudo-Euclidean one.

Proposition 4.3.9. *Let $(\mathfrak{J}, B)$ be a 2-step nilpotent pseudo-Euclidean Jordan algebra (or* **2SNPE-Jordan algebra** *for short), $\mathfrak{h}$ be another 2SN-Jordan algebra and $\pi : \mathfrak{h} \to \mathrm{End}_s(\mathfrak{J})$ be a linear map. Consider the bilinear map $\varphi : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{h}^*$ defined by $\varphi(x,y)(z) = B(\pi(z)(x), y)$, for all $x, y \in \mathfrak{J}, z \in \mathfrak{h}$. Let*

$$\overline{\mathfrak{J}} = \mathfrak{h} \oplus \mathfrak{J} \oplus \mathfrak{h}^*.$$

Define the following product on $\overline{\mathfrak{J}}$:

$$(x+y+f)(x'+y'+f') = xx' + yy' + \pi(x)(y') + \pi(x')(y) + f' \circ R_x + f \circ R_{x'} + \varphi(y, y')$$

for all $x, x' \in \mathfrak{h}, y, y' \in \mathfrak{J}, f, f' \in \mathfrak{h}^*$. Then $\overline{\mathfrak{J}}$ is a 2SN-Jordan algebra if and only if π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$. Moreover, $\overline{\mathfrak{J}}$ is pseudo-Euclidean with the bilinear form

$$\overline{B}(x+y+f, x'+y'+f') = B(y,y') + f(x') + f'(x), \ \forall\, x, x' \in \mathfrak{h}, y, y' \in \mathfrak{J}, f, f' \in \mathfrak{h}^*.$$

In this case, we say that $\overline{\mathfrak{J}}$ is a **2-step nilpotent double extension** *(or* **2SN-double extension***) of $\mathfrak{J}$ by $\mathfrak{h}$ by means of π.*

Proof. If $\overline{\mathfrak{J}}$ is 2-step nilpotent then the product is commutative and $((x+y+f)(x'+y'+f'))(x''+y''+f'') = 0$ for all $x, x', x'' \in \mathfrak{h}, y, y', y'' \in \mathfrak{J}, f, f', f'' \in \mathfrak{h}^*$. By a straightforward computation, one has that π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$.

Conversely, assume that π is a 2SN-admissible representation of $\mathfrak{h}$ in $\mathfrak{J}$. First, we set the extension $\mathfrak{J} \oplus \mathfrak{h}^*$ of $\mathfrak{J}$ by $\mathfrak{h}^*$ with the product:

$$(y+f)(y'+f') = yy' + \varphi(y, y'), \ \forall\, y, y' \in \mathfrak{J}, \ f, f' \in \mathfrak{h}^*.$$

Since $\pi(z) \in \mathrm{End}_s(\mathfrak{J})$ and $\pi(z)(yy') = 0$, for all $z \in \mathfrak{h}, \ y, y' \in \mathfrak{J}$, then one has φ symmetric and $\varphi(yy', y'') = 0$, for all $y, y', y'' \in \mathfrak{J}$. By Definition 4.3.3, $\mathfrak{J} \oplus \mathfrak{h}^*$ is a 2SN-central extension of $\mathfrak{J}$ by $\mathfrak{h}^*$.

Next, we consider the direct sum $\pi \oplus R^*$ of two representations: π and R^* of $\mathfrak{h}$ in $\mathfrak{J} \oplus \mathfrak{h}^*$ (see Definition 4.3.6). By a straightforward computation, we check that $\pi \oplus R^*$ satisfies the conditions of Proposition 4.3.7 then the semi-direct product of $\mathfrak{J} \oplus \mathfrak{h}^*$ by $\mathfrak{h}$ by means of $\pi \oplus R^*$ is 2-step nilpotent. Finally, the product defined in $\overline{\mathfrak{J}}$ is exactly the product defined by the semi-direct product in Proposition 4.3.7. Therefore we obtain the necessary and sufficient conditions.

As a consequence of Definition 4.1.17, $\overline{B}$ is an associative scalar product of $\overline{\mathfrak{J}}$, then $\overline{\mathfrak{J}}$ is a 2SNPE-Jordan algebra. □

The notion of 2SN-double extension **does not characterize** all 2SNPE-Jordan algebras: there exist 2SNPE-Jordan algebras that can be not described in terms of 2SN-double extensions, for example, the 2SNPE-Jordan algebra $\mathfrak{J} = \mathbb{C}a \oplus \mathbb{C}b$ with $a^2 = b$ and $B(a,b) = 1$, zero otherwise. Therefore, we need a better characterization given by the proposition below, its proof is a matter of a simple calculation.

Proposition 4.3.10. *Let $(\mathfrak{J},B)$ be a 2SNPE-Jordan algebra. Let $(D,x_0) \in \mathrm{End}_s(\mathfrak{J}) \times \mathfrak{J}$ be a 2SN-admissible pair with $B(x_0,x_0) = 0$ and let $(\mathfrak{t} = \mathbb{C}x_1 \oplus \mathbb{C}y_1, B_{\mathfrak{t}})$ be a quadratic vector space satisfying*

$$B_{\mathfrak{t}}(x_1,x_1) = B_{\mathfrak{t}}(y_1,y_1) = 0,\ B_{\mathfrak{t}}(x_1,y_1) = 1.$$

Fix α in $\mathbb{C}$ and consider the vector space

$$\overline{\mathfrak{J}} = \mathfrak{J} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with the product

$$y_1 \star y_1 = x_0 + \alpha x_1,\ y_1 \star x = x \star y_1 = D(x) + B(x_0,x)x_1,\ x \star y = xy + B(D(x),y)x_1$$

and $x_1 \star \overline{\mathfrak{J}} = \overline{\mathfrak{J}} \star x_1 = 0$, for all $x,y \in \mathfrak{J}$. Then $\overline{\mathfrak{J}}$ is a 2SNPE-Jordan algebra with the bilinear form $\overline{B} = B + B_{\mathfrak{t}}$.

In this case, $(\overline{\mathfrak{J}},\overline{B})$ is called a **generalized double extension** *of $\mathfrak{J}$ by means of (D,x_0,α).*

Proposition 4.3.11. *Let $(\mathfrak{J},B)$ be a 2SNPE-Jordan algebra. If $\mathfrak{J}$ is non-Abelian then it is obtained from an Abelian algebra by a sequence of generalized double extensions.*

Proof. Assume that $(\mathfrak{J},B)$ is a 2SNPE-Jordan algebra and $\mathfrak{J}$ is non-Abelian. By Proposition 4.1.14, $\mathfrak{J}$ has a reduced ideal $\mathfrak{l}$ that is still 2-step nilpotent. That means $\mathfrak{l}^2 \neq \mathfrak{l}$, so $\mathrm{Ann}(\mathfrak{l}) \neq \{0\}$. Therefore, we can choose a non-zero $x_1 \in \mathrm{Ann}(\mathfrak{l})$ such that $B(x_1,x_1) = 0$. Then there exists an isotropic element $y_1 \in \mathfrak{J}$ such that $B(x_1,y_1) = 1$. Let $\mathfrak{J} = (\mathbb{C}x_1 \oplus \mathbb{C}y_1) \overset{\perp}{\oplus} W$ where $W = (\mathbb{C}x_1 \oplus \mathbb{C}y_1)^{\perp}$. We have that $\mathbb{C}x_1$ and $x_1^{\perp} = \mathbb{C}x_1 \oplus W$ are ideals of $\mathfrak{J}$ as well.

Let $x,y \in W$, $xy = \beta(x,y) + \alpha(x,y)x_1$ where $\beta(x,y) \in W$ and $\alpha(x,y) \in \mathbb{C}$. It is easy to check that W with the product $W \times W \to W$, $(x,y) \mapsto \beta(x,y)$ is a 2SN-Jordan algebra. Moreover, it is also pseudo-Euclidean with the bilinear form $B_W = B|_{W \times W}$.

Now, we show that $\mathfrak{J}$ is a generalized double extension of (W,B_W). Indeed, let $x \in W$ then $y_1x = D(x) + \varphi(x)x_1$ where D is an endomorphism of W and $\varphi \in W^*$. Since $y_1(y_1x) = y_1(xy) = (y_1x)y = 0$, for all $x,y \in W$ we get $D^2(x) = D(x)y = D(xy) = 0$, for all $x,y \in W$. Moreover, $B(y_1x,y) = B(x,y_1y) = B(y_1,xy)$, for all $x,y \in W$ implies that $D \in \mathrm{End}_s(W)$ and $\alpha(x,y) = B_W(D(x),y)$, for all $x,y \in W$.

Since B_W is non-degenerate and $\varphi \in W^*$ then there exists $x_0 \in W$ such that $\varphi = B_W(x_0,.)$. Assume that $y_1^2 = \mu y_1 + y_0 + \lambda x_1$. The equality $B(y_1^2,x_1) = 0$ implies $\mu = 0$. Moreover, $y_0 = x_0$ since $B(y_1x,y_1) = B(x,y_1^2)$, for all $x \in W$. Finally, $D(x_0) = 0$ is obtained by $y_1^3 = 0$ and this is enough to conclude that $\mathfrak{J}$ is a generalized double extension of (W,B_W) by means of (D,x_0,λ). □

4.3.2 T^*-extensions of pseudo-Euclidean 2-step nilpotent

Given a 2SN-Jordan algebra $\mathfrak{J}$ and a symmetric bilinear map $\theta : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}^*$ such that $R^*(z)(\theta(x,y)) + \theta(xy,z) = 0$, for all $x,y,z \in \mathfrak{J}$, then by Proposition 4.3.2, $\overline{\mathfrak{J}} = \mathfrak{J} \oplus \mathfrak{J}^*$ is also a 2SN-Jordan algebra. Moreover, if θ is cyclic (that is, $\theta(x,y)(z) = \theta(y,z)(x)$, for all $x,y,z \in \mathfrak{J}$), then $\overline{\mathfrak{J}}$ is a pseudo-Euclidean Jordan algebra with the bilinear form defined by

$$B(x+f,y+g) = f(y)+g(x),\ \forall\, x,y \in \mathfrak{J}, f,g \in \mathfrak{J}^*.$$

In a more general framework, we can define:

Definition 4.3.12. Let $\mathfrak{a}$ be a complex vector space and $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ a cyclic symmetric bilinear map. Assume that θ is non-degenerate, i.e. if $\theta(x,\mathfrak{a}) = 0$ then $x = 0$. Consider the vector space $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{a}^*$ equipped with the product

$$(x+f)(y+g) = \theta(x,y)$$

and the bilinear form

$$B(x+f,y+g) = f(y)+g(x)$$

for all $x+f, y+g \in \mathfrak{J}$. Then $(\mathfrak{J},B)$ is a 2SNPE-Jordan algebra and it is called the *T^*-extension* of $\mathfrak{a}$ by θ.

Lemma 4.3.13. *Let $\mathfrak{J}$ be a T^*-extension of $\mathfrak{a}$ by θ. If $\mathfrak{J} \neq \{0\}$ then $\mathfrak{J}$ is reduced.*

Proof. Since θ is non-degenerate, it is easy to check that $\mathrm{Ann}(\mathfrak{J}) = \mathfrak{a}^*$ is totally isotropic by the above definition. □

Proposition 4.3.14. *Let $(\mathfrak{J},B)$ be a 2SNPE-Jordan algebra. If $\mathfrak{J}$ is reduced then $\mathfrak{J}$ is i-isomorphic to some T^*-extension.*

Proof. Assume that $\mathfrak{J}$ is a reduced 2SNPE-Jordan algebra. Then one has $\mathrm{Ann}(\mathfrak{J}) = \mathfrak{J}^2$, so $\dim(\mathfrak{J}^2) = \frac{1}{2}\dim(\mathfrak{J})$. Let $\mathfrak{J} = \mathrm{Ann}(\mathfrak{J}) \oplus \mathfrak{a}$ where $\mathfrak{a}$ is a totally isotropic subspace of $\mathfrak{J}$. Then $\mathfrak{a} \simeq \mathfrak{J}/\mathfrak{J}^2$ as an Abelian algebra. Since $\mathfrak{a}$ and $\mathrm{Ann}(\mathfrak{J})$ are maximal totally isotropic subspaces of $\mathfrak{J}$, we can identify $\mathrm{Ann}(\mathfrak{J})$ to $\mathfrak{a}^*$ by the isomorphism: $\varphi : \mathrm{Ann}(\mathfrak{J}) \to \mathfrak{a}^*$, $\varphi(x)(y) = B(x,y)$, for all $x \in \mathrm{Ann}(\mathfrak{J}),\ y \in \mathfrak{a}$. Define $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ by $\theta(x,y) = \varphi(xy)$, for all $x,y \in \mathfrak{a}$.

Now, set $\alpha : \mathfrak{J} \to \mathfrak{a} \oplus \mathfrak{a}^*$ by $\alpha(x) = p_1(x) + \varphi(p_2(x))$, for all $x \in \mathfrak{J}$ where $p_1 : \mathfrak{J} \to \mathfrak{a}$ and $p_2 : \mathfrak{J} \to \mathrm{Ann}(\mathfrak{J})$ are canonical projections. Then α is i-isomorphic. □

Theorem 4.3.15. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be two T^*-extensions of $\mathfrak{a}$ by θ_1 and θ_2 respectively. Then:*

(1) there exists a Jordan algebra isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exist an isomorphism A_1 of $\mathfrak{a}$ and an isomorphism A_2 of $\mathfrak{a}^*$ satisfying:

$$A_2(\theta_1(x,y)) = \theta_2(A_1(x),A_1(y)),\ \forall\, x,y \in \mathfrak{a}.$$

(2) there exists an i-isomorphism between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ if and only if there exists an isomorphism A_1 of $\mathfrak{a}$ satisfying

$$\theta_1(x,y) = \theta_2(A_1(x),A_1(y)) \circ A_1,\ \forall\, x,y \in \mathfrak{a}.$$

Proof.

(1) Let $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ be a Jordan algebra isomorphism. Since $\mathfrak{a}^* = \mathrm{Ann}(\mathfrak{J}_1) = \mathrm{Ann}(\mathfrak{J}_2)$ is stable by A then there exist linear maps $A_1 : \mathfrak{a} \to \mathfrak{a}$, $A_1' : \mathfrak{a} \to \mathfrak{a}^*$ and $A_2 : \mathfrak{a}^* \to \mathfrak{a}^*$ such that:
$$A(x+f) = A_1(x) + A_1'(x) + A_2(f),\ \forall\, x+f \in \mathfrak{J}_1.$$
Since A is an isomorphism one has A_2 also isomorphic. We show that A_1 is an isomorphism of $\mathfrak{a}$. Indeed, if $A_1(x_0) = 0$ with some $x_0 \in \mathfrak{a}$ then $A(x_0) = A_1'(x_0)$ and
$$0 = A(x_0)\mathfrak{J}_2 = A(x_0 A^{-1}(\mathfrak{J}_2)) = A(x_0\mathfrak{J}_1).$$
That implies $x_0\mathfrak{J}_1 = 0$ and so $x_0 \in \mathfrak{a}^*$. That means $x_0 = 0$, i.e. A_1 is an isomorphism of $\mathfrak{a}$.
For all x and y in $\mathfrak{a}$, one has $A(xy) = A(\theta_1(x,y)) = A_2(\theta_1(x,y))$ and
$$A(x)A(y) = (A_1(x) + A_1'(x))(A_1(y) + A_1'(y)) = A_1(x)A_1(y) = \theta_2(A_1(x), A_1(y)).$$
Therefore, $A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y))$, for all $x, y \in \mathfrak{a}$.
Conversely, if there exist an isomorphism A_1 of $\mathfrak{a}$ and an isomorphism A_2 of $\mathfrak{a}^*$ satisfying:
$$A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y)),\ \forall\, x, y \in \mathfrak{a},$$
then we define $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ by $A(x+f) = A_1(x) + A_2(f)$, for all $x + f \in \mathfrak{J}_1$. It is easy to see that A is a Jordan algebra isomorphism.

(2) Assume that $A : \mathfrak{J}_1 \to \mathfrak{J}_2$ is a Jordan algebra i-isomorphism then there exist A_1 and A_2 defined as in (1). Let $x \in \mathfrak{a}, f \in \mathfrak{a}^*$, one has:
$$B'(A(x), A(f)) = B(x,f) \Rightarrow A_2(f)(A_1(x)) = f(x).$$
Hence, $A_2(f) = f \circ A_1^{-1}$, for all $f \in \mathfrak{a}^*$. Moreover, $A_2(\theta_1(x,y)) = \theta_2(A_1(x), A_1(y))$ implies that
$$\theta_1(x,y)) = \theta_2(A_1(x), A_1(y)) \circ A_1,\ \forall\, x, y \in \mathfrak{a}.$$
Conversely, define $A(x+f) = A_1(x) + f \circ A_1^{-1}$, for all $x + f \in \mathfrak{J}_1$ then A is an i-isomorphism.

□

Example 4.3.16. We keep the notations as above. Let $\mathfrak{J}'$ be the T^*-extension of $\mathfrak{a}$ by $\theta' = \lambda\theta, \lambda \neq 0$ then $\mathfrak{J}$ and $\mathfrak{J}'$ is i-isomorphic by $A : \mathfrak{J} \to \mathfrak{J}'$ defined by
$$A(x+f) = \frac{1}{\alpha}x + \alpha f,\ \forall\, x + f \in \mathfrak{J}$$
where $\alpha \in \mathbb{C},\ \alpha^3 = \lambda$.

For a non-degenerate cyclic symmetric map θ of $\mathfrak{a}$, define a 3-form

$$I(x,y,z) = \theta(x,y)z, \ \forall\, x,y,z \in \mathfrak{a}.$$

Then $I \in \mathscr{S}^3(\mathfrak{a})$, the space of symmetric 3-forms on $\mathfrak{a}$. The non-degenerate condition of θ is equivalent to $\frac{\partial I}{\partial p} \neq 0$, for all $p \in \mathfrak{a}^*$.

Conversely, let $\mathfrak{a}$ be a complex vector space and $I \in \mathscr{S}^3(\mathfrak{a})$ such that $\frac{\partial I}{\partial p} \neq 0$ for all $p \in \mathfrak{a}^*$. Define $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ by $\theta(x,y) = I(x,y,.)$, for all $x,y \in \mathfrak{a}$ then θ is symmetric and non-degenerate. Moreover, since I is symmetric, then θ is cyclic and we obtain a reduced pseudo-Euclidean 2-step nilpotent Jordan algebra $T^*_\theta(\mathfrak{a})$ defined by θ. Therefore, there is an one-to-one map from the set of all T^*-extensions of a complex vector space $\mathfrak{a}$ onto the subset $\{I \in \mathscr{S}^3(\mathfrak{a}) \mid \frac{\partial I}{\partial p} \neq 0, \ \forall\, p \in \mathfrak{a}^*\}$, such elements are also called *non-degenerate*.

Corollary 4.3.17. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be T^*-extensions of $\mathfrak{a}$ with respect to I_1 and I_2 non-degenerate. Then $\mathfrak{J}$ and $\mathfrak{J}'$ are i-isomorphic if and only if there exists an isomorphism A of $\mathfrak{a}$ such that*

$$I_1(x,y,z) = I_2(A(x),A(y),A(z)), \ \forall\, x,y,z \in \mathfrak{a}.$$

In particular, $\mathfrak{J}$ and $\mathfrak{J}'$ are i-isomorphic if and only if there is an isomorphism ${}^tA$ on $\mathfrak{a}^*$ which induces the isomorphism on $\mathscr{S}^3(\mathfrak{a})$, also denoted by ${}^tA$ such that ${}^tA(I_1) = I_2$. In this case, we say that I_1 and I_2 are *equivalent*.

Example 4.3.18. Let $\mathfrak{a} = \mathbb{C}e$ be a one-dimensional vector space then $\mathscr{S}^3(\mathfrak{a}) = \mathbb{C}(e^*)^3$. By Example 4.3.16, T^*-extensions of $\mathfrak{a}$ by $(e^*)^3$ and $\lambda(e^*)^3$, $\lambda \neq 0$, are i-isomorphic (also, these 3-forms are equivalent). Hence, there is only one i-isomorphic class of T^*-extensions of $\mathfrak{a}$, that is $\mathfrak{J} = \mathbb{C}e \oplus \mathbb{C}f$ with $e^2 = f$ and $B(e,f) = 1$, the other are zero.

Now, let $\mathfrak{a} = \mathbb{C}x \oplus \mathbb{C}y$ be 2-dimensional vector space then

$$\mathscr{S}^3(\mathfrak{a}) = \{a_1(x^*)^3 + a_2(x^*)^2y^* + a_3x^*(y^*)^2 + a_4(y^*)^3, \ a_i \in \mathbb{C}\}.$$

It is easy to prove that every bivariate homogeneous polynomial of degree 3 is reducible. Therefore, by a suitable basis choice (certainly isomorphic), a non-degenerate element $I \in \mathscr{S}^3(\mathfrak{a})$ has the form $I = ax^*y^*(bx^* + cy^*), \ a,b \neq 0$. Replace x^* by αx^* with $\alpha^2 = ab$ to get the form of $I_\lambda = x^*y^*(x^* + \lambda y^*), \lambda \in \mathbb{C}$.

Next, we will show that I_0 and $I_\lambda, \lambda \neq 0$ are not equivalent. Indeed, assume the contrary, i.e. there is an isomorphism ${}^tA$ such that ${}^tA(I_0) = I_\lambda$. We can write

$${}^tA(x^*) = a_1x^* + b_1y^*, \ {}^tA(y^*) = a_2x^* + b_2y^*, \ a_1,a_2,b_1,b_2 \in \mathbb{C}.$$

Then

$$\begin{aligned}{}^tA(I_0) = (a_1x^* + b_1y^*)^2(a_2x^* + b_2y^*) = a_1^2a_2(x^*)^3 + (a_1^2b_2 + 2a_1a_2b_1)(x^*)^2y^* +\\ (2a_1b_1b_2 + a_2b_1^2)x^*(y^*)^2 + b_1^2b_2(y^*)^3.\end{aligned}$$

Comparing the coefficients we will get a contradiction. Therefore, I_0 and $I_\lambda, \ \lambda \neq 0$ are not equivalent.

However, two forms I_{λ_1} and I_{λ_2} where $\lambda_1, \lambda_2 \neq 0$ are equivalent by the isomorphism ${}^tA$ satisfying ${}^tA(I_{\lambda_1}) = I_{\lambda_2}$ defined by:

$${}^tA(x^*) = \alpha y^*,\ {}^tA(y^*) = \beta x^*$$

where $\alpha, \beta \in \mathbb{C}$ such that $\alpha^3 = \lambda_1 \lambda_2^2$ and $\beta^3 = \frac{1}{\lambda_1^2 \lambda_2}$ and satisfying ${}^tA(I_{\lambda_1}) = I_{\lambda_2}$. That implies that there are only two i-isomorphic classes of T^*-extensions of $\mathfrak{a}$.

Example 4.3.19. Let $\mathfrak{J}_0 = \text{span}\{x, y, e, f\}$ be a T^*-extension of a 2-dimensional vector space $\mathfrak{a}$ by $I_0 = (x^*)^2 y^*$, with $e = x^*$ and $f = y^*$, that means $B(x,e) = B(y,f) = 1$, the other are zero. It is easy to compute the product in $\mathfrak{J}_0$ defined by $x^2 = f$, $xy = e$. Let $I_\lambda = x^* y^* (x^* + \lambda y^*)$, $\lambda \neq 0$ and $\mathfrak{J}_\lambda = \text{span}\{x, y, e, f\}$ be two T^*-extensions of the 2-dimensional vector space $\mathfrak{a}$ by I_λ. The products on $\mathfrak{J}_\lambda$ are $x^2 = f$, $xy = e + \lambda f$ and $y^2 = \lambda e$. These two algebras are neither i-isomorphic nor isomorphic. Indeed, if there is $A : \mathfrak{J}_0 \to \mathfrak{J}_\lambda$ an isomorphism. Assume $A(y) = \alpha_1 x + \alpha_2 y + \alpha_3 e + \alpha_4 f$ then

$$0 = A(y^2) = (\alpha_1 x + \alpha_2 y + \alpha_3 e + \alpha_4 f)^2 = \alpha_1^2 x^2 + 2\alpha_1 \alpha_2 xy + \alpha_2^2 y^2.$$

We obtain $(\lambda \alpha_2^2 + 2\alpha_1 \alpha_2)e + (2\lambda \alpha_1 \alpha_2 + \alpha_1^2) f = 0$. Hence, $\alpha_1 = \pm \lambda \alpha_2$. Both cases imply $\alpha_1 = \alpha_2 = 0$ (a contradiction).

We can also conclude that there are only two isomorphic classes of T^*-extensions of $\mathfrak{a}$.

4.4 Symmetric Novikov algebras

Definition 4.4.1. An algebra $\mathfrak{N}$ over $\mathbb{C}$ with a bilinear product $\mathfrak{N}\times\mathfrak{N}\to\mathfrak{N}$, $(x,y)\mapsto xy$ is called a *left-symmetric algebra* if it satisfies the identity:

$$(xy)z - x(yz) = (yx)z - y(xz),\ \forall\, x,y,z\in\mathfrak{N}. \tag{III}$$

or in term of associators

$$(x,y,z) = (y,x,z),\ \forall\, x,y,z\in\mathfrak{N}.$$

It is called a *Novikov algebra* if in addition

$$(xy)z = (xz)y \tag{IV}$$

holds for all $x,y,z\in\mathfrak{N}$. In this case, the commutator $[x,y]=xy-yx$ of $\mathfrak{N}$ defines a Lie algebra, denoted by $\mathfrak{g}(\mathfrak{N})$, which is called the *sub-adjacent Lie algebra* of $\mathfrak{N}$. It is known that $\mathfrak{g}(\mathfrak{N})$ is a solvable Lie algebra [Bur06]. Conversely, let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\ ,\ ]$. If there exists a bilinear product $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},(x,y)\mapsto xy$ that satisfies (III), (IV) and $[x,y]=xy-yx$, for all $x,y\in\mathfrak{J}$ then we say that $\mathfrak{g}$ admits a *Novikov structure*.

Example 4.4.2. Every 2-step nilpotent algebra $\mathfrak{N}$ satisfying $(xy)z=x(yz)=0$ for all $x,y,z\in\mathfrak{N}$, is a Novikov algebra.

For $x\in\mathfrak{N}$, denote by L_x and R_x respectively the left and right multiplication operators $L_x(y)=xy$, $R_x(y)=yx$, for all $y\in\mathfrak{N}$. The condition (III) is equivalent to $[L_x,L_y]=L_{[x,y]}$ and (IV) is equivalent to $[R_x,R_y]=0$. In the other words, the left-operators form a Lie algebra and the right-operators commute.

It is easy to check two Jacobi-type identities:

Proposition 4.4.3. *Let $\mathfrak{N}$ be a Novikov algebra then for all $x,y,z\in\mathfrak{N}$:*

$$[x,y]z+[y,z]x+[z,x]y=0,$$

$$x[y,z]+y[z,x]+z[x,y]=0$$

Definition 4.4.4. Let $\mathfrak{N}$ be a Novikov algebra. A bilinear form $B:\mathfrak{N}\times\mathfrak{N}\to\mathbb{C}$ is called *associative* if

$$B(xy,z)=B(x,yz),\ \forall\, x,y,z\in\mathfrak{N}.$$

We say that $\mathfrak{N}$ is a *symmetric Novikov algebra* if it is endowed with a non-degenerate associative symmetric bilinear form B.

Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra and S be a subspace of $\mathfrak{N}$. Denote by $S^\perp$ the set $\{x\in\mathfrak{N}\mid\ B(x,S)=0\}$. If $B|_{S\times S}$ is non-degenerate (resp. degenerate) then we say that S is *non-degenerate* (resp. *degenerate*).

Lemma 4.4.5. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra and I be a two-sided ideal (or simply an ideal) of $\mathfrak{N}$ then*

(1) $I^\perp$ is also a two-sided ideal of $\mathfrak{N}$ and $II^\perp=I^\perp I=\{0\}$

(2) If I is non-degenerate then so is $I^{\perp}$ and $\mathfrak{N} = I \overset{\perp}{\oplus} I^{\perp}$.

Proof.

(1) Since $B(x\mathfrak{N},I) = B(x,\mathfrak{N}I) = 0$ and $B(I,\mathfrak{N}x) = B(I\mathfrak{N},x) = 0$, for all $x \in I^{\perp}$ then $I^{\perp}$ is also an ideal of $\mathfrak{N}$. Let $x \in II^{\perp}$, i.e. $x = yz$ with $y \in I, z \in I^{\perp}$ then $B(x,\mathfrak{N}) = B(yz,\mathfrak{N}) = B(y,z\mathfrak{N}) = 0$. Therefore $x = 0$. That means $II^{\perp} = \{0\}$. Similarly, one gains $I^{\perp}I = \{0\}$.

(2) Assume that $I^{\perp}$ is degenerate, that means there is a non-zero $x \in I^{\perp}$ such that $B(x,I^{\perp}) = 0$. Therefore, $x \in I$. However, $B(x,I) = 0$ since $x \in I^{\perp}$ so I is degenerate (that is a contradiction). Hence, $I^{\perp}$ must be non-degenerate.

Let $x \in I \cap I^{\perp}$ then $B(x,I) = 0$. Since I is non-degenerate, one has $x = 0$. That means $I \cap I^{\perp} = \{0\}$ and $\mathfrak{N} = I \overset{\perp}{\oplus} I^{\perp}$.

□

Proposition 4.4.6. *Let* $\mathsf{Z}(\mathfrak{N}) = \{x \in \mathfrak{N} \mid\ xy = yx,\ \forall\ y \in \mathfrak{N}\}$ *the center of* $\mathfrak{N}$ *and denote by* $\mathrm{As}(\mathfrak{N}) = \{x \in \mathfrak{N} \ \mid\ (x,y,z) = 0,\ \forall\ y,z \in \mathfrak{N}\}$. *One has*

(1) If $\mathfrak{N}$ is a Novikov algebra then $\mathsf{Z}(\mathfrak{N}) \subset N(\mathfrak{N})$ where $N(\mathfrak{N}) = \{x \in \mathfrak{N} \mid\ (x,y,z) = (y,x,z) = (y,z,x) = 0,\ \forall\ y,z \in \mathfrak{N}\}$ is the nucleus of $\mathfrak{N}$ (see Definition 4.1.11 (3)). Moreover, if $\mathfrak{N}$ is also commutative then $N(\mathfrak{N}) = \mathfrak{N} = \mathrm{As}(\mathfrak{N})$ (that means $\mathfrak{N}$ is an associative algebra).

(2) If $(\mathfrak{N},B)$ is a symmetric Novikov algebra then

(i) $\mathsf{Z}(\mathfrak{N}) = [\mathfrak{g}(\mathfrak{N}),\mathfrak{g}(\mathfrak{N})]^{\perp}$

(ii) $N(\mathfrak{N}) = \mathrm{As}(\mathfrak{N}) = (\mathfrak{N},\mathfrak{N},\mathfrak{N})^{\perp}$.

(iii) $L\,\mathrm{Ann}(\mathfrak{N}) = R\,\mathrm{Ann}(\mathfrak{N}) = \mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}^2)^{\perp}$.

Proof.

(1) Assume that x is an element in $\mathsf{Z}(\mathfrak{N})$. For all $y,z \in \mathfrak{N}$, one has:

$$(x,y,z) = (xy)z - x(yz) = (yx)z - x(yz) = (yz)x - x(yz) = 0.$$

By (III), we also have $(y,x,z) = 0$. Moreover, since

$$(y,z,x) = (yz)x - y(zx) = x(yz) - y(xz)$$

$$= (xy)z - (x,y,z) - (yx)z + (y,x,z) = (xy)z - (yx)z = 0$$

one obtains $\mathsf{Z}(\mathfrak{N}) \subset N(\mathfrak{N})$. If $\mathfrak{N}$ is commutative then $\mathsf{Z}(\mathfrak{N}) = \mathfrak{N}$. Certainly, $\mathsf{Z}(\mathfrak{N}) = N(\mathfrak{N}) = \mathfrak{N}$.

(2) Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra.

(i) It is obvious since $\mathfrak{g}(\mathfrak{N})$ is a quadratic Lie algebra and $\mathsf{Z}(\mathfrak{N})$ is its center.

(ii) For all $x,y,z,t \in \mathfrak{N}$, one has

$$B((x,y,z),t) = B((xy)z - x(yz),t) = B(x, y(zt) - (yz)t) = -B(x,(y,z,t)).$$

Therefore, $\mathrm{As}(\mathfrak{N}) = (\mathfrak{N},\mathfrak{N},\mathfrak{N})^{\perp}$. To prove $N(\mathfrak{N}) = \mathrm{As}(\mathfrak{N})$, we fix $x \in \mathrm{As}(\mathfrak{N})$ and let $y,z,t \in \mathfrak{N}$. Since $(x,z,t) = 0$ and (III) one has $(z,x,t) = 0$. Moreover, $B((y,z,x),t) = -B(y,(z,x,t))$ and B non-degenerate imply $(y,z,x) = 0$, for all $y,z \in \mathfrak{N}$. Hence, $N(\mathfrak{N}) = \mathrm{As}(\mathfrak{N})$.

(iii) Let $x \in \mathrm{LAnn}(\mathfrak{N})$ then $B(x\mathfrak{N},\mathfrak{N}) = B(x,\mathfrak{N}^2) = 0$. It means that $x \in (\mathfrak{N}^2)^{\perp}$. Conversely, if $x \in (\mathfrak{N}^2)^{\perp}$ then since $B(x\mathfrak{N},\mathfrak{N}) = B(x,\mathfrak{N}^2) = 0$ one has $x \in \mathrm{LAnn}(\mathfrak{N})$. It implies that $\mathrm{LAnn}(\mathfrak{N}) = (\mathfrak{N}^2)^{\perp}$. Similarly, we obtain $\mathrm{LAnn}(\mathfrak{N}) = \mathrm{RAnn}(\mathfrak{N}) = \mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}^2)^{\perp}$.

□

Proposition 4.4.7. *Let $\mathfrak{N}$ be a Novikov algebra then*

(1) $\mathsf{Z}(\mathfrak{N})$ *is a commutative subalgebra.*

(2) $\mathrm{As}(\mathfrak{N})$, $N(\mathfrak{N})$ *are ideals.*

Proof.

(1) Let $x,y \in \mathsf{Z}(\mathfrak{N})$ then $(xy)z = (xz)y = (zx)y = z(xy) + (z,x,y) = z(xy)$, for all $z \in \mathfrak{N}$. Therefore, $xy \in \mathsf{Z}(\mathfrak{N})$ and then $\mathsf{Z}(\mathfrak{N})$ is a subalgebra of $\mathfrak{N}$. Certainly, $\mathsf{Z}(\mathfrak{N})$ is commutative.

(2) Let $x \in \mathrm{As}(\mathfrak{N}), y,z,t \in \mathfrak{N}$. By the equality

$$(xy,z,t) = ((xy)z)t - (xy)(zt) = ((xz)t)y - (x(zt))y = (x,z,t)y = 0,$$

one has $xy \in \mathrm{As}(\mathfrak{N})$. Moreover,

$$(yx,z,t) = ((yx)z)t - (yx)(zt) = (y(xz))t - y(x(zt))$$

$$= (y,xz,t) + y((xz)t) - y(x(zt)) = y(x,z,t) = 0$$

since $xz \in \mathrm{As}(\mathfrak{N})$. Therefore $\mathrm{As}(\mathfrak{N})$ is an ideal of $\mathfrak{N}$.

Similarly, let $x \in N(\mathfrak{N}), y,z,t \in \mathfrak{N}$ one has:

$$(y,z,xt) = (yz)(xt) - y(z(xt)) = ((yz)x)t - (yz,x,t) - y((zx)t - (z,x,t))$$

$$= ((yz)x)t - (y(zx))t + (y,zx,t) = (y,z,x)t = 0$$

and

$$(y,z,tx) = (yz)(tx) - y(z(tx)) = ((yz)t)x - (yz,t,x) - y((zt)x - (z,t,x))$$

$$= ((yz)x)t - y((zx)t) = (y,z,x)t + (y,zx,t) = 0.$$

Then $N(\mathfrak{N})$ is also an ideal of $\mathfrak{N}$.

□

Lemma 4.4.8. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then $[L_x,L_y] = L_{[x,y]} = 0$ for all $x,y \in \mathfrak{N}$. Consequently, for a symmetric Novikov algebra, the Lie algebra formed by the left-operators is Abelian.*

Proof. It follows the proof of Lemma II.5 in [AB10]. Fix $x,y \in \mathfrak{N}$, for all $z,t \in \mathfrak{N}$ one has

$$B([L_x,L_y](z),t) = B(x(yz) - y(xz),t) = B((tx)y - (ty)x,z) = 0.$$

Therefore, $[L_x,L_y] = L_{[x,y]} = 0$, for all $x,y \in \mathfrak{N}$. □

Corollary 4.4.9. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then the the sub-adjacent Lie algebra $\mathfrak{g}(\mathfrak{N})$ of $\mathfrak{N}$ with the bilinear form B becomes a 2-step nilpotent quadratic Lie algebra.*

Proof. One has

$$B([x,y],z) = B(xy - yx,z) = B(x,yz) - B(x,zy) = B(x,[y,z]),\ \forall\, x,y,z \in \mathfrak{N}.$$

Hence, $\mathfrak{g}(\mathfrak{N})$ is quadratic. By Lemma 4.4.8 and 2(iii) of Proposition 4.4.6, one has $[x,y] \in \mathrm{LAnn}(\mathfrak{N}) = \mathrm{Ann}(\mathfrak{N})$, for all $x,y \in \mathfrak{N}$. That implies $[[x,y],z] \in \mathrm{Ann}(\mathfrak{N})\mathfrak{N} = \{0\}$, for all $x,y \in \mathfrak{N}$, i.e. $\mathfrak{g}(\mathfrak{N})$ is 2-step nilpotent. □

It results that the classification of 2-step nilpotent quadratic Lie algebras in [Ova07] and Chapter 2 is closely related to the classification of symmetric Novikov algebras. For instance, by Remark 2.2.10, every 2-step nilpotent quadratic Lie algebra of dimension ≤ 5 is Abelian so that every symmetric Novikov algebra of dimension ≤ 5 is commutative. In general, in the case of dimension ≥ 6, there exists a non-commutative symmetric Novikov algebra by Proposition 4.4.11 below.

Definition 4.4.10. Let $\mathfrak{N}$ be a Novikov algebra. We say that $\mathfrak{N}$ is an *anti-commutative Novikov algebra* if

$$xy = -yx,\ \forall\, x,y \in \mathfrak{N}.$$

Proposition 4.4.11. *Let $\mathfrak{N}$ be a Novikov algebra. Then $\mathfrak{N}$ is anti-commutative if and only if $\mathfrak{N}$ is a 2-step nilpotent Lie algebra with the Lie bracket defined by $[x,y] = xy$, for all $x,y \in \mathfrak{N}$.*

Proof. Assume that $\mathfrak{N}$ is a Novikov algebra such that $xy = -yx$, for all $x,y \in \mathfrak{N}$. Since the commutator $[x,y] = xy - yx = 2xy$ is a Lie bracket, so the product $(x,y) \mapsto xy$ is also a Lie bracket. The identity (III) of Definition 4.4.1 is equivalent to $(xy)z = 0$, for all $x,y,z \in \mathfrak{N}$. It shows that $\mathfrak{N}$ is a 2-step nilpotent Lie algebra.

Conversely, if $\mathfrak{N}$ is a 2-step nilpotent Lie algebra then we define the product $xy = [x,y]$, for all $x,y \in \mathfrak{N}$. It is obvious that the identities (III) and (IV) of Definition 4.4.1 are satisfied since $(xy)z = 0$, for all $x,y,z \in \mathfrak{N}$. □

By the above proposition, the study of anti-commutative Novikov algebras is reduced to the study of 2-step nilpotent Lie algebras. Moreover, the formula in this proposition also can be used to define a 2-step nilpotent symmetric Novikov algebra from a 2-step nilpotent quadratic Lie algebra. Recall that there exists only one non-Abelian 2-step nilpotent quadratic Lie algebra of dimension 6 up to isomorphisms (Remark 2.4.21) then there is only one anti-commutative symmetric Novikov algebra of dimension 6 up to isomorphisms. However, there exist non-commutative symmetric Novikov algebras that are not 2-step nilpotent [AB10]. For example, let $\mathfrak{N} = \mathfrak{g}_6 \overset{\perp}{\oplus} \mathbb{C}c$ where $\mathfrak{g}_6$ is the 6-dimensional elementary quadratic Lie algebra in Proposition 2.2.29 and $\mathbb{C}c$ is a pseudo-Euclidean simple Jordan algebra with the bilinear form $B_c(c,c) = 1$ (obviously, this algebra is a symmetric Novikov algebra and commutative). Then $\mathfrak{N}$ becomes a symmetric Novikov algebra with the bilinear form defined by $B = B_{\mathfrak{g}_6} + B_c$ where $B_{\mathfrak{g}_6}$ is the bilinear form on $\mathfrak{g}_6$. We can extend this example for the case $\mathfrak{N} = \mathfrak{g} \overset{\perp}{\oplus} \mathfrak{J}$ where $\mathfrak{g}$ is a 2-step nilpotent quadratic Lie algebra and $\mathfrak{J}$ is a symmetric Jordan-Novikov algebra defined below. However, these algebras are decomposable. An example in the indecomposable case of dimension 7 can be found in the last part of this section.

Proposition 4.4.12. *Let $\mathfrak{N}$ be a Novikov algebra. Assume that its product is commutative, that means $xy = yx$, for all $x,y \in \mathfrak{N}$. Then the identities (III) and (IV) of Definition 4.4.1 are equivalent to the only condition:*

$$(x,y,z) = (xy)z - x(yz) = 0, \ \forall\, x,y,z \in \mathfrak{N}.$$

That means that $\mathfrak{N}$ is an associative algebra. Moreover, $\mathfrak{N}$ is also a Jordan algebra. In this case, we say that $\mathfrak{N}$ is a **Jordan-Novikov algebra**. *In addition, if $\mathfrak{N}$ has a non-degenerate associative symmetric bilinear form, then we say that $\mathfrak{N}$ is a* **symmetric Jordan-Novikov algebra**.

Proof. Assume $\mathfrak{N}$ is a commutative Novikov algebra. By (1) of Proposition 4.4.6, the product is also associative. Conversely, if one has the condition:

$$(xy)z - x(yz) = 0, \ \forall\, x,y,z \in \mathfrak{N}$$

then (III) is zero and (IV) is obtained by $(yx)z = y(xz)$, for all $x,y,z \in \mathfrak{N}$. □

Example 4.4.13. Recall the pseudo-Euclidean Jordan algebra $\mathfrak{J}$ in Example 4.2.10 spanned by $\{x, x_1, y_1\}$ where the commutative product on $\mathfrak{J}$ is defined by:

$$y_1^2 = y_1, y_1 x = x, y_1 x_1 = x_1, x^2 = x_1.$$

It is easy to check that this product is also associative. Therefore, $\mathfrak{J}$ is a symmetric Jordan-Novikov algebra with the bilinear form B defined by $B(x_1,y_1) = B(x,x) = 1$ and the other zero.

Example 4.4.14. Pseudo-Euclidean 2-step nilpotent Jordan algebras are symmetric Jordan-Novikov algebras.

Remark 4.4.15.

(1) By Lemma 4.4.8, if the symmetric Novikov algebra $\mathfrak{N}$ has $\mathrm{Ann}(\mathfrak{N}) = \{0\}$ then $[x,y] = xy - yx = 0$, for all $x,y \in \mathfrak{N}$. It implies that $\mathfrak{N}$ is commutative and then $\mathfrak{N}$ is a symmetric Jordan-Novikov algebra.

(2) If the product on $\mathfrak{N}$ is associative then it may not be commutative. An example can be found in the next part.

(3) Let $\mathfrak{N}$ be a Novikov algebra with unit element e, that is $ex = xe = x$, for all $x \in \mathfrak{N}$. Then $xy = (ex)y = (ey)x = yx$, for all $x, y \in \mathfrak{N}$ and therefore $\mathfrak{N}$ is a Jordan-Novikov algebra.

(4) The algebra given in Example 4.4.13 is also a Frobenius algebra, that is a finite-dimensional associative algebra with unit element equipped having a non-degenerate associative bilinear form.

A well-known result is that every associative algebra $\mathfrak{N}$ is Lie-admissible and Jordan-admissible, that is, if $(x,y) \mapsto xy$ is the product of $\mathfrak{N}$ then the products

$$[x,y] = xy - yx \quad \text{and} \quad [x,y]_+ = xy + yx$$

define respectively a Lie algebra structure and a Jordan algebra structure on $\mathfrak{N}$. There exist algebras satisfying each one of these properties. For example, the non-commutative Jordan algebras are Jordan-admissible [Sch55] or the Novikov algebras are Lie-admissible. However, remark that a Novikov algebra may not be Jordan-admissible by the following example:

Example 4.4.16. Let the 2-dimensional algebra $\mathfrak{N} = \mathbb{C}a \oplus \mathbb{C}b$ such that $ba = -a$, zero otherwise. Then $\mathfrak{N}$ is a Novikov algebra [BMH02]. One has $[a,b] = a$ and $[a,b]_+ = -a$. For $x \in \mathfrak{N}$, denote by ad^+_x the endomorphism of $\mathfrak{N}$ defined by $\mathrm{ad}^+_x(y) = [x,y]_+ = [y,x]_+$, for all $y \in \mathfrak{N}$. It is easy to see that

$$\mathrm{ad}^+_a = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}^+_b = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $x = \lambda a + \mu b \in \mathfrak{N}$, $\lambda, \mu \in \mathbb{C}$, one has $[x,x]_+ = -2\lambda\mu a$ and therefore:

$$\mathrm{ad}^+_x = \begin{pmatrix} -\mu & -\lambda \\ 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}^+_{[x,x]_+} = \begin{pmatrix} 0 & 2\lambda\mu \\ 0 & 0 \end{pmatrix}.$$

Since $[\mathrm{ad}^+_x, \mathrm{ad}^+_{[x,x]_+}] \neq 0$ if $\lambda, \mu \neq 0$, then $\mathfrak{N}$ is not Jordan-admissible.

We will give a condition for a Novikov algebra to be Jordan-admissible as follows:

Theorem 4.4.17. *Let $\mathfrak{N}$ be a Novikov algebra satisfying*

$$(x,x,x) = 0, \ \forall\, x \in \mathfrak{N}. \tag{V}$$

Define on $\mathfrak{N}$ the product $[x,y]_+ = xy + yx$, for all $x, y \in \mathfrak{N}$ then $\mathfrak{N}$ is a Jordan algebra with this product. In this case, it is called the **associated Jordan algebra** *of $\mathfrak{N}$ and denoted by $\mathfrak{J}(\mathfrak{N})$.*

Proof. Let $x, y \in \mathfrak{N}$ then we can write $x^3 = x^2x = xx^2$. One has

$$[[x,y]_+, [x,x]_+]_+ = [xy + yx, 2x^2]_+ = 2(xy)x^2 + 2(yx)x^2 + 2x^2(xy) + 2x^2(yx)$$
$$= 2x^3y + 2(yx)x^2 + 2x^2(xy) + 2x^2(yx)$$

and

$$[x, [y, [x,x]_+]_+]_+ = [x, 2yx^2 + 2x^2y]_+ = 2x(yx^2) + 2x(x^2y) + 2(yx^2)x + 2(x^2y)x$$
$$= 2x(yx^2) + 2x(x^2y) + 2(yx)x^2 + 2x^3y.$$

Therefore, $[[x,y]_+,[x,x]_+]_+ = [x,[y,[x,x]_+]_+]_+$ if and only if $x^2(xy)+x^2(yx) = x(yx^2)+x(x^2y)$. Remark that we have following identities:

$$x^2(xy) = x^3y-(x^2,x,y) = x^3y-(x,x^2,y),$$

$$x^2(yx) = (x^2y)x-(x^2,y,x) = x^3y-(y,x^2,x),$$
$$x(yx^2) = (xy)x^2-(x,y,x^2) = x^3y-(y,x,x^2),$$
$$x(x^2y) = x^3y-(x,x^2,y).$$

It means that we have only to check the formula $(y,x^2,x) = (y,x,x^2)$. It is clear by the identities (III) and (V). Then we can conclude that $\mathfrak{J}(\mathfrak{N})$ is a Jordan algebra. □

Corollary 4.4.18. *If $(\mathfrak{N},B)$ is a symmetric Novikov algebra satisfying (V) then $(\mathfrak{J}(\mathfrak{N}),B)$ is a pseudo-Euclidean Jordan algebra.*

Proof. It is obvious since $B([x,y]_+,z) = B(xy+yx,z) = B(x,yz+zy) = B(x,[y,z]_+)$, for all $x,y,z \in \mathfrak{J}(\mathfrak{N})$. □

Remark 4.4.19. Obviously, Jordan-Novikov algebras are power-associative but in general this is not true for Novikov algebras. Indeed, if Novikov algebras were power-associative then they would satisfy (V). That would imply they were Jordan-admissible. But, that is a contradiction as shown in Example 4.4.16.

Lemma 4.4.20. *Let $\mathfrak{N}$ be a Novikov algebra then $[x,yz]_+ = [y,xz]_+$, for all $x,y,z \in \mathfrak{N}$.*

Proof. By (III), for all $x,y,z \in \mathfrak{N}$ one has $(xy)z+y(xz) = x(yz)+(yx)z$. Combined with (IV), we obtain:

$$(xz)y+y(xz) = x(yz)+(yz)x.$$

That means $[x,yz]_+ = [y,xz]_+$, for all $x,y,z \in \mathfrak{N}$. □

Proposition 4.4.21. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then following identities:*

(1) $x[y,z] = [y,z]x = 0$*, and therefore* $[x,yz]_+ = [x,zy]_+$*,*

(2) $[x,y]_+z = [x,z]_+y$*,*

(3) $[x,yz]_+ = [xy,z]_+ = x[y,z]_+ = [x,y]_+z$*,*

(4) $x[y,z]_+ = [y,z]_+x$*,*

hold for all $x,y,z \in \mathfrak{N}$.

Proof. Let x,y,z and t be elements in $\mathfrak{N}$.

(1) By Proposition 4.4.6 and Lemma 4.4.8, $L_{[y,z]} = 0$ so one has (1).

(2) We have $B([x,y]_+z,t) = B(y,[x,zt]_+) = B(y,[z,xt]_+) = B([z,y]_+x,t)$. Therefore, $[x,y]_+z = [z,y]_+x$. Since the product $[\ ,\ ]_+$ is commutative then $[y,x]_+z = [y,z]_+x$.

(3) By (1) and Lemma 4.4.20, $[x,yz]_+ = [x,zy]_+ = [z,xy]_+ = [xy,z]_+$.
Since B is associative with respect to the product in $\mathfrak{N}$ and in $\mathfrak{J}(\mathfrak{N})$ then

$$B(t,[xy,z]_+) = B([t,xy]_+,z) = B([t,yx]_+,z) = B([y,tx]_+,z) = B(tx,[y,z]_+) = B(t,x[y,z]_+).$$

It implies that $[xy,z]_+ = x[y,z]_+$. Similarly, one has:

$$B([x,y]_+z,t) = B(x,[y,zt]_+) = B(x,[y,tz]_+) = B(x,[t,yz]_+) = B([x,yz]_+,t).$$

So $[x,y]_+z = [x,yz]_+$.

(4) By (2) and (3), $x[y,z]_+ = [x,y]_+z = [y,x]_+z = [y,z]_+x$.

□

Corollary 4.4.22. *Let $(\mathfrak{N},B)$ be a symmetric Novikov algebra then $(\mathfrak{J}(\mathfrak{N}),B)$ is a symmetric Jordan-Novikov algebra.*

Proof. We will show that $[[x,y]_+,z]_+ = [x,[y,z]_+]_+$, for all $x,y,z \in \mathfrak{N}$. Indeed, By Proposition 4.4.21 one has

$$[[x,y]_+,z]_+ = [2xy,z]_+ = 2[z,xy]_+ = 2[x,yz]_+ = [x,[y,z]_+]_+.$$

Hence, the product $[\ ,\ ]_+$ are both commutative and associative. That means $\mathfrak{J}(\mathfrak{N})$ be a Jordan-Novikov algebra. □

It results that if a Novikov algebra is symmetric then it is Jordan-admissible. In fact, we have the much stronger result as follows:

Proposition 4.4.23. *Let $\mathfrak{N}$ be a symmetric Novikov algebra then the product on $\mathfrak{N}$ is associative, that is $x(yz) = (xy)z$, for all $x,y,z \in \mathfrak{N}$.*

Proof. First, we need the lemma:

Lemma 4.4.24. *Let $\mathfrak{N}$ be a symmetric Novikov algebra then $\mathfrak{N}^2 \subset \mathrm{Z}(\mathfrak{N})$.*

Proof. By Lemma 4.4.8, one has $[x,y] = xy - yx \in \mathrm{Ann}(\mathfrak{N}) \subset \mathrm{Z}(\mathfrak{N})$, for all $x,y \in \mathfrak{N}$. Also, by (4) of Proposition 4.4.21, $x[y,z]_+ = [y,z]_+x$, for all $x,y,z \in \mathfrak{N}$, that means $[x,y]_+ = xy + yx \in \mathrm{Z}(\mathfrak{N})$, for all $x,y \in \mathfrak{N}$. Hence, $xy \in \mathrm{Z}(\mathfrak{N})$, for all $x,y \in \mathfrak{N}$, i.e. $\mathfrak{N}^2 \subset \mathrm{Z}(\mathfrak{N})$. □

Let $x,y,z \in \mathfrak{N}$. By above Lemma, one has $(yz)x = x(yz)$. Combined with (IV), $(yx)z = x(yz)$. On the other hand, $[x,y] \in \mathrm{Ann}(\mathfrak{N})$ implies $(yx)z = (xy)z$. Therefore, $(xy)z = x(yz)$. □

A general proof of the above proposition can be found in [AB10], Lemma II.4 which holds for all symmetric left-symmetric superalgebras.

By Corollary 4.4.9, if $\mathfrak{N}$ is a symmetric Novikov algebra then $\mathfrak{g}(\mathfrak{N})$ is 2-step nilpotent. However, $\mathfrak{J}(\mathfrak{N})$ is not necessarily 2-step nilpotent, for example the one-dimensional Novikov algebra $\mathbb{C}c$ with $c^2 = c$ and $B(c,c) = 1$. If $\mathfrak{N}$ is a symmetric 2-step nilpotent Novikov algebra then $(xy)z = 0$, for all $x,y,z \in \mathfrak{N}$. So $[[x,y]_+,z]_+ = 0$, for all $x,y,z \in \mathfrak{N}$. That implies $\mathfrak{J}(\mathfrak{N})$ is also a 2-step nilpotent Jordan algebra. The converse is also true.

Proposition 4.4.25. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{J}(\mathfrak{N})$ is a 2-step nilpotent Jordan algebra then $\mathfrak{N}$ is a 2-step nilpotent Novikov algebra.*

Proof. Since (4) of Proposition 4.4.21, if $x,y,z \in \mathfrak{N}$ then one has

$$[[x,y]_+,z]_+ = [x,y]_+z + z[x,y]_+ = 2[x,y]_+z = 0.$$

It means $[x,y]_+ = xy + yx \in \mathrm{Ann}(\mathfrak{N})$. On the other hand, $[x,y] = xy - yx \in \mathrm{Ann}(\mathfrak{N})$ then $xy \in \mathrm{Ann}(\mathfrak{N})$, for all $x,y \in \mathfrak{N}$. Therefore, $\mathfrak{N}$ is 2-step nilpotent. □

By Proposition 4.4.11, since the lowest dimension of non-Abelian 2-step nilpotent quadratic Lie algebras is six then examples of non-commutative symmetric Novikov algebras must be at least six dimensional. One of those can be found in [ZC07] and it is also described in term of double extension in [AB10]. We recall this algebra as follows:

Example 4.4.26. First, we define the *character matrix* of a Novikov algebra $\mathfrak{N} = \mathrm{span}\{e_1,\dots,e_n\}$ by

$$\begin{pmatrix} \sum_k c_{11}^k e_k & \dots & \sum_k c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_k c_{n1}^k e_k & \dots & \sum_k c_{nn}^k e_k \end{pmatrix},$$

where c_{ij}^k are the *structure constants* of $\mathfrak{N}$, i. e. $e_ie_j = \sum_k c_{ij}^k e_k$.

Now, let $\mathfrak{N}_6$ be a 6-dimensional vector space spanned by $\{e_1,...,e_6\}$ then $\mathfrak{N}_6$ is a non-commutative symmetric Novikov algebras with character matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 & 0 & 0 \end{pmatrix}$$

and the bilinear form B defined by:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, in this case, $\mathfrak{N}_6$ is a 2-step nilpotent Novikov algebra with $\mathrm{Ann}(\mathfrak{N}) = \mathfrak{N}^2$. Moreover, $\mathfrak{N}_6$ is indecomposable since it is non-commutative and all of symmetric Novikov algebras up to dimension 5 are commutative.

In fact, in the next proposition, we prove that all non-commutative symmetric Novikov algebras of dimension 6 are 2-step nilpotent. We need the following lemma:

Lemma 4.4.27. *Let $\mathfrak{N}$ be a non-Abelian symmetric Novikov algebra then $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ where $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{N})$ and $\mathfrak{l}$ is a reduced symmetric Novikov algebra, that means $\mathfrak{l} \neq \{0\}$ and $\mathrm{Ann}(\mathfrak{l}) \subset \mathfrak{l}^2$.*

Proof. Let $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{N}) \cap \mathfrak{N}^2$, $\mathfrak{z}$ is a complementary subspace of $\mathfrak{z}_0$ in $\mathrm{Ann}(\mathfrak{N})$ and $\mathfrak{l} = \mathfrak{z}^\perp$. If x is an element in $\mathfrak{z}$ such that $B(x,\mathfrak{z}) = 0$ then $B(x,\mathfrak{N}^2) = 0$ since $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}^2)^\perp$. As a consequence, $B(x,\mathfrak{z}_0) = 0$ and then $B(x,\mathrm{Ann}(\mathfrak{N})) = 0$. Hence, x must be in $\mathfrak{N}^2$ since $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}^2)^\perp$. It shows that $x = 0$ and $\mathfrak{z}$ is non-degenerate. By Lemma 4.4.5, $\mathfrak{l}$ is a non-degenerate ideal and $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$.

Since $\mathfrak{N}$ is non-Abelian then $\mathfrak{l} \neq \{0\}$. Moreover, $\mathfrak{l}^2 = \mathfrak{N}^2$ implies $\mathfrak{z}_0 \subset \mathfrak{l}^2$. It is easy to see that $\mathfrak{z}_0 = \mathrm{Ann}(\mathfrak{l})$ and the lemma is proved. □

Proposition 4.4.28. *Let $\mathfrak{N}$ be a non-commutative symmetric Novikov algebras of dimension 6 then $\mathfrak{N}$ is 2-step nilpotent.*

Proof. Let $\mathfrak{N} = \mathrm{span}\{x_1,x_2,x_3,z_1,z_2,z_3\}$. By Remark 2.4.21, there exists only one non-Abelian 2-step nilpotent quadratic Lie algebra of dimension 6 (up to isomorphisms) then $\mathfrak{g}(\mathfrak{N}) = \mathfrak{g}_6$. We can choose the basis such that $[x_1,x_2] = z_3$, $[x_2,x_3] = z_1$, $[x_3,x_1] = z_2$ and the bilinear form $B(x_i,z_i) = 1,\ i = 1,2,3$, the other are zero.

Recall that $\mathsf{Z}(\mathfrak{N}) = \{x \in \mathfrak{N} \mid xy = yx,\ \forall\, y \in \mathfrak{N}\}$ then $\mathsf{Z}(\mathfrak{N}) = \{x \in \mathfrak{N} \mid [x,y] = 0,\ \forall\, y \in \mathfrak{N}\}$. Therefore, $\mathsf{Z}(\mathfrak{N}) = \mathrm{span}\{z_1,z_2,z_3\}$ and $\mathfrak{N}^2 \subset \mathsf{Z}(\mathfrak{N})$ by Lemma 4.4.24. Consequently, $\dim(\mathfrak{N}^2) \leq 3$.

By Lemma 4.4.27, if $\mathfrak{N}$ is not reduced then $\mathfrak{N} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ with $\mathfrak{z} \subset \mathrm{Ann}(\mathfrak{N})$ is a non-degenerate ideal and $\mathfrak{z} \neq \{0\}$. It implies that $\mathfrak{l}$ is a symmetric Novikov algebra having dimension ≤ 5 and then $\mathfrak{l}$ is commutative. This is a contradiction since $\mathfrak{N}$ is non-commutative. Therefore, $\mathfrak{N}$ must be reduced and $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}^2$. Moreover, $\dim(\mathfrak{N}^2) + \dim(\mathrm{Ann}(\mathfrak{N})) = 6$ so we have $\mathfrak{N}^2 = \mathrm{Ann}(\mathfrak{N}) = \mathsf{Z}(\mathfrak{N})$. It shows that $\mathfrak{N}$ is 2-step nilpotent. □

In this case, the character matrix of $\mathfrak{N}$ in the basis $\{x_1,x_2,x_3,z_1,z_2,z_3\}$ is given by:

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a 3×3-matrix defined by the structure constants $x_i x_j = \sum_k c_{ij}^k z_k$, $1 \leq i,j,k \leq 3$, and B has the matrix:

$$\begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}.$$

Since $B(x_i x_j, x_k) = B(x_i, x_j x_k) = B(x_j, x_k x_i)$ then one has $c_{ij}^k = c_{jk}^i = c_{ki}^j$, $1 \leq i,j,k \leq 3$.

Next, we give some simple properties for symmetric Novikov algebras as follows:

Proposition 4.4.29. *Let $\mathfrak{N}$ be a non-commutative symmetric Novikov algebra. If $\mathfrak{N}$ is reduced then*

$$3 \leq \dim(\mathrm{Ann}(\mathfrak{N})) \leq \dim(\mathfrak{N}^2) \leq \dim(\mathfrak{N}) - 3.$$

Proof. By Lemma 4.4.24, $\mathfrak{N}^2 \subset \mathsf{Z}(\mathfrak{N})$. Moreover, $\mathfrak{N}$ non-commutative implies that $\mathfrak{g}(\mathfrak{N})$ is non-Abelian and by Remark 2.2.10, $\dim([\mathfrak{N},\mathfrak{N}]) \geq 3$. Therefore, $\dim(\mathsf{Z}(\mathfrak{N})) \leq \dim(\mathfrak{N}) - 3$ since $\mathsf{Z}(\mathfrak{N}) = [\mathfrak{N},\mathfrak{N}]^{\perp}$. Consequently, $\dim(\mathfrak{N}^2) \leq \dim(\mathfrak{N}) - 3$ and then $\dim(\mathrm{Ann}(\mathfrak{N})) \geq 3$. □

Corollary 4.4.30. *Let $\mathfrak{N}$ be a non-commutative symmetric Novikov algebra of dimension 7. If $\mathfrak{N}$ is 2-step nilpotent then $\mathfrak{N}$ is not reduced.*

Proof. Assume that $\mathfrak{N}$ is reduced then $\dim(\mathrm{Ann}(\mathfrak{N})) = 3$ and $\dim(\mathfrak{N}^2) = 4$. It implies that there must have a non-zero element $x \in \mathfrak{N}^2$ such that $x\mathfrak{N} \neq \{0\}$ and then $\mathfrak{N}$ is not 2-step nilpotent. □

Now, we give a more general result for symmetric Novikov algebra of dimension 7 as follows:

Proposition 4.4.31. *Let $\mathfrak{N}$ be a non-commutative symmetric Novikov algebra of dimension 7. If $\mathfrak{N}$ is reduced then there are only two cases:*

(1) $\mathfrak{N}$ is 3-step nilpotent and indecomposable.

(2) $\mathfrak{N}$ is decomposable by $\mathfrak{N} = \mathbb{C}x \overset{\perp}{\oplus} \mathfrak{N}_6$ where $x^2 = x$ and $\mathfrak{N}_6$ is a non-commutative symmetric Novikov algebra of dimension 6.

Proof. Assume that $\mathfrak{N}$ is reduced then $\dim(\mathrm{Ann}(\mathfrak{N})) = 3$, $\dim(\mathfrak{N}^2) = 4$ since $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}^2$ and $\mathrm{Ann}(\mathfrak{N}) = (\mathfrak{N}^2)^{\perp}$. By [Bou59], $\mathrm{Ann}(\mathfrak{N})$ is totally isotropic then there exist a totally isotropic subspace V and a non-zero x of $\mathfrak{N}$ such that

$$\mathfrak{N} = \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x \oplus V,$$

where $\mathrm{Ann}(\mathfrak{N}) \oplus V$ is non-degenerate, $B(x,x) \neq 0$ and $x^{\perp} = \mathrm{Ann}(\mathfrak{N}) \oplus V$. As a consequence, $\mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x = (\mathrm{Ann}(\mathfrak{N}))^{\perp} = \mathfrak{N}^2$.

Consider the left-multiplication operator $L_x : \mathbb{C}x \oplus V \to \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x$, $L_x(y) = xy$, for all $y \in \mathbb{C}x \oplus V$. Denote by p the projection $\mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x \to \mathbb{C}x$.

- If $p \circ L_x = 0$ then $(\mathfrak{N}\mathfrak{N})\mathfrak{N} = x\mathfrak{N} \subset \mathrm{Ann}(\mathfrak{N})$. Therefore, $((\mathfrak{N}\mathfrak{N})\mathfrak{N})\mathfrak{N} = \{0\}$. That implies $\mathfrak{N}$ is 3-nilpotent. If $\mathfrak{N}$ is decomposable then $\mathfrak{N}$ must be 2-step nilpotent. This is in contradiction to Corollary 4.4.30.

- If $p \circ L_x \neq 0$ then there is a non-zero $y \in \mathbb{C}x \oplus V$ such that $xy = ax + z$ with $0 \neq a \in \mathbb{C}$ and $z \in \mathrm{Ann}(\mathfrak{N})$. In this case, we can choose y such that $a = 1$. It implies that $(x^2)y = x(xy) = x^2$.

  If $x^2 = 0$ then $0 = B(x^2, y) = B(x, xy) = B(x,x)$. This is a contradiction. Therefore, $x^2 \neq 0$. Since $x^2 \in \mathrm{Ann}(\mathfrak{N}) \oplus \mathbb{C}x$ then $x^2 = z' + \mu x$ where $z' \in \mathrm{Ann}(\mathfrak{N})$ and $\mu \in \mathbb{C}$ must be non-zero. By setting $x' := \frac{x}{\mu}$ and $z'' = \frac{z'}{\mu^2}$, we get $(x')^2 = z'' + x'$. Let $x_1 := (x')^2$, one has:

  $$x_1^2 = (x')^2(x')^2 = (z'' + x')(z'' + x') = x_1.$$

  Moreover, for all $t = \lambda x + v \in \mathbb{C}x \oplus V$, we have $t(x^2) = (x^2)t = x(xt) = \lambda\mu(x^2)$. It implies that $\mathbb{C}x^2 = \mathbb{C}x_1$ is an ideal of $\mathfrak{N}$.

  Since $B(x_1, x_1) \neq 0$, by Lemma 4.4.5 one has $\mathfrak{N} = \mathbb{C}x_1 \overset{\perp}{\oplus} (x_1)^{\perp}$. Certainly, $(x_1)^{\perp}$ is a non-commutative symmetric Novikov algebra of dimension 6.

□

Proposition 4.4.32. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{g}(\mathfrak{N})$ or $\mathfrak{J}(\mathfrak{N})$ is reduced then $\mathfrak{N}$ is reduced.*

Proof. Assume that $\mathfrak{N}$ is not reduced then there is a non-zero $x \in \mathrm{Ann}(\mathfrak{N})$ such that $B(x,x) = 1$. Since $[x,\mathfrak{N}] = [x,\mathfrak{N}]_+ = 0$ then $\mathfrak{g}(\mathfrak{N})$ and $\mathfrak{J}(\mathfrak{N})$ are not reduced. □

Corollary 4.4.33. *Let $\mathfrak{N}$ be a symmetric Novikov algebra. If $\mathfrak{g}(\mathfrak{N})$ is reduced then $\mathfrak{N}$ must be 2-step nilpotent.*

Proof. Since $\mathfrak{g}(\mathfrak{N})$ is reduced then $\mathrm{Ann}(\mathfrak{N}) \subset \mathfrak{N}^2$. On the other hand, $\dim(\mathsf{Z}(\mathfrak{N})) = \dim([\mathfrak{N},\mathfrak{N}]) = \frac{1}{2}\dim(\mathfrak{N})$ so $\dim(\mathrm{Ann}(\mathfrak{N})) = \dim(\mathfrak{N}^2)$. Therefore, $\mathrm{Ann}(\mathfrak{N}) = \mathfrak{N}^2$ and $\mathfrak{N}$ is 2-step nilpotent. □

Example 4.4.34. By Example 4.4.2, every 2-step nilpotent algebra is Novikov then we will give here an example of non-commutative symmetric Novikov algebras of dimension 7 which is 3-step nilpotent. Let $\mathfrak{N} = \mathbb{C}x \oplus \mathfrak{N}_6$ be a 7-dimensional vector space where $\mathfrak{N}_6$ is the symmetric Novikov algebra of dimension 6 in Example 4.4.26. Define the product on $\mathfrak{N}$ by

$$xe_4 = e_4x = e_1, e_4e_4 = x, e_4e_5 = e_3, e_5e_6 = e_1, e_6e_4 = e_2,$$

and the symmetric bilinear form B defined by

$$\begin{aligned} B(x,x) = B(e_1,e_4) = B(e_2,e_5) = B(e_3,e_6) = 1 \\ B(e_4,e_1) = B(e_5,e_2) = B(e_6,e_3) = 1, \\ 0 \ \text{ otherwise.} \end{aligned}$$

Note that in above Example, $\mathfrak{g}(\mathfrak{N})$ is not reduced since $x \in \mathsf{Z}(\mathfrak{N})$.

Appendix A

In this appendix, we recall some facts on skew-symmetric maps used in this thesis. Nothing here is new, but short proofs are given for the sake of completeness.

Throughout this section, let V be a vector space endowed with a non-degenerate bilinear form B_ε (quadratic or symplectic) and C be a skew-symmetric map in $\mathfrak{g}_\varepsilon$ then one has the useful identity $\ker(C) = (\mathrm{Im}(C))^\perp$.

Lemma A.1. *There exist subspaces W and N of V such that:*

(1) $N \subset \ker(C)$, $C(W) \subset W$ and $V = W \overset{\perp}{\oplus} N$.

(2) Let $B_W = B_\varepsilon|_{W\times W}$ and $C_W = C|_W$. Then B_W is non-degenerate, $C_W \in \mathfrak{g}_\varepsilon(W, B_W)$ and $\ker(C_W) \subset \mathrm{Im}(C_W) = \mathrm{Im}(C)$.

Proof. We follow the proof of Proposition 2.1.5. Let $N_0 = \ker(C) \cap \mathrm{Im}(C)$ and let N be a complementary subspace of N_0 in $\ker(C)$, $\ker(C) = N_0 \oplus N$. Since $\ker(C) = (\mathrm{Im}(C))^\perp$, we have $B_\varepsilon(N_0, N) = \{0\}$ and $N \cap N^\perp = \{0\}$. So, if $W = N^\perp$, one has $V = W \overset{\perp}{\oplus} N$. From $C(N) = \{0\}$, we deduce that $C(W) \subset W$.

It is clear that B_ε is non-degenerate on W and that $C_W \in \mathfrak{g}_\varepsilon(W)$. Moreover, since $C(W) \subset W$ and $C(N) = \{0\}$, then $\mathrm{Im}(C) = \mathrm{Im}(C_W)$. It is immediate that $\ker(C_W) = N_0$, so $\ker(C_W) \subset \mathrm{Im}(C_W)$. □

Lemma A.2. *Assume that* $\ker(C) \subset \mathrm{Im}(C)$. *Denote by $L = \ker(C)$. Let $\{L_1, \dots, L_r\}$ be a basis of L.*

(1) If $\dim(V)$ *is even, there exist subspaces L' with basis $\{L'_1, \dots, L'_r\}$, U with basis $\{U_1, \dots, U_s\}$ and U' with basis $\{U'_1, \dots, U'_s\}$ such that $B_\varepsilon(L_i, L'_j) = \delta_{ij}$, for all $1 \le i, j \le r$, L and L' are totally isotropic, $B_\varepsilon(U_i, U'_j) = \delta_{ij}$, for all $1 \le i, j \le s$, U and U' are totally isotropic and*

$$V = (L \oplus L') \overset{\perp}{\oplus} (U \oplus U').$$

Moreover $\mathrm{Im}(C) = L \overset{\perp}{\oplus} (U \oplus U')$ *and* $C : L' \overset{\perp}{\oplus} (U \oplus U') \to L \overset{\perp}{\oplus} (U \oplus U')$ *is a bijection.*

(2) If $\varepsilon = 1$ and $\dim(V)$ *is odd, there exist subspaces L', U and U' as in (1) and $v \in V$ such that $B_\varepsilon(v, v) = 1$ and*

$$V = (L \oplus L') \overset{\perp}{\oplus} \mathbb{C}v \overset{\perp}{\oplus} (U \oplus U').$$

Moreover $\mathrm{Im}(C) = L \overset{\perp}{\oplus} \mathbb{C}v \overset{\perp}{\oplus} (U \oplus U')$ *and* $C : L' \overset{\perp}{\oplus} \mathbb{C}v \overset{\perp}{\oplus} (U \oplus U') \to L \overset{\perp}{\oplus} \mathbb{C}v \overset{\perp}{\oplus} (U \oplus U')$ *is a bijection.*

(3) *If* $\varepsilon = 1$*, in both cases,* $\mathrm{rank}(C)$ *is even.*

Proof. Since $(\ker(C))^{\perp} = \mathrm{Im}(C)$, L is isotropic.

(1) If $\dim(V)$ is even, there exist maximal isotropic subspaces W_1 and W_2 such that $V = W_1 \oplus W_2$ [Bou59] and $L \subset W_1$. Let U be a complementary subspace of L in W_1, $W_1 = L \oplus U$ and $\{U_1, \dots, U_s\}$ a basis of U. Consider the isomorphism $\Psi : W_2 \to W_1^*$ defined by $\Psi(w_2)(w_1) = B_\varepsilon(w_2, w_1)$, for all $w_1 \in W_1$, $w_2 \in W_2$. Define $L'_i = \psi^{-1}(L_i^*)$, $1 \leq i \leq r$, $L' = \mathrm{span}\{L'_1, \dots, L'_r\}$, $U'_j = \psi^{-1}(U_j^*)$, $1 \leq j \leq s$, $U' = \mathrm{span}\{U'_1, \dots, U'_s\}$. Then $B_\varepsilon(L_i, L'_j) = \delta_{ij}$, $1 \leq i, j \leq r$, L and L' are isotropic, $B_\varepsilon(U_i, U'_j) = \delta_{ij}$, for all $1 \leq i, j \leq s$, U and U' are isotropic and

$$V = (L \oplus L') \overset{\perp}{\oplus} (U \oplus U').$$

Since $\mathrm{Im}(C) = L^{\perp}$, we have $\mathrm{Im}(C) = L \overset{\perp}{\oplus} (U \oplus U')$. Finally, if $v \in L' \overset{\perp}{\oplus} (U \oplus U')$ and $C(v) = 0$, then $v \in L$. So $v = 0$. Therefore C is one to one from $L' \overset{\perp}{\oplus} (U \oplus U')$ into $L \overset{\perp}{\oplus} (U \oplus U')$ and since the dimensions are the same, C is a bijection.

(2) There exist maximal isotropic subspaces W_1 and W_2 such that $V = (W_1 \oplus W_2) \overset{\perp}{\oplus} \mathbb{C}v$, with $v \in V$ such that $B_\varepsilon(v, v) = 1$ and $L \subset W_1$ [Bou59]. Then the proof is essentially the same as in (1).

(3) Assume that $\dim(V)$ is even. Define a bilinear form Δ on $L' \overset{\perp}{\oplus} (U \oplus U')$ by $\Delta(v_1, v_2) = B_\varepsilon(v_1, C(v_2))$, for all v_1, $v_2 \in L' \overset{\perp}{\oplus} (U \oplus U')$. Since $C \in \mathfrak{o}(V)$, Δ is skew-symmetric. Let $v_1 \in L' \overset{\perp}{\oplus} (U \oplus U')$ such that $\Delta(v_1, v_2) = 0$, for all $v_2 \in L' \overset{\perp}{\oplus} (U \oplus U')$. Then $B_\varepsilon(v_1, w) = 0$, for all $w \in L \overset{\perp}{\oplus} (U \oplus U')$. It follows that $B_\varepsilon(v_1, w) = 0$, for all $w \in V$, so $v_1 = 0$ and Δ is non-degenerate. So $\dim(L' \overset{\perp}{\oplus} (U \oplus U')$ is even. Therefore $\dim(L') = \dim(L)$ is even and $\mathrm{rank}(C)$ is even. If V is odd-dimensional, the proof is completely similar.

□

Corollary A.3. *If* $C \in \mathfrak{o}(V)$*, then* $\mathrm{rank}(C)$ *is even.*

Proof. By Lemma A.1, $\mathrm{Im}(C) = \mathrm{Im}(C_W)$ and $\mathrm{rank}(C_W)$ is even by the preceding Lemma. □

For instance, if $C \in \mathfrak{o}(V)$ and C is invertible, then $\dim(V)$ must be even. But this can also be proved directly: when C is invertible, then the skew-symmetric form Δ_C on V defined by $\Delta_C(v_1, v_2) = B_\varepsilon(v_1, C(v_2))$, for all v_1, $v_2 \in V$, is clearly non-degenerate.

When C is semisimple (i.e. diagonalizable), we have $V = \ker(C) \overset{\perp}{\oplus} \mathrm{Im}(C)$ and $C|_{\mathrm{Im}(C)}$ is invertible. So semisimple elements are completely described by:

Lemma A.4. *Assume that C is semisimple and invertible. Then there is a basis $\{e_1,\ldots,e_p, f_1,\ldots,f_p\}$ of V such that $B_\varepsilon(e_i,e_j) = B_\varepsilon(f_i,f_j) = 0$, $B_\varepsilon(e_i,f_j) = \delta_{ij}$, $1 \leq i,j \leq p$. For $1 \leq i \leq p$, there exist non-zero $\lambda_i \in \mathbb{C}$ such that $C(e_i) = \lambda_i e_i$ and $C(f_i) = -\lambda_i f_i$.*

Moreover, if Λ denotes the spectrum of C, then $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$, λ and $-\lambda$ have the same multiplicity.

Proof. We prove the result by induction on $\dim(V)$. Assume that $\dim(V) = 2$. Let $\{e_1,e_2\}$ be an eigenvector basis of V corresponding to eigenvalues λ_1 and λ_2. We have $B_\varepsilon(C(v),v') = -B_\varepsilon(v,C(v'))$ and C is invertible, so $B_\varepsilon(e_1,e_1) = B_\varepsilon(e_2,e_2) = 0$, $B_\varepsilon(e_1,e_2) \neq 0$ and $\lambda_2 = -\lambda_1$. Let $f_1 = \dfrac{1}{B_\varepsilon(e_1,e_2)} e_2$, then the basis $\{e_1,f_1\}$ is a convenient basis.

Assume that the result is true for vector spaces of dimension n with $n \leq 2(p-1)$. Assume $\dim(V) = 2p$. Let $\{e_1,\ldots,e_{2p}\}$ be an eigenvector basis with corresponding eigenvalues $\lambda_1,\ldots,\lambda_{2p}$. As before, $B_\varepsilon(e_i,e_i) = 0$, $1 \leq i \leq 2p$, so there exists j such that $B_\varepsilon(e_1,e_j) \neq 0$. Then $\lambda_j = -\lambda_1$. Let $f_1 = \dfrac{1}{B_\varepsilon(e_1,e_j)} e_j$. Then $B_\varepsilon|_{\mathrm{span}\{e_1,f_1\}}$ is non-degenerate, so $V = \mathrm{span}\{e_1,f_1\} \overset{\perp}{\oplus} V_1$ where $V_1 = \mathrm{span}\{e_1,f_1\}^\perp$. But C maps V_1 into itself, so we can apply the induction assumption and the result follows. □

As a consequence, we have this classical result, used in Chapter 1:

Lemma A.5.

(1) Let C be a semisimple element of $\mathfrak{o}(n)$. Then C belongs to the $\mathrm{SO}(n)$-adjoint orbit of an element of the standard Cartan subalgebra of $\mathfrak{o}(n)$ (i.e., an element with matrix $\mathrm{diag}_{2p}(\lambda_1,\ldots,\lambda_p,-\lambda_1,\ldots,-\lambda_p)$ if $n = 2p$ and $\mathrm{diag}_{2p+1}(\lambda_1,\ldots,\lambda_p,0,-\lambda_1,\ldots,-\lambda_p)$ if $n = 2p+1$ in a canonical basis of $\mathbb{C}^n$).

(2) Let C be a semisimple element of $\mathfrak{sp}(2p)$. Then C belongs to the $\mathrm{Sp}(2p)$-adjoint orbit of an element of the standard Cartan subalgebra of $\mathfrak{sp}(2p)$ (i.e., an element with matrix $\mathrm{diag}_{2p}(\lambda_1,\ldots,\lambda_p,-\lambda_1,\ldots,-\lambda_p)$).

(3) Let C and C' be semisimple elements of $\mathfrak{g}_\varepsilon$. Then C and C' are in the same I_ε-adjoint orbit if and only if they have the same spectrum, with same multiplicities.

Proof.

(1) We have $\mathbb{C}^n = \ker(C) \overset{\perp}{\oplus} \mathrm{Im}(C)$ and $\mathrm{rank}(C)$ is even. So $\dim(\ker(C))$ is even if $n = 2p$ and odd, if $n = 2p+1$. Then apply Lemma A.4 to $C|_{\mathrm{Im}(C)}$ to obtain the result.

(2) The proof is similar to (1) with n even.

(3) If C and C' have the same spectrum and their eigenvalues having same multiplicities, they are I_ε-conjugate to the same element of the standard Cartan subalgebra.

□

Remark A.6.

(1) Caution: $O(n)$-adjoint orbits are generally not the same as $SO(n)$-adjoint orbits.

(2) Lemma A.5 (1) is a particular case of a general and classical result on semisimple Lie algebras: any semisimple element of a semisimple Lie algebra belongs to a Cartan subalgebra and all Cartan subalgebras are conjugate under the adjoint action [Sam80]. Here, $\mathfrak{g}_\varepsilon$ are semisimple Lie algebras and the corresponding adjoint groups are $SO(n)$ and $Sp(2p)$.

Appendix B

Here we prove:

Lemma B.1. *Let $(\mathfrak{g}, B)$ be a non-Abelian 5-dimensional quadratic Lie algebra. Then $\mathfrak{g}$ is a singular quadratic Lie algebra.*

Proof.

- We assume $\mathfrak{g}$ is not solvable and then we write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{r}$ is the radical of $\mathfrak{g}$ [Bou71]. Then $\mathfrak{s} \simeq \mathfrak{sl}(2)$ and $B|_{\mathfrak{s}\times\mathfrak{s}} = \lambda\kappa$ where κ is the Killing form.

  If $\lambda = 0$, consider $\Psi : \mathfrak{s} \to \mathfrak{r}^*$ defined by $\Psi(S)(R) = B(S,R)$, for all $S \in \mathfrak{s}$, $R \in \mathfrak{r}$. Then Ψ is one-to-one and $\Psi(\mathrm{ad}(X)(S)) = \mathrm{ad}^*(X)(\psi(S))$, for all X, $S \in \mathfrak{s}$. So Ψ must be a homomorphism from the representation $(\mathfrak{s}, \mathrm{ad}|_{\mathfrak{s}})$ of $\mathfrak{s}$ into the representation $(\mathfrak{r}^*, \mathrm{ad}^*|_{\mathfrak{s}})$, so $\Psi = 0$, a contradiction.

  So $\lambda \neq 0$. Then $B|_{\mathfrak{s}\times\mathfrak{s}}$ is non-degenerate. Therefore $\mathfrak{g} = \mathfrak{s} \overset{\perp}{\oplus} \mathfrak{s}^\perp$ and $\mathrm{ad}(\mathfrak{s})|_{\mathfrak{s}^\perp}$ is an orthogonal 2-dimensional representation of $\mathfrak{s}$. Hence, $\mathrm{ad}(\mathfrak{s})|_{\mathfrak{s}^\perp} = 0$ and $[\mathfrak{s}, \mathfrak{s}^\perp] = 0$. We have $B(X,[Y,Z]) = B([X,Y],Z) = 0$, for all $X \in \mathfrak{s}$, $Y \in \mathfrak{s}^\perp$, $Z \in \mathfrak{g}$. It follows that $\mathfrak{s}^\perp$ is an ideal of $\mathfrak{g}$ and therefore a quadratic 2-dimensional Lie algebra. So $\mathfrak{s}^\perp$ is Abelian. Finally, $\mathfrak{g} = \mathfrak{s} \overset{\perp}{\oplus} \mathfrak{s}^\perp$ with $\mathfrak{s}^\perp$ a central ideal of $\mathfrak{g}$, so $\mathrm{dup}(\mathfrak{g}) = \mathrm{dup}(\mathfrak{s}) = 3$.

- We assume that $\mathfrak{g}$ is solvable and we write $\mathfrak{g} = \mathfrak{l} \overset{\perp}{\oplus} \mathfrak{z}$ with $\mathfrak{z}$ a central ideal of $\mathfrak{g}$ (Proposition 2.1.5). Then $\dim(\mathfrak{l}) \geq 3$. If $\dim(\mathfrak{l}) = 3$ or 4, then it is proved in Proposition 2.2.15 that $\mathfrak{l}$ is singular, so $\mathfrak{g}$ is singular. So we can assume that $\mathfrak{g}$ is reduced, i.e. $\mathscr{Z}(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$. It results that $\dim(\mathscr{Z}(\mathfrak{g})) = 1$ or 2 (Proposition 2.1.5 (3) and Remark 2.2.10).

  - If $\dim(\mathscr{Z}(\mathfrak{g})) = 1$, $\mathscr{Z}(\mathfrak{g}) = \mathbb{C}X_0$. Then $\dim([\mathfrak{g},\mathfrak{g}]) = 4$ and $[\mathfrak{g},\mathfrak{g}] = X_0^\perp$. We can choose Y_0 such that $B(X_0,Y_0) = 1$ and $B(Y_0,Y_0) = 0$. Let $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^\perp$. Then $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$. If X, $X' \in \mathfrak{q}$, then $B(X_0,[X,X']) = B([X_0,X],X') = 0$, so $[X,X'] \in X_0^\perp$. Write $[X,X'] = \lambda(X,X')X_0 + [X,X']_{\mathfrak{q}}$ with $[X,X']_{\mathfrak{q}} \in \mathfrak{q}$. Remark that $[X,[X',X'']] = \lambda(X,[X',X'']_{\mathfrak{q}})X_0 + [X,[X',X'']_{\mathfrak{q}}]_{\mathfrak{q}}$, for all X, X', $X'' \in \mathfrak{q}$. So $[\cdot,\cdot]_{\mathfrak{q}}$ satisfies the Jacobi identity. Moreover $B([X,X'],X'') = -B(X',[X,X'']_{\mathfrak{q}})$. But also $B([X,X'],X'') = B([X,X']_{\mathfrak{q}},X'')$. So $(\mathfrak{q},[\cdot,\cdot]_{\mathfrak{q}},B|_{\mathfrak{q}\times\mathfrak{q}})$ is a 3-dimensional quadratic Lie algebra.

    If $\mathfrak{q}$ is an Abelian Lie algebra, then $[X,X'] \in \mathbb{C}X_0$, for all $X, X' \in \mathfrak{q}$. Write $B(Y_0,[X,X'])$ $= B([Y_0,X],X')$ to obtain $[X,X'] = B(\mathrm{ad}(Y_0)(X),X')X_0$, for all X, $X' \in \mathfrak{q}$. Since

$\dim(\mathfrak{q}) = 3$ and $\mathrm{ad}(Y_0)|_{\mathfrak{q}}$ is skew-symmetric, there exists $Q_0 \in \mathfrak{q}$ such that $\mathrm{ad}(Y_0)(Q_0) = 0$. It follows that $Q_0 \in \mathscr{Z}(\mathfrak{g})$ and that is a contradiction since $\dim(\mathscr{Z}(\mathfrak{g})) = 1$.

Therefore $(\mathfrak{q}, [\cdot,\cdot]_{\mathfrak{q}}) \simeq \mathfrak{sl}(2)$. Consider

$$0 \to \mathbb{C}X_0 \to X_0^{\perp} \to \mathfrak{q} \to 0.$$

Then there is a section $\sigma : \mathfrak{q} \to X_0^{\perp}$ such that $\sigma([X,X']_{\mathfrak{q}}) = [\sigma(X), \sigma(X')]$, for all X, $X' \in \mathfrak{q}$ [Bou71]. Then $\sigma(\mathfrak{q})$ is a Lie subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2)$ and that is a contradiction since $\mathfrak{g}$ is solvable.

– If $\dim(\mathscr{Z}(\mathfrak{g})) = 2$, then we choose a non-zero $X_0 \in \mathscr{Z}(\mathfrak{g})$ and $Y_0 \in \mathfrak{g}$ such that $B(X_0,Y_0) = 1$ and $B(Y_0,Y_0) = 0$. Let $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^{\perp}$. Then $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ and as in the preceding case, $[X,X'] \in X_0^{\perp}$, for all X, $X' \in \mathfrak{q}$. Write $[X,X'] = \lambda(X,X')X_0 + [X,X']_{\mathfrak{q}}$ with $[X,X']_{\mathfrak{q}} \in \mathfrak{q}$. Same arguments as in the preceding case allow us to conclude that $[\cdot,\cdot]_{\mathfrak{q}}$ satisfies the Jacobi identity and that $B|_{\mathfrak{q}\times\mathfrak{q}}$ is invariant. So $(\mathfrak{q}, [\cdot,\cdot]_{\mathfrak{q}}, B|_{\mathfrak{q}\times\mathfrak{q}})$ is a 3-dimensional quadratic Lie algebra.

If $\mathfrak{q} \simeq \mathfrak{sl}(2)$, then apply the same reasoning as in the preceding case to obtain a contradiction with $\mathfrak{g}$ solvable.

If $\mathfrak{q}$ is an Abelian Lie algebra, then $[X,X'] \in \mathbb{C}X_0$, for all X, $X' \in \mathfrak{q}$. Again, as in the preceding case, $[X,X'] = B(\mathrm{ad}(Y_0)(X),X')X_0$, for all X, $X' \in \mathfrak{q}$. Then it is easy to check that $\mathfrak{g}$ is the double extension of the quadratic vector space $\mathfrak{q}$ by $\overline{C} = \mathrm{ad}(Y_0)|_{\mathfrak{q}}$. By Proposition 2.2.28, $\mathfrak{g}$ is singular.

□

Remark B.2. Let us give a list of all non-Abelian 5-dimensional quadratic Lie algebras:

- $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{o}(3) \overset{\perp}{\oplus} \mathbb{C}^2$ with $\mathbb{C}^2$ central, $\mathfrak{o}(3)$ equipped with bilinear form $\lambda\kappa$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and κ the Killing form. We have $\mathrm{dup}(\mathfrak{g}) = 3$.

- $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}_4 \overset{\perp}{\oplus} \mathbb{C}$ with $\mathbb{C}$ central, $\mathfrak{g}_4$ the double extension of $\mathbb{C}^2$ by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathfrak{g}$ is solvable, non-nilpotent and $\mathrm{dup}(\mathfrak{g}) = 3$.

- $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}_5$, the double extension of $\mathbb{C}^3$ by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathfrak{g}$ is nilpotent and $\mathrm{dup}(\mathfrak{g}) = 3$.

See Proposition 2.2.29 for the definition of $\mathfrak{g}_4$ and $\mathfrak{g}_5$. Remark that $\mathfrak{g}_4 \overset{\perp}{\oplus} \mathbb{C}$ is actually the double extension of $\mathbb{C}^3$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Appendix C

We will classify (up to isomorphisms) 3-forms on a vector space V with $1 \leq \dim(V) \leq 5$ that can be applied in the classification of quadratic solvable or 2-step nilpotent Lie algebras of low dimension in Chapter 2. The method is based only on changes of basis in the dual space V^*.

Let $I \in \mathscr{A}^3(V)$ be a 3-form on V. It is obvious that $I = 0$ if $\dim(V) = 1$ or 2.

Case 1: $\dim(V) = 3$

If $I \neq 0$ then there exists a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of V^* such that $I = a\alpha_1 \wedge \alpha_2 \wedge \alpha_3$. Replace α_1 by $\frac{1}{a}\alpha_1$, we get the result

$$I = \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

Case 2: $\dim(V) = 4$

We will show that every 3-form on V is decomposable. Hence $\mathscr{A}^3(V)$ has only a non-zero 3-form (up to isomorphisms). Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of V^*. Then I has the following form:

$$I = a\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + b\alpha_1 \wedge \alpha_2 \wedge \alpha_4 + c\alpha_1 \wedge \alpha_3 \wedge \alpha_4 + d\alpha_2 \wedge \alpha_3 \wedge \alpha_4,$$

where $a, b, c, d \in \mathbb{C}$. We rewrite:

$$I = \alpha_1 \wedge \alpha_2 \wedge (a\alpha_3 + b\alpha_4) + (c\alpha_1 + d\alpha_2) \wedge \alpha_3 \wedge \alpha_4.$$

If $a\alpha_3 + b\alpha_4 = 0$ or $c\alpha_1 + d\alpha_2 = 0$ then I is decomposable. If $a\alpha_3 + b\alpha_4 \neq 0$ and $c\alpha_1 + d\alpha_2 \neq 0$, then we can assume that $a \neq 0$ and $c \neq 0$. We replace $\frac{b}{a}$ by b' and $\frac{d}{c}$ by d' to get:

$$I = a\alpha_1 \wedge \alpha_2 \wedge (\alpha_3 + b'\alpha_4) + c(\alpha_1 + d'\alpha_2) \wedge \alpha_3 \wedge \alpha_4.$$

We change the basis of V^* as follows:

$$\beta_1 = \alpha_1 + d'\alpha_2,\ \beta_2 = \alpha_2,\ \beta_3 = \alpha_3 + b'\alpha_4,\ \beta_4 = \alpha_4.$$

Then $I = a\beta_1 \wedge \beta_2 \wedge \beta_3 + c\beta_1 \wedge \beta_3 \wedge \beta_4$. It means that

$$I = \beta_1 \wedge (a\beta_2 - c\beta_4) \wedge \beta_3.$$

Therefore, I is decomposable.

Case 3: $\dim(V) = 5$

Proposition C.1. *If I is an indecomposable 3-form on V then there exists a basis* $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ *of* V^* *such that I has the form*

$$I = \alpha_1 \wedge (a\alpha_2 \wedge \alpha_3 + b\alpha_4 \wedge \alpha_5),$$

where $a, b \in \mathbb{C}$.

Proof. First, we prove the following lemma:

Lemma C.2. *Let* V_1 *be a 4-dimensional vector space and J be a 2-form on* V_1. *Then there exists a basis* $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ *of* V_1^* *such that*

$$J = p\beta_1 \wedge \beta_2 + q\beta_3 \wedge \beta_4,$$

where $p, q \in \mathbb{C}$.

Proof. Let $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a basis of V_1^* then J has the form:

$$J = \beta_1 \wedge (a\beta_2 + b\beta_3 + c\beta_4) + \beta_2 \wedge (d\beta_3 + e\beta_4) + f\beta_3 \wedge \beta_4,$$

where $a, b, c, d, e, f \in \mathbb{C}$.

(1) If $a\beta_2 + b\beta_3 + c\beta_4 = 0$ and $d\beta_3 + e\beta_4 = 0$ then the result follows.

(2) If $a\beta_2 + b\beta_3 + c\beta_4 = 0$ and $d\beta_3 + e\beta_4 \neq 0$ then we can assume $d \neq 0$. Replace with $\beta_3' = \beta_3 + \frac{e}{d}\beta_4$ and $\beta_2' = \beta_2 - \frac{f}{d}\beta_4$ then we have the result.

(3) If $a\beta_2 + b\beta_3 + c\beta_4 \neq 0$ and $d\beta_3 + e\beta_4 = 0$ then we can assume that $a \neq 0$ because if $a = 0$ then we return (2). Replace with $\beta_2' = \beta_2 + \frac{b}{a}\beta_3 + \frac{c}{a}\beta_4$ one has the result.

(4) If $f = 0$ we can assume $d \neq 0$. Replace with $\beta_3' = \beta_3 + \frac{e}{d}\beta_4$ then we return (3).

(5) If $a\beta_2 + b\beta_3 + c\beta_4 \neq 0$ and $d\beta_3 + e\beta_4 \neq 0$ then we can assume that $d \neq 0$. Replace with $\beta_3' = \beta_3 + \frac{e}{d}\beta_4$ then we return the case (4). Therefore, J has only the form

$$J = p\beta_1 \wedge \beta_2 + q\beta_3 \wedge \beta_4,$$

where $p, q \in \mathbb{C}$.

□

By the above Lemma we can choose $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ a basis of V^* such that I has the form:

$$I = \alpha_1 \wedge \Omega + a\alpha_2 \wedge \alpha_3 \wedge \alpha_4 + b\alpha_2 \wedge \alpha_3 \wedge \alpha_5 + c\alpha_2 \wedge \alpha_4 \wedge \alpha_5 + d\alpha_3 \wedge \alpha_4 \wedge \alpha_5,$$

where $\Omega = p\alpha_2 \wedge \alpha_3 + q\alpha_4 \wedge \alpha_5$ and $a, b, c, d, p, q \in \mathbb{C}$. Now, we need the next lemma:

Lemma C.3. *The element I always has the form:*

$$I = \alpha_1 \wedge \Omega + I_1,$$

where $I_1 = 0$ or I_1 decomposable. More particularly, I has the following possible forms:

a) $I = \alpha_1 \wedge \Omega$.

b) $I = \alpha_1 \wedge \Omega + \alpha_2 \wedge \alpha_3 \wedge (a\alpha_4 + b\alpha_5),\ a \neq 0$.

c) $I = \alpha_1 \wedge \Omega + (c\alpha_2 + d\alpha_3) \wedge \alpha_4 \wedge \alpha_5,\ c \neq 0$.

d) $I = \alpha_1 \wedge \Omega + \alpha_2 \wedge (a\alpha_3 - c\alpha_5) \wedge \alpha_4,\ a, c \neq 0$

Proof. We rewrite $I = \alpha_1 \wedge \Omega + \alpha_2 \wedge \alpha_3 \wedge (a\alpha_4 + b\alpha_5) + (c\alpha_2 + d\alpha_3) \wedge \alpha_4 \wedge \alpha_5$. If $a\alpha_4 + b\alpha_5 = 0$ or $c\alpha_2 + d\alpha_3 = 0$ then I has the form a), b) or c). If $a\alpha_4 + b\alpha_5 \neq 0$ and $c\alpha_2 + d\alpha_3 \neq 0$ then we can assume that $a \neq 0$ and $c \neq 0$. Replace with $b' = \frac{b}{a}$, $d' = \frac{d}{c}$ and

$$\alpha_2' = \alpha_2 + d'\alpha_3,\ \alpha_4' = \alpha_4 + b'\alpha_5.$$

Note that our change keeps the form of Ω. Therefore, one has:

$$I = \alpha_1 \wedge \Omega + a\alpha_2' \wedge \alpha_3 \wedge \alpha_4' + c\alpha_2' \wedge \alpha_4' \wedge \alpha_5.$$

It means that I has the form d). □

Clearly, the forms b) and c) of I are equivalent then we only consider the form b). We rewrite the form b) as follows:

$$I = \alpha_1 \wedge (p\alpha_2 \wedge \alpha_3 + q\alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_3 \wedge (a\alpha_4 + b\alpha_5)$$

$$= \alpha_2 \wedge \alpha_3 \wedge (p\alpha_1 + a\alpha_4 + b\alpha_5) + q\alpha_1 \wedge \alpha_4 \wedge \alpha_5.$$

Replace with $p' = \frac{p}{a}$, $b' = \frac{b}{a}$ and after that $\alpha_4' = p'\alpha_1 + \alpha_4 + b'\alpha_5$ then we obtain

$$I = \alpha_4' \wedge (a\alpha_2 \wedge \alpha_3 - q\alpha_1 \wedge \alpha_5).$$

For the form d), we rewirite

$$I = \alpha_2 \wedge \alpha_3 \wedge (p\alpha_1 + a\alpha_4) + (q\alpha_1 + c\alpha_2) \wedge \alpha_4 \wedge \alpha_5.$$

If $p = q = 0$ then $I = \alpha_2 \wedge \alpha_3 \wedge (-a\alpha_3 + c\alpha_5)$ decomposable.

If $p \neq 0$, let $a' = \frac{a}{p}$ and set $\alpha_1' = \alpha_1 + a'\alpha_4$ then

$$I = p\alpha_2 \wedge \alpha_3 \wedge \alpha_1' + (q\alpha_1' + c\alpha_2) \wedge \alpha_4 \wedge \alpha_5.$$

Let $q' = \frac{q}{c}$ and $\alpha_2' = q\alpha_1' + \alpha_2$ then

$$I = \alpha_2' \wedge (p\alpha_3 \wedge \alpha_1' + c\alpha_4 \wedge \alpha_5).$$

□

Corollary C.4. *Let V be a vector space, V^* its dual space and I be a 3-form on V. Define*

$$V_I = \{\alpha \in V^* \mid \alpha \wedge I = 0\} \text{ and } \mathrm{dup}(V) = \dim(V_I).$$

Then $\mathrm{dup}(V) \neq 0$ *if* $1 \leq \dim(V) \leq 5$.

Proof. It is easy to see that if $I = 0$ then $V_I = V^*$. If I is a non-zero decomposable 3-form then $\mathrm{dup}(V) = 3$. Assume that I is indecomposable. Since $\dim(V) \leq 5$ then it happens only in the case $\dim(V) = 5$. In this case I has the form as in Proposition C.1 with $a, b \neq 0$. Therefore one has $\mathrm{dup}(V) = 1$. □

Consequently, we obtain the result given in Appendix B as follows:

Corollary C.5. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie algebra such that* $\dim[\mathfrak{g},\mathfrak{g}] \leq 5$. *Then $\mathfrak{g}$ is singular.*

Proof. Define the element I as in Proposition 2.2.1 then $I \in \bigwedge^3(\mathcal{W}_I)$ where $\mathcal{W}_I = \phi([\mathfrak{g},\mathfrak{g}])$ (Corollary 2.2.6). Since $\dim[\mathfrak{g},\mathfrak{g}] \leq 5$ then $\mathrm{dup}(\mathfrak{g}) \neq 0$ and therefore $\mathfrak{g}$ is singular. □

Corollary C.6. *Let $\mathfrak{g}$ be a non-Abelian quadratic solvable Lie algebra such that* $\dim(\mathfrak{g}) \leq 6$. *Then $\mathfrak{g}$ is singular.*

Proof. Since $\mathfrak{g}$ is solvable then $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$ and therfore $\dim[\mathfrak{g},\mathfrak{g}] \leq 5$. Apply the above corollary, we get the result. □

In the case of higher dimensions, we have the following proposition:

Proposition C.7. *Let V be a vector space such that* $\dim(V) \geq 6$. *Then there exists an element $I \in \mathscr{A}^3(V)$ satisfying $\iota_x(I) \neq 0$ for all non-zero x in V.*

Proof. We denote by $n = \dim(V)$ and fix a basis $\{\alpha_1, \ldots, \alpha_n\}$ of V^*. Then the element I is defined in the following cases:

- If $n = 3k$ then we set

$$I = \alpha_1 \wedge \alpha_2 \wedge \alpha_2 + \cdots + \alpha_{n-2} \wedge \alpha_{n-1} \wedge \alpha_n.$$

- If $n = 3k+1 = 3(k-2)+7$ then we set

$$\begin{aligned} I = \ & \alpha_1 \wedge \alpha_2 \wedge \alpha_2 + \cdots + \alpha_{n-9} \wedge \alpha_{n-8} \wedge \alpha_{n-7} \\ & + \alpha_{n-6} \wedge (\alpha_{n-5} \wedge \alpha_{n-4} + \alpha_{n-3} \wedge \alpha_{n-2} + \alpha_{n-1} \wedge \alpha_n). \end{aligned}$$

- If $n = 3k+2 = 3(k-1)+5$ then we set

$$\begin{aligned} I = \ & \alpha_1 \wedge \alpha_2 \wedge \alpha_2 + \cdots + \alpha_{n-7} \wedge \alpha_{n-6} \wedge \alpha_{n-5} \\ & + \alpha_{n-4} \wedge (\alpha_{n-3} \wedge \alpha_{n-2} + \alpha_{n-1} \wedge \alpha_n). \end{aligned}$$

□

Appendix D

In the last appendix, we will prove a result needed to Chapter 4 as follows:

Lemma D.1. *Let (V,B) be a quadratic vector space, C be an invertible endomorphism of V such that*

(1) $B(C(x),y) = B(x,C(y))$, *for all* $x,y \in V$,

(2) $3C - 2C^2 = \mathrm{Id}$.

Then there is an orthogonal basis $\{e_1,...,e_n\}$ of B such that C is diagonalizable with eigenvalues 1 and $\frac{1}{2}$.

Proof. First, one has (2) equivalent to $C(C-\mathrm{Id}) = \frac{1}{2}(C-\mathrm{Id})$. Therefore, if x is a vector in V such that $C(x)-x \neq 0$ then $C(x)-x$ is an eigenvector with respect to eigenvalue $\frac{1}{2}$. We prove the result by induction on $\dim(V)$. If $\dim(V) = 1$, let $\{e\}$ be an orthogonal basis of V and assume $C(e) = \lambda e$ for some $\lambda \in \mathbb{C}$. Then by (2) one has $\lambda = 1$ or $\lambda = \frac{1}{2}$.

Assume that the result is true for quadratic vector spaces of dimension n, $n \geq 1$. Assume $\dim(V) = n+1$. If $C = \mathrm{Id}$ then the result follows. If $C \neq \mathrm{Id}$ then there exists $x \in V$ such that $C(x)-x \neq 0$. Let $e_1 = C(x)-x$ then $C(e_1) = \frac{1}{2}e_1$.

If $B(e_1,e_1) = 0$ then there is $e_2 \in V$ such that $B(e_2,e_2) = 0$, $B(e_1,e_2) = 1$ and $V = \mathrm{span}\{e_1,e_2\} \overset{\perp}{\oplus} V_1$ where $V_1 = \mathrm{span}\{e_1,e_2\}^\perp$. Since $\frac{1}{2} = B(C(e_1),e_2) = B(e_1,C(e_2))$ one has $C(e_2) = \frac{1}{2}e_2 + y + \beta e_1$ with $y \in V_1, \beta \in \mathbb{C}$. Let $f_1 = C(e_2) - e_2 = -\frac{1}{2}e_2 + y + \beta e_1$ then $C(f_1) = \frac{1}{2}f_1$ and $B(e_1,f_1) = -\frac{1}{2}$. If $B(f_1,f_1) \neq 0$ then let $\widetilde{e_1} = f_1$. If $B(f_1,f_1) = 0$ then let $\widetilde{e_1} = e_1 + f_1$. In the bold cases, we have $B(\widetilde{e_1},\widetilde{e_1}) \neq 0$ and $C(\widetilde{e_1}) = \frac{1}{2}\widetilde{e_1}$. Let $V = \mathbb{C}\widetilde{e_1} \overset{\perp}{\oplus} \widetilde{e_1}^\perp$ then $\widetilde{e_1}^\perp$ is non-degenerate, C maps $\widetilde{e_1}^\perp$ into itself. Therefore the result follows by induction. □

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Index

Abstract

In this thesis, we defind a new invariant of quadratic Lie algebras and quadratic Lie superalgebras and give a complete study and classification of singular quadratic Lie algebras and singular quadratic Lie superalgebras, i.e. those for which the invariant does not vanish. The classification is related to adjoint orbits of Lie algebras $\mathfrak{o}(m)$ and $\mathfrak{sp}(2n)$. Also, we give an isomorphic characterization of 2-step nilpotent quadratic Lie algebras and quasi-singular quadratic Lie superalgebras for the purpose of completeness. We study pseudo-Euclidean Jordan algebras obtained as double extensions of a quadratic vector space by a one-dimensional algebra and 2-step nilpotent pseudo-Euclidean Jordan algebras, in the same manner as it was done for singular quadratic Lie algebras and 2-step nilpotent quadratic Lie algebras. Finally, we focus on the case of a symmetric Novikov algebra and study it up to dimension 7.

Key-words: quadratic Lie algebras, quadratic Lie superalgebras, pseudo-Eucliean Jordan algebras, symmetric Novikov algebras, invariant, adjoint orbits, Lie algebra $\mathfrak{o}(m)$, Lie algebra $\mathfrak{sp}(2n)$, solvable Lie algebras, 2-step nilpotent, double extensions, T^*-extension, generalized double extension, Jordan-admissible.