THESIS

Perturbations of Partially Hyperbolic Automorphisms on Heisenberg Nilmanifold

By: Yi SHI

Advisors: Christian BONATTI and Lan WEN

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy

Peking University
&
Université de Bourgogne

Acknowledgements

First I want to thank my parents who taught me the love and courage of life. Without these, I would never be able to study my doctorate in mathematics and finish this thesis.

Both of Lan Wen and Christian Bonatti, my advisors, are significant and essential to my study of dynamical systems during these years. I still remember the scene of Lan Wen taught me dynamical systems from zero in the first year of my doctorate, which opening the gate of my research in this field. I want to thank his patience and encouragements, which is so crucial and helpful to me, both in mathematics and daily life. To Christian, I seriously don't know how to express my thanks and gratitude in language. Without him, this thesis would be definitely mission impossible. I could never even try to list all what I learned from him, mathematics, the French culture, music, painting, even French cheese.

I want to thank Shaobo Gan. He is an example for me, not only as a mathematical researcher, but also an honest and decent person. I greatly appreciate Amie Wilkinson and Sylvain Crovisier for being the reporter of this thesis, which is an honor for me. This thesis is based on a lot of suggestions and comments from them.

I want to thank Bin Yu who taught me a lot of knowledge of topology. The time we cooked together in France are wonderful memory for me. I owe Sébastien Alvarez a lot for helping me during my staying in Dijon, especially for translating the introduction of this thesis into French. I want to thank Rafael Potrie, Dawei Yang, Baolin He, Katsutoshi Shinohara, Doris Bohnet, Ana Rechtman, my classmates and friends for a lot discussion in mathematics.

This work is mainly finished during my stay in IMB, Dijon, which is supported by the Chinese Scholarship Council. I want to thank them and the Chinese Embassy of France. I also supported by the ANR project of France.

Abstract

In this thesis, we show that all the partially hyperbolic automorphisms on the Heisenberg nilmanifold can be C^1-approximated by structurally stable C^∞ diffeomorphisms which exhibit one attractor and one repeller. This implies that all these automorphisms are not robustly transitive. Our constructions of attractors and repellers need the analysis of dynamical invariant contact structures and fiber isotopic invariant Birkhoff sections for these automorphisms. As a corollary, the holonomy maps of stable and unstable foliations of the approximating diffeomorphisms are twisted quasiperiodically forced circle homeomorphisms which are transitive but non-minimal and satisfying certain fiberwise regularity properties.

Perturbations des automorphismes partiellement hyperboliques sur la nilvariété de Heisenberg

Resumè

Dans cette thèse, nous démontrons que les automorphismes partiellement hyperboliques de la nilvariété non Abélienne de dimension 3 peuvent tous être approchés dans la topologie C^1 par des difféomorphismes structurellement stables, chacun possédant un attracteur et un répulseur comme seuls ensembles récurrents par chaîne. Cela implique que ces automorphismes partiellement hyperboliques ne sont pas robustement transitifs. Nos constructions des attracteurs et répulseurs requiérent une analyse des structures de contact invariantes, et des sections de Birkhoff invariante à isotopie dans les fibres près pour ces automorphismes. Comme corollaire, nous en déduisons que les holonomies des feuilletages stables et instables des difféomorphismes approximants sont des homéomorphismes quasi-périodiquement forcés twistés du cercle, qui sont transitifs mais pas minimaux, qui satisfont à certaines propriétés de régularité dans les fibres.

Contents

Notations

M^d will denote a compact connected Riemannian manifold without boundary of dimension $d \in \mathbb{N}$ and $m(\cdot)$ the Lebesgue measure on M^d. For any two points $x, y \in M^d$, we denote by $d(x, y)$ the distance between x and y. Sometimes, we just denote the manifold by M, ignoring the dimension d.

For a subset $K \subset M$, we denote $T_K M = \bigcup_{x \in K} T_x M$, where the topology is induced from the tangent bundle TM. We denote $\mathrm{Int}(K)$, $\mathrm{Cl}(K)$, ∂K, K^c be the interior, closure, frontier and complement of K respectively. For another subset $L \subset M$, we denote $K \backslash L = \{x : x \in K, x \notin L\}$.

$\mathbf{Diff}^r(M)$ $(r \geq 0)$ denote the set of C^r diffeomorphisms (homeomorphisms if $r = 0$) of M with C^r-topology. Moreover, we denote by $m(\cdot)$ the Lebesgue measure on M, and $\mathbf{Diff}^r_m(M)$ denote the set of Lebesgue measure preserving C^r diffeomorphisms (homeomorphisms if $r = 0$) of M with C^r-topology. For any $f, g \in \mathbf{Diff}^r(M)$ or $\mathbf{Diff}^r_m(M)$, we shall denote $d_{C^r}(f, g)$ the C^r-distance between f and g.

For $f \in \mathbf{Diff}^1(M)$, we denote as $D_x f : T_x M \to T_{f(x)} M$ the derivative of f over the point x, and sometimes just Df when the base point x is obvious.

We call $f \in \mathbf{Diff}^r(M)$ **transitive** if for any two open set $U, V \subset M$, there exists some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. f is C^r**-robustly transitive** if there exists an open neighborhood $\mathcal{U} \subset \mathbf{Diff}^r(M)$ of f, such that any $g \in \mathcal{U}$ is transitive. Usually, we say f is robustly transitive if it is C^1-robustly transitive.

We call $f \in \mathbf{Diff}^r_m(M)$ **ergodic** if for any two set $E, F \subset M$ both with positive Lebesgue measure, there exists some $n \in \mathbb{N}$ such that $m(f^n(E) \cap F) > 0$. We call $f \in \mathbf{Diff}^2_m(M)$ **stably ergodic** if there exists a C^1-neighborhood $\mathcal{U}$ of f, such that any $g \in \mathcal{U} \cap \mathbf{Diff}^2_m(M)$ is ergodic.

S^1 will denote the unit circle $\mathbb{R}/\mathbb{Z}$, and $\mathbb{T}^d$ will denote the flat d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ with the metric induced by the canonical covering map $\pi : \mathbb{R}^d \to \mathbb{T}^d$ and the Euclidean metric on $\mathbb{R}^d$.

We will denote by $\mathbb{H}$ the 3-dimensional real Heisenberg group, and Γ the integer lattice of $\mathbb{H}$, that is the 3-dimensional real Heisenberg group with integer elements. Since $\mathbb{H}$ is a Lie group, we denote $\mathfrak{h}$ be the Lie algebra of $\mathbb{H}$. We use $\mathbf{Aut}(\mathbb{H})$ denote the set of all Lie group automorphisms of $\mathbb{H}$, and $\mathbf{Aut}(\mathfrak{h})$ the set of all Lie algebra automorphisms of $\mathfrak{h}$. Moreover, we denote $\mathbf{Aut}_\Gamma(\mathbb{H})$ the set of all Lie group automorphisms of $\mathbb{H}$ which also preserving Γ invariant. Finally, we denote $\mathcal{H} = \mathbb{H}/\Gamma$ be the Heisenberg nilmanifold. More accurate definitions will be in the introduction.

Let $f \in \mathbf{Diff}^r(M)$, the **chain recurrent set** $\mathcal{R}(f)$ of f is defined as: $x \in \mathcal{R}(f)$ if for any $\varepsilon > 0$, there exists a sequence of points $\{x_0, x_1, \cdots, x_n\}$ such that $x = x_0 = x_n$, and $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, \cdots, n$.

For any map $f : X \to Y$ and $K \subset X$ is a subset. We denote by $f|_K : K \to f(K) \subset Y$ the map f restricted to K.

If E is a tangent bundle over some manifold M, and $X_1, \cdots, X_n$ are vector fields on M, then $\langle X_1, \cdots, X_n \rangle$ will denote the subbundle generated by $X_1, \cdots, X_n$.

For two maps $f_i : X_i \to Y_i$, $i = 1, 2$, we denote by $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ the product map:

$$f_1 \times f_2 \ (x_1, x_2) \ = \ (f_1(x_1), f_2(x_2)).$$

For a Riemannian manifold M and two bundle field $E_1, E_2 \subset TM$, i.e. for any $x \in M$, $E_i(x) = E_i \cap T_xM$ is a linear subspace of T_xM, $i = 1, 2$, we define the angle between $E_1(x)$ and $E_2(x)$ as

$$\measuredangle(E_1(x), E_2(x)) = \max\{d_{T_xM}(v_1, v_2) : \ v_1 \in E_1(x), v_2 \in E_2(x), \|v_1\| = \|v_2\| = 1\}.$$

And the angle between E_1 and E_2 is defined as

$$\measuredangle(E_1, E_2) \ = \ \sup_{x \in M}\{ \ \measuredangle(E_1(x), E_2(x)) \ \}.$$

We use the symbol $\Box$ to denote the end of a proof of a Theorem, Lemma, Proposition, Claim, or Corollary.

Chapter 1

Introduction

1.1 Introduction (Français)

L'éude des systèmes dynamiques hyperboliques [1] remonte l'éude faite par J.Hadamard dans les années 1890 [19] sur le flot géodéique des surfaces à courbure négative. Il a introduit alors les notions de variété stables et instables, et, grâce au théorème de récurrence de Poincaré en a déduit que les orbites périodiques sont denses dans le fibré unitaire tangent d'une telle surface.

Quelques quarante ans plus tard, E. Hopf trouva ce que l'on appelle de nos jours l'argument de Hopf, et prouva l'ergodicité du flot géodéique ϕ_t par rapport à la mesure de Liouville.

La même année, S. Smale [32] et D.V. Anosov [1] publièrent leurs travaux pionniers sur les dynamiques hyperboliques, prouvant en particulier leur stabilité structurelle. De nos jours, les systèmes possédant une structure hyperbolique globale sont connus sous le nom de systèmes d'Anosov.

L'exemple le plus classique de difféomorphisme d'Anosov est l'application du chat d'Arnold:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 ,$$

qui est également un automorphisme des groupes de Lie commutatifs $\mathbb{R}^2$ et $\mathbb{T}^2$. La structure hyperbolique définie sur le fibré tangent de $\mathbb{T}^2$, c'est à dire sur l'algèbre de Lie, correspond aux espaces propres de la matrice.

[1] Nous ne prétendons pas donner en détails l'histoire de l'étude des systèmes dynamiques, mais plutôt quelques résultats, questions, progrès historiques qui ont motiv?cette thèse.

Anosov a démontré que les difféomorphismes d'Anosov sont structurellement stables. C'est à dire qu'il existe un voisinage $\mathcal{U} \subset \mathbf{Diff}^1(\mathbb{T}^2)$ de A, tel que pour tout $f \in \mathcal{U}$, il existe un homéomorphisme h_f de $\mathbb{T}^2$, vrifiant

$$h_f \circ f \;=\; A \circ h_f \;.$$

L'application h est appelé la conjugaison topologique, cela entraîne en particulier que les orbites de f et celles de A ont le même comportement.

De plus, nous savons d'après les travaux de R. Mañé [29] et S. Hayashi [20], que la stabilité structurelle est en fait équivalente à l'hyperbolicité.

La stabilité structurelle garantit la persistence de certaines propriétés dynamiques. Par exemple, remarquons que A est transitive, elle est donc robustement transitive, puisque la transitivité est préservé par conjugaison topologique. En utilisant l'argument de Hopf, Anosov a également prouver dans [1] que les systèmes d'anosov conservatifs de classe C^2 sont également stablement ergodiques.

Les notions d'ergodicité et de transitivité sont assez similaires. La première est une propriété topologique, et la seconde est unbe propriété de théorie de la mesure, mais les deux traduisent certaines propriétés de mélange. Ceci est également le cas des propriétés de robuste transitivité et d'ergodicité stable.

Les systèmes conservatifs ergodiques sont encore transitifs puisque la mesure de Lebesgue charge les ouverts. La réciproque en revanche est fausse. Furstenberg [14] donne l'exemple d'un difféomorphisme analytique du tore $\mathbb{T}^2$, qui préserve la mesure de Lebesgue, est minimal, mais pas ergodique.

Après les travaux de Mañé et Hayashi, les chercheurs se sont appliqués aller au-delà de l'hyperbolicité uniforme, et plus particulièrement chercher quelles sont les propriétés des dynamiques uniformément hyperboliques qui restent vraies dans le cadre non hyperbolique.

Il est vrai que les propriétés de persistances entraînent certaines propriétés faibles d'hyperbolicité Mañé [28] a prouvé que les difféomorphismes robustement transitifs des surfaces sont des difféomorphismes d'Anosov du tore. Puis C. Bonatti, L.J. Díaz, E. Pujals, et R. Ures [12] [5], généralisant les techniques de Mañé, ont prouvé que les difféomorphismes robustement transitifs des variété de dimension plus grande doivent être volume partiellement hyperbolique.

Ils existe également des systèmes non-hyperboliques possédant des propriétés persistentes. Dans les années 90, M. Grayson, C. Pugh, and M. Shub [16](voir également [33]) ont prouvé que le temps 1 du flot géodésique d'une surface hyperbolique est stablement ergodique, ce qui donnait le premier exemple de systéme non ergodique stablement ergodique.

Peu de temps après, C. Bonatti et L.J. Diaz [4] montraient que le temps 1 de n'importe quel flot d'Anosov transitifs peut être approché dans la topologie C^∞ par des systémes robustement transitifs non hyperboliques. Bien entendu, ces systèmes incluent les temps 1 considérés dans [16] et [33].

Puisque tout système conservatif ergodique est transitif, ces deux résultats laissaient à penser que les systèmes stablement ergodiques sont également robustement transitifs.

Dans cette thèse, nous proposons d'étudier la relation qu'entretiennent robuste transitivité et stable ergodicité.

Remarquons que les deux résultats importants de [4] et [16], posent également le problème difficile suivant, qui est une motivation importante de cette thèse:

Le temps 1 map du flot géodésique d'une surface close courbe négativement est-il robustement transitif?

Nous renvoyons à [34] pour plus de détails sur ce problème. Nous devons mentionner le beau travail de C. Bonatti et N. Guelman [8] traitant de cette question difficile. Ils prouvent l'existence de difféomorphismes partiellement hyperboliques sur le fibré tangent de telles surfaces, qui sont conjugués dans les feuilles au temps 1 du flot géodésique, et pourtant ne sont pas transitifs. Dans ce travail, ils donnent une construction appelé DA centrale pour séparer la dynamique, qui joue un rôle crucial dans cette thèse.

Nous étudions une sorte de difféomorphismes qui peut être vue comme un modèle simplifié du temps 1 des flots géodésiques, ce sont les automorphismes partiellement hyperboliques des nilvariété de Heisenberg.

Considérons le groupe de Heisenberg réel de dimension 3 $\mathbb{H}$, qui est le group de Lie non commutatif le plus simple. Nous étudions les automorphismes de groupe de $\mathbb{H}$ qui préservent le réseau entier Γ. Ces automorphismes induisent des difféomorphismes sur la nilvariété quotient $\mathcal{H} = \mathbb{H}/\Gamma$ qui est compacte. De plus, nous demandons ce que ces automorphismes soient partiellement hyperboliques.

La nilvariété $\mathcal{H}$ fibre en cercles au dessus du tore $\mathbb{T}^2$, avec un nombre d'Euler 1. Tout automorphisme partiellement hyperbolique de $\mathcal{H}$ a la fibration en cercle pour feuilletage central, et la somme des fibré stable et instable forme une structure de contact invariante sur $\mathcal{H}$. Cela entraine que ces automorphismes sont des contactomorphismes, comme le temps 1 d'un flot géodésique.

Récemment les difféomorphismes partiellement hyperboliques sur $\mathcal{H}$ a été grandement étudiés, donnant plusieurs jolis résultats. F. Rodriguez Hertz, J. Rodriguez Hertz, et R. Ures ont prouvé que les difféomorphismes partiellement hyperboliques conservatifs de classe C^2 sur $\mathcal{H}$ sont ergodiques [22]. Ainsi, les automorphismes partiellement hyperboliques doivent être stablement ergodiques. Cette propriété de mélange persistente découle de propriétés topologiques de $\mathcal{H}$.

Plus tard, A. Hammerlindl et R. Potrie [17] [18] ont prouvé que les difféomorphismes partiellement hyperboliques de $\mathcal{H}$ sont conjugués dans les feuilles aux automorphismes partiellement hyperboliques, c'est à dire qu'ils admettent un fibré en cercles en tant que feuilletage central, et qu'en passant au quotient par le feuilletage central ce sont des homéomorphismes topologiquement Anosov sur le tore.

Notre résultat principal est le suivant:

Théorème. *Pour tout automorphisme partiellement hyperbolique $f_A : \mathcal{H} \longrightarrow \mathcal{H}$, il existe une suite de difféomorphismes de classe C^∞ $\{f_n\}$ convergeant vers f_A dans la topologie C^1, qui sont structurellement stables, et dont les ensemble de récurrence par chaînes sont réduits à un attracteur et un répulseur.*

Ce théorème entraîne que f_A n'est pas robustement transitifs, donnant ainsi le premier exemple de dynamique stablement ergodique qui n'est pas robustement transitif.

De plus, remarquons que la minimalité de l'un des feuilletages stable ou instable d'un difféomorphisme partiellement hyperbolique implique la transifivité de celui-ci. Les deux feuilletages stables et instables de f_A sont minimaux, et f_A est stablement accessible [18]. Nous pouvons donc prouver que f_A est le premier exemple satisfaisant ces deux propriétés, sans être robustement transitif. Cela donne une réponse négative au *Problème 50* de [21].

Comme application, en analysant les holonomies des feuilletages stables et instables de f_n nous obtenons le corollaire suivant:

Corollaire. Pour tout $1 \leq r < \infty$, il existe des homéomorphismes du cercle forcés quasi-périodiquement:

$$h^r \ : \ \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \ , \quad (\theta, t) \longmapsto (\theta + \omega_r, h^r_\theta(t)) \ ,$$

qui sont homotopes à un twist de Dehn, tels que h^r est transitifs mais non minimal, et chaque homéomorphisme induit sur les fibres en cercles h^r_θ sont des difféomorphismes de classe C^r.

Nous renvoyons le lecteur à [3] pour des constructions de tels homéomorphismes homotopes à l'identité. Pour plus de détails sur ces systèmes, nous renvoyons également à [26] [27].

1.2 Historical Account

The study of hyperbolic dynamics[2] could be traced back to J. Hadamard in about the 1890's [19] who studied of the geodesic flows on negatively curved surfaces. Hadamard introduced the notions of stable manifolds and unstable manifolds, which combined with the Poincaré recurrence allows one to deduce that the periodic orbits are dense in the unit tangent bundle of these surfaces.

About forty years later, E. Hopf applied what we called the Hopf argument now in [24], which showed that the geodesic flow ϕ_t is ergodic with respect to the Liouville measure.

In the same year, S. Smale [32] and D.V. Anosov [1] both published their milestone works on hyperbolic dynamics concerning their structural stability. Nowadays, we call the systems that admitting the global hyperbolic structure on the tangent space of manifolds, the Anosov systems.

For instance, the most classical example of Anosov diffeomorphisms is the Arnold's cat map:

$$A \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \;:\; \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \;\longrightarrow\; \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \;,$$

which is also a Lie group automorphism on the commutable Lie groups $\mathbb{R}^2$ and $\mathbb{T}^2$. And the hyperbolic structure defined on the tangent bundle of $\mathbb{T}^2$, which is the Lie algebra, corresponds to the eigenspaces of the matrix.

Anosov showed that Anosov diffeomorphisms must be structurally stable. That is there exists a neighborhood $\mathcal{U} \subset \mathbf{Diff}^1(\mathbb{T}^2)$ of A, such that for any $f \in \mathcal{U}$, there exists a homeomorphism h_f of $\mathbb{T}^2$, satisfying

$$h_f \circ f \;=\; A \circ h_f \;.$$

Here h is called the topological conjugation, which implies that f admits the same orbit structure with A.

Moreover, from the work of R. Mañé [29] and S. Hayashi [20], we know that actually structural stability is equivalent to hyperbolicity for dynamical systems.

The structural stability guarantees some persistence properties of hyperbolic dynamics. Notice that A is transitive, thus it is robustly transitive since transitivity is preserved by topological conjugation. By applying the Hopf argument, Anosov also showed in [1] that the C^2 conservative Anosov systems must be ergodic, thus also stably ergodic.

[2] We do not intend to give a complete and accurate historical story of the study of dynamical systems, but some historical results, questions, and progresses which motivate this thesis.

We can see that from the definitions of transitivity and ergodicity, they are quite similar. One is from the topology viewpoint, the other one is from the measure viewpoint, but both concerning the mixing property of dynamics. The same to robust transitivity and stable ergodicity.

For a conservative system, if it is ergodic, then it must be transitive since open sets have positive Lebesgue measure. However, the contrary is not true. Furstenberg [14] gave an example of an analytic diffeomorphism on $\mathbb{T}^2$, which preserves the Lebesgue measure, is minimal, but is not ergodic.

After the work of Mañé and S. Hayashi, researchers turned to focus on the dynamics beyond uniformly hyperbolicity, especially whether some properties of hyperbolic dynamics also holds for the non-hyperbolic systems.

However, it has been found that the persistent property also implies some hyperbolicity. Mañé [28] showed that robustly transitive diffeomorphisms on 2-dimensional manifolds must be Anosov diffeomorphisms on the torus. Then Bonatti, Díaz, Pujals, and Ures [12] [5] generalized the techniques of Mañé showed that the robustly transitive diffeomorphisms on higher dimensional manifolds should be volume partially hyperbolic.

There are also some examples of non-hyperbolic systems admitting the persistent properties. In the nineties of last century, M. Grayson, C. Pugh, and M. Shub [16](see also [33]) proved that the time-1 map of the geodesic flow on closed surface with constant negatively curvature is stably ergodic, which is the first non-hyperbolic system that was shown to be stably ergodic.

Very soon, C. Bonatti and L.J. Diaz [4] showed that the time-1 map of any transitive Anosov flow could be C^∞-approximated by non-hyperbolic robustly transitive systems. Of course, this includes the time-1 map appeared in [16] and [33].

Since the ergodic systems must be transitive, these two results convinced people to tend to believe that stably ergodic systems need to be robustly transitive.

In this thesis, we will try to discuss the relation between robust transitivity and stable ergodicity these two persistent mixing properties of dynamical systems.

Notice that in the two great results [4] and [16], both concern another difficult problem, which is also an important motivation of this thesis:

Is the time-1 map of the geodesic flow on closed surface with constant negative curvature is robustly transitive?

For this problem, we refer to [34] for more backgrounds. There is a more general open question about the time-1 map of Anosov flows. J. Palis and C. Pugh asked ([30]) whether the time-1 map of Anosov flow can be approximated by an Axiom-A diffeomorphism. Even for the suspension of an Anosov diffeomorphism, we just knew the explicit answer when the roof function of suspension is constant. It has been showed that [10] for any C^2 volume preserving Anosov flow on a 3-manifold, its time-1 map is stably ergodic if and only if it is not a suspension flow with constant roof function. This result implies that the question of Palis and Pugh would be very difficult.

We have to mention the beautiful work of C. Bonatti and N. Guelman [8] concerning this difficult question, which is also the only known partial result. They showed that there exist partially hyperbolic diffeomorphisms on the unit tangent bundle of such surfaces, which are leaf conjugate to the time-1 maps of geodesic flows and not transitive. Their work shows that there are no topological obstructions for the existence of partially hyperbolic structually stable diffeomorphisms on the 3-manifold supporting transitive Anosov flows, where the partially hyperbolic structurally stable diffeomorphisms are leaf conjugacy to the Anosov flows. In their work, they provide what we called central DA-constructions to separate the dynamics, which plays a crucial role in this thesis.

1.3 Heisenberg Nilmanifold and Partial Hyperbolicity

We first introduce the manifold we deal with and the known results of partially hyperbolic diffeomorphisms on it.

Consider the 3-dimensional Heisenberg group

$$\mathbb{H} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{R}\}$$

with the usual matrix operation. We can also denote $\mathbb{H} = \{(x, y, z) : \ \ x, y, z \in \mathbb{R}\}$ with the operation

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + ay).$$

The integer lattice of $\mathbb{H}$ is quite natural:

$$\Gamma = \{(x, y, z) \in \mathbb{H} : \ x, y, z \in \mathbb{Z}\}.$$

And the homogeneous space $\mathcal{H} = \mathbb{H}/\Gamma$ is defined as $\mathbb{H}$ modulo the equivalent relationship $\sim$: $(a, b, c) \sim (x, y, z)$ if and only if there exists $(k, l, m) \in \Gamma$ such that $(a, b, c) = (k, l, m) \cdot (x, y, z)$. If we view it in $\mathbb{R}^3$, and consider a fundamental domain

$$\{(x, y, z) \in \mathbb{H} : \ 0 \leq x, y, x \leq 1\},$$

on its boundary, then we have the following equivalent relationship:

$$\begin{aligned} (x, y, z) \ &\sim \ (1, 0, 0) \cdot (x, y, z) \ = \ (x + 1, y, z + y) \\ &\sim \ (0, 1, 0) \cdot (x, y, z) \ = \ (x, y + 1, z) \\ &\sim \ (0, 0, 1) \cdot (x, y, z) \ = \ (x, y, z + 1) \end{aligned}$$

From this we can see that $\mathcal{H} = \mathbb{H}/\Gamma$ is an S^1-bundle over $\mathbb{T}^2$ with Euler number 1.

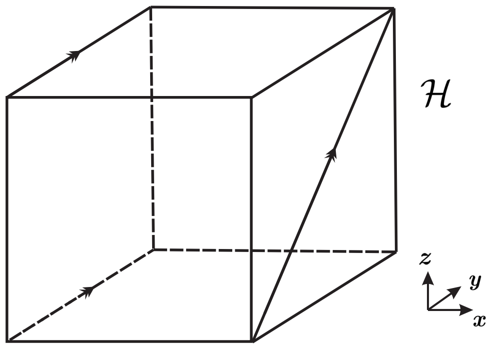

Figure 1.1: *Heisenberg Nilmanifold: constructed from a cube by identifying left and right faces by a Dehn twist, and the other faces are identified by standard translations.*

Actually, any lattice of $\mathbb{H}$ is isomorphic to

$$\Gamma_k = \{(x, y, z) \in \mathbb{H} : \ x, y, \in \mathbb{Z}, z \in \frac{1}{k}\mathbb{Z}\},$$

k is a positive integer (*See Section 4.3.1[21]*). And the homogeneous space $\mathcal{H}_k = \mathbb{H}/\Gamma_k$ could be defined similarly as above. $\mathcal{H}_k$ is an S^1-bundle over $\mathbb{T}^2$ with Euler number k. Thus $\mathcal{H}$ is a k-cover of $\mathcal{H}_k$. Together with the 3-dimensional torus $\mathbb{T}^3$, these gave all the nilmanifolds in dimension 3.

For the simplicity of notations, we will restrict ourselves in the case $\mathcal{H}$, but all our results also holds for any $\mathcal{H}_k$.

For the Heisenberg group $\mathbb{H}$, we denote by $\mathbf{Aut}(\mathbb{H})$ the set of all Lie group automorphisms. That is for any $\widetilde{f} \in \mathbf{Aut}(\mathbb{H})$, $\widetilde{f} : \mathbb{H} \longrightarrow \mathbb{H}$ is a diffeomorphism which preserve the group operation:

$$\widetilde{f}(g_1)\widetilde{f}(g_2) = \widetilde{f}(g_1 g_2), \qquad \forall g_1, g_2 \in \mathbb{H}.$$

Moreover, if the automorphism $\widetilde{f}$ satisfies $\widetilde{f}(\Gamma) = \Gamma$, we denote by $\widetilde{f} \in \mathbf{Aut}_\Gamma(\mathbb{H})$. This allowed us to define a diffeomorphism f on $\mathcal{H} = \mathbb{H}/\Gamma$. That is for any $g \in \mathbb{H}$, and we denote $\Gamma \cdot g \in \mathcal{H}$, we have

$$f(\Gamma \cdot g) \ = \ \Gamma \cdot \widetilde{f}(g) \ .$$

Here f is a well-defined diffeomorphism on $\mathcal{H}$ since $\widetilde{f}(\Gamma) = \Gamma$. This definition makes the following diagram commutable:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\widetilde{f}} & \mathbb{H} \\ \downarrow & & \downarrow \\ \mathcal{H} = \mathbb{H}/\Gamma & \xrightarrow{f} & \mathcal{H} = \mathbb{H}/\Gamma \end{array}$$

We call a diffeomorphism f on $\mathcal{H}$ is partially hyperbolic, if the tangent bundle $T\mathcal{H}$ admits a Df-invariant splitting

$$T\mathcal{H} \;=\; E^s \oplus E^c \oplus E^u \;,$$

and there exists an integer $k > 0$ and a constant $0 < \mu < 1$, such that for any $p \in \mathcal{H}$, and unit vectors $v^s \in E^s(p)$, $v^c \in E^c(p)$, and $v^u \in E^u(p)$, we have

$$\|Df^k(v^s)\| \;<\; \mu < \; \|Df^k(v^c)\| \;<\; \mu^{-1} \;<\; \|Df^k(v^u)\| \;.$$

In chapter 2, we will give a very detailed descriptions of partially hyperbolic automorphisms. We will see that all the partially hyperbolic automorphisms on $\mathcal{H}$ preserve the S^1-fiber structure of $\mathcal{H}$. The S^1-fibers are tangent to the central bundle E^c, and are isometries restricted on each fiber. Thus we can modulo the S^1-fibers, and the automorphism will induce a linear action A on $\mathcal{H}/S^1 = \mathbb{T}^2$. $A \in GL(2,\mathbb{Z})$ is a hyperbolic matrix(the absolute values of eigenvalues not equal to 1). To be more precisely, $f_A : \mathcal{H} = \mathbb{T}^2 \widetilde{\times} S^1 \longrightarrow \mathbb{T}^2 \widetilde{\times} S^1$ could be represented as

$$f_A(x,y,z) \;=\; (\; A(x,y) \;,\; \psi_{x,y}(z) \;) \;, \qquad (x,y,z) \in \mathbb{T}^2 \widetilde{\times} S^1 \;.$$

Here $A \in GL(2,\mathbb{Z})$ is a hyperbolic action, and each $\psi_{x,y}$ is a circle isometry (*see theorem 2.2.2*). Moreover, the invariant bundle $E^s \oplus E^u$ is a contact plane field on $\mathcal{H}$ which transverse to S^1-fibers of $\mathcal{H}$. Thus the partially hyperbolic automorphisms are contactomorphisms.

Recently, the study of partially hyperbolic diffeomorphisms has achieved great progress. In [22], F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures proved that all the C^2 partially hyperbolic volume preserving diffeomorphisms of $\mathcal{H}$ are ergodic. This surprising result strongly relies on the topological property of $\mathcal{H}$.

After that, A. Hammerlindl and R. Potrie [17],[18] showed that every partially hyperbolic diffeomorphisms on $\mathcal{H}$ is leaf conjugate to some partially hyperbolic diffeomorphism. This results gave very accurate descriptions of partially hyperbolic diffeomorphisms on $\mathcal{H}$.

For any partially hyperbolic automorphism f_A of $\mathcal{H}$, its invariant plane field $E^s \oplus E^u$ is a contact plane field (theorem 2.2.2). This implies that the accessible class of any point in $\mathcal{H}$ is an open set, the connectedness of $\mathcal{H}$ ensures that f_A is accessible and stably accessible (*this actually holds for all partially hyperbolic diffeomorphisms of $\mathcal{H}$, see [18]*). Since the central bundle of f_A is one dimensional, it automatically satisfies the center bunching condition. This implies that f_A is stably ergodic [11]. From this observation, we can see that the invariant plane field $E^s \oplus E^u$ is contact is a basic fact that guarantee f_A is stably ergodic. Moreover, the partially hyperbolic automorphisms on torus $\mathbb{T}^3$ could be perturbed to be structurally stable, just because its invariant plane field $E^s \oplus E^u$ is integrable (*naive example in chapter 3*). So all these analysis tell us that the invariant contact structure $E^s \oplus E^u$ of f_A is the main obstruction for breaking the transitivity of f_A.

1.4 Main Results and Corollary

In this thesis, we will prove the following result.

Main Theorem. *Let $\widetilde{f}_A \in \mathbf{Aut}_\Gamma(\mathbb{H})$ be partially hyperbolic and $f_A : \mathcal{H} \longrightarrow \mathcal{H}$ be the diffeomorphism induced on $\mathcal{H}$, there exists a sequence of C^∞-diffeomorphisms $\{f_n\}$ converging to f_A in C^1-topology, such that each f_n is structurally stable and the chain recurrent set of f_n consists of one attractor and one repeller.*

We want to point out that the construction of f_n comes from the perturbation of f_A. All our perturbations are along the S^1-fibers. So the f_n still preserves the S^1-fibers structure of $\mathcal{H}$ and induce the same linear action A on $\mathbb{T}^2 = \mathcal{H}/S^1$.

We now give several remarks about the dynamical consequence of this theorem.

Remark.

- *Combined with [22], this theorem shows that all the partially hyperbolic automorphisms on $\mathcal{H}$ are stably ergodic in the conservative category, but not robustly transitive from the topological viewpoint, which is the first example been found. This also answers one question in [21](Problem 49).*

- *Notice that the strong stable foliation of each partially hyperbolic automorphisms on $\mathcal{H}$ is minimal, and all the partially hyperbolic diffeomorphisms on $\mathcal{H}$ are stably accessible([22],[17],[18]). So we answer a question in [21](Problem 50), show that minimality of stable foliation and stable accessibility does not implies robust transitivity. See [6] for more discussions on the minimality of stable and unstable foliations for robustly transitive partially hyperbolic diffeomorphisms.*

- *Recall the time-1 map of geodesic flow on surfaces with constant negative curvature, is also a partially hyperbolic contactomorphism. That is $E^s \oplus E^u$ are invariant contact structure and the derivative is isometry on E^c. So our partially hyperbolic automorphisms could be seen as a simplified model of it. Our result gives strong evidence that the time-1 map of geodesic flows are not robustly transitive.*

- *In our theorem, the approximation only works in C^1-topology. The author tend to believe that the partially hyperbolic automorphism f_A should be C^2-robustly transitive. But there are no strong evidence to support this point. Actually, we seriously know very few things about robustly transitive systems, especially in higher regularities. In C^1-topology, all the known examples are admitting the whole manifolds as a homoclinic class of the systems. So this relates to another conjecture: the C^1-robustly transitive system must admit a hyperbolic periodic orbit. Our f_A could be seen as a good candidate for robustly transitive system without hyperbolic periodic orbits,* *however we showed it is not C^1-robustly transitive.*

For the strong stable and unstable foliations of f_n, their holonomy maps will also admit some special properties. We first introduce the quasiperiodically forced systems. A homeomorphism is called a quasiperiodically forced circle homeomorphism if

$$h \ : \ \mathbb{T}^2 \ \longrightarrow \ \mathbb{T}^2 \ , \qquad (\theta, t) \ \longmapsto \ (\theta + \omega, h_\theta(t)) \ ,$$

where ω is irrational, and the fiber maps h_θ are all orientation preserving circle homeomorphisms. Such homeomorphisms have been seen as a natural generalization of the circle homeomorphisms, and been widely studied for the case where the homeomorphism is homotopic to identity. We refer to [26] and [27] for more information.

Now we consider an embedded torus $\mathbb{T}^2_0$ in $\mathcal{H}$. Lifting in $\mathbb{H}$ and under the coordinates of $\mathbb{R}^3$, this torus could be represented as

$$\{ \ (x, y, z) \ : \ x = 0, \ y, z \in [0, 1] \ \}.$$

Recall that when we project $\mathcal{H}$ to $\mathbb{T}^2$, the partially hyperbolic automorphism f_A will be the linear action A on torus. This implies that the center stable and unstable foliations of f_A are the lift of the stable and unstable foliations of A on $\mathbb{T}^2$ to $\mathcal{H}$, that is times the S^1-fibers. Since our perturbations of f_n are along S^1-fibers, which implies f_n admits the same center stable and unstable foliations of f_A. From this, we deduce that the center stable and unstable foliations of f_A and f_n are transverse to $\mathbb{T}^2_0$.

Since for each connected component of a center stable manifold of f_A (also f_n) intersecting with $\mathbb{T}^2_0$ is a central S^1-fiber, this implies that the strong stable foliations of f_A and f_n is transverse to $\mathbb{T}^2_0$. Moreover, the angle between the central S^1-fibers and the strong stable foliations of f_A and f_n are uniformly bounded from zero. This implies that $\mathbb{T}^2_0$ admits a global holonomy map of the strong stable foliations of f_A and f_n. The same is true for unstable foliations.

We use the coordinate (θ, t) instead of (y, z). Since the central S^1-fibers are also the S^1-fibers of $\mathbb{T}^2_0$, this implies the center stable and unstable foliations intersect $\mathbb{T}^2_0$ get the S^1-fibers structure $\{\theta \times S^1 : \theta \in S^1\}$.

Recall both f_A and f_n will project into the same linear hyperbolic action A on the base $\mathbb{T}^2$, where the stable and unstable foliations of A are linear irrational foliations on torus. Thus the holonomy map of the stable foliation $h^s : \mathbb{T}^2_0 \longrightarrow \mathbb{T}^2_0$ must be a quasiperiodically forced circle homeomorphism:

$$h^s(\theta, t) \ = \ (\theta + \omega_s, h^s_\theta(t)) \ .$$

Notice that h^s must homotopic to a Dehn twist due to the topology of $\mathcal{H}$. The same to unstable foliations.

In [3], the authors constructed examples of quasiperiodically forced systems homotopic to identiy map, that are transitive but non-minimal. But the fiber circle homeomorphisms could only be C^1. And it is also an open question whether the transitivity of these homeomorphisms with higher smoothness implies minimality([3],[26]).

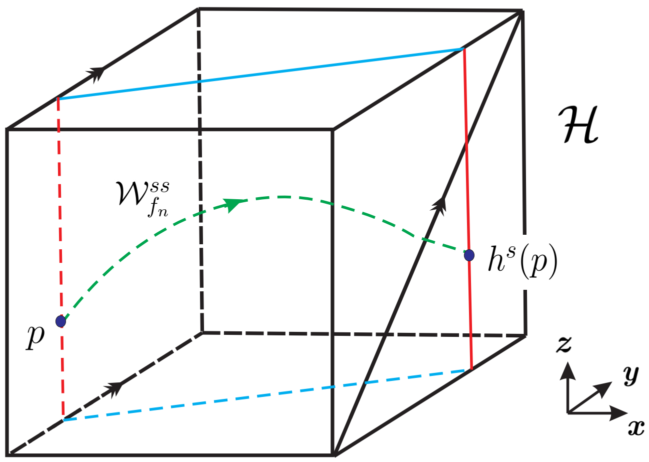

Figure 1.2: *Holonomy Map*

The holonomy maps of stable and unstable foliations of our f_n gives the following corollary.

Corollary. *For any* $0 < r < \infty$*, there exists a quasiperiodically forced circle homeomorphism*

$$h^r \ : \ \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \ , \quad (\theta, t) \longmapsto (\theta + \omega_r, h^r_\theta(t)) \ ,$$

which is homotopic to a Dehn twist, such that h^r *is transitive but non-minimal, and each fiber circle homeomorphism* h^r_θ *is a* C^r*-diffeomorphism.*

Proof. We first show that the holonomy maps of stable foliations of f_n associated to $\mathbb{T}^2_0$ are transitive but non-minimal. Then we prove that the fiber map could be arbitrarily smooth as n tend to infinity.

The transitivity of such homeomorphisms has been proved in section 6 of [18]. Since f_n admit a hyperbolic repeller, which is a stable saturated set, the repeller of f_n intersects $\mathbb{T}^2_0$ in a minimal invariant set of the holonomy map. This proves the non-minimality of holonomy maps.

For the smoothness of fiber maps, we first point out that the fiber maps of holonomy map are the holonomy map of strong stable foliations restricted to center stable manifolds. As f_n will converge to f_A in C^1-topology, the norm of central derivative $\|Df_n|_{E^c}\|$ will converge to 1.

This allows us to apply Theorem 3.2 of [21] showing that for any $0 < r < \infty$, there exists some n, such that the the holonomy map of strong stable foliations of f_n restricted in each center stable manifold is C^r.

This finishes the proof of corollary. $\square$

1.5 Ideas and Sketch of Proof

In this section, we try to illustrate the ideas of our construction and give the organization of this thesis.

The Lie structure of the Heisenberg group and the fact that f_A is a group automorphism makes the invariant bundle $E^s \oplus E^u$ is a contact plane field defined on $\mathcal{H}$. This is the main reason that f_A is stably ergodic, and also the main obstruction for our perturbations to break the transitivity of f_A.

Since f_A is partially hyperbolic and admits the S^1-fibers as its central foliations, so from the structural stability of central foliations (*the central foliation of f_A are smooth, we can applying Theorem 7.1 of [25]*), our perturbations only focuses on the direction of S^1-fibers.

However, $\mathcal{H}$ admits neither any closed surfaces nor any foliations transverse to the S^1-fibers. What we could have is only the Birkhoff sections, that is the compact surfaces whose interior transverse to S^1-fibers, and the boundary consists of finitely many S^1-fibers.

The central DA-construction in [8] allows us choose two parallel such kind Birkhoff sections to be the candidates of our attractor and repeller of new diffeomorphism. However, there are two difficulties here.

- One is that we need to require the Birkhoff section Σ we choose to be dynamically invariant: $f_A(\Sigma)$ is fiber isotopic to Σ.
- The other one is we want some control of the tangent plane field of Σ, which is necessary for estimating the C^1-distance of our future perturbations.

These two difficulties will be managed in theorem 4.0.9, which can be stated in the following way:

There exists a sequence of open book decompositions whose pages are f_A-invariant up to fiber isotopy, and the tangent plane field of each page will approximate the dynamical invariant contact structure $E^s \oplus E^u$ of f_A.

Here the open book decomposition means we fix a Birkhoff section and rotate along the S^1-fibers to get a decomposition of $\mathcal{H}$. It satisfies the Giroux [15] correspondence to the invariant contact structure $E^s \oplus E^u$.

We can see that this result deserve its own interests in the geometric topology field. The work of W. Thurston and Y. Eliashberg shows that the tangent plane field of a 2-dimensional foliation could be approximated by a contact plane field in dimension 3. The converse could not to be true. For example, in our case, there even does not exists any foliation transverse to S^1-fibers of $\mathcal{H}$.

Theorem 4.0.9 actually gives us a sequence of open book decompositions of $\mathcal{H}$, which are all f_A-invariant up to fiber isotopy. Moreover, there exists a sequence of corresponding subsets of $\mathcal{H}$, whose Lebesgue measure will converge to full measure of $\mathcal{H}$, such that for any point in the subsets, the angle between the tangent plane of the page of corresponding open book decomposition and the contact plane at this point will uniformly converge to 0 as the sequence tend to infinity. So we actually construct a sequence of foliations on a sequence of subsets, where the subsets converge to $\mathcal{H}$ and the foliations converges to invariant plane field.

We want to point out that for the time-1 maps of geodesic flows, if we can prove theorem 4.0.9 also works in this situation, then we almost finish the proof of the open question. Here the main difficulty is the unit tangent bundles and the invariant contact structures are more complicated then the nilmanifold case. However, the easier part is the time-1 map is isotopic to identity map, so there are no algebraic obstructions.

Now the new diffeomorphism can be constructed in the following way:

- When far from the boundary fibers, the diffeomorphism on the two sections is one central contracting, the other one is central expanding.

- When close to the boundary fibers, we apply the central DA-construction in [8].

Then we try to glue these two parts together and get the new diffeomorphism which is structurally stable, admits one hyperbolic attractor and one hyperbolic repeller.

Organization of the Paper.

In chapter 2, we give a detailed description of the partially hyperbolic automorphism f_A on $\mathcal{H}$, including what the invariant bundle $E^s \oplus E^u$ associated to f_A looks like.

In chapter 3, we will introduce the definition of Birkhoff sections, and give some examples. Moreover, we will discuss the fiber isotopic class of Birkhoff sections.

In chapter 4, we will give the proof of theorem 4.0.9, which states the existence of invariant Birkhoff sections, and the estimations of their tangent plane fields.

We will prove the main theorem in chapter 5 by admitting the central-DA construction, that is proposition 5.2.1.

Finally, we will give a proof of proposition 5.2.1 in chapter 6.

Chapter 2

Partially Hyperbolic Automorphisms

In this chapter, we will first study all the partially hyperbolic automorphisms on $\mathcal{H}$, including give the explicit formula for such kind automorphisms and their invariant tangent bundles. All these parts are simple Lie groups calculations, the reader could also find them in [21] *Section 4.3.1*. We include them just for completeness. The main results we will need in the future are contained in theorem 2.2.2. Then we show some basic properties of the invariant contact plane field $E^s \oplus E^u$.

2.1 Automorphisms on $\mathbb{H}$ and $\mathcal{H}$

We first state some basic facts about the automorphisms on the Heisenberg group $\mathbb{H}$. Notice that $\mathbb{H}$ is a simply connected Lie group, where $\mathbb{R}^3$ is a global coordinate of it. So we have a one-to-one correspondence between automorphisms of its Lie algebra and automorphisms of $\mathbb{H}$.

For $e = (0,0,0) \in \mathbb{H}$, we choose a basis in the tangent space $T_e\mathbb{H}$ as $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$. Then by the left action, we get three left invariant vector fields on $\mathbb{H}$, which can be represented in $\mathbb{R}^3$ as:

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x \cdot \frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z}\ .$$

Notice they forms a basis of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$. Actually, if view $\mathbb{H}$ as a Lie subgroup of $GL(3,\mathbb{Z})$ and $\mathfrak{h}$ form a Lie sub-algebra of $\mathfrak{gl}(3,\mathbb{Z})$, then we can represent X, Y, Z by the matrix as:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \qquad \text{and} \qquad Z = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

The Lie bracket operation is quite simple:

$$[X, Y] = Z, \qquad [Y, Z] = [Z, X] = 0.$$

So all the automorphisms on $\mathfrak{h}$ are the linear transformation on $\mathbb{R}^3$ and preserve the Lie brackets operation, which means any automorphism φ acting on X, Y, Z must be

$$\varphi(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}) = \begin{pmatrix} a & c & p \\ b & d & q \\ 0 & 0 & ad-bc \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} .$$

Here we require $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$ and $p, q \in \mathbb{R}$. That is, if we identify $\mathfrak{h} \cong \mathbb{R}^3$ under the basis of $\{X, Y, Z\}$, we have

$$\varphi \ = \ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ p & q & ad-bc \end{pmatrix} \ : \ \mathbb{R}^3 \ \longrightarrow \ \mathbb{R}^3 \ .$$

Now applying the exponential map, we could see that any automorphism $f_\varphi \in \mathbf{Aut}(\mathbb{H})$ which associated to $\varphi \in \mathbf{Aut}(\mathfrak{h})$ as above, and any $(x, y, z) \in \mathbb{H}$,

$$\begin{aligned} f_\varphi(x,y,z) &= \exp \circ\, \varphi \,\circ\ \exp^{-1}\ (x,\ y,\ z) = \exp \circ\, \varphi\ (x,\ y,\ z - \frac{xy}{2}) \\ &= \exp\ (ax+by,\ cx+dy,\ px+qy+(ad-bc)(z-\frac{xy}{2})) \\ &= (ax+by,\ cx+dy,\ (ad-bc)z + \frac{1}{2}acx^2 + \frac{1}{2}bdy^2 + bcxy + px + qy). \end{aligned}$$

Actually, from the representation above, we can view that any automorphisms on $\mathbb{H}$ is a lift $A \in GL(2,\mathbb{R})$, which defines an action on $\mathbb{R}^2$. So in the future, we will denote the automorphisms on $\mathbb{H}$ by $\widetilde{f}_A$ to emphasis the matrix A acting on $\mathbb{R}^2$.

If we further require that the automorphism $\widetilde{f}_A \in \mathbf{Aut}_\Gamma(\mathbb{H})$, which could define a diffeomorphism on $\mathcal{H}$. Then we can get more information about such kind automorphisms. Since $\widetilde{f}_A$ is a group automorphism, it must preserve the centralizer of $\mathbb{H}$:

$$C(\mathbb{H}) = \{(0,0,z) \in \mathbb{H} :\ z \in \mathbb{R}\}.$$

This implies $\widetilde{f}_A|_{C(\mathbb{H})}$ is a group automorphism on the real line and preserve all the integers. So $\widetilde{f}_A|_{C(\mathbb{H})}$ could only to be Id or $-$Id. So we must have $|\det(A)| = |ad-bc| = 1$. Moreover, we can see that $\widetilde{f}_A$ induce an automorphism on $\mathbb{Z}^2 = \Gamma/\Gamma \cap C(\mathbb{H})$, which shows that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}).$$

If we still use the Lie algebra automorphisms to represent automorphisms in $\mathrm{Aut}_\Gamma(\mathbb{H})$, then we can associated each $\widetilde{f}_A \in \mathrm{Aut}_\Gamma(\mathbb{H})$ the matrix which acting on the Lie algebra $\mathfrak{h}$. Here we identify $\mathfrak{h} = \mathbb{R}^3$ on the basis of X, Y, and Z, so what we get is the transpose matrix:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2}+p & \frac{bd}{2}+q & ad-bc \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$ and $p,q \in \mathbb{Z}$.

From another point of view, any automorphism on Γ could uniquely extended to an automorphism on $\mathbb{H}$ [2]. The elements $(1,0,0)$ and $(0,1,0)$ will generate Γ, and the first two coordinates of their images are determined by the action of $A \in GL(2,\mathbb{Z})$. Then their images in the third coordinates are two degrees of freedom chosen in $\mathbb{Z}^2$, which corresponding to $p,q \in \mathbb{Z}$ in the above.

2.2 Partially Hyperbolic Automorphisms

Assume $f_A : \mathcal{H} \to \mathcal{H}$ is a partially hyperbolic automorphism with the invariant splitting $T\mathcal{H} = E^s \oplus E^c \oplus E^u$, and $\widetilde{f}_A$ be its lift on $\mathbb{H}$. Recall that we have three left invariant vector fields

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x \cdot \frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z},$$

which form a basis of the Lie algebra $\mathfrak{h}$. It could see that X, Y, and Z are also to be smooth vector fields defined on $\mathcal{H}$, so we can represent the partially hyperbolic invariant bundle by them.

We first consider on the group $\mathbb{H}$, where the lift $\widetilde{f}_A$ also have partially hyperbolic splitting. Denote by $\mathcal{L}_g$ the left action by g on $\mathbb{H}$. Since $\widetilde{f}_A(e) = e$, recall $e = (0,0,0)$, so $D\widetilde{f}_A(T_e\mathbb{H}) = T_e\mathbb{H}$. Assume that E_e is an invariant bundle in $T_e\mathbb{H}$ by $D\widetilde{f}_A$. Then for any $g \in \mathbb{H}$, we define

$$E_g = D\mathcal{L}_g(E_e) \subset T_g\mathbb{H}.$$

Then we could see that $E = \sqcup_{g\in\mathbb{H}} E_g$ is a smooth vector bundle on $\mathbb{H}$. Moreover, it is $D\widetilde{f}_A$-invariant:

$$\begin{aligned} D\widetilde{f}_A(E_g) &= D\widetilde{f}_A \circ D\mathcal{L}_g(E_e) = D(\widetilde{f}_A \circ \mathcal{L}_g)(E_e) \\ &= D(\mathcal{L}_{\widetilde{f}_A(g)} \circ \widetilde{f}_A)(E_e) = D\mathcal{L}_{\widetilde{f}_A(g)}(E_e) \\ &= E_{\widetilde{f}_A(g)} . \end{aligned}$$

Furthermore, if $E_e = E^s_e$ is uniformly contracting by $D\widetilde{f}_A$, i.e. there exists $k > 0$, and $0 < \mu < 1$, such that

$$\|D\widetilde{f}^k_A|_{E_e}\| < \mu,$$

then $D\widetilde{f}_A|_E$ is also uniformly contracting:

$$\begin{aligned} \|D\widetilde{f}^k_A|_{E_g}\| &= \|D(\widetilde{f}^k_A \circ \mathcal{L}_g \circ \mathcal{L}_{g^{-1}})|_{E_g}\| = \|D(\widetilde{f}^k_A \circ \mathcal{L}_g)|_{E_e}\| \\ &= \|D\mathcal{L}_{\widetilde{f}^k_A(g)} \circ D\widetilde{f}^k_A|_{E_e}\| \le \|D\mathcal{L}_{\widetilde{f}^k_A(g)}|_{T_e\mathbb{H}}\| \cdot \|D\widetilde{f}^k_A|_{E_e}\| \\ &\le \mu . \end{aligned}$$

Here we use the fact that for any $g \in \mathbb{H}$, $\mathcal{L}_g$ is an isometry on $\mathbb{H}$.

Similar argument works for the expanding bundle and the relation for dominated splitting. This gives us the following lemma:

Lemma 2.2.1. *For any automorphism $\widetilde{f}_A \in \mathbf{Aut}(\mathbb{H})$, it is partially hyperbolic if and only if $D\widetilde{f}_A$ restricted on $T_e\mathbb{H}$ is a partially hyperbolic linear transformation. Moreover, for any $g \in \mathbb{H}$, we have*

$$E_g^\sigma \;=\; \mathcal{L}_g(E_e^\sigma), \qquad \sigma = s,\ c,\ u.$$

It is also holds for any $\widetilde{f}_A \in \mathbf{Aut}_\Gamma(\mathbb{H})$ and the projection $f_A \in \mathbf{Aut}(\mathcal{H})$.

Now we try to give a more detailed description of the stable, unstable, and central invariant bundle of the partially hyperbolic automorphism $f_A \in \mathbf{Aut}(\mathcal{H})$. Then we show that the union of stable and unstable bundle $E^s \oplus E^u$ form a Df_A-invariant contact plane field.

As stated in lemma 2.2.1, $f_A \in \mathbf{Aut}(\mathcal{H})$ is partially hyperbolic if and only if Df_A acting on $T_e\mathcal{H}$ is partially hyperbolic. We still assume that on the basis of $\{X, Y, Z\}$, Df_A could be represented as matrix:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2}+p & \frac{bd}{2}+q & ad-bc \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$ and $p, q \in \mathbb{Z}$.

Notice that the bundle generated by Z:

$$\langle\ Z\ \rangle \overset{\triangle}{=} \{v \in T_g\mathcal{H} :\ g \in \mathcal{H},\ v = t \cdot Z_g,\ \text{where } t \in \mathbb{R}\}.$$

is invariant by Df_A, and we must have $|ad - bc| = 1$. This implies $E^c = \langle\ Z\ \rangle$, and we must require the matrix A to be hyperbolic to get the hyperbolicity of Df_A.

Denote one of the eigenvalues of A is λ, where $|\lambda| > 1$, then the other one is $(ad - bc)/\lambda$ with modulo smaller than 1. It could easily see that when we projects E^s and E^u on the first two coordinates, that is the plane generated by X and Y, the images would be the eigenspaces of A acting on $\mathbb{R}^2$. We will not try to give the explicit formula of E^s and E^u respectively, but $E^s \oplus E^u$.

We can assume that $E^s \oplus E^u$ is equal to the linear space generated by $X + \alpha\cdot Z$ and $Y + \beta\cdot Z$ for some $\alpha, \beta \in \mathbb{R}$. Then by the invariance of $E^s \oplus E^u$, we have:

$$Df_A(\langle\ X + \alpha Z,\ Y + \beta Z\ \rangle) \;=\; \langle\ X + \alpha Z,\ Y + \beta Z\ \rangle.$$

This deduce two equalities:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2}+p & \frac{bd}{2}+q & ad-bc \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} a \\ c \\ \frac{ac}{2}+p+(ad-bc)\alpha \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix},$$

and

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2}+p & \frac{bd}{2}+q & ad-bc \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix} = \begin{pmatrix} b \\ d \\ \frac{bd}{2}+q+(ad-bc)\beta \end{pmatrix} = b\begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix}.$$

This reduce to the following:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (ad-bc) \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \frac{ac}{2}+p \\ \frac{bd}{2}+q \end{pmatrix}.$$

Since now $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a hyperbolic matrix, we can see that the determinant of the matrix

$$\begin{pmatrix} a-(ad-bc) & c \\ b & d-(ad-bc) \end{pmatrix}$$

is a non-zero integer. We denote it by $m = \det(A - \det A \cdot I) \in \mathbb{Z} \setminus \{0\}$.

Thus we can formulate α, β as:

$$\begin{aligned} \alpha &= \frac{1}{m}[\frac{cd}{2}(a-b) - \frac{ac}{2} + (d-(ad-bc))p - cq] , \\ \beta &= \frac{1}{m}[\frac{ab}{2}(d-c) - \frac{bd}{2} - bp + (a-(ad-bc))q] . \end{aligned}$$

Actually, here the accurate formulas of α and β are not so important for us in the future work. We only need to remember that for partially hyperbolic automorphism f_A,

$$E^s \oplus E^u = \langle\, X + \frac{k}{2m}Z,\ Y + \frac{l}{2m}Z \,\rangle, \qquad k,l \in \mathbb{Z}.$$

Notice that $E^s \oplus E^u$ is a contact plane field defined on $\mathcal{H}$. We will deal with its properties in next subsection.

Remark. *It seems a little bit confusing that all our calculation is restricted on $T_e\mathbb{H}$, but our formulas for the invariant bundles could defined on all $\mathbb{H}$ and $\mathcal{H}$. This is just because all the vector fields X, Y, Z, and all the invariant bundles E^s, E^c, E^u are left invariant. So we can extend the formula to all the group and nilmanifold.*

Now we can summarize all the descriptions about the partially hyperbolic automorphisms on the Heisenberg group $\mathbb{H}$ and nilmanifold $\mathcal{H}$ as the following theorem.

Theorem 2.2.2. *For any partially hyperbolic automorphism $f_A \in \mathbf{Aut}(\mathcal{H})$ with partially hyperbolic $T\mathcal{H} = E^s \oplus E^c \oplus E^u$, and denote its lift $\widetilde{f}_A \in \mathbf{Aut}_\Gamma(\mathbb{H})$. If we denote*

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z},$$

to be a basis of the Lie algebra $\mathfrak{h}$, then the automorphism on $\mathfrak{h}$ induced by $\widetilde{f}_A$ could be represented as the matrix:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2}+p & \frac{bd}{2}+q & ad-bc \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$*, and* $p, q \in \mathbb{Z}$*. Moreover, for any* $(x,y,z) \in \mathbb{H}$*,*

$$f_A(x,y,z) \;=\; (\; ax+by,\; cx+dy,\; (ad-bc)z + \psi_{p,q}(x,y)\;)\;.$$

where $\psi_{p,q}(x,y) = \frac{1}{2}acx^2 + \frac{1}{2}bdy^2 + bcxy + (\frac{ac}{2}+p)x + (\frac{bd}{2}+q)y$*, for some* $p, q \in \mathbb{Z}$*.*

Furthermore, since X*,* Y*, and* Z *are also smooth vector fields defined on* $\mathcal{H}$*, then the invariant bundles satisfy*

$$E^c = \langle\; Z \;\rangle\;, \qquad \text{and} \qquad E^s \oplus E^u = \langle\; X + \frac{k}{2m}\cdot Z,\; Y + \frac{l}{2m}\cdot Z \;\rangle\;,$$

where $m = \det(A - \det A \cdot I) \in \mathbb{Z}\setminus\{0\}$ *and* $k, l \in \mathbb{Z}$*.*

2.3 Invariant Contact Structure

In this subsection, we will focus on studying some properties of $E^s \oplus E^u$ as an invariant contact plane field.

Recall that

$$E^s \oplus E^u \;=\; \langle\; X + \frac{k}{2m}\cdot Z,\; Y + \frac{l}{2m}\cdot Z \;\rangle\;.$$

So the Lie bracket operation

$$[X + \frac{k}{2m}\cdot Z,\; Y + \frac{l}{2m}\cdot Z] = Z,$$

which does not belong to the plane field. This implies that the plane field $E^s \oplus E^u$ is not integrable everywhere.

Actually, if we represent the two vector fields which generated $E^s \oplus E^u$ in $\mathbb{R}^3$ coordinate, then we get

$$E^s \oplus E^u \;=\; \langle\; \frac{\partial}{\partial x} + \frac{k}{2m}\frac{\partial}{\partial z}\;,\; \frac{\partial}{\partial y} + (x + \frac{l}{2m})\frac{\partial}{\partial z} \;\rangle\;.$$

Notice that these two vector fields $\partial/\partial x + k/2m \cdot \partial/\partial z$ and $\partial/\partial y + (x + l/2m)\cdot\partial/\partial z$ are well defined on both $\mathbb{H}$ and $\mathcal{H}$.

Now we consider the 1-form

$$\alpha \;=\; \mathrm{d}z - \frac{k}{2m}\cdot\mathrm{d}x - (x + \frac{l}{2m})\cdot\mathrm{d}y$$

on $\mathbb{H}$. Notice that it can also be projected on $\mathcal{H}$ which also defined a smooth 1-form (still denoted by α) on $\mathcal{H}$. Easy calculation shows that

$$\ker\,\alpha \;=\; E^s \oplus E^u.$$

Moreover, we have

$$\mathrm{d}\alpha = -\mathrm{d}x\wedge\mathrm{d}y, \qquad \text{and} \qquad \alpha\wedge\mathrm{d}\alpha = -\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \neq 0.$$

This implies α is a contact 1-form defined on both $\mathbb{H}$ and $\mathcal{H}$, and $E^s \oplus E^u$ is its kernel, thus a contact plane field.

In the rest of this subsection, we will state a lemma about the twisting property of piecewise smooth curves which tangent to $E^s \oplus E^u$.

First we recall some symbols. We misuse $\pi : \mathbb{H} \longrightarrow \mathbb{R}^2$ denote the projection to the first two coordinates, and also $\pi : \mathcal{H} \longrightarrow \mathbb{T}^2$ the projection along the S^1-fibers.

Now we consider a piecewise smooth simple closed curve $\gamma : [0,1] \longrightarrow \mathbb{R}^2$, with $\gamma(0) = \gamma(1)$. Moreover, we require that γ has positive orientation in $\mathbb{R}^2$. Since γ is a Jordan curve and piecewise smooth, it will bound a region $\mathbb{D}_\gamma$ with finite area $A(\mathbb{D}_\gamma)$.

Since γ is piecewise smooth, so for any $t \in [0,1)$, we have a well defined $\gamma'_+(t) \in T_{\gamma(t)}\mathbb{R}^2$. For any

$$p \in \pi^{-1}(\gamma) = \{(x,y,z) \in \mathbb{H} : \ (x,y) \in \gamma([0,1])\},$$

it exist a unique vector

$$v_p \ \in \ d\pi^{-1}(\gamma'_+(\pi(p))) \ \cap \ E^s_p \oplus E^u_p \ .$$

This is just because $E^s_p \oplus E^u_p$ transverse to $\langle Z \rangle_p$.

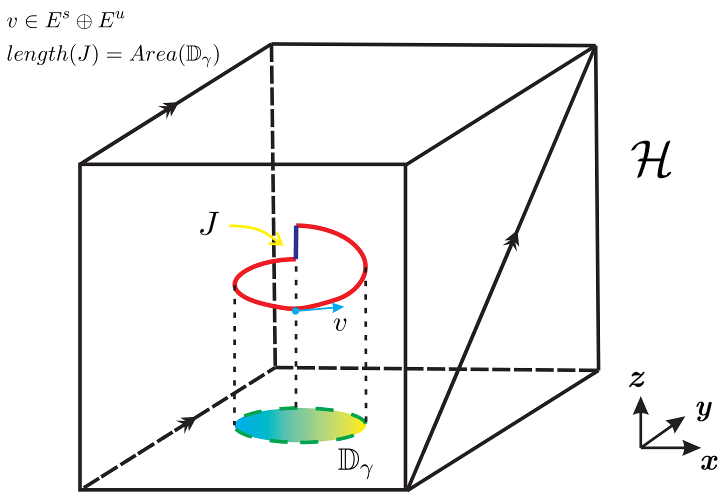

Figure 2.1: *Twisting of Contact Structure*

Lemma 2.3.1. *For any piecewise smooth curve* $\widetilde{\gamma} : [0,1] \longrightarrow \mathbb{H} \cong \mathbb{R}^3$ *satisfying*

- $\widetilde{\gamma}$ *tangent to* $E^s \oplus E^u$ *everywhere.*
- $\gamma = \pi \circ \widetilde{\gamma}$ *is a positively oriented simple closed curve in* $\mathbb{R}^2$*, which bounds a region with area* $A(\mathbb{D}_\gamma)$*.*

- *π is an injection on $\widetilde{\gamma}((0,1))$, and we have $\pi \circ \widetilde{\gamma}(0) = \pi \circ \widetilde{\gamma}(1)$.*

If we denote by $\widetilde{\gamma}(0) = (x_0, y_0, z_0)$ and $\widetilde{\gamma}(1) = (x_0, y_0, z_1)$, then the twisting height

$$z_1 \;-\; z_0 \;=\; A(\mathbb{D}_\gamma)\ .$$

Proof. The proof is applying the fact that

$$E^s \oplus E^u = \langle\ \frac{\partial}{\partial x} + \frac{k}{2m}\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + (x + \frac{l}{2m})\frac{\partial}{\partial z}\ \rangle\ ,$$

then do the basic Riemann integration in $\mathbb{R}^3$.

$\square$

Chapter 3

Birkhoff Sections

In this section, we will introduce the Birkohff sections in $\mathcal{H}$, which will play the central role in our future construction of attractors and repellers of new diffeomorphisms. For showing the necessary of such notion, we first consider a trivial example.

Naive Example. Consider the simplest partially hyperbolic automorphisms of commutative Lie groups

$$A \times \mathrm{Id} \ : \ \mathbb{T}^3 = \mathbb{T}^2 \times S^1 \ \longrightarrow \ \mathbb{T}^3 = \mathbb{T}^2 \times S^1.$$

Notice that such kind diffeomorphisms are not transitive but chain-transitive. We can break the chain-transitivity very easily. Just choose a sequence of Morse-Smale diffeomorphisms $\{g_n\} \subset$ $\mathbf{Diff}^\infty(S^1)$ such that $g_n \to \mathrm{Id}$ in $\mathcal{C}^\infty$ topology. Then $f_n = A \times g_n$ are hyperbolic systems approximating $A \times \mathrm{Id}$ with attractors and repellers.

In this naive example, the attractors and repellers we built for f_n are actually the integral tori of $E^s \oplus E^u$, which are transverse to S^1-fibers.

In the situation of Heisenberg nilmanifold $\mathcal{H}$, things becomes a little subtle. Since the absolute value of Euler number of $\mathcal{H}$ as a fibre bundle is large than the absolute value of Euler number of the base surface $\mathbb{T}^2$, Milnor-Wood inequality [35] shows that there do not exist either any closed surfaces or foliations transverse to the S^1-fibers.

This requires we that we find something else to substitute for them. That is the Birkhoff sections.

3.1 Definition and Half Helicoids

Definition 3.1.1. *A smooth embedded surface $\Sigma \hookrightarrow \mathcal{H}$ is called a Birkhoff section associated to S^1-fiber, if it satisfies*

- *The boundary of Σ consists of finitely many S^1-fibers:*

$$\partial\Sigma = S_{p_1} \cup S_{p_2} \cup \cdots \cup S_{p_k} \ ,$$

where $p_i \in \mathbb{T}^2$ and $S_{p_i} = \pi^{-1}(p_i)$, for $i = 1, \cdots, k$.

- *The interior of Σ is transverse to the S^1-fiber of $\mathcal{H}$:*

$$T_x\mathcal{H} = T_x\Sigma \oplus T_xS^1, \qquad \forall x \in \mathrm{Int}(\Sigma)\ .$$

Remark.

• *The name "Birkhoff sections" comes from G. Birkhoff who defined similar sections for the geodesic flows. The flows do not always admits global sections, but sometimes they have sections whose interior transverse to the flow and boundary to to be some periodic orbits. See [13] for the Birkhoff sections of the transitive Anosov flows. The role of flow lines is similar to our S^1-fibers here.*

• *From the definition of Birkhoff sections, we could see that the interior of Σ is a covering surface of $\mathbb{T}^2 \setminus \{p_1, \cdots, p_k\}$, so there exists $l > 0$ such that for any $p \in \Sigma_g \setminus \{p_1, \cdots, p_k\}$, the fiber S_p intersects Σ with exactly l points.*

• *The most well-known surface looks like a Birkhoff section is the half helicoid $\Sigma_H \subset \mathbb{R}^2 \times S^1$, which is given by the equations:*

$$\begin{cases} x = \rho \cdot \cos 2\pi\theta\ , \\ y = \rho \cdot \sin 2\pi\theta\ , \\ z = \theta \pmod 1\ . \end{cases}$$

Here $\theta \in \mathbb{R}$, and $\rho \geq 0$. We can see that the boundary of Σ_0 is $(0,0) \times S^1$, and its interior transverse to the the vector field $\partial/\partial z$, thus the S^1-fibers.

Before we give the examples of Birkhoff sections, we need to spend some time on the helicoid and its deformations, which will be our future model in a neighborhood of the boundary fibers of Birkhoff sections.

Since for the partially hyperbolic automorphism $f_A : \mathcal{H} \to \mathcal{H}$, the matrix $A \in GL(2, \mathbb{Z})$ is hyperbolic, so there exists a non-degenerate matrix P with $\det(P) > 0$ such that

$$P^{-1} \circ A \circ P \ = \ \begin{pmatrix} \det(A) \cdot \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}.$$

Here λ satisfies $|\lambda| > 1$. Then for the half helicoid $\Sigma_H \subset \mathbb{R}^2 \times S^1$, we consider the image $P \times \mathrm{Id}(\Sigma_H)$, which we will show admits the same boundary as Σ_H and whose interior is also transverse to S^1-fibers.

Actually, here P induces a smooth diffeomorphism on the unit circle $C_0 = \{(x, y) :\ x^2 + y^2 = 1\}$,

$$P :\ (x, y) \ \longmapsto \ \frac{P(x, y)}{\|P(x, y)\|}, \qquad \forall (x, y) \in C_0.$$

If we consider it in the polarizing coordinate, for $C_0 = \{(\rho, \theta) : \ \rho = 1, \theta \in \mathbb{R} \pmod 1\}$, P defines a diffeomorphism $\widetilde{p} : C_0 \to C_0$, where for any $\theta \in \mathbb{R} \pmod 1$,

$$P(\cos 2\pi\theta, \sin 2\pi\theta) \ = \ (\cos 2\pi\widetilde{p}(\theta), \sin 2\pi\widetilde{p}(\theta)) \ \in \ C_0 \ .$$

Moreover, since $\widetilde{p} : C_0 \to C_0$ is a diffeomorphism, so there exist some $\theta_0 \in (0, 1)$ satisfying $\widetilde{p}(\theta_0) = 1/2$.

Thus we can present the new surface $P \times \mathrm{Id}(\Sigma_H)$ by using the formula:

$$\begin{cases} x = \rho \cdot \cos(2\pi \cdot \widetilde{p}(\theta)) \ , \\ y = \rho \cdot \sin(2\pi \cdot \widetilde{p}(\theta)) \ , \\ z = \theta \pmod 1 \ . \end{cases}$$

And it can be easily checked that $\partial(P \times \mathrm{Id}(\Sigma_H)) = (0, 0) \times S^1$, and the interior is transverse to S^1-fibers of $\mathbb{R}^2 \times S^1$.

3.2 Examples of Birkhoff Sections

In this subsection, we give several examples of Birkhoff sections in $\mathcal{H}$ and show how to build them. Especially, we will introduce the affine Birkhoff sections, which will be our future candidates of attractors and repellers.

3.2.1 Section in $[0, 1]^3$

We first try to construct some surfaces in $[0, 1]^3$, which will be the basic stones and bricks for our future constructions.

Consider an imbedded surface $\Sigma_0 \subset [0, 1]^3$ satisfying the following properties:

1. The boundary $\partial\Sigma_0 = S_p \cup S_I \cup S_c$, where
   - $S_p = \{(\frac{1}{2}, \frac{1}{2})\} \times [0, 1]$,
   - $S_I = [0, \frac{1}{2}] \times \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$,
   - $S_c = \{0\} \times [0, \frac{1}{2}] \times \{0\} \ \cup \ \{0\} \times [\frac{1}{2}, 1] \times \{1\} \ \cup \ [0, 1] \times \{(0, 0), (1, 1)\}$
     $\cup \ \{(1, t, t) : \ t \in [0, 1]\}$.
2. The interior of Σ_0 is the image of a smooth function

   $$\phi : (0, 1) \times (0, 1) \setminus (0, \frac{1}{2}] \times \{\frac{1}{2}\} \ \longrightarrow \ (0, 1),$$

   which can extend smoothly to the boundary. Moreover, $\mathrm{Int}(\Sigma_0)$ is transverse to the z-axis, that is $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are bounded everywhere.

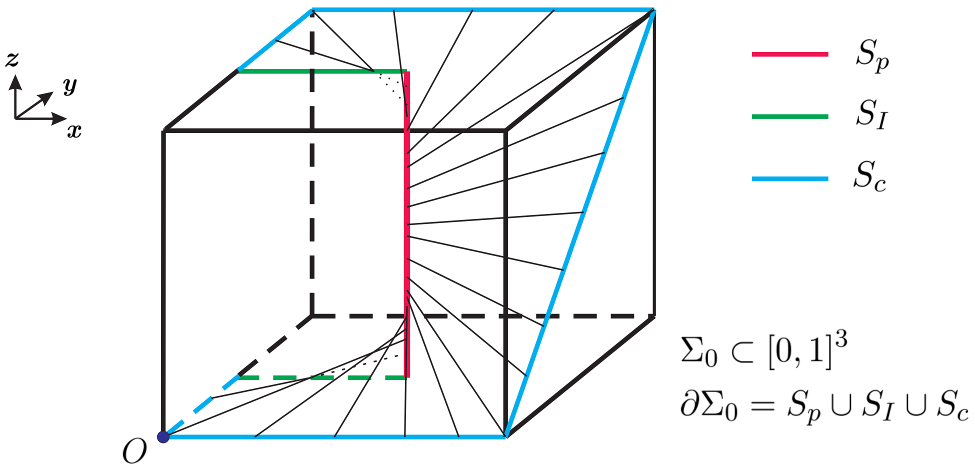

Figure 3.1: *Surface* Σ_0 *in* $[0,1]^3$

3. There exists $0 < \delta \ll 1/2$ such that Σ_0 restricted to

$$B(S_p, 2\delta) \triangleq \{(x,y) \in [0,1]^2 : \ d((x,y), (\frac{1}{2}, \frac{1}{2})) < 2\delta\} \times [0,1]$$

is the image of the helicoid under the action of $P \times Id$:

$$\begin{cases} x = \rho \cdot \cos[2\pi \cdot \widetilde{p}(\theta + \theta_0)] + 1/2, \\ y = \rho \cdot \sin[2\pi \cdot \widetilde{p}(\theta + \theta_0)] + 1/2, \\ z = \theta. \end{cases}$$

Here $\theta \in [0,1]$, and $0 \le \rho \le 2\delta$. Notice that $\widetilde{p}(\theta_0) = 1/2$, so close to the boundary, i.e. $\Sigma_0 \cap B(S_p, 2\delta)$ intersects $[0,1]^2 \times \{0,1\}$ with two segments $[\frac{1}{2} - 2\delta, \frac{1}{2}] \times \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$, which are contained in S_I.

4. Since Σ_0 is smoothly extended to its boundary, we can define the tangent space of Σ_0 on its boundary. Moreover, for any $(t, \frac{1}{2}, 0), (t, \frac{1}{2}, 1) \in S_I$ and $i, j \in \mathbb{N}$, we require

$$\lim_{(x,y,\phi(x,y)) \to (t,\frac{1}{2},0)} \frac{\partial \phi^{i+j}}{\partial x^i \partial y^j}(x,y) \ = \ \lim_{(x,y,\phi(x,y)) \to (t,\frac{1}{2},1)} \frac{\partial \phi^{i+j}}{\partial x^i \partial y^j}(x,y) \ .$$

5. For any point $(x,y,z) \in S_c$, we have

$$T_{(x,y,z)}\Sigma_0 \ = \ \langle \ \frac{\partial}{\partial x} \ , \ \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \ \rangle \ .$$

This tells us on the boundary part S_c, Σ_0 is tangent to the canonical contact plane field generated by X and Y. Obviously, this required that for points in S_I, we have

$$\lim_{t \to 0} T_{(t,\frac{1}{2},0)}\Sigma_0 \ = \ \lim_{t \to 0} T_{(t,\frac{1}{2},1)}\Sigma_0 \ = \ \langle \ \frac{\partial}{\partial x} \ , \ \frac{\partial}{\partial y} \ \rangle \ .$$

Remark. *It seems that our construction of Σ_0 is a little bit cumbersome. However, our future Birkhoff sections are fully relies on Σ_0, which will be achieved by gluing the image of Σ_0 by some affine maps, just like the small chambers of honeycomb. So it is worthy for us to describe it very carefully. From now on, when we talking about Σ_0, we refer to a fixed surface in $[0,1]^3$ which satisfying the above properties.*

3.2.2 Single Boundary Sections in $\mathcal{H}$

Now we can give the first example of Birkhoff sections in $\mathcal{H}$, denoted by Σ_1. Remember that $[0,1]^3$ is a fundamental domain of $\mathcal{H}$, and on the boundary of $[0,1]^3$, we have the identification $\sim$:

$$(x, y, z) \sim (x+1, y, z+y) \sim (x, y+1, z) \sim (x, y, z+1).$$

Lemma 3.2.1. *For the section $\Sigma_0 \hookrightarrow [0,1]^3$, $\Sigma_1 = \Sigma_0/\sim$ is a Birkhoff section with single boundary fiber in $\mathcal{H} = [0,1]^3/\sim$.*

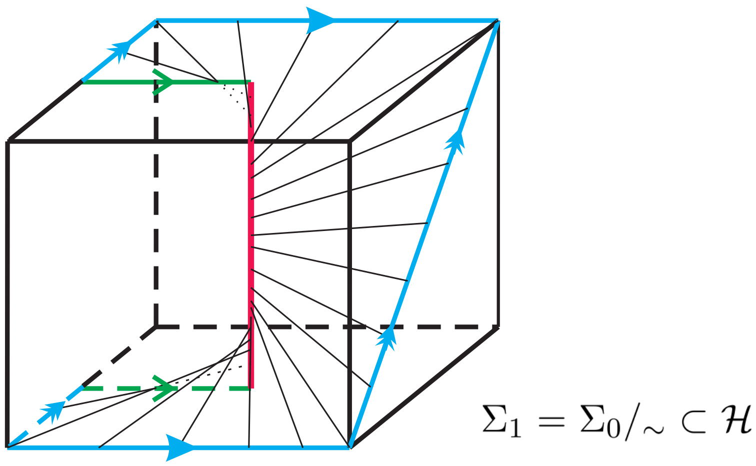

Figure 3.2: *Birkhoff Section with Single Boundary Fiber*

Proof. Considering $\Sigma_0/\sim$, we have two parts of identification. First for S_I, it can see that

$$(t, \frac{1}{2}, 0) \sim (t, \frac{1}{2}, 1), \qquad \forall t \in [0, \frac{1}{2}].$$

Since we already assumed in property 4 of Σ_0, that $T_{(t,\frac{1}{2},0)}\Sigma_0 = T_{(t,\frac{1}{2},1)}\Sigma_0$, so we can glue $[0, \frac{1}{2}] \times \{(\frac{1}{2}, 0)\}$ to $[0, \frac{1}{2}] \times \{(\frac{1}{2}, 1)\}$ smoothly. This implies $(S_I/\sim) \subseteq \mathrm{Int}(\Sigma_0/\sim)$.

Second consider the identification for S_c. We have

$$(0, t, 0) \sim (0, t, 1) \sim (1, t, t), \qquad \text{and} \qquad (t, 0, 0) \sim (t, 1, 1), \qquad \forall t \in [0, 1].$$

The smoothness of gluing comes from property 5 of Σ_0, that Σ_0 is tangent to $\langle X, Y\rangle$ when restricted on S_c, where $\langle X, Y\rangle$ is also a smooth plane field on $\mathcal{H}$.

Combining these two parts together, we get a smooth imbedded surface $\Sigma_1 = (\Sigma_0/\sim) \hookrightarrow \mathcal{H}$. The boundary of Σ_1 consists of $S_p/\sim = \pi^{-1}(\{(1/2, 1/2)\})$, which is a S^1-fiber. And the interior of Σ_1 is transverse to the S^1-fibers. This gives us the most simplest example of Birkhoff sections. $\square$

3.2.3 Multiple Boundaries Sections in $\mathcal{H}$

We are ready to construct some more complicated Birkhoff sections in $\mathcal{H}$. The way are somehow similar to the single boundary one. We plan to use affine maps to imbedding a lot of $[0, 1]^3$ into $\mathcal{H}$, and glue them together, which makes the image of all Σ_0 will be our Birkhoff sections with multiple boundary fibers.

Fix an integer $n_0 \in \mathbb{N}$, and denote

$$[\mathbb{Z}/n_0]^2 \cap \mathbb{T}^2 \ = \ \{(i/n_0, j/n_0) \in \mathbb{T}^2 : \ i, j \in \{0, 1, \cdots, n_0 - 1\}\}.$$

Lemma 3.2.2. *There exists a Birkhoff section $\Sigma_{n_0} \hookrightarrow \mathcal{H}$, such that*

$$\partial\Sigma_{n_0} \ = \ \pi^{-1}([\mathbb{Z}/n_0]^2 \cap \mathbb{T}^2).$$

Proof. We will need a new fundamental domain

$$\mathcal{H} \ = \ [-\frac{1}{2n_0}, \ 1 - \frac{1}{2n_0}]^2 \times [0, 1]/\sim .$$

First we define the two dimensional **skeleton** in $[-1/2n_0, \ 1 - 1/2n_0]^2$ as:

$$\mathbf{Sk_2} \ \triangleq \ \{(\frac{i}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} - \frac{1}{2n_0}) : \ i, j \in \{0, 1, \cdots, n_0\}\} \times [-\frac{1}{2n_0}, \ 1 - \frac{1}{2n_0}]$$
$$\bigcup \ [-\frac{1}{2n_0}, \ 1 - \frac{1}{2n_0}] \times \{(\frac{i}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} - \frac{1}{2n_0}) : \ i, j \in \{0, 1, \cdots, n_0\}\},$$

which cuts $[-1/2n_0, \ 1 - 1/2n_0]^2$ into n_0^2 small squares:

$$\{[\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}] : \ i, j \in \{0, 1, \cdots, n_0 - 1\}\}.$$

Then for each i, j, we try to cut

$$[\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}] \times S^1$$

into n_0^2 small cubes, and imbedding $[0, 1]^3$ inside, which satisfying simultaneously that all the images of Σ_0 can also glue smoothly. Define

$$\psi_{i,j} \ : \ [\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}] \ \longrightarrow \ [0, 1],$$

$$(x, y) \longmapsto \frac{i}{n_0} \cdot (y + \frac{1}{2n_0}).$$

We can get a cube

$$\Delta_{i,j,0} = \{\ (x,y,z):\ (x,y) \in [\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}],$$
$$\text{and } z \in [\psi_{i,j}(x,y), \psi_{i,j}(x,y) + \frac{1}{n_0^2}] (\text{mod } 1)\ \}.$$

Rotate $\Delta_{i,j,0}$ through the z-axis over $1/n_0^2$, we get a new cube $\Delta_{i,j,1}$, here notice that we may need modulo 1 if necessary. Repeat this process n_0^2-times, we cut $[\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}] \times S^1$ into n_0^2 small cubes. Thus we separate $\mathcal{H}$ into n_0^4 small cubes, and labeled by $i, j \in \{0, 1, \cdots, n_0 - 1\}$ and $k \in \{0, 1, \cdots, n_0^2 - 1\}$, which is defined as

$$\Delta_{i,j,k} = \{\ (x,y,z):\ (x,y) \in [\frac{i}{n_0} - \frac{1}{2n_0}, \frac{i}{n_0} + \frac{1}{2n_0}] \times [\frac{j}{n_0} - \frac{1}{2n_0}, \frac{j}{n_0} + \frac{1}{2n_0}],$$
$$\text{and } z \in [\psi_{i,j}(x,y) + \frac{k}{n_0^2}, \psi_{i,j}(x,y) + \frac{k+1}{n_0^2}] (\text{mod } 1)\ \}.$$

Now we try to define the affine map

$$\Psi_{i,j,k}\ :\ [0,1]^3 \longrightarrow \Delta_{i,j,k} \hookrightarrow \mathcal{H}.$$

As for any $(x, y, z) \in [0, 1]^3$,

$$\Psi_{i,j,k}(\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T) = \begin{pmatrix} 1/n_0 \cdot x + (2i-1)/2n_0 \\ 1/n_0 \cdot y + (2j-1)/2n_0 \\ 1/n_0^2 \cdot y + i/n_0 \cdot (y + 1/2n_0) + k/n_0^2 \ (\text{mod } 1) \end{pmatrix}^T.$$

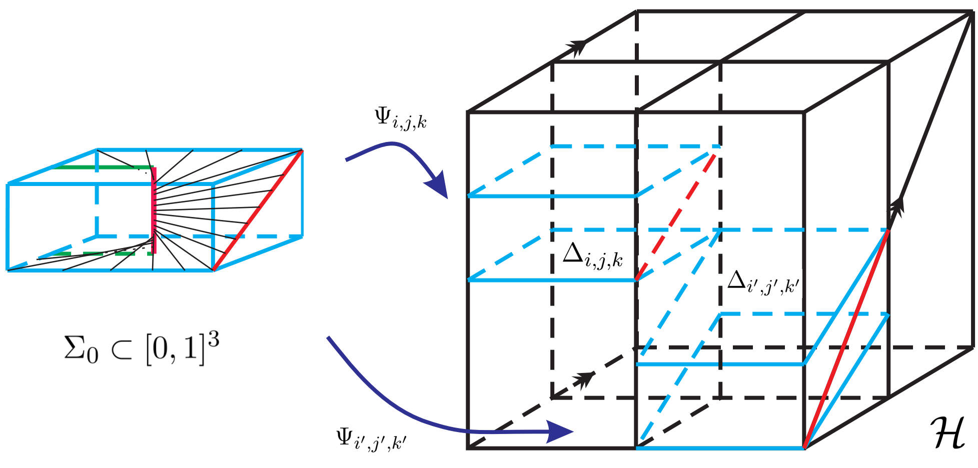

Figure 3.3: *Birkhoff Section with Multiple Boundaries*

Finally, we get the new Birkhoff sections Σ_{n_0} as follows:

$$\Sigma_{n_0} \;=\; \bigsqcup_{i,j,k} \Psi_{i,j,k}(\Sigma_0).$$

Furthermore, we denote the skeleton of Σ_{n_0} to be:

$$\mathbf{Sk}(\Sigma_{n_0}) \;=\; \bigsqcup_{i,j,k} \Psi_{i,j,k}(S_c(\Sigma_0)),$$

which one can easily check that $\mathbf{Sk}(\Sigma_{n_0}) = \Sigma_{n_0} \cap \pi^{-1}(\mathbf{Sk_2})$.

Here Σ_{n_0} is a Birkhoff section comes from the way we cut the small cube $\Delta_{i,j,k}$, the affine map $\Psi_{i,j,k}$, and the boundary properties of Σ_0 in $[0,1]^3$. All these guarantee that $\Psi_{i,j,k}(\Sigma_0)$ could be glued smoothly with the images of Σ_0 in the cubes surrounding it. And the only boundary part after gluing for Σ_{n_0} would be

$$\partial\Sigma_{n_0} \;=\; \pi^{-1}(\{(\frac{i}{n_0}, \frac{j}{n_0}) \in \mathbb{T}^2 :\ i,j \in \{0,1,\cdots,n_0-1\}\}).$$

The fiber transversal property comes from the affine map preserve the z-axis. Thus Σ_{n_0} is a Birkhoff section with n_0^2-fibers boundary.

□

3.2.4 Different Birkhoff Sections with Same Boundary

In the last subsection, we have construct a Birkhoff section Σ_{n_0} with n_0^2-fibers boundary. It is obviously that if we fix the section Σ_0 in $[0,1]^3$, then the new section just relies on the way how we cut $\mathcal{H}$ into small cubes. The rest is just imbedding $[0,1]^3$ into these cubes by affine maps, and check the images of all Σ_0s could be smoothly glued together to achieving the new Birkhoff section. The gluing procedures between different cubes are mostly at the skeleton of Σ_{n_0}. So we call the Birkhoff sections constructed by this way to be **affine Birkhoff sections**. Moreover, if a Birkhoff section Σ_{n_0} is affine, then its way for cutting $\mathcal{H}$ into small cubes is determined by $\mathbf{Sk}(\Sigma_{n_0})$, so does Σ_{n_0}.

We try to consider this construction in a different point of view. In $[-1/2n_0,\ 1-1/2n_0]^2$, which is a fundamental domain of $\mathbb{T}^2$, we consider two segments:

$$[-\frac{1}{2n_0},\ 1-\frac{1}{2n_0}] \times \{-\frac{1}{2n_0}\}, \qquad \text{and} \qquad \{-\frac{1}{2n_0}\} \times [-\frac{1}{2n_0},\ 1-\frac{1}{2n_0}].$$

Notice that in $\mathbb{T}^2$, they will be two simple closed curves, which form a generator of $\pi_1(\mathbb{T}^2)$. Denote these two curves by l_1 and l_2, and $\mathbb{T}_{l_i} = \pi^{-1}(l_i)$ is the torus consists of S^1-fibers for $i = 1, 2$. We can see that

$$\partial\Sigma_{n_0} \ \cap\ \mathbb{T}_{l_i} \;=\; \emptyset, \qquad i = 1, 2.$$

Under the coordinates of the fundamental domain $[-1/2n_0,\ 1-1/2n_0]^2\times[0,1]$, we could see that

$$\Sigma_{n_0}\ \cap\ \mathbb{T}_{l_1}\ =\ [-\frac{1}{2n_0},\ 1-\frac{1}{2n_0}]\times\{-\frac{1}{2n_0}\}\times\{\ 0,\frac{1}{n_0^2},\cdots,\frac{n_0^2-1}{n_0^2}\ \},$$

and

$$\Sigma_{n_0}\ \cap\ \mathbb{T}_{l_2}\ =\ \{-\frac{1}{2n_0}\}\times[-\frac{1}{2n_0},\ 1-\frac{1}{2n_0}]\times\{\ 0,\frac{1}{n_0^2},\cdots,\frac{n_0^2-1}{n_0^2}\ \}.$$

Then, our affine Birkhoff section Σ_{n_0} is determined by these two family of simple closed curves.

Actually, since we do not want to distinguish Σ_{n_0} to another Birkhoff section $R_\alpha(\Sigma_{n_0})$, which is rotate Σ_{n_0} along all the S^1-fibers with angle α. So we just need to remember $\Sigma_{n_0}\cap\mathbb{T}_{l_1}$ is tangent to the vector field X in $\mathcal{H}$, and $\Sigma_{n_0}\cap\mathbb{T}_{l_2}$ is tangent to the vector field $Y+\frac{1}{2n_0}Z$. Then we could see that $\Sigma_{n_0}\cap\pi^{-1}(\mathbf{Sk_2})$ are tangent to $\langle X, Y+\frac{1}{2n_0}Z\rangle$ everywhere. The last thing is guarantee that all these curves need to intersect appropriately on the S^1-fibers of lattice points in $\mathbf{Sk_2}$. That makes $\Sigma_{n_0}\cap\pi^{-1}(\mathbf{Sk_2})$ is still a cover of $\mathbf{Sk_2}$. This fixed the skeleton $\mathbf{Sk}(\Sigma_{n_0})$, thus Σ_{n_0}.

Lemma 3.2.3. *There exists infinitely many different affine Birkhoff sections, which admits the same boundary of Σ_{n_0}, and are not equal to the rotation of Σ_{n_0} along the S^1-fibers.*

Proof. We try to create the new Birkhoff section Σ'_{n_0} from the way stated above. The new one admits the same boundary and the same two dimensional skeleton $\mathbf{Sk_2}(\Sigma_{n_0})$ with Σ_{n_0}.

For any $(k_0,l_0)\neq(0,0)\in\mathbb{Z}^2$, choose two family of simple closed curves in $\mathbb{T}_{l_1}$ and $\mathbb{T}_{l_2}$ respectively. In $\mathbb{T}_{l_1}$, these curves intersect each S^1-fiber exactly n_0^2-points with equal distance $1/n_0^2$ one by one, and tangent to $X+\frac{k_0}{n_0^2}Z$. In $\mathbb{T}_{l_2}$, these curves also intersect each S^1-fiber exactly n_0^2-points with equal distance $1/n_0^2$ one by one, and tangent to $Y+(\frac{1}{2n_0}+\frac{l_0}{n_0^2})Z$. Then the same way extended the two families to the whole $\pi^{-1}(\mathbf{Sk_2}(\Sigma_{n_0}))$, which gives us a new skeleton. This allowed us to construct a new affine Birkhoff section Σ'_{n_0} which depends on two integers k_0 and l_0.

Notice here Σ'_{n_0} could not deformed from Σ_{n_0} by rotations. This is because for any $R_\alpha(\Sigma_{n_0})$, it intersection curves with $\mathbb{T}_{l_i}$ will have the same homology with $\Sigma_{n_0}\cap\mathbb{T}_{l_i}$. But this is impossible for Σ'_{n_0}.

□

3.3 Fiber Isotopy Class of Birkhoff Sections

We have showed that for some family of S^1-fibers, there are infinitely many different affine Birkhoff sections which admits them to be the boundary. Here the different we means they could not deform to each other by rotations along the S^1-fibers. But how could we define "different" for general Birkhoff sections? We need the following definition.

Definition 3.3.1. *For any two Birkhoff sections Σ and Σ' in $\mathcal{H}$, we say that they are* **fiber isotopic** *if there exists a family of diffeomorphisms $F_t : \mathcal{H} \to \mathcal{H}$, $t \in [0,1]$, which satisfying:*

- *F_t continuously depends on t, and $F_0 = \mathrm{Id}|_{\mathcal{H}}$.*
- *F_t preserve each S^1-fiber invariant: $F_t(S_p) = S_p$, for any $p \in \mathbb{T}^2$.*
- *$\Sigma' = F_1(\Sigma)$.*

Roughly speaking, Σ' is fiber isotopic to Σ if it can be deformed form Σ along the S^1-fibers of $\mathcal{H}$. We call F_t be a fiber isotopy function.

From the definition, we can see that for the affine Birkhoff sections, they are still different in the meaning of fiber isotopy. In this section, we will try to give the conditions when two Birkhoff sections are fiber isotopic. The most obvious one is they need to have the same boundary fibers.

3.3.1 Boundary Conditions

Now we will consider the local homology of a boundary fiber. Denote by

$$\partial\Sigma \ = \ \bigcup_{i=1}^{n} S_{p_i}, \qquad p_i \in \mathbb{T}^2.$$

There exists $0 < \varepsilon \ll 1$ such that for any $i \in \{1, \cdots, n\}$, the ε-neighborhood $B(p_i, \varepsilon)$ of p_i in $\mathbb{T}^2$ do not contain any p_j for $j \neq i$. Then we consider the local trivial bundle

$$D(S_{p_i}, \varepsilon) \stackrel{\triangle}{=} \pi^{-1}(B(p_i, \varepsilon)) = B(p_i, \varepsilon) \times S^1.$$

Since $D(S_{p_i}, \varepsilon)$ is a solid torus, for its boundary torus $\mathbb{T}_{p_i}$, there exists a unique homology element in $H_1(\mathbb{T}_{p_i}, \mathbb{Z})$ which representative closed curve could bound a disk in $D(S_{p_i}, \varepsilon)$, and also admits the positive orientation on $\mathbb{T}^2$ when projected down. We denote this homology element by $< med >$ which means the meridian direction.

On the other hand, we know that $\mathbb{T}_{p_i} = \pi^{-1}(\partial B(p_i, \varepsilon))$. So it could naturally define the S^1-fibers in $\mathbb{T}_{p_i}$ represent the longitude direction $< long >$ in $H_1(\mathbb{T}_{p_i}, \mathbb{Z})$. Here the orientation is the same as the fiber orientation.

Since we have assume that $D(S_{p_i}, \varepsilon)$ contains a single boundary fiber S_{p_i}, it implies $\mathbb{T}_{p_i} \cap \Sigma$ is a simple closed curve η_i, and if we further assume $\pi(\eta_i)$ has positive orientation in $\mathbb{T}^2$, then the homology of η_i could only to be

$$< \eta_i >= l \cdot < med > + < long >, \qquad \text{or} \qquad < \eta >= l \cdot < med > - < long > .$$

We define the corresponding local twisting number of the boundary S_{p_i} as

$$\tau(p_i, \Sigma) = 1/l, \qquad \text{or} \qquad \tau(p_i, \Sigma) = -1/l.$$

Lemma 3.3.2. *The sum of all the twisting number over all the boundary fibers of any Birkhoff section Σ is equal to the Euler number of the circle bundle $\mathcal{H}$:*

$$\sum_{i=1}^{k} \tau(p_i, \Sigma) = \chi(\mathcal{H}) = 1.$$

Remark. *For any 3-manifold which is an S^1-bundle over closed surface, we can similar define the Birkhoff sections and the local twisting number of boundary fibers. Then it also has the same formula holds. The most trivial way is that if the bundle is a trivial bundle, then we can find a Birkhoff section without boundary, that is a transversal surface. And of course the sum of twisting number is zero, equal to the Euler number of trivial bundles.*

Proof. We prove this lemma by induction. First we look at the case where the Birkhoff section Σ admits only one boundary fiber. From the definition of Euler number, if we consider $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ and a disk $\mathbb{D}^2 \subset \mathbb{T}^2$, and do the Dehn surgery(meridian direction to meridian plus fiber direction) on the solid torus $\mathbb{D}^2 \times S^1$, then we get $\mathcal{H}$. Thus if we consider a $\mathbb{T}^2 \subset \mathbb{T}^3$ which transverse to S^1-fibers and intersect on each fiber only once, then the Dehn surgery acting on this torus will give us the Birkhoff section with single boundary, and the local twisting number of boundary fiber must be 1.

Now we consider the case where all the local twisting number on the boundary fibers are positive. Assume that if Σ admit l-boundary fibers and all with positive local twisting number, then each no-boundary S^1-fiber intersects Σ at l-points, and all local twisting numbers are $1/l$.

For the Birkhoff section Σ admit $l+1$-boundary fibers and all with positive local twisting number, then it can be achieved by consider the union of two Birkhoff sections. One is with l-boundary fibers and all with twisting number $1/l$. The other one admits single boundary and twisting number 1. Then we consider the union of them and do the operations in [13] to get the Birkhoff section Σ. It can be shown that each interior fiber will intersect Σ with $l+1$ points, and all the local twisting number of boundary fibers are $1/(l+1)$.

For the case there exist some boundary fibers with negative twisting number $-1/l$, we need do some operation to demolish these negative ones.

Consider any disk $\mathbb{D}^2 \subset \mathbb{T}^2$, where $\mathbb{D}^2 \times S^1 \subset \mathcal{H}$ containing two boundary fibers of Σ, one admits positive twisting number, the other one is negative. Then $\partial(\mathbb{D}^2 \times S^1)$ must intersect Σ with l parallel circles which transverse to S^1-fibers. This allowed us to substitute $\Sigma \cap \mathbb{D}^2 \times S^1$ by l-disks which all transverse to S^1-fibers, and get a new Birkhoff section.

Repeating such procedures, we will get a new Birkhoff section Σ' without any boundary fibers with negative twisting numbers. During this procedures, the number of positive and negative boundary fibers that been demolished are equal. Since Σ' satisfies the equation in the lemma, so does Σ. This finishes the proof of the lemma.

□

Lemma 3.3.3. *If two Birkhoff sections Σ and Σ' are fiber isotopic, then they must have the same boundary fibers, and they admit the same local twisting number in each boundary fiber.*

Proof. The same boundary part is obvious. So assume that $\partial\Sigma = \partial\Sigma' = \bigcup_{i=1}^{n} S_{p_i}$. Since Σ is fiber isotopic to Σ', for each i, we can define the surrounding torus $\mathbb{T}_{p_i}$ as before. The intersecting curve $\eta_i = \Sigma \cap \mathbb{T}_{p_i}$ is also fiber isotopic $\eta_i' = \Sigma' \cap \mathbb{T}_{p_i}$, which means η_i is isotopic to η_i' in $\mathbb{T}_{p_i}$. So they must admit the same homology as

$$\tau(p_i, \Sigma) \;=\; \tau(p_i, \Sigma') \; .$$

□

We say that two Birkhoff sections admit the same boundary conditions, if they have the same boundary fibers, and their local twisting number are equal in each boundary fiber.

3.3.2 Global Conditions

In this subsection, we will give the necessary and sufficient conditions for two Birkhoff sections, which have the same boundary conditions, will be fiber isotopic.

Lemma 3.3.4. *Assume that Σ and Σ' are two Birkhoff sections have the same boundary conditions:*

$$\partial\Sigma = \partial\Sigma' = \bigcup_{i=1}^{n} S_{p_i} \; , \qquad \text{and} \qquad \tau(p_i, \Sigma) \;=\; \tau(p_i, \Sigma') \; .$$

Then Σ and Σ' are fiber isotopic, if and only if:

For any two simple closed curves $\gamma_i \subset \mathbb{T}^2$, $i = 1, 2$, which could generate $\pi_1(\mathbb{T}^2)$ and do not intersect $\pi(\partial\Sigma) = \{p_1, \cdots, p_n\}$, the simple closed curves contained in $\Sigma \cap \pi^{-1}(\gamma_i)$ have the same homology type with the curves contained in $\Sigma' \cap \pi^{-1}(\gamma_i)$, for $i = 1, 2$.

Remark. *Notice here both $\Sigma \cap \pi^{-1}(\gamma_i)$ may be consists of several simple closed curves. But these curves must be parallel, that is they have the same homology type. The same holds to $\Sigma' \cap \pi^{-1}(\gamma_i)$. The condition here is that all these curves need to define the same homology element in $H_1(\pi^{-1}(\gamma_i), \mathbb{Z})$.*

Proof. The "only if" part is exactly the same with lemma 3.3.3, just substitute the surrounding torus by $\mathbb{T}_{\gamma_i} = \pi^{-1}(\gamma_i)$.

On the other hand, since γ_1 and γ_2 could generate $\mathbb{T}^2$, choose them appropriately, we can assume

$$\mathbb{T}^2 \setminus (\gamma_1 \cup \gamma_2) \;=\; \bigcup \; \Delta_j \; ,$$

where the number of Δ_j is finite, and each Δ_j is a contractible open region contained in $\mathbb{T}^2$. This means

$$\pi^{-1}(\Delta_j) \;=\; \Delta_j \times S^1$$

is a trivial bundle.

Claim. *There exists a global isotopy function between* Σ *and* Σ' *when restricted on* $\pi^{-1}(\gamma_1 \cup \gamma_2)$.

Proof of the Claim. Fix a point $p \in \gamma_1 \cap \gamma_2$ and the fiber S_p. Choose two points $y \in \Sigma \cap S_p$ and $y' \in \Sigma \cap S_p$. The intersecting curves in $\Sigma \cap \pi^{-1}(\gamma_i)$ have the same homology type with $\Sigma' \cap \pi^{-1}(\gamma_i)$, for $i = 1, 2$, implies there exists a unique fiber isotopy function

$$F_t^i \;:\; \pi^{-1}(\gamma_i) \times [0,1] \;\longrightarrow\; \pi^{-1}(\gamma_i), \qquad i = 1, 2,$$

such that

- $F_0^i \;=\; \mathrm{Id}|_{\pi^{-1}(\gamma_i)}$;
- $F_1^i(\Sigma \cap \pi^{-1}(\gamma_i)) \;=\; \Sigma' \cap \pi^{-1}(\gamma_i)$;
- $F_1^i(y) \;=\; y'$.

Here "unique" means for any $x \in \Sigma$, $F_1^i(x) \in \Sigma'$ has been uniquely determined.

If $\gamma_1 \cap \gamma_2 = \{p\}$, then modify the isotopy function of F_t^i on a small neighborhood of the fiber S_p, such that $F_t^1|_{S_p} \equiv F_t^2|_{S_p}$. Here we can do this modification since we just care about the F_1^i-image of the points contained in $\Sigma \cap S_p$. The property $F_1^1(y) = F_1^2(y) = y'$ guarantee that for any point $z \in \Sigma \cap S_p$, we have $F_1^1(z) = F_1^2(z) \in \Sigma' \cap S_p$. Then we just define the isotopy function on $\pi^{-1}(\gamma_1 \cup \gamma_2)$ as the union of F_t^1 and F_t^2, and we are done.

Otherwise, for some $q \in \gamma_1 \cap \gamma_2 \setminus \{p\}$, we need to verify that the isotopy functions F_t^1 and F_t^2 are coincide when restricted on the fiber S_q. However, here we just need to show that for any $z \in \Sigma \cap S_q$, it must have $F_1^1(z) = F_1^2(z) \in \Sigma' \cap S_q$. Then modify $F_t^1|_{t\in(0,1)}$ and $F_t^2|_{t\in(0,1)}$ on a neighborhood of S_q can guarantee that they coincide on S_q.

To prove this, we can assume that both p and q are in the boundary of Δ_j, and we can separate $\partial\Delta_j$ into two segments $\sigma_i \subset \gamma_i$ for $i = 1, 2$, both with end points p and q. Then there exists fixed curves

$$\widetilde{\sigma}_i \subset \Sigma \cap \pi^{-1}(\sigma_i), \qquad \text{and} \qquad \widetilde{\sigma}'_i \subset \Sigma' \cap \pi^{-1}(\sigma_i), \qquad i = 1, 2,$$

where $y \in \widetilde{\sigma}_i$ is an endpoint of $\widetilde{\sigma}_i$, and $y' \in \widetilde{\sigma}'_i$ is an endpoint of $\widetilde{\sigma}'_i$ for $i = 1, 2$.

If $\Delta_j \cap \partial\Sigma = \Delta_j \cap \partial\Sigma' = \emptyset$, then the other endpoint (not y) of $\widetilde{\sigma}_1$ coincides with the other endpoint (not y) of $\widetilde{\sigma}_2$. The same is true to $\widetilde{\sigma}'_i$, i.e. the other endpoint (not y') of $\widetilde{\sigma}'_1$ coincides with the other endpoint (not y') of $\widetilde{\sigma}'_2$.

Otherwise Σ and Σ' admit the same boundary conditions, this implies the number of points contained in $S_q \cap \Sigma$ between the other two endpoints (not y) of $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$, is equal to the number of points contained in $S_q \cap \Sigma'$ between the other two endpoints (not y') of $\widetilde{\sigma}'_1$ and $\widetilde{\sigma}'_2$, Figure 3.4. This number plus 1 is equal to the number of boundary fibers of Σ (also Σ') with positive local twisting number minus the number of boundary fibers of Σ (also Σ') with negative local twisting number.

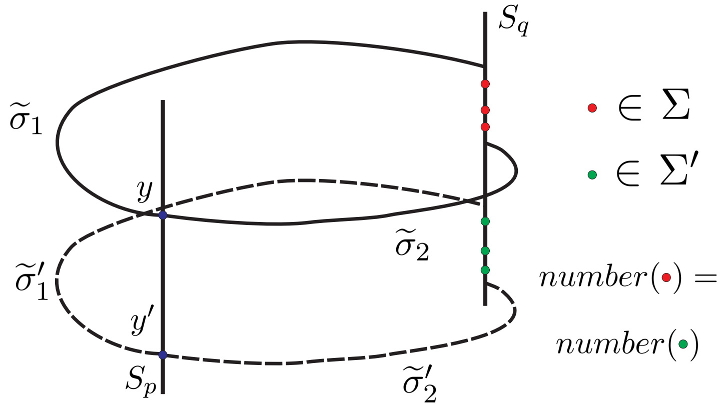

Figure 3.4: *The red points are the points contained in $S_q \cap \Sigma$ between the other two endpoints (not y) of $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$; the green points are the points contained in $S_q \cap \Sigma'$ between the other two endpoints (not y') of $\widetilde{\sigma}'_1$ and $\widetilde{\sigma}'_2$.*

Since F^i_1 maps the endpoints of $\widetilde{\sigma}_i$ to the endpoints of $\widetilde{\sigma}'_i$ for $i = 1, 2$, it must have F^1_1 also maps the endpoints of $\widetilde{\sigma}_2$ to the endpoints of $\widetilde{\sigma}'_2$. The reverse is also true. This implies that we have

$$F^1_1|_{\Sigma \cap S_q} \equiv F^2_1|_{\Sigma \cap S_q} .$$

This finishes on the fiber S_q.

Then we repeat this procedure to all the points contained in $\gamma_1 \cap \gamma_2$. Here the set $\gamma_1 \cap \gamma_2$ is a finite set since we can assume that γ_1 intersects γ_2 transversely and they are smooth. Thus we could define the isotopy functions satisfying

$$F^1_t|_{\pi^{-1}(\gamma_1 \cap \gamma_2)} \equiv F^2_t|_{\pi^{-1}(\gamma_1 \cap \gamma_2)} , \qquad \forall t \in [0, 1] .$$

Finally we can define a global isotopy function between Σ and Σ' is equal to the union of F^1_t and F^2_t when restricted to $\pi^{-1}(\gamma_1 \cup \gamma_2)$. This finishes the proof of the claim.

□

For any Δ_j, we have defined the isotopy function on $\pi^{-1}(\partial \Delta_j)$. Applying the contractibility of Δ_j, this fiber isotopy function could extended to the whole $\pi^{-1}(\Delta_j)$. Glue all these fiber isotopy functions restricted on $\pi^{-1}(\Delta_j)$ for all j together, we get a fiber isotopy function defined on $\mathcal{H}$. This proves that Σ and Σ' are fiber isotopic to each other.

□

3.3.3 Global Twisting

Definition 3.3.5. *A Birkhoff section $\Sigma \hookrightarrow \mathcal{H}$ is called an equidistant Birkhoff section if for any S^1-fiber $S_q \subset \mathcal{H} \setminus \partial\Sigma$, $S_q \setminus \Sigma$ are the union of finitely many intervals with equal lengths.*

Remark. *From the definition of Birkhoff sections, we know that if $\partial\Sigma = S_{p_1} \cup S_{p_2} \cup \cdots \cup S_{p_k}$, then $\mathrm{Int}(\Sigma)$ is an l-cover of $\mathbb{T}^2 \setminus \{p_1, \cdots, p_k\}$ for some $l \in \mathbb{N}$. So if Σ is equidistant, then Σ cuts S_q into l intervals with length $1/l$.*

Lemma 3.3.6. *We have the following simple facts:*

- *Any Birkhoff section Σ could be fiber isotopic to some equidistant Birkhoff section.*
- *The affine Birkhoff sections are all equidistant.*
- *The image of an equidistant Birkhoff section by a partially hyperbolic automorphism is also an equidistant Birkhoff section.*

We lift a Birkhoff section $\Sigma \subset \mathcal{H}$ to a surface $\widetilde{\Sigma} \subset \mathbb{H}^3$. If $\partial\Sigma = S_{p_1} \cup S_{p_2} \cup \cdots \cup S_{p_k}$, and for simplicity also denote $\{p_1, \cdots, p_k\} \subset [0,1) \times [0,1)$ which is a fundamental domain of $\mathbb{T}^2$, we can easily see that

$$\partial\widetilde{\Sigma} = (\{p_1, \cdots, p_k\} + \mathbb{Z}^2) \times \{z : \ z \in \mathbb{R}\}.$$

And of course $\pi(\partial\widetilde{\Sigma}) = \{p_1, \cdots, p_k\} + \mathbb{Z}^2 \subset \mathbb{R}^2$.

So for some $\widetilde{p} = p_i + (m,n) \in \pi(\partial\widetilde{\Sigma})$, where $(m,n) \in \mathbb{Z}^2$, we can also define the local twisting number

$$\tau(\widetilde{p}, \widetilde{\Sigma}) = \tau(p_i, \Sigma).$$

Now if Σ is equidistant and

$$\{(x_0, y_0, z) : \ z \in \mathbb{R}\} \cap \partial\widetilde{\Sigma} = \emptyset,$$

then $\widetilde{\Sigma}$ cuts $\{(x_0, y_0, z) : \ z \in \mathbb{R}\}$ into infinitely many intervals all with length $1/l$.

Now we can state a lemma, which shows the twist of curves in $\widetilde{\Sigma}$. This lemma looks quite similar to lemma 2.3.1, and perfectly explained where the name local twisting number of boundary fibers came from.

Lemma 3.3.7. *For any piecewise smooth curve $\widetilde{\gamma} : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma})$, which satisfying*

- *$\gamma = \pi \circ \widetilde{\gamma}$ is a positive oriented simple closed curve in $\mathbb{R}^2$, which bounds a region $\mathbb{D}_\gamma$.*
- *π is a injective on $\widetilde{\gamma}((0,1))$, and $\pi \circ \widetilde{\gamma}(0) = \pi \circ \widetilde{\gamma}(1)$.*

If denote by $\widetilde{\gamma}(0) = (x_0, y_0, z_0)$ and $\widetilde{\gamma}(1) = (x_0, y_0, z_1)$, then the twisting height

$$z_1 \ - \ z_0 \ = \sum_{\widetilde{p} \in \mathbb{D}_\gamma} \tau(\widetilde{p}, \widetilde{\Sigma}) \ .$$

Remark. *The proof of this lemma is quite simple, just recall the definition of local twisting number for boundary fibers. From this lemma, we can see that if we lift a simple closed curve in the base space to the equidistant Birkhoff sections, its twisting height depends on the boundary fibers it bounds.*

Notice this lemma is quite similar to lemma 2.3.1, both concerning lifting some simple closed curve to $\mathbb{H}$*, but one is tangent to* $E^s \oplus E^u$*, the other is contained in some Birkhoff section. And it shows our idea that use the Birkhoff sections to approximate the contact structure.*

Chapter 4

Invariant Birkhoff Sections

In this chapter, we will show the existence of invariant Birkhoff sections associated to a partially hyperbolic automorphism f_A, and give the estimations of their tangent plane fields. These Birkhoff sections will be our candidates of attractors and repellers for our structurally stable hyperbolic diffeomorphisms.

As we promised before, it will see that such invariant Birkhoff sections will approximate the invariant contact structure $E^s \oplus E^u$ of f_A. This is the key fact that we needed for the estimation of the C^1-distance of our perturbations.

First we define the invariant Birkhoff sections.

Definition 4.0.8. *Let f_A be a partially hyperbolic automorphism on $\mathcal{H}$. We call a Birkhoff section Σ is fiber isotopic invariant by f_A, if $f_A(\Sigma)$ is fiber isotopic to Σ. For shortly, we call Σ is invariant by f_A.*

Recall that for a fixed partially hyperbolic automorphism $f_A \in \mathbf{Aut}(\mathcal{H})$, where $A \in GL(2,\mathbb{Z})$ is hyperbolic, we denote

$$m = \det(A - \det(A) \cdot I) \ \in \ \mathbb{Z} \setminus \{0\}.$$

Then for the corresponding partially hyperbolic splitting $T\mathcal{H} = E^s \oplus E^c \oplus E^u$ of f_A, we know that E^c is tangent to the S^1-fibers of $\mathcal{H}$, and

$$E^s \oplus E^u \ = \ \langle\ X + \frac{k}{2m} \cdot Z,\ Y + \frac{l}{2m} \cdot Z\ \rangle,$$

here $k, l \in \mathbb{Z}$ are fixed integers. For any $\delta > 0$, we denote $B(\partial\Sigma, \delta) \subset \Sigma$ the set of points which is contained in the δ-neighborhood of $\partial\Sigma$.

Theorem 4.0.9. *There exists a sequence of affine Birkhoff section $\{\Sigma_n\}_{n>1}$, such that:*

- $\partial\Sigma_n = \pi^{-1}([\mathbb{Z}/(2m)^n]^2 \cap \mathbb{T}^2)$, *and on each boundary fiber, the local twisting number is* $1/(2m)^{2n}$.

- Σ_n *is f_A invariant, i.e. $f_A(\Sigma_n)$ is fiber isotopic to Σ_n.*

- *For the tangent plane of* Σ_n*, we have*

$$\lim_{n\to\infty} \max_{x\in\Sigma\setminus B(\partial\Sigma_n,\frac{1}{n\cdot(2m)^n})} \measuredangle\ (\ T_x\Sigma_n\ ,\ E^s(x)\oplus E^u(x)\ )\ =\ 0\ .$$

Remark. *The third item of this theorem means that for any* x *which is not too close to the boundary of* Σ_n*,* $T_x\Sigma_n$ *uniformly converge to* $E^s(x)\oplus E^u(x)$*. Moreover, from the affine point of view, here* x *could be chose more and more close to the boundary fibers.*

The proof of the first two items of this theorem is in theorem 4.3.1, the estimation in the third item is proved in lemma 4.4.2.

4.1 Homology Invariants

In lemma 3.3.4, we have showed that the fiber isotopic class of Birkhoff sections with fixed boundary conditions, is determined by the homology type of the intersecting curves of the Birkhoff sections with two vertical tori which are not homotopic.

Now we will consider the case where the boundary conditions of Birkhoff sections are described as theorem 4.0.9. That is we consider affine Birkhoff section Σ_n, with

$$\partial\Sigma_n = \pi^{-1}([\mathbb{Z}/(2m)^n]^2\cap\mathbb{T}^2)\ , \qquad \text{and} \qquad \tau(p,\Sigma_n) = \frac{1}{(2m)^{2n}}\ ,$$

for any $p\in\pi(\partial\Sigma_n)$.

For describe the homology type of Birkhoff sections intersect with some vertical torus, we need to introduce some invariants that are helpful for our future computations. In $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, we denote

$$\gamma_1 : S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{T}^2 \ \text{ with } \ \gamma(t) = (t,0)\in\mathbb{T}^2\ ,$$
$$\gamma_2 : S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{T}^2 \ \text{ with } \ \gamma(t) = (0,t)\in\mathbb{T}^2\ ,$$

are two simple closed curve which generate $\pi_1(\mathbb{T}^2)=H_1(\mathbb{T}^2,\mathbb{Z})$. We can see that their homology form a basis of $\mathbb{Z}^2\cong H_1(\mathbb{T}^2,\mathbb{Z})$.

Consider $\gamma : S^1 \longrightarrow \mathbb{T}^2\setminus\pi(\partial\Sigma_n)$ is a simple closed curve with the homology type

$$<\gamma>\ =\ p_1\cdot<\gamma_1>\ +\ p_2\cdot<\gamma_2>\ ,$$

where $p_i = p_i(\gamma)\in\mathbb{Z}$, and p_1, p_2 are coprime since γ is simple closed. The assumption that $\gamma\cap\pi(\partial\Sigma_n)=\emptyset$, implies that Σ_n intersects $\mathbb{T}_\gamma=\pi^{-1}(\gamma)$ with a union of finitely many parallel simple closed curves.

Since we have assumed that Σ_n is an affine Birkhoff section, thus it is equidistant. We still denote it lifts to $\widetilde{\Sigma}_n$ in the universal cover $\mathbb{H}$. Notice that for $\gamma \subset \mathbb{T}^2$, it will have infinitely many different lifts in $\mathbb{R}^2$. Choose $\overline{\gamma}$ be one of these lifts with $\overline{\gamma}(0) = (x_0, y_0) \in \mathbb{R}^2$, then we must have $\overline{\gamma}(1) = (x_0 + p_1, y_0 + p_2) \in \mathbb{R}^2$.

Then we consider the segments contained in

$$\pi^{-1}(\overline{\gamma}([0,1])) \ \cap \ \widetilde{\Sigma}_n \ = \ \overline{\gamma}([0,1]) \times \mathbb{R} \ \cap \ \widetilde{\Sigma}_n \ .$$

It could be checked that if we use the coordinates $\mathbb{H} = \mathbb{R}^3$, this intersection can be formulated as

$$\overline{\gamma}([0,1]) \times \mathbb{R} \cap \widetilde{\Sigma}_n \ = \ \{ \ (\overline{\gamma}(t), z(t) + \frac{k}{(2m)^{2n}}) \in \mathbb{R}^3 : \ t \in [0,1], k \in \mathbb{Z} \ \}.$$

Here $z(t)$ is a smooth function from $[0,1]$ to $\mathbb{R}$. Moreover, for each $k \in \mathbb{Z}$,

$$\{ \ (\overline{\gamma}(t), z(t) + \frac{k}{(2m)^{2n}}) \in \mathbb{R}^3 : \ t \in [0,1] \ \}$$

is a connect component of $\overline{\gamma}([0,1]) \times \mathbb{R} \cap \widetilde{\Sigma}_n$.

Lemma 4.1.1. *There exists some integer $k_n \in \mathbb{Z}$ which decided only by Σ_n and $\gamma(0) = (x_0, y_0)$, such that*

$$z(1) \ - \ z(0) \ = \ p_1 \cdot y_0 \ + \ \frac{k_n}{(2m)^{2n}} \ .$$

Moreover, the homology of the curves contained in $\mathbb{T}_\gamma \cap \Sigma_n$ is uniquely determined by the integer k_n.

Proof. Notice that the two vertical lines $(x_0, y_0) \times \mathbb{R}$ and $(x_0 + p_1, y_0 + p_2) \times \mathbb{R}$ will be projected into the same S^1-fibers in $\mathcal{H}$. So Σ_n will intersect this fiber with exactly $(2m)^{2n}$-points with mutually distance $1/(2m)^{2n}$. And both two points $(x_0, y_0, z(0))$ and $(x_0 + p_1, y_0 + p_2, z(1))$ will be projected into two of these $(2m)^{2n}$-points. By the equivalent relationship that define $\mathcal{H}$ from $\mathbb{R}^3$, we get some integer k_n satisfies the equation in the lemma. Notice that here the term $p_1 \cdot y_0$ comes from the geometry of Heisenberg group, where the equivalence relationship in $\mathbb{R}^3$ is $(x_0, y_0, z(0)) \sim (x_0 + p_1, y_0 + p_2, z(0) + p_1 \cdot y_0)$.

To prove that k_n is the invariant for deciding the homology of intersecting curves in $\mathbb{T}_\gamma \cap \Sigma_n$, we just need to fix a basis in $H_1(\mathbb{T}_\gamma, \mathbb{Z})$. The longitude direction we still choose the fiber circles as $< long >$. For the meridian direction, we consider the segment in $\overline{\gamma}([0,1]) \times \mathbb{R}$ which homeomorphic to $\overline{\gamma}([0,1])$ by projection π, and connecting two points $(x_0, y_0, z(0))$, $(x_0 + p_1, y_0 + p_2, z(0) + p_1 \cdot y_0)$, which are the same point in $\mathcal{H}$. Then this segment will define a simple closed curve in $\mathbb{T}_\gamma$, and its homology is independent of $< long >$. We denote its homology by $< med >$. We want to point out that here the choice of $< med >$ depends on (x_0, y_0).

In $H_1(\mathbb{T}_\gamma, \mathbb{Z})$, we can check that the homology of the curves contained in $\mathbb{T}_\gamma \cap \Sigma_n$ is

$$\frac{(2m)^{2n}}{((2m)^{2n}, |k_n|)} \cdot < med > \ + \ \frac{k_n}{((2m)^{2n}, |k_n|)} \cdot < long > \ ,$$

where $((2m)^{2n}, |k_n|)$ is the biggest common factor of $(2m)^{2n}$ and $|k_n|$. Thus we can see that k_n determines the homology of the simple closed curves contained in $\mathbb{T}_\gamma \cap \Sigma_n$.

$\square$

Remark. *Notice that this lemma just require the Birkhoff section Σ_n is equidistant. And the difference of $z(1) - z(0)$ just depends on two things, one is the homology of curves in $\mathbb{T}_\gamma \cap \Sigma_n$; the other is the homology of γ and starting point $\gamma(0)$.*

In this lemma, we can see that the integer k_n depends both on the fiber isotopy class of Σ_n, and the choice of starting point $\gamma(0) = (x_0, y_0) \in \mathbb{R}^2$. So if we fix the Birkhoff section Σ_n, then we can view the integer $k_n = k_n(x_0, y_0)$ is a continuous function defined on the lifting of γ in $\mathbb{R}^2$, since we can choose any point in the lifting set as the starting point of $\overline{\gamma}$.

However, if we lift the simple closed curve γ to $\mathbb{R}^2$, its universal cover are infinitely many parallel infinite curves in $\mathbb{R}^2$. More precisely, we have denote $\overline{\gamma} : [0,1] \longrightarrow \mathbb{R}^2$ is one path curve of the lift of γ with $\overline{\gamma}(1) = (x_0 + p_1, y_0 + p_2)$, then all the lift set of γ in $\mathbb{R}^2$ could be represented as

$$\begin{aligned} &\bigcup_{r\in\mathbb{Z}} \bigcup_{q\in\mathbb{Z}} \{\overline{\gamma}(t) + (p_1 q, p_2 q) + (r, 0) \in \mathbb{R}^2 :\ t \in [0,1]\} \ , \qquad \text{if } p_2 \neq 0 \ ; \\ &\bigcup_{r\in\mathbb{Z}} \bigcup_{q\in\mathbb{Z}} \{\overline{\gamma}(t) + (p_1 q, p_2 q) + (0, r) \in \mathbb{R}^2 :\ t \in [0,1]\} \ , \qquad \text{if } p_1 \neq 0 \ . \end{aligned}$$

Since $\overline{\gamma}(1) = (x_0 + p_1, y_0 + p_2)$, we know that for any fixed $r \in \mathbb{Z}$, the set

$$\begin{aligned} &\bigcup_{q\in\mathbb{Z}} \{\overline{\gamma}(t) + (p_1 q, p_2 q) + (r, 0) \in \mathbb{R}^2 :\ t \in [0,1]\} \ , \qquad \text{if } p_2 \neq 0 \ ; \\ &\bigcup_{q\in\mathbb{Z}} \{\overline{\gamma}(t) + (p_1 q, p_2 q) + (0, r) \in \mathbb{R}^2 :\ t \in [0,1]\} \ , \qquad \text{if } p_1 \neq 0 \ . \end{aligned}$$

is one connected component of the lifting of γ in $\mathbb{R}^2$. By the continuity, we can see that k_n is a constant integer in each connected components.

For the case where the simple closed curves are canonical generator of $H_1(\mathbb{T}^2, \mathbb{Z})$, we can get more accurate estimation of the central difference by applying the boundary properties of Σ_n.

Lemma 4.1.2. *Consider a curve $\widetilde{\gamma}_0 : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ which projects down on $\mathbb{R}^2$ as:*

$$\pi \circ \widetilde{\gamma}_0 : [0,1] \longrightarrow \mathbb{R}^2 \setminus [\mathbb{Z}/(2m)^n]^2, \qquad \text{with} \qquad \pi \circ \widetilde{\gamma}_0(t) = (x_0 + t, y_0),$$

and we denote $\pi^c \circ \widetilde{\gamma}_0(1) - \pi^c \circ \widetilde{\gamma}_0(0) = y_0 + k_n/(2m)^{2n}$. Then for any parallel curves $\widetilde{\gamma}_1 : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ with $\pi \circ \widetilde{\gamma}_1(t) = (x_1 + t, y_1)$, we have

- *if the interval between y_0 and y_1 in $\mathbb{R}$ does not intersect $\mathbb{Z}/(2m)^n$, it will admit*

$$\pi^c \circ \widetilde{\gamma}_1(1) - \pi^c \circ \widetilde{\gamma}_1(0) \ = \ y_1 + \frac{k_n}{(2m)^{2n}} \ ;$$

- *if $y_1 = y_0 + q_1/(2m)^n$ for some integer $q_1 \in \mathbb{Z}$, it will admit*

$$\pi^c \circ \widetilde{\gamma}_1(1) - \pi^c \circ \widetilde{\gamma}_1(0) \;=\; y_0 + \frac{k_n}{(2m)^{2n}} \;=\; y_1 + \frac{k_n - q_1 \cdot (2m)^n}{(2m)^{2n}} \;.$$

Proof. We choose a curve $\sigma_0 : [0,1] \longrightarrow \mathbb{R}^2 \setminus [\mathbb{Z}/(2m)^n]^2$, which connect $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ as its endpoints. Then the curve $\sigma_1 : [0,1] \longrightarrow \mathbb{R}^2 \setminus [\mathbb{Z}/(2m)^n]^2$ which defined as $\sigma_1(t) = \sigma_0(t) + (1,0)$ will admit $(x_0 + 1, y_0)$ and $(x_1 + 1, y_1)$ as its endpoints.

We first assume that $y_0 \leq y_1$. Then we can see that the curve $\pi \circ \widetilde{\gamma}_0$, σ_1, $-\pi \circ \widetilde{\gamma}_1$, $-\sigma_0$ bound a closed region in $\mathbb{R}^2$. Moreover, the number of the boundary fibers of $\widetilde{\Sigma}_n$ contained in this region is equal to

$$(2m)^n \times \sharp\{\mathbb{Z}/(2m)^n \cap (y_0, y_1) \subset \mathbb{R}\} \;.$$

Notice that in the first item, $\sharp\{\mathbb{Z}/(2m)^n \cap (y_0, y_1) \subset \mathbb{R}\}$ is zero; in the second item, it is equal to q_1.

Finally, $\pi^{-1}(\sigma_0) \cap \widetilde{\Sigma}_n$ and $\pi^{-1}(\sigma_1) \cap \widetilde{\Sigma}_n$ would be projected into the same set in $\mathcal{H}$, and we apply lemma 3.3.7 to get this lemma. The case $y_0 > y_1$ is the same.

$\square$

Similarly, we also have these properties for the curves generating another canonical element in $H_1(\mathbb{T}^2, \mathbb{Z})$.

Lemma 4.1.3. *Consider a curve $\widetilde{\gamma}_2 : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ which projects down on $\mathbb{R}^2$ as:*

$$\pi \circ \widetilde{\gamma}_2 : [0,1] \longrightarrow \mathbb{R}^2 \setminus [\mathbb{Z}/(2m)^n]^2, \qquad \text{with} \qquad \pi \circ \widetilde{\gamma}_2(t) = (x_2, y_2 + t),$$

and we denote $\pi^c \circ \widetilde{\gamma}_2(1) - \pi^c \circ \widetilde{\gamma}_2(0) = l_n/(2m)^{2n}$. Then for any parallel curves $\widetilde{\gamma}_3 : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ with $\pi \circ \widetilde{\gamma}_3(t) = (x_3, y_3 + t)$, we have

- *if the interval between x_2 and x_3 in $\mathbb{R}$ does not intersect $\mathbb{Z}/(2m)^n$, it will admit*

$$\pi^c \circ \widetilde{\gamma}_3(1) - \pi^c \circ \widetilde{\gamma}_3(0) \;=\; \frac{l_n}{(2m)^{2n}} \;;$$

- *if $x_3 = x_2 + q_2/(2m)^n$ for some integer $q_2 \in \mathbb{Z}$, it will admit*

$$\pi^c \circ \widetilde{\gamma}_3(1) - \pi^c \circ \widetilde{\gamma}_3(0) \;=\; \frac{l_n + q_2 \cdot (2m)^n}{(2m)^{2n}} \;.$$

4.2 Homology Equations of Invariant Sections

Since we have fixed the boundary properties of Σ_n, thus from lemma 3.3.4, the fiber isotopy class of Σ_n is determined by the homology of the intersecting curves of Σ_n with two vertical tori, which do not intersect $\partial\Sigma_n$ and projected into two generator of $H_1(\mathbb{T}^2, \mathbb{Z})$.

Our plan for proving the existence of invariant Birkhoff sections is, first choose two simple closed curves $\gamma_1, \gamma_2 \subset \mathbb{T}^2 \setminus \pi(\partial\Sigma_n)$, and see the homology of curves in $\Sigma_n \cap \mathbb{T}_{\gamma_1}, \Sigma_n \cap \mathbb{T}_{\gamma_2}$. Then we calculate the homology of curves in $\Sigma_n \cap \mathbb{T}_{A\gamma_1}$ and $\Sigma_n \cap \mathbb{T}_{A\gamma_2}$. Finally we need to show that there exists some homology type of curves in $\Sigma_n \cap \mathbb{T}_{\gamma_1}, \Sigma_n \cap \mathbb{T}_{\gamma_2}$, such that the homology of curves in $\Sigma_n \cap \mathbb{T}_{A\gamma_1}, \Sigma_n \cap \mathbb{T}_{A\gamma_2}$ are equal to the homology of curves in $f_A(\Sigma_n \cap \mathbb{T}_{\gamma_1}), f_A(\Sigma_n \cap \mathbb{T}_{\gamma_2})$. The Birkhoff section Σ_n decided by this homology is an invariant Birkhoff section.

All our calculation will use the homology invariants introduced in last section, which will be helpful for our future estimations about the tangent plane field of Σ_n.

We consider two curves $\widetilde{\gamma}_{n,1}, \widetilde{\gamma}_{n,2} : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ such that

$$\begin{aligned} \gamma_{n,1}(t) &= \pi \circ \widetilde{\gamma}_{n,1}(t) = (t , \frac{1}{2(2m)^n}) \in \mathbb{R}^2 , \\ \gamma_{n,2}(t) &= \pi \circ \widetilde{\gamma}_{n,2}(t) = (\frac{1}{2(2m)^n} , t) \in \mathbb{R}^2 . \end{aligned}$$

Moreover, we assume that

$$\begin{aligned} \pi^c \circ \widetilde{\gamma}_{n,1}(1) - \pi^c \circ \widetilde{\gamma}_{n,1}(0) &= \frac{k_n}{(2m)^{2n}} + \frac{1}{2(2m)^n} ; \\ \pi^c \circ \widetilde{\gamma}_{n,2}(1) - \pi^c \circ \widetilde{\gamma}_{n,2}(0) &= \frac{l_n}{(2m)^{2n}} . \end{aligned}$$

As we explained before and lemma 4.1.1, the two integers k_n, l_n decided the fiber isotopy class of Σ_n.

On the other hand, give any two integers k_n, l_n, as section 3.2.4, we have showed there exists an affine Birkhoff section Σ_n admitting the boundary property we named, and satisfies these two equations.

Now we need to calculate for fixed k_n, l_n, what is the homology invariants of the curves contained in the intersections of $A\gamma_{n,1} \times \mathbb{R}$ and $A\gamma_{n,2} \times \mathbb{R}$ with $\widetilde{\Sigma}_n$.

We could see that the two curves $A\gamma_{n,1}, A\gamma_{n,2} : [0,1] \longrightarrow \mathbb{R}^2 \setminus \pi(\partial\widetilde{\Sigma}_n)$ could be expressed as

$$\begin{aligned} A\gamma_{n,1}(t) &= (\frac{b}{2(2m)^n} + a \cdot t , \frac{d}{2(2m)^n} + c \cdot t) , \\ A\gamma_{n,2}(t) &= (\frac{a}{2(2m)^n} + b \cdot t , \frac{c}{2(2m)^n} + d \cdot t) . \end{aligned}$$

And we denote $A\widetilde{\gamma}_{n,i} : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ to be the curve which is one connected component of $A\gamma_{n,i} \times \mathbb{R} \cap \widetilde{\Sigma}_n$, such that for $i = 1, 2$:

$$\pi \circ A\widetilde{\gamma}_{n,i}(t) = A\gamma_{n,i}(t) \in \mathbb{R}^2 \setminus \pi(\partial\widetilde{\Sigma}_n) , \qquad \forall t \in [0,1].$$

Since the two curves $A\gamma_{n,1}$ and $A\gamma_{n,2}$ project on $\mathbb{T}^2$ are two simple closed curves, we could see that there exists two integers k'_n, l'_n such that

$$\pi^c \circ A\widetilde{\gamma}_{n,1}(1) \;-\; \pi^c \circ A\widetilde{\gamma}_{n,1}(0) \;=\; a \cdot \frac{d}{2(2m)^n} \;+\; \frac{k'_n}{(2m)^{2n}} \;,$$
$$\pi^c \circ A\widetilde{\gamma}_{n,2}(1) \;-\; \pi^c \circ A\widetilde{\gamma}_{n,2}(0) \;=\; b \cdot \frac{c}{2(2m)^n} \;+\; \frac{l'_n}{(2m)^{2n}} \;.$$

Now we will try to calculate k'_n, l'_n from k_n, l_n. That is the following lemma.

Lemma 4.2.1. *There exists two sequence of integers $\{\iota_{n,1}\}$ and $\{\iota_{n,2}\}$, which all admitting m as a factor and satisfying*

$$\lim_{n\to\infty} \frac{\iota_{n,1}}{(2m)^{2n}} \;=\; \lim_{n\to\infty} \frac{\iota_{n,2}}{(2m)^{2n}} \;=\; 0 \;,$$

such that, the two integers k'_n and l'_n could be given by the following equations:

$$\frac{k'_n}{(2m)^{2n}} \;+\; \mathrm{Sign}(ac) \cdot \frac{ac}{2} \;+\; \frac{\iota_{n,1}}{(2m)^{2n}} \;=\; a \cdot \frac{k_n}{(2m)^{2n}} \;+\; c \cdot \frac{l_n}{(2m)^{2n}} \;+\; ac \;;$$
$$\frac{l'_n}{(2m)^{2n}} \;+\; \mathrm{Sign}(bd) \cdot \frac{bd}{2} \;+\; \frac{\iota_{n,2}}{(2m)^{2n}} \;=\; b \cdot \frac{k_n}{(2m)^{2n}} \;+\; d \cdot \frac{l_n}{(2m)^{2n}} \;+\; bd \;.$$

Proof. This lemma is possible since we have showed that the fiber isotopic class of Σ_n is determined by k_n and l_n, which make that calculate k'_n and l'_n is possible. We will give a complete proof of first formula, the second one is the same.

For the first formula, there are two cases:

Case I. For the matrix A, $a = 0$. Since $A \in GL(2, \mathbb{Z})$, we must have $|b| = |c| = 1$. In this case, we just need to apply lemma 4.1.3, then we get

$$\pi^c \circ A\widetilde{\gamma}_{n,1}(1) - \pi^c \circ A\widetilde{\gamma}_{n,1}(0) \;=\; c \cdot \frac{l_n}{(2m)^{2n}} \;+\; \frac{b-1}{2} \cdot \frac{(2m)^n}{(2m)^{2n}} \;.$$

This implies we get the equation

$$\frac{k'_n}{(2m)^{2n}} \;+\; \frac{1-b}{2} \cdot \frac{(2m)^n}{(2m)^{2n}} \;=\; c \cdot \frac{l_n}{(2m)^{2n}} \;.$$

In this case, we can easily check that it satisfies the formula in the lemma, where $\iota_{n,1} = (1 - b)(2m)^n/2$.

Case II. For the matrix A, $a \neq 0$. This case is a little complicated, we mainly need to applying lemma 3.3.7 for calculation.

The equation of strict line containing the segment $A\widetilde{\gamma}_{n,1}$ in $\mathbb{R}^2$ is

$$y \;=\; \frac{c}{a} \cdot \left(x - \frac{b}{2(2m)^n}\right) \;+\; \frac{d}{2(2m)^n} \;.$$

We fix an irrational number $0 < r_1 \ll 1$, then this strict line will intersect with the line $y = r_1/(2m)^n$ at the point $(\frac{2ar_1-1}{2c(2m)^n}, \frac{r_1}{(2m)^n})$. Notice that the matrix $A \in SL(2,\mathbb{Z})$ is hyperbolic guarantees that $b, c \neq 0$.

We consider the compact region $\Delta_{n,1}$ bounded by three lines in $\mathbb{R}^2$:

$$y = \frac{c}{a} \cdot (x - \frac{b}{2(2m)^n}) + \frac{d}{2(2m)^n} \ , \quad y = \frac{r_1}{(2m)^n} \ , \text{ and } \ x = \frac{2ar_1 - 1}{2c(2m)^n} + a \ .$$

We will find a curve in $\mathrm{Int}(\widetilde{\Sigma}_n)$ which will project down as the boundary of $\Delta_{n,1}$, then try to use lemma 3.3.7 and the homology invariants k_n, l_n to give the formula of the invariant k'_n of $A\widetilde{\gamma}_{n,1}$.

First we have the following lemma for showing the number of boundary fibers in $\Delta_{n,1}$. Notice that if we choose the irrational number r_1 small enough, then the number of $\mathbb{Z}^2 \cap \Delta_{n,1}$ is a fixed integer does not depend on n.

Lemma 4.2.2. *If we denote $T_1 = \sharp\{\mathbb{Z}^2 \cap \Delta_{n,1}\}$, then we must have $0 \le T_1 \le ac$. Moreover, we have*

$$T_{n,1} \ \overset{\triangle}{=} \ \sharp\{[\mathbb{Z}/(2m)^n]^2 \cap \Delta_{n,1}\} \ = \ T_1 \cdot (2m)^n + \frac{1}{2}ac \cdot (2m)^n[(2m)^n - 1] \ .$$

Remark. *Notice that here $T_{n,1}$ admits m as a factor. Moreover, we have*

$$\lim_{n\to\infty} \frac{T_{n,1}}{(2m)^{2n}} \ = \ \frac{ac}{2} \ = \ \mathrm{Area}(\Delta_{n,1}) \ .$$

To calculate the homology invariant of $\widetilde{\Sigma}_n$ restricted on the line $x = \frac{2ar_1-1}{2c(2m)^n} + a$, it needs to apply lemma 4.1.2. First we need to calculate the number of points contained in $\mathbb{Z}/(2m)^{2n}$ intersecting with interval between $\frac{1}{2(2m)^n}$ and $\frac{2ar_1-1}{2c(2m)^n} + a$. Since we have required that r_1 is small enough, this number must equal to an integer $a(2m)^n + u_1$, where $|u_1| \le 1$.

Now we apply lemma 3.3.7, which consider a piecewise smooth curve contained in $\widetilde{\Sigma_n}$ and projets down as the boundary of $\Delta_{n,1}$. The local twisting property of Birkhoff section gives us the following equation:

$$\frac{k'_n}{(2m)^{2n}} \ + \ \mathrm{Sign}(ac) \cdot \frac{T_{n,1}}{(2m)^{2n}} \ = \ a \cdot \frac{k_n}{(2m)^{2n}} \ + \ c \cdot \frac{l_n + [a \cdot (2m)^n + u_1](2m)^n}{(2m)^{2n}} \ .$$

Here $T_{n,1}$ is the number of boundary fibers contained in the region $\Delta_{n,1}$, and $l_n + [a \cdot (2m)^n + u_1](2m)^n$ is the homology invariant of Σ_n restricted on the line $x = \frac{2ar_1-1}{2c(2m)^n} + a$.

Since $T_{n,1}$ admit m as a factor, and $\lim_{n\to\infty} T_{n,1}/(2m)^{2n} = ac/2$, so the integer

$$\iota_{n,1} = \mathrm{Sign}(a \cdot c) \cdot [\ T'_{n,1} - m \cdot ac(2m)^{n-1}] - c \cdot u_1(2m)^n$$

will admit m as a factor, and $\lim_{n\to\infty} \iota_{n,1}/(2m)^{2n} = 0$. Moreover, we have the equation

$$\frac{k'_n}{(2m)^{2n}} \ + \ \mathrm{Sign}(ac) \cdot \frac{ac}{2} \ + \ \frac{\iota_{n,1}}{(2m)^{2n}} \ = \ a \cdot \frac{k_n}{(2m)^{2n}} \ + \ c \cdot \frac{l_n}{(2m)^{2n}} \ + \ ac \ .$$

This finishes the proof of the first formula.

Similarly, we can get a formula for l'_n, this finishes the proof of this lemma.

□

4.3 Existence of Invariant Sections

Now we can prove the first part of theorem 4.0.9, which shows the existence of invariant Birkhoff sections.

Theorem 4.3.1. *There exists a sequence of affine Birkhoff section $\{\Sigma_n\}_n$, where $n > 1$, which satisfying:*

- $\partial\Sigma_n = \pi^{-1}([\mathbb{Z}/(2m)^n]^2 \cap \mathbb{T}^2)$, *and on each boundary fiber, the local twisting number is* $1/(2m)^{2n}$.
- Σ_n *is* f_A *invariant, i.e.* $f_A(\Sigma_n)$ *is fiber isotopic to* Σ_n.

Proof. By the invariance of the lattice $[\mathbb{Z}/(2m)^n]^2 \cap \mathbb{T}^2$ under the action of A on $\mathbb{T}^2$. Thus we need to show that the Birkhoff section $f_A(\Sigma_n)$ admits the same homology invariants associated to Σ_n. Lifted on the universal cover $\mathbb{R}^3$, we denote $\widetilde{f}_A$ the lift of f_A.

Consider the intersection $A\gamma_{n,i} \times \mathbb{R} \cap \widetilde{f}_A(\widetilde{\Sigma}_n)$, we could see that Σ_n is an invariant Birkhoff section if and only if

$$\begin{aligned}
\pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,1}(1)) \;-\; \pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,1}(0)) \;&=\; \pi^c \circ A\widetilde{\gamma}_{n,1}(1) \;-\; \pi^c \circ A\widetilde{\gamma}_{n,1}(0)\ ;\\
\pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,2}(1)) \;-\; \pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,2}(0)) \;&=\; \pi^c \circ A\widetilde{\gamma}_{n,2}(1) \;-\; \pi^c \circ A\widetilde{\gamma}_{n,2}(0)\ .
\end{aligned}$$

In these two equations, the left side are the homology invariants of $f_A(\Sigma_n)$ restricted on $A\gamma_{n,i}\times\mathbb{R}$ for $i = 1, 2$, and the right side are the corresponding homology invariants of Σ_n.

Notice that when we restricted on $\mathcal{H}$, f_A maps the vertical torus $\pi^{-1}(\gamma_{n,i})$ into $\pi^{-1}(A\gamma_{n,i})$. So it must map the simple closed curves contained in $\pi^{-1}(\gamma_{n,i})$ into simple closed curves contained in $\pi^{-1}(A\gamma_{n,i})$. In other words, this observation is equivalent to $\widetilde{f}_A(\Gamma) = \Gamma$.

Moreover, f_A restricted on each S^1 fibers are isometries. If $\det(A) = 1$, then they are rotations; otherwise, they are the combinations of rotations and reflections. Thus we have the following claim:

Claim. *There exists two integers $K_{n,1}$ and $K_{n,2}$ such that*

$$\begin{aligned}
\pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,1}(1)) \;-\; \pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,1}(0)) \;&=\; a\cdot\frac{d}{2(2m)^n} \;+\; K_{n,1} \;+\; \det(A)\cdot\frac{k_n}{(2m)^{2n}}\ ;\\
\pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,2}(1)) \;-\; \pi^c \circ \widetilde{f}_A(\widetilde{\gamma}_{n,2}(0)) \;&=\; b\cdot\frac{c}{2(2m)^n} \;+\; K_{n,2} \;+\; \det(A)\cdot\frac{l_n}{(2m)^{2n}}\ .
\end{aligned}$$

Moreover, when n is large enough, these two integers $K_{n,1}, K_{n,2}$ do not depend on n.

Proof of the Claim. First we notice that the two points $(0, \frac{1}{2(2m)^n}, 0)$, $(1, \frac{1}{2(2m)^n}, \frac{1}{2(2m)^n})$ in $\mathbb{H}$ will be the same points in $\mathcal{H}$. This implies their $\widetilde{f}_A$-images will also be projected in the same point in $\mathcal{H}$. Thus from lemma 4.1.1, we must have

$$\pi^c \circ \widetilde{f}_A(1, \frac{1}{2(2m)^n}, \frac{1}{2(2m)^n}) - \pi^c \circ \widetilde{f}_A(0, \frac{1}{2(2m)^n}, 0) \ = \ a \cdot \frac{d}{2(2m)^n} + K_{n,1}$$

holds for some integer $K_{n,1}$.

Since we know that $\widetilde{f}_A$ restricted on the central direction would be isometry, i.e. it preserve orientation if $\det(A) = 1$; otherwise, it reverse the orientation. So from the assumption that

$$\pi^c \circ \widetilde{\gamma}_{n,1}(1) \ - \ \pi^c \circ \widetilde{\gamma}_{n,1}(0) \ = \ \frac{k_n}{(2m)^{2n}} + \frac{1}{2(2m)^n} \ ,$$

we get the first equation in the claim.

For $K_{n,1}$ will be constant when n large enough, we just need to notice that

$$(0, \frac{1}{2(2m)^n}, 0) \longrightarrow (0,0,0) \qquad \text{and} \qquad (1, \frac{1}{2(2m)^n}, \frac{1}{2(2m)^n}) \longrightarrow (1,0,0)$$

as $n \to \infty$. So from the continuity of $\widetilde{f}_A$ and $K_{n,1}$ would be integer, we know that they will be constant when n large enough.

The proof of second equality and $K_{n,2}$ is exactly the same. □

This implies that there exists some invariant Birkhoff section Σ_n admitting the boundary property we assumed before, if and only if there exists two integers k_n and l_n satisfying the following equations

$$\begin{pmatrix} K_{n,1} \\ K_{n,2} \end{pmatrix} + \frac{\det(A)}{(2m)^{2n}} \cdot \begin{pmatrix} k_n \\ l_n \end{pmatrix} \ = \ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} k_n/(2m)^{2n} \\ l_n/(2m)^{2n} \end{pmatrix} + \begin{pmatrix} m \cdot L_{n,1}/(2m)^{2n} \\ m \cdot L_{n,2}/(2m)^{2n} \end{pmatrix} .$$

Here $L_{n,1}, L_{n,2}$ are two integers, and the equations are equivalent to

$$\begin{pmatrix} a - \det(A) & c \\ b & d - \det(A) \end{pmatrix} \cdot \begin{pmatrix} k_n \\ l_n \end{pmatrix} \ = \ \begin{pmatrix} K_{n,1} \cdot (2m)^{2n} - m \cdot L_{n,1} \\ K_{n,2} \cdot (2m)^{2n} - m \cdot L_{n,2} \end{pmatrix} .$$

Notice that we have assumed that $|\det(A^T - \det(A) \cdot I)| = m$, so this implies there exists two integer k_n and l_n satisfies this equation, and we get an invariant Birkhoff section Σ_n.

□

Remark. *Notice that* $|\det(A^T - \det(A) \cdot I)| \neq 0$ *implies we can always solve some rational numbers satisfies this equation. But if the solution are not integers, then do not get the imbedded Birkhoff sections, but the immersed surfaces. For example, when* $m > 1$*, then there does not exist any invariant Birkhoff sections with single boundary fiber. That is the reason that we need to choose the boundary fibers very carefully.*

The following corollary state the properties of the curves contained in the invariant Birkhoff sections, which is crucial for our future estimations of tangent plane fields of invariant Birkhoff sections. It is a direct consequence of lemma 4.2.1 and the claim contained in the proof of theorem 4.3.1.

Corollary 4.3.2. *If Σ_n is an invariant Birkhoff section, and consider two curves $\widetilde{\gamma}_{n,1}, \widetilde{\gamma}_{n,2} : [0,1] \longrightarrow \mathrm{Int}(\widetilde{\Sigma}_n)$ where*

$$\gamma_{n,1}(t) = \pi \circ \widetilde{\gamma}_{n,1}(t) = (t, \frac{1}{2(2m)^n}) \in \mathbb{R}^2 , \qquad \gamma_{n,2}(t) = \pi \circ \widetilde{\gamma}_{n,2}(t) = (\frac{1}{2(2m)^n}, t) \in \mathbb{R}^2 .$$

Then the endpoints of these two curves must satisfy the following equations:

$$\det(A) \cdot \begin{pmatrix} \pi^c \circ \widetilde{\gamma}_{n,1}(1) - \pi^c \circ \widetilde{\gamma}_{n,1}(0) \\ \pi^c \circ \widetilde{\gamma}_{n,2}(1) - \pi^c \circ \widetilde{\gamma}_{n,2}(0) \end{pmatrix} + \begin{pmatrix} \mathrm{Sign}(ac) \cdot ac/2 \\ \mathrm{Sign}(bd) \cdot bd/2 \end{pmatrix} + \begin{pmatrix} K_{n,1} \\ K_{n,2} \end{pmatrix}$$
$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} \pi^c \circ \widetilde{\gamma}_{n,1}(1) - \pi^c \circ \widetilde{\gamma}_{n,1}(0) \\ \pi^c \circ \widetilde{\gamma}_{n,2}(1) - \pi^c \circ \widetilde{\gamma}_{n,2}(0) \end{pmatrix} + \begin{pmatrix} ac \\ bd \end{pmatrix} + \begin{pmatrix} \iota'_{n,1}/(2m)^{2n} \\ \iota'_{n,2}/(2m)^{2n} \end{pmatrix}$$

Here the two integers $\iota'_{n,1}$ and $\iota'_{n,2}$ satisfy $\lim_{n\to\infty} \iota'_{n,1}/(2m)^{2n} = \lim_{n\to\infty} \iota'_{n,2}/(2m)^{2n} = 0$.

4.4 Estimation of Tangent Spaces

The rest of our task is to get the estimation of the tangent plane field of the invariant Birkhoff section. We first show that for the sequence of affine invariant Birkhoff sections we proved in theorem 4.3.1, their tangent plane field restricted on the skeleton will uniformly converge to $E^s \oplus E^u$.

Lemma 4.4.1. *For the affine invariant Birkhoff sections Σ_n in theorem 4.3.1, they will satisfy*

$$\lim_{n\to\infty} \max_{x \in \mathbf{Sk}(\Sigma_n)} \measuredangle \left(T_x\Sigma_n \ , \ E^s(x) \oplus E^u(x) \right) = 0 .$$

Proof. From the definition of affine Birkhoff sections, to estimate the tangent plane of Σ_n at the skeleton $\mathbf{Sk}(\Sigma_n)$, we just need to see that at the two curves $\widetilde{\gamma}_{n,1}$ and $\widetilde{\gamma}_{n,2}$, how their tangent line field close to $E^s \oplus E^u$.

If we consider two curves $\widehat{\gamma}_{n,i} : [0,1] \longrightarrow \mathbb{R}^3$, $i = 1, 2$, which satisfying for any $t \in [0,1]$:

- $\pi \circ \widehat{\gamma}_{n,i}(t) = \gamma_{n,i}(t)$;
- $\widehat{\gamma}'_{n,i}(t) \in E^s(\widehat{\gamma}_{n,i}(t)) \oplus E^u(\widehat{\gamma}_{n,i}(t))$.

Then by the contact property of $E^s \oplus E^u$ which is preserved by Df_A, and f_A is an isometry on the central direction, these two curves must satisfy

$$\begin{pmatrix} \pi^c \circ f_A(\widehat{\gamma}_{n,1}(1)) - \pi^c \circ f_A(\widehat{\gamma}_{n,1}(0)) \\ \pi^c \circ f_A(\widehat{\gamma}_{n,2}(1)) - \pi^c \circ f_A(\widehat{\gamma}_{n,2}(0)) \end{pmatrix} + \begin{pmatrix} \mathrm{Sign}(ac) \cdot ac/2 \\ \mathrm{Sign}(bd) \cdot bd/2 \end{pmatrix} =$$
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} \pi^c \circ \widehat{\gamma}_{n,1}(1) - \pi^c \circ \widehat{\gamma}_{n,1}(0) \\ \pi^c \circ \widehat{\gamma}_{n,2}(1) - \pi^c \circ \widehat{\gamma}_{n,2}(0) \end{pmatrix} + \begin{pmatrix} ac \\ bd \end{pmatrix} .$$

Actually, recall we have denote $E^s \oplus E^u = \langle\ X + \frac{k}{2m} \cdot Z,\ Y + \frac{l}{2m} \cdot Z\ \rangle$, then

$$\begin{aligned}
\pi^c \circ \widehat{\gamma}_{n,1}(1) \ - \ \pi^c \circ \widehat{\gamma}_{n,1}(0) \ &= \ \frac{k}{2m} \ + \ \frac{1}{2(2m)^n} \ ; \\
\pi^c \circ \widehat{\gamma}_{n,2}(1) \ - \ \pi^c \circ \widehat{\gamma}_{n,2}(0) \ &= \ \frac{l}{2m} \ .
\end{aligned}$$

Recall that the curve which is tangent to the contact structure also admits the local twisting property (lemma 2.3.1). Notice that the curves $f_A(\widehat{\gamma}_{n,1})$ and $f_A(\widehat{\gamma}_{n,2})$ are also tangent to the contact plane field. So we can formulate the equations like corollary 4.3.2, which the only difference is the local twisting term of the equations for Birkhoff sections are the sum of local twisting numbers, but for the curves tangent contact plane field are the area of bounded regions. This shows that two integers $k/2m$ and $l/2m$ satisfy

$$\det(A) \cdot \begin{pmatrix} k/2m \\ l/2m \end{pmatrix} + \begin{pmatrix} \mathrm{Sign}(ac) \cdot ac/2 \\ \mathrm{Sign}(bd) \cdot bd/2 \end{pmatrix} + \begin{pmatrix} K_{n,1} \\ K_{n,2} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} k/2m \\ l/2m \end{pmatrix} + \begin{pmatrix} ac \\ bd \end{pmatrix} \ .$$

Notice that here we proved again that $K_{n,1}$ and $K_{n,2}$ are constant integers.

Comparing with the formula in corollary 4.3.2, let $n \to \infty$, since $\lim_{n\to\infty} \iota'_{n,1}/(2m)^{2n} = \lim_{n\to\infty} \iota'_{n,2}/(2m)^{2n} = 0$, we know that

$$\begin{aligned}
\lim_{n\to\infty} \pi^c \circ \widetilde{\gamma}_{n,1}(1) - \pi^c \circ \widetilde{\gamma}_{n,1}(0) \ &= \ \lim_{n\to\infty} k_n/(2m)^{2n} \ = \ k/2m \ , \\
\lim_{n\to\infty} \pi^c \circ \widetilde{\gamma}_{n,2}(1) - \pi^c \circ \widetilde{\gamma}_{n,2}(0) \ &= \ \lim_{n\to\infty} l_n/(2m)^{2n} \ = \ l/2m \ .
\end{aligned}$$

This convergence guarantees that we can construct Σ_n satisfying the tangent line field of $\pi^{-1}(\gamma_{n,i}) \cap \Sigma_n$ will converge to $E^s \oplus E^u|_{\pi^{-1}(\gamma_{n,i})}$, for $i = 1, 2$. Form the affine property of Σ_n, we get

$$\lim_{n\to\infty} \ \max_{x\in \mathbf{Sk}(\Sigma_n)} \ \measuredangle\ (\ T_x\Sigma_n\ ,\ E^s(x) \oplus E^u(x)\ ) \ = \ 0 \ .$$

$\square$

The next lemma shows that the estimation of tangent plane fields on the skeleton can be extended to almost the whole Birkhoff section.

Lemma 4.4.2. *If the sequence of Birkhoff sections Σ_n satisfies*

$$\lim_{n\to\infty} \ \max_{p\in \mathbf{Sk}(\Sigma_n)} \ \measuredangle\ (\ T_p\Sigma_n\ ,\ E^s(p) \oplus E^u(p)\ ) \ = \ 0 \ .$$

Then it must admit

$$\lim_{n\to\infty} \ \max_{q\in \Sigma\setminus B(\partial\Sigma_n, \frac{1}{n\cdot(2m)^n})} \ \measuredangle\ (\ T_q\Sigma_n\ ,\ E^s(q) \oplus E^u(q)\ ) \ = \ 0 \ .$$

Proof. Recall that in our definition of the imbedded surface $\Sigma_0 \hookrightarrow [0,1]^3$, we can see that there exists some constant L_0, such that for any $q' = (x,y,z) \in \Sigma_0$ satisfying

$$d((x,y),(\frac{1}{2},\frac{1}{2})) \;\geq\; \frac{1}{n} \;,$$

it will admit

$$\measuredangle\;(\;T_{q'}\Sigma_0\;,\;\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle\;)\;<\;n\cdot L_0\;.$$

This is from the property that close to $(\frac{1}{2},\frac{1}{2}) \times [0,1]$, Σ_0 is a linear transformation of a helicoid.

Now we consider the construction of affine Birkhoff section Σ_n. Recall that there exists a family of affine maps

$$\Psi^n_{i,j,k}\;:\;[0,1]^3\;\longrightarrow\;\Delta^n_{i,j,k}\;\hookrightarrow\;\mathcal{H}\;,$$

where $i,j \in \{0,1,\cdots,(2m)^n - 1\}$ and $k \in \{0,1,\cdots,(2m)^{2n}-1\}$, such that

$$\Sigma_n\;=\;\bigsqcup_{i,j,k}\Psi_{i,j,k}(\Sigma_0)\;.$$

Here the small cube $\Delta^n_{i,j,k}$ is determined by the skeleton $\mathbf{Sk}(\Sigma_n)$ and i,j,k.

The assumption that the tangent space of Σ_n restricted on $\mathbf{Sk}(\Sigma_n)$ will converge to $E^s \oplus E^u$ implies we have

$$\lim_{n\to\infty}\measuredangle\;(\;D\Psi^n_{i,j,k}(\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle)\;,\;E^s\oplus E^u\;)\;=\;0\;.$$

On the other hand, notice that the affine map $\Psi^n_{i,j,k}$ compress much more strong along the $\partial/\partial z$ direction, which implies

$$\measuredangle\;(\;D\Psi^n_{i,j,k}(T_{q'}\Sigma_0)\;,\;D\Psi^n_{i,j,k}(\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle)\;)\;=\;\frac{(2m)^n}{(2m)^{2n}}\cdot\measuredangle\;(\;T_{q'}\Sigma_0\;,\;\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle\;)\;.$$

Thus we get for any $q = \Psi^n_{i,j,k}(q') \in \Sigma \setminus B(\partial\Sigma_n, \frac{1}{n\cdot(2m)^n})$, we must have

$$\begin{aligned}\measuredangle\;(\;T_q\Sigma_n\;,\;E^s(q)\oplus E^u(q)\;)\;\leq\;&\measuredangle\;(\;T_q\Sigma_n\;,\;D\Psi^n_{i,j,k}(\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle)\;)\;+\\&\measuredangle\;(\;D\Psi^n_{i,j,k}(\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle)\;,\;E^s(q)\oplus E^u(q)\;)\;.\end{aligned}$$

Notice that $T_q\Sigma_n = D\Psi^n_{i,j,k}(T_{q'}\Sigma_0)$, and we have

$$\measuredangle\;(\;T_q\Sigma_n\;,\;D\Psi^n_{i,j,k}(\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\rangle)\;)\;\leq\;\frac{1}{(2m)^n}\cdot n\cdot L_0\;.$$

Combining with the convergence on the skeleton, we have

$$\lim_{n\to\infty}\;\max_{q\in\Sigma\setminus B(\partial\Sigma_n,\frac{1}{n\cdot(2m)^n})}\;\measuredangle\;(\;T_q\Sigma_n\;,\;E^s(q)\oplus E^u(q)\;)\;=\;0\;.$$

$\square$

Chapter 5

Construction of Diffeomorphisms

In this chapter, we will give the proof of the main theorem assuming the existence of central DA-construction on the boundary fibers. Actually, all our constructions and perturbations of the diffeomorphisms preserve the S^1-fibers. That is all these diffeomorphisms project on $\mathbb{T}^2$ would be equal to the linear Anosov map A. So our perturbations are all through the S^1 fibers.

The construction of f_n consists of two steps. First we perturb f_A on a neighborhood of the boundary fibers of the invariant Birkhoff section Σ_n to get the diffeomorphism g_n, where g_n admits some product structure close the the boundary fibers. Our g_n will converge to f_A in C^1-topology as $n \to \infty$.

Then we separate the nilmanifold $\mathcal{H}$ as the union of two open sets, called E_n and B_n. Both of them are saturated by the S^1-fibers. And we try to construct f_n on E_n and B_n respectively. Since our perturbations are all preserve S^1-fibers, we will have:

$$\begin{aligned} f_A(E_n) &= g_n(E_n) = f_n(E_n) \ , \\ f_A(B_n) &= g_n(B_n) = f_n(B_n) \ . \end{aligned}$$

Actually, we will construct $f_{n,ext}$ on E_n in this section, and the C^1-distance between $f_{n,ext}$ and $g_n|_{E_n}$ will tend to 0 as $n \to \infty$. Then we admit the existence of unit model $f_{n,mod}$ defined on B_n, and also $f_{n,mod}$ tend to $g_n|_{B_n}$. We require that

$$f_{n,ext}|_{E_n \cap B_n} = f_{n,mod}|_{E_n \cap B_n},$$

which allow us to define $f_n = f_{n,ext} \sqcup f_{n,mod}$, and consequently C^1-distance between f_n and g_n will converge to 0. Thus f_n will C^1-approximate f_A.

Finally, we will prove that f_n is structurally stable with one attractor and one repeller as its chain recurrent set.

5.1 Product Structure on Boundary Fibers

In this section, we will perturb f_A to g_n to get the local product representations on a neighborhood of boundary fibers of Σ_n. We will show that the perturbations could be C^1-small.

Actually, here g_n is mainly used for estimating the C^1-distance between f_n and f_A.

We first fix some notations. We will usually denote by $\widetilde{p} = (x, y, z)$ a point belongs $\mathcal{H}$ or $\mathbb{H}$ with the coordinates $\mathbb{R}^3$. Denote by p, q points belong $\mathbb{R}^2$ or $\mathbb{T}^2$, and $\delta > 0$, we will denote by $B_\delta(p)$ the δ-neighborhood of the point p in $\mathbb{R}^2$ or $\mathbb{T}^2$.

Now we recall some properties of the invariant Birkhoff sections Σ_n. Σ_n is an affine Birkhoff section, which means there exists a family of affine maps

$$\Psi^n_{i,j,k} \ : \ [0,1]^3 \longrightarrow \ \Delta^n_{i,j,k} \ \hookrightarrow \ \mathcal{H},$$

where $i, j \in \{0, 1, \cdots, (2m)^n - 1\}$, and $k \in \{0, 1, \cdots, (2m)^{2n} - 1\}$. Notice that the Birkhoff section Σ_n satisfying

$$\Sigma_n \cap \Delta^n_{i,j,k} \ = \ \Psi^n_{i,j,k}(\Sigma_0).$$

Here $\Sigma_0 \subset [0,1]^3$ was defined at the introduction of Birkhoff sections.

Moreover, from theorem 4.3.1 and continuity, we know that there exists $\epsilon_n > 0$ such that for any $\widetilde{p} \in \mathrm{Int}(\Psi^n_{i,j,k}([0,1]^2 \times \{0\}))$, we have

$$\measuredangle \ (\ T_{\widetilde{p}}\Psi^n_{i,j,k}([0,1]^2 \times \{0\}), \ E^s(\widetilde{p}) \oplus E^u(\widetilde{p}) \ ) \ < \ \epsilon_n.$$

And $\epsilon_n \to 0$, as $n \to \infty$.

We first fix some notations. for any $p \in \mathbb{R}^2$ or $\mathbb{T}^2$, and $\delta > 0$, we will denote by $B_\delta(p)$ the δ-neighborhood of the point p in $\mathbb{R}^2$ or $\mathbb{T}^2$.

Fix $n \in \mathbb{N}$ and for any $i, j \in \{0, 1, \cdots, (2m)^n - 1\}$, we pick a fixed $k \in \{0, 1, \cdots, (2m)^{2n} - 1\}$, and consider $p = \pi(\Psi^n_{i,j,k}(\frac{1}{2}, \frac{1}{2}, 0)) \in \pi(\partial\Sigma_n) \subset \mathbb{T}^2$. Then we denote the disk

$$\mathbb{D}(p, \frac{\delta}{(2m)^n}) = \Psi^n_{i,j,k}(B_\delta((\frac{1}{2}, \frac{1}{2})) \times \{0\}) \hookrightarrow \mathcal{H},$$

which is an imbedded disk in $\mathcal{H}$.

In the rest of this paper, we will give a coordinate of the disk $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ by identify p be the original point in $\mathbb{R}^2$, and by the projection

$$\pi(\mathbb{D}(p, \frac{\delta}{(2m)^n})) = B_{\frac{\delta}{(2m)^n}}(p) \subset \mathbb{T}^2,$$

which $B_{\frac{\delta}{(2m)^n}}(p)$ could also be seen as a disk in $\mathbb{R}^2$, and we move p to the original point.

Then we can also give a coordinate of

$$\bigcup_{q \in \mathbb{D}(p, \delta/(2m)^n)} S^1_q \ = \ \mathbb{D}(p, \frac{\delta}{(2m)^n}) \times S^1 \ \subset \ \mathcal{H}.$$

Here every point in $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ is the zero point of its S^1 fiber.

From the construction of affine Birkhoff sections, we can see that $\Sigma_n \cap \mathbb{D}(p, \frac{\delta}{(2m)^n}) \times S^1$ could be parameterized as the helicoid

$$\begin{cases} x = \rho \cdot \cos[2\pi \cdot \widetilde{p}((2m)^{2n}\theta + \theta_0)], \\ y = \rho \cdot \sin[2\pi \cdot \widetilde{p}((2m)^{2n}\theta + \theta_0)], \\ z = \theta \pmod 1. \end{cases}$$

Here $\theta \in \mathbb{R}$, and $0 \leq \rho < \delta/(2m)^n$.

Now we can state the following lemma.

Lemma 5.1.1. *There exists a sequence of diffeomorphisms $\{g_n\}_{n\in\mathbb{N}}$ which satisfying:*

1. *$\pi \circ g_n = A : \mathbb{T}^2 \to \mathbb{T}^2$, and g_n is an isometry restricted on every S^1-fiber, i.e. $\|D^c g_n\| \equiv 1$.*
2. *For the constant $K_0 > \max\{\|A\|, \|A^{-1}\|\}$, $g_n|_{\mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1}$ could be represented as*

$$\begin{aligned} g_n \ &: \ \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1 \ \longrightarrow \ \mathbb{D}(A(p), \frac{\delta}{(2m)^n}) \times S^1 \ , \\ g_n(q,t) \ &= \ (\ A(q) \ , \ \det(A) \cdot t + \frac{s_{p,n}}{(2m)^{2n}} \pmod 1 \ ) \ . \end{aligned}$$

   Here $(q,t) \in \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1$, and $s_{p,n} \in \mathbb{Z}$ is a fixed integer.

3. *The diffeomorphisms g_n converge to f_A in C^1-topology as $n \to \infty$.*

Remark. *Notice that here our choice of the disk is not unique. Actually, if we find a disk $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ for g_n, then rotate $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ along the S^1 fibers $i/(2m)^{2n}$ for any $i \in \mathbb{Z}$ is still a disk satisfying all our requirements. And this corresponding to another k for the affine map $\Psi^n_{i,j,k}$.*

Proof. The proof relies on the facts that the tangent plane of the disk $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ will converge to $E^s \oplus E^u$ as $n \to \infty$. For simplicity, we do not distinguish the disk $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ and its projection on $\mathbb{T}^2$.

The perturbation of f_A to get g_n is just combine f_A with some rotations along the S^1-fibers. That is we define a real function $\theta_n : \mathbb{T}^2 \to \mathbb{R}$, and

$$g_n = R_{\theta_n} \circ f_A.$$

For any $(q,t) \in \mathbb{T}^2 \widetilde{\times} S^1 = \mathcal{H}$, if $f_A(q,t) = (A(q), s) \in \mathcal{H}$, then

$$g_n(q,t) = (A(q), s + \theta_n(A(q))).$$

So to prove that $g_n \to f_A$, we just need to show $\theta_n \to 0$ in C^1-topology.

Recall that we required that $\delta \ll 1$, so we pick a constant $K_1 \gg K_0$ which satisfying $4K_1\delta \ll 1$. So on the boundary fiber S_p for Σ_n, we can similar define the disk $\mathbb{D}(p, \frac{2K_1\delta}{(2m)^n})$ as before. Then we will try to construct g_n restricted on $\mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n}) \times S^1$, and we have $f_A(\mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n}) \times S^1) \subset \mathbb{D}(A(p), \frac{2K_1\delta}{(2m)^n}) \times S^1$.

For the fixed boundary fiber S_p, we can represent f_A locally as:

$$f_A \ : \ \mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n}) \times S^1 \ \longrightarrow \ \mathbb{D}(A(p), \frac{2K_1\delta}{(2m)^n}) \times S^1,$$

and for any $(q,t) \in \mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n}) \times S^1$,

$$f_A(q,t) \ = \ (\ A(q)\ ,\ \omega(A(q)) + \det(A)\cdot t \ (\mathrm{mod}\ 1)\ )\ .$$

Here $\omega : A(\mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n})) \to S^1$ is smooth and its graph is equal to $f_A(\mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n}) \times \{0\})$. Denote the two coordinates are x and y on $A(\mathbb{D}(p, \frac{2K_1\delta}{K_0(2m)^n})) \subset \mathbb{R}^2$, we have the following claim:

Claim. *The partial derivatives of the function ω satisfying*

$$\|\frac{\partial \omega}{\partial x}\| < 2K_0\epsilon_n\ , \qquad \text{and} \qquad \|\frac{\partial \omega}{\partial y}\| < 2K_0\epsilon_n\ .$$

As a consequence, for any $q \in A(\mathbb{D}_{\frac{\delta}{K_0(2m)^n}}(p))$, we have

$$|\omega(q) - \omega(0)| \ < \ \frac{2\delta K_0\epsilon_n}{(2m)^n}.$$

Proof of the Claim. We proof the claim by some symbolic computation. Notice that the tangent plane at the point $f_A(q,0)$ generated by

$$\frac{\partial}{\partial x} + \frac{\partial \omega}{\partial x}\cdot\frac{\partial}{\partial z}, \qquad \text{and} \qquad \frac{\partial}{\partial y} + \frac{\partial \omega}{\partial y}\cdot\frac{\partial}{\partial z}$$

is equal to $Df_A(\langle\ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\ \rangle|_{(q,0)})$.

There exists two smooth function α and β defined on $\mathbb{D}(p, \frac{\delta}{(2m)^n})$, such that for any point $\widetilde{q} = (q,t) \in \mathbb{D}(p, \frac{\delta}{(2m)^n}) \times S^1$, we have

$$E^s(\widetilde{q}) \oplus E^u(\widetilde{q}) \ = \ \langle\ \frac{\partial}{\partial x} + \alpha(q)\cdot\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + \beta(q)\cdot\frac{\partial}{\partial z}\ \rangle\ .$$

Since

$$\measuredangle\ (\ \langle\ \frac{\partial}{\partial x}|_{\widetilde{q}}, \frac{\partial}{\partial y}|_{\widetilde{q}}\ \rangle\ ,\ E^s(\widetilde{q}) \oplus E^u(\widetilde{q})\ )\ <\ \epsilon_n,$$

we know that $|\alpha| < \epsilon_n$, and $|\beta| < \epsilon_n$. Here the constant ϵ_n comes from the beginning of this section.

So from the equality

$$Df_A(\ \langle\ \frac{\partial}{\partial x} + \alpha(q)\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + \beta(q)\frac{\partial}{\partial z}\ \rangle\ )\ =\ \langle\ \frac{\partial}{\partial x} + \alpha(A(q))\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + \beta(A(q))\frac{\partial}{\partial z}\ \rangle\ ,$$

we know that

$$\|\frac{\partial \omega}{\partial x}\| < (\|A\|+1)\cdot \epsilon_n < 2K_0\epsilon_n, \qquad \text{and} \qquad \|\frac{\partial \omega}{\partial y}\| < (\|A\|+1)\cdot \epsilon_n < 2K_0\epsilon_n.$$

Finally,

$$|\omega(q)-\omega(0)| \le \max\{\|\frac{\partial \omega}{\partial x}\|, \|\frac{\partial \omega}{\partial y}\|\}\cdot \|q-0\| < \frac{2\delta K_0\epsilon_n}{(2m)^n}\ .$$

This finishes the proof of the claim.

□

Now assume that for some integer $s_{p,n}$, it has $\frac{s_{p,n}-1}{(2m^{2n})} < \omega(0) \le \frac{s_{p,n}}{(2m^{2n})}$. Then we define

$$\theta_n|_{A(\mathbb{D}(p,\frac{\delta}{K_0(2m)^n}))} = \frac{s_{p,n}}{(2m^{2n})} - \omega,$$

with the estimations

$$\|\theta_n|_{A(\mathbb{D}(p,\frac{\delta}{K_0(2m)^n}))}\| \ \le \ \frac{2\delta K_0\epsilon_n}{(2m)^n} + \frac{1}{(2m)^{2n}};$$

$$\|\frac{\partial \theta_n}{\partial x}|_{A(\mathbb{D}(p,\frac{\delta}{K_0(2m)^n}))}\| < 2K_0\epsilon_n\ , \qquad \text{and} \qquad \|\frac{\partial \theta_n}{\partial y}|_{A(\mathbb{D}(p,\frac{\delta}{K_0(2m)^n}))}\| < 2K_0\epsilon_n\ .$$

Extending θ_n smoothly to $\mathbb{D}(A(p),\frac{2K_1\delta}{(2m)^n})$, where on the boundary of this region, $\theta_n \equiv 0$. Then we can still get

$$\|\theta_n|_{\mathbb{D}(A(p),\frac{2K_1\delta}{(2m)^n})}\| \ \le \ \frac{2\delta K_0\epsilon_n}{(2m)^n} + \frac{1}{(2m)^{2n}}.$$

For the estimation of the partial derivatives, it becomes a little bit complicated. Actually,

$$\begin{aligned}\|\frac{\partial \theta_n}{\partial x}|_{\mathbb{D}(A(p),\frac{2K_1\delta}{(2m)^n})}\| &\le \max\ \{\ 2K_0\epsilon_n\ ,\ \frac{\|\theta_n|_{\mathbb{D}(A(p),\frac{2K_1\delta}{(2m)^n})}\|}{\frac{2K_1\delta}{K_0(2m)^n}-\frac{\delta}{(2m)^n}}\ \}\ ,\\ &\le \max\ \{\ 2K_0\epsilon_n\ ,\ 2K_0\epsilon_n + \frac{\delta}{(2m)^n}\ \}\ ,\\ &\le 2K_0\epsilon_n\ .\end{aligned}$$

Similarly we have

$$\|\frac{\partial \theta_n}{\partial y}|_{\mathbb{D}(A(p),\frac{2K_1\delta}{(2m)^n})}\| \ \le \ 2K_0\epsilon_n\ .$$

Finally, we do this process for all $p \in \pi(\partial\Sigma_n)$ and define $\theta_n \equiv 0$ when restricted on

$$\mathbb{T}^2 \setminus \bigcup_{p\in\pi(\partial\Sigma_n)} B_{\frac{2K_1\delta}{(2m)^n}}(p).$$

It could check that $g_n = R_{\theta_n}\circ f_A$ satisfies the first and second items of the lemma. And since θ_n will converge to 0 in C^1-topology, g_n will also converge to f_A as $n\to\infty$.

□

5.2 Unit Models on Boundary Fibers

In last section, we built the diffeomorphisms g_n, which admitted some kind local product structure. We will state the construction near the boundary fibers, which we called the central DA-construction first appeared at [8].

In the sketch of ideas for our construction, we say that the two parallel Birkhoff sections Σ_n and Σ_n' will be our candidate for attractor and repeller. So we need to separate them on their intersection $\partial\Sigma_n = \partial\Sigma_n'$.

First consider the space to be $\mathbb{R}^2 \times S^1$ with the natural coordinates x, y, z. Then for any $n \in \mathbb{N}$, we can define a sequence of deformed half helicoid surfaces $S_n \subset \mathbb{R}^2 \times S^1$ as

$$\begin{cases} x = \rho \cdot \cos[2\pi \cdot \widetilde{p}((2m)^{2n}\theta + \theta_0)], \\ y = \rho \cdot \sin[2\pi \cdot \widetilde{p}((2m)^{2n}\theta + \theta_0)], \\ z = \theta \pmod 1). \end{cases}$$

Here the parameter $\theta \in \mathbb{R}$ and $\rho \geq 0$.

Then rotate S_n along the S^1-fibers with distance $\frac{1}{2\cdot(2m)^{2n}}$, we get another deformed half helicoid $S_n' \subset \mathbb{R}^2 \times S^1$. Notice that $S_n \cap S_n' = \{(0,0)\} \times S^1$, and for any $(x,y) \neq (0,0)$ in $\mathbb{R}^2$, $\{(x,y)\} \times S^1$ intersects S_n and S_n' alternatively with the distance $\frac{1}{2\cdot(2m)^{2n}}$ for adjacent points in S_n and S_n' respectively.

Since all the diffeomorphisms we will handle are preserve the S^1-fibers, so for any diffeomorphism f of $\mathbb{R}^2 \times S^1$, we denote the central derivative of f at point p is $D^c f(p) = Df_p|_{T_pS^1}$. Moreover, for any $p \in \mathbb{R}^2 \times S^1$ and $0 < s, t < 1/2$, we use $[p - s, p + t]^c$ denote the interval contained in the S^1-fiber of p, with two endpoints to p with distance s and t. The orientation is the same with the natural orientation of S^1-fibers.

We will denote the rotation through the S^1-fibers with the angle θ by R_θ, in both the case $\mathbb{R}^2 \times S^1$ and $\mathcal{H}$ which is the S^1-fiber over $\mathbb{T}^2$.

First we introduce a family of interval diffeomorphisms. Fix a constant $0 < \alpha < 1$, we call a smooth diffeomorphism

$$\Theta_\alpha : \ I = [0,1] \ \longrightarrow \ I = [0,1],$$

is a model map associated to α of the interval $[0,1]$, if it satisfies:

1. $\Theta_\alpha(t) = \alpha \cdot t$, for $t \in [0, 1/2]$;
2. $\Theta_\alpha(t) = \alpha^{-1} \cdot (t-1) + 1$, for $t \in [1 - \alpha/2, 1]$;
3. $\Theta_\alpha(t)$ is smooth on $[1/2, 1-\alpha/2]$, and $\alpha \leq \Theta_\alpha'(t) \leq \alpha^{-1}$.

Moreover, we can extended Θ_α to the $[-1,1]$ by defining $\Theta_\alpha(t) = -\Theta_\alpha(-t)$ for any $t \in [-1, 0]$, and still call it the model map.

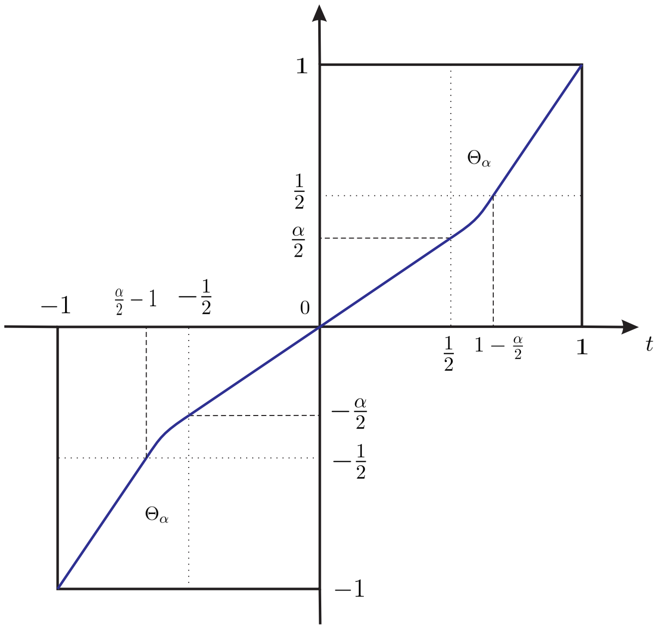

Figure 5.1: *The Model Map* Θ_α

Now we can state the main technical proposition and will prove it after we finish the whole construction of f_n.

Proposition 5.2.1. *Consider the diffeomorphism $f_0 : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ which defined as $f_0(q,t) = (A(q), \det(A) \cdot t)$. There exists a sequence of diffeomorphisms $f_{n,mod} : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$, and*

- *a sequence of real numbers $0 < \alpha_n < 1$, where $\lim_{n\to\infty} \alpha_n = 1$,*
- *a sequence of model maps $\Theta_n : [-1,1] \to [-1,1]$ associated to α_n,*

which satisfying the following properties:

1. *Every $f_{n,mod}$ preserves the S^1-fibers, and $\pi \circ f_{n,mod} = A : \mathbb{R}^2 \to \mathbb{R}^2$.*
2. *There exists two disjoint closed region $U_{n,mod}, V_{n,mod} \subset \mathbb{R}^2 \times S^1$, where $U_{n,mod}$ is strictly invariant by $f_{n,mod}$: $f_{n,mod}(U_{n,mod}) \subset \mathrm{Int}(U_{n,mod})$; and $V_{n,mod}$ is strictly invariant by $f^{-1}_{n,mod}$: $f^{-1}_{n,mod}(V_{n,mod}) \subset \mathrm{Int}(V_{n,mod})$.*

3. *Denote the region* $M_n = \{(x,y,z) : \sqrt{x^2+y^2} \geq \frac{1}{n^2(2m)^n}\}$*, then*

$$\begin{aligned} U_{n,mod} \cap M_n &= \bigcup_{p \in S_n \cap M_n} [\ p - \frac{1}{4(2m)^{2n}}\ ,\ p + \frac{1}{4(2m)^{2n}}\ ]^c, \\ V_{n,mod} \cap M_n &= \bigcup_{p \in S'_n \cap M_n} [\ p - \frac{\alpha_n}{4(2m)^{2n}}\ ,\ p + \frac{\alpha_n}{4(2m)^{2n}}\ ]^c. \end{aligned}$$

4. *The restriction of* $f_{n,mod}$ *on the fixed fiber* $(0,0) \times S^1$ *is a Morse-Smale diffeomorphism of the circle having* $4 \cdot (2m)^{2n}$ *periodic points,* $2(2m)^{2n}$ *of them are in* $U_{n,mod}$ *and the others are in* $V_{n,mod}$*.*

5. $f_{n,mod}(S_n \cap M_n) \subset S_n$*, and* $f_{n,mod}(S'_n \cap M_n) \subset S'_n$*. Moreover, for any* $p \in S_n \cap M_n$*, if we parameterize* $[p - \frac{1}{2(2m)^{2n}}, p + \frac{1}{2(2m)^{2n}}]^c$ *naturally to be* $[-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$*, and the same to* $[f_{n,mod}(p) - \frac{1}{2(2m)^{2n}}, f_{n,mod}(p) + \frac{1}{2(2m)^{2n}}]^c$*, then for all* $t \in [-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$*, we have*
   - $f_{n,mod}|_{[p-\frac{1}{2(2m)^{2n}}, p+\frac{1}{2(2m)^{2n}}]^c}(t) = \frac{1}{2(2m)^{2n}} \cdot \Theta_n(\ 2(2m)^{2n} \cdot t\ )$, *if* $\det(A) = 1$*;*
   - $f_{n,mod}|_{[p-\frac{1}{2(2m)^{2n}}, p+\frac{1}{2(2m)^{2n}}]^c}(t) = \frac{1}{2(2m)^{2n}} \cdot \Theta_n(\ -2(2m)^{2n} \cdot t\ )$, *if* $\det(A) = -1$*.*

6. *The central derivative* $\|D^c f_{n,mod}\|$ *and* $\|D^c f^{-1}_{n,mod}\|$ *are small or equal to* α_n *in* $U_{n,mod}$ *and* $V_{n,mod}$ *respectively.*

7. *For any integer* $k_0 \in \mathbb{Z}$*,* $f_{n,mod}$ *is commutable with the rotation* $R_{\frac{k_0}{(2m)^{2n}}}$ *through the* S^1*-fibers:*

$$R_{\frac{k_0}{(2m)^{2n}}} \circ f_{n,mod} = f_{n,mod} \circ R_{\frac{k_0}{(2m)^{2n}}}.$$

8. $f_{n,mod}$ *converge to* f_0 *uniformly in* C^1*-topology as* $n \to \infty$*.*

Remark. *We can see that the sequence of diffoemorphisms* $\{f_{n,mod}\}$ *admits some kind flexibility. Actually, for any sequence of integers* l_n*, if* $\frac{l_n}{(2m)^{2n}} \to 0$ *as* $n \to \infty$*. Then the new diffeomorphism sequence* $\{\ R_{\frac{l_n}{(2m)^{2n}}} \circ f_{n,mod}\ \}$ *also satisfies all the properties in the proposition.*

5.3 Building the Diffeomorphisms $f_{n,ext}$

In this section, we try to construct the diffeomorphism $f_{n,ext}$ which defined on the region

$$E_n = \mathbb{T}^2 \setminus \bigcup_{p \in \pi(\partial\Sigma_n)} \pi^{-1}(B_{\frac{1}{n(2m)^n}}(p)) = \mathbb{T}^2 \setminus \bigcup_{p \in \pi(\partial\Sigma_n)} \mathbb{D}(p, \frac{1}{n(2m)^n}) \times S^1.$$

The idea is quite simple. Denote the invariant Birkhoff section Σ'_n which is derived form rotate Σ_n along the S^1-fibers with angle $\frac{1}{2(2m)^{2n}}$. We will construct $f_{n,ext}$ preserve the Birkhoff sections $\Sigma_n \cap E_n$ and $\Sigma'_n \cap E_n$ invariant, and on Σ_n the central direction is contracting, on Σ'_n is expanding.

Here the two parallel Birkhoff sections Σ_n and Σ'_n would be our future candidates of attractor and repeller for f_n.

5.3.1 Invariance of Birkhoff sections

Lemma 5.3.1. *There exists a sequence of smooth functions*

$$\vartheta_n : \ \pi(g_n(E_n)) = A(\pi(E_n)) \ \longrightarrow \ \mathbb{R},$$

which satisfying:

- *The diffeomorphism* $R_{\vartheta_n} \circ g_n|_{E_n} : \ E_n \to \mathcal{H}$ *preserve the Birkhoff sections* Σ_n *and* Σ'_n *invariant:*

$$R_{\vartheta_n} \circ g_n|_{E_n}(\Sigma_n \cap E_n) \subset \Sigma_n \ , \qquad \text{and} \qquad R_{\vartheta_n} \circ g_n|_{E_n}(\Sigma'_n \cap E_n) \subset \Sigma'_n \ .$$

- $\lim_{n\to\infty} \|\vartheta_n - 0\|_{C^1} = 0$, *and consequently, we have*

$$\lim_{n\to\infty} d_{C^1}(R_{\vartheta_n} \circ g_n|_{E_n} \ , \ g_n|_{E_n}) \ = \ 0 \ .$$

Proof. Since the Birkhoff section Σ'_n is achieved from Σ_n through rotation, and g_n restricted on S^1-fibers are all isometries, we know that if $R_{\vartheta_n} \circ g_n|_{E_n}$ preserve $\Sigma_n \cap E_n$ invariant, then it must also preserve $\Sigma'_n \cap E_n$ invariant.

Recall that Σ_n is an invariant Birkhoff section, Σ_n is fiber isotopic to $f_A(\Sigma_n)$, and also to $g_n(\Sigma_n)$. This implies we can define a global function

$$\vartheta_n : \ \pi(g_n(E_n)) = A(\pi(E_n)) \ \longrightarrow \ \mathbb{R},$$

such that

$$R_{\vartheta_n}(g_n|_{E_n}(\Sigma_n \cap E_n)) \ = \ \Sigma_n \cap g_n(E_n).$$

Moreover, it could easily see that here ϑ_n is not unique for preserving Σ_n invariant. For any integer $i \in \mathbb{Z}$, it will also have

$$R_{\vartheta_n + \frac{i}{(2m)^{2n}}}(g_n|_{E_n}(\Sigma_n \cap E_n)) \ = \ \Sigma_n \cap g_n(E_n).$$

We need to show that ϑ_n could be chosen C^1-converge to 0.

First we fix some $q_n \in \pi(g_n(E_n))$, then we can require that $0 \le \vartheta_n(q_n) < \frac{1}{(2m)^{2n}}$.

Since g_n is isometries on each S^1-fiber, so it commutes with the constant rotation R_{t_0}:

$$R_{t_0} \circ g_n \ = \ g_n \circ R_{t_0},$$

for any $t_0 \in \mathbb{R}$. This implies the constant rotation R_{t_0} preserve the plane field $E^s_{g_n} \oplus E^u_{g_n}$:

$$DR_{t_0}(E^s_{g_n} \oplus E^u_{g_n}) \ = \ E^s_{g_n} \oplus E^u_{g_n}.$$

For any $\widetilde{p} \in \Sigma_n \cap E_n$, we have

$$\begin{aligned} &\measuredangle(\ E^s_{g_n}(\widetilde{p}) \oplus E^u_{g_n}(\widetilde{p})\ ,\ T_{\widetilde{p}}\Sigma_n\ ) \\ \leq\ &\measuredangle(\ E^s_{g_n}(\widetilde{p}) \oplus E^u_{g_n}(\widetilde{p})\ ,\ E^s_{f_A}(\widetilde{p}) \oplus E^u_{f_A}(\widetilde{p})\ )\ +\ \measuredangle(\ E^s_{f_A}(\widetilde{p}) \oplus E^u_{f_A}(\widetilde{p})\ ,\ T_{\widetilde{p}}\Sigma_n\ ). \end{aligned}$$

Then $d_{C^1}(g_n, f_A) \longrightarrow 0$ implies $\measuredangle(\ E^s_{g_n}(\widetilde{p}) \oplus E^u_{g_n}(\widetilde{p})\ ,\ E^s_{f_A}(\widetilde{p}) \oplus E^u_{f_A}(\widetilde{p})\ )$ converge to 0. And we have

$$\measuredangle(\ E^s_{f_A}(\widetilde{p}) \oplus E^u_{f_A}(\widetilde{p})\ \leq\ \epsilon_n\ \longrightarrow\ 0\ .$$

Thus there exists $\kappa_n \to 0$, as $n \to \infty$, such that .

$$\measuredangle(\ E^s_{g_n}(\widetilde{p}) \oplus E^u_{g_n}(\widetilde{p})\ ,\ T_{\widetilde{p}}\Sigma_n\ )\ \leq\ \kappa_n\ .$$

Moreover, for any $\widetilde{q} \in f_A(\Sigma_n \cap E_n)$, it admits

$$\measuredangle(\ E^s_{g_n}(\widetilde{q}) \oplus E^u_{g_n}(\widetilde{q})\ ,\ T_{\widetilde{q}}g_n(\Sigma_n \cap E_n)\ )\ \leq\ K_0 \cdot \kappa_n.$$

Here recall that $K_0 \leq \max\{\|A\|, \|A^{-1}\|\}$ is a constant.

For any $q \in \pi(g_n(E_n))$, locally we choose a very small neighborhood $V_q \subset \pi(g_n(E_n))$ of q, then $\Sigma_n \cap \pi^{-1}(V_q)$ and $g_n(\Sigma_n) \cap \pi^{-1}(V_q)$ are both $(2m)^{2n}$-cover of V_q. We choose one connected component for each of them, denoted by $\Sigma_n(V_q)$ and $g_n(\Sigma_n)(V_q)$ respectively. Notice that $\Sigma_n(V_q)$ and $g_n(\Sigma_n)(V_q)$ intersect each S^1-fiber in $\pi^{-1}(V_q)$ with exact one point. This allowed us define a function $\Sigma_n(V_q) - g_n(\Sigma_n)(V_q)$ which denote the oriented distance from the point in $\Sigma_n(V_q)$ to the point in $g_n(\Sigma_n)(V_q)$ in each S^1-fiber. It could see that for some integer $i_q \in \mathbb{Z}$, we have

$$\vartheta_n|_{V_q} = \Sigma_n(V_q) - g_n(\Sigma_n)(V_q) + \frac{i_q}{(2m)^{2n}}.$$

Since we have know that the constant rotation preserve $E^s_{g_n} \oplus E^u_{g_n}$, we get the estimation

$$\begin{aligned} \|\frac{\partial \vartheta_n}{\partial x}\|\ &\leq\ \|\frac{\partial t_0}{\partial x}\| + \measuredangle(E^s_{g_n} \oplus E^u_{g_n}, Tg_n(\Sigma_n \cap E_n)) + \measuredangle(E^s_{g_n} \oplus E^u_{g_n}, T\Sigma_n) \\ &\leq\ 0 + \kappa_n + K_0 \cdot \kappa_n\ =\ (K_0+1)\kappa_n\ . \end{aligned}$$

Here t_0 is the constant rotation distance at the base point where we take the partial derivative. Similarly, we have $\|\partial\vartheta_n/\partial y\| \leq (K_0+1)\kappa_n$.

These two estimations deduce that for any $q \in \pi(g_n(E_n))$, we will have

$$\begin{aligned} |\vartheta_n(q) - \vartheta_n(q_n)|\ &\leq\ C \cdot \max\{\|\frac{\partial \vartheta_n}{\partial x}\|, \|\frac{\partial \vartheta_n}{\partial y}\|\} \cdot |q - q_n| \\ &\leq\ C' \cdot (K_0+1) \cdot \kappa_n \end{aligned}$$

Thus we have $\|\vartheta_n\|\ \leq\ 1/(2m)^{2n} + C' \cdot (K_0+1) \cdot \kappa_n\ \to\ 0$, as $n \to \infty$. Combining with $\max\{\|\partial\vartheta_n/\partial x\|, \|\partial\vartheta_n/\partial y\|\} \leq (K_0+1)\kappa_n$, we proved that

$$\lim_{n\to\infty} \|\vartheta_n - 0\|_{C^1}\ =\ 0.$$

This finishes the proof of the lemma.

$\square$

5.3.2 Building $f_{n,ext}$ on central fibers

Now we will define the last perturbation of $R_{\vartheta_n} \circ g_n|_{E_n}$ on the S^1-fibers for constructing $f_{n,ext}$.

For each S^1-fiber $S_q \subset g_n(E_n) = R_{\vartheta_n} \circ g_n|_{E_n}(E_n)$, we know that S_q intersects Σ_n with $(2m)^{2n}$ points with neighboring distance $1/(2m)^{2n}$. $S_q \cap \Sigma'_n$ is equal to rotate $S_q \cap \Sigma_n$ with distance $1/2(2m)^{2n}$. So for any $\widetilde{q} \in S_q \cap \Sigma_n$, we can find an interval

$$I_{\widetilde{q}} = [\ \widetilde{q} - \frac{1}{2(2m)^{2n}}\ ,\ \widetilde{q} + \frac{1}{2(2m)^{2n}}\ ] \subset S_q\ ,$$

where $I_{\widetilde{q}} \cap \Sigma_n = \{\widetilde{q}\}$, and $I_{\widetilde{q}} \cap \Sigma'_n = \partial I_{\widetilde{q}}$. Parameterize $I_{\widetilde{q}}$ by its length, and denote $\widetilde{q}$ is the zero point, then $I_{\widetilde{q}} = [-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$.

We want point out that, we have known that $R_{\vartheta_n} \circ g_n|_{E_n}(x) \in \Sigma_n$. If we also consider the central interval

$$I_{R_{\vartheta_n} \circ g_n|_{E_n}(\widetilde{q})} = [\ R_{\vartheta_n} \circ g_n|_{E_n}(\widetilde{q}) - \frac{1}{2(2m)^{2n}}\ ,\ R_{\vartheta_n} \circ g_n|_{E_n}(\widetilde{q}) + \frac{1}{2(2m)^{2n}}\ ]\ ,$$

also with the parameter identification $I_{R_{\vartheta_n} \circ g_n|_{E_n}(\widetilde{q})} = [-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$, then we can see that

- if $\det(A) = 1$, then $R_{\vartheta_n} \circ g_n|_{I_x} = id$ defined on : $[-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$,
- if $\det(A) = -1$, then $R_{\vartheta_n} \circ g_n|_{I_x} = -id$ defined on $[-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$.

Define the diffeomorphism $h_n|_{I_{\widetilde{q}}} : I_{\widetilde{q}} = [-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}] \to I_{\widetilde{q}}$,

$$h_n|_{I_{\widetilde{q}}}(t) \ = \ \frac{1}{2(2m)^{2n}} \cdot \Theta_n(2(2m)^{2n} \cdot t)\ , \qquad \forall t \in I_{\widetilde{q}}.$$

Here Θ_n is the model map of the interval $[-1, 1]$ defined in proposition 5.2.1. Moreover, we can see that the derivative satisfying

$$\alpha_n \ \leq \ (h_n|_{I_{\widetilde{q}}})'(t) \ \leq \ \alpha_n^{-1}\ , \qquad \forall t \in I_{\widetilde{q}}.$$

Since $g_n(E_n) = \bigcup_{\widetilde{q} \in \Sigma_n \cap g_n(E_n)} I_x$ and $h_n|_{I_{\widetilde{q}}}$ fix the end points of $I_{\widetilde{q}}$, we can define a diffeomorphism

$$h_n = \bigsqcup_{\widetilde{q} \in \Sigma_n \cap g_n(E_n)} h_n|_{I_{\widetilde{q}}} :\ g_n(E_n) \ \longrightarrow \ g_n(E_n)\ .$$

Actually, we have the following lemma.

Lemma 5.3.2. *The sequence of maps $h_n : g_n(E_n) \to g_n(E_n)$ are smooth diffeomorphisms, which satisfying the following properties:*

1. $\pi \circ h_n = id : \pi(g_n(E_n)) \to \pi(g_n(E_n))$;

2. $h_n(\Sigma_n \cap g_n(E_n)) = \Sigma_n \cap g_n(E_n)$, and $h_n(\Sigma'_n \cap g_n(E_n)) = \Sigma'_n \cap g_n(E_n)$;

3. $\lim_{n\to\infty} d_{C^1}(h_n, id_{g_n(E_n)}) = 0$.

Proof. The first item comes from h_n maps each S^1-fiber to itself. The second one comes from $h_n|_{I_{\widetilde{q}}}$ keep $\widetilde{q}$ and the the end points of $I_{\widetilde{q}}$ invariant. We need to show that $d_{C^1}(h_n, id_{g_n(E_n)}) \to 0$ as $n \to \infty$.

The map h_n preserve interval I_x invariant, and the length of $I_{\widetilde{q}}$ tends to 0 allows us to get $d_{C^0}(h_n, id_{g_n(E_n)}) \to 0$ as $n \to \infty$. For the smoothness of h_n and the estimation of its C^1-norm, we need some analysis of $g_n(E_n)$.

Notice that $\{R_t(\Sigma_n \cap g_n(E_n)) : t \in \mathbb{R}\}$ defines an C^∞ foliation of $g_n(E_n)$. From the definition of h_n, we can see that it preserve the foliation structure. i.e. h_n maps leaves to leaves, where $\Sigma_n \cap g_n(E_n)$ and $\Sigma'_n \cap g_n(E_n)$ are two invariant leaves.

For any fixed t, we can define two smooth vector field $\{\partial/\partial\widetilde{x}_n\}, \{\partial/\partial\widetilde{y}_n\} \subset TR_t(\Sigma_n \cap g_n(E_n))$ on $R_t(\Sigma_n \cap g_n(E_n))$, such that these two vector fields projected down by $D\pi$ will be the canonical vector field basis $\{\partial/\partial x\}, \{\partial/\partial y\}$ of $T\pi(g_n(E_n))$ on $\pi(g_n(E_n)) \subset \mathbb{T}^2$.

Combined with the vector field $\{\partial/\partial\widetilde{z}_n = \partial/\partial z\}$ which are unit vectors tangent to S^1-fibers with positive orientation, we defined a smooth base filed on $Tg_n(E_n)$. Under this base field, Dh_n at the point $t \in I_{\widetilde{q}} \subset g_n(E_n)$ could be represented as the following matrix function on $g_n(E_n)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (h_n|_{I_{\widetilde{q}}})'(t) \end{pmatrix}$$

Here we have $\alpha_n \leq (h_n|_{I_x})'(t) \leq \alpha_n^{-1}$ and $\lim_{n\to\infty} \alpha_n = 1$.

Since we already know that the tangent plane field of the foliation $\{R_t(\Sigma_n \cap g_n(E_n)) : t \in \mathbb{R}\}$ will converge to the invariant contact plane field. This implies for any point $\widetilde{q} \in g_n(E_n)$, we have

$$\frac{\partial}{\partial\widetilde{x}_n} \longrightarrow \frac{\partial}{\partial x} + \frac{k}{2m}\cdot\frac{\partial}{\partial z}\,, \qquad \frac{\partial}{\partial\widetilde{y}_n} \longrightarrow \frac{\partial}{\partial y} + (x + \frac{l}{2m})\cdot\frac{\partial}{\partial z}\,,$$

as n tend to infinity. Combining with the fact that $\partial/\partial\widetilde{z}_n = \partial/\partial z$, we know that under the fixed base field on $g_n(E_n)$:

$$\{\ \frac{\partial}{\partial x} + \frac{k}{2m}\cdot\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + (x + \frac{l}{2m})\cdot\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial z}\ \},$$

we have Dh_n uniformly converge to the identity matrix at each point of $g_n(E_n)$. Thus we showed $d_{C^1}(h_n, id_{g_n(E_n)}) \to 0$ as $n \to \infty$. This finishes the proof of this lemma.

□

5.3.3 Definition of $f_{n,ext}$ and basic properties

Now we can formally define the diffeomorphism $f_{n,ext} : E_n \to f_A(E_n) \subset \mathcal{H}$ as

$$f_{n,ext} \stackrel{\triangle}{=} h_n \circ R_{\vartheta_n} \circ g_n .$$

From the properties of R_{ϑ_n} and h_n, we can summarize the basic properties of $f_{n,ext}$ as the following lemma.

Lemma 5.3.3. *The sequence of diffeomorphisms satisfy the following properties:*

1. $\pi \circ f_{n,ext} = A : \ \pi(E_n) \longrightarrow A(\pi(E_n))$.
2. $f_{n,ext}(\Sigma_n \cap E_n) = \Sigma_n \cap f_A(E_n)$, *and* $f_{n,ext}(\Sigma'_n \cap E_n) = \Sigma'_n \cap f_A(E_n)$.
3. *If we denote* $I_{\widetilde{q}} \subset S^1(\widetilde{q})$ *be the segment centered at* $\widetilde{q} \in \Sigma_n$, *where* $I_{\widetilde{q}} \cap \Sigma'_n = \partial I_{\widetilde{q}}$, *then for any* $\widetilde{q} \in \Sigma_n \cap E_n$,

$$f_{n,ext}|_{I_{\widetilde{q}}} \ : \ I_{\widetilde{q}} = [-1/2(2m)^{2n}, 1/2(2m)^{2n}] \longrightarrow I_{f_{n,ext}(\widetilde{q})} = [-1/2(2m)^{2n}, 1/2(2m)^{2n}]$$

   is defined as for $t \in I_{\widetilde{q}}$*:*

   - *if* $\det(A) = 1$, *then* $f_{n,ext}|_{I_{\widetilde{q}}}(t) = \frac{1}{2(2m)^{2n}} \cdot \Theta_n(2(2m)^{2n} \cdot t)$,
   - *if* $\det(A) = -1$, *then* $f_{n,ext}|_{I_{\widetilde{q}}}(t) = \frac{1}{2(2m)^{2n}} \cdot \Theta_n(-2(2m)^{2n} \cdot t)$.

4. *Denote*

$$\begin{aligned} U_{n,ext} &= \cup_{\widetilde{q} \in \Sigma_n \cap E_n} \ [\ \widetilde{q} - 1/4(2m)^{2n} \ , \ \widetilde{q} + 1/4(2m)^{2n} \ ] \ , \\ V_{n,ext} &= \cup_{\widetilde{q} \in \Sigma'_n \cap E_n} \ [\ \widetilde{q} - \alpha_n/4(2m)^{2n} \ , \ \widetilde{q} + \alpha_n^{-1}/4(2m)^{2n} \ ] \ . \end{aligned}$$

   Then we have

$$\begin{aligned} f_{n,ext}(U_{n,ext}) \cap E_n &\subset \mathrm{Int}(U_{n,ext}) \ , \\ f^{-1}_{n,ext}(V_{n,ext}) \cap E_n &\subset \mathrm{Int}(V_{n,ext}) \ . \end{aligned}$$

5. $\lim_{n\to\infty} d_{C^1}(\ f_{n,ext}\ ,\ g_n|_{E_n}\ ) = 0$.

The proof of this lemma is the direct consequence of lemma 5.3.1 and lemma 5.3.2.

5.4 Construction of f_n

Now we can gluing the model map $f_{n,mod}$ defined on the neighborhood of the boundary fibers to $f_{n,ext}$, this will finish our construction of f_n. We will also prove that the diffeomorphisms f_n will converge to f_A in C^1-topology as $n \to \infty$.

Recall the for every boundary fiber $S_p \subset \Sigma_n$, we have defined an embedding disk $\mathbb{D}(p, \frac{\delta}{(2m)^n})$ which allow us to parameterize the neighborhood $\pi^{-1}(B_{\frac{\delta}{(2m)^n}}(p))$ as $\mathbb{D}(p, \frac{\delta}{(2m)^n}) \times S^1$. Moreover, $g_n : \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1 \longrightarrow \mathbb{D}(A(p), \frac{\delta}{(2m)^n}) \times S^1$ could be represented as

$$g_n(q,t) \;=\; (\ A(q)\ ,\ \det(A) \cdot t + s_{p,n}/(2m)^{2n}\ )\ ,$$

where $(q,t) \in \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1$, and $s_{p,n} \in \mathbb{Z}$. Moreover, under this coordinate, the two Birkhoff sections satisfy

$$\begin{aligned}
\Sigma_n \cap \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1 \;&=\; S_n \cap B_{\frac{\delta}{K_0(2m)^n}}(0) \times S^1, \\
\Sigma'_n \cap \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1 \;&=\; S'_n \cap B_{\frac{\delta}{K_0(2m)^n}}(0) \times S^1.
\end{aligned}$$

Now we try to glue $f_{n,ext}$ to $f_{n,mod}$ on every boundary fibers. Consider the annulus

$$Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \;\triangleq\; \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \;-\; \mathbb{D}(p, \frac{\delta}{n(2m)^n}),$$

we will focus on $f_{n,ext}$ restrict on $Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \times S^1$, and glue to $f_{n,mod}$.

Actually, identifying $Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \subset \mathbb{R}^2$ and $\mathbb{D}(A(p)\frac{\delta}{(2m)^n}) \subset \mathbb{R}^2$, we have

$$f_{n,ext} = R_{\frac{t_{p,n}}{(2m)^{2n}}} \circ f_{n,mod} :\; Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \times S^1 \;\longrightarrow\; \mathbb{D}(A(p), \frac{\delta}{(2m)^n}) \times S^1,$$

where $t_{p,n} \in \mathbb{Z}$.

This express comes from the fact that restricted on $Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \times S^1$, both $f_{n,ext}$ and $f_{n,mod}$ preserve the helicoid S_n and S'_n invariant. Moreover, on the intervals contained in S^1-fibers, which centered at points in S_n, and bounded by neighboring points in S'_n, they are also equal. This shows that the difference between $f_{n,ext}$ and $f_{n,mod}$ is just a rotation with the angle is an integer $t_{p,n}$ times $\frac{1}{(2m)^{2n}}$.

Furthermore, since

$$g_n|_{Annu(\frac{\delta}{n(2m)^n}, \frac{\delta}{K_0(2m)^n}) \times S^1}(q,t) \;=\; (\ A(q)\ ,\ \det(A) \cdot t + s_{p,n}/(2m)^{2n}\ )\ ,$$

we must have

$$\begin{aligned}
|\frac{s_{p,n}}{(2m^{2n})} - \frac{t_{p,n}}{(2m^{2n})}| \;&\leq\; C \cdot (d_{C^1}(f_{n,ext}, g_n|_{E_n}) \;+\; d_{C^1}(f_{n,mod}, f_0)), \\
&\longrightarrow\; 0 \qquad \text{as} \qquad n \to \infty.
\end{aligned}$$

Thus we can define that for any $p \in \pi(\Sigma_n)$,

$$f_n|_{\mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1} \triangleq R_{\frac{t_{p,n}}{(2m)^{2n}}} \circ f_{n,mod} \;:\; \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1 \longrightarrow \mathbb{D}(A(p), \frac{\delta}{(2m)^n}) \times S^1,$$

and

$$f_n|_{E_n} \overset{\triangle}{=} f_{n,ext} \ : \ E_n \longrightarrow f_A(E_n).$$

We have the following lemma.

Lemma 5.4.1. *The diffeomorphism f_n is well defined on $\mathcal{H}$, and*

$$\lim_{n\to\infty} d_{C^1}(f_n, f_A) = 0.$$

Proof. f_n is well defined since $f_{n,ext}$ is coincide with $R_{t_{p,n}/(2m)^{2n}} \circ f_{n,mod}$ on the intersection of their defining domains.

For the estimation of C^1-distance between, we have

$$\begin{aligned} d_{C^1}(f_A, f_n) \ &\leq \ d_{C^1}(f_A, g_n) \ + \ d_{C^1}(g_n, f_n), \\ &\leq \ d_{C^1}(f_A, g_n) \ + \ \max\{d_{C^1}(g_n|_{E_n}, f_{n,ext}), d_{C^1}(f_{n,mod}, f_0)\} \\ &\quad + \max_{p\in\pi(\partial\Sigma_n)}\{|\frac{s_{p,n}}{(2m)^{2n}} - \frac{t_{p,n}}{(2m)^{2n}}|\}, \\ &\longrightarrow \ 0 \qquad \text{as} \qquad n \longrightarrow \infty. \end{aligned}$$

This proved that f_n converge to f_A in C^1-topology as $n \to \infty$.

□

5.5 Hyperbolic Properties of f_n

We have construct the smooth diffeomorphisms f_n and show that they can C^1-approximate f_A as $n \to \infty$. Now we will prove that f_n is structurally stable. The proof is almost exactly the same to the case in [8], we just sketch it. Similarly, the chain recurrent set of is one attractor and one repeller we left after we finish the unit model of boundary fibers.

Proposition 5.5.1. *f_n satisfies Axiom-A and strong transversality condition, thus structurally stable.*

Proof. Recall that on the neighborhood of each boundary fiber S_p, we have the local coordinate $\mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1$. Under this coordinate and the way we define f_n, we can check that

$$\begin{aligned} U_{n,mod}|_{\mathbb{D}(p,\frac{\delta}{K_0(2m)^n})\times S^1} \cap E_n \ &= \ U_{n,ext} \cap \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1, \\ V_{n,mod}|_{\mathbb{D}(p,\frac{\delta}{K_0(2m)^n})\times S^1} \cap E_n \ &= \ V_{n,ext} \cap \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1, \end{aligned}$$

This allowed us to define the attracting region and repelling region:

$$\begin{aligned} U_n \ &= \ \bigcup_{p\in\pi(\Sigma_n)} U_{n,mod}|_{\mathbb{D}(p,\frac{\delta}{K_0(2m)^n})\times S^1} \bigcup U_{n,ext} \ , \\ V_n \ &= \ \bigcup_{p\in\pi(\Sigma_n)} V_{n,mod}|_{\mathbb{D}(p,\frac{\delta}{K_0(2m)^n})\times S^1} \bigcup V_{n,ext} \ . \end{aligned}$$

Thus U_n and V_n are disjoint compact sets. Moreover, we can check that $f_n(U_n) \subset \mathrm{Int}(U_n)$ and $f_n(V_n) \subset \mathrm{Int}(V_n)$. We denote $A_n = \cap_{i\in\mathbb{Z}} f_n^i(U_n)$, and $R_n = \cap_{i\in\mathbb{Z}} f_n^i(V_n)$.

Claim. *The chain recurrent set $\mathcal{R}(f_n)$ is contained in $A_n \cup R_n$.*

Proof of the Claim. By the contracting of U_n and repelling of V_n, we know that $\mathcal{R}(f_n) \cap U_n \subset A_n$, and $\mathcal{R}(f_n) \cap V_n \subset R_n$.

By the construction of $f_{n,ext}$, we know that for any point $x \in \mathcal{H} \setminus \cup_{p\in\pi(\partial\Sigma_n)} \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1$, if $x \notin U_n \cup V_n$, then $f_n(x) \in U_n$. So it is impossible that this point $x \in \mathcal{R}(f_n)$. This implies

$$\mathcal{R}(f_n) \cap (\mathcal{H} \setminus \cup_{p\in\pi(\partial\Sigma_n)} \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1) \subset A_n \cup R_n \ .$$

On the other hand, the maximal invariant set contained in $\cup_{p\in\pi(\partial\Sigma_n)} \mathbb{D}(p, \frac{\delta}{K_0(2m)^n}) \times S^1$ is equal to $\partial\Sigma_n$. For any point $x \in \mathcal{R}(f_n) \cap \partial\Sigma_n$, since f_n restrict on each boundary fiber is Morse-Smale, so ω-limit set of x is a periodic orbit in $\partial\Sigma_n$ which also in U_n. This implies $x \in U_n$. This finishes the proof of the claim. □

We continue to prove the proposition. Since the norm of central derivative $D^c f_n$ and $D^c f_n^{-1}$ are small or equal to α_n in U_n and V_n respectively, we can see that A_n and R_n are both hyperbolic sets with stable dimension 2 and 1. This implies $\mathcal{R}(f_n)$ is hyperbolic. So f_n is Axiom-A and has no cycle.

Furthermore, for any two hyperbolic set K and L of f_n, such that $W^u(K) \cap W^s(L) \neq \emptyset$, then

- either $K \cup L \subset U_n$,
- or $K \cup L \subset V_n$,
- or $K \subset V_n$ and $L \subset U_n$.

In all these three cases, we gets $\dim W^u(K) + \dim W^s(L) \geq 3 = \dim \mathcal{H}$. By the partial hyperbolicity and dynamical coherence of f_n, this guarantees the strong transversality property of f_n.

□

Chapter 6

Central DA-Construction

In this chapter, we will give a proof of proposition 5.2.1. That is construct a family of diffeomorphisms $\{f_{n,mod}\}_{n\in\mathbb{N}}$, which will be the stand models for our hyperbolic diffeomorphisms when close to the boundary fibers of the Birkhoff sections.

Actually, it can be seen that all these diffeomorphisms are derived from the DA-construction along the central direction of a fixed partially hyperbolic diffeomorphism. Such kind construction first appeared in the paper of Bonatti and Guelman [8]. However, they did not require any estimations about the C^1-distance of the stand models with the original partial hyperbolic diffeomorphism, which is a significant task and demand for us.

6.1 Proof of Proposition 5.2.1

We will first state a simplified technical lemma, and give the proof of proposition 5.2.1 by admitting this lemma.

Recall some notions and symbols. For the classical helicoid $\Sigma_H \subset \mathbb{R}^2 \times S^1$, we rotate Σ_H along the S^1-fibers with distance $1/2$, we get a parallel helicoid Σ'_H. We can see the formula of Σ'_H is

$$\begin{cases} x = \rho \cdot \cos 2\pi \cdot (\theta + 1/2) \ , \\ y = \rho \cdot \sin 2\pi \cdot (\theta + 1/2) \ , \\ z = \theta \pmod 1 \ . \end{cases}$$

For the hyperbolic matrix $A \in GL(2,\mathbb{Z})$, there exists a matrix P with $\det(P) > 0$, such that $P^{-1} \circ A \circ P = \mathrm{Diag}\{\det(A)\cdot\lambda, 1/\lambda\}$. Here $\det(A)\cdot\lambda, 1/\lambda$ are eigenvalues of A, and $|\lambda| > 1$. We fix the constant $T_0 \geq \max\{\|P\|, \|P^{-1}\|\}$. Since we will also consider diffeomorphisms on $\mathbb{R}^2 \times S^1$, so we will denote the central segments and central derivatives as before.

Lemma 6.1.1 (Technical Lemma)**.** *For any constant $\lambda > 1$, there exists a sequence of diffeomorphisms $F_n : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ and real numbers $0 < \alpha_n < 1$ where $\lim_{n\to\infty} \alpha_n = 1$, such that:*

1. *F_n preserve the S^1-fibers, and $\pi \circ F_n(x, y) = (\lambda \cdot x, 1/\lambda \cdot y)$ is a linear hyperbolic diffeomorphism on $\mathbb{R}^2$.*

2. *There exists two disjoint closed region $U^n, V^n \subset \mathbb{R}^2 \times S^1$, where U^n is strictly invariant by F_n: $F_n(U^n) \subset \mathrm{Int}(U^n)$; and V^n is strictly invariant by F_n^{-1}: $F_n^{-1}(V^n) \subset \mathrm{Int}(V^n)$.*

3. *Denote the region $Q_n = \{(x, y, z) : \sqrt{x^2 + y^2} \geq \frac{(2m)^n}{T_0 \cdot n}\}$, then*
$$U^n \cap Q_n = \bigcup_{p \in \Sigma_H \cap Q_n} [\ p - \frac{1}{2}\ ,\ p + \frac{1}{2}\ ]^c, \qquad V^n \cap Q_n = \bigcup_{p \in \Sigma'_H \cap Q_n} [\ p - \frac{\alpha_n}{2}\ ,\ p + \frac{\alpha_n}{2}\ ]^c.$$

4. *The restriction of F_n on the fixed fiber $(0, 0) \times S^1$ is a Morse-Smale diffeomorphism of the circle having four periodic points, two of them are in U^n with distance $1/2$, and two are in V^n also with distance $1/2$.*

5. *$F_n(\Sigma_H \cap Q_n) \subset \Sigma_H$, and $F_n(\Sigma'_H \cap Q_n) \subset \Sigma'_H$. Moreover, for any $p \in \Sigma_H \cap Q_n$, if we parameterize $[p - 1/2, p + 1/2]^c$ naturally to be $[-1/2, 1/2]$, and the same to $[F_n(p) - 1/2, F_n(p) + 1/2]^c$, then we have*
$$F_n|_{[p-\frac{1}{2}, p+\frac{1}{2}]^c}(t) = \ \frac{1}{2} \cdot \Theta_n(\ 2 \cdot t\ )\ ,$$
for all $t \in [-1/2, 1/2]$.

6. *The central derivative $D^c F_n$ and $D^c F_n^{-1}$ are uniformly contracting when restricted on U^n and V^n respectively.*

7. *The norm of partial derivatives $\|\partial F_{n,z}/\partial x\|$ and $\|\partial F_{n,z}/\partial y\|$ are uniformly bounded on $\mathbb{R}^2 \times S^1$, and the upper bound is independent on n .*

8. *The central derivative $D^c F_n$ uniformly converge to 1 on $\mathbb{R}^2 \times S^1$ as $n \to \infty$.*

We will first do some normalization of this lemma.

For every n, we consider the space $\mathbb{R}^2 \times (\mathbb{R}/(2m)^{2n}\mathbb{Z})$, which is naturally a $(2m)^{2n}$-cover of $\mathbb{R}^2 \times S^1$. So we will have the lift of half helicoid Σ_H and diffeomorphisms F_n on $\mathbb{R}^2 \times (\mathbb{R}/(2m)^{2n}\mathbb{Z})$, and the corresponding lift attracting region U^n and lift repelling region V^n. (Here we do not change the symbols on the $(2m)^{2n}$-cover $\mathbb{R}^2 \times (\mathbb{R}/(2m)^{2n}\mathbb{Z})$.)

Define the homothety $H_n : \mathbb{R}^2 \times (\mathbb{R}/(2m)^{2n}\mathbb{Z}) \longrightarrow \mathbb{R}^2 \times S^1$,
$$H_n(x, y, z) \ = \ (\frac{1}{n(2m)^{2n}} \cdot x,\ \frac{1}{n(2m)^{2n}} \cdot y,\ \frac{1}{(2m)^{2n}} \cdot z).$$
Then we can see that the image of half helicoid $H_n(\Sigma_H)$ could be represented as:
$$\begin{cases} x = \rho \cdot \cos 2\pi \cdot (2m)^{2n}\theta\ , \\ y = \rho \cdot \sin 2\pi \cdot (2m)^{2n}\theta\ , \\ z = \theta \ (\mathrm{mod}\ 1)\ . \end{cases}$$

Where $\theta \in \mathbb{R}$, $\rho \geq 0$, and we have $H_n(\Sigma'_H) = R_{\frac{1}{2(2m)^{2n}}} \circ H_n(\Sigma_H)$. Furthermore, recall the deformed half helicoid $S_n, S'_n \subset \mathbb{R}^2 \times S^1$, if we denote $P_0 = P \times \mathrm{Id} : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$, then we can see that

$$S_n = R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma_H), \qquad \text{and} \qquad S'_n = R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma'_H).$$

Lemma 6.1.2. *The sequence of diffeomorphisms*

$$H_n \circ F_n \circ H_n^{-1} : \mathbb{R}^2 \times S^1 \longrightarrow \mathbb{R}^2 \times S^1.$$

satisfies the following properties:

- *$H_n \circ F_n \circ H_n^{-1}$ preserve the S^1-fibers, and $\pi \circ H_n \circ F_n \circ H_n^{-1}(x, y) = (\lambda \cdot x, 1/\lambda \cdot y)$ is a linear hyperbolic diffeomorphism on $\mathbb{R}^2$.*
- *The two disjoint closed region $H_n(U^n), H_n(V^n) \subset \mathbb{R}^2 \times S^1$ satisfy*

$$H_n \circ F_n \circ H_n^{-1}(H_n(U^n)) = H_n \circ F_n(U^n) \subset H_n(\mathrm{Int}(U^n)) = \mathrm{Int}(H_n(U^n)),$$
$$(H_n \circ F_n \circ H_n^{-1})^{-1}(H_n(V^n)) = H_n \circ F_n^{-1}(V^n) \subset H_n(\mathrm{Int}(V^n)) = \mathrm{Int}(H_n(V^n)).$$

- *For the region $H_n(Q_n) = \{(x, y, z) : \sqrt{x^2 + y^2} \geq \frac{1}{T_0 \cdot n^2 (2m)^n}\}$, then*

$$H_n(U^n) \cap H_n(Q_n) = \bigcup_{p \in H_n(\Sigma_H) \cap H_n(Q_n)} [\, p - \frac{1}{4(2m)^{2n}} \,,\, p + \frac{1}{4(2m)^{2n}} \,]^c,$$
$$H_n(V^n) \cap H_n(Q_n) = \bigcup_{p \in H_n(\Sigma'_H) \cap H_n(Q_n)} [\, p - \frac{\alpha_n}{4(2m)^{2n}} \,,\, p + \frac{\alpha_n}{4(2m)^{2n}} \,]^c.$$

- *The restriction of $H_n \circ F_n \circ H_n^{-1}$ on the fixed fiber $(0, 0) \times S^1$ is a Morse-Smale diffeomorphism of the circle having $4 \cdot (2m)^{2n}$ periodic points, $2(2m)^{2n}$ of them are in $H_n(U^n)$ with neighboring distance $1/2(2m)^{2n}$, and the others are in $H_n(V^n)$ also with neighboring distance $1/2(2m)^{2n}$.*
- *For $H_n(\Sigma_H)$ and $H_n(\Sigma'_H)$, we have the invariant property:*

$$H_n \circ F_n \circ H_n^{-1}(\ H_n(\Sigma_H) \cap H_n(Q_n)\ ) = H_n \circ F_n(\Sigma_H \cap Q_n) \ \subset\ H_n(\Sigma_H),$$
$$H_n \circ F_n \circ H_n^{-1}(\ H_n(\Sigma'_H) \cap H_n(Q_n)\ ) = H_n \circ F_n(\Sigma'_H \cap Q_n) \ \subset\ H_n(\Sigma'_H).$$

- *For any $p \in H_n(\Sigma_H) \cap H_n(Q_n)$ If we parameterize $[p - \frac{1}{2(2m)^{2n}}, p + \frac{1}{2(2m)^{2n}}]^c$ naturally to be $[-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$, and the same to $[H_n \circ F_n \circ H_n^{-1}(p) - \frac{1}{2(2m)^{2n}}, H_n \circ F_n \circ H_n^{-1}(p) + \frac{1}{2(2m)^{2n}}]^c$, then we have*

$$H_n \circ F_n \circ H_n^{-1}|_{[p - \frac{1}{2(2m)^{2n}}, p + \frac{1}{2(2m)^{2n}}]^c}(t) = \frac{1}{2(2m)^{2n}} \cdot \Theta_n(\ 2(2m)^{2n} \cdot t\ )\ ,$$

for all $t \in [-\frac{1}{2(2m)^{2n}}, \frac{1}{2(2m)^{2n}}]$.

- *There exists a sequence of real numbers* $\{\alpha_n\}$, *where* $0 < \alpha_n < 1$ *and* $\lim_{n\to\infty}\alpha_n = 1$,*such that the central derivative* $D^c H_n \circ F_n \circ H_n^{-1}$ *and* $D^c H_n \circ F_n \circ H_n^{-1}$ *are small or equal to* α_n *in* $H_n(U^n)$ *and* $H_n(V^n)$ *respectively.*
- *For any integer* $k_0 \in \mathbb{Z}$, $H_n \circ F_n \circ H_n^{-1}$ *is commutable with the rotation* $R_{\frac{k_0}{(2m)^{2n}}}$ *through the* S^1*-fibers:*
$$R_{\frac{k_0}{(2m)^{2n}}} \circ (H_n \circ F_n \circ H_n^{-1}) \;=\; (H_n \circ F_n \circ H_n^{-1}) \circ R_{\frac{k_0}{(2m)^{2n}}}.$$
- $H_n \circ F_n \circ H_n^{-1}$ *uniformly converges to*
$$F_0 = \mathrm{Diag}\{\lambda, 1/\lambda\} \times \mathrm{Id} \;:\; \mathbb{R}^2 \times S^1 \;\longrightarrow\; \mathbb{R}^2 \times S^1 \;,$$
in the C^1*-topology as* $n \to \infty$.

Proof. We will focus on the last item, all the others could be translated directly from the technical lemma.

Since the central derivative of $H_n \circ F_n \circ H_n^{-1}$ converge to 1 uniformly, $H_n \circ F_n \circ H_n^{-1}$ converges to F_0 in the C^0-distance could be deduced from the fact that $\pi \circ H_n \circ F_n \circ H_n^{-1} = \mathrm{Diag}\{\lambda, 1/\lambda\}$, and
$$H_n \circ F_n \circ H_n^{-1}(\mathbb{R}^2 \times 0) \subset \mathbb{R}^2 \times [-\frac{1}{(2m)^{2n}}, \frac{1}{(2m)^{2n}}].$$

Since the differential matrix of DF_n could be represented as
$$DF_n \;=\; \begin{pmatrix} \partial F_{n,x}/\partial x & \partial F_{n,y}/\partial x & \partial F_{n,z}/\partial x \\ \partial F_{n,x}/\partial y & \partial F_{n,y}/\partial y & \partial F_{n,z}/\partial y \\ \partial F_{n,x}/\partial z & \partial F_{n,y}/\partial z & \partial F_{n,z}/\partial z \end{pmatrix} \;=\; \begin{pmatrix} \lambda & 0 & \partial F_{n,z}/\partial x \\ 0 & 1/\lambda & \partial F_{n,z}/\partial y \\ 0 & 0 & D^c F_n \end{pmatrix}$$

So we can see that
$$\begin{aligned} D(H_n \circ F_n \circ H_n^{-1}) \;&=\; DH_n \cdot DF_n \cdot DH_n^{-1} \\ &= \begin{pmatrix} \frac{1}{n(2m)^{2n}} & 0 & 0 \\ 0 & \frac{1}{n(2m)^{2n}} & 0 \\ 0 & 0 & \frac{1}{(2m)^{2n}} \end{pmatrix} \cdot DF_n \cdot \begin{pmatrix} n(2m)^{2n} & 0 & 0 \\ 0 & n(2m)^{2n} & 0 \\ 0 & 0 & (2m)^{2n} \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 & 1/n \cdot \partial F_{n,z}/\partial x \\ 0 & 1/\lambda & 1/n \cdot \partial F_{n,z}/\partial y \\ 0 & 0 & D^c F_n \end{pmatrix} . \end{aligned}$$

From the technical lemma, we know that $\|\partial F_{n,z}/\partial x\|$ and $\|\partial F_{n,z}/\partial y\|$ are uniformly bounded. Combined with the fact $D^c F_n$ tends to 1, we know that
$$\lim_{n\to\infty} D(H_n \circ F_n \circ H_n^{-1}) \;=\; \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; DF_0 \;.$$

This proves that $H_n \circ F_n \circ H_n^{-1}$ converge to F_0 in C^1-topology as $n \to \infty$.

□

Now we can prove proposition 5.2.1 from the technical lemma and its normalized version.

Proof of Proposition 5.2.1. We need to separate into three cases. The first two cases are $\det(A) = 1$, one is the eigenvales of A are positive, the other is the eigenvales of A are negative. The third case is where $\det(A) = -1$.

Case I. The two eigenvalues of A are both positive, denoted by $\lambda > 1$ and $0 < 1/\lambda < 1$.

Since we have already known that for the deformed helicoid S_n and S'_n, we have

$$S_n \;=\; R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma_H), \qquad \text{and} \qquad S'_n \;=\; R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma'_H)$$

where $P_0 = P \times \mathrm{Id} : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ and the matrix P satisfies $P^{-1} \circ A \circ P = \mathrm{Diag}\{\lambda, 1/\lambda\}$.

So we define $f_{n,mod}$ as follows:

$$f_{n,mod} \triangleq R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n \circ F_n \circ H_n^{-1} \circ P_0^{-1} \circ R_{-\frac{2\theta_0+1}{2(2m)^{2n}}} \;:\; \mathbb{R}^2 \times S^1 \;\longrightarrow\; \mathbb{R}^2 \times S^1.$$

Now we can check that $f_{n,mod}$ satisfies all the properties stated in proposition 5.2.1 one by one.

1. All the diffeomorphisms appeared in the definition preserve S^1-fibers, so does $f_{n,mod}$. Moreover, for $\pi \circ f_{n,mod} : \mathbb{R}^2 \to \mathbb{R}^2$, we have

$$\begin{aligned}
\pi \circ f_{n,mod} &= \mathrm{Id} \circ \pi(P_0) \circ \pi(H_n) \circ \pi(F_n) \circ \pi(H_n^{-1}) \circ \pi(P_0^{-1}) \circ \mathrm{Id} \\
&= P \circ (\frac{1}{n(2m)^{2n}} \cdot \mathrm{Id}) \circ \mathrm{Diag}\{\lambda, 1/\lambda\} \circ (n(2m)^{2n} \cdot \mathrm{Id}) \circ P^{-1} \\
&= P \circ \mathrm{Diag}\{\lambda, 1/\lambda\} \circ P^{-1} \\
&= A\,.
\end{aligned}$$

2. For the attracting region and repelling region, we define

$$U_{n,mod} \;\triangleq\; R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(U^n), \qquad V_{n,mod} \;\triangleq\; R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(V^n).$$

   Then from the property that $F_n(U^n) \subset \mathrm{Int}(U^n)$ and $F_n^{-1}(V^n) \subset \mathrm{Int}(V^n)$, we get $U_{n,mod}$ and $V_{n,mod}$ are strictly contracting by $f_{n,mod}$ and $f_{n,mod}^{-1}$ respectively.

3. Since we know that $T_0 \geq \|P^{-1}\|$ and $H_n(Q_n) = \{(x, y, z) : \sqrt{x^2 + y^2} \geq \frac{1}{T_0 \cdot n^2(2m)^n}\}$, we have

$$M_n = \{(x, y, z) : \sqrt{x^2 + y^2} \geq \frac{1}{n^2(2m)^n}\} \subset \; P_0 \circ H_n(Q_n).$$

Combined with $S_n = R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma_H)$, we get

$$\begin{aligned} U_{n,mod} \cap M_n &= \bigcup_{p\in S_n\cap M_n} [\, p - \frac{1}{4(2m)^{2n}} \,,\, p + \frac{1}{4(2m)^{2n}} \,]^c, \\ V_{n,mod} \cap M_n &= \bigcup_{p\in S'_n\cap M_n} [\, p - \frac{\alpha_n}{4(2m)^{2n}} \,,\, p + \frac{\alpha_n}{4(2m)^{2n}} \,]^c. \end{aligned}$$

4. $f_{n,mod}|_{(0,0)\times S^1} = R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ H_n \circ F_n \circ H_n^{-1} \circ R_{-\frac{2\theta_0+1}{2(2m)^{2n}}}|_{(0,0)\times S^1}$. From lemma 6.1.2, we know that $H_n \circ F_n \circ H_n^{-1}$ is Morse-Smale and have $4 \cdot (2m)^{2n}$ periodic points on $(0,0) \times S^1$. Moreover, notice that

$$\begin{aligned} U_{n,mod}|_{(0,0)\times S^1} &= R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ H_n(U^n)|_{(0,0)\times S^1}, \\ V_{n,mod}|_{(0,0)\times S^1} &= R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ H_n(V^n)|_{(0,0)\times S^1}. \end{aligned}$$

So this shows $2(2m)^{2n}$ of these periodic points are in $U_{n,mod}$, and the others are in $V_{n,mod}$, and the neighboring distance both are $1/2(2m)^{2n}$.

5. The invariance of S_n and S'_n comes from the way we define $f_{n,mod}$ and $M_n \subset P_0 \circ H_n(Q_n)$. For the action of $f_{n,mod}$ restricted on central fibers, we just notice when we guarantee S_n is invariant by $f_{mod,n}$, that is we fix the zero point of the central segment, P_0 and the rotation $R_{\frac{2\theta_0+1}{2(2m)^{2n}}}$ does not change the dynamics restricted on the central segment.

6. The central derivatives $D^c f_{n,mod}|_{U_{n,mod}}$ and $D^c f^{-1}_{n,mod}|_{V_{n,mod}}$ is the same to $D^c(H_n \circ F_n \circ H_n^{-1})|_{H_n(U^n)}$ and $D^c(H_n \circ F_n \circ H_n^{-1})^{-1}|_{H_n(V^n)}$, respectively. So from lemma 6.1.2, α_n is their upper bound.

7. Notice that $H_n \circ F_n \circ H_n^{-1}$ is commutable with $R_{\frac{k_0}{(2m)^{2n}}}$ for any $k_0 \in \mathbb{Z}$. The same property holds to P_0 and $R_{\frac{2\theta_0+1}{2(2m)^{2n}}}$, so $f_{n,mod}$ is commutable to $R_{\frac{k_0}{(2m)^{2n}}}$.

8. Since $H_n \circ F_n \circ H_n^{-1}$ converges to $\mathrm{Diag}\{\lambda, 1/\lambda\} \times \mathrm{Id}$, we get in C^1-topology,

$$\begin{aligned} \lim_{n\to\infty} P_0 \circ H_n \circ F_n \circ H_n^{-1} \circ P_0^{-1} &= P_0 \circ (\mathrm{Diag}\{\lambda, 1/\lambda\} \times \mathrm{Id}) \circ P_0^{-1} \\ &= A \times \mathrm{Id}\ . \end{aligned}$$

Finally, $\frac{2\theta_0+1}{2(2m)^{2n}}$ tends to zero as $n \to \infty$, we get $f_{n,mod}$ converges to $A \times \mathrm{Id}$.

Case II. The two eigenvalues of A are both negative, denoted by $-\lambda < -1$ and $-1 < -1/\lambda < 0$.

We consider $-A\times\mathrm{Id}$, then it could reduce to case I, and we denote $\overline{f}_{n,mod}$ the diffeomorphisms satisfying the properties of this proposition with respect to $-A\times\mathrm{Id}$, and $\overline{U}_{n,mod}$, $\overline{V}_{n,mod}$ be the corresponding attracting and repelling regions.

Notice that both surfaces S_n and S'_n is invariant under the symmetric action of

$$\begin{aligned} Sym_n \ : \ \mathbb{R}^2\times S^1 \ &\longrightarrow \ \mathbb{R}^2\times S^1 \ , \\ (x,y,z) \ &\longmapsto \ (-x,-y,z+\frac{1}{2\cdot(2m)^{2n}}) \ . \end{aligned}$$

Then we can define

$$f_{n,mod} \triangleq Sym_n\circ\overline{f}_{n,mod},$$

and $U_{n,mod} = Sym_n(\overline{U}_{n,mod})$, $V_{n,mod} = Sym_n(\overline{V}_{n,mod})$. Repeat the process in case I, it can prove that $f_{n,mod}$ satisfies all the properties we required. We omit it here, just remark that the 4th item need to use the neighboring distance of periodic points in $\overline{U}_{n,mod}$ and $\overline{V}_{n,mod}$ are $1/2(2m)^{2n}$.

Case III. Now $\det(A) = -1$, one of the eigenvalue is positive, the other one is negative. We can assume the two eigenvalues of A are $\lambda > 1$ and $-1 < -1/\lambda < 0$.

Now there exists a matrix P with $\det(P) > 0$, such that $P^{-1}\circ A\circ P = \mathrm{Diag}\{\lambda,-1/\lambda\}$. We define the reflection map $Refl:\mathbb{R}^2\times S^1\to\mathbb{R}^2\times S^1$ as:

$$Refl \ : \ (x,y,z) \ \longmapsto \ (x,-y,-z) \ .$$

Then we can see that

$$Refl\circ P_0^{-1}\circ(A\times -\mathrm{Id})\circ P_0 \ = \ F_0 \ = \ \mathrm{Diag}\{\lambda,1/\lambda\}\times\mathrm{Id}.$$

Notice that the reflection map $Refl$ preserve both the half helicoids Σ_H and Σ'_H invariant. And it is also commutable with the homothety H_n, which implies preserve $H_n(\Sigma_H)$ and $H_n(\Sigma'_H)$ invariant.

The same construction like case I., we define $f_{n,mod}:\mathbb{R}^2\times S^1\to\mathbb{R}^2\times S^1$ as follows:

$$f_{n,mod}\triangleq R_{\frac{2\theta_0+1}{2(2m)^{2n}}}\circ P_0\circ Refl\circ H_n\circ F_n\circ H_n^{-1}\circ P_0^{-1}\circ R_{-\frac{2\theta_0+1}{2(2m)^{2n}}}.$$

Then we can check the items in propostition 5.2.1 one by one, which is almost exactly the same of case I. Here we just point out the key fact is that the reflection map $Refl$ preserve the helicoid $H_n(\Sigma_H)$ invariant. The convergence comes from the fact that

$$\begin{aligned} \lim_{n\to\infty} P_0\circ Refl\circ H_n\circ F_n\circ H_n^{-1}\circ P_0^{-1} \ &= \ P_0\circ Refl\circ F_0\circ P_0^{-1} \\ &= \ A\times -\mathrm{Id} \ . \end{aligned}$$

This finishes the proof of the proposition.

$\square$

6.2 Proof of Lemma 6.1.1

For the rest part of this chapter, we will give the proof of the technical lemma 6.1.1. Actually, our constructions of the sequence of diffeomorphisms is quite similar to [8]. However, we need do some estimation about the Jacobian derivative of these diffeomorphisms.

For completeness, we repeat the constructions appeared in [8], and do the estimation step by step.

We first want to make some remarks about the technical lemma. Notice that the lemma requires to construct a sequence of diffeomorphisms $F_n : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$, satisfying some properties. We will assume that n is large enough, since from [8], it is no difficulties to construct a single, or finitely many diffeomorphisms like the central DA-construction. Our main task is to get the control of the derivatives of these diffeomorphisms.

6.2.1 Strategy and Sketch of Constructions

Our F_n need to preserve the S^1-fibers and satisfying $\pi \circ F_n(x,y) = (\lambda \cdot x, \frac{1}{\lambda} \cdot y)$, so we can separate $\mathbb{R}^2 \times S^1$ into quadrants which will be invariant by F_n:

$$\begin{aligned} C^{++} &= \{(x,y,z) : x \geq 0, y \geq 0\}\ , \qquad & C^{+-} &= \{(x,y,z) : x \geq 0, y \leq 0\}\ , \\ C^{-+} &= \{(x,y,z) : x \leq 0, y \geq 0\}\ , \qquad & C^{--} &= \{(x,y,z) : x \leq 0, y \leq 0\}\ . \end{aligned}$$

We will first do some surgery of the helicoid Σ_H and Σ'_H, which intends to separate them and make the new branching surfaces' tubular neighborhoods be our attracting and repelling regions.

Then we will construct $F_n^{\pm\pm}$ on each quadrants, and gluing them to the Morse-Smale diffeomorphisms $F_{n,0}$ defined on the invariant fiber $(0,0) \times S^1$, which guarantees that the gluing diffeomorphisms $F_{n,0}^{\pm\pm}$ coincide with $F_{n,0}$ on a neighborhood of $(0,0) \times S^1$, and equal to $F_n^{\pm\pm}$ when far from the center fiber.

The last and most difficult part is gluing $F_{n,0}^{\pm\pm}$ on the intersection of their definition domains, and all these steps need us handle carefully to estimation the derivative of diffeomorphisms.

We define a fixed C^∞ bump function $\psi : (-\infty, +\infty) \to [0,1]$, which satisfying the following properties:

- $\psi(t)|_{(-\infty,2]} \equiv 1$, $\psi(t)|_{[3,+\infty)} \equiv 0$, and $\psi(5/2) = 1/2$;
- the derivatives ψ' admits $-2 < \psi'(t)|_{(2,3)} < 0$;
- $\psi''(t)|_{(-\infty,5/2]} \leq 0$, and $\psi''(t)|_{[5/2,+\infty)} \geq 0$.

It allowed us to define a sequence of bump functions $\psi_n : [0,\infty) \to [0,1]$, which becomes more and more flat as $n \to \infty$:

$$\psi_n(t) = \psi(t^{\frac{2}{n}}), \qquad \forall\ t \in [0,\infty)\ .$$

We can see that ψ_n satisfying the following properties:

- $\psi_n(t) = 1$ for every $t \in [0, 2^{\frac{n}{2}}]$;
- $\psi_n(t) = 0$ for every $t \in [3^{\frac{n}{2}}, \infty)$;
- there exists some constant K, such that for any $t \geq 0$, we have
$$|\psi_n(t) - \psi_n(\lambda \cdot t)| \leq \frac{K \cdot (\lambda - 1)}{n}.$$

The last item is achieved by applying the mean value theorem. Since we have explained that we focus on the case where n is large enough, so we will always assuming that $\lambda \cdot (3^{\frac{n}{2}} + 1) < \frac{(2m)^n}{T_0 \cdot n}$ in the future.

6.2.2 Surgeries on Σ_H and Σ'_H

From the definition of helicoid, we know that $\Sigma_{H,-} = \Sigma_H \cap \{y \leq 0\}$ is diffeomorphic to $[1/2, 1] \times [0, +\infty)$. It intersects with the annulus $\{y = 0\} \subset \mathbb{R}^2 \times S^1$ is equal to

$$\{(x,y,z) : x \leq 0, y = 0, z = -\frac{1}{2}\} \ \cup\ \{(x,y,z) : x = y = 0, z \in [-\frac{1}{2}, 0]\}$$
$$\cup\ \{(x,y,z) : x \geq 0, y = z = 0\}.$$

Since our aim is to deform Σ_H and Σ'_H in order to separate them, so the region where need to do surgery is mainly on the neighborhood of the fiber $(0,0) \times S^1$. We will make a convex sum of $\Sigma_{H,-}$ and the half plane $\{(x,y,z) : y \leq 0, z = -\frac{1}{4}\}$. More accurately, the new surface derived from $\Sigma_{H,-}$ will be equal to $\{(x,y,z) : y \leq 0, z = -\frac{1}{4}\}$ when close to $(0,0) \times S^1$, and no change when the radius $r = \sqrt{x^2 + y^2}$ large enough.

As before, we need do a sequence of different surgeries. Denote

$$\Sigma^-_{A,n} \ = \ \{(\ x,\ y,\ z - \psi_n(r)(\frac{1}{4} + z)\ ) :\ (x,y,z) \in \Sigma_{H,-},\ r = \sqrt{x^2 + y^2}\}.$$

So it can be checked that $\Sigma_{A,n}$ is smooth and satisfying

- $\Sigma^-_{A,n} \cap \{r \leq 2^{\frac{n}{2}}\} \ = \ \{(x,y,z) : y \leq 0, z = -\frac{1}{4}\} \cap \{r \leq 2^{\frac{n}{2}}\}$;
- $\Sigma^-_{A,n} \cap \{r \geq 3^{\frac{n}{2}}\} \ = \ \Sigma_{H,-} \cap \{r \geq 3^{\frac{n}{2}}\}$.

Similarly, for $\Sigma_{H,+} = \Sigma_H \cap \{y \geq 0\} \subset \{(x,y,z) : z \in [0, \frac{1}{2}]\}$, we can define the convex sum of it with the half plane $\{(x,y,z) : y \geq 0, z = \frac{1}{4}\}$:

$$\Sigma^+_{A,n} \;=\; \{(\; x,\; y,\; z + \psi_n(r)(\frac{1}{4} - z)\;) : \; (x,y,z) \in \Sigma_{H,+},\; r = \sqrt{x^2+y^2}\}.$$

Notice that

$$\Sigma^-_{A,n} \cap \Sigma^+_{A,n} = \{y = 0, z = \frac{1}{2} = -\frac{1}{2} \in S^1, x \leq -3^{\frac{n}{2}}\} \cup \{y = 0, z = 0 \in S^1, x \geq 3^{\frac{n}{2}}\}.$$

We define $\Sigma_{A,n} = \Sigma^-_{A,n} \cup \Sigma^+_{A,n}$, then it satisfies the following properties:

1. $\Sigma_{A,n}$ is a branched surface with boundary and corners, its interior is smooth.
2. $\partial\Sigma_{A,n} \subset \{y = 0, x \in [-3^{\frac{n}{2}}, 3^{\frac{n}{2}}]\}$ consists of two segments:
   - $\{z = -\frac{1}{4}\psi_n(x), x \in [0, 3^{\frac{n}{2}}]\} \cup \{z = \frac{1}{4}\psi_n(-x) - \frac{1}{2}, x \in [-3^{\frac{n}{2}}, 0]\} \;\subset\; \partial\Sigma^-_{A,n}$;
   - $\{z = \frac{1}{4}\psi_n(x), x \in [0, 3^{\frac{n}{2}}]\} \cup \{z = -\frac{1}{4}\psi_n(-x) + \frac{1}{2}, x \in [-3^{\frac{n}{2}}, 0]\} \;\subset\; \partial\Sigma^+_{A,n}$.
3. The angle between the tangent plane field of $T\Sigma_{A,n}$, where it could be defined, and the x,y-plane tend to zero uniformly as $n \to \infty$. Notice that for any point in the half helicoid, its tangent plane will converge to the x,y-plane when its distance to the original fiber $(0,0) \times S^1$ tend to infinity. Both $\Sigma^-_{A,n}$ and $\Sigma^+_{A,n}$ are the convex sum of the half helicoid with some half plane parallel to the x,y-plane. Moreover, the regions of $\Sigma^-_{A,n}$ and $\Sigma^+_{A,n}$ that are not parallel to the x,y-plane will be uniformly far away from the original fiber $(0,0) \times S^1$. And the convex sum of two surface whose plane fields are close the x,y-plane field will be also close to the x,y-plane field. This shows that angle between the tangent plane field of $T\Sigma_{A,n}$ and x,y-plane uniformly converge to 0 as n tends to infinity.

In the same way, we surgery Σ'_H, but along the y-direction. That is make the convex sum of $\Sigma'_{H,-} = \Sigma'_H \cap \{x \leq 0\}$ and $\Sigma'_{H,+} = \Sigma'_H \cap \{x \geq 0\}$ to the planes $\{z = 0\}$ and $\{z = 1/2\}$ respectively.

For $(x,y,z) \in \Sigma'_{H,-} \subset \{(x,y,z) : z \in [-\frac{1}{4}, \frac{1}{4}]\}$, define:

$$\Sigma^-_{R,n} \;=\; \{(\; x,\; y,\; z - \psi_n(r)z\;) : \; (x,y,z) \in \Sigma'_{H,-},\; r = \sqrt{x^2+y^2}\}.$$

For $(x,y,z) \in \Sigma'_{H,+} \subset \{(x,y,z) : z \in [\frac{1}{4}, \frac{3}{4}]\}$, define:

$$\Sigma^+_{R,n} \;=\; \{(\; x,\; y,\; z + \psi_n(r)(\frac{1}{2} - z)\;) : \; (x,y,z) \in \Sigma'_{H,+},\; r = \sqrt{x^2+y^2}\}.$$

Then we have

$$\Sigma^-_{R,n} \cap \Sigma^+_{R,n} = \{x = 0, z = \frac{1}{4} \in S^1, y \leq -3^{\frac{n}{2}}\} \cup \{x = 0, z = \frac{3}{4} \in S^1, y \geq 3^{\frac{n}{2}}\}.$$

Denote $\Sigma_{R,n} = \Sigma^-_{R,n} \cup \Sigma^+_{R,n}$, then it satisfies the following properties:

1. $\Sigma_{R,n}$ is a branched surface with boundary and corners, its interior is smooth.

2. $\partial\Sigma_{R,n} \subset \{x=0, y\in[-3^{\frac{n}{2}}, 3^{\frac{n}{2}}]\}$ consists of two segments:

   - $\{z=\frac{1}{4}\psi_n(y)-\frac{1}{4}, y\in[0,3^{\frac{n}{2}}]\}\cup\{z=-\frac{1}{4}\psi_n(-y)+\frac{1}{4}, y\in[-3^{\frac{n}{2}},0]\} \subset \partial\Sigma^-_{R,n}$;
   - $\{z=-\frac{1}{4}\psi_n(x)+\frac{3}{4}, x\in[0,3^{\frac{n}{2}}]\}\cup\{z=\frac{1}{4}\psi_n(-x)+\frac{1}{4}, x\in[-3^{\frac{n}{2}},0]\} \subset \partial\Sigma^+_{R,n}$.

3. The angle between the tangent plane field of $T\Sigma_{R,n}$, where it could be defined, and the x,y-plane tend to zero uniformly as $n\to\infty$.

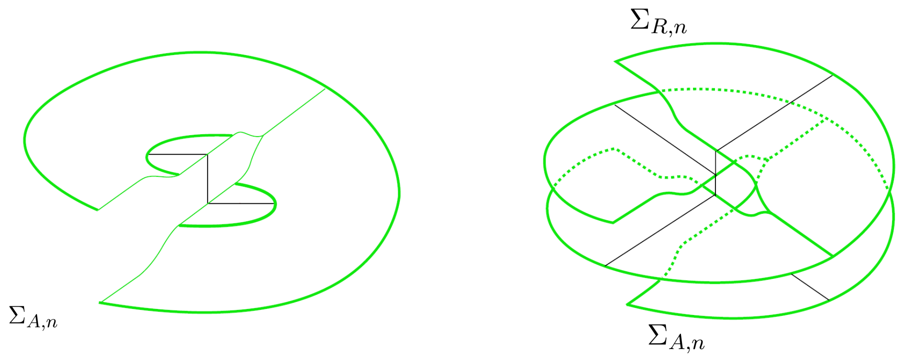

Figure 6.1: *Separating* $\Sigma_{A,n}$ *and* $\Sigma_{R,n}$

Lemma 6.2.1. *For any n, the surfaces $\Sigma_{A,n}$ and $\Sigma_{R,n}$ are disjoint.*

This is lemma 7.1 of [8], we sketch the proof for completeness.

Proof. Notice that two annulus $\{x=0\}$ and $\{y=0\}$ cut $\mathbb{R}^2\times S^1$ into four disjoint regions, which are the interior of $C^{\pm\pm}$. On the invariant fiber $(0,0)\times S^1$, $\Sigma_{A,n}$ intersect it at $z=\frac{1}{4}$ and $z=\frac{3}{4}$, $\Sigma_{R,n}$ intersect it at $z=0$ and $z=\frac{1}{2}$.

For $\{x>0, y=0\}$, we have

$$\Sigma_{A,n}\cap\{x>0,y=0\}\subset\{x>0,y=0,z\in[-\frac{1}{4},\frac{1}{4}]\},$$

and $\Sigma_{R,n}\cap\{x>0,y=0\}$ is equal to $\{x>0,y=0,z=\frac{1}{2}\}$, so they are disjoint. Similarly results hold for $\{x>0,y=0\}$, $\{x=0,y>0\}$, and $\{x=0,y<0\}$.

We just need do deal with inside the regions $\mathrm{Int}(C^{\pm\pm})$. $\mathrm{Int}(C^{++})$, for instance, we can check that

$$\mathrm{Int}(C^{++})\cap\Sigma_{A,n} \ \subset \ \{\ x>0, y>0, z\in[0,\frac{1}{4}]\ \}\ ,$$
$$\mathrm{Int}(C^{++})\cap\Sigma_{R,n} \ \subset \ \{\ x>0, y>0, z\in[\frac{1}{2},\frac{3}{4}]\ \}\ .$$

This implies $\mathrm{Int}(C^{++}) \cap \Sigma_{A,n}$ and $\mathrm{Int}(C^{++}) \cap \Sigma_{R,n}$ are disjoint. The same analysis works for other three regions.

□

We now define a projection map from $\Sigma_{A,n} \setminus (0,0) \times S^1$ to $\Sigma_H \setminus (0,0) \times S^1$ which will be needed in the last part of this paper.

Definition 6.2.2. *We define the projection map*

$$\pi_{\Sigma_{A,n}} \;:\; \Sigma_{A,n} \setminus (0,0) \times S^1 \;\longrightarrow\; \Sigma_H \setminus (0,0) \times S^1$$

as

- *for* $x \in \Sigma^+_{A,n} \setminus (0,0) \times S^1$, $\pi_{\Sigma_{A,n}}(x)$ *is the intersecting point of the* S^1*-fiber containing* x *with* Σ_H*;*
- *for* $x \in \Sigma^-_{A,n} \setminus (0,0) \times S^1$, $\pi_{\Sigma_{A,n}}(x)$ *is the intersecting point of the* S^1*-fiber containing* x *with* Σ_H*;*

Then we can see that $\pi_{\Sigma_{A,n}}$ *is an injection when restricted on Int*$(\Sigma_{A,n})$*; and maps two points into one point when restricted on* $\partial\Sigma_{A,n} \setminus (0,0) \times S^1$*.*

6.2.3 Central segments cut by $\Sigma_{A,n}$ and $\Sigma_{R,n}$

We will give some estimations about the segments cut by $\Sigma_{A,n}$ and $\Sigma_{R,n}$ for each S^1-fiber.

For any $(x,y) \times S^1 \subset C^{++}$, it intersects with $\Sigma^+_{A,n}$ and $\Sigma^+_{R,n}$ with exactly one point respectively. Denote them by $p^{++}_n(x,y) \in \Sigma^+_{A,n}$ and $q^{++}_n(x,y) \in \Sigma^+_{R,n}$. Then we can define the central interval with positive orientation:

$$I^{++}_n(x,y) = [p^{++}_n(x,y), q^{++}_n(x,y)]^c \,, \qquad \text{and} \qquad J^{++}_n(x,y) = [q^{++}_n(x,y), p^{++}_n(x,y)]^c \,.$$

From the way we do surgeries, it can see that these two intervals satisfying $\frac{1}{4} \le |I^{++}_n(x,y)| \le \frac{1}{2}$, and $\frac{1}{2} \le |J^{++}_n(x,y)| \le \frac{3}{4}$. Moreover, there exists a sequence of real numbers $0 < \beta_n < 1$, where $\beta_n \to 1$ as $n \to \infty$, such that

$$\beta_n \le \frac{|I^{++}_n(\lambda x, \frac{1}{\lambda}y)|}{|I^{++}_n(x,y)|} \le \frac{1}{\beta_n} \,, \qquad \text{and} \qquad \beta_n \le \frac{|J^{++}_n(\lambda x, \frac{1}{\lambda}y)|}{|J^{++}_n(x,y)|} \le \frac{1}{\beta_n} \,.$$

This properties can achieved by the fact that the bump function ψ_n we used to make the convex sum of surface satisfying $|\psi_n(t) - \psi_n(\lambda \cdot t)| \le \frac{8(\lambda-1)}{n}$.

Similarly, for other three quadrants, we have

- For $(x,y) \times S^1 \subset C^{+-}$, we consider it intersects with $\Sigma^+_{A,n}$ and $\Sigma^-_{R,n}$ at $p^{+-}_n(x,y)$ and $q^{+-}_n(x,y)$ respectively. Similarly define $I^{+-}_n(x,y)$ and $J^{+-}_n(x,y)$, then $\frac{1}{2} \le |I^{+-}_n(x,y)| \le \frac{3}{4}$, and $\frac{1}{4} \le |J^{+-}_n(x,y)| \le \frac{1}{2}$.

- For $(x, y) \times S^1 \subset C^{-+}$, we consider it intersects with $\Sigma^-_{A,n}$ and $\Sigma^+_{R,n}$ at $p^{-+}_n(x, y)$ and $q^{-+}_n(x, y)$ respectively. Similarly define $I^{-+}_n(x, y)$ and $J^{-+}_n(x, y)$, then $\frac{1}{2} \le |I^{-+}_n(x, y)| \le \frac{3}{4}$, and $\frac{1}{4} \le |J^{-+}_n(x, y)| \le \frac{1}{2}$.
- For $(x, y) \times S^1 \subset C^{--}$, we consider it intersects with $\Sigma^-_{A,n}$ and $\Sigma^-_{R,n}$ at $p^{--}_n(x, y)$ and $q^{--}_n(x, y)$ respectively. Similarly define $I^{--}_n(x, y)$ and $J^{--}_n(x, y)$, then $\frac{1}{4} \le |I^{--}_n(x, y)| \le \frac{1}{2}$, and $\frac{1}{2} \le |J^{--}_n(x, y)| \le \frac{3}{4}$.

And we have the following lemma.

Lemma 6.2.3. *There exists a sequence of real numbers $0 < \beta_n < 1$ which satisfying $\lim_{n\to\infty} \beta_n = 1$, such that on each quadrant where we can define the cutting central interval $I^{\pm\pm}_n(x, y)$ and $J^{\pm\pm}_n(x, y)$, we have*

$$\beta_n \le \frac{|I^{\pm\pm}_n(\lambda x, \frac{1}{\lambda} y)|}{|I^{\pm\pm}_n(x, y)|} \le \frac{1}{\beta_n}\ , \qquad \text{and} \qquad \beta_n \le \frac{|J^{\pm\pm}_n(\lambda x, \frac{1}{\lambda} y)|}{|J^{\pm\pm}_n(x, y)|} \le \frac{1}{\beta_n}\ .$$

Moreover, the norm of partial derivatives for their length

$$\|\frac{\partial |I^{\pm\pm}_n(x, y)|}{\partial x}\|,\ \|\frac{\partial |I^{\pm\pm}_n(x, y)|}{\partial y}\|, \qquad \text{and} \qquad \|\frac{\partial |J^{\pm\pm}_n(x, y)|}{\partial x}\|,\ \|\frac{\partial |J^{\pm\pm}_n(x, y)|}{\partial y}\|,$$

are uniformly converge to zero as n tend to infinity.

As remarked before, we focus on the case where n large enough, so it can be assumed that β_n is very close to 1.

Notice that the four quadrants have some intersections, and for the intersecting sets, we have the following lemma.

Lemma 6.2.4. *In the intersecting set of different quadrants, we have*

- *If $x \ge 3^{\frac{n}{2}}, y = 0$, then $p^{++}_n(x, y) = p^{+-}_n(x, y) \in \Sigma^+_{A,n} \cap \Sigma^-_{A,n}$, and $q^{++}_n(x, y) = q^{+-}_n(x, y) \in \Sigma^+_{R,n}$.*
- *If $x = 0, y \ge 3^{\frac{n}{2}}$, then $p^{-+}_n(x, y) = p^{++}_n(x, y) \in \Sigma^+_{A,n}$, and $q^{-+}_n(x, y) = q^{++}_n(x, y) \in \Sigma^+_{R,n} \cap \Sigma^-_{R,n}$.*
- *If $x \le -3^{\frac{n}{2}}, y = 0$, then $p^{--}_n(x, y) = p^{-+}_n(x, y) \in \Sigma^+_{A,n} \cap \Sigma^-_{A,n}$, and $q^{--}_n(x, y) = q^{-+}_n(x, y) \in \Sigma^-_{R,n}$.*
- *If $x = 0, y \le -3^{\frac{n}{2}}$, then $p^{+-}_n(x, y) = p^{--}_n(x, y) \in \Sigma^-_{A,n}$, and $q^{+-}_n(x, y) = q^{--}_n(x, y) \in \Sigma^+_{R,n} \cap \Sigma^-_{R,n}$.*

6.2.4 A family of segment diffeomorphisms

First we state a lemma about the existence of a smooth family of interval diffeomorphisms, which we will admit it directly.

Lemma 6.2.5. *There is a smooth function* $\sigma : [0,1] \times (0,+\infty)^2 \to [0,1]$ *such that for any* $a, b > 0$*, the map* $\sigma_{a,b} : [0,1] \to [0,1]$ *is an increasing diffeomorphism satisfying:*

- $\sigma_{a,b}(x) = ax$*, for* $0 \leq x \ll 1$.
- $\sigma_{a,b}(x) = 1 - b(1-x)$*, for* $0 \leq (1-x) \ll 1$.
- $\sigma_{1,1} = \mathrm{Id}|_{[0,1]}$.
- $\max\{|\sigma'(t) - 1| : t \in [0,1]\} \leq 2\max\{|a-1|, |b-1|\}$.
- $\sigma_{a,a^{-1}} = \sigma^{-1}_{a^{-1},a}$*, for any* $a > 0$.

Then for two segment I, J, we consider the diffeomorphism defined as

$$\sigma_{a,b,I,J} = \Phi_J^{-1} \circ \sigma_{a\frac{l(I)}{l(J)}, b\frac{l(I)}{l(J)}} \circ \Phi_I \ : \ I \longrightarrow J \ ,$$

where $\Phi_I : I \to [0,1]$ and $\Phi_J : J \to [0,1]$ are the canonical affine diffeomorphisms. Then $\sigma_{a,b,I,J}$ satisfying the following properties:

- The derivative of $\sigma_{a,b,I,J}$ at the origin of I is a, at the end point of I is b.
- The derivative of $\sigma_{a,b,I,J}$ uniformly tends to 1 as a, b and $l(I)/l(J)$ tend to 1.

Since for our construction of diffeomorphisms, we also need to prove the boundedness of partial derivatives, we need the following lemma.

Lemma 6.2.6. *For any constant* $L_0 > 1$*, we consider two family of intervals* $\{I(s), J(s) : s \in (-\delta,\delta)\}$*, where the lengths* $l(I(s))$ *and* $l(J(s))$ *various smoothly with the parameter* s*, and two smooth function* $a(s), b(s) : (-\delta,\delta) \to (0,+\infty)$*, which satisfies the following properties:*

- $1/L_0 < l(J(s))/l(I(s)) < L_0$*, and* $l(I(s)), l(J(s)), l(I(s))', l(J(s))' < L_0$*;*
- $1/L_0 < a(s), b(s) < L_0$*, and* $a'(s), b'(s) < L_0$*;*

then there exists a constant $K = K(\sigma, L_0)$*, such that for the diffeomorphism*

$$G(s,t) \ \triangleq \ \sigma_{a(s),b(s),I(s),J(s)}(t) \ : \ (-\delta,\delta) \times I(s) \ \longrightarrow \ (-\delta,\delta) \times J(s),$$

where $s \in (-\delta,\delta)$ *and* $t \in I(s)$*, it admits*

$$\| \frac{\partial G(s,t)}{\partial t} \| \ \leq \ K \qquad \text{and} \qquad \| \frac{\partial G(s,t)}{\partial s} \| \ \leq \ K \ .$$

Proof. We can see that this lemma is the consequence of a compactness argument if we assume that length $l(I(s))$ and $l(J(s))$ are uniformly bounded and away from zero. Our main difficulties comes from the analysis when $l(I(s))$ and $l(J(s))$ tend to zero.

From the definition of $\sigma_{a,b,I,J}$, we can see that it composed by following maps:

- $G_I(s,t) : (-\delta,\delta) \times I(s) \to (-\delta,\delta) \times [0,1]$ is the canonical affine maps on each vertical intervals from $I(s)$ to $[0,1]$;
- $G_0(s,t) = \sigma_{a(s)\frac{l(I(s))}{l(J(s))}, b(s)\frac{l(I(s))}{l(J(s))}}(t) : (-\delta,\delta) \times [0,1] \to (-\delta,\delta) \times [0,1]$;
- $G_J^{-1}(s,t) : (-\delta,\delta) \times [0,1] \to (-\delta,\delta) \times I(s)$ is the canonical affine maps on each vertical intervals from $[0,1]$ to $J(s)$.

Then we have $G(s,t) = G_J^{-1}(s,t) \circ G_0(s,t) \circ G_I(s,t)$.

Now we try to give some estimation of the vectors $\partial/\partial t$ and $\partial/\partial s$ acting by the differential operators of above smooth maps.

For the estimation of $\partial G(s,t)/\partial t$, it just need to notice that G preserve the vertical segments for each steps of mapping, and $\frac{l(I(s))}{l(J(s))}, a(s)\frac{l(I(s))}{l(J(s))}, b(s)\frac{l(I(s))}{l(J(s))}$ are all uniformly bounded and away from zero. So it is a result of compactness of the domain for $\sigma_{a,b}$.

For the estimation of $\partial G(s,t)/\partial s$, it is more complicated. First, there exists some constant $K_1 = K_1(L_0)$ such that

$$\|DG_I(\frac{\partial}{\partial s})|_{(s,t)}\| \;\leq\; \|\frac{\partial}{\partial s}\| \;+\; \frac{K_1}{l(I(s))} \cdot \|\frac{\partial}{\partial t}\|.$$

Then there exists some constant $K_2 = K_2(L_0)$ such that $\|DG_0(\frac{\partial}{\partial t})|_{(s,t)}\| \;\leq\; K_2 \cdot \|\frac{\partial}{\partial t}\|$. This is because DG_0 preserve the vertical segments and the vertical derivatives only depends on $a(s)\frac{l(I(s))}{l(J(s))}$ and $b(s)\frac{l(I(s))}{l(J(s))}$ which are both belong to the range $[1/L_0^2, L_0^2]$.

Moreover, we have

$$\begin{aligned}
\|DG_0(\frac{\partial}{\partial s})|_{(s,t)}\| &\leq \|\frac{\partial \sigma_{a,b}}{\partial a}\frac{d(a(s)\frac{l(I(s))}{l(J(s))})}{ds}\| \cdot \|\frac{\partial}{\partial t}\| + \|\frac{\partial \sigma_{a,b}}{\partial b}\frac{d(b(s)\frac{l(I(s))}{l(J(s))})}{ds}\| \cdot \|\frac{\partial}{\partial t}\| + \|\frac{\partial}{\partial s}\| \\
&\leq 2K_2 \cdot [\; L_0^2 + L_0 \cdot \frac{d(l(I(s))/l(J(s)))}{ds} \;] \cdot \|\frac{\partial}{\partial t}\| + \|\frac{\partial}{\partial s}\| \\
&\leq \frac{K_3}{l(I(s))} \cdot \|\frac{\partial}{\partial t}\| + \|\frac{\partial}{\partial s}\|.
\end{aligned}$$

Here the constant K_3 also only relies on L_0. The first inequality is the chain rule. The second and third inequalities all came from the boundedness of $a(s), b(s), a'(s), b'(s)$, and $\frac{l(I(s))}{l(J(s))}$.

Combine these two steps of estimations, we have

$$\|DG_0 \circ DG_I(\frac{\partial}{\partial s})|_{(s,t)}\| \;\leq\; \frac{K_1K_2 + K_3}{l(I(s))} \cdot \|\frac{\partial}{\partial t}\| \;+\; \|\frac{\partial}{\partial s}\|.$$

And finally, we have some constant K_4, such that

$$\begin{aligned}\|DG_J^{-1}(\frac{\partial}{\partial s})|_{(s,t)}\| \ &\leq \ L_0 \cdot \|\frac{\partial}{\partial t}\| \ + \ \|\frac{\partial}{\partial s}\| \ , \\ \|DG_J^{-1}(\frac{\partial}{\partial t})|_{(s,t)}\| \ &\leq \ K_4 \cdot l(J(s)) \cdot \|\frac{\partial}{\partial t}\| \ .\end{aligned}$$

Thus we applying the boundedness of $l(J(s))/l(I(s))$ again, which shows that there exists some constant K satisfying

$$\|DG(\frac{\partial}{\partial s})|_{(s,t)}\| \ \leq \ K \cdot \|\frac{\partial}{\partial t}\| \ + \ \|\frac{\partial}{\partial s}\| \ .$$

This finishes the proof of boundedness of partial derivatives of $\partial G(s,t)/\partial s$.

□

Remark. *For our future constructions, they are all from central fibers to central fibers with the segments diffeomorphisms like $\sigma_{a,b,I,J}$. And it is unavoidable to dealing with the cases where $l(I)$ and $l(J)$ tend to zero. This lemma tell us that for the estimation of partial derivatives, we don't need to worry this problem, just guarantee that $l(J)/l(I)$ and the partial derivative of $l(J), l(I)$ is uniformly bounded is enough.*

Lemma 6.2.7. *For any sequence of real numbers $\{\beta_n\}$, which satisfying $0 < \beta_n < 1$ and $\lim_{n\to\infty}\beta_n = 1$, there exists $\{\alpha_n\}$ such that*

- $0 < \alpha_n < \beta_n^3 < 1$, *and* $\lim_{n\to\infty}\alpha_n = 1$.
- $\lim_{n\to\infty}(1-\beta_n)/(1-\alpha_n) = 0$.

Now we can define a family of segments diffeomorphisms.

Definition 6.2.8. *Let $I = [0,a]$, $J = [0,b]$ be two segments where $\beta_n < b/a < 1/\beta_n$. We denote by $\Psi^+_{n,I,J} : I \to J$ the diffeomorphism defined as follows:*

- *For $t \in U^+_{n,I} = [0, \frac{a}{2}]$, $\Psi^+_{n,I,J}(t) = \alpha_n t$.*
- *For $t \in V^+_{n,I} = [a - \frac{a\alpha_n}{2}, a]$, $\Psi^+_{n,I,J}(t) = b - (\alpha_n^{-1}(a-t))$.*
- *Denote $I^+_n = [\frac{a}{2}, a - \frac{a\alpha_n}{2}]$, $J^+_n = [\frac{a\alpha_n}{2}, b - \frac{a}{2}]$. For $t \in I^+_n$, one defines $\Psi^+_{n,I,J}(t) = \sigma_{\alpha_n,\alpha_n^{-1},I^+_n,J^+_n}(t)$.*

Here we require the constant α_n satisfying the above lemma. And we can similarly define $\Psi^-_{n,I,J} : I \to J$ as follows:

- *For $t \in V^-_{n,I} = [0, \frac{a\alpha_n}{2}]$, $\Psi^-_{n,I,J}(t) = \alpha_n^{-1}t$.*
- *For $t \in U^-_{n,I} = [\frac{a}{2}, a]$, $\Psi^+_{n,I,J}(t) = b - (\alpha_n(a-t))$.*

- *Denote* $I_n^- = [\frac{a\alpha_n}{2}, \frac{a}{2}]$, $J_n^- = [\frac{a}{2}, b - \frac{a\alpha_n}{2}]$. *For* $t \in I_n^-$, *one defines* $\Psi^+_{n,I,J}(t) = \sigma_{\alpha_n^{-1},\alpha_n,I_n^-,J_n^-}(t)$.

Actually, here $U^{\pm}_{n,I}$ and $V^{\pm}_{n,I}$ will be our attracting and repelling regions restricted on the center fibers. Notice that their definition do not depend on J, only n and I itself. Then it can seen that $\Psi^{\pm}_{n,I,J}$ satisfying properties.

Lemma 6.2.9. *The derivative of* $\Psi^+_{n,I,J}$ *and* $\Psi^-_{n,I,J}$ *converge to 1 uniformly as* n *tend to infinity. Moreover, we have*

$$\Psi^{\pm}_{n,I,J}(U^{\pm}_{n,I}) \subset \mathrm{Int}(U^{\pm}_{n,J}) \ , \qquad \text{and} \qquad (\Psi^{\pm}_{n,I,J})^{-1}(V^{\pm}_{n,J}) \subset \mathrm{Int}(V^{\pm}_{n,I}) \ .$$

Proof. The convergence of derivatives comes from the fact that $\alpha_n \to 1$, and $\lim_{n\to\infty}(1-\beta_n)/(1-\alpha_n) = 0$ guarantees that $\lim_{n\to\infty} l(J_n^{\pm})/l(I_n^{\pm}) = 1$. The second part is correct since $\alpha_n < \beta_n^3$ and $\beta_n < b/a < 1/\beta_n$. □

Remark. *From the definition of* $\Psi^{\pm}_{n,I,J}$, *we can see that if we consider a family of such kind segment diffeomorphisms, then the partial derivatives with respect to the parameters of the family are uniformly bounded, if it satisfies*

- $l(J)/l(I)$ *are uniformly bounded and away from zero;*
- *the partial derivatives of* $l(I), l(J), \alpha_n$ *with respect to the parameters of the family are uniformly bounded;*
- $l(J_n^{\pm})/l(I_n^{\pm})$ *are uniformly bounded and away from zero, which is guaranteed by the fact that* $\lim_{n\to\infty}(1-\beta_n)/(1-\alpha_n) = 0$.

This implies we can also deal the case where $l(I), l(J), l(I_n^{\pm})$, *and* $l(J_n^{\pm})$ *tend to zero. The proof is exactly the same with lemma 6.2.6.*

6.2.5 Diffeomorphisms on the quadrants

Definition 6.2.10. *We define a sequence of diffeomorphisms* $F_n^{++} : C^{++} \to C^{++}$ *as follows, for every* $(x,y) \in \pi(C^{++}) = \{x \geq 0, y \geq 0\}$:

- *for* $p_n^{++}(x,y) \in \Sigma^+_{A,n} \cap C^{++}$, *we name* $F_n^{++}(p_n^{++}(x,y)) = p_n^{++}(\lambda x, \frac{1}{\lambda} y) \in \Sigma^+_{A,n}$.
- *for* $q_n^{++}(x,y) \in \Sigma^+_{R,n} \cap C^{++}$, *we name* $F_n^{++}(q_n^{++}(x,y)) = q_n^{++}(\lambda x, \frac{1}{\lambda} y) \in \Sigma^+_{R,n}$.
- *for* $I_n^{++}(x,y) = [p_n^{++}(x,y), q_n^{++}(x,y)]^c$, *and* $J_n^{++}(x,y) = [q_n^{++}(x,y), p_n^{++}(x,y)]^c$, *we define*
  - $F_n^{++}|_{I_n^{++}(x,y)} = \Psi^+_{n,I_n^{++}(x,y),I_n^{++}(\lambda x,\frac{1}{\lambda}y)} : I_n^{++}(x,y) \longrightarrow I_n^{++}(\lambda x, \frac{1}{\lambda} y)$;
  - $F_n^{++}|_{J_n^{++}(x,y)} = \Psi^-_{n,J_n^{++}(x,y),J_n^{++}(\lambda x,\frac{1}{\lambda}y)} : J_n^{++}(x,y) \longrightarrow J_n^{++}(\lambda x, \frac{1}{\lambda} y)$.

From the definition of F_n^{++} restricted on $I_n^{++}(x,y)$ and $J_n^{++}(x,y)$, we can define two closed disjoint regions U_n^{++} and V_n^{++} contained in C^{++} as follows:

$$\begin{aligned} U_n^{++} &= \bigcup_{(x,y)\in\pi(C^{++})} U^+_{n,I_n^{++}(x,y)} \cup U^-_{n,J_n^{++}(x,y)}, \\ V_n^{++} &= \bigcup_{(x,y)\in\pi(C^{++})} V^+_{n,I_n^{++}(x,y)} \cup V^-_{n,J_n^{++}(x,y)}. \end{aligned}$$

Lemma 6.2.11. *The map F_n^{++} is a well defined diffeomorphism on C^{++}, and we have*

$$F_n^{++}(U_n^{++}) \subset \mathrm{Int}(U_n^{++}), \qquad \text{and} \qquad (F_n^{++})^{-1}(V_n^{++}) \subset \mathrm{Int}(V_n^{++}).$$

Furthermore, there exists a constant $K_0 > 0$ such that

- *for each n, we always have* $\sup_{C^{++}}\{\ \|\partial F^{++}_{n,z}/\partial x\|,\ \|\partial F^{++}_{n,z}/\partial y\|\ \} \leq K_0$.
- $\lim_{n\to\infty} D^c F_n^{++} = 1$.

Proof. F_n^{++} is a smooth diffeomorphism since the two surface $\Sigma^+_{A,n}$ and $\Sigma^+_{R,n}$ are smooth, and the central derivative of F_n^{++} restricted on them are α_n and α_n^{-1} respectively. The attracting and repelling region comes from the definition and the properties of diffeomorphisms on central segments.

Since the angle between $\partial/\partial z$ and the tangent plane fields of $\Sigma^+_{A,n}$, $\Sigma^+_{R,n}$ are uniformly bounded away from zero, which implies the partial derivatives two smooth functions $l(I_n^{++}(x,y))$ and $l(J_n^{++}(x,y))$ are uniformly bounded on C^{++} and for n. Moreover, here we also have the property that $\lim_{n\to\infty}(1-\beta_n)/(1-\alpha_n) = 0$, so from lemma 6.2.6 and the remark of the definition of the segments diffeomorphisms, we manage to show that the partial derivatives of $F^{++}_{n,z}$ are uniformly bounded with respect to C^{++} and n.

The estimation of central derivatives comes from α_n and β_n both converge to 1. For the partial derivatives, it can be controlled by the estimation of tangent plane fields of $\Sigma^+_{A,n}$ and $\Sigma^+_{R,n}$, which actually becomes more and more flat as $n \to \infty$.

$\square$

Now we can define analogously the diffeomorphisms sequences $\{F_n^{+-}\}$, $\{F_n^{-+}\}$, and $\{F_n^{--}\}$ on the C^{+-}, C^{-+}, and C^{--}, respectively, with corresponding attracting regions sequences $\{U_n^{+-}\}$, $\{U_n^{-+}\}$, and $\{U_n^{--}\}$, and repelling regions sequences $\{V_n^{+-}\}$, $\{V_n^{-+}\}$, and $\{V_n^{--}\}$. All these sequences of diffeomorphisms satisfy that the partial derivatives are uniformly bounded, and the central derivatives converge to 1.

Notice that they are not coincide on their intersecting domains. The rest of our task is to gluing them together. We first look at what happens on the invariant fiber.

6.2.6 On the invariant fiber $(0,0)\times S^1$

Recall that we have

$$\begin{aligned}\Sigma_{A,n}\cap(0,0)\times S^1 &= \{(0,0,1/4),(0,0,3/4=-1/4)\}\ ,\\ \Sigma_{R,n}\cap(0,0)\times S^1 &= \{(0,0,0),(0,0,1/2)\}\ .\end{aligned}$$

We consider the diffeomrophisms $f_{0,n}:S^1\to S^1$ defined as follows:

- For $I_0=[0,1/4]$, we state $f_{0,n}|_{I_0}=\Psi^-_{n,I_0,I_0}:I_0\longrightarrow I_0$;
- For $I_1=[1/4,1/2]$, we state $f_{0,n}|_{I_1}=\Psi^+_{n,I_1,I_1}:I_1\longrightarrow I_1$;
- For $I_2=[1/2,3/4]$, we state $f_{0,n}|_{I_2}=\Psi^-_{n,I_2,I_2}:I_2\longrightarrow I_2$;
- For $I_3=[3/4,1=0]$, we state $f_{0,n}|_{I_3}=\Psi^+_{n,I_3,I_3}:I_3\longrightarrow I_3$.

It can be seen that $f_{0,n}$ is a Morse-Smale diffeomorphism on S^1 with exactly 4 fixed points, $(0,0,1/4)$ and $(0,0,3/4)$ are two sinks, $(0,0,0)$ and $(0,0,1/2)$ are two sources. Notice that $f_{0,n}$ will converge to identity map on S^1 as $n\to\infty$.

Then we define

$$\begin{aligned}F_{0,n}\ :\ \mathbb{R}^2\times S^1 &\longrightarrow \mathbb{R}^2\times S^1,\\ (\ x\ ,\ y\ ,\ z\ ) &\longmapsto (\ \lambda\cdot x\ ,\ \frac{1}{\lambda}\cdot y\ ,\ f_{0,n}(z)\ ).\end{aligned}$$

It is obviously $F_{0,n}$ has 4 saddle fixed points. This will be the diffeomorphisms when the domain restricted on the neighborhood of invariant fiber $(0,0)\times S^1$.

6.2.7 Gluing $F_{0,n}$ with $F_n^{\pm\pm}$

Now we will try to glue $F_{0,n}$ with each $F_n^{\pm\pm}$ to get new diffeomorphisms $F_{0,n}^{\pm\pm}$ on $C^{\pm\pm}$, which are coincide on the neighborhood of invariant fiber.

Notice that $\Sigma^+_{A,n}\cap C^{++}$ contains the intersection of horizontal disk $\{\sqrt{x^2+y^2}\le 2^{\frac{n}{2}}, z=1/4\}$ with C^{++}; and $\Sigma^+_{R,n}\cap C^{++}$ contains the intersection of horizontal disk $\{\sqrt{x^2+y^2}\le 2^{\frac{n}{2}}, z=1/2\}$ with C^{++}. This implies F_n^{++} coincide with $F_{0,n}$ on $\{\sqrt{x^2+y^2}\le 2^{\frac{n}{2}}, z\in[\frac{1}{8},\frac{1}{2}+\frac{\alpha_n}{8}]\}\cap C^{++}$.

Definition 6.2.12. *We define the diffeomorphisms $F_{0,n}^{++}:C^{++}\to C^{++}$ as follows:*

- *if $p\in\Sigma^+_{A,n}\cup\Sigma^+_{R,n}$, then $F_{0,n}^{++}(p)=F_n^{++}(p)$;*
- *if $p\in\bigcup_{x,y\ge 0}I_n^{++}(x,y)$, then $F_{0,n}^{++}(p)=F_n^{++}(p)$;*
- *if $p\in\bigcup_{\sqrt{x^2+y^2}\ge 2}J_n^{++}(x,y)$, then $F_{0,n}^{++}(p)=F_n^{++}(p)$;*

- *for any* $x, y \geq 0$ *and* $\sqrt{x^2+y^2} \leq 2$, *notice that the*

$$F_n^{++}(J_n^{++}(x,y)) = F_{0,n}(J_n^{++}(x,y)) = J_n^{++}(\lambda x, \frac{1}{\lambda} y) = (\lambda x, \frac{1}{\lambda} y) \times [-1/2, 1/4].$$

  So this allowed us to define the convex sum of F_n^{++} *and* $F_{0,n}$ *restricted on* $J_n^{++}(x,y) = (x,y) \times [-1/2, 1/4]$, *for* $r = \sqrt{x^2+y^2}$, *and* $t \in [-1/2, 1/4]$,

$$F_{0,n}^{++}|_{J_n^{++}(x,y)}(t) = \psi(r+1) \cdot F_{0,n}|_{J_n^{++}(x,y)}(t) + (1 - \psi(r+1)) \cdot F_n^{++}|_{J_n^{++}(x,y)}(t).$$

Lemma 6.2.13. *The map* $F_{0,n}^{++}$ *is a well defined smooth diffeomorphism of* C^{++}. *Moreover, it satisfies*

- *restricted on the domain* $C^{++} \cap \{\sqrt{x^2+y^2} \leq 1\}$, *it admits* $F_{0,n}^{++} = F_{0,n}$;
- *restricted on the domain* $C^{++} \cap \{\sqrt{x^2+y^2} \geq 2\}$, *it admits* $F_{0,n}^{++} = F_n^{++}$;
- *the partial derivative of* $F_{0,n}^{++}$ *are uniformly bounded on* C^{++} *and for all* n;
- $\lim_{n\to\infty} D^c F_{0,n}^{++} = 1$, *here the convergence are uniformly on* C^{++}.

Proof. The first two items came from the definition of $F_{0,n}^{++}$. For the last two items, it could see that both $F_{0,n}$ and F_n^{++} satisfy the boundedness of partial derivatives and the convergence of central derivatives, so we only need check this for their convex sum.

Restricted on the region where $F_{0,n}^{++}$ is defined by convex sum, we have

$$\begin{aligned} D^c F_{0,n}^{++} &= D^c(\psi(r+1) \cdot F_{0,n}) \ + \ D^c((1-\psi(r+1)) \cdot F_n^{++}) \\ &= \psi(r+1) \cdot D^c F_{0,n} \ + \ (1-\psi(r+1)) \cdot D^c F_n^{++} \\ &\longrightarrow 1 \qquad \text{as} \qquad n \longrightarrow \infty. \end{aligned}$$

For the partial derivatives,

$$\begin{aligned} \left\|\frac{\partial F_{0,n}^{++}(t)}{\partial x}\right\| &\leq \left\|\frac{\partial(\psi(r+1)F_{0,n}|_{J_n^{++}(x,y)}(t))}{\partial x}\right\| + \left\|\frac{\partial((1-\psi(r+1))F_n^{++}|_{J_n^{++}(x,y)}(t))}{\partial x}\right\| \\ &\leq \left\|\frac{\partial \psi(\sqrt{x^2+y^2}+1)}{\partial x}\right\| \cdot \left(\left\|\frac{\partial F_{0,n}|_{J_n^{++}(x,y)}(t)}{\partial x}\right\| + \left\|\frac{\partial F_n^{++}|_{J_n^{++}(x,y)}(t)}{\partial x}\right\|\right). \end{aligned}$$

Notice that the end points of $J_n^{++}(x,y)$ are contained in $\Sigma_{R,n}^+$ and $\Sigma_{A,n}^+$, which various s-moothly with respect to x, y, and the partial derivatives are uniformly bounded. This implies $\|\partial F_{0,n}^{++}(t)/\partial x\|$ are uniformly bounded on C^{++} and the upper bounds are independent of n. The same property holds for $\partial F_{0,n}^{++}(t)/\partial y$. This finishes the proof of the lemma.

□

Now we need to define the attracting region and repelling region of $F_{0,n}^{++}$ very carefully. We first define a series of smooth bump function $s_n : [0, +\infty) \to [1/8, 3/8]$ as

$$s_n(t) \;=\; \frac{3}{8} - \frac{1}{4} \cdot \psi[(\frac{1}{2} \cdot t)^{\frac{2}{n-2}} + 1].$$

As before, here we require $n > 2$, and it can be checked that $s_n|_{[0,2]} \equiv 1/8$, $s_n|_{[2^{\frac{n}{2}},\infty)} \equiv 3/8$, and $s_n(\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}) = 1/4$. Moreover, it admits similar flatness properties as ψ_n we defined before, and $s_n'(t)$ is decreasing on $[0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$.

Definition 6.2.14. *We define $U_{0,n}^{++} \subset C^{++}$ as follows:*

- $U_{0,n}^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \geq 2^{\frac{n}{2}}\} = U_n^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \geq 2^{\frac{n}{2}}\}$;
- $U_{0,n}^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \leq 2^{\frac{n}{2}}\} = \bigcup_{r=\sqrt{x^2+y^2} \leq 2^{\frac{n}{2}}} (x, y) \times [\frac{1}{4} - s_n(r), \frac{3}{8}]$.

And we define $V_{0,n}^{++}$ as:

- $V_{0,n}^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \geq 2^{\frac{n}{2}}\} = V_n^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \geq 2^{\frac{n}{2}}\}$;
- $V_{0,n}^{++} \cap \{x, y \geq 0, \sqrt{x^2+y^2} \leq 2^{\frac{n}{2}}\} = \bigcup_{r=\sqrt{x^2+y^2} \leq 2^{\frac{n}{2}}} (x, y) \times [\frac{1}{2} - \frac{\alpha_n}{8}, \frac{1}{2} + \alpha_n \cdot s_n(r)]$.

Remark. *We can see that $U_{0,n}^{++} \subset U_n^{++}$ is actually the union of intervals which generate a tubular neighborhood of $\Sigma_{A,n}^{+}$, and $V_{0,n}^{++} \subset V_n^{++}$ is the union of intervals which generate a tubular neighborhood of $\Sigma_{R,n}^{+}$. If we describe them in another point of view, it can be seen that for $U_{0,n}^{++}$*

- $U_{0,n}^{++} \;\cap\; \bigcup_{x,y\geq 0} I_n^{++}(x, y) \;=\; \bigcup_{x,y\geq 0}[\; p_n^{++}(x, y) \;,\; p_n^{++}(x, y) + \frac{1}{2} \cdot l(I_n^{++}(x, y)) \;]^c$;
- $U_{0,n}^{++} \;\cap\; \bigcup_{x,y\geq 0} J_n^{++}(x, y) \;=\; \bigcup_{x,y\geq 0}[\; p_n^{++}(x, y) - s_n(\sqrt{x^2+y^2}) \;,\; p_n^{++}(x, y) \;]^c$.

Here recall that $p_n^{++}(x, y) \in \Sigma_{A,n}^{+}$, and $l(I_n^{++}(x, y))$ is the length of $I_n^{++}(x, y)$. In the same spirit, we have

- $V_{0,n}^{++} \;\cap\; \bigcup_{x,y\geq 0} I_n^{++}(x, y) \;=\; \bigcup_{x,y\geq 0}[\; q_n^{++}(x, y) - \frac{\alpha_n}{2} \cdot l(I_n^{++}(x, y)) \;,\; q_n^{++}(x, y) \;]^c$;
- $V_{0,n}^{++} \;\cap\; \bigcup_{x,y\geq 0} J_n^{++}(x, y) \;=\; \bigcup_{x,y\geq 0}[\; q_n^{++}(x, y) \;,\; q_n^{++}(x, y) + \alpha_n \cdot s_n(\sqrt{x^2+y^2}) \;]^c$.

Here also have $q_n^{++}(x, y) \in \Sigma_{R,n}^{+}$.

Lemma 6.2.15. *$F_{0,n}^{++}$ coincide with F_n^{++} when restricted on the two closed disjoint regions $U_{0,n}^{++}$ and $V_{0,n}^{++}$. Moreover, we have*

$$F_{0,n}^{++}(U_{0,n}^{++}) \subset \mathrm{Int}(U_{0,n}^{++}) \;, \qquad \text{and} \qquad (F_{0,n}^{++})^{-1}(V_{0,n}^{++}) \subset \mathrm{Int}(V_{0,n}^{++}) \;.$$

Proof. The first part of the lemma comes from the definition of $F_{0,n}^{++}$. For the second part, notice that $U_{0,n}^{++} \subset U_n^{++}$ and $V_{0,n}^{++} \subset V_n^{++}$ implies $D^c F_{0,n}^{++}|_{U_{0,n}^{++}} = D^c (F_{0,n}^{++})^{-1}|_{V_{0,n}^{++}} \equiv \alpha_n$.

From the fact that $F_{0,n}^{++}(\Sigma_{A,n}^{+} \cap C^{++}) = \Sigma_{A,n}^{+} \cap C^{++}$, we just need analysis $F_{0,n}^{++}$ acting on each S^1-fiber intersecting with $U_{0,n}^{++}$. Notice that as n tend to infinity, both functions $l(I_n^{++}(x,y))$ and $s_n(\sqrt{x^2+y^2})$ become more and more flat. Actually, we can still denote by $0 < \beta_n < 1$ a sequence of real numbers with $\lim_{n\to\infty} \beta_n = 1$, such that

$$\beta_n \le \frac{|I_n^{++}(\lambda x, \frac{1}{\lambda} y)|}{|I_n^{++}(x,y)|} \le \frac{1}{\beta_n}, \qquad \text{and} \qquad \beta_n \le \frac{s_n(\sqrt{(\lambda x)^2 + (\frac{1}{\lambda} y)^2})}{s_n(\sqrt{x^2+y^2})} \le \frac{1}{\beta_n}.$$

This implies for every $(x,y) \in \pi(C^{++})$, we have

$$F_{0,n}^{++}(U_{0,n}^{++} \cap S^1_{(x,y)}) \subset \operatorname{Int}(U_{0,n}^{++} \cap S^1_{(\lambda x, \frac{1}{\lambda} y)}).$$

This proves that $U_{0,n}^{++}$ is an attracting region for $F_{0,n}^{++}$. The same argument shows that $V_{0,n}^{++}$ is an repelling region for $F_{0,n}^{++}$.

□

In the analogous way, we can define all the diffeomorphisms $F_{0,n}^{+-}$, $F_{0,n}^{-+}$, and $F_{0,n}^{--}$ on the other three quadrants respectively, such that they coincide with $F_{0,n}$ on $\{\sqrt{x^2+y^2} \le 1\}$ on each quadrants, and coincide with F_n^{+-}, F_n^{-+}, and F_n^{--} on $\{\sqrt{x^2+y^2} \ge 2\}$ respectively. Moreover, restricted on the attracting regions $U_{0,n}^{+-}$, $U_{0,n}^{-+}$, and $U_{0,n}^{--}$, their central derivatives are all equal to α_n; And the central derivatives of their reverse on the repelling regions $V_{0,n}^{+-}$, $V_{0,n}^{-+}$, and $V_{0,n}^{--}$ are also equal to α_n.

6.3 Gluing Diffeomorphisms

In last section, we have defined the four diffeomorphisms on the four quadrants, and they coincide with $F_{0,n}$ on the neighborhood of invariant fiber. In this section, we will try to glue them mutually with each others to manage our final constructions F_n.

The main difficulties for gluing is on the control of attracting and repelling regions, and control the central derivatives of the gluing diffeomorphisms simultaneously.

We will focus on gluing $F_{0,n}^{++}$ with $F_{0,n}^{+-}$ which will be defined on $C^{++} \cup C^{+-}$, and dealing with other gluing procedures in the same way. Denote $C^{+\pm} = C^{++} \cap C^{+-} = \{x \ge 0, y = 0\}$, and we first define a diffeomorphism $F_{0,n}^{+\pm}$ on $C^{+\pm}$.

6.3.1 Constructing $F_{0,n}^{+\pm}$ on $C^{+\pm}$

For the cylinder $C^{+\pm} = \{x \ge 0, y = 0\}$, it can checked that $F_{0,n}^{++}$ and $F_{0,n}^{+-}$ satisfying the following properties:

- $F^{++}_{0,n}|_{\{0\leq x\leq 1, y=0\}\cup\{x\geq 3^{\frac{n}{2}}, y=0\}} = F^{+-}_{0,n}|_{\{0\leq x\leq 1, y=0\}\cup\{x\geq 3^{\frac{n}{2}}, y=0\}}$;
- for the attracting and repelling regions, we have

$$\begin{aligned} U^{++}_{0,n}\cap\{x\geq 3^{\frac{n}{2}}, y=0\} &= U^{+-}_{0,n}\cap\{x\geq 3^{\frac{n}{2}}, y=0\}, \\ V^{++}_{0,n}\cap\{x\geq 3^{\frac{n}{2}}, y=0\} &= V^{+-}_{0,n}\cap\{x\geq 3^{\frac{n}{2}}, y=0\}; \end{aligned}$$

- for the union of two attracting regions, we have

$$\begin{aligned} U^{+\pm}_{0,n} &\triangleq U^{++}_{0,n}\cup U^{+-}_{0,n}|_{C^{+\pm}} \\ &= \{\ x\in[0,\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}], y=0, z\in[-\frac{3}{8}, s_n(x)-\frac{1}{4}]\cup[\frac{1}{4}-s_n(x),\frac{3}{8}]\ \} \\ &\quad\cup\{\ x\in[\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}, +\infty), y=0, z\in[-\frac{1}{4}-\frac{1}{8}\cdot\psi_n(x), \frac{1}{4}+\frac{1}{8}\cdot\psi_n(x)]\ \}; \end{aligned}$$

- for the intersection of two repelling regions, we have

$$V^{+\pm}_{0,n} \triangleq V^{++}_{0,n}\cap V^{+-}_{0,n}|_{C^{+\pm}} = \{x\geq 0, y=0, z\in[\frac{1}{2}-\frac{\alpha_n}{8}\cdot\psi_n(x), \frac{1}{2}+\frac{\alpha_n}{8}\cdot\psi_n(x)]\}.$$

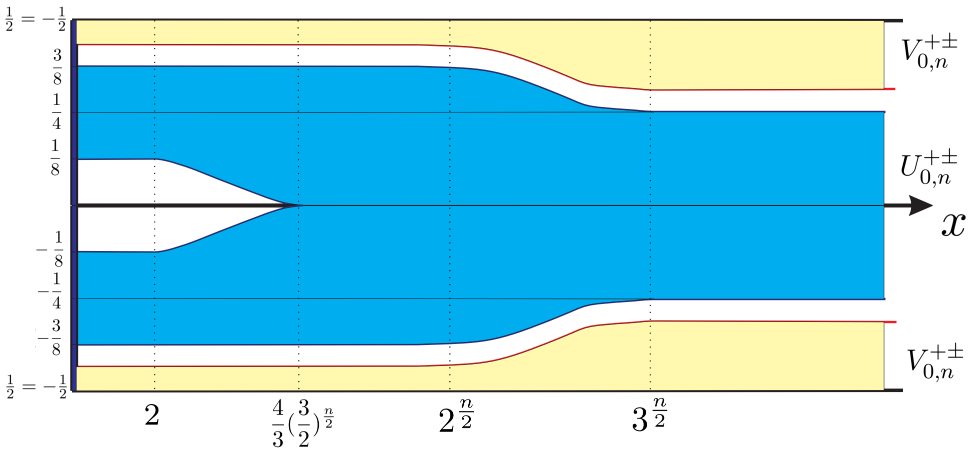

Figure 6.2: $U^{+\pm}_{0,n}$ *and* $V^{+\pm}_{0,n}$ *in* $C^{+\pm}$

Lemma 6.3.1. *There exists a sequence of smooth diffeomorphisms* $F^{+\pm}_{0,n}$ *on* $C^{+\pm}=\{x\geq 0, z\in S^1\}$ *admitting the following properties:*

- $F^{+\pm}_{0,n}(\{x\}\times S^1)=\{\lambda\cdot x\}\times S^1$ *for all* $x\geq 0$*;*

- $F^{+\pm}_{0,n}$ *coincide with* $F^{++}_{0,n}$ *and* $F^{+-}_{0,n}$ *on the set* $\{x \in [0,1] \cup [3^{\frac{n}{2}}, +\infty), z \in S^1\}$*;*
- $F^{+\pm}_{0,n}(U^{+\pm}_{0,n}) \subset \mathrm{Int}(U^{+\pm}_{0,n})$*, and* $D^cF^{+\pm}_{0,n}$ *restricted on* $U^{+\pm}_{0,n}$ *is uniformly contracting;*
- $(F^{+\pm}_{0,n})^{-1}(V^{+\pm}_{0,n}) \subset \mathrm{Int}(V^{+\pm}_{0,n})$*, and* $D^c(F^{+\pm}_{0,n})^{-1}$ *restricted on* $V^{+\pm}_{0,n}$ *is uniformly contracting;*
- *the partial derivatives* $\partial F^{+\pm}_{0,n,z}/\partial x$ *is uniformly bounded on* $C^{+\pm}$ *and for all* n*, the central derivatives* $D^cF^{+\pm}_{0,n}$ *uniformly converge to 1 as* $n \to \infty$*.*

Remark. *This lemma is the most significant part of our construction. Since we already build the diffeomorphisms on each quadrants, and they coincide when close to the invariant fiber and on the region very far from invariant fiber, the only problem is to glue them together. Here the diffeomorphism* $F^{+\pm}_{0,n}$ *is the key part for their gluing, and we will make the convex sum of it with the diffeomorphisms on quadrants, to get the constructions we desired.*

Proof of Lemma 6.3.1. By the symmetry of $U^{+\pm}_{0,n}$ and $V^{+\pm}_{0,n}$ with respect to the two rays $\{z = 0\}$ and $\{z = 1/2\}$, we just need to construct the diffeomorphisms on $[0, +\infty) \times [0, 1/2]$ and for any $(x, z) \in [0, +\infty) \times \{0, 1/2\}$, we have $F^{+\pm}_{0,n}(x, z) = (\lambda \cdot x, z)$.

The main difficulty of the construction is how to keep the attracting region $U^{+\pm}_{0,n}$ positive invariant by the action of $F^{+\pm}_{0,n}$.

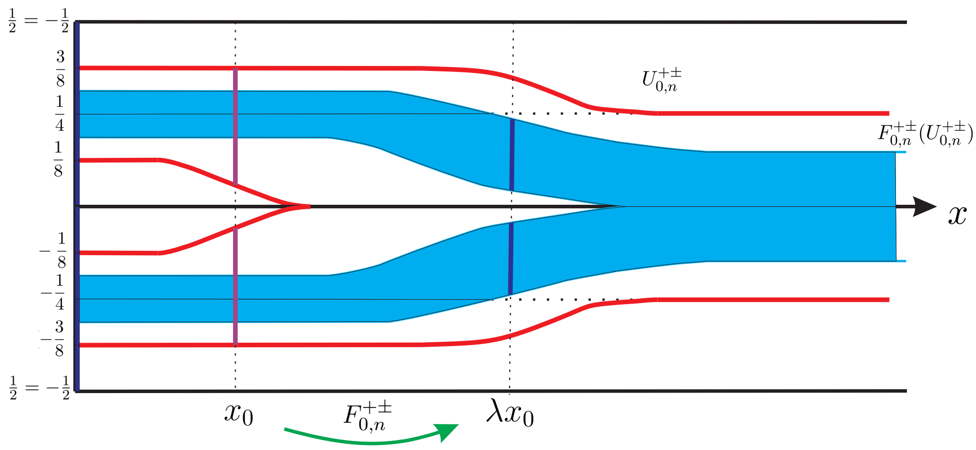

Figure 6.3: $F^{+\pm}_{0,n}(U^{+\pm}_{0,n}) \subset \mathrm{Int}(U^{+\pm}_{0,n})$

We first look at $U^{+\pm}_{0,n}$ restricted on $\{x \in [0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}], z \in [0, \frac{1}{2}]\}$. Recall that

$$U^{+\pm}_{0,n} \cap ([0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \times [0, 1/2]) = \{\ x \in [0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}], z \in [\frac{1}{4} - s_n(x), \frac{3}{8}]\ \}\ .$$

We will map the upper boundary $\{x \in [0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}], z = \frac{3}{8}\}$ to its image by $F^{++}_{0,n}$, which equal to $\{x \in [0, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}], z = \frac{1}{4} + \frac{\alpha_n}{8}\}$. For the image of lower boundary, we need the following claim.

Claim. 1. *There exists a sequence of smooth function* $r_n : [0, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \to [0, 1/2]$ *and real numbers* $T_n \in [2, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$*, such that*

1. *for each* $x \in [0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$*, we have* $\frac{1}{4} - s_n(x) < r_n(x) < \frac{1}{4} + \frac{\alpha_n}{8}$*;*
2. *for each* $x \in [0, 2]$*, we have* $r_n(\lambda \cdot x) = \frac{1}{4} - \frac{\alpha_n}{8}$*;*
3. *for each* $x \in [T_n, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$*, we have* $r_n(\lambda \cdot x) = \frac{2+\alpha_n}{3} \cdot [\frac{1}{4} - s_n(x)]$*;*
4. $\lim_{n\to\infty} r'_n(x) = 0$*;*
5. *for each* $x \in [0, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$*, we have* $\frac{(2+\alpha_n)/8 - r_n(\lambda x)}{(2+\alpha_n)/8 - r_n(x)} < 1$*;*
6. $\lim_{n\to\infty} \inf_{x\in[0,\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]}\{\frac{(2+\alpha_n)/8 - r_n(\lambda x)}{(2+\alpha_n)/8 - r_n(x)}\} \geq \lim_{n\to\infty} \alpha_n = 1$*;*
7. $\lim_{n\to\infty} \sup_{x\in[0,\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]}\{\frac{r_n(\lambda x)}{1/4 - s_n(x)}, \frac{1/4 - s_n(x)}{r_n(\lambda x)}\} = 1.$

We first make some remarks about this claim.

Remark. *Since we want the image if* r_n *will be the* $F^{+\pm}_{0,n}$*-image of lower boundary of* $U^{+\pm}_{0,n}$*, so in the claim:*

- *item 1-3 are used to guarantee the positive invariance of* $U^{+\pm}_{0,n}$*.*
- *item 4 is used for the estimation of partial derivatives* $\|\partial F^{+\pm}_{0,n,z}/\partial x\|$*.*
- *item 5 and 6 aim to insure that the the length of central segments in* $U^{+\pm}_{0,n}$ *are large than the length of its* $F^{+\pm}_{0,n}$*-image, and the ratio will converge to 1 as n tend to infinite. This allowed us to build* $F^{+\pm}_{0,n,z}$ *is central uniformly contracting on* $U^{+\pm}_{0,n}$*, and the central derivatives converge to 1.*
- *item 7 guarantees that the central segments between* $z = 0$ *and lower boundary of* $U^{+\pm}_{0,n}$ *will be mapped by* $F^{+\pm}_{0,n}$ *to a central segments almost have the same length. This is for estimating the central derivatives of* $F^{+\pm}_{0,n}$ *in the region between* $z = 0$ *and lower boundary of* $U^{+\pm}_{0,n}$*.*

Proof of Claim 1. Recall that the lower boundary of $U^{+\pm}_{0,n} \cap [0, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \times [0, 1/2]$ is the image of the function $z = \frac{1}{4} - s_n(x) = \frac{1}{4} \cdot \psi[(\frac{1}{2} \cdot t)^{\frac{2}{n-2}} + 1] - \frac{1}{8}$. And we define the smooth function $e_n : [0, \frac{4}{3\lambda}(\frac{3}{2})^{\frac{n}{2}}] \to \mathbb{R}$ as

$$e_n(x) = [\frac{1}{4} - s_n(\lambda \cdot x)] - \frac{2+\alpha_n}{3} \cdot [\frac{1}{4} - s_n(x)] \ .$$

Then for the function $e_n(x)$, we can see that $e_n|_{[0,2/\lambda]} \equiv \frac{1-\alpha_n}{24} > 0$, and $e_n(\frac{4}{3\lambda}(\frac{3}{2})^{\frac{n}{2}}) = -\frac{2+\alpha_n}{3} \cdot [\frac{1}{4} - s_n(\frac{4}{3\lambda}(\frac{3}{2})^{\frac{n}{2}})] < 0$. Moreover, since $\psi'(t)$ is decreasing on $[0, 5/2]$, some calculation shows that $e_n'(t) \leq 0$ on its definition domain, and there exists a unique $T_n \in [2, \frac{4}{3\lambda}(\frac{3}{2})^{\frac{n}{2}}]$ such that $e_n(T_n - 1) = 0$, that is

$$\frac{1}{4} - s_n(\lambda \cdot (T_n - 1)) \;=\; \frac{2+\alpha_n}{3} \cdot [\frac{1}{4} - s_n(T_n - 1)].$$

Actually, from the flatness of s_n, we know that $T_n \to \infty$ as $n \to \infty$.

From the analysis above, we can denote that

$$\varrho_n \;=\; \frac{2+\alpha_n}{3} \cdot [\frac{1}{4} - s_n(T_n)] \;-\; [\frac{1}{4} - s_n(\lambda \cdot (T_n))] \;>\; 0 \;.$$

Moreover, here we can have $\lim_{n\to\infty} \varrho_n/[\frac{1}{4} - s_n(\lambda \cdot (T_n))] = 0$. On the other hand, we denote

$$\varpi_n \;=\; (\frac{1}{4} - \frac{\alpha_n}{8}) - [\frac{1}{4} - s_n(2\lambda)] \;>\; 0,$$

then we have $\lim_{n\to\infty} \varpi_n/[\frac{1}{4} - s_n(2\lambda)] = 0$.

Notice that the smooth function $z = \frac{1}{4} - s_n(x)$ is decreasing on $[0, \lambda(T_n + 1)]$, this allowed us to define a smooth function $\kappa_n : [2\lambda, \lambda T_n] \to [0, 1]$ such that $x \in [2\lambda, \lambda T_n]$, we have

$$\frac{1}{4} - s_n(x) = \kappa_n(x) \cdot [\frac{1}{4} - s_n(2\lambda)] + (1 - \kappa_n(x)) \cdot [\frac{1}{4} - s_n(\lambda \cdot T_n)].$$

We define a smooth curve contained $[0, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \times [0, 1/2]$ as the image of the function $r_n : [0, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \to [0, 1/2]$, where

$$r_n(x) = \begin{cases} 1/4 - \alpha_n/8 \;, & x \in [0, 2\lambda] \;, \\ 1/4 - s_n(x) + \kappa_n(x) \cdot \varpi_n + (1 - \kappa_n(x)) \cdot \varrho_n \;, & x \in [2\lambda, \lambda T_n] \;, \\ (2+\alpha_n)/3 \cdot (1/4 - s_n(x/\lambda)) \;, & x \in [\lambda T_n, \lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \;. \end{cases}$$

From the construction of r_n, we check it satisfies the properties in the claim one by one.

The first three items came from the definition of r_n. The fourth item came from the fact that the derivatives of $s_n(x)$ and $\kappa_n(x)$ are all uniformly converge to zero on their domain.

Item 5 can be shown by the monotonous decreasing of $r_n(x)$. Item 6 is a little bit tricky. Notice that when $x \in [0, 2]$, $\frac{(2+\alpha_n)/8 - r_n(\lambda x)}{(2+\alpha_n)/8 - r_n(x)}$ is just equal to α_n. But when $x \geq T_n$, it will be equal to $(2 + \alpha_n)/3 > \alpha_n$. Notice that we build r_n as some kind convex sum , which will make $\frac{(2+\alpha_n)/8 - r_n(\lambda x)}{(2+\alpha_n)/8 - r_n(x)}$ is monotonous increasing. This proves item 6.

The last item just the results that α_n will converge to 1 as n tend to infinity.

□

Continue proving Lemma 6.3.1

Now we can define the sequence of diffeomorphisms $F^{+\pm}_{0,n}$ on $[0,+\infty)\times[0,1/2]$. The construction of $F^{+\pm}_{0,n}$ will be separated into three parts, $[0,2]\times[0,1/2]$, $[2,3^{\frac{n}{2}}]\times[0,1/2]$, and $[3^{\frac{n}{2}},+\infty)\times[0,1/2]$.

The first and third parts are very clearly. we define

- $F^{+\pm}_{0,n}|_{[0,2]\times[0,1/2]} = F^{++}_{0,n}|_{[0,2]\times[0,1/2]}$;
- $F^{+\pm}_{0,n}|_{[3^{\frac{n}{2}},+\infty)\times[0,1/2]} = F^{++}_{0,n}|_{[3^{\frac{n}{2}},+\infty)\times[0,1/2]} = F^{+-}_{0,n}|_{[3^{\frac{n}{2}},+\infty)\times[0,1/2]}$.

Here we want to point out that by the symmetry, $F^{+\pm}_{0,n}|_{[0,2]\times[1/2,1]}$ will equal to $F^{+-}_{0,n}|_{[0,2]\times[0,1/2]}$. And for $F^{+\pm}_{0,n}|_{[3^{\frac{n}{2}},+\infty)\times[1/2,1]}$, it coincide with both $F^{++}_{0,n}$ and $F^{+-}_{0,n}$.

We define $F^{+\pm}_{0,n}|_{[2,3^{\frac{n}{2}}]\times[0,1/2]} : [2,3^{\frac{n}{2}}]\times[0,1/2] \longrightarrow [2\lambda,3^{\frac{n}{2}}\lambda]\times[0,1/2]$ as following constructions.

- Restricted on $V^{+\pm}_{0,n}$, we define $F^{+\pm}_{0,n} = F^{++}_{0,n}$.

- To describe $U^{+\pm}_{0,n}\cap[2,3^{\frac{n}{2}}]\times[0,1/2]$, for every $x\in[2,3^{\frac{n}{2}}]$, we denote

$$\begin{aligned} L_n(x) &= \inf\{z:\ (x,z)\in U^{+\pm}_{0,n}\cap\{x\}\times[0,1/2]\}\ . \\ U_n(x) &= \sup\{z:\ (x,z)\in U^{+\pm}_{0,n}\cap\{x\}\times[0,1/2]\}\ . \end{aligned}$$

  Notice that when $x\in[2,\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$, $L_n(x)=\frac{1}{4}-s_n(x)$; and for $x\in[\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}},3^{\frac{n}{2}}]$, $L_n(x)=0$. $U_n(x)\equiv\frac{1}{4}+\frac{1}{8}\cdot\psi_n(x)$.

  For every $x\in[\lambda 2,\lambda 3^{\frac{n}{2}}]$, we denote

$$\begin{aligned} L'_n(x) &= \begin{cases} r_n(x)\ , & if\ \ x\le\lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}\ ; \\ 0\ , & if\ \ x\ge\lambda\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}\ . \end{cases} \\ U'_n(x) &= \pi^c\circ F^{++}_{0,n}(x/\lambda,U_n(x/\lambda))\ . \end{aligned}$$

  This allowed us to define

$$F^{+\pm}_{0,n}(x,L_n(x)) = (\lambda x, L'_n(\lambda x))\ , \qquad \text{and} \qquad F^{+\pm}_{0,n}(x,U_n(x)) = (\lambda x, U'_n(\lambda x))\ .$$

  And we define $F^{+\pm}_{0,n}|_{[L_n(x),U_n(x)]^c} : [L_n(x),U_n(x)]^c \longrightarrow [L'_n(\lambda x),U'_n(\lambda x)]^c$ be the affine map. From item 5 and 6 of the claim, we know that

$$\alpha_n\ \le\ \alpha'_n(x)\ \triangleq\ \frac{U'_n(\lambda x)-L'_n(\lambda x)}{U_n(x)-L_n(x)}\ <\ 1\ .$$

  This gave the definition of $F^{+\pm}_{0,n}$ on $U^{+\pm}_{0,n}$.

- In the region between $U_{0,n}^{+\pm}$ and $V_{0,n}^{+\pm}$, every $x \in [2, 3^{\frac{n}{2}}]$, we denote $K_n(x)$ be the central segment between $U_{0,n}^{+\pm}$ and $V_{0,n}^{+\pm}$. Since we have already know the $F_{0,n}^{+\pm}$-images of $U_{0,n}^{+\pm}$ and $V_{0,n}^{+\pm}$, we actually already know that

$$F_{0,n}^{+\pm}(K_n(x)) \;=\; F_{0,n}^{++}(K_n(x)) \; .$$

And the derivatives of $F_{0,n}^{+\pm}$ on the end points of $K_n(s)$ are $\alpha_n'(x)$ and α_n respectively. So we define $F_{0,n}^{+\pm}|_{K_n(x)} : K_n(x) \longrightarrow F_{0,n}^{++}(K_n(x))$ as

$$F_{0,n}^{+\pm}|_{K_n(x)}(z) \;\overset{\triangle}{=}\; \sigma_{\alpha_n'(x),\alpha_n,K_n(x),F_{0,n}^{++}(K_n(x))}(z), \qquad \forall\, z \in K_n(x) \; .$$

It could easily check that $F_{0,n}^{+\pm}$ defined on this region can smoothly glue to $F_{0,n}^{+\pm}$ restricted on $U_{0,n}^{+\pm}$ and $V_{0,n}^{+\pm}$.

- The last part we need to deal with is on the region between $\{z = 0\}$ and $U_{0,n}^{+\pm}$. This region could be expressed as $[2, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \times [0, U_n(x)]$, and for each $x \in [2, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}]$, we have

$$F_{0,n}^{+\pm}(\{x\} \times [0, U_n(x)]) \;=\; \{\lambda \cdot x\} \times [0, U_n'(\lambda \cdot x)] \; .$$

Notice that at the point $(2, 0)$, the central derivative of $F_{0,n}^{+\pm}$ is α_n^{-1}; at the point $(\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}, 0)$, the central derivative of $F_{0,n}^{+\pm}$ is $(2 + \alpha_n)/3$. We define a smooth decreasing function $\varsigma_n : (2 - \varepsilon, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}} + \varepsilon) \to \mathbb{R}$ such that

$$\varsigma_n|_{(2-\varepsilon,2]} \equiv \alpha_n^{-1} \qquad \text{and} \qquad \varsigma_n|_{[\frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}+\varepsilon)} \equiv (2 + \alpha_n)/3.$$

So we can define that $F_{0,n}^{+\pm}|_{\{x\}\times[0,U_n(x)]} : [0, U_n(x)]^c \longrightarrow [0, U_n'(\lambda \cdot x)]^c$ as

$$F_{0,n}^{+\pm}|_{\{x\}\times[0,U_n(x)]}(z) = \sigma_{\varsigma_n(x),\alpha_n'(x),[0,U_n(x)]^c,[0,U_n'(\lambda\cdot x)]^c}(z), \qquad \forall\, z \in [0, U_n(x)]^c \; .$$

Now we can finish the proof of this lemma by the following claim.

Claim. 2. *The diffeomorphisms $F_{0,n}^{+\pm}$ is well defined and smooth on $C^{+\pm}$. Moreover, they satisfying all the properties stated in the lemma.*

Proof of Claim 2. For proving $F_{0,n}^{+\pm}$ is well defined and smooth, the only difficulty appears at we need to glue the region $[2, \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}] \times [0, U_n(x)]$ to $U_{0,n}^{+\pm}$. Here we just notify that for $T_n \leq x \leq \frac{4}{3}(\frac{3}{2})^{\frac{n}{2}}$, on their intersection part, they are all equal to the map

$$(x, z) \;\longmapsto\; (\lambda \cdot z, \frac{2 + \alpha_n}{3} \cdot z) \; .$$

So they can smoothly glue together.

For all the properties in the lemma, the coincide with $F^{++}_{0,n}$ and $F^{+-}_{0,n}$ can be seen from the definition of $F^{+\pm}_{0,n}$. The attracting and repelling region is also from the construction. We only need to take care the estimations of central derivatives and partial derivatives.

All these estimations came from lemma 6.2.6. We just need to realize that all the boundary of these central segments are varying smoothly, so does the central derivatives on these boundary points. All these partial derivatives are uniformly bounded.

□

This claim closed the proof of lemma.

□

6.3.2 Extension of $F^{+\pm}_{0,n}$

In this subsection, we want to extend $F^{+\pm}_{0,n}$ in a neighborhood of $C^{+\pm}$. We first define the neighborhood of $C^{+\pm}$. Fix a small constant $0 < \varepsilon \ll 1$, and we consider the closed region

$$W^{+\pm}_{\varepsilon} \;=\; \{x \geq 0,\ -\varepsilon \leq y \leq \varepsilon,\ z \in S^1\}\ .$$

We want to define the diffeomorphisms which are extensions of $F^{+\pm}_{0,n}$ on $W^{+\pm}_{\varepsilon}$. Recall that we can express $F^{+\pm}_{0,n}$ defined on $C^{+\pm}$ as

$$F^{+\pm}_{0,n}\ :\ (\ x,\ 0,\ z\ )\ \longrightarrow\ (\ \lambda\cdot x,\ 0,\ \pi^c\circ F^{+\pm}_{0,n}(x,z)\ )\ .$$

So the most natural way for extension is map the point (x,y,z) to $(\lambda x, \frac{1}{\lambda}y, \pi^c F^{+\pm}_{0,n}(x,z))$.

However, this definition does not coincide with the diffeomorphisms we required in the technical lemma, since when $x \geq 3^{\frac{n}{2}}$, the contracting region of this diffeomorphism are not the union of intervals centered at the helicoid, but centered at the plane $\{z = 0\}$. Beside this, $F^{+\pm}_{0,n}$ satisfies all the properties we required.

Recall that for the surface $\Sigma^{+}_{R,n}$, every S^1-fiber $(x,y)\times S^1 \subset W^{+\pm}_{\varepsilon}$ intersects with $\Sigma^{+}_{R,n}$ exactly one point, denote by $q^{+\pm}_n(x,y)$. Moreover, we denote the central segment $L^{+\pm}_n(x,y) = [q^{+\pm}_n(x,y), q^{+\pm}_n(x,y)]^c$ to be the closure of $(x,y)\times S^1 \setminus q^{+\pm}_n(x,y)$, which is a closed segment of length 1, and we will identify it with the interval $[0,1]$.

Definition 6.3.2. *We define the extension diffeomrophism $F^{+\pm}_{0,n} : W^{+\pm}_{\varepsilon} \longrightarrow W^{+\pm}_{\varepsilon/\lambda} \subset W^{+\pm}_{\varepsilon}$ as the following way:*

- $F^{+\pm}_{0,n}(x,y,q^{+\pm}_n(x,y)) = (\lambda x, \frac{1}{\lambda}y, q^{+\pm}_n(\lambda x, \frac{1}{\lambda}y))$;
- $F^{+\pm}_{0,n}(L^{+\pm}_n(x,y)) = L^{+\pm}_n(\lambda x, \frac{1}{\lambda}y)$, *and* $F^{+\pm}_{0,n}|_{L^{+\pm}_n(x,y)} = F^{+\pm}_{0,n}|_{L^{+\pm}_n(x,0)}$.

Notice that this definition relies on $F^{+\pm}_{0,n}$ has already been defined on $C^{+\pm}$. Then we define the attracting and repelling regions $U^{+\pm}_{\varepsilon,n}, V^{+\pm}_{\varepsilon,n} \subset W^{+\pm}_{\varepsilon}$ as

$$U^{+\pm}_{\varepsilon,n} \cap L^{+\pm}_n(x,y) = U^{+\pm}_{0,n} \cap L^{+\pm}_n(x,0), \qquad V^{+\pm}_{\varepsilon,n} \cap L^{+\pm}_n(x,y) = V^{+\pm}_{0,n} \cap L^{+\pm}_n(x,0).$$

Here we actually identify the interval $L_n^{+\pm}(x,y)$ to $L_n^{+\pm}(x,0)$, and pull the attracting and repelling sets $U_{0,n}^{+\pm}, V_{0,n}^{+\pm}$ back on $L_n^{+\pm}(x,y)$ by this identification.

Then we can summarize all the properties of these diffeomorphisms and regions into the following lemma.

Lemma 6.3.3. *For the diffeomorphism $F_{0,n}^{+\pm} : W_\varepsilon^{+\pm} \longrightarrow W_{\varepsilon/\lambda}^{+\pm} \subset W_\varepsilon^{+\pm}$ and the corresponding regions $U_{\varepsilon,n}^{+\pm}$ and $V_{\varepsilon,n}^{+\pm}$, they admit the following properties:*

- $F_{0,n}^{+\pm}((x,y)\times S^1) = (\lambda x, \frac{1}{\lambda}y)\times S^1$*;*
- $F_{0,n}^{+\pm}(\Sigma_{R,n}^{+} \cap W_\varepsilon^{+\pm}) = \Sigma_{R,n}^{+} \cap W_{\varepsilon/\lambda}^{+\pm}$*;*
- $F_{0,n}^{+\pm}$ *coincide with* $F_{0,n}^{++}$ *and* $F_{0,n}^{+-}$ *on the intersections of* $\{x \in [0,1] \cup [3^{\frac{n}{2}}+1, +\infty)\}$ *with where their definition domain intersecting respectively;*
- $F_{0,n}^{+\pm}(U_{\varepsilon,n}^{+\pm}) \subset \mathrm{Int}(U_{\varepsilon,n}^{+\pm} \cap W_{\varepsilon/\lambda,n}^{+\pm})$*, and* $D^c F_{0,n}^{+\pm}$ *restricted on* $U_{\varepsilon,n}^{+\pm}$ *is uniformly contracting;*
- $(F_{0,n}^{+\pm})^{-1}(V_{\varepsilon,n}^{+\pm} \cap W_{\varepsilon/\lambda,n}^{+\pm}) \subset \mathrm{Int}(V_{\varepsilon,n}^{+\pm})$*, and* $D^c(F_{0,n}^{+\pm})^{-1}$ *restricted on* $V_{\varepsilon,n}^{+\pm} \cap W_{\varepsilon/\lambda,n}^{+\pm}$ *is uniformly contracting;*
- *the partial derivatives* $\partial F_{0,n,z}^{+\pm}/\partial x$ *is uniformly bounded on* $C^{+\pm}$ *and for all* n*, the central derivatives* $D^c F_{0,n}^{+\pm}$ *uniformly converge to 1 as* $n \to \infty$*.*

All these properties are came from the definitions, and we have prove similar results several times. So we skip the proof here.

Similarly, we can also define the diffeomorphisms in a neighborhood of the other intersection parts of each quadrants. We can define the diffeomorphism $F_{0,n}^{-\pm}$ on the ε-neighborhood of $C^{-\pm} = C^{-+} \cap C^{--}$. Actually here we can see that

$$W_\varepsilon^{-\pm} \;=\; \{(x,y,z):\ (-x,-y,z) \in W_\varepsilon^{-\pm}\}.$$

And we can define $F_{0,n}^{-\pm} : W_\varepsilon^{-\pm} \longrightarrow W_{\varepsilon/\lambda}^{-\pm} \subset W_\varepsilon^{-\pm}$ as:

$$F_{0,n}^{-\pm}(x,y,z) \;=\; (\ \lambda x\ ,\ \frac{1}{\lambda}y\ ,\ \pi^c F_{0,n}^{+\pm}(-x,-y,z) + \frac{1}{2}\ )\ .$$

This definition is purely by the symmetry, we cam similar define the attracting region and repelling region in this way, which could be verified that satisfying all the properties in lemma 6.3.3.

For the diffeomorphisms $F_{0,n}^{\pm+}$ and $F_{0,n}^{\pm-}$ defined on the neighborhood of $C^{\pm+} = C^{++} \cap C^{-+}$ and $C^{\pm-} = C^{+-} \cap C^{--}$, we just need to consider the inverse of $F_{0,n}^{+\pm}$ and $F_{0,n}^{-\pm}$ respectively. Then get two diffeomorphisms

$$F_{0,n}^{\pm+} : W_{\varepsilon/\lambda}^{\pm+} \longrightarrow W_\varepsilon^{\pm+}, \qquad \text{and} \qquad F_{0,n}^{\pm-} : W_{\varepsilon/\lambda}^{\pm-} \longrightarrow W_\varepsilon^{\pm-},$$

which both admitting the corresponding attracting and repelling regions, and satisfying all the properties stated in lemma 6.3.3.

6.3.3 Gluing all these diffeomorphisms

Now we have defined the diffeomorphisms $F_{0,n}^{++}$, $F_{0,n}^{+-}$, $F_{0,n}^{--}$, and $F_{0,n}^{-+}$ on each quadrants, and the diffeomorphisms $F_{0,n}^{+\pm}$, $F_{0,n}^{-\pm}$, $F_{0,n}^{\pm+}$, and $F_{0,n}^{\pm-}$ on the neighborhood of the intersection parts of these quadrants.

It's time to glue all these diffeomorphisms together. As before, we just focus on illustrating the gluing of $F_{0,n}^{++}$, $F_{0,n}^{+-}$, and $F_{0,n}^{+\pm}$, which will be defined on $C^{++} \cup C^{+-}$. The other parts could be defined by symmetry and the inverses of diffeomorphisms.

Let us first state some observations of the diffeomorphisms $F_{0,n}^{++}$, $F_{0,n}^{+-}$, $F_{0,n}^{+\pm}$ and their attracting repelling regions, which will be helpful in our gluing process.

- All the three diffeomorphisms preserve the S^1-fibers. Moreover, they all keep the surface $\Sigma_{R,n}^{+}$ invariant. This allowed us to represent for any $(x,y) \times S^1 \subset W_{\varepsilon}^{+\pm}$, we can express these three diffeomorphisms restricted on $(x,y) \times S^1$ to be diffeomorphisms from $L_n^{+\pm}(x,y) = [q_n^{+\pm}(x,y), q_n^{+\pm}(x,y)]^c$ to $L_n^{+\pm}(\lambda x, \frac{1}{\lambda}y) = [q_n^{+\pm}(\lambda x, \frac{1}{\lambda}y), q_n^{+\pm}(\lambda x, \frac{1}{\lambda}y)]^c$, here $q_n^{+\pm}(x,y) \in (x,y) \times S^1 \cap \Sigma_{R,n}^{+}$ and $q_n^{+\pm}(\lambda x, \frac{1}{\lambda}y) \in (\lambda x, \frac{1}{\lambda}y) \times S^1 \cap \Sigma_{R,n}^{+}$. Notice all these intervals could be identified with $[0,1]$.

- $F_{0,n}^{++}$ coincide with $F_{0,n}^{+\pm}$ on the region
$$\{0 \le x \le 1, 0 \le y \le \varepsilon, z \in S^1\} \cup \{x \ge 3^{\frac{n}{2}} + 1, 0 \le y \le \varepsilon, z \in S^1\};$$
  $F_{0,n}^{+-}$ coincide with $F_{0,n}^{+\pm}$ on the region
$$\{0 \le x \le 1, -\varepsilon \le y \le 0, z \in S^1\} \cup \{x \ge 3^{\frac{n}{2}} + 1, -\varepsilon \le y \le 0, z \in S^1\}.$$

- For any $x \ge 0, 0 \le y \le \varepsilon$, we have $V_{\varepsilon,n}^{+\pm} \cap (x,y) \times S^1 \subseteq V_{0,n}^{++} \cap (x,y) \times S^1$;
  For any $x \ge 0, -\varepsilon \le y \le 0$, we have $V_{\varepsilon,n}^{+\pm} \cap (x,y) \times S^1 \subseteq V_{0,n}^{+-} \cap (x,y) \times S^1$.

- For any $x \ge 0, 0 \le y \le \varepsilon$, we have $U_{\varepsilon,n}^{+\pm} \cap (x,y) \times S^1 \supseteq U_{0,n}^{++} \cap (x,y) \times S^1$;
  For any $x \ge 0, -\varepsilon \le y \le 0$, we have $U_{\varepsilon,n}^{+\pm} \cap (x,y) \times S^1 \supseteq U_{0,n}^{+-} \cap (x,y) \times S^1$.

Similarly properties hold for the other diffeomorphisms on each quadrants and the neighborhood of their intersection parts.

Definition 6.3.4. *We define the diffeomorphisms $F_n : \mathbb{R}^2 \times S^1 \longrightarrow \mathbb{R}^2 \times S^1$ as follows:*

- *If* (x,y,z) *does not belong to*

$$W_\varepsilon \triangleq W_\varepsilon^{+\pm}\cup W_\varepsilon^{-\pm}\cup W_{\varepsilon/\lambda}^{\pm+}\cup W_{\varepsilon/\lambda}^{\pm-},$$

then $F_n(x,y,z)$ *is equal to* $F_{0,n}^{++}(x,y,z)$, $F_{0,n}^{+-}(x,y,z)$, $F_{0,n}^{--}(x,y,z)$, *or* $F_{0,n}^{-+}(x,y,z)$ *according to the point* (x,y,z) *belong to which quadrant.*

- *If* $(x,y,z)\in W_\varepsilon^{+\pm}$, *we separate into three cases*
  - *for* $(x,y,z)\in W_{\varepsilon/\lambda}^{+\pm}$, $F_n(x,y,z)=F_{0,n}^{+\pm}(x,y,z)$;
  - *for* $\varepsilon/\lambda\le y\le\varepsilon$, *we consider* $F_n|_{(x,y)\times S^1}=F_n|_{L_n^{+\pm}(x,y)}:L_n^{+\pm}(x,y)\longrightarrow L_n^{+\pm}(\lambda x,\frac{1}{\lambda}y)$ *as*

$$F_n(z)|_{L_n^{+\pm}(x,y)}=\zeta(y)\cdot F_{0,n}^{+\pm}|_{L_n^{+\pm}(x,y)}(z)+(1-\zeta(y))\cdot F_{0,n}^{++}|_{L_n^{+\pm}(x,y)}(z).$$

    Here the bump function $\zeta(y)=\psi(\frac{y-\frac{\varepsilon}{\lambda}}{\varepsilon-\frac{\varepsilon}{\lambda}}+2)$;
  - *for* $-\varepsilon\le y\le-\varepsilon/\lambda$, *we consider* $F_n|_{(x,y)\times S^1}=F_n|_{L_n^{+\pm}(x,y)}:L_n^{+\pm}(x,y)\longrightarrow L_n^{+\pm}(\lambda x,\frac{1}{\lambda}y)$ *as*

$$F_n(z)|_{L_n^{+\pm}(x,y)}=\zeta(y)\cdot F_{0,n}^{+\pm}|_{L_n^{+\pm}(x,y)}(z)+(1-\zeta(y))\cdot F_{0,n}^{+-}|_{L_n^{+\pm}(x,y)}(z);$$

    Here the bump function $\zeta(y)=\psi(\frac{|y|-\frac{\varepsilon}{\lambda}}{\varepsilon-\frac{\varepsilon}{\lambda}}+2)$.

- *If* $(x,y,z)\in W_\varepsilon^{-\pm}$, *then we define* $F_n(x,y,z)=(\lambda x,\frac{1}{\lambda}y,\pi^cF_n(-x,-y,z)+\frac{1}{2})$. *Here we use the fact that* F_n *already been defined on* $W_\varepsilon^{+\pm}$.

- *If* $(x,y,z)\in W_{\varepsilon/\lambda}^{\pm+}\cup W_{\varepsilon/\lambda}^{\pm-}$, *then we define*

$$F_n(x,y,z)=(\ \lambda x\ ,\ \frac{1}{\lambda}y\ ,\ \pi^c\circ F_n^{-1}(y,x,z)-\frac{1}{4}\ )\ .$$

Here we applying $F_n(W_\varepsilon^{+\pm}\cup W_\varepsilon^{-\pm})=W_{\varepsilon/\lambda}^{+\pm}\cup W_{\varepsilon/\lambda}^{-\pm}$ *is already defined, and we can define its inverse map.*

Lemma 6.3.5. *For any* n, *the map* F_n *is a well defined smooth diffeomorphism on* $\mathbb{R}^2\times S^1$. *Moreover, they satisfy*

- $\|\partial\pi^cF_n/\partial x\|$ *and* $\|\partial\pi^cF_n/\partial y\|$ *are uniformly bounded on* $\mathbb{R}^2\times S^1$, *and the upper bounds are independent on* n;
- $\lim_{n\to\infty}D^cF_n=1$, *the convergence is uniform on* $\mathbb{R}^2\times S^1$.

Proof. To show that F_n is well defined smooth diffeomorphism, we just need to take care about the gluing construction of $F_{0,n}^{++}$, $F_{0,n}^{+-}$, and $F_{0,n}^{+\pm}$. Since for $\varepsilon/\lambda\le y\le\varepsilon$, we have

$$F_n(z)|_{L_n^{+\pm}(x,y)}=\zeta(y)\cdot F_{0,n}^{+\pm}|_{L_n^{+\pm}(x,y)}(z)+(1-\zeta(y))\cdot F_{0,n}^{++}|_{L_n^{+\pm}(x,y)}(z).$$

From the definition of $\zeta(y)$, we can see that when $y = \varepsilon/\lambda$, this definition shows that $F_n = F_{0,n}^{+\pm}$; and when $y = \varepsilon$, $F_n = F_{0,n}^{++}$. by the smoothness of $\zeta(y)$, we know that F_n is smooth. Moreover, for the central derivatives, we have

$$D^c F_n|_{L_n^{+\pm}(x,y)} = \zeta(y) \cdot D^c F_{0,n}^{+\pm}|_{L_n^{+\pm}(x,y)} + (1-\zeta(y)) \cdot D^c F_{0,n}^{++}|_{L_n^{+\pm}(x,y)}.$$

This shows F_n is diffeomorphisms on each S^1-fibers, thus on $\mathbb{R}^2 \times S^1$. In the meanwhile, since we know that both $D^c F_{0,n}^{+\pm}$ and $D^c F_{0,n}^{++}$ converge to 1 uniformly, so does $D^c F_n$.

The last mission is to verify the uniform boundedness of partial derivatives. We just check for $\partial \pi^c F_n/\partial y$:

$$\begin{aligned}\|\frac{\partial \pi^c F_n}{\partial y}\| &\leq \|\frac{\partial(\zeta \cdot \pi^c F_{0,n}^{+\pm})}{\partial y}\| + \|\frac{\partial((1-\zeta)\cdot \pi^c F_{0,n}^{++})}{\partial y}\| \\ &\leq (\|\zeta\|+1)\cdot(\|\frac{\partial \pi^c F_{0,n}^{+\pm}}{\partial y}\| + \|\frac{\partial \pi^c F_{0,n}^{++}}{\partial y}\|) + K \cdot \|\frac{\partial \zeta}{\partial y}\|\end{aligned}$$

Notice that the function ζ does not depend on n, both $\|\partial \pi^c F_{0,n}^{+\pm}/\partial y\|$ and $\|\partial \pi^c F_{0,n}^{++}/\partial y\|$ are uniformly bounded with respect to $\mathbb{R}^2 \times S^1$ and n. This shows the uniformly boundedness of partial derivatives.

□

6.3.4 Contracting and repelling regions

Now we can define the attracting and repelling regions for F_n.

Definition 6.3.6. *We define the two closed regions $U^n, V^n \subset \mathbb{R}^2 \times S^1$ as follows:*

- *U^n restricted on $\mathbb{R}^2 \times S^1 \setminus \mathrm{Int}(W_\varepsilon)$ is equal to $(U_{0,n}^{++} \cup U_{0,n}^{+-} \cup U_{0,n}^{--} \cup U_{0,n}^{-+}) \setminus \mathrm{Int}(W_\varepsilon)$. V^n restricted on $\mathbb{R}^2 \times S^1 \setminus \mathrm{Int}(W_\varepsilon)$ is equal to $(V_{0,n}^{++} \cup V_{0,n}^{+-} \cup V_{0,n}^{--} \cup V_{0,n}^{-+}) \setminus \mathrm{Int}(W_\varepsilon)$.*
- *For $U^n \cap W_\varepsilon^{+\pm}$, we have $U^n \cap W_{\varepsilon/\lambda}^{+\pm} = U_{\varepsilon,n}^{+\pm} \cap W_{\varepsilon/\lambda}^{+\pm}$; and $U_n \cap (W_\varepsilon^{+\pm} \setminus W_{\varepsilon/\lambda}^{+\pm}) = (U_{0,n}^{++} \cup U_{0,n}^{+-}) \cap (W_\varepsilon^{+\pm} \setminus W_{\varepsilon/\lambda}^{+\pm})$. We define $V^n \cap W_\varepsilon^{+\pm} = V_{\varepsilon,n}^{+\pm}$.*
- *For $U^n \cap W_\varepsilon^{-\pm}$ and $V^n \cap W_\varepsilon^{-\pm}$, we also defined by symmetry:*

$$\begin{aligned}U^n \cap W_\varepsilon^{-\pm} &= \{\ (x,y,z):\ (-x,-y,z+\frac{1}{2}) \in U_n \cap W_\varepsilon^{+\pm}\ \};\\ V^n \cap W_\varepsilon^{-\pm} &= \{\ (x,y,z):\ (-x,-y,z+\frac{1}{2}) \in V_n \cap W_\varepsilon^{+\pm}\ \}.\end{aligned}$$

- *For $U^n \cap (W_{\varepsilon/\lambda}^{\pm+} \cup W_{\varepsilon/\lambda}^{\pm-})$ and $V^n \cap (W_{\varepsilon/\lambda}^{\pm+} \cup W_{\varepsilon/\lambda}^{\pm-})$, we defined them as:*

$$\begin{aligned}U^n \cap (W_{\varepsilon/\lambda}^{\pm+} \cup W_{\varepsilon/\lambda}^{\pm-}) &= \{\ (x,y,z):\ (y,x,z+\frac{1}{4}) \in F_n(V_n \cap (W_\varepsilon^{+\pm} \cup W_\varepsilon^{-\pm}))\ \};\\ V^n \cap (W_{\varepsilon/\lambda}^{\pm+} \cup W_{\varepsilon/\lambda}^{\pm-}) &= \{\ (x,y,z):\ (y,x,z+\frac{1}{4}) \in F_n(U_n \cap (W_\varepsilon^{+\pm} \cup W_\varepsilon^{-\pm}))\ \}.\end{aligned}$$

For U^n and V^n, we have the following lemma holds associated to F_n.

Lemma 6.3.7. *The two closed regions U^n and V^n are disjoint. Moreover, U^n is positively invariant: $F_n(U^n) \subset \mathrm{Int}(U^n)$, and $D^cF_n|_{U^n}$ is uniformly contracting; V_n is negatively invariant: $F_n^{-1}(V^n) \subset \mathrm{Int}(V^n)$, and $D^cF_n^{-1}|_{V^n}$ is uniformly contracting.*

Proof. To prove this lemma, we just need to check the invariant properties of U^n,V^n, and the central contracting and expanding properties restricted on each S^1-fibers.

Furthermore, from the way how we define U^n,V^n, and F_n, we only need to check that $F_n|_{W^{+\pm}_{\lambda\cdot\varepsilon}}$ maps $U^n \cap W^{+\pm}_{\lambda\cdot\varepsilon}$ positively invariant and central contracting, and $F_n^{-1}|_{W^{+\pm}_{\varepsilon}}$ maps $V^n \cap W^{+\pm}_{\varepsilon}$ positively invariant and central contracting.

First we look at U^n. Form the definition of U^n, we know that when we convex sum of $F^{++}_{0,n}$ and $F^{+\pm}_{0,n}$, or $F^{+-}_{0,n}$ and $F^{+\pm}_{0,n}$, they restricted on U^n are all central contracting. This shows that $F_n|_{U^n}$ is central contracting. We only need to verify the positively invariant. Notice that

$$U^n|_{W^{+\pm}_{\varepsilon}\setminus W^{+\pm}_{\varepsilon/\lambda}} = (U^{++}_{0,n} \cup U^{+-}_{0,n})|_{W^{+\pm}_{\varepsilon}\setminus W^{+\pm}_{\varepsilon/\lambda}} ,$$

this guarantees that

$$F_n(U^n \cap (W^{+\pm}_{\lambda\cdot\varepsilon} \setminus W^{+\pm}_{\varepsilon})) \subset \mathrm{Int}(U^n \cap (W^{+\pm}_{\varepsilon} \setminus W^{+\pm}_{\varepsilon/\lambda})) .$$

For $U^n \cap (W^{+\pm}_{\varepsilon} \setminus W^{+\pm}_{\varepsilon/\lambda})$, things became a little bit tricky. Since F_n are defined as convex sum in this region. But we know that this region coincide with $U^{++}_{0,n}$ or $U^{+-}_{0,n}$, whose image by $F^{++}_{0,n}$ and $F^{+-}_{0,n}$ are contained in $U^{+\pm}_{\varepsilon,n} \cap W^{+\pm}_{\varepsilon/\lambda}$. In the meanwhile, itself also contained in $U^{+\pm}_{\varepsilon,n}$, so positively invariant by $F^{+\pm}_{0,n}$. Some simple calculation shows that the convex sum of diffeomorphisms also map $U^n \cap (W^{+\pm}_{\varepsilon} \setminus W^{+\pm}_{\varepsilon/\lambda})$ into the interior of $U^{+\pm}_{\varepsilon,n} \cap W^{+\pm}_{\varepsilon/\lambda}$. This show the positive invariance of F_n acting on U^n.

Then we look at $F_n|_{V^n}$. Notice that for any $(x, y) \times S^1 \subset W^{+\pm}_{\varepsilon}$ and $y \geq 0$, we have

$$V^n \cap (x, y) \times S^1 = V^{+\pm}_{\varepsilon,n} \cap (x, y) \times S^1 \subseteq V^{++}_{0,n} \cap (x, y) \times S^1,$$

which is the interval centered at $q^{+\pm}_n(x, y) \in \Sigma^+_{R,n}$. And both $(F^{++}_{0,n})^{-1}$ and $(F^{+\pm}_{0,n})^{-1}$ restricted on $V^n \cap (x, y) \times S^1$ maps the central point to central point contained in $\Sigma^+_{R,n}$, with the contracting rate α_n, which both admit the invariant property with respect to V_n. So the same properties holds for their convex sum. This guarantees the negatively invariant and central expanding of F_n acting on V^n.

□

6.3.5 Proof of lemma 6.1.1

Now we can close this section by give a proof of the technical lemma 6.1.1.

Proof of Lemma 6.1.1. We have already defined the diffeomorphisms F_n and two disjoint regions U^n and V^n. We check they satisfy all the properties listed in the lemma one by one.

Item 1 came from the definition of F_n. Item 2 and 6 concerned about the invariant region U^n and V^n, which has been showed in lemma 6.3.7. Item 3 and 5 are the consequence we have assumed $\lambda \cdot (3^{\frac{n}{2}} + 1) < \frac{(2m)^n}{T_0 \cdot n}$. Thus all the diffeomorphisms we defined satisfied these two properties, so does F_n. Item 4 is the result of our definition of diffeomorphisms $F_{0,n}$ on a neighborhood of invariant fiber $(0,0) \times S^1$. Item 7 and 8 have been verified in lemma 6.3.5. This finishes the proof of lemma 6.1.1.

□

6.4 Transitivity of Attractor and Repeller

In this section, we will discuss the transitivity of the maximal invariant sets A_n and R_n contained in the attracting and repelling regions U_n and V_n. This implies that the chain recurrent set $\mathcal{R}(f_n)$ of f_n consists of one hyperbolic attractor and one hyperbolic repeller.

The proof is exactly the same with [8], we just sketch it for completeness.

We will focus on the attracting region U_n and the corresponding Birkhoff section Σ_n. The repelling region V_n and the transitivity of R_n is exactly the same.

If we collapsing each boundary component of Σ_n into one point(singularity), then we get a closed surface P_n with $(2m)^{2n}$-singularities. We will show that f_n restricted on the maximal invariant set A_n of U_n is semi-conjugate to a pseudo-Anosov map on P_n. Especially, the stable and unstable foliations of this pseudo-Anosov map is induced by the intersections of the center stable and center unstable manifold of f_A (the same with f_n) with Σ_n.

Finally the minimality of unstable foliations of the pseudo-Anosov map will help us get the minimality of unstable foliations of A_n, thus the transitivity of A_n.

First we illustrate the pseudo-Anosov map on the closed surface P_n. Recall that in our construction of f_n, when far from the boundary fibers (restricted on $E_n \subset \mathcal{H}$), we have defined the diffeomorphism $f_{n,ext}$ maps the Birkhoff section $\Sigma_n|_{E_n}$ into Σ_n. This means that we have defined a unique fiber isotopy function $F_t : \mathcal{H} \times [0,1] \to \mathcal{H}$ such that:

- $F_0 = f_A$;
- $F_1(\Sigma_n) = \Sigma_n$;
- $F_1(\Sigma_n \cap E_n) = f_n(\Sigma_n \cap E_n) \subset \Sigma_n$.

Claim. *If we collapsing each boundary components of Σ_n into singularities to get a closed surface P_n, then F_1 defines a pseudo-Anosov map $PA_n : P_n \to P_n$. Moreover, the stable and unstable* *foliations of P_n is induced by the intersections of the center stable and center unstable manifold of f_A with Σ_n.*

Proof of the Claim. First we can see that the intersections of the center stable and center unstable manifold of f_A with Σ_n defines two family of foliations on Σ_n which are transverse to each other. When we collapse one boundary component into one singularity, we can see that both these two foliations has $2(2m)^{2n}$ prolongs at each singularity.

Since the F_1 is fiber isotopic to f_A, which preserve the center stable and center unstable foliations of $\mathcal{H}$, this implies the map $PA_n : P_n \to P_n$ induced by F_1 preserve two family of foliations above. Moreover, since if we modulo the center S^1-fibers, F_1 is still the linear Anosov action A on $\mathbb{T}^2$. Thus it contracts the intersection of center stable foliations with Σ_n, and expands the intersection of center unstable foliations with Σ_n. This proved that PA_n is a pseudo-Anosov map.

□

Now we can state the main proposition of this section, which will closed the proof of the main theorem.

Proposition 6.4.1. *For each f_n, it admits exactly two basic pieces, where one is the maximal invariant set A_n of U_n which is a connected mixing hyperbolic attractor, the other is is the maximal invariant set R_n of V_n which is a connected mixing hyperbolic repeller.*

Moreover, there exist a continuous surjective projection $\pi_{A_n} : A_n \to P_n$ which induces a semi-conjugacy between $f_n|_{A_n}$ and PA_n, such that

- *For any $x \in P_n$ which does not belong to the unstable manifold of any singularities, $\pi_{A_n}^{-1}(x)$ is a single point;*
- *For any $x \in P_n$ which belongs to the unstable separatrix of a singularity, $\pi_{A_n}^{-1}(x)$ consists of exactly two points;*
- *For any $x \in P_n$ is a singularity, $\pi_{A_n}^{-1}(x)$ consists of $2(2m)^{2n}$ periodic points of A_n which belongs to one boundary fiber of Σ_n.*

As we said at the beginning of this section, the key fact of the proof relies on the semi-conjugacy. Now we try to construct the conjugate projection.

Recall that in the central DA-construction where we proof the technical lemma, the first step of our proof is deforming the half helicoid Σ_H into a branch surface with boundary and corners $\Sigma_{A,n}$. Mapping this deformation into the nilmanifold $\mathcal{H}$, we can get a branch surface $B_{\Sigma,n}$ with boundary and corners. For any $p \in \partial\Sigma_n$, restricted on the neighborhood of a boundary fiber S_p, we have

$$B_{\Sigma,n}|_{\mathbb{D}(p,\frac{\delta}{(2m)^{2n}})\times S^1} = R_{\frac{t_{p,n}}{(2m)^{2n}}} \circ R_{\frac{2\theta_0+1}{2(2m)^{2n}}} \circ P_0 \circ H_n(\Sigma_{A,n})|_{\mathbb{D}(p,\frac{\delta}{(2m)^{2n}})\times S^1}.$$

And for from the boundary fibers(restricted in $E_n \subset \mathcal{H}$), $B_{\Sigma,n}|_{E_n}$ is equal to $\Sigma_n|_{E_n}$.

It is clear that $B_{\Sigma,n}$ contained in U_n. Now we can define a new region $U_{\Sigma,n}$ as the union of central segment components in U_n which contained a point in $B_{\Sigma,n}$.

Lemma 6.4.2. *We have $A_n \subset U_{\Sigma,n} \subset U_n$.*

Proof. From the central DA-construction, we know that the periodic points in the boundary fibers are also boundary periodic points. That means if the stable manifolds of these periodic points minus the center S^1-fiber will consist of two components, and one of these two components will not intersect the maximal invariant set A_n. This also holds for the points contained in the unstable manifolds of these boundary periodic points.

From the construction of U_n in last section, we can see that $U_{\Sigma,n}$ is the set U_n minus the region in these components, thus the part been deleted does not intersect the maximal invariant set A_n.

□

Now we can construct a projection from $U_{\Sigma,n}$ to P_n. We denote the projection by $\pi_{U_{\Sigma,n}}$: $U_{\Sigma,n} \to P_n$.

We can first define a projection

$$\pi_{B_{\Sigma,n}} \;:\; B_{\Sigma,n} \;\longrightarrow\; P_n$$

in the following way. When far from the boundary fibers, that is restricted on $E_n \subset \mathcal{H}$, $B_{\Sigma,n} \cap E_n = \Sigma_n \cap E_n = P_n \cap E_n$. So the projection is identity.

When close to the boundary fibers, that is in the unit model (section 6.2.2), we have defined in Definition 6.2.2,

$$\pi_{\Sigma_{A,n}} \;:\; \Sigma_{A,n} \setminus (0,0) \times S^1 \;\longrightarrow\; \Sigma_H \setminus (0,0) \times S^1.$$

Here we ignore the linear transformation for simplicity. And the interior of Σ_n is equal to P_n minus all the singularities. For the points contained in $\partial B_{\Sigma,n}$ intersects with boundary fibers of Σ_n (the boundary periodic points of f_n in A_n), $\pi_{U_{\Sigma,n}}$ maps them into the singularities which collapsed from the boundary fibers.

From this definition, we have the following claim holds.

Claim. *$\pi_{B_{\Sigma,n}}$ is a continuous surjective projection from $B_{\Sigma,n}$ to P_n. It is injection when restricted on the interior of $B_{\Sigma,n}$. When restricted on $\partial B_{\Sigma,n}$, the points in boundary fibers of Σ_n be mapped into the corresponding singularities, the image points of the other boundary points all have exactly two preimages.*

Thus we defined the projection from $B_{\Sigma,n}$ to P_n. Then $\pi_{U_{\Sigma,n}}$ is defined as mapping the central segment containing point $x \in B_{\Sigma,n}$ to $\pi_{\Sigma_{A,n}}(x) \in P_n$.

Lemma 6.4.3. *The projection* $\pi_{U_{\Sigma,n}}$ *is a continuous surjective map, and induces a semi-conjugacy between* f_n *restricted on* $B_{\Sigma,n}$ *and the pseudo-Anosov homeomorphism* PA_n.

Proof. The semi-conjugacy property comes from the fact that the projection $\pi_{\Sigma_{A,n}}$ induce a semi-conjugacy between f_n acting on $B_{\Sigma,n}$ and the pseudo-Anosov map P_N, and it maps the central segment in $U_{\Sigma,n}$ into the interior of another central segment in $U_{\Sigma,n}$. □

Proof of Proposition 6.4.1. First, there must exists a transitive attractor $\Lambda_n \subset A_n$, which could not be a periodic orbit. For any $p \in \Lambda_n$, we have $W^u(p) \subset \Lambda_n \subset U_{\Sigma,n}$. If we further require that p does not belong to the unstable manifolds of boundary periodic points and denote $\Lambda_{n,0}$ be the closure of $W^u(p)$, then $\Lambda_{n,0}$ is a mixing component of Λ_n.

Moreover, $\pi_{U_{\Sigma,n}}(W^u(p))$ is a regular leaf of the unstable foliation of the pseudo-Anosov map PA_n in P_N, which implies $\pi_{U_{\Sigma,n}}(\Lambda_{n,0}) = P_N$. Thus each central segment in $U_{\Sigma,n}$ contains at least one point in $\Lambda_{n,0}$.

Now for any compact set $K \subset A_n \subset U_{\Sigma,n}$ which is invariant by f_n^l for some integer l. There must exists two points $x \in K$ and $y \in \Lambda_{n,0}$ which contained in one central segment of $U_{\Sigma,n}$. Iterate this segment by f_n^l, which the length will tend to zero. This implies $K \cap \Lambda_{n,0} \neq \emptyset$. So $\Lambda_{n,0}$ is the unique mixing component in A_n. This implies $A_n = \Lambda_{n,0}$ is a mixing connected attractor.

We define the semi-conjugacy π_{A_n} from $f_n|_{A_n}$ to PA_n as $\pi_{A_n} = \pi_{U_{\Sigma,n}}|_{A_n}$. So it is continuous and surjective.

For the analysis preimages, first we look at the boundary periodic points. Since we collapse the boundary fibers of Σ_n into singularities of P_N, so the preimage of these singularities is $2(2m)^{2n}$ periodic points.

For the unstable manifolds of these periodic points, each one separatrix of the neighboring two periodic points will be asymtoptic, and will be projected into one separatrix of the unstable manifold of a singularity.

So we only need to show that the for any p does not belong to the unstable manifolds of boundary periodic points, π_{A_n} is injection on p. Assume that $\pi_{A_n}(q) = \pi_{A_n}(p)$, then they are in the same central component J of $U_{\Sigma,n}$. Since both p, q do not belong to the unstable manifolds of boundary periodic points, then $f_n^{-l}(J)$ intersect $U_{\Sigma,n}$ could only be in one component, for any $l > 0$. This implies the $f_n^{-l}(p)$ and $f_n^{-l}(q)$ are always in the same central component of $U_{\Sigma,n}$, thus with uniformly bounded central distance. By the uniformly central contracting of f_n in $U_{\Sigma,n}$, we have

$$d^c(p,q) \;\leq\; \lim_{l\to\infty} \alpha_n^l \cdot d^c(f_n^{-l}(p), f_n^{-l}(q)) \;=\; 0\ .$$

This proves that $p = q$ and π_{A_n} is injective on p. □

Bibliography

[1] D.V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, *Trudy Mat. Inst. Steklov.*, **90** (1967), 209 pp.

[2] L. Auslander, Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups, *Ann. of Math.*, **71** (1960), 579-590.

[3] F. Béguin, S. Crovisier, T. Jäger, F. Le Roux, Denjoy constructions for fibered homeomorphisms of the torus, *Trans. Amer. Math. Soc.*, **361** (2009), no. 11, 5851-5883.

[4] Ch. Bonatti, and L. J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms, *Ann. of Math.*, **143** (1996), no. 2, 357-396.

[5] Ch. Bonatti, L. J. Díaz, and E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, *Ann. of Math.*, **158** (2003), no. 2, 355-418.

[6] Ch. Bonatti, L. J. Díaz, and R. Ures, Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms, *J. Inst. Math. Jussieu*, **1** (2002), no. 4, 513-541.

[7] Ch. Bonatti, and N. Guelman, Transitive Anosov flows and Axiom-A diffeomorphisms, *Ergodic Theory Dynam. Systems*, **29** (2009), no. 3, 817-848.

[8] Ch. Bonatti, and N. Guelman, Axiom A diffeomorphisms derived from Anosov flows, *J. Mod. Dyn.*, **4** (2010), no. 1, 1-63.

[9] Ch. Bonatti, and A. Wilkinson, Transitive partially hyperbolic diffeomorphisms on 3-manifolds, *Topology*, **44** (2005), no. 3, 475-508.

[10] K. Burns, C.Pugh, and A. Wilkinson, Stable ergodicity and Anosov flows, *Topology*, **39** (2000), no. 1, 149-159.

[11] K. Burns, and A. Wilkinson, On the ergodicity of partially hyperbolic systems, *Ann. of Math.*, **171** (2010), no. 1, 451-489.

[12] L. J. Díaz, E. Pujals, and R.Ures, Partial hyperbolicity and robust transitivity, *Acta Math.*, **183** (1999), no. 1, 1-43.

[13] D. Fried, Transitive Anosov flows and pseudo-Anosov maps, *Topology*, **22** (1983), no. 3, 299-303.

[14] H. Furstenberg, Strict ergodicity and transformation of the torus, *Amer. J. Math.*, **83** (1961), 573-601.

[15] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, *Proceedings of the International Congress of Mathematicians*, Vol. II, (Beijing, 2002), 405-414.

[16] M. Grayson, C. Pugh, and M. Shub, Stably ergodic diffeomorphisms, *Ann. of Math.*, **140** (1994), no. 2, 295-329.

[17] A. Hammerlindl, Partial hyperbolicity on 3-dimensional nilmanifolds, *Discrete Contin. Dyn. Syst.*, **33** (2013), no. 8, 3641-3669.

[18] A. Hammerlindl, and R. Potrie, Pointwise partial hyperbolicity in 3-dimensional nilmanifolds, Preprints (2013), arXiv:1302.0543.

[19] J. Hadamard, Les surfaces á courbures opposées et leurs lignes géodésiques, *Journal de Math. Pures et Appliquées*, **IV** (1898), 27-73.

[20] S. Hayashi, Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows, *Ann. of Math.*, **145** (1997), no. 1, 81-137.

[21] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, A survey of partially hyperbolic dynamics, *Partially hyperbolic dynamics, laminations, and Teichmüller flow*, 35-87, Fields Inst. Commun., 51, AMS, 2007.

[22] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, Partial hyperbolicity and ergodicity in dimension three, *J. Mod. Dyn.*, **2** (2008), no. 2, 187-208.

[23] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle, *Invent. Math.*, **172** (2008), no. 2, 353-381.

[24] E. Hopf, Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung, *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, **91** (1939), 261-304.

[25] M. Hirsch, C. Pugh, and M. Shub, Invariant manifolds, *Lecture Notes in Mathematics*, Vol. 583. Springer-Verlag, Berlin-New York, 1977. ii+149 pp.

[26] T. Jäger, G. Keller, The Denjoy type of argument for quasiperiodically forced circle diffeomorphisms, *Ergodic Theory Dynam. Systems*, **26** (2006), no. 2, 447-465.

[27] T. Jäger, J. Stark, Towards a classification for quasiperiodically forced circle homeomorphisms, *J. London Math. Soc.*, **73** (2006), no. 3, 727-744.

[28] R. Mañé, An ergodic closing lemma, *Ann. of Math.*, **116** (1982), no. 3, 503-540.

[29] R. Mañé, A proof of the C^1 stability conjecture, *Inst. Hautes Études Sci. Publ. Math.*, No. 66 (1988), 161-210.

[30] J. Palis, and C. Pugh, Fifty problems in dynamical systems, *Lecture Notes in Math.*, Vol. 468, Springer, Berlin, (1975), pp. 345-353.

[31] J. Plante, Anosov flows, *Amer. J. Math.*, **94** (1972), 729-754.

[32] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747-817.

[33] A. Wilkinson, Stable ergodicity of the time-one map of a geodesic flow, *Ergodic Theory Dynam. Systems*, **18** (1998), no. 6, 1545-1587.

[34] A. Wilkinson, Conservative partially hyperbolic dynamics, *Proceedings of the International Congress of Mathematicians*, Vol. III, (New Delhi, 2010), 1816-1836.

[35] J. Wood, Bundles with totally disconnected structure group, *Comment. Math. Helv.*, **46** (1971), 257-273.

Yi SHI

School of Mathematical Sciences,
Peking University, Beijing 100871, China
&
Institut de Mathématiques de Bourgogne,
Université de Bourgogne, Dijon 21000, France

Email: polarbearsy@gmail.com