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Equivalence relations among homology 3-spheres and the Johnson filtration

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Contents

Introduction

1 Introduction (in English)

The term "low-dimensional topology" usually refers to the study of manifolds of dimension less than 4, up to some "deformations". Here, a manifold is a topological space locally homeomorphic to an Euclidean space. The classification of compact oriented 1-dimensional manifolds is a simple exercise, while the one of compact oriented 2-dimensional manifolds, i.e. of surfaces, is more involved: the homeomorphism type of a compact connected oriented surface is given by its genus and the number of its boundary components. In contrast, dimension 3 is really rich and not so well-understood. Low-dimensional topology includes the study of homeomorphisms between low-dimensional objects, but also of embeddings of manifolds in higher-dimensional spaces (knots, braids...). All these topics are intricately linked, and a vast literature appeared over the last century. A lot of geometric, group-theoretic, and even analytic questions are also related to low-dimensional topology. A striking example of this fact was the proof of the Poincaré conjecture by Perelman in 2003: the natural question of knowing if a simply connected closed 3-manifold is homeomorphic to the 3-sphere appeared to be extremely complex.

One of the modern goals of low-dimensional topology is to get a better understanding of the classification of closed oriented 3-manifolds up to homeomorphims. A 3-manifold can be described by using knots, through the notion of Dehn surgery, but also by using the mapping class group of surfaces, through the notions of Heegaard splittings and surgery along handlebodies, as we shall detail below. Hence, this matter is, in particular, connected to knot theory and to the study of mapping class groups. In this thesis, we are mainly interested in the latter. The mapping class group of a surface is the group of isotopy classes of homeomorphisms of the surface which fix the boundary components (the isotopies fixing the boundary components). The mapping class group of a surface of genus g with one boundary component will be denoted $\mathcal{M}_{g,1}$ in the sequel.

Regardless of how one presents two given manifolds, it is in general delicate to determine if they are homeomorphic. A way to show that they are *not*, however, is to produce *invariants* that do not have the same values on the two manifolds. By an invariant we mean a map on some set of manifolds whose value on a manifold depends only on its homeomorphism type. Since the discovery in the 1980's of the Jones polynomial for knots, numerous invariants of knots and 3-manifolds have been produced, whose construction often involved ideas from mathematical physics. We refer here for example to *quantum invariants*, i.e. invariants derived from certain quantum groups representations, and to *finite-type invariants*, i.e. invariants with a "polynomial behaviour" with respect to certain surgery operations. These two types of invariants are related: invariants of the first type can, under certain circumstances, yield invariants of the second type via a process of expansion into power series. A tremendous amount of effort has been put in the study of these invariants, and they have been organized, in some sense, via the Konsevitch integral (for knots) and the LMO invariant (for 3-manifolds). Unfortunately, these invariants are often built in a complicated and indirect way, and a current challenge of low-dimensional topology is to give them more classical topological meanings. Pursuing this goal, we should be looking for refinements or extensions of some invariants, but also for *surgery formulas*, i.e. descriptions of how an invariant changes when one modifies a given manifold. Working with 3-manifolds up to homeomorphisms is not always what we seek. We sometimes study some weaker equivalence relations, or reduce the set of manifolds we work with: we shall often restrict ourselves to *homology 3-spheres*. Recall that a homology 3-sphere is an oriented 3-manifold with the same homology groups as the sphere S^3.

We now set some notations, in order to state precisely our results. We fix a handlebody V_g of genus g (i.e. a ball with g handles) whose boundary is a surface Σ_g of genus g. The handlebody is drawn in Figure 1, where the red curves bound blue disks in V_g but not in Σ_g. The surface Σ_g minus a disk (suggested by grey stripes on Figure 1) will be the surface with one boundary component $\Sigma_{g,1}$. Most of the time this surface is simply considered as an abstract surface and not as the boundary of the handlebody. We shall not specify the genus and the number of boundary components in the notations when it is clear from context. Set $\pi := \pi_1(\Sigma_{g,1}, x_0)$ where x_0 is a point on the boundary of $\Sigma_{g,1}$, and $H := H_1(\Sigma_{g,1})$ the abelianization of π. We denote $\pi' := \pi_1(V_g, x_0)$ the fundamental group of the handlebody of genus g, and $\mathbb{A}$ the kernel of the projection $\pi \to \pi'$ induced by the inclusion of $\Sigma_{g,1}$ in V_g. The abelianization of π', corresponding to the first homology group of the handlebody, is denoted H' and A stands for the kernel of the projection from H to H' induced by the inclusion of $\Sigma_{g,1}$ in V_g.

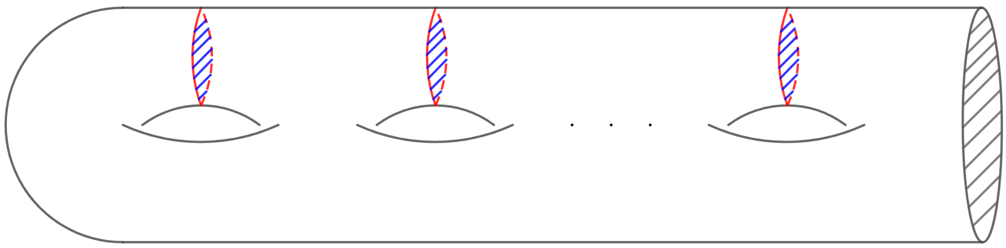


Figure 1: The handlebody V_g with boundary Σ_g

The *handlebody group* $\mathcal{A}_{g,1}$ is the subgroup of the mapping class group consisting of elements extending to the whole handlebody. It coincides with the subgroup of elements of $\mathcal{M}$ preserving $\mathbb{A}$ [12]. The mapping class group acts naturally on H, preserving the symplectic form ω induced by the intersection form of the surface. This defines an action $\mathcal{M} \to \mathrm{Sp}(H)$, where $\mathrm{Sp}(H)$ stands for the subgroup of $\mathrm{Aut}(H)$ of elements leaving invariant the symplectic form. The kernel of this action, i.e. the subgroup of $\mathcal{M}_{g,1}$ of elements acting trivially in homology, is the *Torelli group* $\mathcal{I}_{g,1}$. Let $\mathcal{V}(3)$ and $\mathcal{S}(3)$ be respectively the set of all closed oriented 3-manifolds and all closed oriented homology 3-spheres up to oriented homeomorphisms.

A *Heegaard splitting* of genus g of a 3-manifold M is an inclusion of the handlebody V_g in M such that the complement of the interior of V_g is also a handlebody (of genus g). Any closed oriented 3-manifold admits a Heegaard splitting, provided we allow the genus to be big enough, and the homeomorphism type of the manifold obtained by gluing two handlebodies only depends on the isotopy class of the gluing map (see Theorem 1.1 below). Hence, the study of 3-manifolds is strongly related to the understanding of $\mathcal{M}$. Also, considering a splitting of a manifold, one is allowed to "twist" it (i.e. compose the gluing map) by an element of $\mathcal{M}$. This is what we call a *surgery*. Notice that this notion somehow generalizes the notion of *Dehn surgery* which consists in removing the neighborhood of a knot in a 3-manifold and regluing it in a different manner. The purpose of this PhD thesis is precisely to study some specific kinds of surgeries, by combining the use of filtrations on $\mathcal{M}$, and topological invariants of 3-manifolds.

Let us precise the notions above. There is, up to isotopy, and for each genus g, a unique Heegaard splitting of S^3 [57]. Let us fix an orientation-preserving homeomorphism ι_g of Σ_g such that $S^3 = V_g \cup_{\iota_g} (-V_g)$. Here $(-V_g)$ is the notation for the handlebody V_g with reversed orientation. We set $\mathcal{B}_{g,1} := \iota_g \mathcal{A}_{g,1} \iota_g^{-1}$ and we denote by S^3_φ the 3-manifold $V_g \underset{\iota \circ \varphi}{\cup} (-V_g)$ for any element $\varphi \in \mathcal{M}_{g,1}$ (extended to Σ_g by the identity on the removed disk).

This procedure defines, for any $g \geq 1$, a map from $\mathcal{M}_{g,1}$ to $\mathcal{V}(3)$. We also have stabilization maps $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, compatible with the other maps, in the sense that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{M}_{g,1} \arrow[r] \arrow[d, hook] & \mathcal{V}(3) \\ \mathcal{M}_{g+1,1} \arrow[ur] & \end{tikzcd} .$$

When one composes the gluing map on the right by an element of $\mathcal{B}_{g,1}$ or to the left by an element of $\mathcal{A}_{g,1}$, the resulting manifold is the same up to homeomorphism. Indeed, similarly to the fact that the elements of $\mathcal{A}_{g,1}$ extend to V_g, the elements of $\mathcal{B}_{g,1}$ can be identified with the elements of $\mathcal{M}$ extending to the complementary of the interior of V_g in S^3. This allows us to state the following refinement of the famous Reidemeister-Singer theorem [52, 54], where the second statement is a consequence of [41, Corollary of Theorem 2].

Theorem 1.1 (Reidemeister-Singer)**.** *There is a bijection*

$$\begin{aligned} \lim_{g \to +\infty} \mathcal{A}_{g,1} \setminus \mathcal{M}_{g,1}/\mathcal{B}_{g,1} &\longrightarrow \mathcal{V}(3) \\ \varphi &\longmapsto S^3_\varphi \end{aligned}$$

which actually restricts to a bijection $\lim_{g \to +\infty} \mathcal{A}_{g,1} \setminus \mathcal{I}_{g,1}/\mathcal{B}_{g,1} \longrightarrow \mathcal{S}(3)$.

Notice that, among other things, the second part of Theorem 1.1 states that when twisting the splitting of S^3 by an element of $\mathcal{I}$, one gets a homology 3-sphere, and any homology 3-sphere can be obtained in this way. Be aware that the notation $\mathcal{A}_{g,1} \setminus \mathcal{I}_{g,1}/\mathcal{B}_{g,1}$ designates here the image of the Torelli group in the double coset $\mathcal{A}_{g,1} \setminus \mathcal{M}_{g,1}/\mathcal{B}_{g,1}$. Now, roughly, if one has a *surgery formula* for an invariant, i.e. if one knows how the invariant varies when applying a surgery, then one can get useful information on $\mathcal{M}$ by defining maps on the mapping class group which measure the variation of the invariant when twisting some Heegaard splitting of a fixed manifold (e.g. S^3). We will do so with the Casson invariant, using a surgery formula given by Morita [42]. This formula is a central and recurring theme of these PhD dissertation. We give more details in Section 1.2.

1.1 The Johnson filtration and the Johnson homomorphisms

Let us put aside 3-manifolds for a moment and focus on the mapping class group $\mathcal{M}_{g,1}$. By the Dehn-Nielsen theorem, an element of $\mathcal{M}$ is completely determined by its action on π. In other words $\mathcal{M}$ imbeds in $\mathrm{Aut}(\pi)$, the automorphism group of the free group π. This reduces the study of the topological object $\mathcal{M}$ to a purely algebraic question, but the group $\mathrm{Aut}(\pi)$ is, however, quite complicated. Thus, a promising method (see e.g. [1, 28, 26, 42, 44]) to study $\mathcal{M}$ using this point of view is to consider simplified versions of this action, namely the action of $\mathcal{M}$ on the nilpotent quotients of π:

$$\rho_k : \mathcal{M} \to \mathrm{Aut}(N_k)$$

where $N_k := \pi/\Gamma_{k+1}\pi$ for $k \geq 1$ is the *k-th nilpotent quotient of* π, and $\Gamma_k\pi$ is the k-th term of the lower central series of π. Here, the *lower central series* of a group G is defined inductively by $\Gamma_1 G := G$ and $\Gamma_{k+1}G := [\Gamma_k G, G]$.

Then there is a well-known exact sequence

$$0 \longrightarrow \mathcal{L}_{k+1}(H) \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow 0$$

where $\mathcal{L}(H)$ stands for the graded free Lie algebra generated by H in degree 1, and the first non-trivial arrow is given by the identification between $\mathcal{L}_{k+1}(H)$ and $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$. This sequence, in turn, induces the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)) \longrightarrow \mathrm{Aut}(N_{k+1}) \longrightarrow \mathrm{Aut}(N_k). \tag{1.1}$$

The subgroup J_k of $\mathcal{M}$ is then defined as the kernel of the homomorphism ρ_k. In particular, by the Hurewicz theorem, the group J_1 is the Torelli group, that we denoted $\mathcal{I}$, and which takes place in the following exact sequence:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{M} \longrightarrow \mathrm{Sp}(H) \longrightarrow 0. \tag{1.2}$$

We also use the notation $\mathcal{K}$ for J_2, and call it the *the Johnson kernel.* Not much is known about the subgroups J_k, even though it is known that the k-th commutator subgroup of the Torelli group $\Gamma_k\mathcal{I}$ is included in J_k. Generators of $\mathcal{I}$ [51, 3, 28, 24] and $\mathcal{K}$ [26] are known. The abelianization of $\mathcal{I}$ was also determined in [27]. More recently, progress has been made in the determination of the rational abelianization of $\mathcal{K}$ [7, 46], in the slightly different case of a closed surface. The *Johnson filtration* $(J_k)_{k\geq 1}$ is a separating filtration of the Torelli group: $\bigcap\limits_{k\geq 1} J_k = \{\mathrm{Id}\}$. This motivates the study of the associated graded space $\mathrm{Gr}^J(\mathcal{I}) := \bigoplus_{k\geq 1} J_k/J_{k+1}$. By the exact sequence (1.1), the restriction of ρ_{k+1} to J_k induces a morphism:

$$\tau_k : J_k \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$$

which we refer to as the k-th *Johnson homomorphism*, and whose kernel is, by definition, J_{k+1}. The mapping class group acts on itself by conjugation, which induces via the exact sequence (1.2) an action of the symplectic group $\mathrm{Sp}(H)$ on the quotient J_k/J_{k+1}. Each τ_k is then $\mathrm{Sp}(H)$-equivariant with respect to this action. Furthermore, the graded space associated to the Johnson filtration has a Lie structure, its bracket being induced by the commutator in $\mathcal{M}$. Any derivation is determined by its values on generators, and $\mathcal{L}(H)$ is generated by H. Hence, the target space of τ_k can be identified with the space of *derivations of degree* k, i.e. derivations of the Lie algebra $\mathcal{L}(H)$ mapping $H = \mathcal{L}_1(H)$ to $\mathcal{L}_{k+1}(H)$. We denote by $D_k(H)$ the subspace of *symplectic derivations* of degree k, which consists of derivations of $\mathcal{L}(H)$ of degree k vanishing on $\tilde{\omega} \in \Lambda^2 H \simeq \mathcal{L}_2(H)$, the bivector dual to the symplectic form ω. As an element of $\mathcal{M}$ fixes the boundary of Σ, we can further restrict the target of τ_k to $D_k(H)$. Furthermore, $D_k(H)$ can be inserted in a short exact sequence:

$$0 \longrightarrow D_k(H) \longrightarrow H \otimes \mathcal{L}_{k+1}(H) \longrightarrow \mathcal{L}_{k+2}(H) \longrightarrow 0$$

where the arrow from $H \otimes \mathcal{L}_{k+1}(H)$ to $\mathcal{L}_{k+2}(H)$ is the bracket of the free Lie algebra.

The spaces $(D_k(H))_{k\geq 1}$ assemble in a graded Lie algebra $D(H)$ (the bracket of two derivations d_1 and d_2 being classically defined as the difference of compositions $d_1 d_2 - d_2 d_1$). The family $(\tau_k)_{k\geq 1}$, in turn, assembles in a map:

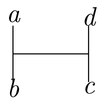
$$\tau : \bigoplus_{k\geq 1} J_k/J_{k+1} \longrightarrow D(H)$$

which is an $\mathrm{Sp}(H)$-equivariant graded Lie morphism.

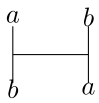
The map τ is injective by definition, which implies that the characterization of its image in $D(H)$ is tantamount to the determination of $\mathrm{Gr}^J(\mathcal{I})$. Infortunately, we only know precisely this image in degree 1 [21] and 2 [42, 58]. When tensoring with the field of rational numbers, $\mathrm{Im}(\tau_k) \otimes \mathbb{Q}$ becomes a $\mathrm{Sp}(H_\mathbb{Q})$-module [2], where $H_\mathbb{Q} := H \otimes \mathbb{Q}$. The rational representation theory of the symplectic group then allows to compute its irreducible decomposition in low degrees (see e.g. [45]). In this case, a result of Hain [18] implies that the space $\mathrm{Gr}^J(\mathcal{I}) \otimes \mathbb{Q}$ is generated in degree 1. Remains the highly non-trivial question, as τ_1 is onto $D_1(H)$, of determining the Lie algebra generated by $D_1(H_\mathbb{Q})$ in $D(H_\mathbb{Q})$.

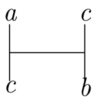
In Chapter 1, we are interested in the handlebody version of the Johnson filtration, i.e. the sequence $(\mathcal{A} \cap J_k)_{k\geq 1}$ of subgroups of $\mathcal{A}$. The intersection of the Johnson filtration with the handlebody group is also separating, thus understanding the graded space $\bigoplus_{k\geq 1} \frac{\mathcal{A}\cap J_k}{\mathcal{A}\cap J_{k+1}}$ is relevant for the study of the inclusion $\mathcal{A} \subset \mathcal{M}$. We shall also see that this is a natural object when studying 3-manifolds via Heegaard splittings. Omori gave in [48] a generating set of $\mathcal{A} \cap \mathcal{I}$, verifying the computation of $\frac{\mathcal{A}\cap J_1}{\mathcal{A}\cap J_2}$ by Morita [42]. We shall tackle the next step, determining explicitly $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3}$.

First, making use of the work of Levine [33], we recall a description of $D_2(H)$ in terms of a certain module of trees whose leaves are colored with elements of H. It consists of integral trees, generating a subspace $D_2'(H)$ of $D_2(H)$, and half of symmetric integral trees denoted $a \odot b$ for $a, b \in H$. A symplectic derivation is uniquely associated to any linear combination of such trees.

Proposition 1.2. *$D_2(H)$ has the following presentation: it is generated by trees* 

for a, b, c and d in H and elements $a \odot b$ for $a, b \in H$, subject to the following relations:

- *AS, IHX, and multilinearity with respect to the labels for all trees (see Figure 2).*
- *$a \odot a = 0$ and $a \odot b = b \odot a$ for all $(a, b) \in H \times H$.*
- $2(a \odot b) =$ 

.
- $(a + b) \odot c = a \odot c + b \odot c +$ 

.

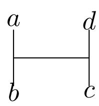
AS IHX

Figure 2: The AS and IHX relations

Then, we introduce two new trace-like operators Tr^{as} and Tr^{sym}, defined respectively on $D_2(H)$ and on $D_2'(H)$. Making use of the computations of Morita [42] and Yokomizo [58], we prove that they induce isomorphisms $D_2(H)/\operatorname{Im}(\tau_2) \simeq \operatorname{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2)$ and $D_2'(H)/\tau_2([\mathcal{I}, \mathcal{I}]) \simeq \operatorname{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$, respectively. In other terms, the maps Tr^{as} and Tr^{sym} allow us to characterize $\operatorname{Im}(\tau_2)$ in $D_2(H)$ and $\tau_2([\mathcal{I}, \mathcal{I}])$ in $D_2'(H)$, respectively. These "traces" are both inspired by Morita's trace [42].

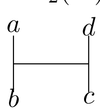
Theorem 1.3. *For any $g \geq 2$, the following homomorphisms*

$$D_2'(H) \xrightarrow{\mathrm{Tr}^{sym}} S^2(H/2H)$$



$$\longmapsto \omega(a, d)bc + \omega(a, c)bd + \omega(b, d)ac + \omega(b, c)ad$$

$$D_2(H) \xrightarrow{\mathrm{Tr}^{as}} \Lambda^2(H/2H)$$



$$\longmapsto \omega(a, d)b \wedge c + \omega(a, c)b \wedge d + \omega(b, d)a \wedge c + \omega(b, c)a \wedge d$$

$$a \odot b \longmapsto (1 + \omega(a, b))a \wedge b$$

are well-defined, $\mathrm{Sp}(H)$-equivariant, and induce the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}/J_3 \arrow[r, "\tau_2"] & D_2(H) \arrow[r, "\mathrm{Tr}^{as}"] & \operatorname{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{[\mathcal{I},\mathcal{I}]}{J_3 \cap [\mathcal{I},\mathcal{I}]} \arrow[r, "\tau_2"] \arrow[u, hook] & D_2'(H) \arrow[r, "\mathrm{Tr}^{sym}"] \arrow[u] & \operatorname{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{1.3}$$

where the up arrow on the right is induced by the canonical projection $S^2(H/2H) \to \Lambda^2(H/2H)$.

To describe $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3} \simeq \tau_2(\mathcal{A}\cap J_2)$ in $\mathrm{Im}(\tau_2)$, we shall use the map Tr^{as} described above. Recall that H' is the first homology group of the handlebody and A is the kernel of the projection from H to H'. Levine [34] first observed that $\tau_2(\mathcal{A}\cap J_2)$ is contained in the kernel of the canonical projection from $D_2(H)$ to $D_2(H')$. The symplectic form ω induces, via restriction and projection, a pairing $\omega' : A \otimes H' \to \mathbb{Z}$. This pairing is non-singular and allows us to define another trace-like operator, denoted Tr^A, that vanishes on $\tau_2(\mathcal{A}\cap J_2)$, but not on the subgroup $\mathrm{Ker}(D_2(H) \to D_2(H'))$ proposed by Levine. The definition of the map Tr^A, which is valid for any degree k, only depends on a Lagrangian subspace A of H (i.e. a maximal isotropic subspace of H for the symplectic form ω). In degree $k = 2$, the map Tr^A happens to be related to the Casson invariant through the notion of surgery. We give details in the next section.

Theorem 1.4. *There exists a non-trivial homomorphism* Tr^A*, defined on* $\mathrm{Ker}(D_k(H) \to D_k(H'))$*, and such that for* $k = 2$ *and* $g \geq 4$*, we have*

$$\mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Im}(\tau_2) = \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^{as}) = \tau_2(\mathcal{A}\cap J_2).$$

Note that the intersections in Theorem 1.4 take place in the space $D_2(H)$.

We shall also give a description of the image by τ_2 of the intersection of the *Goeritz group* with J_2. Consider the standard Heegaard splitting of genus g of the 3-sphere: $S^3 = V_g \cup_{\iota_g} (-V_g)$. The Goeritz group $\mathcal{G}$ of S^3 is the group of isotopy classes of orientation-preserving homeomorphisms of S^3 preserving this Heegaard splitting. One can show that we actually have an identification between $\mathcal{G}$ and the subgroup $\mathcal{A}\cap\mathcal{B}$ of $\mathcal{M}$.

We define a second Lagrangian subgroup B associated to the projection from H to the first homology group of the "outer" handlebody $S^3 - \mathrm{int}(V_g)$. This produces a second trace-like operator Tr^B, and with the help of Theorem 1.4, we shall be able to deduce:

Proposition 1.5. *For* $g \geq 4$*, we have* $\tau_2(\mathcal{G}\cap J_2) = \mathrm{Ker}(\mathrm{Tr}^{as}) \cap \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^B)$.

With the help of the previous computation, we are able to give precise statements about the description of homology 3-spheres by Heegaard splittings with gluing maps lying in the second or the third term of the Johnson filtration of the gluing surface, thus improving Theorem 1.1. In the following proposition, the subscript $\mathcal{G}_{g,1}$ designates the quotient by the conjugation action of the Goeritz group on the Torelli group.

Proposition 1.6. *Denote* $\mathcal{K}_{g,1} := J_2(\Sigma_{g,1})$ *and* $\mathcal{L}_{g,1} := J_3(\Sigma_{g,1})$*. There are well-defined isomorphisms*

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1}\cap\mathcal{K}_{g,1})\backslash\mathcal{K}_{g,1}/(\mathcal{B}_{g,1}\cap\mathcal{K}_{g,1})\big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}(3).$$

and

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1}\cap\mathcal{L}_{g,1})\backslash\mathcal{L}_{g,1}/(\mathcal{B}_{g,1}\cap\mathcal{L}_{g,1})\big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}(3).$$

This kind of results may be useful, for example, when one tries to build invariants using Pitsch's method [49, Theorem 2]. Roughly, if one can define trivial cocycles on $\mathcal{I}_{g,1}$ with good properties, these cocycles will derive from maps on the Torelli group which will reassemble in an invariant of homology 3-spheres (see also [53]).

Finally, in Appendix 1.A of Chapter 1, we show that $\tau_2(\mathcal{G}\cap J_2)\otimes\mathbb{Q}$ admits an action of $\mathrm{GL}(g,\mathbb{Q})$, and we give its decomposition in irreducible modules.

1.2 Equivalence relations among 3-manifolds

We now give more precise statements about surgeries in 3-manifolds. Dehn surgeries and Kirby calculus have been used extensively to connect the study of 3-manifolds and the study of knots and links, and notably to construct invariants of the former from invariants of the latter. Now one can perform surgeries not only along knots, but also along handlebodies (or, equivalently, thickenings of a surfaces with boundary). More precisely, performing a surgery along an embedded surface S with one boundary component in an oriented 3-manifold M means choosing an element s of the mapping class group $\mathcal{M}(S)$ of the surface S, identifying a regular neighborhood of S with $S \times [-1, 1]$, and doing the move

$$M \rightsquigarrow M_s := (M \setminus \operatorname{int}(S \times [-1,1])) \cup_{\tilde{s}} (S \times [-1,1])$$

where $\tilde{s}$ is the map from $\partial(S \times [-1,1])$ to itself defined by $(\mathrm{Id} \times (-1)) \cup (\mathrm{Id} \times \partial S) \cup (s \times 1)$. The move hence consists in removing and regluing a handlebody with a "twist". By this construction, a filtration of the mapping class group or the Torelli group might define equivalence relations among 3-manifolds. There are two such filtrations of the Torelli group of importance: the lower central series of the Torelli group $(\Gamma_k \mathcal{I})_{k \geq 1}$ and the Johnson filtration $(J_k)_{k \geq 1}$, whose definition has been recalled in Section 1.1.

Definition 1.7. *The Y_k-equivalence and J_k-equivalence relations are defined by:*

$$M \overset{Y_k}{\sim} M' \Leftrightarrow \exists S \subset M,\ \exists s \in \Gamma_k \mathcal{I}(S)\ s.t.\ M' = M_s$$

$$M \overset{J_k}{\sim} M' \Leftrightarrow \exists S \subset M, \exists s \in J_k(S)\ s.t.\ M' = M_s.$$

It can be shown quite easily (see e.g. [39]) that these actually define equivalence relations (the only non-trivial part of this affirmation being the transitivity). Also, these equivalence relations are preserved by stabilization of the surface. This means that if the surgery is performed along $s \in \Gamma_k \mathcal{I}(S)$, one can embed S in a surface of greater genus S', and perform the surgery (extending s by the identity on $S' \setminus S$). We will get that M_s is equivalent to M through a surgery of greater genus. We remind here some facts about these relations, but a better survey can be found in [37], where the reader can also learn about the link between these relations and finite-type invariants of 3-manifolds. Note also that if the move defining surgeries consists in removing a handlebody and gluing it back differently, it can not be considered as a "Heegaard splitting twisting" as described earlier. Indeed, the neighborhood of a surface with boundary *is* a handlebody, but its complement might not be. In the case of Y-equivalences and J-equivalences, supposing that the complement is a handlebody or not does not make any difference (see e.g. [39, Lemma 2.1]).

The Y_k-equivalence relation can be studied using the surgery techniques of "clasper calculus" introduced by Goussarov [11, 13] and Habiro [15]. For the case of $\mathcal{V}(3)$, the Y_1 [41] and Y_2-equivalences [35] between closed 3-manifolds are classified by well-known invariants. The Y_2 and the J_2-equivalence [38], as well as the Y_3-equivalence and the J_3-equivalence [39], have been characterized among homology cylinders.

It is also known, thanks to results of Morita [43] and Pitsch [49], that two homology 3-spheres are always at least J_3-equivalent. Hence the J_1, the J_2 and the J_3-equivalence are trivial on $\mathcal{S}(3)$. It was not known whether it was the case or not for the J_4-equivalence, and it could hardly have been shown by brute force calculation as it was done in [43] and [49]. However, an alternative proof of the fact that the J_3-equivalence is trivial on $\mathcal{S}(3)$ can be found in [39], where a way to address the question for J_4 is proposed [39, Rem. 6.4]. This approach relates, through a clasper calculus argument, the question of the triviality of the J_4-equivalence to a question about the Casson invariant.

The Casson invariant $\lambda : \mathcal{S}(3) \to \mathbb{Z}$ is an invariant of oriented homology 3-spheres. It was originally defined by counting, with some signs, the number of irreducible representations of the fundamental group of the homology 3-sphere into $SU(2)$. It is a lift of the Rokhlin invariant $\mu : \mathcal{S}(3) \to \mathbb{Z}_2$. We recall here that the Rokhlin invariant of a homology 3-sphere M is the signature, divided by 8 and modulo 2, of any spin 4-manifold with boundary M.

A *Heegaard embedding* $j : \Sigma_{g,1} \to S^3$ is an embedding such that gluing a disk along the boundary of the image of j yields a Heegaard splitting of S^3. For such an embedding j, and any $\varphi \in \mathcal{M}$, one can define a manifold $S^3_{j,\varphi}$, by removing an open neighborhood of the surface and regluing it after composing with the map φ. Any invariant F of 3-manifolds valued in some abelian group C then induces for any j a map F_j on $\mathcal{M}$:

$$\begin{aligned} F_j : \mathcal{M} &\longrightarrow C \\ \varphi &\longmapsto F(S^3_{j,\varphi}) - F(S^3). \end{aligned}$$

In the case of the Casson invariant, we want to stay inside the space of homology 3-spheres $\mathcal{S}(3)$. A Mayer-Vietoris argument implies that if we pick φ in $\mathcal{I}$, then $S^3_{j,\varphi}$ is a homology 3-sphere and it makes sense to define:

$$\begin{aligned} \lambda_j : \mathcal{I} &\longrightarrow \mathbb{Z} \\ \varphi &\longmapsto \lambda(S^3_{j,\varphi}) \end{aligned}$$

The maps λ_j are not homomorphisms. Nevertheless, Morita [42] proved that their restrictions to $\mathcal{K} = J_2$ *are.* Furthermore he proved that this restriction of λ_j has the following decomposition:

$$-\lambda_j = \frac{1}{24} d + q_j : \mathcal{K} \to \mathbb{Z}.$$

The homomorphism d is called the *core of the Casson invariant.* Notice that it is independent of the Heegaard embedding j. By definition, the homomorphism q_j factorizes through the second Johnson homomorphism, which implies that for any j, it vanishes on J_3. Consequently, the Casson invariant induces a well-defined homomorphism λ defined on J_k for any $k \geq 3$. Concretely, this points out that the value of the Casson invariant on $S^3_{j,\varphi}$ is independent of j when $\varphi \in J_k$ and $k \geq 3$. The core of the Casson invariant is not completely understood, but it is known that Dehn twists along bounding simple closed curves (abbreviated BSCC in the sequel) of genus 1 and 2 generate $\mathcal{K}$ [26] and that the value of d on a Dehn twist along a BSCC of genus h is $4h(h-1)$ [42].

Morita claimed in [42] that $\lambda(J_3) = \mathbb{Z}$ in genus $g \geq 2$, and Massuyeau and Meilhan [39] gave the explicit computation. Using Habiro's clasper calculus [15] (which allows one to show that Y_3-equivalence among homology 3-spheres is classified by λ), Massuyeau and Meilhan [39, Theorem C] reproved that the J_3-equivalence is trivial on $\mathcal{S}(3)$. This method generalizes to the case of the J_4-equivalence, as we shall prove.

Theorem 1.8. *For any genus $g \geq 2$, the restriction of $\lambda : J_3 \to \mathbb{Z}$ to J_4 is surjective.*

Theorem 1.9. *The J_4-equivalence is trivial on $\mathcal{S}(3)$.*

Theorem 1.8 is proven by constructing explicitly an element of J_4, in genus $g = 2$, whose Casson invariant is equal to 1 (thus exhibiting a homology 3-sphere in the J_4-equivalence class of S^3 with Casson invariant equal to 1). The computations involved in the construction use the so-called *infinitesimal Johnson homomorphisms*, and more specifically a formula by Kawazumi and Kuno in [31]. A more general point of view for this formula is given in [40]. We also use a SageMath computer program to carry on these computations, and give the code in Appendix 2.A. Theorem 1.8 is also interesting for the study of the mapping class group in itself. Indeed, Hain [17] proved that $\lambda(J_k) \neq \{0\}$ for $k \geq 3$. Knowing if the restriction of λ to J_k remains surjective for $k \geq 5$ would be of great interest concerning the study of J_k-equivalence in general. One could start with the strictly simpler question about the Rokhlin invariant:

Question 1.10. *Is there some $k \geq 5$ such that $\mu(J_k) = 0$?*

Nevertheless, it is likely that knowing the values of λ on J_5 will not be enough to study the J_5-equivalence. Indeed, the method of Massuyeau and Meilhan uses the classification of the Y_k-equivalence by Habiro [15], and other finite-type invariants of higher degree are involved in the case $k = 5$.

We now discuss another method for tackling the question of the J_k-equivalence, which needs to be investigated further. All the proofs will be given in Appendix A. We first define another equivalence relation, whose definition is suggested by the work of Levine in [34]. Levine defines a Lagrangian filtration $(L_k)_{k\geq 1}$ which is a non-separating filtration of the mapping class group. It is thus *not* helpful to get an approximation of the mapping class group of the surface, but it is fitted to the study of 3-manifolds presented by Heegaard splittings. The definition of Levine depends on the choice of the inclusion $\Sigma \subset V$ in some handlebody V such that $\partial V \setminus \Sigma$ is a disk. When needed, this will be specified in the notation (e.g. $L_k(V)$ for a handlebody V). In particular, it depends on the Lagrangian subspace $A \subset H$, the kernel of the projection $H = H_1(\Sigma_{g,1}) \to H' = H_1(V_g)$. We denote p the projection from $\pi = \pi_1(\Sigma_{g,1})$ to $\pi' = \pi_1(V_g)$, and $\mathbb{A}$ the kernel of p. Of course A is the image of $\mathbb{A}$ in H via the abelianization map. We denote $D_k(H')$ the set of positive symplectic derivations of degree k of $\mathcal{L}(H')$. Also, whenever f is an element of the mapping class group, $f_* \in \mathrm{Sp}(H)$ stands for the action of f on H. We still write abusively f for the action of f on the fundamental group.

Definition 1.11. *The Lagrangian Torelli group is defined by:*

$$\mathcal{I}^L := \{h \in \mathcal{M} : h_*(A) \subset A \text{ and } h_* \text{ is the identity on } A\}.$$

Definition 1.12. *For $k \geq 1$, the group $L_k = L_k(V)$ is defined by:*

$$L_k := \Big\{h \in \mathcal{I}^L \;\Big|\; p\big(h(\mathbb{A})\big) \subset \Gamma_{k+1}\pi'\Big\}.$$

Levine proved that it is indeed a subgroup of $\mathcal{M}$ [32].

We now define the *L_k-equivalence*, and shall prove in Appendix A that it is indeed an equivalence relation.

Definition 1.13. *Two oriented manifolds M and M' are said to be L_k-equivalent if M' can be obtained from M by removing a handlebody V and regluing it by twisting with an element of $L_k(V)$ (extended by the identity on the capping disk).*

It is clear that $J_k \subset L_k$, as subgroups of $\mathcal{M}$, for all $k \geq 1$. Thus, the L_k-equivalence is weaker than the J_k-equivalence. As the latter is, by Theorem 1.9, trivial up to degree 4 among homology 3-spheres, it is also true for the L_k-equivalence. We can naturally ask the following, at least for low degrees.

Question 1.14. *For general 3-manifolds, and $k \geq 1$, does the J_k-equivalence and the L_k-equivalence coincide ?*

This question is driven by the fact that the answer is true for $k = 1, 2$.

Proposition 1.15. *For $k = 1, 2$, any two manifolds that are L_k-equivalent are J_k-equivalent.*

A result of Levine [34] allows us to give a simple proof of Proposition 1.15, which uses Johnson homomorphisms. Pointing out that

$$J_k \cdot L_\infty \subset L_k, \text{ with } L_\infty := \bigcap_{k\geq 1} L_k,$$

Levine asked the following question:

Question 1.16. *Do we have $L_k = J_k \cdot L_\infty$ for all k ?*

This is motivated, in our context, by the next lemma.

Lemma 1.17. *If $L_k = J_k \cdot L_\infty$, then L_k-equivalence is the same as J_k-equivalence.*

Levine also showed the following lemma (see [34, Lemma 6.2] for a detailed proof).

Lemma 1.18. *Suppose $L_k = J_k \cdot L_\infty$, then $L_{k+1} = J_{k+1} \cdot L_\infty$ if and only if $\mathrm{Im}(\tau_k) \cap \mathrm{Ker}(D_k(H) \to D_k(H')) = \tau_k(\mathcal{A} \cap J_k)$.*

It is easy to show that $L_1 = J_1 \cdot \mathcal{A}$ (see, for example, [34, Lemma 6.3]). Computations by Morita in [42] then shows that $\mathrm{Ker}(D_1(H) \to D_1(H')) = \tau_1(\mathcal{A} \cap J_1)$ (remind that τ_1 is surjective). Hence, for $k = 1, 2$, we get a positive answer to Levine's question, and to Question 1.14. As for the $k = 3$ case, the equality necessary for the induction step is no longer true, by Theorem 1.4:

$$\tau_2(\mathcal{A} \cap J_2) \subsetneq \mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H')).$$

This answers negatively to Question 1.16, but not necessarily to Question 1.14. The difference between these two submodules of $D_2(H)$ gives us good candidates to perform some L_3-surgeries on some manifolds, in order to give two L_3-equivalent manifolds that are not J_3-equivalent. This is what we do in Appendix A, using the classification of the J_3-equivalence from [39].

Proposition 1.19. *Among all closed oriented 3-manifolds, L_3-equivalence is strictly weaker than J_3-equivalence.*

Notice, though, that L_k-equivalence and J_k-equivalence could be the same for all $k \geq 1$ for homology 3-spheres. This may help studying J_k-equivalence among homology 3-spheres. We hence formulate the two following questions.

Question 1.20. *Up to which $k \geq 5$ is the L_k-equivalence relation trivial on $\mathcal{S}(3)$?*

Question 1.21. *Up to which $k \geq 5$ is the J_k-equivalence relation trivial on $\mathcal{S}(3)$?*

1.3 Contents and organization of the dissertation

This PhD dissertation consists of two articles and an appendix which can be read independently. In each of the articles, the reader can find a more detailed introduction. In the first article [8], *The handlebody group and the images of the second Johnson homomorphism*, reproduced in Chapter 1, the reader will find the proofs of Theorems 1.3, 1.4, and Propositions 1.5 and 1.6. The second article [9], *Triviality of the J_4-equivalence among homology 3-spheres*, is reproduced in Chapter 2 and contains the proofs of Theorems 1.8 and 1.9. Proofs of Propositions 1.15 and 1.19 can be found in Appendix A.

2 Introduction (en Français)

Le terme « topologie de basse dimension » fait référence à l'étude des variétés de dimensions inférieures à 4, à « déformation » près. Ici, on entend par variété un espace topologique localement homéomorphe à un espace euclidien. La classification des variétés compactes orientées de dimension 1 est un simple exercice, tandis que celle des variétés compactes orientées de dimension 2, c'est à dire des surfaces, est plus complexe : le type d'homéomorphisme d'une surface compacte connexe orientée est donnée par son genre et le nombre de ses composantes de bord. En revanche, la situation en dimension 3 est très riche et moins bien comprise. La topologie de basse dimension inclut l'étude des homéomorphismes entre objets de basses dimensions, mais aussi des plongements de variétés dans des variétés de plus grandes dimensions (nœuds, tresses...). Tous ces sujets sont liés de façon complexe, et une vaste littérature les concernant est apparue au cours du siècle dernier. De nombreuses questions géométriques, de théorie des groupes, ou même analytiques sont aussi reliées à la topologie de basse dimension. Un exemple frappant de ce fait a été la preuve de la conjecture de Poincaré par Perelman en 2003 : la question de savoir si une variété orientée de dimension 3 compacte, fermée et simplement connexe est homéomorphe à une sphère s'est révélée extrêmement difficile.

L'un des buts modernes de la topologie de basse dimension est d'avoir une meilleure compréhension de la classification des 3-variétés compactes orientées à homéomorphisme près. Une 3-variété peut être décrite en utilisant les nœuds, à travers les notions de chirurgie de Dehn, mais aussi en utilisant le groupe d'homéotopie des surfaces, à travers les notions de scindements de Heegaard et de chirurgies sur des corps en anses, comme nous allons le détailler. Ainsi, ce sujet est en particulier lié à la théorie des nœuds et à l'étude du groupe d'homéotopie des surfaces. Dans cette thèse, nous nous intéressons principalement à ce groupe. Le groupe d'homéotopie d'une surface est le groupe des classes d'isotopies des homéomorphismes de la surface qui fixent les composantes de bord (les isotopies fixant également le bord). Le groupe d'homéotopie d'une surface de genre g avec une composante de bord sera noté $\mathcal{M}_{g,1}$.

Quelle que soit la façon de présenter deux variétés données, il est en général délicat de déterminer si elles sont homéomorphes. Une façon de montrer qu'elles ne le sont *pas*, cependant, est de produite des *invariants* dont les valeurs diffèrent sur les deux variétés. Par un invariant, nous entendons une application définie sur un ensemble de variétés dont la valeur sur une variété ne dépend que de son type d'homéomorphisme. Depuis la découverte dans les années 80 du polynôme de Jones pour les nœuds, de nombreux invariants de nœuds et de 3-variétés ont été produits, leur construction impliquant souvent des idées issues de la physique mathématique. Nous faisons ici référence aux *invariants quantiques*, i.e. des invariants dérivés de certaines représentations de groupes quantiques, et aux *invariants de type fini*, i.e. des invariants ayant un « comportement polynomial » par rapport à certaines opérations chirurgicales. Ces 2 types d'invariants sont liés : les invariants du premier type peuvent, dans certaines circonstances, produire des invariants du second type par des procédés de développement en séries. Beaucoup d'efforts ont été investis dans l'étude de ces invariants, et ils ont pu être organisés, en un certain sens, via l'intégrale de Konsevitch (pour les nœuds), et l'invariant LMO (pour les 3-variétés).

Malheureusement, ces invariants ont souvent des constructions complexes et indirectes, et un problème fréquemment rencontré en topologie de basse dimension est de leur donner un sens plus topologique. Pour atteindre ce but, il est utile de chercher des raffinements ou des extensions de certains invariants, mais aussi des *formules de chirurgie*, c'est à dire des formules décrivant la variation d'un invariant quand on modifie une variété donnée. On ne cherche pas toujours à travailler avec l'ensemble des 3-variétés à homéomorphismes près. On étudie parfois des relations d'équivalences plus faibles, ou on réduit le problème à des variétés particulières : on se restreindra souvent aux *3-sphères d'homologie entière*. Une 3-sphère d'homologie entière est une 3-variété orientée avec les mêmes groupes d'homologie que la sphère S^3. On note respectivement $\mathcal{V}(3)$ et $\mathcal{S}(3)$ l'ensemble des 3-variétés compactes fermées et orientées et l'ensemble des 3-sphères d'homologie entière, à homéomorphisme près.

Nous fixons maintenant quelques notations, pour pouvoir exprimer clairement nos résultats. Nous considérons un corps en anses V_g de genre g (i.e. une boule avec g anses) dont le bord est une surface Σ_g de genre g sans bord. Le corps en anses est dessiné sur la figure Figure 3, où les courbes rouges bordent dans V_g mais pas dans Σ_g. La surface Σ_g privée d'un disque (suggéré par des rayures grises sur la Figure 3) est la surface avec une composante de bord $\Sigma_{g,1}$. La plupart du temps cette surface sera simplement considérée comme une surface abstraite et non pas comme le bord d'un corps en anses. Nous ne spécifierons pas le genre et le nombre de composantes de bord en indice lorsque que le contexte est suffisamment clair. Posons $\pi := \pi_1(\Sigma_{g,1}, x_0)$ où x_0 est un point sur le bord de $\Sigma_{g,1}$, et $H := H_1(\Sigma_{g,1})$ l'abélianisé de π. On note $\pi' := \pi_1(V_g, x_0)$ le groupe fondamental du corps en anses de genre g, et $\mathbb{A}$ le noyau de la projection $\pi \to \pi'$ induite par l'inclusion de $\Sigma_{g,1}$ dans V_g. L'abélianisé de π', correspondant au premier groupe d'homologie du corps en anses, est noté H' et A est le noyau de la projection de H vers H' induite par l'inclusion de $\Sigma_{g,1}$ dans V_g.

Le *groupe d'homéotopie du corps en anses* $\mathcal{A}_{g,1}$ est le sous-groupe du groupe d'homéotopie de la surface constitué des éléments s'étendant au corps en anses. Il coïncide avec le sous-groupe des éléments de $\mathcal{M}$ qui préservent $\mathbb{A}$ [12]. Le groupe d'homéotopie de la surface agit sur H, en préservant la forme symplectique ω induite par la forme d'intersection de la surface. Cela définit une action $\mathcal{M} \to \mathrm{Sp}(H)$, où $\mathrm{Sp}(H)$ est le sous-groupe de $\mathrm{Aut}(H)$

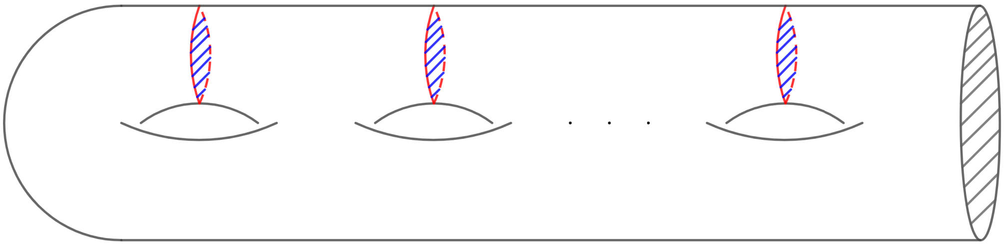


Figure 3: Le corps en anses V_g de bord Σ_g

des éléments laissant invariant la forme symplectique. Le noyau de cette action, i.e. le sous groupe des éléments de $\mathcal{M}_{g,1}$ agissant trivialement en homologie, est le *groupe de Torelli* $\mathcal{I}_{g,1}$.

Un *scindement de Heegaard* de genre g d'une 3-variété M est l'inclusion d'un corps en anses V_g dans M telle que le complémentaire de l'intérieur de V_g est aussi un corps en anses (de genre g). Toute 3-variété fermée compacte orientée admet un scindement de Heegaard, d'un certain genre suffisamment grand, et le type d'homéomorphisme de la variété obtenue en recollant deux corps en anses ne dépend que de la classe d'isotopie de l'application de recollement (voir Théorème 2.1 ci-dessous). Ainsi, l'étude des 3-variétés est fortement liée à la compréhension de $\mathcal{M}$. De plus, si l'on considère un scindement de Heegaard, on peut le « tordre » (i.e. composer l'application de recollement) par un élément de $\mathcal{M}$. C'est ce que l'on appellera ici une *chirurgie*. Notons que cette notion généralise d'une certaine façon la notion de *chirurgie de Dehn* qui consiste à retirer le voisinage d'un nœud dans une 3-variété et à le recoller différemment. Le but de cette thèse de doctorat est précisément d'étudier certains types de chirurgies, en combinant l'utilisation de filtrations sur $\mathcal{M}$, et des invariants topologiques de 3-variétés.

Précisons encore les notions ci-dessus. Il y a, à isotopie près, et pour tout genre g, un unique scindement de Heegaard de S^3 [57]. Fixons un homéomorphisme orienté ι_g de Σ_g tel que $S^3 = V_g \cup_{\iota_g} (-V_g)$. Ici $(-V_g)$ est le corps en anses V_g avec l'orientation opposée. On pose $\mathcal{B}_{g,1} := \iota_g \mathcal{A}_{g,1} \iota_g^{-1}$ et on note S^3_φ la 3-variété $V_g \underset{\iota\circ\varphi}{\cup} (-V_g)$ pour tout élément $\varphi \in \mathcal{M}_{g,1}$ (étendu à Σ_g par l'identité sur le disque extérieur). Cette procédure définit, pour tout $g \geq 1$, une application de $\mathcal{M}_{g,1}$ vers $\mathcal{V}(3)$. Il existe aussi des applications de stabilisation $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, compatibles avec les autres, au sens où le diagramme suivant est commutatif :

$$\begin{tikzcd} \mathcal{M}_{g,1} \arrow[r] \arrow[d, hook] & \mathcal{V}(3) \\ \mathcal{M}_{g+1,1} \arrow[ur] & \end{tikzcd} .$$

Quand on compose l'application de recollement à droite par un élément de $\mathcal{B}_{g,1}$ ou à gauche par un élément de $\mathcal{A}_{g,1}$, la variété obtenue ne change pas à homéomorphisme près. En effet, comme les éléments de $\mathcal{A}_{g,1}$ s'étendent à V_g, les éléments de $\mathcal{B}_{g,1}$ peuvent être identifiés aux éléments de $\mathcal{M}$ s'étendant au complémentaire de l'intérieur de V_g dans S^3. Cela nous permet d'exprimer un raffinement du fameux théorème de Reidemeister-Singer [52, 54], où la seconde affirmation est une conséquence de [41, Corollaire du Théorème 2].

Théorème 2.1 (Reidemeister-Singer)**.** *Il existe une bijection*

$$\begin{aligned} \lim_{g\to+\infty} \mathcal{A}_{g,1} \setminus \mathcal{M}_{g,1}/\mathcal{B}_{g,1} &\longrightarrow \mathcal{V}(3) \\ \varphi &\longmapsto S^3_\varphi \end{aligned}$$

qui se restreint à une bijection $\lim_{g\to+\infty} \mathcal{A}_{g,1} \setminus \mathcal{I}_{g,1}/\mathcal{B}_{g,1} \longrightarrow \mathcal{S}(3)$.

Remarquons que, entre autres, la deuxième partie du Théorème 2.1 affirme que lorsque l'on tord un scindement de Heegaard de S^3 par un élément de $\mathcal{I}$, on obtient une sphère d'homologie entière, et que toute sphère d'homologie entière peut être obtenue ainsi. Précisons aussi que la notation $\mathcal{A}_{g,1} \setminus \mathcal{I}_{g,1}/\mathcal{B}_{g,1}$ désigne l'image du groupe de Torelli dans l'espace quotient double $\mathcal{A}_{g,1} \setminus \mathcal{M}_{g,1}/\mathcal{B}_{g,1}$. Maintenant, grossièrement, si l'on dispose d'une formule de chirurgie pour un invariant, i.e. si l'on sait comment l'invariant varie au cours d'une chirurgie, alors on peut obtenir des informations utiles sur $\mathcal{M}$ en définissant une application sur le groupe d'homéotopie qui mesure la variation de l'invariant lorsque l'on tord un scindement de Heegaard d'une variété fixée (par exemple S^3). C'est ce que nous allons faire avec l'invariant de Casson, en utilisant une formule de chirurgie due à Morita [42]. Cette formule est un thème récurrent de cette dissertation. Nous donnons plus de détails dans la Section 2.2.

2.1 La filtration de Johnson et les homomorphismes de Johnson

Mettons momentanément de côté les variétés de dimension 3 pour nous concentrer sur le groupe d'homéotopie $\mathcal{M}_{g,1}$. Par le théorème de Dehn-Nielsen, un élément de $\mathcal{M}$ est entièrement déterminé par son action sur π. En d'autres termes $\mathcal{M}$ s'injecte dans $\mathrm{Aut}(\pi)$, le groupe des automorphismes du groupe libre π. Ceci réduit l'étude de l'objet topologique $\mathcal{M}$ à une question purement algébrique, mais le groupe $\mathrm{Aut}(\pi)$ est, cependant, assez complexe. Ainsi une méthode prometteuse (voir par exemple [1, 28, 26, 42, 44]) pour étudier $\mathcal{M}$ de ce point de vue est de considérer des versions simplifiées de cette action, nommément l'action de $\mathcal{M}$ sur les quotients nilpotents de π :

$$\rho_k : \mathcal{M} \to \mathrm{Aut}(N_k)$$

où $N_k := \pi/\Gamma_{k+1}\pi$ pour $k \geq 1$ est le *k-ème quotient nilpotent de* π, et $\Gamma_k\pi$ est le k-ème terme de la suite centrale descendante de π. Ici, la *suite centrale descendante* d'un groupe G est définie inductivement par $\Gamma_1 G := G$ et $\Gamma_{k+1}G := [\Gamma_k G, G]$.

On a ensuite une suite exacte courte bien connue

$$0 \longrightarrow \mathcal{L}_{k+1}(H) \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow 0$$

où $\mathcal{L}(H)$ est l'algèbre de Lie libre graduée engendrée par H en degré 1, et la première flèche non-triviale est donnée par l'identification entre $\mathcal{L}_{k+1}(H)$ et $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$. Cette suite exacte, à son tour, induit la suite exacte courte

$$0 \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)) \longrightarrow \mathrm{Aut}(N_{k+1}) \longrightarrow \mathrm{Aut}(N_k). \tag{2.1}$$

Le sous-groupe J_k de $\mathcal{M}$ est alors défini comme le noyau de l'homomorphisme ρ_k. En particulier, par le théorème de Hurewicz, le groupe J_1 est le groupe de Torelli, que nous avons noté $\mathcal{I}$, et qui prend place dans la suite exacte suivante :

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{M} \longrightarrow \mathrm{Sp}(H) \longrightarrow 0. \tag{2.2}$$

Nous utilisons aussi la notation $\mathcal{K}$ pour J_2, qui est aussi appelé *le noyau de Johnson.* On connaît peu de choses à propos des sous-groupes J_k, même s'il est connu que le sous-groupe $\Gamma_k\mathcal{I}$ des k-commutateurs du groupe de Torelli est inclus dans J_k. Des générateurs de $\mathcal{I}$ [51, 3, 28, 24] et de $\mathcal{K}$ [26] sont connus. L'abélianisé de $\mathcal{I}$ a aussi été déterminé dans [27]. Plus récemment, des progrès ont été faits dans la détermination de l'abélianisé rationnel de $\mathcal{K}$ [7, 46], dans le cas légèrement différent d'une surface fermée. La *filtration de Johnson* $(J_k)_{k\geq 1}$ est une filtration séparante du groupe de Torelli : $\bigcap_{k\geq 1} J_k = \{\mathrm{Id}\}$. Cela motive l'étude de l'espace gradué associé $\mathrm{Gr}^J(\mathcal{I}) := \bigoplus_{k\geq 1} J_k/J_{k+1}$. Par la suite exacte (2.1), la restriction de ρ_{k+1} à J_k induit un morphisme :

$$\tau_k : J_k \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$$

que l'on appellera le k-ème *homomorphisme de Johnson*, et dont le noyau est, par définition, J_{k+1}. Le groupe d'homéotopie agit sur lui-même par conjugaison, ce qui induit via la suite exacte (2.2) une action du groupe symplectique $\mathrm{Sp}(H)$ sur le quotient J_k/J_{k+1}. Chaque τ_k

est alors $\mathrm{Sp}(H)$-équivariant par rapport à cette action. De plus, l'espace gradué associé à la filtration de Johnson a une structure de Lie, son crochet étant induit par le commutateur dans $\mathcal{M}$. Toute dérivation est déterminée par ses valeurs sur des générateurs, et $\mathcal{L}(H)$ est engendré par H. Ainsi, l'espace d'arrivée de τ_k peut être identifié avec l'espace des *dérivations de degré* k, i.e. les dérivations de l'algèbre de Lie $\mathcal{L}(H)$ envoyant $H = \mathcal{L}_1(H)$ dans $\mathcal{L}_{k+1}(H)$. On note $D_k(H)$ l'espace des *dérivations symplectiques* de degré k, constitué des dérivations de $\mathcal{L}(H)$ de degré k s'annulant sur $\tilde{\omega} \in \Lambda^2 H \simeq \mathcal{L}_2(H)$, le bivecteur dual de la forme symplectique ω. En utilisant le fait qu'un élément de $\mathcal{M}$ fixe le bord de Σ, on peut restreindre l'espace d'arrivée de τ_k à $D_k(H)$. De plus, $D_k(H)$ peut être inséré dans une suite exacte :

$$0 \longrightarrow D_k(H) \longrightarrow H \otimes \mathcal{L}_{k+1}(H) \longrightarrow \mathcal{L}_{k+2}(H) \longrightarrow 0$$

où la flèche de $H \otimes \mathcal{L}_{k+1}(H)$ vers $\mathcal{L}_{k+2}(H)$ est le crochet de l'algèbre de Lie libre.

Les espaces $(D_k(H))_{k\geq 1}$ s'assemblent en une algèbre de Lie graduée $D(H)$ (le crochet de deux dérivations d_1 et d_2 étant défini de façon standard comme la différence de compositions $d_1d_2 - d_2d_1$). La famille $(\tau_k)_{k\geq 1}$, à son tour, s'assemble en une application :

$$\tau : \bigoplus_{k\geq 1} J_k/J_{k+1} \longrightarrow D(H)$$

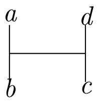
qui est un morphisme $\mathrm{Sp}(H)$-équivariant d'algèbre de Lie graduée.

L'application τ est injective par définition, ce qui implique que la caractérisation de son image dans $D(H)$ est équivalente à la détermination de $\mathrm{Gr}^J(\mathcal{I})$. Malheureusement, nous ne connaissons précisément cette image qu'en degré 1 [21] et 2 [42, 58]. Lorsque l'on tensorise avec le corps des nombres rationnels, $\mathrm{Im}(\tau_k) \otimes \mathbb{Q}$ devient un $\mathrm{Sp}(H_{\mathbb{Q}})$-module [2], où $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$. La théorie (rationnelle) des représentations du groupe symplectique nous permet alors de calculer sa décomposition en facteurs irréductibles en bas degré (voir e.g. [45]). Dans cette situation, un résultat de Hain [18] implique que l'espace $\mathrm{Gr}^J(\mathcal{I}) \otimes \mathbb{Q}$ est engendré en degré 1. De plus τ_1 est surjective sur $D_1(H)$. Reste la question, hautement non-triviale, de la détermination de l'algèbre de Lie engendrée par $D_1(H_{\mathbb{Q}})$ dans $D(H_{\mathbb{Q}})$.

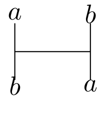
Dans le Chapitre 1, nous nous intéressons à la version « corps en anses » de la filtration de Johnson, i.e. à la suite $(\mathcal{A}\cap J_k)_{k\geq 1}$ de sous-groupes de $\mathcal{A}$. L'intersection de la filtration de Johnson avec le groupe d'homéotopie du corps en anses est aussi séparante, ainsi comprendre le gradué associé $\bigoplus_{k\geq 1} \frac{\mathcal{A}\cap J_k}{\mathcal{A}\cap J_{k+1}}$ est pertinent pour l'étude de l'inclusion $\mathcal{A} \subset \mathcal{M}$. Nous verrons aussi que c'est un objet naturel quand on étudie les 3-variétés via les scindements de Heegaard. Omori [48] a donné des générateurs de $\mathcal{A} \cap \mathcal{I}$, confirmant le calcul de $\frac{\mathcal{A}\cap J_1}{\mathcal{A}\cap J_2}$ fait par Morita [42]. Nous nous attaquons à l'étape suivante, en déterminant explicitement $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3}$.

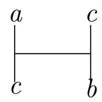
D'abord, en utilisant les travaux de Levine [33], nous rappelons une description de $D_2(H)$ en termes d'un certain module d'arbres dont les feuilles sont coloriées par des éléments de H. Ce module est constitué d'arbres dits *entiers*, qui engendre un sous-espace $D_2'(H)$ de $D_2(H)$, et de moitiés d'arbres dits *symétriques*, noté $a \odot b$ pour $a, b \in H$. Une dérivation symplectique est uniquement associée à une combinaison linéaire de tels arbres.

Proposition 2.2. *$D_2(H)$ admet la présentation suivante : il est engendré par des arbres*



pour a, b, c et d dans H et des éléments $a \odot b$ pour $a, b \in H$, sujets aux relations suivantes :

- *AS, IHX, et multilinéarité des étiquettes pour les arbres (voir Figure 4).*
- *$a \odot a = 0$ et $a \odot b = b \odot a$ pour tout $(a, b) \in H \times H$.*
- *$2(a \odot b) =$*  *.*

- $(a+b)\odot c = a\odot c + b\odot c +$ 

.

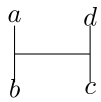
AS IHX

Figure 4: Les relations AS et IHX

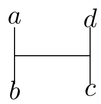
Ensuite, nous introduisons deux nouveaux opérateurs de type « trace » Tr^{as} et Tr^{sym}, définis respectivement sur $D_2(H)$ et sur $D_2'(H)$, et en utilisant les calculs de Morita [42] et Yokomizo [58], nous montrons qu'ils induisent des isomorphismes $D_2(H)/\operatorname{Im}(\tau_2) \simeq \operatorname{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2)$ et $D_2'(H)/\tau_2([\mathcal{I},\mathcal{I}]) \simeq \operatorname{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$, respectivement. En d'autres termes, les applications Tr^{as} et Tr^{sym} permettent de caractériser $\operatorname{Im}(\tau_2)$ dans $D_2(H)$ et $\tau_2([\mathcal{I},\mathcal{I}])$ dans $D_2'(H)$, respectivement. Ces « traces » sont toutes les deux inspirées par la trace de Morita [42].

Théorème 2.3. *Pour tout $g \geq 2$, les homomorphismes suivants*

$$\left\{\begin{array}{ccl} D_2'(H) & \xrightarrow{\mathrm{Tr}^{sym}} & S^2(H/2H) \\ \text{[diagramme]} & \longmapsto & \omega(a,d)bc + \omega(a,c)bd + \omega(b,d)ac + \omega(b,c)ad \end{array}\right.$$



$$\left\{\begin{array}{ccl} D_2(H) & \xrightarrow{\mathrm{Tr}^{as}} & \Lambda^2(H/2H) \\ \text{[diagramme]} & \longmapsto & \omega(a,d)b\wedge c + \omega(a,c)b\wedge d + \omega(b,d)a\wedge c + \omega(b,c)a\wedge d \\ a\odot b & \longmapsto & (1+\omega(a,b))a\wedge b \end{array}\right.$$



sont bien définis, $\mathrm{Sp}(H)$*-équivariants, et induisent le diagramme commutatif de suites exactes suivant :*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}/J_3 \arrow[r, "\tau_2"] & D_2(H) \arrow[r, "\mathrm{Tr}^{as}"] & \operatorname{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{[\mathcal{I},\mathcal{I}]}{J_3\cap[\mathcal{I},\mathcal{I}]} \arrow[r, "\tau_2"] \arrow[u, hook] & D_2'(H) \arrow[r, "\mathrm{Tr}^{sym}"] \arrow[u] & \operatorname{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{2.3}$$

où la flèche de bas en haut à droite est induite par la projection canonique $S^2(H/2H) \to \Lambda^2(H/2H)$.

Pour décrire $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3} \simeq \tau_2(\mathcal{A}\cap J_2)$ dans $\operatorname{Im}(\tau_2)$, nous utilisons l'application Tr^{as} décrite ci-dessus. Rappelons que H' est le premier groupe d'homologie du corps en anses et que A est le noyau de la projection de H vers H'. Levine [34] a d'abord observé que $\tau_2(\mathcal{A}\cap J_2)$ est contenu dans le noyau de la projection canonique de $D_2(H)$ vers $D_2(H')$. La forme symplectique ω induit, via restriction et projection, un appariement $\omega' : A\otimes H' \to \mathbb{Z}$. Cet appariement est non-dégénéré et nous permet de définir un autre opérateur de type « trace » noté Tr^A, qui s'annule sur $\tau_2(\mathcal{A}\cap J_2)$, mais pas sur le sous-groupe $\operatorname{Ker}(D_2(H) \to D_2(H'))$ proposé par Levine. La définition de l'application Tr^A, qui vaut en tout degré k, ne dépend que d'un sous-espace Lagrangien A de H (i.e. un sous-espace isotrope maximal de H pour la forme symplectique ω). Pour $k=2$, l'application Tr^A est en fait lié à l'invariant de Casson à travers la notion de chirurgie. Nous donnons plus de détails à ce sujet dans la section suivante.

Théorème 2.4. *Il existe un homomorphisme non-trivial* Tr^A*, défini sur* $\mathrm{Ker}(D_k(H) \to D_k(H'))$*, et tel que pour* $k = 2$ *et pour* $g \geq 4$*, nous ayons*

$$\mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Im}(\tau_2) = \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^{as}) = \tau_2(\mathcal{A} \cap J_2).$$

Remarquons que les intersections dans le Théorème 2.4 ont lieu dans l'espace $D_2(H)$.

Nous donnons aussi une description de l'image par τ_2 de l'intersection du *groupe de Goeritz* avec J_2. Considérons le scindement de Heegaard standard de genre g de la 3-sphère: $S^3 = V_g \cup_{\iota_g} (-V_g)$. Le groupe de Goeritz $\mathcal{G}$ de S^3 est le groupe des classes d'isotopies d'homéomorphismes orientés de S^3 préservant ce scindement de Heegaard. On peut montrer qu'il y a en fait une identification entre $\mathcal{G}$ et le sous-groupe $\mathcal{A} \cap \mathcal{B}$ de $\mathcal{M}$.

On définit un second Lagrangien B associé à la projection de H vers le premier groupe d'homologie du corps en anses « extérieur » $S^3 - \mathrm{int}(V_g)$. Cela nous donne un second opérateur de type « trace » Tr^B, et avec l'aide du Théorème 2.4, on peut alors déduire :

Proposition 2.5. *Pour* $g \geq 4$*, nous avons* $\tau_2(\mathcal{G} \cap J_2) = \mathrm{Ker}(\mathrm{Tr}^{as}) \cap \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^B)$.

Avec l'aide des calculs précédents, nous sommes capables de donner des énoncés précis à propos de la description des 3-sphères d'homologie entière par des scindements de Heegaard ayant des applications de recollement dans le second ou le troisième terme de la filtration de Johnson de la surface de recollement, améliorant ainsi le Théorème 2.1. Dans la proposition suivante, l'indice $\mathcal{G}_{g,1}$ désigne le quotient par l'action par conjugaison du groupe de Goeritz sur le groupe de Torelli.

Proposition 2.6. *Notons* $\mathcal{K}_{g,1} := J_2(\Sigma_{g,1})$ *et* $\mathcal{L}_{g,1} := J_3(\Sigma_{g,1})$*. Il existe des isomorphismes bien définis*

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1} \cap \mathcal{K}_{g,1}) \backslash \mathcal{K}_{g,1} / (\mathcal{B}_{g,1} \cap \mathcal{K}_{g,1}) \big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}(3),$$

et

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1} \cap \mathcal{L}_{g,1}) \backslash \mathcal{L}_{g,1} / (\mathcal{B}_{g,1} \cap \mathcal{L}_{g,1}) \big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}(3).$$

Ce type de résultat pourrait être utile, par exemple, si l'on voulait essayer de construire de nouveaux invariants avec la méthode de Pitsch [49, Théorème 2]. Grossièrement, si l'on arrive à définir des cocycles triviaux sur $\mathcal{I}_{g,1}$ avec de bonnes propriétés, ces cocycles dérivent d'applications sur le groupe de Torelli qui s'assemblent en un invariant des 3-sphères d'homologie entière (voir aussi [53]).

Enfin, dans l'Appendice 1.A du Chapitre 1, nous montrons que $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ admet une action de $\mathrm{GL}(g, \mathbb{Q})$, et nous en donnons une décomposition en modules irréductibles.

2.2 Relation d'équivalence sur l'espace des 3-variétés

Nous donnons maintenant des énoncés précis à propos des chirurgies sur les 3-variétés. Les chirurgies de Dehn et le calcul de Kirby ont beaucoup été utilisés pour connecter l'étude des 3-variétés et l'étude des nœuds et des entrelacs, et ce notamment pour construire des invariants de 3-variétés à partir d'invariants de nœuds ou d'entrelacs. Enfin si l'on peut faire des chirurgies sur des nœuds, on peut aussi en faire sur des corps en anses (ou, de manière équivalente, une surface à bord épaissie). Plus précisément, pratiquer une chirurgie sur une surface S avec une composante de bord dans une 3-variété orientée M signifie choisir un élément s du groupe d'homéotopie $\mathcal{M}(S)$ de la surface S, identifier un voisinage régulier de S avec $S \times [-1, 1]$, et faire le mouvement

$$M \rightsquigarrow M_s := (M \backslash \mathrm{int}(S \times [-1, 1])) \cup_{\tilde{s}} (S \times [-1, 1])$$

où $\tilde{s}$ est l'application de $\partial(S\times[-1,1])$ vers lui-même défini par $(\mathrm{Id}\times(-1))\cup(\mathrm{Id}\times\partial S)\cup(s\times 1)$. Le mouvement consiste donc à enlever puis recoller un corps en anse de manière « tordue ». À travers cette construction, une filtration du groupe d'homéotopie ou du groupe de Torelli peut parfois induire des relations d'équivalence sur les 3-variétés. Il existe sur le groupe de Torelli deux telles filtrations d'importance : la suite centrale descendante $(\Gamma_k\mathcal{I})_{k\geq 1}$ et la filtration de Johnson $(J_k)_{k\geq 1}$, dont la définition a été rappelée dans la Section 2.1.

Définition 2.7. *Les relations de Y_k-équivalence et de J_k-équivalence sont définies par :*

$$M \overset{Y_k}{\sim} M' \Leftrightarrow \exists S\subset M,\ \exists s\in\Gamma_k\mathcal{I}(S)\ s.t.\ M'=M_s$$

$$M \overset{J_k}{\sim} M' \Leftrightarrow \exists S\subset M, \exists s\in J_k(S)\ s.t.\ M'=M_s.$$

On peut montrer assez facilement (voir par exemple [39]) que ceci définit en fait des relations d'équivalence (la partie non-triviale de cette affirmation étant l'hypothèse de transitivité). De plus, ces relations d'équivalence sont préservées par stabilisation de la surface. Cela signifie que si la chirurgie est faite via $s\in\Gamma_k\mathcal{I}(S)$, on peut plonger S dans une surface de genre plus grand S', et faire la chirurgie (en étendant s par l'identité sur $S'\setminus S$). On obtiendra que M_s est équivalente à M par une chirurgie de genre plus grand. Nous rappelons ici certains faits à propos de ces relations d'équivalence, mais un meilleur résumé peut être trouvé dans [37], où le lecteur pourra aussi s'informer sur les liens entre ces relations et les invariants de type fini des 3-variétés. Notons aussi que si le mouvement définissant les chirurgies consiste à retirer puis à recoller différemment un corps en anses, on ne peut pas dire que cela correspond à une « torsion d'un scindement de Heegaard » tel que décrit plus haut. En effet, si le voisinage d'une surface à bord *est* un corps en anses, son complémentaire ne l'est pas forcément. En fait, dans les cas de la Y_k-équivalence et de la J_k-équivalence, supposer ou non que que ce complémentaire est un corps en anses ne fait aucune différence (voir e.g. [39, Lemme 2.1]).

Les relations de Y_k-équivalence peuvent être étudiées en utilisant les techniques de « calcul de claspers » introduites par Goussarov [11, 13] et Habiro [15]. Dans le cas de $\mathcal{V}(3)$, la Y_1 [41] et la Y_2-équivalence [35] entre 3-variétés sont classifiées par des invariants bien connus. La Y_2 et la J_2-équivalence [38], ainsi que la Y_3-équivalence et la J_3-équivalence [39], ont été caractérisées dans le cas des cylindres d'homologie.

Il est aussi connu, grâce à des résultats de Morita [43] et Pitsch [49], que deux 3-sphères d'homologie sont toujours J_3-équivalentes. Ainsi, la J_1, la J_2 et la J_3-équivalence sont triviales sur $\mathcal{S}(3)$. Le cas de la J_4-équivalence n'était pas connue, et il aurait difficilement pu être traité par un calcul direct comme cela avait été fait dans [43] et [49]. Cependant, une preuve alternative du fait que la J_3-équivalence est triviale sur $\mathcal{S}(3)$ peut être trouvée dans [39], où une façon de traiter la question pour J_4 est proposée [39, Rem. 6.4]. Cette approche relie, à travers un argument de calcul de claspers, la question de la trivialité de la J_4-équivalence à une question à propos de l'invariant de Casson.

L'invariant de Casson $\lambda:\mathcal{S}(3)\to\mathbb{Z}$ est un invariant des sphères d'homologie entière orientées de dimension 3. Il est défini à l'origine comme le décompte, avec certains signes, du nombre de représentations irréductibles du groupe fondamental de la 3-sphère d'homologie à valeur dans $SU(2)$. C'est un relèvement de l'invariant de Rokhlin $\mu:\mathcal{S}(3)\to\mathbb{Z}_2$. Rappelons que l'invariant de Rokhlin d'une 3-sphère d'homologie M est la signature, divisée par 8 puis considérée modulo 2, d'une 4-variété spinorielle quelconque dont le bord est M.

Un *plongement de Heegaard* $j:\Sigma_{g,1}\to S^3$ est un plongement tel que recoller un disque le long du bord de l'image de j donne un scindement de Heegaard de S^3. Pour un tel plongement j, et pour tout $\varphi\in\mathcal{M}$, on peut définir $S^3_{j,\varphi}$, en retirant un voisinage régulier de la surface et en le recollant après avoir composé avec l'application φ. Tout invariant F de 3-variétés à valeurs dans un groupe abélien C induit alors pour tout j une application F_j sur $\mathcal{M}$:

$$\begin{aligned} F_j:\mathcal{M}&\longrightarrow C\\ \varphi&\longmapsto F(S^3_{j,\varphi})-F(S^3). \end{aligned}$$

Dans le cas de l'invariant de Casson, nous voulons rester dans l'espace des 3-sphères d'homologie entière $\mathcal{S}(3)$. Un argument de Mayer-Vietoris implique que si l'on choisit φ dans $\mathcal{I}$, alors $S^3_{j,\varphi}$ est une 3-sphère d'homologie et l'on peut définir :

$$\begin{aligned} \lambda_j : \mathcal{I} &\longrightarrow \mathbb{Z} \\ \varphi &\longmapsto \lambda(S^3_{j,\varphi}) \end{aligned}$$

Les applications λ_j ne sont pas des homomorphismes. Cependant, Morita [42] a prouvé que leurs restrictions à $\mathcal{K} = J_2$ *le sont*. De plus il a prouvé que la restriction de λ_j admet la décomposition suivante :

$$-\lambda_j = \frac{1}{24}d + q_j : \mathcal{K} \to \mathbb{Z}.$$

L'homomorphisme d est appelée le *cœur de l'invariant de Casson*. Remarquons qu'il est indépendant du plongement de Heegaard j. Par définition, l'homomorphisme q_j factorise à travers le second homomorphisme de Johnson, ce qui implique que pour tout j, il s'annule sur le sous-groupe J_3. En conséquence, l'invariant de Casson définit de manière intrinsèque un homomorphisme λ sur J_k pour tout $k \geq 3$. Concrètement, cela souligne que la valeur de de l'invariant de Casson sur $S^3_{j,\varphi}$ est indépendante de j quand $\varphi \in J_k$ et $k \geq 3$. Le cœur de l'invariant de Casson n'est pas complètement compris, mais on sait que les twists de Dehn le long de courbes fermées simples et bordantes (abrégées BSCC dans la suite) de genre 1 et 2 engendrent $\mathcal{K}$ [26] et que la valeur de d sur un twist de Dehn le long d'une BSCC de genre h est $4h(h-1)$ [42].

Morita affirme dans [42] que $\lambda(J_3) = \mathbb{Z}$ en genre $g \geq 2$, et Massuyeau et Meilhan [39] explicitent le calcul. En utilisant le calcul de claspers de Habiro [15] (qui permet de montrer que la Y_3-équivalence parmi les 3-sphères d'homologie est classifiée par λ), Massuyeau et Meilhan [39, Théorème C] ont reprouvé que la J_3-équivalence est triviale sur $\mathcal{S}(3)$. Cette méthode se généralise au cas de la J_4-équivalence, comme nous allons le montrer.

Théorème 2.8. *Pour tout genre $g \geq 2$, la restriction de $\lambda : J_3 \to \mathbb{Z}$ à J_4 est surjective.*

Théorème 2.9. *La J_4-équivalence est triviale sur $\mathcal{S}(3)$.*

Le Théorème 2.8 est prouvé en construisant explicitement un élément de J_4, en genre $g = 2$, dont l'invariant de Casson est égal à 1 (produisant ainsi une 3-sphère d'homologie entière dans la classe de J_4-équivalence de S^3 avec un invariant de Casson égal à 1). Les calculs utilisés dans cette construction utilisent les *homomorphismes de Johnson infinitésimaux*, et plus spécifiquement une formule de Kawazumi et Kuno dans [31]. Un point de vue plus général pour cette formule est donné dans [40]. Nous utilisons aussi en pratique un programme informatique SageMath pour faire les calculs, et le code est donné dans l'Appendice 2.A.

Le Théorème 2.8 est aussi intéressant pour l'étude du groupe d'homéotopie de la surface en soi. En effet, Hain [17] a prouvé que $\lambda(J_k) \neq \{0\}$ pour $k \geq 3$. Savoir si la restriction de λ à J_k reste surjective pour $k \geq 5$ serait très intéressant pour l'étude de la J_k-équivalence en général. On pourrait commencer par la question suivante, strictement plus simple, à propos de l'invariant de Rokhlin :

Question 2.10. *Existe-t-il un $k \geq 5$ tel que $\mu(J_k) = 0$?*

Cependant, il y a fort à parier que la connaissance des valeurs de λ sur J_5 ne serait pas suffisante pour étudier la J_5-équivalence. En effet, la méthode de Massuyeau et Meilhan utilise la classification de la Y_k-équivalence par Habiro [15], et des invariants de type fini de plus haut degré sont impliqués pour le cas $k = 5$.

Nous introduisons maintenant une autre méthode pour attaquer la question de la J_k-équivalence, qui nécessiterait de plus amples investigations. Cela fera l'objet de l'Appendice A. Nous définissons d'abord une autre relation d'équivalence, suggérée par les travaux de Levine dans [34]. En effet, Levine y définit une filtration $(L_k)_{k\geq 1}$ qui est une filtration non-séparante du groupe d'homéotopie. Elle n'est ainsi *pas* suffisante pour obtenir une approximation du groupe d'homéotopie de la surface, mais elle semble mieux adaptée à l'étude des 3-variétés présentées par scindements de Heegaard. La définition de Levine dépend du choix d'une inclusion $\Sigma \subset V$ dans un corps en anses V telle que $\partial V \setminus \Sigma$ est un disque. Lorsque cela est nécessaire, la notation sera plus précise (e.g. $L_k(V)$ pour un corps en anses V). En particulier, ceci dépend d'un sous-espace Lagrangien $A \subset H$, le noyau de la projection $H = H_1(\Sigma_{g,1}) \to H' = H_1(V_g)$. On note p la projection de $\pi = \pi_1(\Sigma_{g,1})$ vers $\pi' = \pi_1(V_g)$, et $\mathbb{A}$ le noyau de p. Bien sûr A est l'image de $\mathbb{A}$ dans H à travers l'application d'abélianisation de π. On note aussi $D_k(H')$ l'espace des dérivations symplectiques de degré k de $\mathcal{L}(H')$. Enfin, quand f est un élément du groupe d'homéotopie de la surface, $f_* \in \mathrm{Sp}(H)$ désigne l'action de f sur H. Nous écrirons abusivement f pour l'action de f sur le groupe fondamental.

Définition 2.11. *Le groupe de Torelli Lagrangien est défini par :*

$$\mathcal{I}^L := \{h \in \mathcal{M} : h_*(A) \subset A \textit{ et } h_* \textit{ est l'identité sur } A\}.$$

Définition 2.12. *Pour $k \geq 1$, le groupe $L_k = L_k(V)$ est défini par :*

$$L_k := \Big\{ h \in \mathcal{I}^L \;\Big|\; p\big(h(\mathbb{A})\big) \subset \Gamma_{k+1}\pi' \Big\}.$$

Levine a prouvé que c'est en effet un sous-groupe de $\mathcal{M}$ [32].

Nous donnons maintenant la définition de la L_k-*équivalence*, et prouverons dans l'Appendice A que c'est bel et bien une relation d'équivalence.

Définition 2.13. *Deux 3-variétés orientées M et M' sont dites L_k-équivalentes si M' peut être obtenue de M en retirant un corps en anses V et en le recollant en tordant par un élément de $L_k(V)$ (étendu par l'identité sur un disque fermant la surface de recollement).*

Il est clair que $J_k \subset L_k$, en tant que sous-groupes de $\mathcal{M}$, pour tout $k \geq 1$. Ainsi, la L_k-équivalence est plus faible que la J_k-équivalence. Comme cette dernière est, par le Théorème 2.9, triviale jusqu'en degré 4 pour les 3-sphères d'homologie entière, ceci reste vraie pour la L_k-équivalence. Il est donc naturel de poser la question suivante, au moins en bas degrés.

Question 2.14. *Pour les 3-variétés en général, et $k \geq 1$, la J_k-équivalence et la L_k-équivalence coïncident-t-elle ?*

Cette question est d'autant plus justifiée que la réponse est positive pour $k = 1, 2$.

Proposition 2.15. *Pour $k = 1, 2$, deux 3-variétés orientées quelconques qui sont L_k-équivalentes sont aussi J_k-équivalentes.*

Un résultat de Levine [34] permet de donner une preuve simple de la Proposition 2.15, qui utilise les homomorphismes de Johnson. En soulignant que

$$J_k \cdot L_\infty \subset L_k, \text{ avec } L_\infty := \bigcap_{k \geq 1} L_k,$$

Levine pose la question suivante :

Question 2.16. *A-t-on $L_k = J_k \cdot L_\infty$ pour tout k ?*

Ceci est motivé, dans notre situation, par le lemme suivant.

Lemma 2.17. *Si $L_k = J_k \cdot L_\infty$, alors la L_k-équivalence coïncide avec la J_k-équivalence.*

Levine a aussi prouvé le lemme suivant (voir [34, Lemme 6.2] pour une preuve détaillée).

Lemme 2.18. *Supposons que $L_k = J_k \cdot L_\infty$, alors $L_{k+1} = J_{k+1} \cdot L_\infty$ si et seulement si* $\mathrm{Im}(\tau_k) \cap \mathrm{Ker}(D_k(H) \to D_k(H')) = \tau_k(\mathcal{A} \cap J_k)$.

Il est facile de montrer que $L_1 = J_1 \cdot \mathcal{A}$ (voir, par exemple, [34, Lemme 6.3]). Des calculs de Morita dans [42] montrent alors que $\mathrm{Ker}(D_1(H) \to D_1(H')) = \tau_1(\mathcal{A} \cap J_1)$ (rappelons que τ_1 est surjectif). Ainsi, pour $k = 1, 2$, nous obtenons une réponse positive à la question de Levine 2.16, et à la Question 2.14. Quant au cas $k = 3$, l'égalité nécessaire à l'étape d'induction n'est plus vraie, par le Théorème 2.4 :

$$\tau_2(\mathcal{A} \cap J_2) \subsetneq \mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H')).$$

Ceci répond négativement à la Question 2.16, mais pas nécessairement à la Question 2.14. La différence entre ces deux sous-modules de $D_2(H)$ donne de bons candidats pour pratiquer des L_3-chirurgies sur certaines variétés, dans le but de donner deux variétés L_3-équivalentes mais pas J_3-équivalentes. C'est ce que nous faisons dans l'Appendice A, en utilisant la classification de la J_3-équivalence donnée dans [39].

Proposition 2.19. *Parmi les 3-variétés fermées orientées, la L_3-équivalence est strictement plus faible que la J_3-équivalence.*

Remarquons, cependant, que la L_k-équivalence et la J_k-équivalence pourraientt coïncider pour tout $k \geq 1$ pour les 3-sphères d'homologie. Cela pourrait aider à étudier la J_k-équivalence parmi les 3-sphères d'homologie. Nous formulons donc les questions suivantes.

Question 2.20. *Jusqu'à quel $k \geq 5$ la relation L_k est-elle triviale sur $\mathcal{S}(3)$?*

Question 2.21. *Jusqu'à quel $k \geq 5$ la relation J_k est-elle triviale sur $\mathcal{S}(3)$?*

2.3 Contenu et organisation de la dissertation

Cette dissertation est composée de deux articles et d'un appendice qui peuvent être lus séparément. Dans chacun des articles, le lecteur trouvera une introduction plus détaillée. Dans le premier article [8], *The handlebody group and the images of the second Johnson homomorphism*, reproduit dans le Chapitre 1, le lecteur trouvera les preuves des Théorèmes 2.3, 2.4, et des Propositions 2.5 et 2.6. Le second article [9], *Triviality of the J_4-equivalence among homology 3-spheres*, est reproduit dans le Chapitre 2 et contient les preuves des Théorèmes 2.8 and 2.9. Les preuves des Propositions 2.15 and 2.19 figurent dans l'Appendice A.

Chapter 1

The handlebody group and the images of the second Johnson homomorphism

Abstract. Given an oriented surface bounding a handlebody, we study the subgroup of its mapping class group defined as the intersection of the handlebody group and the second term of the Johnson filtration: $\mathcal{A} \cap J_2$. We introduce two trace-like operators, inspired by Morita's trace, and show that their kernels coincide with the images by the second Johnson homomorphism τ_2 of J_2 and $\mathcal{A} \cap J_2$, respectively. In particular, we answer by the negative to a question asked by Levine about an algebraic description of $\tau_2(\mathcal{A} \cap J_2)$. By the same techniques, and for a Heegaard surface in S^3, we also compute the image by τ_2 of the intersection of the Goeritz group $\mathcal{G}$ with J_2.

1 Introduction and notations

We consider an abstract handlebody V_g of genus g whose boundary is a surface Σ_g of genus g. This surface minus a disk will be the surface with non-empty boundary $\Sigma_{g,1}$. We will often forget the indices concerning the genus and the number of boundary components when they are clear from context.

The study of the handlebody group $\mathcal{A}$ is of major importance for the study of the mapping class group of surfaces $\mathcal{M}$, especially in connection with the theory of 3-manifolds and their Heegaard presentations. The reader may find useful information on this topic in the survey by Hensel [19]. It is a non-normal subgroup of the mapping class group of infinite index, which makes its study as a subgroup of $\mathcal{M}$ uneasy. Precisely, $\mathcal{M}$ will be our notation for $\mathcal{M}_{g,1}$, the mapping class group of $\Sigma_{g,1}$, and $\mathcal{A}$ will be our notation for $\mathcal{A}_{g,1}$, the mapping class group of V_g relative to a disk in ∂V_g.

We will denote $\pi := \pi_1(\Sigma_{g,1}, x_0)$, where x_0 is a point on the boundary of $\Sigma_{g,1}$, and $H := H_1(\Sigma_{g,1})$ its abelianization. Recall that π is isomorphic to the free group with $2g$ generators F_{2g}, and hence H is isomorphic to $\mathbb{Z}^{2g}$. The curves $(\alpha_i)_{1\leq i\leq g}$ and $(\beta_i)_{1\leq i\leq g}$ on Figure 1.1 are two cut systems such that each curve in the first one has exactly one intersection point with exactly one curve in the second one, and vice versa. Such a choice is called a system of *meridians* and *parallels*. In particular, it fixes a choice of a basis for $H = \mathbb{Z}\langle a_1, a_2 \dots a_g, b_1, b_2, \dots b_g\rangle$, where a_i (resp. b_i) is the homology class of α_i (resp. β_i). When $\Sigma_{g,1}$ will be regarded as the boundary of V_g (minus a disk), we will suppose that the meridians (i.e. the curves α_i) bound pairwise-disjoint disks in the handlebody. If promoted to elements of the fundamental group π, the curves β_i define generators of $\pi' := \pi_1(V, x_0)$ and the curves α_i normally generate the kernel of the surjection $\pi \to \pi'$ induced by the inclusion of $\Sigma_{g,1}$ in V_g. We denote $\mathbb{A}$ this kernel, so that $\pi' \simeq \pi/\mathbb{A}$. It is well-known that the handlebody group $\mathcal{A}$, which can be thought of as consisting of elements of the mapping class group $\mathcal{M}$ extending to the whole handlebody, coincides with the subgroup of $\mathcal{M}$ preserving $\mathbb{A}$ [19]. We emphasize that, from the point of view of the surface $\Sigma_{g,1}$, this subgroup $\mathcal{A}$ of $\mathcal{M}$ depends on the choice of handlebody V_g.

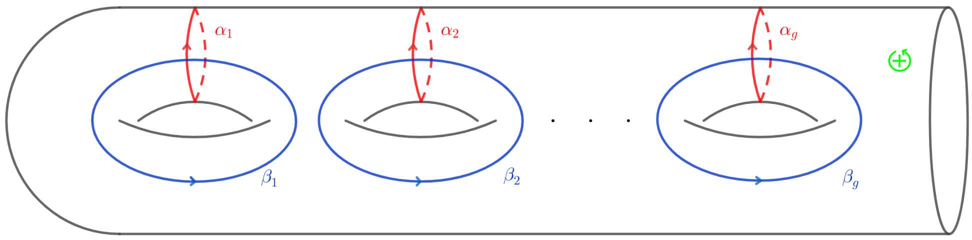


Figure 1.1: Model for $\Sigma_{g,1}$, and a possible choice of system of meridians and parallels

We also consider $H' := H_1(V_g)$ the first homology group of the handlebody. The kernel of the homomorphism $H \to H'$ induced by the inclusion of $\Sigma_{g,1}$ in V_g is denoted A. It is generated in H by the elements a_i. The group $H' \simeq H/A$ is freely generated by the classes of the elements b_i, but should not be thought of as a subgroup of H since there is no canonical way to choose a supplement of A in H. We consider the homological intersection form $\omega : H \otimes H \to \mathbb{Z}$, which induces a non-singular pairing $\omega' : A \otimes H' \to \mathbb{Z}$. We denote by $\mathcal{L}(H) = \bigoplus_{k\geq 1} \mathcal{L}_k(H)$ the graded Lie ring freely generated by H in degree 1. We denote by $T(H)$ the tensor algebra, in which $\mathcal{L}(H)$ can be imbedded. The symmetric algebra $S(H)$ is as usual the quotient of $T(H)$ by its antisymmetric tensors.

In this paper we focus on the study of the group $\mathcal{A} \cap J_2$, where J_2 is the second term of the Johnson filtration $(J_k)_{k\geq 1}$ [25]. Examining the group $\mathcal{A} \cap J_2$ seems natural when one uses Johnson-type homomorphisms to study finite-type invariants of 3-manifolds from the point of view of Heegaard splittings. Besides, the Johnson filtration of $\mathcal{M}$ is separating, and so is its intersection with $\mathcal{A}$: hence the study of the filtration $(\mathcal{A} \cap J_k)_{k\geq 1}$, including the determination of its associated graded $\bigoplus_{k\geq 1} \frac{\mathcal{A}\cap J_k}{\mathcal{A}\cap J_{k+1}}$, is also relevant for the study of the group $\mathcal{A}$ itself. As the Torelli group $\mathcal{I}$ (the subgroup of $\mathcal{M}$ acting trivially at the homological level) is the first term J_1 of the Johnson filtration, the question addressed here is the next natural step after the study of $\mathcal{A} \cap \mathcal{I}$ pursued by Omori in [48], and the earlier computation of $\frac{\mathcal{A}\cap J_1}{\mathcal{A}\cap J_2}$ given by Morita in [42].

The study of the relationship between the Johnson filtration and the handlebody group may cover other aspects. In particular, it was proved independently by Hain [18] and Jorgensen [29] that there exist elements of $\mathcal{M}$ arbitrarily deep in the Johnson filtration that are not in the union of the conjugates of $\mathcal{A}$ in $\mathcal{M}$. Besides, Hain also introduced a filtration of a completion of $\mathcal{M}$ (relative to the symplectic representation), called the *weight filtration* and he introduced in [18] another filtration, the *relative weight filtration* associated to the choice of a handlebody bounded by Σ. The study of the graded spaces associated to these filtrations should be related to the quotients $\frac{\mathcal{A}\cap J_k}{\mathcal{A}\cap J_{k+1}} \otimes \mathbb{Q}$.

In this paper, we work with coefficients in $\mathbb{Z}$ (the only exception will be in Appendix 1.A). To get a more precise grasp of the intersection $\mathcal{A} \cap J_2$, we use the Johnson homomorphisms $(\tau_k)_{k\geq 1}$ introduced in [25], trace-like operators, and the Casson invariant.

The first step is to define a trace-like operator Tr^{as} on the codomain of τ_2 (which is the group of symplectic derivations of degree 2 of $\mathcal{L}(H)$, denoted $D_2(H)$). Using the results of Morita [42] and Yokomizo [58], we prove that the kernel of Tr^{as} is precisely $\tau_2(J_2)$. We also show that $\tau_2([J_1, J_1]) = \mathrm{Ker}(\mathrm{Tr}^{sym})$, where Tr^{sym} is another trace-like map (defined on a subgroup of $D_2(H)$). The codomains of Tr^{as} and Tr^{sym} will be respectively $\mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$ and $\mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2)$. Here, and in the sequel, for any module V, the notation $S^2(V)$ stands for the quotient of $V \otimes V$ by the two-sided ideal generated by the tensors of the form $v \otimes w - w \otimes v$. The module $\Lambda^2(V)$ is the quotient of the same module by the two-sided ideal generated by the tensors of the form $v \otimes w + w \otimes v$ and $v \otimes v$. Notice that this last type of tensor is needed in the definition in the case of $\mathbb{Z}_2$-modules. For example there is a canonical projection from $S^2(H/2H)$ to $\Lambda^2(H/2H)$, given by reducing the classes of elements of the form $v \otimes v$.

The second step is the study of $\tau_2(\mathcal{A} \cap J_2)$, which, by definition of τ_2, is isomorphic to $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3}$. In [34], Levine observed that this image is contained in the kernel of the canonical projection from $D_2(H)$ to $D_2(H')$. He asked whether the intersection of $\mathrm{Ker}(D_2(H) \to D_2(H'))$ with $\mathrm{Im}(\tau_2)$ was equal to $\tau_2(\mathcal{A} \cap J_2)$. We shall define, using the non-singular pairing ω', another trace-like operator Tr^A vanishing on $\tau_2(\mathcal{A} \cap J_2)$, but not on this subgroup proposed by Levine. Therefore, we answer negatively to Levine's question. Furthermore, Tr^{as} and Tr^A will allow us to compute precisely $\tau_2(\mathcal{A} \cap J_2)$, and thus to identify $\frac{\mathcal{A}\cap J_2}{\mathcal{A}\cap J_3}$ with an explicit subgroup of $D_2(H)$.

The paper is organized as follows. In Section 2, we review the definition of the Johnson filtration $(J_k)_{k\geq 1}$ from [25], as well as the definition of the Johnson homomorphisms $(\tau_k)_{k\geq 1}$ from [44]. Then we define the maps Tr^{as} and Tr^{sym} and use them to characterize $\tau_2(J_2)$ and $\tau_2([J_1, J_1])$, respectively. In Section 3, we first review closely related works. Then we recall the definition of the Levine filtration $(L_k)_{k\geq 1}$ from [34], so as to state and motivate precisely the question asked by Levine. In Section 4, we define the map Tr^A, and we prove that it gives a new obstruction for an element of $D_2(H)$ to be in $\tau_2(\mathcal{A} \cap J_2)$, by using Morita's decomposition of the Casson invariant [42]. In Section 5, we compute the image $\tau_2(\mathcal{A} \cap J_2)$ using the algebraic tools introduced in Sections 2 and 4. In Section 6, when Σ is a Heegaard surface of S^3, we compute $\tau_2(\mathcal{G} \cap J_2)$ where $\mathcal{G} \subset \mathcal{M}$ is the Goeritz group defined by this Heegaard splitting. Finally, in Appendix A, we decompose $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ into irreducible $\mathrm{GL}(g, \mathbb{Q})$-modules, and we check the computation of Section 6 for rational coefficients, without using the main result of Section 5.

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2 Image of the second Johnson homomorphism τ_2

2.1 The space of symplectic derivations of degree 2

Here, we review some facts about Johnson homomorphisms and their diagrammatic description. We are especially interested in describing the image of the second Johnson homomorphism.

Johnson homomorphisms and tree-like Jacobi diagrams

The Johnson filtration and the Johnson homomorphisms have been introduced and studied by Johnson and Morita in [25, 44]. Recall that $\pi := \pi_1(\Sigma_{g,1})$ is a free group. For $k \geq 1$, we consider its lower central series $(\Gamma_k\pi)_{k\geq 1}$. We call the quotient $N_k := \pi/\Gamma_{k+1}\pi$ the k-th nilpotent quotient of π. The first nilpotent quotient is canonically isomorphic to $H := H_1(\Sigma_{g,1})$. It is clear that $\mathcal{M}$ acts both on π and all its nilpotent quotients. There is an exact sequence:

$$0 \longrightarrow \mathcal{L}_{k+1}(H) \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow 0$$

where the first non-trivial arrow is given by the identification between $\mathcal{L}_{k+1}(H)$ and the quotient $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$. This sequence induces the short exact sequence :

$$0 \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)) \longrightarrow \mathrm{Aut}(N_{k+1}) \longrightarrow \mathrm{Aut}(N_k).$$

The group J_k is defined as the kernel of the canonical homomorphism $\rho_k : \mathcal{M} \to \mathrm{Aut}(N_k)$. In particular J_1 is called the Torelli group, otherwise denoted $\mathcal{I} = \mathcal{I}_{g,1}$. It consists of elements of the mapping class group acting trivially on the homology of the surface. The alternative notation $\mathcal{K} = \mathcal{K}_{g,1}$ is also sometimes used for J_2.

The restriction of ρ_{k+1} to J_k then induces a morphism:

$$\tau_k : J_k \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)).$$

We call this map the *k-th Johnson homomorphism*. Its kernel is J_{k+1}. Furthermore, the mapping class group acts on itself by conjugation, inducing an action of the symplectic group $\mathrm{Sp}(H)$ on the quotient J_k/J_{k+1}. This group also naturally acts on H. Each τ_k is then $\mathrm{Sp}(H)$-equivariant. It is also known that the graded space induced by the Johnson filtration has a Lie structure, its bracket being induced by the commutator in $\mathcal{M}$. The target space of τ_k can be identified with the space of *derivations of degree k*, i.e. derivations of $\mathcal{L}(H)$ mapping $H = \mathcal{L}_1(H)$ to $\mathcal{L}_{k+1}(H)$. We denote by $D_k(H)$ the subspace of *symplectic derivations* of degree k. It consists of derivations of degree k sending $\tilde{\omega} \in \Lambda^2 H \simeq \mathcal{L}_2(H)$, the bivector dual to ω, to 0. The fact that an element of $\mathcal{M}$ fixes the boundary of $\Sigma_{g,1}$ allows to further restrict the image of τ_k to $D_k(H)$. Also, $D_k(H)$ is determined by the short exact sequence:

$$0 \longrightarrow D_k(H) \longrightarrow H \otimes \mathcal{L}_{k+1}(H) \longrightarrow \mathcal{L}_{k+2}(H) \longrightarrow 0$$

where the arrow from $H \otimes \mathcal{L}_{k+1}(H)$ to $\mathcal{L}_{k+2}(H)$ is the bracket of the free Lie algebra.

With these definitions, the spaces $(D_k(H))_{k\geq 1}$ reassembles in a graded Lie algebra $D(H)$ (the bracket of two derivations d_1 and d_2 being classically defined as $d_1 d_2 - d_2 d_1$). The family $(\tau_k)_{k\geq 1}$ induces a map τ :

$$\tau : \bigoplus_{k\geq 1} J_k/J_{k+1} \longrightarrow D(H)$$

which is an $\mathrm{Sp}(H)$-equivariant graded Lie morphism. The map τ_k is not onto $D_k(H)$ in general, but it is known to be surjective for $k = 1$ [21] and rationally surjective for $k = 2$ [42]. We shall describe in the next subsections the image of τ_2 in a precise way.

We also need to define the spaces of tree-like Jacobi diagrams $\mathcal{A}_k^t(H)$ and rooted tree-like Jacobi diagrams $\mathcal{A}_k^{t,r}(H)$. A tree is a connected graph that is contractible as a topological space. From now on, by "a tree", we mean a uni-trivalent tree T, possibly rooted, whose set of trivalent (or *internal*) vertices is oriented (the orientation being counterclockwise in all the figures), and whose set of univalent (or *external*) vertices, denoted $v_1(T)$, is colored by elements of H. We will also refer to external vertices as *leaves* and internal vertices as *nodes*. The cardinality of the set of trivalent vertices $v_3(T)$ is the *degree* of the tree T. The spaces $\mathcal{A}_k^t(H)$ and $\mathcal{A}_k^{t,r}(H)$ are the $\mathbb{Z}$-modules generated by trees (respectively rooted trees) of degree k subject to some relations: multilinearity of the labels, the *AS relation*, and the *IHX relation*. We specify these relations for $k = 2$ in Figure 1.2, and we refer the reader to [33] for further details about what follows. These spaces assemble in two graded algebras $\mathcal{A}^t(H)$ and $\mathcal{A}^{t,r}(H)$ endowed respectively with a Lie bracket and a quasi-Lie bracket. For the bracket of $\mathcal{A}^t(H)$, take two trees, and sum all the ways to contract an external vertex from the first one with an external vertex from the second one using the symplectic form ω. For $\mathcal{A}^{t,r}(H)$, take two trees, and form a tree by gluing their roots to a rooted binary tree with two leaves.

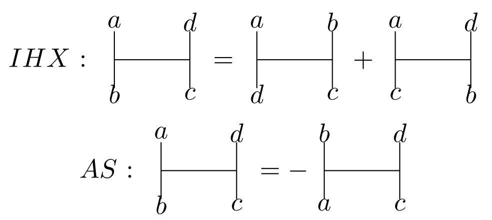
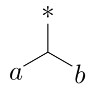


Figure 1.2: Relations in $\mathcal{A}_2^t(H)$

We also define, for any k, maps

$$\begin{aligned} \eta_k : \mathcal{A}_k^t(H) &\longrightarrow D_k(H) \\ T &\longmapsto \sum_{x \in v_1(T)} l_x \otimes T^x \end{aligned}$$

where l_x is the element of H coloring the vertex x and T^x is the rooted tree obtained by setting x to be the root in T, read as an element of $\mathcal{L}_{k+1}(H)$ (which can be done inductively by considering that  corresponds to $[b, a]$). These maps assemble into a graded Lie algebra morphism which we refer to as "the expansion map".

A presentation for $D_2(H)$

The first Johnson homomorphism takes values in $D_1(H)$ which is known to be isomorphic to $\Lambda^3 H$. The map τ_1 is surjective, and η_1 is an isomorphism, thus identifying the quotient J_1/J_2 to $\mathcal{A}_1^t(H)$.

The second Johnson homomorphism takes values in $D_2(H)$. This space is well understood too. Morita [42], using the exact sequence

$$0 \longrightarrow \Lambda^3 H \longrightarrow H \otimes \mathcal{L}_2(H) \longrightarrow \mathcal{L}_3(H) \longrightarrow 0,$$

described it as the image of $(\Lambda^2(H) \otimes \Lambda^2 H)^{\mathfrak{S}_2}$ in the quotient $(H \otimes H \otimes \Lambda^2 H)/H \otimes \Lambda^3 H$, where $\mathcal{L}_2(H)$ has been identified with $\Lambda^2 H$.

We will prefer to use the following description given by Levine [33]. Indeed, a simpler way to think about this space is to use the free *quasi-Lie algebra* $\mathcal{L}'(H) = \bigoplus_{k \geq 1} \mathcal{L}'_k(H)$ on H, which is defined similarly to the free Lie algebra with the alternativity axiom $[x, x] = 0$ (for any $x \in \mathcal{L}$) replaced by the antisymmetry axiom $[x, y] + [y, x] = 0$ (for any $x, y \in \mathcal{L}$). This change adds 2-torsion to the group. We define $D'_k(H)$, similarly to $D_k(H)$, as the kernel of the bracket from $H \otimes \mathcal{L}'_{k+1}(H)$ to $\mathcal{L}'_{k+2}(H)$. We will only use $k = 1$ or 2 in this paper. We have $D'_1(H) \simeq D_1(H)$ and a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & D'_2(H) \arrow[r] \arrow[d] & H \otimes \mathcal{L}'_3(H) \arrow[r] \arrow[d] & \mathcal{L}'_4(H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D_2(H) \arrow[r] & H \otimes \mathcal{L}_3(H) \arrow[r] & \mathcal{L}_4(H) \arrow[r] & 0. \end{tikzcd}$$

Levine also showed that we have the following exact sequence:

$$0 \longrightarrow D'_2(H) \longrightarrow D_2(H) \longrightarrow \Lambda^2(H/2H) \longrightarrow 0. \tag{2.1}$$

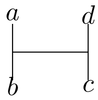
This is helpful for the following reason: $D_2(H)$, which is a free abelian group, can be thought of as a lattice in $D_2(H) \otimes \mathbb{Q}$. By (2.1), to generate $D_2(H)$, one simply needs to add to $D'_2(H)$ expansions of type $\frac{1}{2}\eta(u - u)$ for any rooted tree u with 2 external vertices, that we glue to its copy along their roots. These are indeed elements of $D_2(H)$, i.e. they have integer coefficients. For $x, y \in \Lambda^2 H$ we write $x \leftrightarrow y$ for the element $x \otimes y + y \otimes x$. Also $\Lambda^4 H$ can be embedded in $\left(\Lambda^2 H \leftrightarrow \Lambda^2 H\right) \subset \left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2}$ by sending $a \wedge b \wedge c \wedge d$ to

$$(a \wedge b) \leftrightarrow (c \wedge d) - (a \wedge c) \leftrightarrow (b \wedge d) + (a \wedge d) \leftrightarrow (b \wedge c),$$

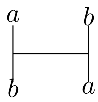
for $a, b, c, d \in H$. It has been proven by Levine in [33] using Morita's work in [42] (see also

[39, Prop. 3.1]) that the map

$$
\begin{aligned}
\frac{(\Lambda^2 H \otimes \Lambda^2 H)^{\mathfrak{S}_2}}{\Lambda^4 H} &\longrightarrow D_2(H)\\
(a\wedge b) \leftrightarrow (c \wedge d) &\longmapsto a\otimes [b,[c,d]] + b\otimes [[c,d],a]\\
&\qquad + c\otimes [d,[a,b]] + d\otimes [[a,b],c]
\end{aligned}
$$

$= \eta_2\Big($  $\Big)$

$$(a\wedge b)\otimes(a\wedge b) \longmapsto a\otimes [b,[a,b]] + b\otimes [[a,b],a]$$

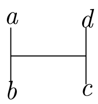
$= \frac{1}{2}\eta_2\Big($  $\Big)$

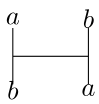
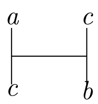
is a well-defined isomorphism that fits in the commutative diagram with exact rows

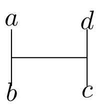
$$
\begin{tikzcd}
0 \arrow[r] & \frac{S^2(\Lambda^2 H)}{\Lambda^4 H} \arrow[r, "\leftrightarrow"] \arrow[d, "\eta'"] & \frac{\left(\Lambda^2 H\otimes\Lambda^2 H\right)^{\mathfrak{S}_2}}{\Lambda^4 H} \arrow[r] \arrow[d] & \frac{\Lambda^2 H}{2\cdot\Lambda^2 H} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & D_2'(H) \arrow[r] & D_2(H) \arrow[r] & \Lambda^2(H/2H) \arrow[r] & 0
\end{tikzcd}
\tag{2.2}
$$

where η' is defined in a way similar to η [33]. To be precise the expansion of a tree is actually an element of $D_2'(H)$, and this defines an isomorphism between $D_2'(H)$ and $\mathcal{A}_2^t(H)$ [33].

From this we deduce the following presentation of the abelian group $D_2(H)$.

Proposition 2.1. *$D_2(H)$ is generated by trees*  *for a, b, c and d in H and elements $a\odot b$ for $a, b\in H$ subject to the following relations:*

- *AS, IHX, and multilinearity with respect to the labels for all trees*
- *$a\odot a = 0$ and $a\odot b = b\odot a$ for all $(a,b)\in H\times H$*
- $2(a\odot b) =$ 
- $(a+b)\odot c = a\odot c + b\odot c +$ 

Proof. Let us momentarily denote by G the group defined by the presentation. We define a homomorphism from G to $\frac{\left(\Lambda^2 H\otimes\Lambda^2 H\right)^{\mathfrak{S}_2}}{\Lambda^4 H}$ by sending $a\odot b$ to the class of $(a\wedge b)\otimes(a\wedge b)$ and any tree  to the element corresponding to its expansion through diagram (2.2), i.e. to $(a\wedge b)\leftrightarrow(c\wedge d)$. We define a converse homomorphism by reversing the previous mappings. It suffices to show that these maps are well-defined to conclude. It is straightforward calculus to check that the relations for G vanish in $\frac{\left(\Lambda^2 H\otimes\Lambda^2 H\right)^{\mathfrak{S}_2}}{\Lambda^4 H}$, noting in particular that it is known that the expansion map sends the IHX relation to 0. Conversely, $\left(\Lambda^2 H\otimes\Lambda^2 H\right)^{\mathfrak{S}_2}$ can be presented in the following way. The group $\left(\Lambda^2 H\otimes\Lambda^2 H\right)^{\mathfrak{S}_2}$ is generated by elements $(a\wedge b)\otimes(a\wedge b)$ and elements $(a\wedge b\leftrightarrow c\wedge d)$ with $a,b,c,d\in H$. The relations are $(a\wedge b)\leftrightarrow (a\wedge b) = 2(a\wedge b)\otimes(a\wedge b)$ and $((a+b)\wedge c)\otimes((a+b)\wedge c) - (a\wedge c)\otimes(a\wedge c) - (b\wedge c)\otimes(b\wedge c) =$

$(a \wedge c \leftrightarrow b \wedge c)$. This presentation is summarized in the short exact sequence

$$0 \longrightarrow S^2(\Lambda^2 H) \longrightarrow \left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2} \longrightarrow \frac{\Lambda^2 H}{2 \cdot \Lambda^2 H} \longrightarrow 0$$

where the last arrow sends $(a \wedge b) \otimes (a \wedge b)$ to $a \wedge b$ and $(a \wedge b) \leftrightarrow (c \wedge d)$ to 0. We can then read these relations in the presentation of G. We finally notice that for any $a, b, c, d \in H$, $(a \wedge b) \leftrightarrow (c \wedge d) - (a \wedge c) \leftrightarrow (b \wedge d) + (a \wedge d) \leftrightarrow (b \wedge c)$ is sent to the IHX relation, up to some antisymmetries. $\square$

Remark 2.2. Elements $a \odot b$ for $a, b \in H$ correspond to halves of symmetric trees (namely $\frac{1}{2}\begin{array}{ccc} a & & b \\ \vert & — & \vert \\ b & & a \end{array}$ for $a, b \in H$) through the inclusion $D_2(H) \subset D_2(H) \otimes \mathbb{Q} \simeq \mathcal{A}_2^t(H) \otimes \mathbb{Q}$. Then, a concise and simple way to summarize the previous discussion, is to say that $D_2(H)$ embeds in the space of trees $\mathcal{A}_2^t(H) \otimes \mathbb{Q}$, and its image is the lattice generated by trees and halves of symmetric trees. This is what we will do, especially in Sections 4 and 5.

2.2 An explicit description of $\mathrm{Im}(\tau_2)$ in $D_2(H)$

We aim at a homomorphism that would be explicitly defined on $D_2(H)$, using the presentation in Proposition 2.1, and whose kernel would be $\mathrm{Im}(\tau_2)$. From now on, we will abuse notation and identify $D'_2(H)$ with $\mathcal{A}_2^t(H)$ and think of its elements as trees.

In [26], Johnson showed that $\mathcal{K}$ is generated by Dehn twists along bounding simple closed curves (called *BSCC* maps) of genus 1 and 2. We will denote T_γ the Dehn twist along a given simple closed curve γ. In the sequel, we will need Morita's computations for the image of a BSCC map by the second Johnson homomorphism [42]:

Lemma 2.3. *Let γ be a BSCC bounding a subsurface F of genus h in Σ, and let $(u_i, v_i)_{1 \le i \le h}$ be any symplectic basis of the first homology group of F, then we have:*

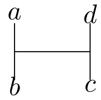
$$\tau_2(T_\gamma) = \left(\sum_{i=1}^{h} u_i \wedge v_i\right)^{\otimes 2} = \sum_{i=1}^{h} u_i \odot v_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{h} \begin{array}{ccc} u_i & & v_j \\ \vert & — & \vert \\ v_i & & u_j \end{array} \in D_2(H).$$

BSCC maps of genus 1 and 2 are all conjugated, by an element of the mapping class group, to one of the Dehn twists T_{γ_1} or $T_{\gamma_{1,2}}$ (see Figure 1.4 in Section 5). Lemma 2.3 then shows that $\mathrm{Im}(\tau_2)$ is generated by elements of type $u \odot v$ with $\omega(u, v) = 1$ and elements of type $\begin{array}{ccc} u_1 & & v_2 \\ \vert & — & \vert \\ v_1 & & u_2 \end{array}$ with $\omega(u_i, v_j) = \delta_{ij}$ and $\omega(u_1, u_2) = \omega(v_1, v_2) = 0$.

We also recall that Morita showed in [42] that the cokernel $D_2(H)/\,\mathrm{Im}(\tau_2)$ is a 2-torsion group. Yokomizo showed that whenever $g \ge 2$, its rank over $\mathbb{Z}_2$ is $(g-1)(2g+1)$ [58]. He gave an explicit basis of the cokernel using the computations of Morita. He also computed that the dimension of $D_2(H)/\tau_2([\mathcal{I}, \mathcal{I}])$, which is also a 2-torsion group, is $4g^2 - 1$. We shall use the computations of Morita and Yokomizo to prove the second statement in the following theorem. We now suppose that $g \ge 2$.

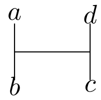
Theorem 2.4. *For any $g \ge 2$, the following homomorphisms*

$$\begin{cases} D_2'(H) & \xrightarrow{\mathrm{Tr}^{sym}} \quad \mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2) \end{cases}$$



$$\longmapsto \quad \omega(a,d)bc + \omega(a,c)bd + \omega(b,d)ac + \omega(b,c)ad$$

$$\begin{cases} D_2(H) & \xrightarrow{\mathrm{Tr}^{as}} \quad \mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2) \end{cases}$$



$$\longmapsto \quad \omega(a,d)b \wedge c + \omega(a,c)b \wedge d + \omega(b,d)a \wedge c + \omega(b,c)a \wedge d$$

$$a \odot b \longmapsto (1 + \omega(a,b))a \wedge b$$

are well-defined, $\mathrm{Sp}(H)$*-equivariant, and induce the following commutative diagram with exact rows :*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}/J_3 \arrow[r, "\tau_2"] & D_2(H) \arrow[r, "\mathrm{Tr}^{as}"] & \mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{[\mathcal{I},\mathcal{I}]}{J_3 \cap [\mathcal{I},\mathcal{I}]} \arrow[r, "\tau_2"] \arrow[u, hook] & D_2'(H) \arrow[r, "\mathrm{Tr}^{sym}"] \arrow[u] & \mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{2.3}$$

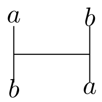
where the up arrow on the right is induced by the canonical projection $S^2(H/2H) \to \Lambda^2(H/2H)$.

Proof. Let us first show that the maps are well-defined. For $D_2(H)$ we use the presentation from Proposition 2.1, and for $D_2'(H)$ the presentation given by the definition of $\mathcal{A}_2(H)$. It is clear that the antisymmetry relation is sent to 0 since we are working modulo $\mathbb{Z}_2$. Multilinearity is also clear by multilinearity of the symplectic form. Hence, for the tree part, the only relation to check is the IHX relation:

$$\begin{aligned} IHX \longmapsto\ & \omega(a,d)bc + \omega(a,c)bd + \omega(b,d)ac + \omega(b,c)ad \\ & \omega(d,c)ab + \omega(d,b)ac + \omega(a,c)db + \omega(a,b)dc \\ & \omega(a,d)cb + \omega(a,b)cd + \omega(c,d)ab + \omega(c,b)ad \end{aligned}$$

which vanishes in $S^2(H/2H)$ and $\Lambda^2(H/2H)$. We have more relations to check for Tr^{as}. The only non-trivial ones are

$$(2(a \odot b) -$$

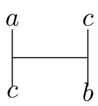


$$) \longmapsto 0 - 2(\omega(a,b)ab) = 0$$

and the one relating halves of symmetric trees with regular trees (Remark 2.2)

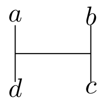
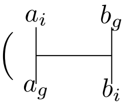
$$\begin{aligned} (a+b) \odot c - a \odot c - b \odot c \longmapsto\ & (1 + \omega((a+b),c))(a \wedge c + b \wedge c) \\ & + (1 + \omega(a,c))a \wedge c \\ & + (1 + \omega(b,c))b \wedge c \\ & = \omega(a,c)b \wedge c + \omega(b,c)a \wedge c \end{aligned}$$

which is also exactly the image of

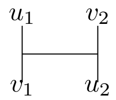


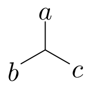
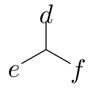
.

It is immediate that Tr^{sym} and Tr^{as} are $\mathrm{Sp}(H)$-equivariant, because ω is, by definition. It is also straightforward to check that they are onto $\mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2)$ and $\mathrm{Ker}(\omega :$

$\Lambda^2(H/2H) \to \mathbb{Z}_2)$, respectively. Indeed, over $\mathbb{Z}_2$ these kernels respectively have dimensions $\binom{2g}{2} + 2g - 1 = (g+1)(2g-1)$ and $\binom{2g}{2} - 1 = (g-1)(2g+1)$. We can easily give explicit generators for these spaces and show the desired surjectivity. The elements a_ib_j, a_ia_j, b_ib_j, a_ia_i, and b_ib_i (for $1 \le i, j \le g$), together with the elements $a_ib_i + a_gb_g$ (for $1 \le i < g$) are generators for $\mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2)$. The projection of these elements in $\Lambda^2(H/2H)$ gives generators for $\mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$. To produce elements mapping to one of these generators cd with $\omega(c, d) = 0$, we do the following. The genus being greater than or equal to two we can always suppose that there exists $a, b \in H$ with $\omega(a, b) = 1$, $\omega(a, c) = \omega(b, d) = 0$ and then $\mathrm{Tr}^{sym}\Big($  $\Big) = cd$. Also $\mathrm{Tr}^{sym}\Big($  $\Big) = a_ib_i + a_gb_g$. The same computations show that Tr^{as} is onto.

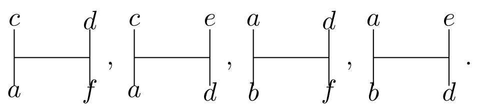
Also, we have from [26] a set of generators of $\mathrm{Im}(\tau_2)$ which is sent to 0 by the map Tr^{as}: for all (u, v) with $\omega(u, v) = 1$ and all (u_1, v_1, u_2, v_2) with $\omega(u_i, v_j) = \delta_{ij}$ and $\omega(u_1, u_2) = \omega(v_1, v_2) = 0$ we have

$$\mathrm{Tr}^{as}(u \odot v) = \mathrm{Tr}^{as}\Big($$  $$\Big) = 0.$$

Hence, $\mathrm{Im}(\tau_2)$ is contained in the kernel of Tr^{as}. For the image of $[\mathcal{I}, \mathcal{I}]$ by τ_2, it is known that the image is $[\Lambda^3 H, \Lambda^3 H]$ by the surjectivity of τ_1 and the fact that τ is a Lie algebra homomorphism. Recall that the bracket in $\mathcal{A}^t(H)$ is given by all the ways to contract external vertices using the symplectic form. Taking the bracket of two elements of form  and , we get 9 trees, which will be sent by Tr^{sym} to 36 terms in $S^2(H/2H)$. For example, the coefficient of the symmetric term ad is

$$\omega(b, e)\omega(c, f) + \omega(b, f)\omega(c, e) + \omega(c, e)\omega(b, f) + \omega(c, f)\omega(b, e)$$

coming from the trees



The above term vanishes, and we thus see that $\tau_2([\mathcal{I}, \mathcal{I}]) \subset \mathrm{Ker}(\mathrm{Tr}^{sym})$.

Finally, the dimensions of the targets of Tr^{as} and Tr^{sym} are equal to the ones given by Yokomizo in [58, Cor.2.2, Cor.3.2] for the dimensions of the cokernels of τ_2; i.e. $(g-1)(2g+1)$ for $D_2(H)/\mathrm{Im}(\tau_2)$ and $(g+1)(2g-1)$ for $D'_2(H)/(\tau_2([\mathcal{I}, \mathcal{I}]))$. This last dimension is not directly given by Yokomizo: it is obtained from the dimension of $D_2(H)/\tau_2([\mathcal{I}, \mathcal{I}])$ by removing $\binom{2g}{2}$, because of the exact sequence (2.1). □

Notice that the kernel of the canonical projection $\mathrm{Ker}(\omega : S^2(H/2H) \to \mathbb{Z}_2) \to \mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$ is isomorphic to $H/2H$, which can be mapped into $S^2(H/2H)$ in the obvious way. Hence, applying the snake lemma to the diagram (2.3) and using (2.1), we get the following description of the image of $\mathcal{K}/[\mathcal{I}, \mathcal{I}]$ under τ_2, i.e. the quotient $\mathcal{K}/([\mathcal{I}, \mathcal{I}] \cdot J_3)$.

Corollary 2.5. *There is a short exact sequence:*

$$0 \longrightarrow H/2H \longrightarrow \tau_2(\mathcal{K})/\tau_2([\mathcal{I}, \mathcal{I}]) \simeq \mathcal{K}/([\mathcal{I}, \mathcal{I}] \cdot J_3) \longrightarrow \Lambda^2(H/2H) \longrightarrow 0.$$

We can relate this short exact sequence to what we know about the abelianization of the Torelli group. For $g \ge 3$, the abelianization of $\mathcal{I}$ is well understood, thanks to the work of Johnson [27]. In [22], he built a homomorphism β (the so-called Birman-Craggs homomorphism) from the Torelli group to a 2-torsion abelian group $B_{\le 3}$ (where $B_{\le k}$ is the filtered space of Boolean polynomial functions of degree at most k on a certain $\mathbb{Z}_2$-affine space), such that the abelianization of $\mathcal{I}$ is isomorphic by (τ_1, β) to a fibered product:

$$\Lambda^3 H \times_{\Lambda^3(H/2H)} B_{\leq 3}.$$

This description implies that $\mathcal{K}/[\mathcal{I},\mathcal{I}]$ is isomorphic to $B_{\leq 2}$ via β. Johnson also claimed that $\beta(J_3) = B_0$ (see [25, p.178], [39, Rem. 3.21], and Remark 4.15 below for a proof). Hence, we have that $J_3/([\mathcal{I},\mathcal{I}] \cap J_3)$ is identified to $B_0 \simeq \mathbb{Z}_2$ by the map β. Therefore, we have $\frac{\mathcal{K}}{[\mathcal{I},\mathcal{I}]\cdot J_3} \overset{\beta}{\simeq} B_{\leq 2}/B_0$. Then, the short exact sequence of Corollary 2.5 fits into the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H/2H \arrow[r] \arrow[d] & \frac{\mathcal{K}}{[\mathcal{I},\mathcal{I}]\cdot J_3} \arrow[r] \arrow[d, "\beta"] & \Lambda^2(H/2H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{\leq 1}/B_0 \arrow[r] & B_{\leq 2}/B_0 \arrow[r] & B_{\leq 2}/B_{\leq 1} \arrow[r] & 0.
\end{tikzcd} \tag{2.4}$$

All vertical arrows are isomorphisms, the left one (respectively the right one) being the inverse of the formal first (respectively second) differential on $B_{\leq 1}$ (respectively $B_{\leq 2}$). We can recover a precise description for the horizontal map $H/2H \to \frac{\mathcal{K}}{[\mathcal{I},\mathcal{I}]\cdot J_3}$ by investigating in detail the connecting homomorphism arising from the snake lemma applied to diagram (2.3). The commutativity of the diagram is not trivial and can be deduced from [58, Prop. 3.3] or [39, Lemma 3.18].

3 Motivations for the study of $\mathcal{A} \cap J_2$

We are particularly interested in the relation of the handlebody group with the Johnson filtration. We explain our interest in this filtration and briefly review previous works on this subject.

Below, $\mathcal{V}(3)$ and $\mathcal{S}(3)$ denote respectively the set of all oriented 3-manifolds and all closed oriented homology 3-spheres up to orientation-preserving homeomorphisms. We firstly remind some facts about Heegaard splittings. Any 3-manifold can be divided (not in a unique way) in two handlebodies of same genus. Equivalently, any 3-manifold can be obtained by gluing two handlebodies together by a homeomorphism between their boundaries. Essentially, this homeomorphism specify where a set of meridians of the second handlebody should be sent on the boundary of the first one, yielding the notion of Heegaard diagrams.

The standard exemple is of course the sphere S^3, where one considers the standard handlebody V_g and glues a copy $-V_g$ with opposite orientation by a map sending its meridians to the curves β_i in Figure 1.1. Then we get for all g a splitting $S^3 := V_g \underset{\iota_g}{\cup} (-V_g)$, where ι_g is a certain orientation-preserving homeomorphism of Σ_g which can be defined by giving its action on π (see Section 6). Note that there is, up to isotopy, a unique Heegaard splitting of S^3 of genus g. We define $\mathcal{B}_{g,1} := \iota_g \mathcal{A}_{g,1} \iota_g^{-1}$. We denote by S^3_φ the 3-manifold $V_g \underset{\iota\circ\varphi}{\cup} (-V_g)$ for any element $\varphi \in \mathcal{M}_{g,1}$(we extend φ to Σ_g by the identity on the remaining disk). The map φ is called the *gluing map*. We also have stabilization maps $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, compatible with the other maps. When one composes the gluing map on the right, by an element of $\mathcal{B} = \mathcal{B}_{g,1}$ or to the left by an element of $\mathcal{A}$, the resulting manifold does not change up to homeomorphism. The following result is a refinement of the Reidemeister-Singer theorem [52, 54].

Theorem 3.1 (Reidemeister-Singer). *There is a bijection*

$$\begin{aligned} \lim_{g\to+\infty} \mathcal{A}_{g,1} \backslash \mathcal{M}_{g,1}/\mathcal{B}_{g,1} &\longrightarrow \mathcal{V}(3) \\ \varphi &\longmapsto S^3_\varphi \end{aligned}$$

which actually restricts to a bijection $\lim_{g\to+\infty} \mathcal{A}_{g,1} \backslash \mathcal{I}_{g,1}/\mathcal{B}_{g,1} \longrightarrow \mathcal{S}(3)$.

The second fact in Theorem 3.1 is written explicitly in [41]. One would expect that considering restrictions to deeper groups of the Johnson filtration would yield other topological conditions on the manifold, but this is not the case in low degrees for homology 3-spheres. We call a homology 3-sphere J_k-*equivalent* to S^3 if it ishomeomorphic to S^3_φ for some φ in J_k. More generally, we say that two 3-manifolds are J_k-*equivalent* if there exists a Heegaard splitting of the first one such that one can compose the gluing map by an element of J_k and get a Heegaard presentation for the second manifold. It is known that J_k-equivalence is an equivalence relation.

Morita [42] has shown that any two homology 3-spheres are J_2-equivalent. Pitsch [49] improved this result to J_3-equivalence. They both used the following.

Lemma 3.2. *Let $l \geq 1$. If for some genus g, we have $\mathrm{Im}(\tau_k) = \tau_k(\mathcal{A} \cap J_k) + \tau_k(\mathcal{B} \cap J_k)$ for all $k \leq l$ then any homology 3-sphere is J_{l+1}-equivalent to S^3.*

Another proof of the fact that any two homology 3-spheres are J_3-equivalent is given in [39, Theorem C]. Unfortunately, using Lemma 3.2 for $l = 3$ seems complicated: the computations could hardly be made by hand, and we do not know how to build all elements of J_3, whereas J_2 has well-known generators. Besides, as the result involves $\mathcal{B}$, this lemma only addresses the question of homology 3-spheres, hence manifolds at least J_1-equivalent to S^3. That is one reason why we want to describe in this paper $\tau_2(\mathcal{A} \cap J_2)$ by polarizing some computations in [49] and by introducing new arguments.

We also know some facts about the first term $\mathcal{IA} := \mathcal{A} \cap J_1$. A generating set was described by Omori in [48]. He gives the following theorem, where HBP stands for “homotopical bounding pair”, and a genus-h HBP-map is the composition $T_c \circ T_d^{-1}$ of two Dehn twists where c and d are essential simple closed curves cobounding a surface of genus h, cobounding an annulus in the handlebody, and not bounding disks in the handlebody.

Theorem 3.3 (Omori)**.** *For $g \geq 3$, $\mathcal{IA}_{g,1}$ is normally generated in $\mathcal{A}_{g,1}$ by a genus-1 HBP-map, and hence it is generated by genus-1 HBP-maps.*

It would be interesting to get the same kind of description for $\mathcal{A} \cap J_2$, but we only give in this paper its image by the second Johnson homomorphism, and formulate some questions (see Remark 5.8).

But our main motivation for the study of $\mathcal{A} \cap J_2$ comes from [34]. In this paper, Levine defines the *Lagrangian filtration* $(L_k)_{k\geq 1}$ which is a non-separating filtration of the mapping class group. It is not helpful to get an approximation of the mapping class group of the surface, but it is natural to study 3-manifolds presented through Heegaard splittings. The definition of this filtration depends on $\mathbb{A}$, the kernel of the projection p from π to $\pi' \simeq \pi/\mathbb{A}$. The Lagrangian subgroup A is the kernel of the projection $H \to H'$ which is the image of $\mathbb{A}$ under the projection from π to H. Also, whenever f is an element of the mapping class group, $f_* \in \mathrm{Sp}(H)$ stands for the action of f on H. We still write abusively f for the action of f on the fundamental group.

Definition 3.4. *The Lagrangian Torelli group is defined by*

$$\mathcal{I}^L := \{h \in \mathcal{M} \mid h_*(A) \subset A \text{ and } h_* \text{ is the identity on } A\}.$$

For $k \geq 1$, an element h of $\mathcal{M}$ belongs to L_k if it is in $\mathcal{I}^L$ and $p(h((\mathbb{A})) \subset \Gamma_{k+1}\pi'$.

Note that $L_1 = \mathcal{I}^L$. We remind the following fact from [34], describing the intersection of this filtration, which is non-empty.

Lemma 3.5. *$L_\infty := \bigcap_{k\geq 1} L_k$ coincides with $\mathcal{A} \cap L_1$.*

It is clear that $J_k \subset L_k$ for all $k \geq 1$.

Question 3.6. *Do we have $L_k = J_k \cdot L_\infty$ for all k ?*

This question can be approached inductively, which leads to the next lemma, given by Levine (see [34, Lemma 6.2] for a proof).

Lemma 3.7. *Suppose $L_k = J_k \cdot L_\infty$, then $L_{k+1} = J_{k+1} \cdot L_\infty$ if and only if*

$$\mathrm{Im}(\tau_k) \cap \mathrm{Ker}(D_k(H) \to D_k(H')) = \tau_k(\mathcal{A} \cap J_k).$$

It is shown in [34, Lemma 6.3] that $L_1 = J_1 \cdot L_\infty$. Furthermore, the following proposition, describing $\frac{\mathcal{A}\cap J_1}{\mathcal{A}\cap J_2}$, was given by Morita in [42, Lemma 2.5]:

Proposition 3.8. *We have* $\mathrm{Ker}(D_1(H) \to D_1(H')) = \tau_1(\mathcal{A} \cap J_1)$.

Recall that τ_1 is surjective, hence this proposition together with Lemma 3.7 implies that the answer to Levine's question is positive for $k = 1, 2$ (as explained in [34, Proposition 6.1]). As for the $k = 3$ case, the equality necessary for the induction step is no longer true, as will be shown in the next section:

$$\tau_2(\mathcal{A} \cap J_2) \subsetneq \mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H')).$$

Therefore the answer to Question 3.6 is "no" for $k = 3$.

4 The A-trace

In this section, we are still working with two "abstract" surfaces $\Sigma_{g,1} \subset \Sigma_g$ bounding a handlebody: $\Sigma_g = \partial V_g$. We consider the subgroup $\mathcal{A}$ of $\mathcal{M}$ consisting of elements of the mapping class group of Σ extending to V. The context differs from [49], where there are two handlebodies defined by a Heegaard splitting of S^3. In this paper, we wish to investigate about $\tau_2(\mathcal{A}\cap J_2)$. Considering that an element of $\mathcal{A}$ globally preserves $\mathbb{A}$, it is not hard to see that the k-th Johnson homomorphism sends an element of $\mathcal{A} \cap J_k$ to the sum of an element in $A \otimes \mathcal{L}_{k+1}(H)$ and an element in $H \otimes \mathrm{Ker}(\mathcal{L}_{k+1}(H) \to \mathcal{L}_{k+1}(H'))$. Hence we certainly have $\tau_k(\mathcal{A} \cap J_k) \subset \tau_k(J_k) \cap \mathrm{Ker}(D_k(H) \to D_k(H'))$. It is not easy to see what could be another necessary condition to be in $\tau_k(\mathcal{A} \cap J_k)$. Hence one could wonder, in relation to Question 3.6, whether $\tau_2(\mathcal{A} \cap J_2)$ coincides with $\mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H'))$. We show in this section that it is not the case.

4.1 Examples of elements of $\mathcal{A} \cap J_2$

Here, we describe three families of examples of elements in $\mathcal{A} \cap J_2$. We start by recalling some facts about the generation of $\mathcal{A}$ and J_2.

First, a Dehn twist along a simple closed curve belongs to the handlebody group if and only if this curve bounds a disk in the handlebody V. Such a meridional twist can also be performed half-way. Furthermore, if two curves δ and δ' cobound a properly embedded annulus in V, one can perform an annulus twist in the handlebody and see that $T_\delta T_{\delta'}^{-1}$ is an element of $\mathcal{A}$. The handlebody group is generated by meridional twists, meridional half-twists and annulus twists. See [19] for more details.

As for the second term in the Johnson filtration, it is generated by BSCC maps [26], i.e. Dehn twists along simple closed curves bounding in the surface. Also, it is a classical fact from [43] that $[J_k, J_l] \subset J_{k+l}$, so any commutator of two elements of the Torelli group are in the Johnson subgroup $\mathcal{K} = J_2$.

Knowing these facts we can build three families of elements in $\mathcal{A} \cap J_2$:

1. Dehn twists along bounding simple closed curves, which also bound disks in the handlebody.

2. Annulus twists along two simple closed curves which are both bounding subsurfaces in the surface (but not necessarily bounding disks in the handlebody).

3. Commutators of the group $\mathcal{A} \cap J_1$, the Torelli handlebody group, for which a generating system is recalled in Theorem 3.3.

We shall now define a map $\mathrm{Tr}^A : \mathrm{Ker}(D_2(H) \to D_2(H')) \to S^2(H')$, and show that it vanishes on all the image of $\mathcal{A} \cap J_2$ under τ_2.

4.2 Definition of the A-trace

We consider the following filtration on $D_k(H)$, which only depends on the Lagrangian subgroup A of H. For $-1 \leq l \leq k+1$, we set:

$$\mathcal{F}_l = \mathrm{Span}\left\langle \begin{array}{c}\text{expansion of trees with } k \text{ nodes (and halves of symmetric trees when } k \text{ is even)}\\ \text{with at least } l+1 \text{ leaves vanishing in } H'\end{array}\right\rangle.$$

Below, for $k=2$, we identify trees and their expansions (see Remark 2.2).

We consider the following diagram, where all vertical arrows are induced by the projection from H to H':

$$\begin{tikzcd} 0 \arrow[r] & A\otimes\mathcal{L}_{k+1}(H) \arrow[r] \arrow[d, two heads] & H\otimes\mathcal{L}_{k+1}(H) \arrow[r] \arrow[d, two heads, "p"] & H'\otimes\mathcal{L}_{k+1}(H) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & A\otimes\mathcal{L}_{k+1}(H') \arrow[r] & H\otimes\mathcal{L}_{k+1}(H') \arrow[r] & H'\otimes\mathcal{L}_{k+1}(H') \arrow[r] & 0. \end{tikzcd}$$

We claim that the following holds:

Lemma 4.1. *Set $K := \mathrm{Ker}(\mathcal{L}_{k+1}(H) \to \mathcal{L}_{k+1}(H'))$. We have:*

(i) $\mathcal{F}_{-1} = D_k(H)$

(ii) $\mathcal{F}_0 \subset D_k(H) \cap p^{-1}(A\otimes\mathcal{L}_{k+1}(H')) = \mathrm{Ker}(D_k(H) \to D_k(H'))$

(iii) $\mathcal{F}_1 \subset D_k(H) \cap \mathrm{Ker}(p) = D_k(H) \cap (H\otimes K)$

Proof. We have seen in Section 2 that expansions of half symmetric trees and expansions of trees lie in $D(H)$. If a tree with k leaves has at least one leaf in A, then after expanding the tree, there will be $k-1$ terms in which such leaf is involved in the free Lie algebra part. This $k-1$ terms will vanish after projecting on $\mathcal{L}(H')$. The remaining term will be a tensor product of the root vanishing in H' and some element in $\mathcal{L}(H)$. This shows that $p(\mathcal{F}_0) \subset A\otimes\mathcal{L}_{k+1}(H')$. If the tree has at least two leaves in A, then the expansion gives k terms such that the part in the free Lie algebra vanish in $\mathcal{L}(H')$. Hence $p(\mathcal{F}_1) = 0$. □

Remark 4.2. In fact all the inclusions in Lemma 4.1 are equalities, but we shall not need this.

Remark 4.3. The graded space associated with the filtration $(\mathcal{F}_l)_{-1\leq l\leq k+1}$ can be identified to the space $\mathcal{A}_k^t(A\oplus H')$ of tree-like Jacobi diagrams colored by $A\oplus H'$ with degree defined by the number of A-colored leaves shifted by 1 (the same space appears with a different grading in [56]).

Besides, the long exact sequence in relative homology for the handlebody

$$0 \longrightarrow H_2(V,\partial V;\mathbb{Z}) \longrightarrow H \simeq H_1(\partial V;\mathbb{Z}) \longrightarrow H' = H_1(V;\mathbb{Z}) \longrightarrow 0$$

gives a canonical isomorphism $H_2(V,\partial V;\mathbb{Z}) \simeq A$. Now, Poincaré-Lefschetz duality

$$H_2(V,\partial V;\mathbb{Z}) \simeq\ H^1(V;\mathbb{Z}) \simeq\ (H')^*$$

gives an intersection pairing

$$\omega' : A\otimes H' \longrightarrow \mathbb{Z}$$

which is also induced by ω in the obvious way. Then, by considering the injection i of $\mathcal{L}(H')$ in the tensor algebra $T(H')$, and the contraction $(\omega')^{1,2}$ of the first two tensors in $A\otimes T(H')$ by ω', we define the following map:

$$\mathrm{Tr}^A : \mathcal{F}_0 \xrightarrow{\ p\ } A\otimes\mathcal{L}_{k+1}(H') \xrightarrow{\ i\ } A\otimes T_{k+1}(H') \xrightarrow{(\omega')^{1,2}} T_k(H') \longrightarrow\!\!\!\!\!\rightarrow S^k(H').$$

Remark 4.4. The definition of the homomorphism Tr^A is inspired by the trace Tr defined by Morita in [44], but the reader should be aware that the following diagram does not commute:

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, hook] \arrow[d, "\mathrm{Tr}^A"] & D_k(H) \arrow[d, "\mathrm{Tr}"] \\
S^k(H') & S^k(H). \arrow[l, two heads]
\end{tikzcd}$$

The first thing to notice about Tr^A is that it vanishes on $\mathcal{F}_1$ as p already vanishes on this space. Hence it can be thought of as a map starting from $\mathcal{F}_0/\mathcal{F}_1$. Therefore, to compute this map, we can consider only trees with one leaf colored by A and the other leaves non-trivial in H'. The map Tr^A is thus defined on the graded space associated with the filtration $\mathcal{F}$, which corresponds to diagrams whose leaves are colored by A or H' (see Remark 4.3). On such a space, a direct computation shows that there is a practical way of computing Tr^A: take the leaf colored by A and consider all possible ways to contract it by ω' with the other leaves in H'. One gets a sum of oriented circles with leaves in H' (the orientation being given by drawing an arrow from the leaf in A to the other leaf). One can read this oriented diagram in $S^k(H')$, the inward leaves contributing with a minus sign. We now denote by x' the class in H' of an element x in H. We will also omit some tensor product notations when it is clear from context.

Example 4.5. For $a \in A$ and $c, d, e \in H$ we have:

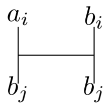
$$\mathrm{Tr}^A\left(\begin{array}{ccc} a & & e' \\ | & \text{---} & | \\ c' & & d' \end{array}\right) = \omega(a,e)d'c' - \omega(a,d)e'c'. \tag{4.1}$$

Indeed, in $S^2(H')$, we have:

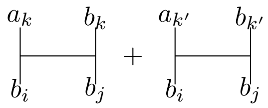
$$\begin{aligned}
(\omega')^{1,2} \circ i \circ p\left(\begin{array}{ccc} a & & e \\ | & \text{---} & | \\ c & & d \end{array}\right) &= (\omega')^{1,2} \circ i(a \otimes [[e', d'], c']) \\
&= (\omega')^{1,2}(ae'd'c' - ad'e'c' - ac'e'd' + ac'd'e') \\
&= \omega(a,c)(d'e' - e'd') - \omega(a,d)e'c' + \omega(a,e)d'c' \\
&= \omega(a,e)d'c' - \omega(a,d)e'c' \quad \in S^2(H').
\end{aligned}$$

Remark 4.6. It is worth noting that the restriction of the Johnson filtration to $\mathcal{A}$ is compatible with the conjugation by elements of $\mathcal{A}$. This induces a $\rho_0(\mathcal{A})$-module structure on the quotients $\mathcal{A} \cap J_k/\mathcal{A} \cap J_{k+1}$, where $\rho_0(\mathcal{A})$ is the image of $\mathcal{A}$ under the representation $\rho_0 : \mathcal{M} \to \mathrm{Sp}(H) \subset \mathrm{Aut}(H)$. The action of $\rho_0(\mathcal{A})$ on H induces an action on H' (and thus on $S^k(H')$). The map Tr^A is equivariant relatively to these actions.

We now focus on the case $k = 2$. One could check by direct computation that this map vanishes on the image by τ_2 of all elements of $\mathcal{A} \cap J_2$ of the three kinds described in Section 4.1. Instead of that, we will show in the next section that the map actually vanishes on the whole of $\tau_2(\mathcal{A} \cap J_2)$. Nevertheless, this map is not trivial on $\mathcal{F}_0 \cap \mathrm{Im}(\tau_2)$, as we shall now see. We fix a choice $(a_i, b_i)_{1 \le i \le g}$ of a symplectic basis for H, such that A is generated by the family $(a_i)_{1 \le i \le g}$. For instance, we consider the basis of H induced by a system of meridians and parallels $(\alpha_i, \beta_i)_{1 \le i \le g}$ as explained in Section 1. Let us define two families of elements in $\mathcal{F}_0$, depending of the previous choice:

$T_1^{ij} :=$ 

$i \neq j,$

$T_2^{kk',ij} :=$ 

$i \neq j, k \neq j, k' \neq j.$

Lemma 4.7. *The elements* T_1^{ij} *and* $T_2^{kk',ij}$ *belong to* $\mathrm{Im}(\tau_2)$ *and*

$$\mathrm{Tr}^A(T_1^{ij}) = b'_j b'_j$$
$$\mathrm{Tr}^A(T_2^{kk',ij}) = 2b'_i b'_j.$$

Proof. By definition of Tr^{as}, we get $\mathrm{Tr}^{as}(T_1^{ij}) = \omega(a_i, b_i) b_j \wedge b_j = 0 \in \Lambda^2(H/2H)$ and $\mathrm{Tr}^{as}(T_2^{kk',ij}) = 2\omega(a_k, b_k) b_i \wedge b_j = 0 \in \Lambda^2(H/2H)$. Therefore, by Theorem 2.4,we have $T_1^{ij}, T_2^{kk',ij} \in \mathrm{Im}(\tau_2)$. For the computation of Tr^A on T_1^{ij} and $T_2^{kk',ij}$, we use formula (4.1). $\square$

We embed $S^2(H')$ in $(H' \otimes H')^{\mathfrak{S}_2}$ by sending $h'_1 h'_2 \in S^2(H')$ to $h'_1 \otimes h'_2 + h'_2 \otimes h'_1 \in (H' \otimes H')^{\mathfrak{S}_2}$. It defines a restriction map $(H' \otimes H')^* \to S^2(H')^*$. Using the duality $A \simeq H'^*$ given by the map $a \mapsto \omega'(a, -)$, we then have an isomorphism from $(A \otimes A)^{\mathfrak{S}_2}$ to $(H'^* \otimes H'^*)^{\mathfrak{S}_2}$ which is a subspace of $(H'^* \otimes H'^*) \simeq (H' \otimes H')^*$. Hence we obtain a well-defined map r from $(A \otimes A)^{\mathfrak{S}_2}$ to $S^2(H')^*$:

$$r : (A \otimes A)^{\mathfrak{S}_2} \longrightarrow (H'^* \otimes H'^*)^{\mathfrak{S}_2} \longrightarrow (H'^* \otimes H'^*) \simeq (H' \otimes H')^* \longrightarrow S^2(H')^* \qquad (4.2)$$

Notice that $r(a_i \otimes a_i) = 2(b'_i b'_i)^*$ and $r(a_i \leftrightarrow a_j) = 2(b'_i b'_j)^*$, which shows that $r/2$ is well-defined, surjective, and hence is an isomorphism. We can now define $\widetilde{\mathrm{Tr}}^A$ as the bilinear map

$$\begin{array}{rcl} \widetilde{\mathrm{Tr}}^A : \mathcal{F}_0 \times (A \otimes A)^{\mathfrak{S}_2} & \longrightarrow & \mathbb{Z} \\ (T, s) & \longmapsto & \frac{1}{2} r(s)(\mathrm{Tr}^A(T)). \end{array}$$

We can also regard $\widetilde{\mathrm{Tr}}^A$ as a bilinear map: $\mathcal{F}_0/\mathcal{F}_1 \times (A \otimes A)^{\mathfrak{S}_2} \to \mathbb{Z}$.

Remark 4.8. Notice that Tr^A depends only on the choice of the Lagrangian subgroup $A \subset H$.

Remark 4.9. Since $r/2$ is an isomorphism, for any $T \in \mathcal{F}_0$, we have that $\widetilde{\mathrm{Tr}}^A(T, s) = 0$ for all $s \in (A \otimes A)^{\mathfrak{S}_2}$ if and only if $\mathrm{Tr}^A(T) = 0$.

4.3 Relating Tr^A with the Casson invariant

In this section, we review Morita's decomposition of the Casson invariant in [42] and use it to show the following:

Theorem 4.10. *The Casson invariant induces a map* $\mu : D_2(H) \times \mathcal{M} \to \mathbb{Z}$*, which is not bilinear. Its restriction to* $\mathcal{F}_0 \times \mathcal{I}^L$ *is bilinear and fits into a commutative diagram*

$$\begin{tikzcd} \mathcal{F}_0 \times \mathcal{I}^L \arrow[r, "\mu"] \arrow[d, two heads, "\downarrow \times \sigma"'] & \mathbb{Z} \\ \mathcal{F}_0/\mathcal{F}_1 \times (A \otimes A)^{\mathfrak{S}_2} \arrow[ur, bend right, "\widetilde{\mathrm{Tr}}^A"'] & \end{tikzcd}.$$

Furthermore, for any $T \in \tau_2(\mathcal{A} \cap J_2)$ *and any* $\varphi \in \mathcal{I}^L$*,* $\mu(T, \varphi) = 0$*. Consequently,* Tr^A *vanishes on* $\tau_2(\mathcal{A} \cap J_2)$*.*

The map σ is defined in the following way. Recall from Definition 3.4 that $\mathcal{I}^L$ is the Lagrangian Torelli group. For $f \in \mathcal{I}^L$ and $h \in H$, the difference $f_*(h) - h$ only depends on the class of h in H', and is in A because of the very definition of $\mathcal{I}^L$. Hence we get a map

$$\mathcal{I}^L \longrightarrow \mathrm{Hom}(H', A) \simeq (H')^* \otimes A \simeq A \otimes A$$

whose target restricts to $(A \otimes A)^{\mathfrak{S}_2}$ because of the symplectic condition. Hence we get a homomorphism $\sigma : \mathcal{I}^L \to (A \otimes A)^{\mathfrak{S}_2}$. Let us describe σ in terms of the symplectic basis described in Section 4.2. It is known that the canonical map from $\mathcal{M}$ to $\mathrm{Sp}(H)$ given by the action in homology is surjective. Using the symplectic basis, we identify $\mathrm{Sp}(H)$ with the group $\mathrm{Sp}(2g, \mathbb{Z})$ of matrices M such that $M^T J M = J$ where $J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$, i.e. matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D satisfy the following equations:

$$\begin{aligned} A^T D - C^T B &= Id \\ A^T C &= C^T A \\ D^T B &= B^T D. \end{aligned} \tag{4.3}$$

The image of $\mathcal{A}$ by $\mathcal{M} \to \mathrm{Sp}(2g, \mathbb{Z})$ consists of all matrices of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ where $A^T D = Id$ and $D^T B$ is symmetric (see [4, Lemma 2.2] or [20]). The image of $\mathcal{I}^L$ by $\mathcal{M} \to \mathrm{Sp}(2g, \mathbb{Z})$ consists of all matrices of type $\begin{pmatrix} Id & S \\ 0 & Id \end{pmatrix}$ where S is symmetric. The matrix S associated in this way to an element φ is the description of $\sigma(\varphi) \in \mathrm{Hom}(H', A)$ in the basis $(b'_i)_{1 \le i \le g}$ and $(a_i)_{1 \le i \le g}$. In particular, σ is surjective. The matrix $S = (S_{i,j})_{1 \le i,j \le g}$ actually corresponds to the symmetric tensor $\sum_{i,j=1}^{g} S_{i,j}(a_i \otimes a_j) \in (A \otimes A)^{\mathfrak{S}_2}$ (via the isomorphism $(A \otimes A) \simeq \mathrm{Hom}(H', A)$ given by ω').

Remark 4.11. The map σ can even be restricted to $\mathcal{I}^L \cap \mathcal{A}$, and will still be onto $(A \otimes A)^{\mathfrak{S}_2}$ (as a consequence of [19, Theorem 7.1]). This has a role to play in the proof of Theorem 4.10.

The following corollary is a consequence of Theorem 4.10.

Corollary 4.12. *For any $g \ge 2$, $\tau_2(\mathcal{A} \cap J_2)$ is strictly included in* $\mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H'))$.

Proof. We have exhibited in Lemma 4.7 elements of $\mathrm{Im}(\tau_2) \cap \mathrm{Ker}(D_2(H) \to D_2(H'))$ on which Tr^A does not vanish. ☐

The rest of this section is dedicated to the proof of Theorem 4.10, and in particular to the construction of μ.

Morita's decomposition of the Casson invariant

Let λ denote the Casson invariant. We consider a Heegaard embedding $j : \Sigma_{g,1} \to S^3$ of our abstract surface $\Sigma_{g,1}$ in S^3. This means that there exists a surface $\overline{\Sigma_g} \subset S^3$ such that $\overline{\Sigma_{g,1}} := j(\Sigma_{g,1})$ is obtained from $\overline{\Sigma_g}$ by removing a small open disk, and such that $\overline{\Sigma_g}$ splits S^3 in two handlebodies $\overline{V_g}$ and $\overline{W_g}$, which are called the "inner" and the "outer" handlebody, respectively. The orientation that j induces on $\overline{\Sigma_{g,1}}$ is supposed to coincide with the one induced by $\overline{V_g}$. Later, we will also suppose that j extends to V_g, and that $j(V_g)$ is the "inner" handlebody $\overline{V_g}$ in the splitting of S^3. Then, the handlebody group $\mathcal{A} = \mathcal{A}_{g,1}$ is identified through j to the mapping class group of $\overline{V_g}$ relative to the disk $\overline{\Sigma_g} \setminus \overline{\Sigma_{g,1}}$.

For every $\varphi \in \mathcal{I}$, one can define the 3-manifold obtained by cutting S^3 along the image of j and gluing back the two handlebodies using the mapping cylinder of φ. In [42], Morita defines $\lambda_j(\varphi)$ as the Casson invariant of the resulting homology 3-sphere $S^3(j,\varphi)$, yielding a map:

$$\begin{aligned}\lambda_j : \mathcal{I} &\longrightarrow \mathbb{Z}\\ \varphi &\longmapsto \lambda(S^3(j,\varphi)).\end{aligned}$$

The above map is *not* a homomorphism, nevertheless Morita showed that its restriction to $\mathcal{K} = J_2$ is a homomorphism. He also showed that it can be expressed as the sum of two homomorphisms. We review their definitions, and refer the reader to [42] or [39] for more details. The first one, d, is called the "core of the Casson invariant" and is independent of j. The second one is not, but is completely determined by the second Johnson homomorphism. Our notation conventions differ slightly from the original ones given in [42], the content being exactly the same.

We do not need to give a precise definition for the map $d : \mathcal{K} \to \mathbb{Z}$, we only need to recall the following facts. Johnson showed [26] that $\mathcal{K}$ is generated by Dehn twists along bounding simple closed curves and Morita proved in [42] that

$$d(T_\gamma) = 4h(h-1)$$

whenever γ is a simple closed curve bounding a subsurface of genus h.

As for the second map, we need to fully review its definition. Let $\mathcal{C}$ be the unital, commutative, and associative algebra with generators $l(u,v)$ for all u and v in H and subject to the relations:

$$\begin{aligned} l\left(n\cdot u + n'\cdot u', v\right) &= n\cdot l(u,v) + n'\cdot l\left(u',v\right)\\ l(v,u) &= l(u,v) + \omega(u,v),\end{aligned}$$

for all $u, u', v \in H$ and for all $n, n' \in \mathbb{Z}$. We denote by lk the linking number in S^3. Let $\varepsilon_j : \mathcal{C} \to \mathbb{Z}$ be the unique algebra homomorphism such that:

$$\varepsilon_j(l(u,v)) := \mathrm{lk}\left(j_*(u), j_*^+(v)\right)$$

where j^+ is an embedding parallel to j, meaning that the image of j^+ is obtained by pushing the image of j towards the outer handlebody. We fix a set of meridians and parallels $(\overline{\alpha},\overline{\beta})$ for the surface $\overline{\Sigma_{g,1}}$ (see Figure 1.3). This defines a system (α,β) of meridians and parallels for Σ given by $\alpha := j^{-1}(\overline{\alpha})$ and $\beta := j^{-1}(\overline{\beta})$. For any $1 \leq i \leq g$, the homology classes of α_i and β_i are denoted respectively by a_i and b_i.

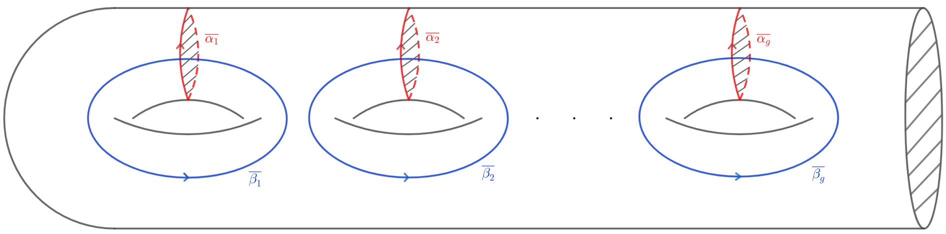


Figure 1.3: A system of meridians and parallels on $\overline{\Sigma_{g,1}} \subset \overline{V_g} \subset S^3$.

Remark 4.13. Considering that $\mathrm{lk}(j_*(a_i), j_*^+(b_j)) = 0$ and $\mathrm{lk}(j_*(b_i), j_*^+(a_j)) = \delta_{ij}$, the matrix associated to the bilinear mapping $\mathrm{lk}(j_*(-), j_*^+(-)) : H \times H \to \mathbb{Z}$ is $\begin{pmatrix} 0 & 0\\ Id & 0\end{pmatrix}$.

Morita also defines a map $\theta : \left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2} \to \mathcal{C}$ determined by:

$$\begin{aligned}&\theta((u\wedge v)\otimes(u\wedge v)) := l(u,u)l(v,v) - l(u,v)l(v,u)\\ &\theta((a\wedge b)\leftrightarrow(c\wedge d)) := l(a,c)l(b,d) - l(a,d)l(b,c) - l(d,a)l(c,b) + l(c,a)l(d,b).\end{aligned}$$

He then defines a map $\bar{d} : \left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2} \to \mathbb{Z}$ by:

$$\begin{aligned}
&\bar{d}((u \wedge v) \otimes (u \wedge v)) := 0 \\
&\bar{d}((a \wedge b) \leftrightarrow (c \wedge d)) := \omega(a,b)\omega(c,d) - \omega(a,c)\omega(b,d) + \omega(a,d)\omega(b,c),
\end{aligned}$$

so that $\overline{q_j} := \varepsilon_j \circ \theta + \frac{1}{3}\bar{d}$ vanishes on $\Lambda^4 H \subset \left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2}$. Hence, it is defined on $D_2(H)$ (see diagram (2.2)). Finally, $q_j := -\overline{q_j} \circ \tau_2 : \mathcal{K} \to \mathbb{Q}$ is such that

$$-\lambda_j = \frac{1}{24} d + q_j : \mathcal{K} \to \mathbb{Z}. \tag{4.4}$$

Here comes the key point of the definition of the map μ:

Lemma 4.14. *For any Heegaard embedding j, there is a well defined map $\mu_j : D_2(H) \times \mathcal{M} \to \mathbb{Z}$ given by*

$$\mu_j([T], \varphi) := (\varepsilon_j - \varepsilon_{j\circ\varphi}) \circ \theta(T)$$

for $\varphi \in \mathcal{M}$ and $T \in (\Lambda^2 H \otimes \Lambda^2 H)^{\mathfrak{S}_2}$ (here $[T]$ denotes the class of T in $D_2(H)$). This map is linear in its left argument and it satisfies:

$$(\lambda_j - \lambda_{j\circ\varphi})(h) = \mu_j(\tau_2(h), \varphi) \tag{4.5}$$

for all $\varphi \in \mathcal{M}$ and $h \in \mathcal{K}$.

Proof. For any $\varphi \in \mathcal{M}$ we have, applying (4.4) both to j and $j \circ \varphi$, that $-(\lambda_j - \lambda_{j\circ\varphi}) = q_j - q_{j\circ\varphi}$. This last part depends only on the second Johnson homomorphism. More precisely, by looking at the definition of $\overline{q_j}$ and $\overline{q}_{j\circ\varphi}$, one can compute that for any element T in $\left(\Lambda^2 H \otimes \Lambda^2 H\right)^{\mathfrak{S}_2}$, whose class in $D_2(H)$ is $[T]$:

$$(\overline{q_j} - \overline{q}_{j\circ\varphi})([T]) = (\varepsilon_j - \varepsilon_{j\circ\varphi}) \circ \theta(T).$$

The result is then straightforward. □

Remark 4.15. Lemma 4.14 shows as explained by Morita in [42, Rem. 6.3], that the homomorphism τ_2 contains all the information about the differences $(\lambda_j - \lambda_{j\circ\varphi})$ with $\varphi \in \mathcal{M}$. Furthermore, when reducing equation (4.5) mod 2, one can deduce that $\beta(J_3) \subset B_0$ as claimed by Johnson in [25, p.178]. Indeed for any $f \in J_3$, and for any $\varphi \in \mathcal{M}$, we have that $\beta(f)(\omega_j) - \beta(f)(\omega_{j\circ\varphi}) = \mu_j(\tau_2(f), \varphi) = 0$, where ω_j and $\omega_{j\circ\varphi}$ are the Sp-quadratic forms defined by the Heegaard embeddings j and $j \circ \varphi$ respectively (see [22] for more details). Hence, $\beta(f)$ is fixed by the action of $\mathrm{Sp}(2g, \mathbb{Z})$. Furthermore, it is not hard to prove from [22] and Lemma 2.3 that there exists a map d^2, with kernel $B_{\leq 1}$ (giving the second formal differential of boolean quadratic functions), and a commutative diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\beta"] \arrow[d, "\tau_2"] & B_{\leq 2} \arrow[d, "d^2"] \\
D_2(H) \arrow[r] & \Lambda^2(H \otimes \mathbb{Z}_2).
\end{tikzcd}$$

This implies that $\beta(J_3) \subset B_{\leq 1}$ which in turn implies that $\beta(f)$ is a constant. Indeed, there is no non-trivial $\mathrm{Sp}(2g, \mathbb{Z})$-invariant boolean affine function on the set of Sp-quadratic forms.

The application μ

We now suppose that j extends to the handlebody V, in such a way that $j(V) = \overline{V}$ is the inner handlebody of the Heegaard splitting of S^3. Once such a j is fixed we simply define $\mu := \mu_j$, where μ_j is defined in Lemma 4.14. We need first the following lemma:

Lemma 4.16. *For any element $T \in \tau_2(\mathcal{A} \cap J_2)$, the map $\mu(T, -)$ vanishes on $\mathcal{A}$.*

Proof. If we choose φ to be in $\mathcal{A}$ and ψ to be in $\mathcal{A} \cap J_2$, we have that $(\lambda_j - \lambda_{j\circ\varphi})(\psi) = \lambda(S^3) - \lambda(S^3) = 0$. Indeed, both $j \circ \psi \circ j^{-1}$ and $(j\circ\varphi)\circ\psi\circ(j\circ\varphi)^{-1}$ extend to the handlebody $\overline{V}$. Hence $\mu(\tau_2(\psi), \varphi) = 0$, by equation (4.5). □

Remark that whenever φ is not in $\mathcal{A}$, then $j \circ \varphi$ does not extend to an embedding on V, and the conclusions of Lemma 4.16 may not be true. Also the fact that j extends to V is needed.

We now compute the map μ explicitly. Let $\varphi \in \mathcal{M}$ be such that $\varphi_*(A) \subset A$. Notice first that

$$\varepsilon_{j\circ\varphi}(l(u,v)) = \mathrm{lk}((j\circ\varphi)_*(u), (j\circ\varphi)_*^+(v)) = \varepsilon_j(l(\varphi_*(u), \varphi_*(v))) \tag{4.6}$$

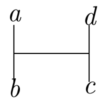
for any $u, v \in H$. We use our chosen basis for H (the one defined by j), and write the action of φ as a matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Then the matrix of the bilinear map $\mathrm{lk}((j\circ\varphi)_*(-), (j\circ\varphi)_*^+(-))$ is given by :

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Id & D^T B \end{pmatrix} \tag{4.7}$$

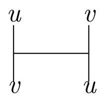
where $S := D^T B$ is a symmetric matrix. We now suppose that $\varphi \in \mathcal{I}^L$, and denote ω_δ and ω_S the pairings $H \times H \to \mathbb{Z}$ corresponding to the matrices $\begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$ through our choice of basis for H, where S is the matrix describing $\sigma(\varphi) \in (A \otimes A)^{\mathfrak{S}_2}$ in the basis $(a_1, \dots, a_g)$. Note that these definitions depend on the choice of Heegaard embedding j.

We then have the following:

Lemma 4.17. *For any a, b, c, d, u, v in H and for any $\varphi \in \mathcal{I}^L$, we have*

$-\mu\Big($  $, \varphi\Big)$

$$\begin{aligned} &= \omega_S(a,c)\omega_S(b,d) + \omega_S(c,a)\omega_S(d,b) \\ &\quad - \omega_S(a,d)\omega_S(b,c) - \omega_S(d,a)\omega_S(c,b) \\ &\quad + \omega_S(a,c)\omega_\delta(b,d) + \omega_S(c,a)\omega_\delta(d,b) \\ &\quad - \omega_S(a,d)\omega_\delta(b,c) - \omega_S(d,a)\omega_\delta(c,b) \\ &\quad + \omega_\delta(a,c)\omega_S(b,d) + \omega_\delta(c,a)\omega_S(d,b) \\ &\quad - \omega_\delta(a,d)\omega_S(b,c) - \omega_\delta(d,a)\omega_S(c,b) \end{aligned}$$

$-\mu\Big(\frac{1}{2}$  $, \varphi\Big)$

$$\begin{aligned} &= \omega_S(u,u)\omega_S(v,v) - \omega_S(u,v)\omega_S(v,u) \\ &\quad + \omega_S(u,u)\omega_\delta(v,v) - \omega_S(u,v)\omega_\delta(v,u) \\ &\quad + \omega_\delta(u,u)\omega_S(v,v) - \omega_\delta(u,v)\omega_S(v,u) \end{aligned}$$

where S is the matrix describing $\sigma(\varphi)$ in the basis $(a_1, \dots, a_g)$.

Proof. The result follows from the definition of $\mu := \mu_j$, from the definition of θ and from:

$$\begin{aligned} (\varepsilon_{j\circ\varphi} - \varepsilon_j)(l(a,c)l(b,d)) &= \varepsilon_{j\circ\varphi}(l(a,c)l(b,d)) - \varepsilon_j(l(a,c)l(b,d)) \\ &= \varepsilon_{j\circ\varphi}(l(a,c))\ \varepsilon_{j\circ\varphi}l(b,d)) - \varepsilon_j(l(a,c))\ \varepsilon_j(l(b,d)) \\ &= (\omega_S + \omega_\delta)(a,c)(\omega_S + \omega_\delta)(b,d) - \omega_\delta(a,c)\omega_\delta(b,d) \\ &= \omega_S(a,c)\omega_S(b,d) + \omega_S(a,c)\omega_\delta(b,d) + \omega_\delta(a,c)\omega_S(b,d) \end{aligned}$$

where the third equality is obtained by (4.6) and (4.7). □

We can express this is in a very compact way. Once again we define a trace-like operator Tr^{ω_S}:

$$\mathrm{Tr}^{\omega_S} : D_2(H) \longrightarrow H \otimes \mathcal{L}_3(H) \xrightarrow{\ i\ } T_4(H) \xrightarrow{(\omega_S)^{1,2}} T_2(H)$$

where $(\omega_S)^{1,2}$ is the contraction of the first two tensors by ω_S. We now need the following lemma.

Lemma 4.18. *For any $a, b, c, d, u, v \in H$, and for any $\varphi \in \mathcal{I}^L$, we have*

$$\begin{aligned}
\mathrm{Tr}^{\omega_S}\Big(\begin{array}{c@{}c@{}c} a & & d \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ b & & c \end{array} \Big) &= \omega_S(a,d)(b \otimes c + c \otimes b) + \omega_S(b,c)(a \otimes d + d \otimes a) \\
&\quad - \omega_S(a,c)(b \otimes d + d \otimes b) - \omega_S(b,d)(a \otimes c + c \otimes a) \\
\mathrm{Tr}^{\omega_S}\Big(\frac{1}{2} \begin{array}{c@{}c@{}c} u & & v \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ v & & u \end{array} \Big) &= \omega_S(u,v)(u \otimes v + v \otimes u) - \omega_S(u,u) v \otimes v - \omega_S(v,v) u \otimes u
\end{aligned}$$

where S is the matrix describing $\sigma(\varphi)$ in the basis $(a_1, \ldots, a_g)$.

Corollary 4.19. *For any $\varphi \in \mathcal{I}^L$, we have*

$$(\frac{1}{2}\omega_S + \omega_\delta) \circ \mathrm{Tr}^{\omega_S} = \mu(-, \varphi)$$

where S is the matrix describing $\sigma(\varphi)$ in the basis $(a_1, \ldots, a_g)$.

Proof of Corollary 4.19. This is a direct computation, together with the fact that the matrix S is symmetric. Set $y := (\frac{1}{2}\omega_S + \omega_\delta) \circ Tr^{\omega_S}\Big(\begin{array}{c@{}c@{}c} a & & d \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ b & & c \end{array} \Big)$, then:

$$\begin{aligned}
y &= (\frac{1}{2}\omega_S + \omega_\delta)(\omega_S(a,d)(b \otimes c + c \otimes b) + \omega_S(b,c)(a \otimes d + d \otimes a) \\
&\quad - \omega_S(a,c)(b \otimes d + d \otimes b) - \omega_S(b,d)(a \otimes c + c \otimes a)) \\
&= \omega_S(a,d)\omega_S(b,c) + \omega_S(b,c)\omega_S(a,d) \\
&\quad - \omega_S(a,c)\omega_S(b,d) - \omega_S(b,d)\omega_S(a,c) \\
&\quad + \omega_\delta(\omega_S(a,d)(b \otimes c + c \otimes b) + \omega_S(b,c)(a \otimes d + d \otimes a) \\
&\quad - \omega_S(a,c)(b \otimes d + d \otimes b) - \omega_S(b,d)(a \otimes c + c \otimes a)) \\
&= \mu\Big(\begin{array}{c@{}c@{}c} a & & d \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ b & & c \end{array}, \varphi \Big)
\end{aligned}$$

where the last equality comes from Lemma 4.17. The equality for halves of symmetric trees can be checked in a similar way. □

Remark 4.20. It is easy to see that the map μ is not linear in the second variable. However, since $\omega_S \circ \mathrm{Tr}^{\omega_S}$ clearly vanishes on $\mathcal{F}_0$, we have that the restriction $\mu_{|\mathcal{F}_0 \times \mathcal{I}^L}$ is bilinear, as stated in Theorem 4.10.

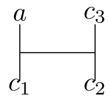
We now prove Theorem 4.10.

Proof of Theorem 4.10. Recall that Tr^A vanishes on $\mathcal{F}_1$. So does μ. Indeed, by Corollary 4.19 and Remark 4.20, we have $\omega_\delta \circ \mathrm{Tr}^{\omega_S} = \mu(-, \varphi)$ for any $\varphi \in \mathcal{I}^L$, with $S = \sigma(\varphi)$. Also, by Lemma 4.18, for any $x_1, x_2 \in A$ and for any $c, d \in H$:

$$\begin{aligned}
\omega_\delta\Big(\mathrm{Tr}^{\omega_S}\Big(\begin{array}{c@{}c@{}c} x_1 & & d \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ x_2 & & c \end{array} \Big)\Big) &= \omega_\delta(0) = 0, \\
\omega_\delta\Big(\mathrm{Tr}^{\omega_S}\Big(\begin{array}{c@{}c@{}c} x_1 & & x_2 \\ \Big| & \rule[0.5ex]{2.5em}{0.4pt} & \Big| \\ c & & d \end{array} \Big)\Big) &= \omega_\delta(\omega_S(c,d)(x_1 \otimes x_2 + x_2 \otimes x_1)) = 0.
\end{aligned}$$

For any symmetric tree T' in $\mathcal{F}_1$, $\mathrm{Tr}^{\omega_S}(\frac{1}{2}T') = \frac{1}{2}\mathrm{Tr}^{\omega_S}(T') = 0$.

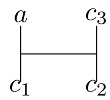
Hence, it is sufficient to compute the maps on trees with only one leaf colored by an element of A. Any half of a symmetric tree in $\mathcal{F}_0$ is actually in $\mathcal{F}_1$, and for any $a \in A$ and $c_1, c_2, c_3 \in H$, we have, once again applying Lemma 4.18:

$\omega_\delta\Big(\mathrm{Tr}^{\omega_S}\Big($ 

$\Big)\Big)$

$$\begin{aligned} &= \omega_S(c_1,c_2)\omega_\delta(a\otimes c_3 + c_3 \otimes a) \\ &\quad - \omega_S(c_1,c_3)\omega_\delta(a\otimes c_2 + c_2\otimes a) \\ &= \omega_S(c_1,c_2)\omega_\delta(c_3,a) - \omega_S(c_1,c_3)\omega_\delta(c_2,a) \\ &= \omega_S(c_1,c_2)\omega'(a,{c_3}') - \omega_S(c_1,c_3)\omega'(a,{c_2}') \\ &= \omega_S(c_1,c_2)\omega(a,c_3) - \omega_S(c_1,c_3)\omega(a,c_2), \end{aligned}$$

and, if $s := \sum_{i,j=1}^{g} S_{i,j}(a_i \otimes a_j)$ is the element of $(A\otimes A)^{\mathfrak{S}_2}$ corresponding to S under the isomorphism $(A \otimes A) \simeq \mathrm{Hom}(H', A)$ given by ω':

$\widetilde{\mathrm{Tr}}^{A}\Big($ 

$, S\Big)$

$$\begin{aligned} &= \frac{1}{2} r(s)(\omega(a,c_3)(c_2c_1) - \omega(a,c_2)(c_3c_1)) \\ &= \omega_S(c_1,c_2)\omega(a,c_3) - \omega_S(c_1,c_3)\omega(a,c_2) \end{aligned}$$

as one can see by using equations (4.1) and (4.2). Indeed, s yields after dualization an element $\sum_{i,j=1}^{g} S_{i,j}(b_i'^* \otimes b_j'^*) \in (H'^* \otimes H'^*)$. This corresponds exactly to the element of $(H' \otimes H')^*$ induced by ω_S. In other words, $r(s)(c_2c_1) = \omega_S(c_2 \otimes c_1 + c_2 \otimes c_1) = 2\omega_S(c_1,c_2)$.

From these equalities, and Corollary 4.19, we can conclude that for all $T \in \mathcal{F}_0$, and $\varphi \in \mathcal{I}^L$, $\widetilde{\mathrm{Tr}}^{A}(T, \sigma(\varphi)) = \mu(T,\varphi)$. To conclude, if a tree T is in $\tau_2(\mathcal{A} \cap J_2)$, for any $\varphi \in \mathcal{I}^L \cap \mathcal{A}$, $\mu(T,\varphi) = 0$ by Lemma 4.16. By Remark 4.11, it is the same as saying that $\mu(T,\varphi) = 0$ for any $\varphi \in \mathcal{I}^L$. Remark 4.9 then implies that $\mathrm{Tr}^A(T) = 0$. The map Tr^A then vanishes on $\tau_2(\mathcal{A} \cap J_2)$. $\square$

Remark 4.21. Note that, while the map $\mu = \mu_j : D_2(H) \times \mathcal{M} \to \mathbb{Z}$ depends on the choice of the Heegaard embedding $j : \Sigma \to S^3$ (extending to V), its restriction to $\mathcal{F}_0 \times \mathcal{I}^L$ only depends on the Lagrangian $A \subset H$, as a consequence of Theorem 4.10.

5 Computing $\tau_2(\mathcal{A} \cap J_2)$

In this section we compute explicitly the image of $\mathcal{A} \cap J_2$ under τ_2. We are going to show that it is detected by $\mathrm{Tr}^A : \mathcal{F}_0 \to S^2(H')$. The hypothesis on the genus in the next result could probably be improved, but it would add a lot of special cases to the computations below.

Theorem 5.1. *For $g \geq 4$, we have $\tau_2(\mathcal{A} \cap J_2) = \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^{as}) = \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Im}(\tau_2)$.*

The inclusion $\tau_2(\mathcal{A} \cap J_2) \subset \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^{as})$ follows from Theorems 2.4 and 4.10. Recall that the elements in $D_2(H)$ are expansions of trees and halves of symmetric trees, as explained in Section 2. As before, identify a tree with 4 leaves with its expansion in $D_2(H)$. A symplectic basis (a_i, b_i) of H is chosen so that the a_i's generate the Lagrangian subgroup $A \subset H$ which is involved in the definition of Tr^A. We denote by B the Lagrangian generated by the b_i's. Now, notice that trees with $0 \leq k \leq 4$ leaves colored by elements among the a_i's and $4 - k$ colored by elements among the b_i's give, after projection, generators of the quotient $\mathcal{F}_{k-1}/\mathcal{F}_k$. We call such trees *trees of type k*. For example $\mathcal{F}_0/\mathcal{F}_1$ is generated by trees of type 1. Also an element of $\mathcal{F}_0$ can be written as a linear combination of elements of type 1 to 4.

We will use several times the following lemma.

Lemma 5.2. *Let*

$$0 \longrightarrow K \longrightarrow F \longrightarrow C \longrightarrow 0$$

be a short exact sequence of finitely generated abelian groups. We suppose that C is a free abelian group or a $\mathbb{Z}_2$-vector space, and that we have a generating family $(f_i)_{0\le i\le f}$ of F and a basis $(c_j)_{0\le j\le c}$ for C such that (f_i) consists of elements of K and lifts of elements of (c_j). Then, K is generated by

$$(\{f_i \mid 0 \le i \le f\} \cap K) \cup (\{f_i - f_j \mid 0 \le i < j \le f\} \cap K)$$

if C is free abelian and by

$$(\{f_i \mid 0 \le i \le f\} \cap K) \cup (\{f_i - f_j \mid 0 \le i < j \le f\} \cap K) \cup \{2f_i \mid 0 \le i \le f\}$$

if C is a $\mathbb{Z}_2$-vector space.

Proof. Let us suppose that C is a free abelian group. Let $p : F \to C$ denote the projection. By our hypothesis, for every $j \in \{0, \dots, c\}$, c_j has a lift among the f_i's. We denote $(k_i)_{0\le i\le \kappa}$ the elements among the f_i's that are in K and $(l_i)_{0\le i\le l}$ the other ones. Now, for any $x \in K$, we can write $x = \sum\limits_{i\le\kappa} \lambda_i k_i + \sum\limits_{j\le l} \mu_j l_j$, with $\lambda_i, \mu_j \in \mathbb{Z}$. We thus have $(\sum\limits_{p(l_j)=c_i} \mu_j) c_i = 0$, hence $\sum\limits_{p(l_j)=c_i} \mu_j = 0$ for every $i \in \{0, \dots, c\}$. Fix $1 \le i \le c$ and consider the l_j's such that $p(l_j) = c_i$, and renumber in a simpler way the elements (denoted by μ' and l' after renumbering) from 0 to n_i such that: $\sum\limits_{p(l_j)=c_i} \mu_j l_j = \sum\limits_{j=0}^{n_i} \mu'_j l'_j = \sum\limits_{j=0}^{n_i} \sum\limits_{s=0}^{j-1} \mu'_j (l'_{s+1} - l'_s)$, where we used that $\sum\limits_{j=0}^{n_i} \mu'_j = 0$. This computation allows us to write x as a linear combination of the k_i's and elements $l_i - l_j$ such that $p(l_i) = p(l_j)$. For the case where C is a $\mathbb{Z}_2$-vector space, the proof can be easily adapted. $\square$

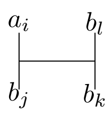
Remark 5.3. The generating family provided by Lemma 5.2 is far from being optimal. For example, given $x, y, z \in F$ with the same image in C, one does not need to take $(x - y)$, $(x - z)$ and $(y - z)$, as the last one is a linear combination of the other two.

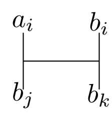
Let T be in $\mathcal{F}_0 \cap \mathrm{Im}(\tau_2)$ and write it as $T_1 + T_{\ge 2}$ where T_1 and $T_{\ge 2}$ are written as some linear combinations of respectively type 1 elements and type 2 to 4 elements. We suppose that $\mathrm{Tr}^A(T_1 + T_{\ge 2}) = 0$ i.e. $\mathrm{Tr}^A(T_1) = 0$. Using the special elements of $\mathcal{A} \cap J_2$ described in Section 4.1 we are going to show that $T \in \tau_2(\mathcal{A} \cap J_2)$.

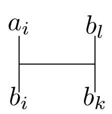
From now on, we refer to the element in $\tau_2(\mathcal{A} \cap J_2)$ as *realizable* elements. We also say that a tree of type 0 to 4 has a *contraction* when at least two of its leaves can be paired non-trivially through ω. Some of the computations below are inspired by computations in [42] and [49].

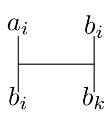
For the sake of preciseness, we emphasize that for two submodules P and Q of a module V, the notation $P \wedge Q$ stands for the image of $P \otimes Q$ under the projection $V \otimes V \to \Lambda^2(V)$. From Section 2 we have that Tr^{as} vanishes on elements of $\mathrm{Im}(\tau_2)$. On an element of type 1 this trace takes value in $(B \wedge B) \otimes \mathbb{Z}_2$ and on other types it takes values in $(A \wedge H) \otimes \mathbb{Z}_2$. Hence, using the decomposition $\Lambda^2 H = (B \wedge B) \oplus (A \wedge H)$, it is clear that $\mathrm{Tr}^{as}(T_1 + T_{\ge 2}) = 0$ implies $\mathrm{Tr}^{as}(T_1) = \mathrm{Tr}^{as}(T_{\ge 2}) = 0$. In the sequel, we shall prove that $T_1 \in \tau_2(\mathcal{A} \cap J_2)$ and, next, we will show that $T_{\ge 2} \in \tau_2(\mathcal{A} \cap J_2)$ using the fact that $\mathrm{Tr}^{as}(T_{\ge 2}) = 0$.

In terms of the symplectic basis (a_i, b_i) of H, the elements of type 1 can be of the following form (up to sign):

$1_{i,j,k,l} :=$ 

$2_{i,j,k} :=$ 

$3_{i,k,l} :=$ 

$4_{i,k} :=$ 

with i different from j, k and l.

Proposition 5.4. *Set* $N := \operatorname{Ker}(\mathrm{Tr}^A : \mathrm{Span}_{\mathbb{Z}}\{\textit{type 1 elements}\} \to S^2(H'))$. *Then* N *is generated by elements of type* ①, ③ *and*

$$②_{i,j,j} - ②_{i',j,j}\ ;\quad ②_{i,j,k} - ②_{i',j,k}\ ;\quad ②_{i,j,k} - ②_{i',k,j}\ ;$$
$$②_{i,j,k} - ④_{j,k}\ ;\quad ②_{i,j,k} - ④_{k,j}\ ;\quad ④_{i,k} - ④_{k,i}\ ;$$

where i *and* i' *must be different from* j, k *and* l, *and* $j \neq k$.

Proof. It is a consequence of Lemma 5.2 applied to the short exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Span}_{\mathbb{Z}}\{\textit{type 1 elements}\} \longrightarrow S^2(H') \longrightarrow 0$$

after computing that

$$\begin{aligned} \mathrm{Tr}^A(①_{i,j,k,l}) &= 0\\ \mathrm{Tr}^A(③_{i,k,l}) &= 0\\ \mathrm{Tr}^A(②_{i,j,k}) &= +b'_k b'_j\\ \mathrm{Tr}^A(④_{i,k}) &= +b'_k b'_i. \end{aligned}$$

Here, the generating family for $\mathrm{Span}_{\mathbb{Z}}\{\textit{type 1 elements}\}$ is the family of type 1 elements, and the basis we use for $S^2(H')$ is $(b'_i b'_j)_{1\le i\le j\le g}$. ☐

We are going to show that $N \subset \tau_2(\mathcal{A} \cap J_2)$, in particular we will have $T_1 \in \tau_2(\mathcal{A} \cap J_2)$. First, $①_{i,j,k,l}$ can be written as $\begin{array}{c|c} a_i & b_l \\ \hline b_j & b_k \end{array} = \left[-\overset{a_i}{\underset{b_j\ \ b_m}{\curlywedge}}\ ,\ \overset{b_l}{\underset{a_m\ \ b_k}{\curlywedge}}\right]$ where m is different from j. Morita has shown, as stated in Proposition 3.8, that (the expansion of) a tree (with three leaves) in $D_1(H)$ is in $\tau_1(\mathcal{A} \cap J_1)$ if and only if one of the leaves vanishes in H' [44], where $D_1(H) = \Lambda^3 H$ has been identified with $\mathcal{A}^t_1(H)$. Hence $①_{i,j,k,l}$ is indeed in $\tau_2(\mathcal{A} \cap J_2)$, obtained as the image by τ_2 of an element of the third family defined in Section 4.1: a commutator of the Torelli handlebody group. This is also true for $③_{i,k,l} = \left[-\overset{a_i}{\underset{b_i\ \ b_m}{\curlywedge}}\ ,\ \overset{b_l}{\underset{a_m\ \ b_k}{\curlywedge}}\right]$ with $m \neq i$. Now, we are left with the generators of N built in Proposition 5.4 from ② and ④ elements. One can check that:

$$②_{i,j,j} - ②_{i',j,j} = \left[-\overset{a_i}{\underset{b_j\ \ b_{i'}}{\curlywedge}}\ ,\ \overset{b_i}{\underset{a_{i'}\ \ b_j}{\curlywedge}}\right]$$

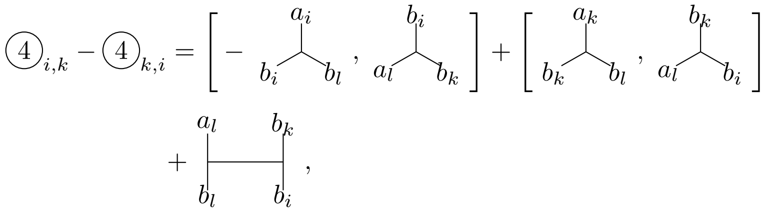
$$②_{i,j,k} - ②_{i',k,j} = \left[-\overset{a_i}{\underset{b_j\ \ b_{i'}}{\curlywedge}}\ ,\ \overset{b_i}{\underset{a_{i'}\ \ b_k}{\curlywedge}}\right]$$

$$②_{i,j,k} - ②_{i',j,k} = \left[-\overset{a_i}{\underset{b_j\ \ b_{i'}}{\curlywedge}}\ ,\ \overset{b_i}{\underset{a_{i'}\ \ b_k}{\curlywedge}}\right] + \begin{array}{c|c} a_{i'} & b_k \\ \hline b_{i'} & b_j \end{array},$$

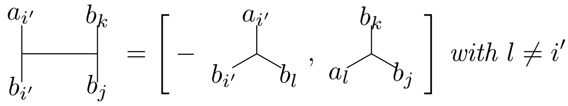
and that:

$$②_{i,j,k} - ④_{j,k} = \left[-\overset{a_i}{\underset{b_j\ \ b_l}{\curlywedge}}\ ,\ \overset{b_i}{\underset{a_l\ \ b_k}{\curlywedge}}\right] + \left[\overset{a_j}{\underset{b_j\ \ b_l}{\curlywedge}}\ ,\ \overset{b_j}{\underset{a_l\ \ b_k}{\curlywedge}}\right]$$

$$②_{i,j,k} - ④_{k,j} = \quad (②_{i,j,k} - ④_{j,k}) + (④_{j,k} - ④_{k,j})$$



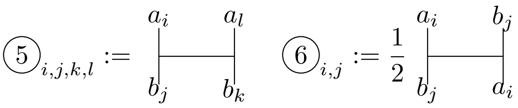
with l always chosen so that it does not add any contraction, which is possible if the genus is greater or equal to 4. We know how to show that each of the terms are in $\tau_2(\mathcal{A} \cap J_2)$, because:



for $i' \neq j, k$; and all of these terms are in the image of elements of the third kind described in Section 4.1. Hence $N \subset \tau_2(\mathcal{A} \cap J_2)$.

Remark 5.5. One can notice that all the trees that have been used above to realize elements of N as linear combination of Lie brackets of elements of $\tau_1(\mathcal{A} \cap J_1)$ are colored by elements of A *and* elements of B (and never only by A or only by B).

We now turn to the element $T_{\geq 2}$. We remark that $(A \wedge H) = (A \wedge A) \oplus (A \wedge B)$, hence if write $T_{\geq 2} = T_2 + T_{\geq 3}$ where T_2 is a linear combination of type two elements and $T_{\geq 3}$ a linear combination of type 3 and 4 elements, then we have $\mathrm{Tr}^{as}(T_2) = \mathrm{Tr}^{as}(T_{\geq 3}) = 0$, because $\mathrm{Tr}^{as}(T_{\geq 2}) = 0$. We will deal first with T_2. By the IHX relation, we can even restrict our type two elements appearing in the writing of T_2 to trees where the two A colors are not "close" to each other, i.e. trees of the form:



with no conditions on the indices. It is known, by Morita's formula in Lemma 2.3, that the elements of the form $⑥_{i,i}$ can be obtained as the image under τ_2 of a Dehn twist along a curve γ_i bounding a subsurface with a_i, b_i forming a symplectic basis of this subsurface. This curve can be chosen to bound a disk in the handlebody (see Figure 1.4) so that the corresponding Dehn twist is an element of the first kind described in Section 4.1. Hence, $⑥_{i,i}$ belongs to $\tau_2(\mathcal{A} \cap J_2)$, and we now suppose $i \neq j$ in the definition of $⑥_{i,j}$.

We show that $⑤_{i,i,j,j}$ is realizable. This will be useful in the computations below. Take disjoints neighborhoods of, respectively, $\alpha_i \cup \beta_i$ and $\alpha_j \cup \beta_j$, and band this two genus 1 surfaces as shown in Figure 1.4. The boundary $\gamma_{i,j}$ of the resulting genus 2 surface is bounding a disk in the handlebody and its image by τ_2 (using Lemma 2.3) is:

$$\tau_2(T_{\gamma_{i,j}}) = ⑥_{i,i} - ⑤_{i,i,j,j} + ⑥_{j,j}$$

which shows that $⑤_{i,i,j,j} \in \tau_2(\mathcal{A} \cap J_2)$.

We divide cases in terms of the number of leaves that contract in $⑤_{i,j,k,l}$. If there is no contraction ($j \neq l$ and $k \neq i$), then $⑤_{i,j,k,l}$ can be easily obtained as a commutator of trees with a leaf in A, supposing $g \geq 4$. If there are 2 contractions, then $k = i$ and $j = l$, which yields two cases: if $i = j$ then we get $-2⑥_{i,i}$, which we have already dealt with; if not, we get an element $⑤_{i,j,i,j} \notin \mathrm{Ker}(\mathrm{Tr}^{as})$. If there is only one contraction, then up to symmetry

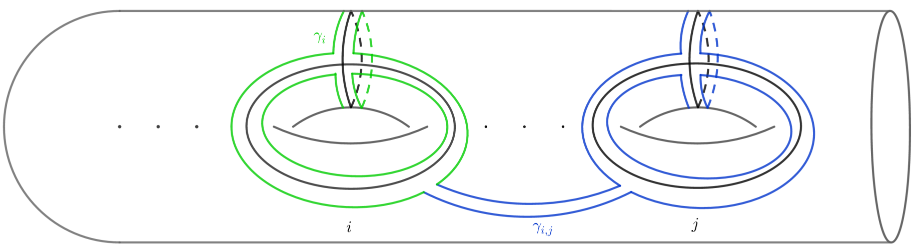


Figure 1.4: Curves γ_i and $\gamma_{i,j}$.

($\textcircled{5}_{i,j,k,l} = \textcircled{5}_{l,k,j,i}$) we can suppose that $k = i$ and $j \neq l$. Hence the remaining element T_2' (the part of T_2 which is not yet proved to be in $\tau_2(\mathcal{A} \cap J_2)$) is a linear combination of trees of the form $\textcircled{5}_{i,j,i,j}$ (with $i \neq j$), $\textcircled{5}_{i,j,i,l}$ (with $l \neq j$) and $\textcircled{6}_{i,j}$ (with $i \neq j$) such that :

$$\begin{aligned}
\mathrm{Tr}^{as}(\textcircled{5}_{i,j,i,j}) &= a_i \wedge b_i + a_j \wedge b_j \quad \textit{with } i \neq j \\
\mathrm{Tr}^{as}(\textcircled{5}_{i,j,i,l}) &= a_l \wedge b_j \quad \textit{with } l \neq j \\
\mathrm{Tr}^{as}(\textcircled{6}_{i,j}) &= a_i \wedge b_j \quad \textit{with } i \neq j,
\end{aligned}$$

and satisfies $\mathrm{Tr}^{as}(T_2') = 0$.

Notice that in $\mathrm{Ker}(\omega : \Lambda^2(H/2H) \to \mathbb{Z}_2)$ the two subspaces $\mathrm{Span}_{\mathbb{Z}_2}\{a_i \wedge b_j \mid i \neq j\}$ and $\mathrm{Span}_{\mathbb{Z}_2}\{a_i \wedge b_i + a_j \wedge b_j \mid i \neq j\}$ have trivial intersection. This allow us to write T_2' as a sum of two elements, say U and V, the element U being in $\mathrm{Span}_{\mathbb{Z}}\left\{\textcircled{5}_{i,j,i,l}, \textcircled{6}_{i,j} \mid j \neq l\right\}$ and V being in $\mathrm{Span}_{\mathbb{Z}}\left\{\textcircled{5}_{i,j,i,j} \mid i \neq j\right\}$, such that $\mathrm{Tr}^{as}(U) = \mathrm{Tr}^{as}(V) = 0$. The element U can be written as a linear combination of

$$\textcircled{5}_{i,j,i,l} \pm \textcircled{5}_{i',j,i',l} \quad ; \quad 2\textcircled{6}_{i,j} \quad ; \quad \textcircled{5}_{i,j,i,l} \pm \textcircled{6}_{l,j}$$

by Lemma 5.2 applied to the short exact sequence

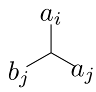
$$0 \longrightarrow K \longrightarrow \mathrm{Span}_{\mathbb{Z}}\left\{\textcircled{5}_{i,j,i,l}\,,\ \textcircled{6}_{i,j} \mid j \neq l\right\} \xrightarrow{\mathrm{Tr}^{as}} \mathrm{Span}_{\mathbb{Z}_2}\{a_i \wedge b_j \mid i \neq j\} \longrightarrow 0 \qquad (5.1)$$

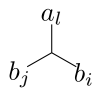
where both the generating family for $\mathrm{Span}_{\mathbb{Z}}\left\{\textcircled{5}_{i,j,i,l}\,,\ \textcircled{6}_{i,j} \mid j \neq l\right\}$ and the basis for $\mathrm{Span}_{\mathbb{Z}_2}\{a_i \wedge b_j \mid i \neq j\}$ are given in their definition. The tree $2\textcircled{6}_{i,j}$ has no contractions and can be realized as a commutator of the Torelli handlebody group. We also have, with $r \neq i, j, l$:

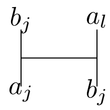
$$\textcircled{5}_{i,j,i,l} + \textcircled{5}_{i',j,i',l} = \left[\begin{array}{c} a_i \\ \curlywedge \\ b_j \quad a_{i'} \end{array}, \begin{array}{c} a_l \\ \curlywedge \\ b_{i'} \quad b_i \end{array}\right] \ \textit{as } l \neq j,$$

$$\begin{aligned}
2\textcircled{5}_{i,j,i,l} = &\left[\begin{array}{c} a_i \\ \curlywedge \\ b_j \quad a_r \end{array}, \begin{array}{c} a_l \\ \curlywedge \\ b_r \quad b_i \end{array}\right] - \left[\begin{array}{c} a_i \\ \curlywedge \\ b_j \quad b_r \end{array}, \begin{array}{c} a_l \\ \curlywedge \\ a_r \quad b_i \end{array}\right] \\
&- \left[\begin{array}{c} a_r \\ \curlywedge \\ b_r \quad a_i \end{array}, \begin{array}{c} a_l \\ \curlywedge \\ b_i \quad b_j \end{array}\right] \ \textit{as } l \neq j,
\end{aligned}$$

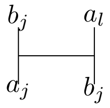
and with the same arguments as above these elements are realizable (using the first family described in Section 4.1). For elements involving $\textcircled{5}$ and $\textcircled{6}$, if $i \neq j$, we have:

$$\textcircled{6}_{l,j} - \textcircled{5}_{i,j,i,l} = \textcircled{6}_{l,j} + \Bigg[-$$ 

$$,$$ 

$$\Bigg] -$$ 

$$.$$

As $\textcircled{6}_{l,j} -$ 

$+\textcircled{6}_{j,j}$ can be obtained from the Dehn twist along the boundary of a neighborhood of $(\alpha_j \sharp \alpha_l^{-1}) \cup \beta_j$ (where $(\alpha_j \sharp \alpha_l^{-1})$ denotes the connected sum of α_j and α_l), and knowing that $\textcircled{6}_{j,j}$ is in $\tau_2(\mathcal{A} \cap J_2)$ we conclude that $\textcircled{6}_{l,j} - \textcircled{5}_{i,j,i,l}$ is realizable for $i \neq j$. If $i = j$, then we write, for some $j' \neq i$:

$$\textcircled{6}_{l,j} - \textcircled{5}_{j,j,j,l} = (\textcircled{6}_{l,j} - \textcircled{5}_{j',j,j',l}) + (\textcircled{5}_{j',j,j',l} - \textcircled{5}_{j,j,j,l}) \tag{5.2}$$

and we have just shown that both terms of this sum are realizable. We conclude that U is realizable.

We need to show that V is also realizable, which will show that T_2', and hence T_2 are also realizable. We need the following.

Lemma 5.6. *The kernel S in the short exact sequence*

$$0 \longrightarrow S \longrightarrow \mathrm{Span}_{\mathbb{Z}} \left\{ \textcircled{5}_{i,j,i,j} \mid i \neq j \right\} \xrightarrow{\mathrm{Tr}^{as}} \mathrm{Span}_{\mathbb{Z}_2} \{ a_i \wedge b_i + a_j \wedge b_j \mid i \neq j \} \longrightarrow 0$$

is generated by the family

$$\left\{ \textcircled{5}_{1,i,1,i} + \textcircled{5}_{i,j,i,j} + \textcircled{5}_{j,1,j,1} \mid i \neq j \right\} \cup \left\{ 2\textcircled{5}_{i,j,i,j} \mid i \neq j \right\}.$$

Proof. It is not hard to see, by sending the family $\left\{ \textcircled{5}_{i,j,i,j} \mid i < j \right\}$ to $(H \otimes \mathbb{Q})^{\otimes 4}$ through the expansion map and the inclusion $\mathcal{L}(H \otimes \mathbb{Q}) \subset T(H \otimes \mathbb{Q})$, that this family is free. Indeed, $\textcircled{5}_{i,j,i,j}$ is sent to a sum of 16 terms, from each of which one can recover i and j. Each of these terms belongs (up to a sign) to the basis of $(H \otimes \mathbb{Q})^{\otimes 4}$ induced by the basis chosen for H. Hence, $\mathrm{Span}_{\mathbb{Z}} \left\{ \textcircled{5}_{i,j,i,j} \mid i \neq j \right\}$ is free and we can apply Lemma 5.2 to the short exact sequence by using the family $\left\{ \textcircled{5}_{i,j,i,j} - \textcircled{5}_{j,1,j,1} \mid i < j \right\} \cup \left\{ \textcircled{5}_{j,1,j,1} \mid 2 \leq j \right\}$ as a generating family for $\mathrm{Span}_{\mathbb{Z}} \left\{ \textcircled{5}_{i,j,i,j} \mid i \neq j \right\}$, and the basis $(a_1 \wedge b_1 + a_i \wedge b_i)_{2 \leq i \leq g}$ for $\mathrm{Span}_{\mathbb{Z}_2} \{ a_i \wedge b_i + a_j \wedge b_j \mid i \neq j \}$. Then S is generated by $\left\{ 2\textcircled{5}_{i,j,i,j} - 2\textcircled{5}_{j,1,j,1} \mid i < j \right\} \cup \left\{ 2\textcircled{5}_{j,1,j,1} \mid 2 \leq j \right\} \cup \left\{ \textcircled{5}_{i,j,i,j} - \textcircled{5}_{j,1,j,1} - \textcircled{5}_{i,1,i,1} \mid i < j \right\} \cup \left\{ \textcircled{5}_{i,j,i,j} - \textcircled{5}_{j,1,j,1} - \textcircled{5}_{i,k,i,k} + \textcircled{5}_{k,1,k,1} \mid i < j < k \right\}$, from which we deduce the simpler generating family

$$\left\{ \textcircled{5}_{1,i,1,i} + \textcircled{5}_{i,j,i,j} + \textcircled{5}_{j,1,j,1} \mid i \neq j \right\} \cup \left\{ 2\textcircled{5}_{i,j,i,j} \mid i \neq j \right\}.$$

Indeed, it is easy to get the elements of the family given by Lemma 5.2 with the elements given right above. For example, for $i < j$:

$$\textcircled{5}_{i,j,i,j} - \textcircled{5}_{j,1,j,1} - \textcircled{5}_{i,1,i,1} = (\textcircled{5}_{1,i,1,i} + \textcircled{5}_{i,j,i,j} + \textcircled{5}_{j,1,j,1}) - 2\textcircled{5}_{1,i,1,i} - 2\textcircled{5}_{j,1,j,1},$$

and for $i < j < k$:

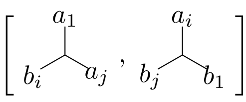
$$\begin{aligned} \textcircled{5}_{i,j,i,j} - \textcircled{5}_{j,1,j,1} - \textcircled{5}_{i,k,i,k} + \textcircled{5}_{k,1,k,1} = {} & (\textcircled{5}_{1,i,1,i} + \textcircled{5}_{i,j,i,j} + \textcircled{5}_{j,1,j,1} - 2\textcircled{5}_{j,1,j,1}) \\ & - (\textcircled{5}_{1,i,1,i} + \textcircled{5}_{i,k,i,k} + \textcircled{5}_{k,1,k,1} - 2\textcircled{5}_{k,1,k,1}). \end{aligned}$$

$\square$

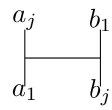
Hence by Lemma 5.6, the element V can be written as a linear combination of

$$5_{1,i,1,i} + 5_{i,j,i,j} + 5_{j,1,j,1} \ ; \quad 2\,5_{i,j,i,j}$$

where $i \neq j$. We compute, for $i, j \neq 1$:



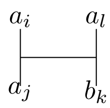
$= 5_{1,i,1,i} + 5_{i,j,i,j} -$



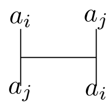
$$= 5_{1,i,1,i} + 5_{i,j,i,j} - 5_{j,1,j,1} + 5_{j,j,1,1}.$$

We know that $5_{j,j,1,1}$ is realizable which shows that $5_{1,i,1,i} + 5_{i,j,i,j} - 5_{j,1,j,1}$ is also realizable. Similarly, the element $5_{i,j,i,j} + 5_{j,1,j,1} - 5_{1,i,1,i}$ is realizable. By summing these two elements, we get that $2\,5_{i,j,i,j}$ is also realizable. We deduce that both $5_{1,i,1,i} + 5_{i,j,i,j} + 5_{j,1,j,1}$ and $2\,5_{i,j,i,j}$ belong to $\tau_2(\mathcal{A} \cap J_2)$ for any $i \neq j$. Therefore V is realizable. We finally turn to $T_{\geq 3}$. We define the elements

$7_{i,j,k,l} :=$



$8_{i,j} := \frac{1}{2}$



with $i \neq j$, then

$$\mathrm{Tr}^{as}(7_{i,j,k,l}) = \delta_{ki} a_j \wedge a_l + \delta_{kj} a_i \wedge a_l$$
$$\mathrm{Tr}^{as}(8_{i,j}) = a_i \wedge a_j$$

The fact that $T_{\geq 3}$ can be realized will follow from the same kind of computations as for T_2. We define $P := \mathrm{Ker}(\mathrm{Tr}^{\mathrm{as}} : \mathrm{Span}_{\mathbb{Z}}\{\textit{type 3 and 4 elements}\} \to (A \wedge A) \otimes \mathbb{Z}_2)$.

Proposition 5.7. *P is generated by trees with 4 leaves colored by A, elements of type 7 with no contractions, elements of type $7_{i,k,k,i}$ and elements*

$$7_{i,k,k,m} \pm 7_{m,k,k,i} \ ; 7_{i,k,k,l} \pm 7_{i,k',k',l} \ ; 7_{i,k,k,l} \pm 8_{i,l}$$

where i must be different from k, k' and l, and $m \neq k$.

Proof. It follows once again from Lemma 5.2 applied to the short exact sequence

$$0 \longrightarrow P \longrightarrow \mathrm{Span}_{\mathbb{Z}}\{\textit{type 3 and 4 elements}\} \xrightarrow{\mathrm{Tr}^{as}} ((A \wedge A) \otimes \mathbb{Z}_2) \longrightarrow 0 \ .$$

Type 3 and 4 elements give a generating family for $\mathrm{Span}_{\mathbb{Z}}\{\textit{type 3 and 4 elements}\}$ and $(a_i \wedge a_j)_{1 \leq i < j \leq g}$ a basis for $((A \wedge A) \otimes \mathbb{Z}_2)$. Note that, according to Lemma 5.2, in our family of generators we should have elements of type $2\,7_{i,j,k,l}$ and $2\,8_{i,j}$ for any $i \neq j$. Nevertheless, these elements are not needed, because if there is no contraction we have the element $7_{i,j,k,l}$ as a generator and if there is one contraction it is easy to obtain both $2\,7_{i,j,j,l}$ and $2\,7_{i,j,i,l} = -2\,7_{j,i,i,l}$ from the generators given in the proposition. This last argument also works for $2\,8_{i,j}$. ☐

We now show that P is contained in $\tau_2(\mathcal{A} \cap J_2)$. Elements of type 4 that are not expansion of half trees in $D_2(H)$ are not worth mentioning: they always have no contractions and are in $\tau_2([\mathcal{A} \cap J_1, \mathcal{A} \cap J_1])$. The same is true for elements of type 7 with no contractions. Once again we check some relations, making sure that any tree with three leaves appearing in the computations below has at least one leaf colored by A:

$$7_{i,k,k,l} + 7_{i,k',k',l} = \left[\begin{matrix} & a_i & \\ & \curlywedge & \\ a_k & & a_{k'} \end{matrix} , \begin{matrix} & a_l & \\ & \curlywedge & \\ b_{k'} & & b_k \end{matrix} \right],$$

$$7_{i,k,k,l} - 7_{l,k',k',i} = \left[- \begin{matrix} & a_i & \\ & \curlywedge & \\ a_k & & b_{k'} \end{matrix} , \begin{matrix} & a_l & \\ & \curlywedge & \\ a_{k'} & & b_k \end{matrix} \right],$$

$$7_{i,k,k,m} - 7_{m,k,k,i} = \begin{matrix} a_k & & a_i \\ & \mathrm{H} & \\ b_k & & a_m \end{matrix}$$

$$= \left[\begin{matrix} & a_k & \\ & \curlywedge & \\ b_k & & a_i \end{matrix} , \begin{matrix} & a_i & \\ & \curlywedge & \\ b_i & & a_m \end{matrix} \right],$$

$$2\,7_{i,k,k,m} = \left[\begin{matrix} & a_i & \\ & \curlywedge & \\ a_k & & a_r \end{matrix} , \begin{matrix} & a_m & \\ & \curlywedge & \\ b_r & & b_k \end{matrix} \right] - \left[\begin{matrix} & a_i & \\ & \curlywedge & \\ a_k & & b_r \end{matrix} , \begin{matrix} & a_m & \\ & \curlywedge & \\ a_r & & b_k \end{matrix} \right]$$

$$- \left[\begin{matrix} & a_r & \\ & \curlywedge & \\ b_r & & a_k \end{matrix} , \begin{matrix} & a_i & \\ & \curlywedge & \\ b_k & & a_m \end{matrix} \right],$$

$$8_{i,l} \pm 7_{i,k,k,l} = (8_{i,l} \pm 7_{i,l,l,l}) \pm (7_{i,k,k,l} \mp 7_{i,l,l,l}).$$

We also consider the Dehn twist along the curve bounding the surface which is a neighborhood of $\alpha_l \cup (\alpha_i \sharp \beta_l^{\pm})$, where $(\alpha_i \sharp \beta_l^{\pm})$ is a connected sum of α_i and β_l with orientation defined by the sign. This element is in $\mathcal{A} \cap J_2$, its image under τ_2 is

$$\frac{1}{2} \begin{matrix} a_i \pm b_l & & a_l \\ & \mathrm{H} & \\ a_l & & a_i \pm b_l \end{matrix} = 8_{i,l} \pm 7_{i,l,l,l} + \frac{1}{2} \begin{matrix} b_l & & a_l \\ & \mathrm{H} & \\ a_l & & b_l \end{matrix}$$

which ultimately shows that $8_{i,l} \pm 7_{i,l,l,l}$ belongs to $\tau_2(\mathcal{A} \cap J_2)$. Therefore, $8_{i,l} \pm 7_{i,k,k,l}$ is also realizable. Finally, elements of type $7_{i,k,k,i}$ can be realized in the following way. Notice that:

$$\frac{1}{2} \begin{matrix} a_i + a_k & & b_k + a_i \\ & \mathrm{H} & \\ b_k + a_k & & a_i + a_k \end{matrix} - \frac{1}{2} \begin{matrix} a_k & & b_k + a_i \\ & \mathrm{H} & \\ b_k + a_i & & a_k \end{matrix} = \frac{1}{2} \begin{matrix} a_i & & b_k + a_i \\ & \mathrm{H} & \\ b_k + a_i & & a_i \end{matrix} + \begin{matrix} a_i & & b_k + a_i \\ & \mathrm{H} & \\ b_k + a_i & & a_k \end{matrix}$$

$$= \frac{1}{2} \begin{matrix} a_i & & b_k \\ & \mathrm{H} & \\ b_k & & a_i \end{matrix} + \begin{matrix} a_i & & b_k + a_i \\ & \mathrm{H} & \\ b_k & & a_k \end{matrix}$$

$$= 6_{i,k} - 5_{i,k,k,k} + 7_{i,k,k,i}.$$

Now, we have already shown that $6_{i,k} - 5_{i,k,k,k} \in \tau_2(\mathcal{A} \cap J_2)$ (by equality (5.2)), because it can be written as $6_{i,k} - 5_{k,k,k,i}$. So we just need to show that the left part of this equality is also in $\tau_2(\mathcal{A} \cap J_2)$ to conclude that $7_{i,k,k,i}$ is realizable. This comes once again from Lemma 2.3 and the fact that the curves bounding $(\alpha_i \sharp \alpha_k) \cup (\beta_k \sharp \alpha_i)$ and $(\alpha_k) \cup (\beta_k \sharp \alpha_i)$ (where $(\alpha_i \sharp \alpha_k)$ and $(\beta_k \sharp \alpha_i)$ are connected sums of the curves involved) are bounding disks in the handlebody. All these computations imply that $P \subset \tau_2(\mathcal{A} \cap J_2)$, so that $T_{\geq 3} \in \tau_2(\mathcal{A} \cap J_2)$.

Consequently, we get that $T \in \tau_2(\mathcal{A} \cap J_2)$ which finishes the proof of Theorem 5.1.

Remark 5.8. The computations in this section actually give generators for $\tau_2(\mathcal{A} \cap J_2)$, which we can write explicitly. Also, it can be noticed that we used only elements of $\mathcal{A} \cap J_2$ of the first and the third kind defined in Section 4.1. This tells us something about the generation of $\mathcal{A} \cap J_2$, but only up to J_3. Naturally the following question arises: is $\mathcal{A} \cap J_2$ generated by elements of the first and the third kind in Section 4.1 ?

Theorem 5.1 allows us to recover the result shown by Pitsch in [49], whose immediate corollary is that any homology 3-sphere is J_3-equivalent to S^3. We even get a slight improvement on the genus condition. With the definitions of $\mathcal{A}$, $\mathcal{B}$, and ι given in Section 3, we get the following result :

Corollary 5.9. *For any $g \geq 4$, $\mathrm{Im}(\tau_2) = \tau_2(\mathcal{A} \cap J_2) + \tau_2(\mathcal{B} \cap J_2)$.*

Proof. Any element T in the image of τ_2 can be written as (an expansion of) a linear combination T_1 of trees with 0 or 1 leaf colored by A and a linear combination T_2 of trees with $2, 3$ or 4 leaves colored by A (here, the term "tree" includes halves of symmetric trees as well). Then it is clear that $\mathrm{Tr}^{as}(T_1) \in B \wedge B$, whereas $\mathrm{Tr}^{as}(T_2) \in A \wedge H$. The spaces $B \wedge B$ and $A \wedge H$ having trivial intersection in $\Lambda^2 H$, both T_1 and T_2 lie in the kernel of Tr^{as}. The term T_2, by definition, also lies in the kernel of Tr^A. We also know (see Section 6) that ι acts on H as the map sending a_i to $(-b_i)$ and b_i to a_i for all i's, and that $\mathcal{B} = \iota \mathcal{A} \iota^{-1}$. Now $\iota_*(T_1)$ lie in the kernel of Tr^A. By Theorem 4.10, we know that there are two mapping classes ψ_1 and ψ_2 in $\mathcal{A} \cap J_2$ such that $T = T_1 + T_2 = \tau_2(\iota \psi_1 \iota^{-1}) + \tau_2(\psi_2)$, which finishes the proof. $\square$

Remark 5.10. In this proof, we used only the fact that $\mathcal{F}_1 \cap \tau_2(J_2) \subset \tau_2(\mathcal{A} \cap J_2)$ which is stricly weaker than the equality $\tau_2(\mathcal{A} \cap J_2) = \mathrm{Ker}(\mathrm{Tr}^A) \cap \tau_2(J_2)$ from Theorem 5.1. In this sense, the computation in this section is more precise than the one from [49].

6 Computing $\tau_2(\mathcal{G} \cap J_2)$

Like in Section 3, we choose a system of meridians and parallels in the boundary of V_g, and we identify S^3 to $V_g \cup_{\iota_g} (-V_g)$. This gives the Heegaard splitting of genus g of the 3-sphere, and we consider the subgroup $\mathcal{B} = \iota \mathcal{A} \iota^{-1}$ of $\mathcal{M}$. We thus have a family of curves $(\alpha_i)_{1 \leq i \leq g}$ with homology classes $(a_i)_{1 \leq i \leq g}$ as in the previous sections, but also a set of curves $(\beta_i)_{1 \leq i \leq g}$ with homology classes $(b_i)_{1 \leq i \leq g}$, defining a Lagrangian $B \subset H$. The map ι can be defined by its action on π. We lift the curves α_i and β_i to elements of π as described in Figure 1.5, and we set

$$\begin{aligned} \iota_* : \pi &\longrightarrow \pi \\ \alpha_i &\longmapsto \beta_i^{-1} \\ \beta_i &\longmapsto \beta_i \alpha_i \beta_i^{-1}. \end{aligned}$$

Indeed by the Dehn-Nielsen theorem, as ι_* fixes the element $\xi := \prod_{i=1}^{g} [\beta_i^{-1}, \alpha_i]$ defined by $-\partial \Sigma$ in π (ξ is described in Figure 1.5), the map ι realizing this action is well-defined.

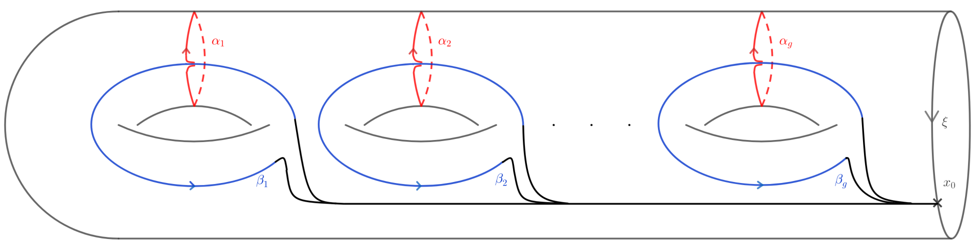


Figure 1.5: The based curves $(\alpha_i)_{1 \leq i \leq g}$ and $(\beta_i)_{1 \leq i \leq g}$

The *Goeritz group* of S^3 is the group of isotopy classes of orientation-preserving homeomorphisms of S^3 preserving this Heegaard splitting (and fixing the disk). We denote it by $\mathcal{G} := \mathcal{G}_{g,1}$. Observe that $\mathcal{G}$ coincides with $\mathcal{A} \cap \mathcal{B}$. The Johnson filtration restricts to a separating filtration on $\mathcal{G}$. In this section, we compute $\tau_1(\mathcal{G} \cap J_1)$ and $\tau_2(\mathcal{G} \cap J_2)$ using, respectively, a refinement of the computations made by Morita in [42] and the computations and results in Section 5. Notice that ι acts on $\mathcal{G}$ by conjugation, and on H by sending a_i to $-b_i$ and b_i to a_i for all i. We also need the following from [55, Section 3].

Lemma 6.1. *The image of $\mathcal{G}$ in $\mathrm{Sp}(2g, \mathbb{Z})$ coincides with*

$$\left\{ \begin{pmatrix} P & 0 \\ 0 & (P^T)^{-1} \end{pmatrix} \middle| \; P \in \mathrm{GL}(g, \mathbb{Z}) \right\}$$

and, so, is canonically isomorphic to $\mathrm{GL}(g, \mathbb{Z})$.

Thus, for all k, $\tau_k(\mathcal{G} \cap J_k)$ is a $\mathrm{GL}(g, \mathbb{Z})$-module.

Proposition 6.2. *For $g \geq 2$, we have $\tau_1(\mathcal{G} \cap J_1) = A \wedge B \wedge H$.*

Proof. We identify once again elements of $\Lambda^3 H$ to trees with three leaves. Any element in $\tau_1(\mathcal{G} \cap J_1)$ must vanish when we reduce its leaves in H/A or H/B. Hence it can be written as a linear combination of trees whose leaves are never colored solely by A or by B. Now, one can check that any tree in $A \wedge B \wedge H$ colored by elements in $\{a_i, b_i \mid 1 \leq i \leq g\}$ is in the $\mathbb{Z}$-module generated by the orbit of $T := \begin{array}{c} a_1 \\ \mid \\ b_1 \quad b_2 \end{array}$ under the actions of ι and $\mathrm{GL}(g, \mathbb{Z})$. Indeed, if such a tree has 2 leaves colored by A, the action of ι allows us to have a tree in the same orbit but with two leaves colored by B. Now, such a tree is always in the orbit of T or $T' := \begin{array}{c} a_1 \\ \mid \\ b_2 \quad b_3 \end{array}$ under the action of $\mathrm{GL}(g, \mathbb{Z})$ (just by renumbering). But T' is also in the $\mathbb{Z}$-module generated by the orbit of T, as one can write $T' = \begin{array}{c} a_1 \\ \mid \\ b_1 + b_2 b_3 \end{array} - T$. Hence, it is sufficient to show that this particular tree is in $\tau_1(\mathcal{G} \cap J_1)$. Actually, if ψ denotes the composition of a right Dehn twist along a simple closed curve corresponding to $[\alpha_2, \beta_2^{-1}][\alpha_1, \beta_1^{-1}]\beta_2 \in \pi$ with the left Dehn twist along a simple closed curve corresponding to $\beta_2 \in \pi$ (as described in Figure 1.6 and in Fig. 3a in [42]), then $\tau_1(\psi) = T$. The map ψ is an annulus twist in the inner handlebody, and the composition of two Dehn twists along curves bounding disks in the outer handlebody. Hence, we have $\psi \in \mathcal{A} \cap \mathcal{B} = \mathcal{G}$. $\square$

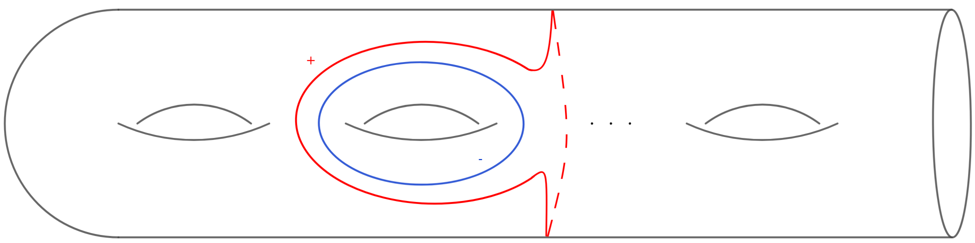


Figure 1.6: The two curves defining ψ

Even though Tr^A and Tr^B are not defined over the same subspaces of $D_2(H)$, their kernels are both included in $D_2(H)$. Hence the following makes sense

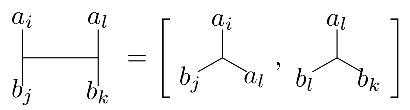
Proposition 6.3. *For $g \geq 4$, we have $\tau_2(\mathcal{G} \cap J_2) = \mathrm{Ker}(\mathrm{Tr}^{as}) \cap \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^B)$.*

Proof. The inclusion from the left to the right is a direct consequence of Theorems 2.4 and 5.1. For the other inclusion, let us take an element $T \in \mathrm{Ker}(\mathrm{Tr}^{as}) \cap \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^B)$. We say, for $0 \leq k, l \leq 4$, that a tree has *type* (k, l) if it has k leaves colored by A and l leaves colored by B. The tree T is then a linear combination of trees of type $(1,3)$, $(2,2)$ and $(3,1)$. This is due to the fact that T must be in the kernel of the projections $D_2(H) \to D_2(H/A)$ and $D_2(H) \to D_2(H/B)$. Hence we decompose T into 3 elements: $T = T_1 + T_2 + T_3$, such that T_i is a linear combination of elements of type $(i, 4-i)$, for $i = 1,2,3$. The images by Tr^{as} of these 3 elements take place in separate summands in $\Lambda^2(H/2H)$. Also, by definition, Tr^A (resp. Tr^B) vanishes on the element of types $(2,2)$ and $(3,1)$ (resp. $(2,2)$ and $(1,3)$). We thus have, for $i = 1,2,3$, $T_i \in \mathrm{Ker}(\mathrm{Tr}^{as}) \cap \mathrm{Ker}(\mathrm{Tr}^A) \cap \mathrm{Ker}(\mathrm{Tr}^B)$. We thus treat these 3 elements separately.

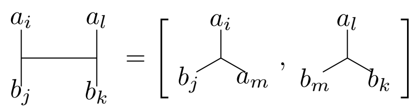
The element T_1 belongs to the space N defined in Proposition 5.4. By Remark 5.5, any element in N can be realized as a linear combination of commutators of trees with three leaves with always at least one leaf in A and one leaf in B. By Proposition 6.2, this implies that $N \subset \tau_2(\mathcal{G} \cap J_2)$. Indeed, we have $[\mathcal{G} \cap J_1, \mathcal{G} \cap J_1] \subset \mathcal{G} \cap J_2$.

The element T_3 is such that $\iota_*(T_3)$ is of type $(1,3)$, hence is in N. We deduce that $\iota_*(T_3)$, and consequently T_3, also belong to $\tau_2(\mathcal{G} \cap J_2)$.

The element T_2 is exactly of the same type as its homonym in Section 5. We want to modify slightly the argument in order to show that it is also in $\tau_2(\mathcal{G} \cap J_2)$. The elements $[\alpha_1, \beta_1^{-1}]$ and $[\alpha_2, \beta_2^{-1}][\alpha_1, \beta_1^{-1}]$ in π (where the curves have been lifted to elements of π as in Figure 1.5) define two simple closed curves bounding disks both in the inner and the outer handlebody. Then the Dehn twists along these two curves are maps in the Goeritz group, but also in J_2. This gives, respectively, that $6_{1,1}$ and $6_{1,1} - 5_{1,1,2,2} + 6_{2,2}$ are in $\tau_2(\mathcal{G} \cap J_2)$. Using the action of $\mathrm{GL}(g,\mathbb{Z})$ (by sending 1 on i and 2 on j), we deduce that for all $1 \leq i \neq j \leq g$, we have $6_{i,i}, 5_{i,i,j,j} \in \tau_2(\mathcal{G} \cap J_2)$. For the trees of type $5_{i,j,k,l}$ with no contraction discussed page 24, we can simply write



if all the indices are different or



with $m \notin \{i,j,k,l\}$ otherwise (which is possible with $g \geq 4$). We conclude, as in Section 5 that T_2 is a sum of an element in $\tau_2(\mathcal{G} \cap J_2)$ and an element $T_2' = U + V$ with $U \in K$ and $V \in S$, where these spaces are respectively defined in the short exact sequence (5.1) and in Lemma 5.6. The computations showing that $V \in \tau_2(\mathcal{A} \cap J_2)$ only involves commutators of trees colored both by A and B, and the element $5_{j,j,1,1}$. By Proposition 6.2, $V \in \tau_2(\mathcal{G} \cap J_2)$. Finally, using once again the same argument, we only need to show that $6_{l,j} - 5_{j,j,j,l} \in \tau_2(\mathcal{G} \cap J_2)$ for $i \neq j$ to show that $U \in \tau_2(\mathcal{G} \cap J_2)$. We notice that the action of $\mathrm{GL}(g,\mathbb{Z})$ corresponding to $b_1 \mapsto b_1 + b_2$ and $a_2 \mapsto a_2 - a_1$ (and fixing the other elements in the basis) sends $6_{1,2}$ to $6_{1,2} - 5_{1,1,2,1}$. Then the action of ι on this element gives $6_{2,1} - 5_{1,1,1,2}$. This proves that $6_{2,1} - 5_{1,1,1,2} \in \tau_2(\mathcal{G} \cap J_2)$, and by action of $\mathrm{GL}(g,\mathbb{Z})$, that $6_{l,j} - 5_{j,j,j,l} \in \tau_2(\mathcal{G} \cap J_2)$ for any $i \neq j$. $\square$

Remark 6.4. The rationalization of $\tau_2(\mathcal{G} \cap J_2)$ is a finite-dimensional $\mathrm{GL}(g,\mathbb{Q})$-module. In Appendix A, we give its decomposition into irreducible modules. This results in a rational version of Proposition 6.3.

Clearly one has that $\tau_k(\mathcal{G} \cap J_k) \subset \tau_k(\mathcal{A} \cap J_k) \cap \tau_k(\mathcal{B} \cap J_k)$. It is not clear if the converse is true in general. As a direct consequence of Proposition 6.2, Proposition 6.3 and Theorem 5.1, we get the following:

Corollary 6.5. *For $g \geq 4$, we have $\tau_1(\mathcal{G} \cap J_1) = \tau_1(\mathcal{A} \cap J_1) \cap \tau_1(\mathcal{B} \cap J_1)$ and $\tau_2(\mathcal{G} \cap J_1) = \tau_2(\mathcal{A} \cap J_2) \cap \tau_2(\mathcal{B} \cap J_2)$.*

In [50], Pitsch already pointed out that the image of $\mathcal{G}$ in $\mathrm{Sp}(2g,\mathbb{Z})$ coincides with the intersection of the images of $\mathcal{A}$ and $\mathcal{B}$ (see Lemma 6.1). Using this fact and the Reidemeister-Singer Theorem (see Theorem 3.1), he showed ([50, Theorem 1]):

Proposition 6.6. *There is a well-defined isomorphism*

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1}\cap\mathcal{I}_{g,1})\backslash\mathcal{I}_{g,1}/(\mathcal{B}_{g,1}\cap\mathcal{I}_{g,1})\big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}^3.$$

where $\mathcal{G}$ acts by conjugation.

This gives an intrinsic description of the equivalence relation given by Reidemeister-Singer Theorem on the Torelli group. The same can be done, using Corollary 6.5, for the second and third term of the Johnson filtration.

Proposition 6.7. *Denote $\mathcal{K}_{g,1} := J_2(\Sigma_{g,1})$ and $\mathcal{L}_{g,1} := J_3(\Sigma_{g,1})$. There are well-defined isomorphisms*

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1}\cap\mathcal{K}_{g,1})\backslash\mathcal{K}_{g,1}/(\mathcal{B}_{g,1}\cap\mathcal{K}_{g,1})\big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}^3.$$

and

$$\lim_{g\to\infty} \big((\mathcal{A}_{g,1}\cap\mathcal{L}_{g,1})\backslash\mathcal{L}_{g,1}/(\mathcal{B}_{g,1}\cap\mathcal{L}_{g,1})\big)_{\mathcal{G}_{g,1}} \simeq \mathcal{S}^3.$$

Proof. The proof is by induction. We already know that the maps are well-defined and surjective (see Section 3. We know by Proposition 6.6 that two gluing maps $\phi\in\mathcal{K}_{g,1}$ and $\psi\in\mathcal{K}_{g,1}$ yield the same homology 3-sphere if and only if, after an eventual stabilization, there exists maps $\xi_a\in\mathcal{A}\cap\mathcal{I}$, $\xi_b\in\mathcal{B}\cap\mathcal{I}$ and $\mu\in\mathcal{G}$ such that $\phi=\mu\xi_a\psi\xi_b\mu^{-1}$. Applying τ_1 to this equality we get that $\tau_1(\xi_a)=-\tau_1(\xi_b)\in\tau_1(\mathcal{A}\cap\mathcal{I})\cap\tau_1(\mathcal{B}\cap\mathcal{I})=\tau_1(\mathcal{G}\cap\mathcal{I})$. Then there exists $\mu'\in\mathcal{G}\cap\mathcal{I}$ such that $\mu^{-1}\phi\mu=\mu'\circ\big(\mu'^{-1}\xi_a\big)\,\psi\,(\xi_b\mu')\circ\mu'^{-1}$, and $\mu'^{-1}\xi_a\in\mathcal{A}\cap\mathcal{K}_{g,1}$, $\xi_b\mu'\in\mathcal{B}\cap\mathcal{K}_{g,1}$. Then a conjugate of ϕ by an element of the Goeritz group is in the same double coset class as ψ. This concludes, as one can get the proof for $\mathcal{L}_{g,1}$ by applying the same method to some elements ϕ and ψ in $\mathcal{L}_{g,1}$. □

Using the methods described by Pitsh in [50], Proposition 6.7 could help to build invariants of homology 3-spheres by using algebraic properties of J_2 and J_3. Infortunately we do not know about generators of $\mathcal{A}\cap J_2$ and $\mathcal{A}\cap J_3$.

1.A Decomposition of $\tau_2(\mathcal{G}\cap J_2)\otimes\mathbb{Q}$

As a consequence of Lemma 6.1, the conjugation action of the Goeritz group $\mathcal{G}$ on itself induces a $\mathrm{GL}(g,\mathbb{Z})$-module structure on $\tau_2(\mathcal{G}\cap J_2)$, the image of the Goeritz group by the second Johnson homomorphism. This action is the restriction of the canonical action of $\mathrm{GL}(g,\mathbb{Z})\subset\mathrm{Sp}(2g,\mathbb{Z})\simeq\mathrm{Sp}(H)$ on $D_2(H)$ to $\tau_2(\mathcal{G}\cap J_2)$. Let $D_2(H^{\mathbb{Q}})$ be the rationalization of the abelian group $D_2(H)$, with $H^{\mathbb{Q}} := H\otimes\mathbb{Q}$. It is clear that $H^{\mathbb{Q}}$ is a $\mathrm{GL}(g,\mathbb{Q})$-module, hence $D_2(H^{\mathbb{Q}})$ is also a $\mathrm{GL}(g,\mathbb{Q})$-module. Then, by standard arguments (see [2] for instance), the $\mathrm{GL}(g,\mathbb{Z})$-module structure on $\tau_2(\mathcal{G}\cap J_2)$ extends to a $\mathrm{GL}(g,\mathbb{Q})$-module structure on $\tau_2(\mathcal{G}\cap J_2)\otimes\mathbb{Q}$.

In this appendix, we fix a genus $g\geq 4$ and we give the decomposition of this module into irreducible $\mathrm{GL}(g,\mathbb{Q})$-modules. We do not use any results from Section 5. Recall from Section 6 that we have a basis $(a_i,b_i)_{1\leq i\leq g}$ for H, inducing a basis for $H^{\mathbb{Q}}$. This also yields a decomposition $H=A^{\mathbb{Q}}\oplus B^{\mathbb{Q}}$, with $A^{\mathbb{Q}}$ and $B^{\mathbb{Q}}$ stable under the action of $\mathrm{GL}(g,\mathbb{Q})$. Specifically, $\mathrm{GL}(A^{\mathbb{Q}})$ acts on $A^{\mathbb{Q}}$ and $(A^{\mathbb{Q}})^*$ in the natural way, $B^{\mathbb{Q}}$ is identified to $(A^{\mathbb{Q}})^*$ via ω and $\mathrm{GL}(A^{\mathbb{Q}})$ is identified to $\mathrm{GL}(g,\mathbb{Q})$ through the basis $(a_1,\dots,a_g)$ of $A^{\mathbb{Q}}$.

Let $D_{i,j}$ be the subspace of $D_2(H^{\mathbb{Q}})$ generated by expansions of trees with i leaves in $A^{\mathbb{Q}}$ and j leaves in $B^{\mathbb{Q}}$, for $0\leq i\leq 4$ and $i+j=4$. We compute the dimensions of these submodules of $D_2(H^{\mathbb{Q}})$.

Lemma 1.A.1. *For any $g \geq 3$, we have:*

$$\dim(D_2(H^{\mathbb{Q}})) = \frac{g^2(2g-1)(2g+1)}{3} \tag{1.A.1}$$

$$\dim(D_{i,j}) = \dim(D_{j,i}) \tag{1.A.2}$$

$$\dim(D_{0,4}) = \frac{g^2(g-1)(g+1)}{12} \tag{1.A.3}$$

$$\dim(D_{1,3}) = \frac{g^2(g-1)(g+1)}{3} \tag{1.A.4}$$

$$\dim(D_{2,2}) = \frac{g^2(g^2+1)}{2} \tag{1.A.5}$$

Proof. Equation (1.A.1) is a consequence of the isomorphism $\frac{(\Lambda^2 H^{\mathbb{Q}} \otimes \Lambda^2 H^{\mathbb{Q}})^{\mathfrak{S}_2}}{\Lambda^4 H^{\mathbb{Q}}} \simeq D_2(H^{\mathbb{Q}})$ (see diagram (2.2)). Equation (1.A.2) is obtained by interchanging the $a_i's$ and $b_i's$. We also notice that $D_{0,4} \simeq D_2(A^{\mathbb{Q}})$, and we obtain equation (1.A.3). The space $D_{1,3}$ is isomorphic to $A^{\mathbb{Q}} \otimes \mathcal{L}_3(B^{\mathbb{Q}})$, and the dimension of $\mathcal{L}_3(V)$ is equal to $\frac{n^3-n}{3}$ for a vector space V of dimension n: this proves equation (1.A.4). Equation (1.A.5) follows from the previous using that $D_2(H^{\mathbb{Q}}) = \bigoplus_{0 \leq i \leq 4} D_{i,4-i}$. One can also get equation (1.A.5) by showing that there is an isomorphism $D_{2,2} \simeq S^2(A^{\mathbb{Q}} \otimes B^{\mathbb{Q}})$. □

We now decompose $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ in 3 submodules and compute their respective dimensions. Denote $\mathrm{Tr}^{A,\mathbb{Q}}$ for $\mathrm{Tr}^A \otimes \mathbb{Q}$ and $\mathrm{Tr}^{B,\mathbb{Q}}$ for $\mathrm{Tr}^B \otimes \mathbb{Q}$. The kernels of these two maps are both regarded as $\mathrm{GL}(g,\mathbb{Q})$-submodules of $D_2(H^{\mathbb{Q}})$.

Corollary 1.A.2. *The space $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ is a subset of*

$$\mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}}) \cap \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}}) = \big(D_{1,3} \cap \mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}})\big) \oplus D_{2,2} \oplus \big(D_{3,1} \cap \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}})\big),$$

and the summands are $\mathrm{GL}(g,\mathbb{Q})$-submodules with respective dimensions $\frac{g(g+1)(2g^2-2g-3)}{6}$, $\frac{g^2(g^2+1)}{2}$ and $\frac{g(g+1)(2g^2-2g-3)}{6}$.

Proof. The inclusion is a consequence of Theorem 4.10, given that $\mathcal{G} = \mathcal{A} \cap \mathcal{B}$. The decomposition is an immediate consequence of the fact that $D_{3,1} \oplus D_{2,2} \subset \mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}})$, and $D_{1,3} \oplus D_{2,2} \subset \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}})$. The maps $\mathrm{Tr}^{A,\mathbb{Q}}$ and $\mathrm{Tr}^{B,\mathbb{Q}}$ respect the action of $\mathrm{GL}(g,\mathbb{Q})$ by Remark 4.6, hence the 3 summands are $\mathrm{GL}(g,\mathbb{Q})$-submodules. The computation of the dimensions is a consequence of the rank theorem and the previous lemma, as $\mathrm{Tr}^{A,\mathbb{Q}}$ is surjective onto $S^2(H/A \otimes \mathbb{Q})$. □

Next, we use the representation theory of $\mathrm{SL}(g,\mathbb{C})$, and exhibit the irreducible modules in $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ by finding heighest weight vectors. Our notation convention for a Young diagram with n rows of type $(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0)$ is $[\lambda_1\lambda_2...\lambda_n]$. To such a diagram is associated an irreducible representation of $\mathrm{SL}(g,\mathbb{Q})$ whenever $n \leq g-1$, as described in [10]. For short, to a Young diagram $\lambda := [\lambda_1\lambda_2...\lambda_{g-1}]$ is associated the subrepresentation of the tensor product $\bigotimes_{i=1}^{g-1} S^{(\lambda_i-\lambda_{i+1})}(\Lambda^i V)$ spanned by $v_\lambda := (e_1)^{\lambda_1-\lambda_2} \otimes (e_1 \wedge e_2)^{(\lambda_2-\lambda_3)} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_{g-1})^{(\lambda_{g-1}-\lambda_g)}$, where $V := \mathbb{Q}^g$ has a basis $e_1, e_2 \ldots e_g$, and $\lambda_g = 0$. This defines both a representation of $\mathrm{GL}(g,\mathbb{Q})$ and $\mathrm{SL}(g,\mathbb{Q})$.

Theorem 1.A.3. *For any $g \geq 4$, we have an isomorphism of $\mathrm{SL}(g,\mathbb{Q})$-modules*

$$\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q} = 2[0] + 2[21^{g-2}] + [42^{g-2}] + [2^21^{g-4}] + [32^{g-3}1] + [1^{g-2}] + [321^{g-3}] + [1^2].$$

Sketch of proof. We simply need to exhibit highest weight vectors in $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ for the action of $\mathrm{SL}(g,\mathbb{Q})$ on $D_2(H^{\mathbb{Q}})$, such that the sum of the dimensions of the modules they generate is the dimension of $\mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}}) \cap \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}})$. We can check this using Lemma

1.A.1, and it is standard representation theory to verify that a given vector is a highest weight vector. Hence we get that $D_{2,2}$ decomposes into

$\bigoplus$	$[0]$	$[0]$	$[21^{g-2}]$	$[21^{g-2}]$	$[42^{g-2}]$	$[2^2 1^{g-4}]$
dim	1	1	g^2-1	g^2-1	$\frac{g^2(g-1)(g+3)}{4}$	$\frac{g^2(g+1)(g-3)}{4}$
HWV	$\tau_2(T_\xi)$	$\sum\limits_{i,j=1}^{g} \begin{matrix} a_i & b_j \\ a_j & b_i \end{matrix}$	$\sum\limits_{i=1}^{g} \begin{matrix} a_1 & a_i \\ b_g & b_i \end{matrix}$	$\sum\limits_{i=1}^{g} \begin{matrix} a_1 & b_g \\ a_i & b_i \end{matrix}$	$\begin{matrix} a_1 & b_g \\ b_g & a_1 \end{matrix}$	$\begin{matrix} a_1 & b_{g-1} \\ a_2 & b_g \end{matrix}$

where T_ξ is the Dehn twist around the boundary component of $\Sigma_{g,1}$, and $\big(D_{1,3} \cap \mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}})\big) \oplus \big(D_{3,1} \cap \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}})\big)$ decomposes into

$\bigoplus$	$[32^{g-3}1]$	$[1^{g-2}]$	$[321^{g-3}]$	$[1^2]$
dim	$\frac{g^2(g-2)(g+2)}{3}$	$\frac{g(g-1)}{2}$	$\frac{g^2(g-2)(g+2)}{3}$	$\frac{g(g-1)}{2}$
HWV	$\begin{matrix} a_1 & b_g \\ b_g & b_{g-1} \end{matrix}$	$\sum\limits_{i=1}^{g} \begin{matrix} a_i & b_{g-1} \\ b_i & b_g \end{matrix}$	$\begin{matrix} a_2 & a_1 \\ a_1 & b_g \end{matrix}$	$\sum\limits_{i=1}^{g} \begin{matrix} a_1 & a_i \\ a_2 & b_i \end{matrix}$

One can also get these decompositions by giving tensorial description of the modules (such as $D_{2,2} \simeq S^2(A^{\mathbb{Q}} \otimes B^{\mathbb{Q}})$) and by using Pieri's formula. It remains to show that the above highest weight vectors are indeed in $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$. The author checked this for $g \geq 4$ and did it in the same spirit as in the proof of Proposition 6.3. $\square$

This already gives a rational version of Proposition 6.3.

Corollary 1.A.4. *For any $g \geq 4$, $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q} = \mathrm{Ker}(\mathrm{Tr}^{A,\mathbb{Q}}) \cap \mathrm{Ker}(\mathrm{Tr}^{B,\mathbb{Q}})$.*

Finally, we turn to the decomposition of $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ into irreducible $\mathrm{GL}(g,\mathbb{Q})$-modules. For any integer $k \geq 0$, we now denote Det^k the kth power of the determinant representation, and Det^{-k}, its dual. Any irreducible rational representation of $\mathrm{GL}(g,\mathbb{C})$ is obtained as the tensor product of an irreducible representation of $\mathrm{SL}(g,\mathbb{C})$ of type λ (for a young diagram λ) with a power of the determinant representation. By looking at the action of the center of $\mathrm{GL}(g,\mathbb{Q})$ on the highest weight vectors given in the proof of Theorem 1.A.3, we get the following:

Theorem 1.A.5. *For any $g \geq 4$, we have an isomorphism of* $\mathrm{GL}(g,\mathbb{Q})$*-modules*

$$\begin{aligned}\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q} = 2[0] + 2[21^{g-2}] \otimes \mathrm{Det}^{-1} + [42^{g-2}] \otimes \mathrm{Det}^{-2} + [2^2 1^{g-4}] \otimes \mathrm{Det}^{-1} \\ + [32^{g-3}1] \otimes \mathrm{Det}^{-2} + [1^{g-2}] \otimes \mathrm{Det}^{-1} + [321^{g-3}] \otimes \mathrm{Det}^{-1} + [1^2].\end{aligned}$$

Proof. We know that each irreducible summand W of the $\mathrm{SL}(g,\mathbb{Q})$-module decomposition of $\tau_2(\mathcal{G} \cap J_2) \otimes \mathbb{Q}$ is isomorphic as a $\mathrm{GL}(g,\mathbb{Q})$-module to $W \otimes \mathrm{Det}^k$, for some $k \in \mathbb{Z}$. We also know that the isomorphism between the "model" representation given by the Young diagram λ and W can be made explicit by sending v_λ to the highest weight vector of our representation. The integer k must be chosen in such a way that this isomorphism lifts to an isomorphism of $\mathrm{GL}(g,\mathbb{Q})$-modules.

We only do the computation for one summand, say $\lambda = [32^{g-3}1]$. The map sending v_λ to $T_\lambda := \begin{matrix} a_1 & b_g \\ b_g & b_{g-1} \end{matrix}$ is an isomorphism of $\mathrm{SL}(g,\mathbb{Q})$-modules, but one can check that for any $d \in \mathbb{Q}$, $(dId) \cdot v_\lambda = d^{2g-2} v_\lambda$, while $(dId) \cdot T_\lambda = \frac{1}{d^2} T_\lambda$. By choosing $k = -2$, we get that the map from $[32^{g-3}1] \otimes \mathrm{Det}^{-2}$ to the $\mathrm{GL}(g,\mathbb{Q})$-module spanned by T_λ, sending $v_\lambda \otimes 1$ to T_λ is a $\mathrm{GL}(g,\mathbb{Q})$-equivariant isormorphism. More generally, one can check that for a Young diagram $\lambda := [\lambda_1 \lambda_2 \ldots \lambda_{g-1}]$ appearing in the irreducible decomposition of $D_{i,j}$, we get $k = \frac{1}{g}(i - j - \sum\limits_{i=1}^{g-1} \lambda_i)$. $\square$

Remark 1.A.6. In the decomposition of Theorem 1.A.5, the action of ι induces the following symmetries:

1. the irreducible summands in $D_{2,2}$ are isomorphic to their own duals,
2. the irreducible summands in $D_{1,3}$ and $D_{3,1}$ are exchanged when dualizing, indeed we have: $([321^{g-3}] \otimes \mathrm{Det}^{-1})^* \simeq [32^{g-3}1] \otimes \mathrm{Det}^{-2}$, and $[1^2]^* \simeq [1^{g-2}] \otimes \mathrm{Det}^{-1}$.

This is an instance of a general fact: for any $k \geq 1$, $\tau_k(\mathcal{G} \cap J_k) \otimes \mathbb{Q}$ is isomorphic to its dual as a $\mathrm{GL}(g, \mathbb{Q})$-module. Indeed, the map ι preserves $\tau_k(\mathcal{G} \cap J_k) \otimes \mathbb{Q}$, and one can see by direct computation that $\forall P \in \mathrm{GL}(g, \mathbb{Q}), \forall X \in D_k(H^{\mathbb{Q}}), \iota(P \cdot X) = (P^T)^{-1} \cdot \iota(X)$. Hence the basis of $H^{\mathbb{Q}}$ induces a $\mathbb{Q}$-module isomorphism between $D_2(H^{\mathbb{Q}})$ and its dual, and the composition of this isomorphism with ι is a $\mathrm{GL}(g, \mathbb{Q})$-module isomorphism between $D_2(H^{\mathbb{Q}})$ and $D_2(H^{\mathbb{Q}})^*$. We conclude that if W is an irreducible module in $\tau_k(\mathcal{G} \cap J_k) \otimes \mathbb{Q}$, then ιW is also an irreducible module in $\tau_k(\mathcal{G} \cap J_k) \otimes \mathbb{Q}$ which is isomorphic as a $\mathrm{GL}(g, \mathbb{Q})$-module to the dual representation W^*.

Chapter 2

Triviality of the J_4-equivalence among homology 3-spheres

ABSTRACT. We prove that all homology 3-spheres are J_4-equivalent, i.e. that any homology 3-sphere can be obtained from one another by twisting one of its Heegaard splittings by an element of the mapping class group acting trivially on the fourth nilpotent quotient of the fundamental group of the gluing surface. We do so by exhibiting an element of J_4, the fourth term of the Johnson filtration of the mapping class group, on which (the core of) the Casson invariant takes the value 1. In particular, this provides an explicit example of an element of J_4 that is not a commutator of length 2 in the Torelli group.

1 Introduction

The study of the mapping class groups of surfaces is related to the study of 3-manifolds through the notion of *Heegaard splitting*, a way of presenting a 3-manifold by specifying a gluing map between the boundaries of two handlebodies. Also, one can take such a presentation and compose the gluing map by an element of the mapping class group, yielding another 3-manifold. This procedure of "twisting" Heegaard splittings is called a "surgery". When we restrict the surgeries to certain subgroups of the mapping class group, it provides us with some equivalence relations among the set of homeomorphism classes of 3-manifolds. This can help to understand and study the topological properties of 3-manifolds invariants, in particular the ones of the family of so-called "finite-type invariants" (see, e.g., [42] for the case of the Casson invariant). For example, one can consider restricting surgeries to the *lower central series* or the *Johnson filtration* of the *Torelli group*. Let $\Sigma := \Sigma_{g,1}$ be a compact connected oriented surface of genus g, with one boundary component, and $\mathcal{M} = \mathcal{M}(\Sigma)$ the mapping class group of Σ relative to the boundary. The Torelli group $\mathcal{I} = \mathcal{I}(\Sigma)$ is the subgroup of $\mathcal{M}$ consisting of elements acting trivially on the first homology group of the surface. Let k be a positive integer. The $(k+1)$-th term of the lower central series of the Torelli group is defined inductively as the commutator subgroup $\Gamma_{k+1}\mathcal{I} := [\Gamma_k\mathcal{I}, \mathcal{I}]$, with $\Gamma_1\mathcal{I} := \mathcal{I}$. The k-th term $J_k = J_k(\Sigma)$ of the Johnson filtration consists of elements acting trivially on the k-th nilpotent quotient of the fundamental group of Σ.

Let us denote by $\mathcal{S}(3)$ the set of homeomorphism classes of homology 3-spheres (to which we restrict ourselves in this paper). For any embedding j of the surface Σ in a homology 3-sphere M, and for any element φ of the Torelli group of the surface, one can define a new homology 3-sphere $M_{j,\varphi}$ by removing from M an open neighborhood $\nu\Sigma$ of Σ, and gluing the mapping cylinder of φ in place of $\nu\Sigma$. It is known that any homology 3-sphere can be obtained from S^3 in this way (see, for example, [41]). Two homology 3-spheres M and M' are said to be J_k*-equivalent* (resp Y_k*-equivalent*) if and only M' is isomorphic to $M_{j,\phi}$ for some embedding $j : \Sigma \to M$ of a surface Σ and some $\phi \in J_k(\Sigma)$ (resp. in $\Gamma_k\mathcal{I}(\Sigma)$). These are known to be equivalence relations, and one can easily see that Y_k-equivalence implies

J_k-equivalence. Moreover, in the above definitions, we can always restrict ourselves to the case where Σ is the intermediate surface of a Heegaard splitting, thus limiting ourselves to the above-mentioned notion of "surgery" (see [39, Lemma 2.1], for instance). Morita [42] and Pitsch [49], successively, have shown that J_2-equivalence and J_3-equivalence are trivial on $\mathcal{S}(3)$. In constrast, the Y_2-equivalence is non-trivial and is classified by Rokhlin's invariant $\mu : \mathcal{S}(3) \to \mathbb{Z}/2\mathbb{Z}$ (see [15] and [38]). The goal of this paper is to show that J_4-equivalence is also trivial on $\mathcal{S}(3)$.

Let us now be more precise. We recall the definition of the Johnson filtration and the *Johnson homomorphisms*, which have been introduced and studied by Johnson and Morita in [25, 44]. Recall that $\pi := \pi_1(\Sigma)$ is a free group. We denote by ζ the element of π corresponding to the oriented boundary $\partial\Sigma$. For $k \geq 1$, we consider the lower central series of the group π, the filtration $(\Gamma_k\pi)_{k\geq 1}$. We call the quotient $N_k := \pi/\Gamma_{k+1}\pi$ the *k-th nilpotent quotient* of π. The first nilpotent quotient is canonically isomorphic to $H := H_1(\Sigma, \mathbb{Z})$. The intersection form of the surface induces a symplectic form ω on the abelian group H. The action of $\mathcal{M}$ on the surface induces the natural $\mathrm{Sp}(H)$-module structure on H, as any transformation of the surface preserves the intersection form. The curves $(\alpha_i)_{1\leq i\leq g}$ and $(\beta_i)_{1\leq i\leq g}$ on Figure 2.1 are two cut systems of Σ such that each curve in the first one has exactly one intersection point with exactly one curve in the second one, and vice versa. Such a choice is called a system of *meridians* and *parallels*. In particular, it fixes a choice of a symplectic basis for $H = \mathbb{Z}\langle a_1, a_2 \dots a_g, b_1, b_2, \dots b_g\rangle$, where a_i (resp. b_i) is the homology class of α_i (resp. β_i).

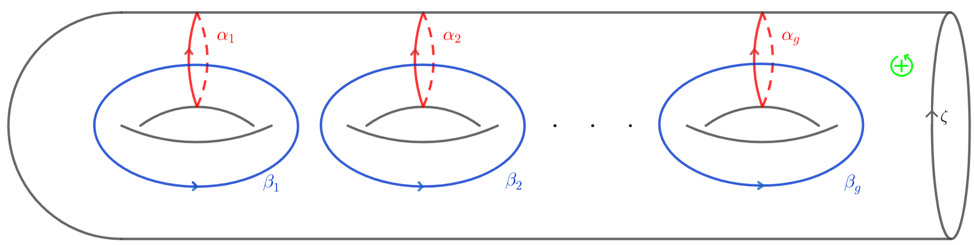


Figure 2.1: Model for $\Sigma_{g,1}$, and a possible choice of system of meridians and parallels

We will also need at some point to lift these curves to a basis of $\pi = \pi_1(\Sigma, x_0)$, with $x_0 \in \partial\Sigma$. We do so as described in Figure 2.2.

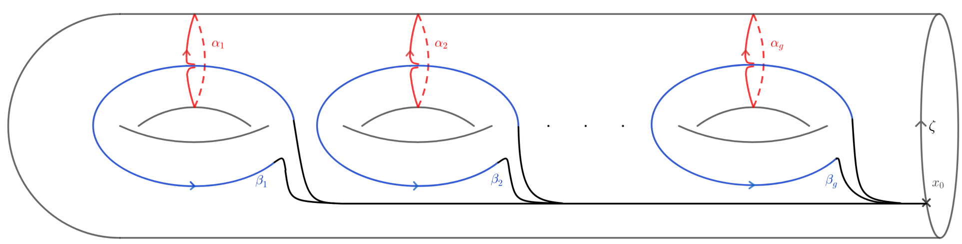


Figure 2.2: Based curves inducing a basis of π

It is clear that $\mathcal{M}$ acts both on π and all its nilpotent quotients. There is an exact sequence:

$$0 \longrightarrow \mathcal{L}_{k+1}(H) \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow 0$$

where $\mathcal{L}(H)$ stands for the graded free Lie algebra generated by H in degree 1, and the first non-trivial arrow is given by the identification between $\mathcal{L}_{k+1}(H)$ and $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$. This sequence induces the short exact sequence:

$$0 \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)) \longrightarrow \mathrm{Aut}(N_{k+1}) \longrightarrow \mathrm{Aut}(N_k).$$

The group J_k is defined as the kernel of the canonical homomorphism $\rho_k : \mathcal{M} \to \mathrm{Aut}(N_k)$. In particular, by the Hurewicz theorem, J_1 is the Torelli group, otherwise denoted $\mathcal{I} = \mathcal{I}(\Sigma)$. The alternative notation $\mathcal{K} = \mathcal{K}(\Sigma)$ is also sometimes used for J_2. The restriction of ρ_{k+1} to J_k then induces a morphism:

$$\tau_k : J_k \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)).$$

This map is called the *k-th Johnson homomorphism*, and its kernel is J_{k+1}. Furthermore, the mapping class group acts on itself by conjugation, inducing an action of the symplectic group $\mathrm{Sp}(H)$ on the quotient J_k/J_{k+1}. Each τ_k is then $\mathrm{Sp}(H)$-equivariant. It is also known that the graded space associated to the Johnson filtration has a Lie structure, its bracket being induced by the commutator in $\mathcal{M}$. The target space of τ_k can be identified with the space of *derivations of degree* k, i.e. derivations of the Lie algebra $\mathcal{L}(H)$ mapping $H = \mathcal{L}_1(H)$ to $\mathcal{L}_{k+1}(H)$. We denote by $D_k(H)$ the subspace of *symplectic derivations* of degree k, which consists of derivations of $\mathcal{L}(H)$ of degree k sending $\tilde{\omega} \in \Lambda^2 H \simeq \mathcal{L}_2(H)$, the bivector dual to ω, to 0. The fact that an element of $\mathcal{M}$ fixes the boundary of Σ allows to further restrict the target of τ_k to $D_k(H)$. Also, $D_k(H)$ can be defined by the short exact sequence:

$$0 \longrightarrow D_k(H) \longrightarrow H \otimes \mathcal{L}_{k+1}(H) \longrightarrow \mathcal{L}_{k+2}(H) \longrightarrow 0$$

where the arrow from $H \otimes \mathcal{L}_{k+1}(H)$ to $\mathcal{L}_{k+2}(H)$ is the bracket of the free Lie algebra.

With these definitions, the spaces $(D_k(H))_{k\geq 1}$ reassembles in a graded Lie algebra $D(H)$ (the bracket of two derivations d_1 and d_2 being classically defined as $d_1 d_2 - d_2 d_1$). The family $(\tau_k)_{k\geq 1}$ induces a map:

$$\tau : \bigoplus_{k\geq 1} J_k/J_{k+1} \longrightarrow D(H)$$

which is an $\mathrm{Sp}(H)$-equivariant graded Lie morphism.

The Casson invariant $\lambda : \mathcal{S}(3) \to \mathbb{Z}$ is an invariant of homology 3-spheres, lifting the Rokhlin invariant $\mu : \mathcal{S}(3) \to \mathbb{Z}_2$, and originally defined by counting in some way the number of irreducible representations of the fundamental group of the homology 3-sphere into $SU(2)$. In [42], Morita explained that the map λ_j induced on $\mathcal{I}$ by the Casson invariant and any Heegaard embedding $j : \Sigma \to S^3$:

$$\begin{aligned} \lambda_j : \mathcal{I} &\longrightarrow \mathbb{Z} \\ \varphi &\longmapsto \lambda(S^3_{j,\varphi}) \end{aligned}$$

is *not* a homomorphism (here, by a Heegaard embedding, we mean that capping the surface $j(\Sigma)$ by a disk yields a Heegaard splitting of S^3). Nevertheless, Morita showed that its restriction to $\mathcal{K} := J_2$ is a homomorphism. He also showed that this restriction can be expressed as the sum of two homomorphisms:

$$-\lambda_j = \frac{1}{24}d + q_j : \mathcal{K} \to \mathbb{Z}. \tag{1.1}$$

The map d, named *the core of the Casson invariant* is independent of j, and the map q_j factorizes through the second Johnson homomorphism. Thus, the Casson invariant induces a well-defined homomorphism λ on J_k for any $k \geq 3$, meaning that the value of the Casson invariant on $S^3_{j,\varphi}$ is independent of j when $k \geq 3$. The map d is rather difficult to understand, but it is known that Dehn twists along bounding simple closed curves (abbreviated BSCC in the rest of this paper) of genus 1 and 2 generate $\mathcal{K}$ and that the value of d on a Dehn twist along a BSCC of genus h is $4h(h-1)$. Recall that the genus of a BSCC is defined as the genus of the subsurface bounding the curve which does not contain $\partial\Sigma$. We will denote $\mathcal{K}'$ (resp. $\mathcal{K}''$) the subgroup of $\mathcal{K}$ generated by twists around BSCC of genus 1 (resp. genus 2).

It was claimed by Morita, also in [42], and written explicitly by Massuyeau and Meilhan in [39], that $\lambda(J_3) = \mathbb{Z}$ in genus $g = 2$. Moreover, according to Habiro [15], Y_3-equivalence among homology 3-spheres is classified by λ. Massuyeau and Meilhan [39, Rem. 6.4] then explained how to reprove from these two facts Pitsch's result stating that any two homology 3-spheres are always J_3-equivalent. Using the same strategy, we shall prove the following.

Theorem A. *For any genus $g \geq 2$, the restriction of $\lambda : J_3 \to \mathbb{Z}$ to J_4 is surjective.*

Theorem B. *The J_4-equivalence is trivial on $\mathcal{S}(3)$. In other words, every homology 3-sphere can be obtained from S^3 by twisting one of its Heegaard splitting by an element of the fourth term of its Johnson filtration.*

A motivation for Theorem A is given by the Dehn-Nielsen Theorem, which states in particular that an element of the mapping class group $\mathcal{M}$ is completely determined by its action on π. Thus, given a map on $\mathcal{M}$, one could hope to compute it by purely algebraic methods, considering only this action. This is also a motivation when using the Johnson homomorphisms. For the case of the Casson invariant, we already know that there is no $k \geq 1$ such that λ can be fully computed from $\rho_k : \mathcal{M} \to \mathrm{Aut}(N_k)$, as Hain [17] proved that $\lambda(J_k) \neq \{0\}$ for $k \geq 3$. Nevertheless, it is still possible that for some $k \geq 5$, the homomorphism λ restricted to J_k is no longer surjective. This question is of course related to the determination of the graded space associated to the Johnson filtration.

In order to prove Theorem A, we will explicitly build an element φ in J_4 whose Casson invariant is equal to 1. This element also seems to be the first explicit example of an element of J_4 which cannot be obtained as a bracket of elements of lower terms of the Johnson filtration. The result of Hain [17] mentioned above implies that there are elements which are not in the commutator subgroup $[\mathcal{I}, \mathcal{K}]$ arbitrarily far down in the Johnson filtration. Theorem B is next deduced from Theorem A, using the classification of the Y_4-equivalence by Habiro [15].

The paper is organized as follows. In Section 2, we recall and use computational tools developed by Morita in [42] and Kawazumi and Kuno in [31], and we build the element φ. We also discuss the complexity of the construction of such an element. In Section 3, we define J_4-equivalence and explain how the fact that $\lambda(J_4) = \mathbb{Z}$ implies the triviality of J_4-equivalence on $\mathcal{S}(3)$. Finally, the computer program used to claim some equalities in Section 2 is given and explained in appendix 2.A.

Acknowledgements. I would like to thank my advisor, Gwénaël Massuyeau, for guiding and supporting me throughout this work, and the members of the forum ask.sagemaths.org for their answers to my questions.

2 A special element of J_4

In this section, we recall a method presented by Kawazumi and Kuno [31] to explicitly compute the action on π of any element of the Torelli group and, in particular, the image of an element in J_k by the k-th Johnson homomorphism. We then use this method to compute $\tau_3(\psi)$, where $\psi \in \mathcal{M}$ is a certain element of J_3 such that $\lambda(\psi) = 1$. This element was first presented in [39]. We then prove that $\tau_3(\psi) \in \tau_3([\mathcal{K}, \mathcal{I}])$, and use this to build an element $\varphi \in J_4$ such that $\lambda(\varphi) = 1$, proving Theorem A.

2.1 The Kawazumi-Kuno formula

All the results used here are from [31], but the reader will find enlightening additional information in [40]. We refer to these papers for further details.

We denote by $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ the rationalization of H and by $\widehat{T} := \prod_{m=0}^{\infty} H_{\mathbb{Q}}^{\otimes m}$ the completed tensor algebra generated by $H_{\mathbb{Q}}$. The algebra $\widehat{T}$ is filtered by the sequence of two-sided ideals $\widehat{T}_p := \prod_{m \geq p}^{\infty} H_{\mathbb{Q}}^{\otimes m}$. It is known that the choice of a Magnus expansion (in the sense of [30]), gives an identification of $\widehat{\mathbb{Q}\pi}$, the completed group algebra of π (with respect to the filtration induced by the augmentation ideal), with $\widehat{T}$. Furthermore, Massuyeau introduced in [36] the notion of *symplectic expansion.* Shortly, a Magnus expansion is a monoid map θ from π to $\widehat{T}$ such that $\theta(x) = 1 + \{x\} + deg_{\geq 2}$ for all $x \in \pi$, where $\{x\}$ is the class of x in $H_{\mathbb{Q}}$. A symplectic expansion is then an expansion taking group-like values in $\widehat{T}$

and sending ζ to $\exp(-\tilde{\omega})$, where $\tilde{\omega}$ is the element of $H_{\mathbb{Q}}^{\otimes 2}$ representing the symplectic form through Poincaré duality.

As any element of $\mathcal{M}$ acts naturally on $\widehat{\mathbb{Q}\pi}$, a Magnus expansion provides, via the identification $\widehat{\mathbb{Q}\pi} \simeq \widehat{T}$, a map $T^\theta : \mathcal{M} \to \mathrm{Aut}(\widehat{T})$. In particular, denoting by T_γ the Dehn twist on a simple closed curve γ on Σ, the map $T^\theta(T_\gamma)$ is an automorphism of $\widehat{T}$. We now fix a Magnus expansion, and we call T^θ the *total Johnson map*, this terminology being justified by the following theorem, derived from [30, Theorem 3.1]. Define, for any $\phi \in \mathcal{I}$, $\tau^\theta(\phi) := T^\theta(\phi)_{|H_{\mathbb{Q}}} - \mathrm{Id}_{|H_{\mathbb{Q}}}$ and write $\tau^\theta = \sum\limits_{k \geq 1} \tau_k^\theta$, with $\tau_k^\theta(\phi) \in \mathrm{Hom}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(k+1)})$. Also, consider the rational version $\tau_{k,\mathbb{Q}} := \tau_k \otimes \mathbb{Q}$ of the k-th Johnson homomorphism.

Theorem 2.1 (Kawazumi)**.** *For any Magnus expansion θ, any $k \geq 1$, and any $\phi \in J_k$, we have*

$$\tau_k^\theta(\phi) = \tau_{k,\mathbb{Q}}(\phi).$$

where $\mathcal{L}_{k+1}(H) \otimes \mathbb{Q}$ is regarded as a subspace of $H_{\mathbb{Q}}^{\otimes(k+1)}$ in the usual way.

We now define $l^\theta := \log \circ \theta$, which is a map from π to $\widehat{T}$, and consider the "cyclicization" map $N : \widehat{T}_1 \to \widehat{T}_1$, defined by $N|_{H_{\mathbb{Q}}^{\otimes p}} = \sum_{m=0}^{p-1} \nu^m$, for $p \geq 1$, where $\nu : H_{\mathbb{Q}}^{\otimes p} \to H_{\mathbb{Q}}^{\otimes p}$ is the map induced by the cyclic permutation of the tensors. We also define, for all x in π,

$$L^\theta(x) := \frac{1}{2} N(l^\theta(x)^2) \in H_{\mathbb{Q}} \otimes \widehat{T}_1 = \widehat{T}_2. \tag{2.1}$$

The value of L^θ on $x \in \pi$ actually only depends on the conjugacy class of x and $L^\theta(x) = L^\theta(x^{-1})$. Thus, we can write abusively $L^\theta(\gamma)$ for any closed curve γ in Σ. Furthermore, identifying $H_{\mathbb{Q}} \otimes \widehat{T}_1$ with $\mathrm{Hom}(H_{\mathbb{Q}}, \widehat{T}_1)$ by Poincaré duality, such an element $L^\theta(\gamma)$ can be regarded as a weakly nilpotent derivation of $\widehat{T}$. By a *weakly nilpotent* derivation, we mean a derivation d such that $d(\widehat{T}_p) \subset \widehat{T}_p$, and for any $p \geq 1$, there is some $n \geq 1$ such that $d^n(\widehat{T}) \subset \widehat{T}_p$. The exponential of such a derivation is a well-defined automorphism of $\widehat{T}$.

Remark 2.2. The reader should be aware that there are two ways to identify H (or $H_{\mathbb{Q}}$) with its dual, and that the conventions chosen in [30, 31] and [36, 40] are different. We chose the convention of the latter, meaning that the isomorphism from H to H^* is given by $x \mapsto \omega(x, -)$.

We now recall a formula due to Kawazumi and Kuno [31, Theorem 1.1.1].

Theorem 2.3 (Kawazumi, Kuno)**.** *Let θ be a symplectic expansion and γ a simple closed curve on Σ. Then we have:*

$$T^\theta\,(T_\gamma) = e^{L^\theta(\gamma)}. \tag{2.2}$$

By degree considerations, making use of equation (2.2), Kawazumi and Kuno also get the following simple formulas, which are extracted from [31, Theorem 6.3.1]:

Corollary 2.4 (Kawazumi, Kuno)**.** *Let θ be a symplectic expansion and γ a bounding simple closed curve. Then we have:*

$$\tau_2^\theta\,(T_\gamma) = L_4^\theta(\gamma)$$
$$\tau_3^\theta\,(T_\gamma) = L_5^\theta(\gamma).$$

Hence, as implied by Theorem 2.1, if φ is given as a product of Dehn twists along BSCC and is in J_2 (resp. J_3), it is easy to compute $\tau_2(\varphi) = \tau_2^\theta(\varphi)$ (resp. $\tau_3(\varphi) = \tau_3^\theta(\varphi)$) by using Corollary 2.4.

In order to do so, we will need an instance of a symplectic expansion θ. As explained in [36], we can build inductively such an expansion and write explicitly its values on generators of π in low degrees. The following proposition shows that for our purposes we only need to know the values of a symplectic expansion up to order 2.

Proposition 2.5. *Let γ be a bounding simple closed curve. In order to compute $L_4^\theta(\gamma)$ and $L_5^\theta(\gamma)$, we only need the expression of l^θ up to degree 2.*

Proof. As γ is separating, any of its representative in π will be a product of commutators. Therefore $l^\theta(\gamma)$ starts in degree 2 and it follows that we only need the expression of l^θ up to order 3 to compute L_4^θ and L_5^θ, due to the square in formula (2.1). Let U, V be elements of π, let u, v be their respective classes in H, and write

$$\theta(U) = 1 + u + \theta_2(U) + \theta_3(U) + deg_{\geq 4} \in \widehat{T}$$
$$\theta(V) = 1 + v + \theta_2(V) + \theta_3(V) + deg_{\geq 4} \in \widehat{T}$$

where $\theta_i(X)$ is homogeneous of degree i. The image by θ of the inverse of an element X in π is given by $\theta(X^{-1}) = \sum_{i=0}^{\infty}(-1)^i(\theta(X)-1)^i$. Then we compute directly

$$\begin{aligned}\theta([U,V]) &= \theta(U)\theta(V)\theta(U)^{-1}\theta(V)^{-1}\\ &= 1 + uv - vu + vuv - uvu + vu^2 - uv^2\\ &\quad + u\theta_2(V) - \theta_2(V)u + \theta_2(U)v - v\theta_2(U) + deg_{\geq 4}\end{aligned}$$

hence

$$\begin{aligned}l^\theta([U,V]) &= uv - vu + vuv - uvu + vu^2 - uv^2\\ &\quad + u\theta_2(V) - \theta_2(V)u + \theta_2(U)v - v\theta_2(U) + deg_{\geq 4}\\ &= uv - vu + vuv - uvu + \frac{1}{2}(u^2v - v^2u - uv^2 + vu^2)\\ &\quad + ul_2^\theta(V) - vl_2^\theta(U) + l_2^\theta(U)v - l_2^\theta(V)u + deg_{\geq 4}\\ &= [u,v] + [u, l_2^\theta(V)] + [l_2^\theta(U), v] + \frac{1}{2}[u,[u,v]] - \frac{1}{2}[v,[v,u]] + deg_{\geq 4}\end{aligned}$$

which depends only on the expression of l^θ up to degree 2. Alternatively, one can use the Baker-Campbell-Hausdorff formula to compute $l^\theta([U,V])$. ☐

Consequently, we will use for our computations a symplectic expansion given up to degree 2. Specifically, we use the truncation of the one given up to degree 4 in [36, Example 2.19]. The fact that it verifies the symplectic condition (up to degree 3) can be checked by hand. The based versions of the curves α_i and β_i defining elements of π are as shown in Figure 2.2.

Proposition 2.6. *There is a symplectic expansion θ of π, which is given in degree ≤ 2 by*

$$l^\theta(\alpha_i) = a_i - \frac{1}{2}[a_i, b_i] + deg_{\geq 3}$$
$$l^\theta(\beta_i) = b_i - \frac{1}{2}[a_i, b_i] + deg_{\geq 3}.$$

Before giving the description of the elements ψ and φ we aim at, we clarify the way we compute in $D_k(H)$, for $1 \leq k \leq 3$.

2.2 Derivations of degree 1, 2 and 3

We recall the description of the spaces $D_k(H)$ in terms of tree-like Jacobi diagrams. More precisely, we consider the spaces of tree-like Jacobi diagram $\mathcal{A}^t(H)$ and rooted tree-like Jacobi diagrams $\mathcal{A}^{t,r}(H)$. A tree is a connected graph that is contractible as a topological space. From now on, by "a tree", we mean a uni-trivalent tree T, possibly rooted, whose trivalent vertices are oriented (the orientation being counterclockwise in all the figures), and whose univalent vertices are colored by elements of H. The cardinality of the set of trivalent vertices $v_3(T)$ is the *degree* of the tree T. The space $\mathcal{A}_k^t(H)$ (resp. $\mathcal{A}_k^{t,r}(H)$) is the $\mathbb{Z}$-module generated by trees (resp. rooted trees) of degree k subject to some relations: multilinearity of the labels, the *AS relation*, and the *IHX relation* (see Figure 2.3).

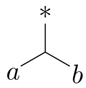
AS $\qquad\qquad$ IHX

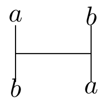
Figure 2.3: The AS and IHX relations

We refer the reader to [33] for further details about what follows. The spaces $\mathcal{A}_k^t(H)$ and $\mathcal{A}_k^{t,r}(H)$ assemble in two graded spaces $\mathcal{A}^t(H)$ and $\mathcal{A}^{t,r}(H)$ endowed respectively with a Lie bracket and a quasi-Lie bracket. For the bracket of $\mathcal{A}^t(H)$, take two trees, and use all the ways to contract external vertices from the first one with the second one using the symplectic form ω. For $\mathcal{A}^{t,r}(H)$, take two trees, glue them along their roots, and add a new root attached to the gluing point (the root must be placed so that the root, the first tree and the second tree are in the clockwise order).

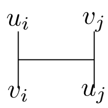
We also define, for $k \geq 1$, maps:

$$\begin{aligned} \eta_k : \mathcal{A}_k^t(H) &\longrightarrow D_k(H) \\ T &\longmapsto \sum_{x \in v_1(T)} l_x \otimes T^x \end{aligned}$$

where $v_1(T)$ is the set of univalent vertices, l_x is the element of H coloring the vertex x and T^x is the rooted tree obtained by setting x to be the root in T. A rooted tree can then be read as an element of $\mathcal{L}_{k+1}(H)$ by setting that  corresponds to $[b, a]$. These maps, to which we refer as "the expansion maps", assemble into a graded Lie algebra morphism. The first expansion map η_1 is an isomorphism, hence any tree with 1 trivalent vertex in the sequel will represent an element of $D_1(H)$.

The rationalization of the expansion $\eta_{k,\mathbb{Q}} : \mathcal{A}_k^t(H) \otimes \mathbb{Q} \to D_k(H) \otimes \mathbb{Q}$ is an isomorphism [14]. Notice that $D_k(H)$ is a free abelian group, and hence is included as a $\mathbb{Z}$-module in $D_k(H) \otimes \mathbb{Q}$. Furthermore, as proved in [33], any element in $D_2(H) \subset D_2(H) \otimes \mathbb{Q}$ can be obtained by linear combination of expansion of elements of $\mathcal{A}_2^t(H)$, and expansion of halves of symmetric trees of degree 2, written $a \odot b := \eta_{2,\mathbb{Q}}\Big(\frac{1}{2}$  $\Big)$. Any element in $D_3(H)$ is obtained by summing expansions of elements of $\mathcal{A}_3^t(H)$ (but η_3 is not injective). We will consequently refer to such trees as elements of $D_k(H) \subset H \otimes \mathcal{L}_{k+1}(H) \subset H_{\mathbb{Q}} \otimes \widehat{T_1}$, for $k = 2, 3$, getting rid of the map η that should appear everywhere. We now recall an important formula from [42].

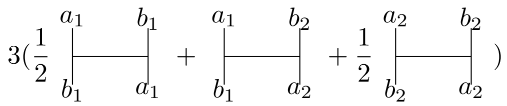
Lemma 2.7 (Morita). *Let γ be a SCC bounding a subsurface F of genus h in Σ, and let $(u_i, v_i)_{1 \leq i \leq h}$ be any symplectic basis of the first homology group of F, then we have:*

$$\tau_2(T_\gamma) = \sum_{i=1}^{h} u_i \odot v_i + \sum_{\substack{i,j=1 \\ i<j}}^{h}$$  $$\in D_2(H).$$

2.3 The map ψ as a product of Dehn twists

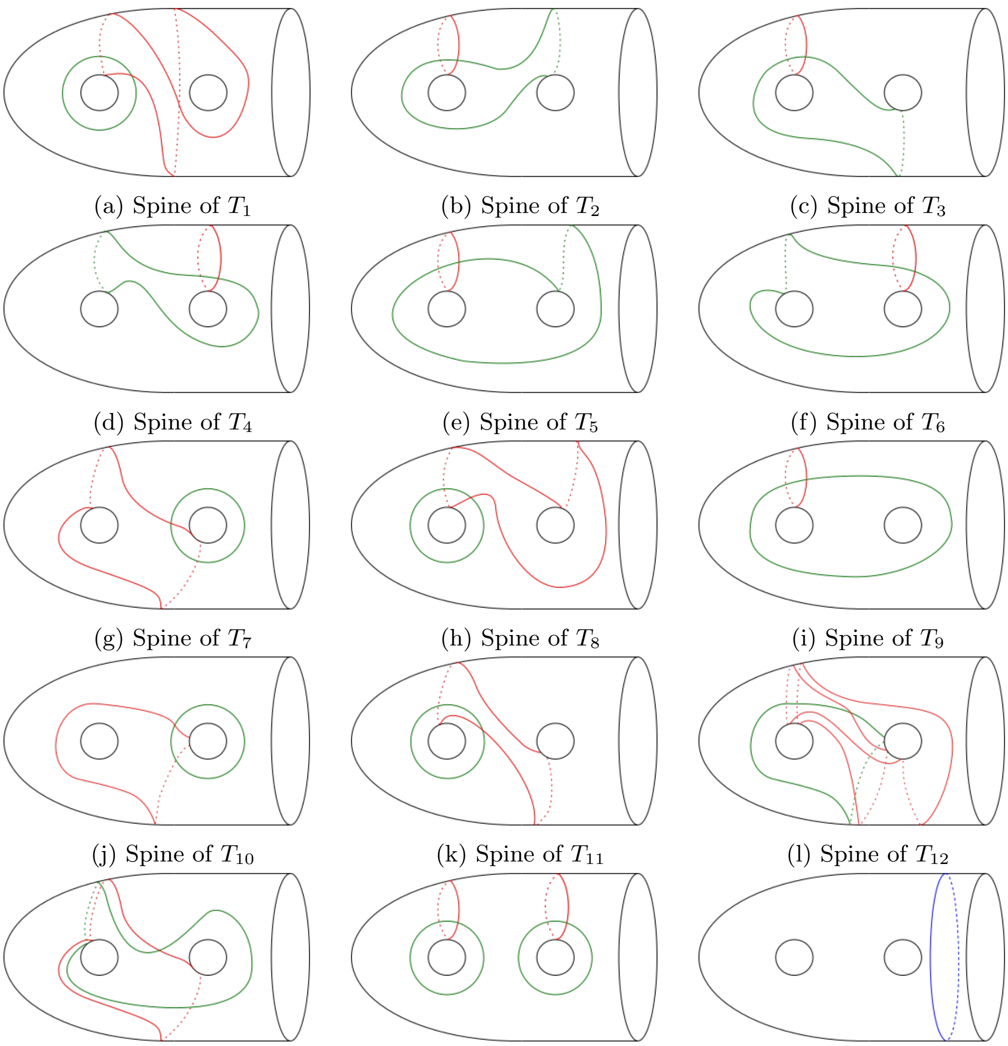
In this section, we take $\Sigma := \Sigma_{2,1}$. Recall that $\mathcal{K}'$ and $\mathcal{K}''$ denote respectively the subgroups of $\mathcal{K}$ generated by BSCC maps of genus 1 and 2. In [39], following a claim by Morita in [42], Massuyeau and Meilhan explicited an element in $3(\tau_2(\mathcal{K}''))$ which is also in

$\tau_2(\mathcal{K}')$. Indeed it can be checked that, in $D_2(H)$:



$$\begin{aligned}
&= 7 \cdot a_1 \odot b_1 + 2 \cdot a_2 \odot b_2 - a_1 \odot (b_1 + b_2) + (b_1 + a_2) \odot b_2 - (a_1 + a_2) \odot b_1 \\
&\quad - (a_1 + b_1 + a_2) \odot b_2 + (a_1 + a_2 + b_2) \odot b_1 + a_1 \odot (b_1 + a_2 + b_2) \\
&\quad - a_2 \odot (a_1 + b_1 + b_2) + 2 \cdot a_1 \odot (b_1 + a_2) + 2 \cdot a_2 \odot (a_1 + b_2) - (a_1 - b_2) \odot b_1 \\
&\quad - a_1 \odot (b_1 - a_2) - (2a_1 + b_2) \odot (b_1 + a_2) + (a_1 + b_1 + a_2) \odot (a_1 + b_1 + b_2).
\end{aligned} \tag{2.3}$$

The right-hand side of equation (2.3) is in $\tau_2(\mathcal{K}')$. Indeed, by Lemma 2.7 applied to the boundary of the neighborhood of two simple closed curves with a single intersection point inducing $u, v \in H$ such that $\omega(u, v) = 1$, we get that $u \odot v \in \tau_2(\mathcal{K}')$. The left-hand side can be obtained as $\tau_2(T^3_{\gamma_2})$ (where γ_2 is drawn in Figure 2.4 and is isotopic to the boundary), hence is in $\tau_2(\mathcal{K}'')$. We deduce that there is a map ψ_1 (not unique), which is a product of Dehn twists on BSCC of genus 1, such that $\psi := T^{-3}_{\gamma_2}\psi_1$ belongs to J_3. Furthermore, we deduce from (1.1) that $\lambda(\psi) = -\frac{1}{24}(d(\psi)) = 1$, considering that $d(T_{\gamma_2}) = 8$ and that d is trivial over $\mathcal{K}'$.



(a) Spine of T_1 (b) Spine of T_2 (c) Spine of T_3

(d) Spine of T_4 (e) Spine of T_5 (f) Spine of T_6

(g) Spine of T_7 (h) Spine of T_8 (i) Spine of T_9

(j) Spine of T_{10} (k) Spine of T_{11} (l) Spine of T_{12}

(m) Spine of T_{13} (n) Spines of T_{s1} and T_{s2} (o) The curve γ_2

Figure 2.4: Twists involved in the definition of ψ

We now exhibit such a $\psi_1 \in \mathcal{K}'$, thus obtaining an application $\psi \in J_3$ whose Casson invariant is 1. A *spine* of the Dehn twist around a BSCC γ of genus 1 designates two simple closed curves (with geometric intersection equal to 1) such that the boundary of a regular neighborghood of the union of the two curves is isotopic to γ. The map ψ_1 is defined as the product of some twists, the spines of which are drawn in Figure 2.4:

$$\psi_1 := T_1^{-1}\, T_2^{-1}\, T_3^2\, T_4^2\, T_5\, T_6^{-1}\, T_7^{-1}\, T_8\, T_9^{-1}\, T_{10}\, T_{11}^{-1}\, T_{12}^{-1}\, T_{13}\, T_{s1}^7\, T_{s2}^2$$

and we set $\psi := T_{\gamma_2}^{-3}\, \psi_1$. This definition of ψ is rather complicated, involving 10 left Dehn twists and 17 right Dehn twists. Counting algebraically (positively for right Dehn twists and negatively for left Dehn twists), we needed -3 twists of genus 2 and 10 twists of genus 1 to create our element ψ. In fact, one needs so many twists to get $\psi \in J_3$ such that $\lambda(\psi) = 1$, as we shall explain in subsection 2.5.

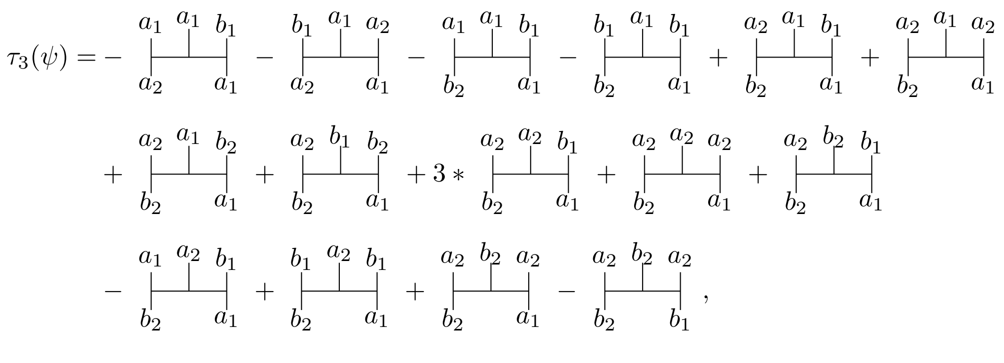
2.4 The map φ

This section is dedicated to the proof of Theorem A. As λ is a homomorphism on J_4, we "simply" need to find an element $\varphi \in J_4$ with $\lambda(\varphi) = 1$. We build φ by "perturbing" the map ψ that has been defined in the previous section. By using the Kawazumi-Kuno formula, we can compute the image of ψ by τ_3 (note here that we actually compute $\tau_{3,\mathbb{Q}}(\psi)$ and not $\tau_3(\psi)$, but the arrow from $D_3(H)$ to its rationalization is injective, hence we indeed get the value of $\tau_3(\psi)$). Indeed, by construction, ψ belongs to J_3, hence $\tau_3^\theta(\psi) = \tau_3(\psi)$. Let us write formally $\psi = \Pi_{i=0}^{27} U_i$, where U_i is either a right or left Dehn twist. Then Corollary 2.4 gives

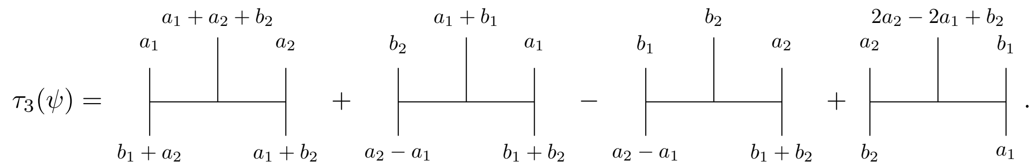
$$\tau_3(\psi) = \sum_{i=0}^{27} \tau_3^\theta(U_i) = \sum_{i=0}^{27} L_5^\theta(U_i).$$

Here we use the fact that $\tau_2^\theta \oplus \tau_3^\theta$ is a homomorphism on J_2. For the proof of this fact, see [36] where it is proven that, for any $k \geq 1$, $\bigoplus_{i \in [k,2k[} \tau_i^\theta$ corresponds to the k-th Morita homomorphism on J_k (which is a certain refinement of τ_k).

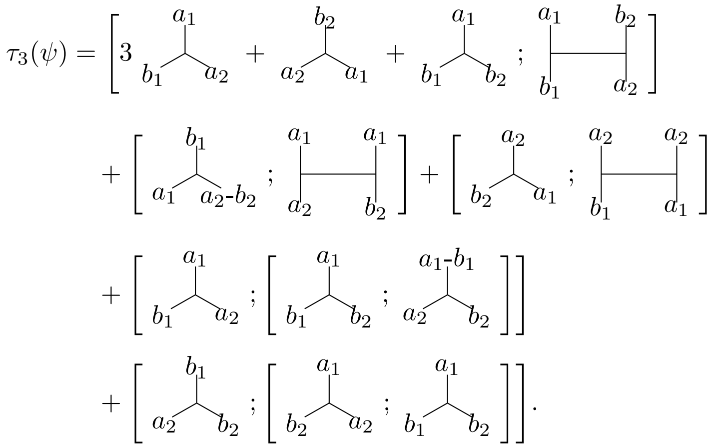
After implementing this formula in a SageMath computer program (see Appendix 2.A), we get



or, in a more compact form:

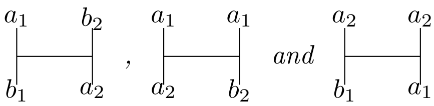


Next we rewrite $\tau_3(\psi)$ as a sum of brackets:

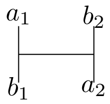


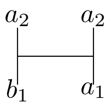
(2.4)

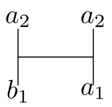
Recall that the trees appearing in the above formulas define derivations through the expansion map η, which is a Lie homomorphism. The trees of degree 2 inside the above brackets actually are in $\tau_2(\mathcal{K})$. To see this, one can simply observe that they are in the kernel of the "antisymmetric" trace that has been defined in [8] to characterize $\tau_2(\mathcal{K})$. We can also show this directly:

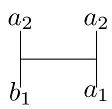
Lemma 2.8. *The trees*  *are in* $\tau_2(\mathcal{K})$*.*

Proof. By the formula in Lemma 2.7 we deduce that

 $= \tau_2(T_{\gamma_2} T_{s1}^{-1} T_{s2}^{-1})$

It is easily seen that the two other trees are in the same orbit under the action of $\mathrm{Sp}(H)$. Hence, it is enough to show that  belongs to $\tau_2(\mathcal{K})$. For this, we note that

 $= a_1 \odot b_1 - a_1 \odot (b_1 + a_2) - (a_1 + a_2) \odot b_1 + (a_1 + a_2) \odot (b_1 + a_2).$

Any element of the form $u \odot v$ with $\omega(u, v) = 1$ being in $\tau_2(\mathcal{K})$, we conclude that  is indeed an element of $\tau_2(\mathcal{K})$. $\square$

By Lemma 2.8, we get that $\tau_3(\psi) \in \tau_3([\mathcal{K}, \mathcal{I}])$. Indeed, Johnson has shown that τ_1 is onto $D_1(H)$ [21], hence any tree of degree 1 is the image by τ_1 of an element in the Torelli group. Pick any element $\psi_2 \in [\mathcal{K}, \mathcal{I}]$, such that $\tau_3(\psi) + \tau_3(\psi_2) = 0$, and define $\varphi := \psi\psi_2 \in J_3$, so that φ belongs to $\mathrm{Ker}(\tau_3) = J_4$. Since $d : \mathcal{K} \to \mathbb{Z}$ is invariant under the conjugacy action of $\mathcal{M}$ (see [43]), we have $d([\mathcal{K}, \mathcal{M}]) = 0$, so that $d(\psi_2) = 0$. We get

$$\lambda(\varphi) = -\frac{1}{24} d(\varphi) = -\frac{1}{24} d(\psi) = 1,$$

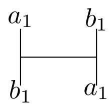
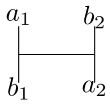
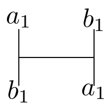
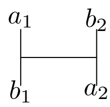
which concludes the proof of Theorem A.

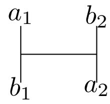
Remark 2.9. Recall that J_4 contains $[J_1, J_3] = [\mathcal{I}, J_3]$ and $[J_2, J_2] = [\mathcal{K}, \mathcal{K}]$ as subgroups. Our map φ seems to be the first explicit example of an element of J_4 which is not in $[\mathcal{I}, J_3]$ nor in $[\mathcal{K}, \mathcal{K}]$ since it is not in $[\mathcal{K}, \mathcal{M}]$ (and none of its powers is, actually). By "explicit", we mean that we are able to decompose the map $\varphi = \psi\psi_2$ as a product of Dehn twists. Specifically, one can build ψ_2 with 22 twists. To do so, we use the decomposition of $\tau_3(\psi) = -\tau_3(\psi_2)$ in brackets given by equation (2.4) and we use the following fact: if $s, t \in \mathcal{M}$ and t is a product of $k \geq 1$ twists, then $[s, t]$ can be written as a product of $2k$ twists. As ψ is a product of 27 twists, we deduce that φ can be given as a product of 49 twists. Nozaki, Sato and Suzuki obtained in [47, Corollary 1.6], in the case of a closed surface and for $g \geq 6$, the existence of an element of J_4, which was not in $[\mathcal{K}, \mathcal{K}]$: this element has a power in $[\mathcal{K}, \mathcal{K}]$, though.

Remark 2.10. Note also that φ does not belong to $[\mathcal{I}, \mathcal{I}]$ neither. Indeed the fact that $\lambda_j(\phi) = 1$ implies that the Birman-Craggs homomorphism (see [5, 22]) does not vanish on φ.

2.5 Complexity of the computation

One could wonder whether it is possible to build elements similar to $\psi \in J_3$ and $\varphi \in J_4$ using fewer twists. In this subsection we explain why so many Dehn twists have been necessary (see Proposition 2.16 below).

Recall that $\mathcal{K}'$ and $\mathcal{K}''$ are the subgroups of $\mathcal{K}$ generated, respectively, by twists around BSCC of genus 1 and 2. We will first give a description of the quotient $\tau_2(\mathcal{K})/\tau_2(\mathcal{K}')$. Here we suppose that the genus g of Σ is at least 2, so that, by a result of Johnson [26], the group $\mathcal{K}$ is normally generated by any BSCC map of genus 1 and any BSCC map of genus 2. This implies, by Lemma 2.7, that $\tau_2(\mathcal{K})$ is generated as a $\mathrm{Sp}(H)$-module by $\frac{1}{2}$  and  . It is also clear that $\tau_2(\mathcal{K}')$ is the $\mathrm{Sp}(H)$-submodule of $D_2(H)$ generated by $\frac{1}{2}$  . Equation (2.3) implies that 3  $\in \tau_2(\mathcal{K}')$, hence we get the following:

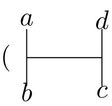
Corollary 2.11. *For any $g \geq 2$, the quotient $\tau_2(\mathcal{K})/\tau_2(\mathcal{K}')$ is a $\mathbb{Z}_3$-module generated as a* $\mathrm{Sp}(H)$*-module by the class of*  .

From this, we get a precise description of the quotient $\tau_2(\mathcal{K})/\tau_2(\mathcal{K}')$.

Proposition 2.12. *For any $g \geq 2$, we have the following* $\mathrm{Sp}(H)$*-module isomorphisms:*

$$\frac{D_2(H)}{\tau_2(\mathcal{K}')} \otimes \mathbb{Z}_3 \simeq \frac{\tau_2(\mathcal{K})}{\tau_2(\mathcal{K}')} \simeq \Lambda^4(H/3H).$$

Proof. We denote by Q the quotient $\frac{D_2(H)}{\tau_2(\mathcal{K}')} \otimes \mathbb{Z}_3$. The first isomorphism is a direct consequence of the fact that $\tau_2(\mathcal{K})$ is of order 2 in $D_2(H)$, and 2 is invertible in $\mathbb{Z}_3$. Indeed, the map from Q to $\frac{\tau_2(\mathcal{K})}{\tau_2(\mathcal{K}')}$ sending an element x to the class of $4x$ in $\frac{\tau_2(\mathcal{K})}{\tau_2(\mathcal{K}')}$ is well-defined (because $D_2(H)/\tau_2(\mathcal{K})$ is a 2-torsion module [42, Prop 1.2]), and has for inverse the map induced by the inclusion of $\tau_2(\mathcal{K})$ in $D_2(H)$.

We then use the presentation of $D_2(H)$ from [8, Prop. 2.1] to define a homomorphism $\kappa : \frac{D_2(H)}{\tau_2(\mathcal{K}')} \otimes \mathbb{Z}_3 \to \Lambda^4(H/3H)$, by putting $\kappa(a \odot b) := 0$, for any $a, b \in H$, and $\kappa($  $) :=$

$a \wedge b \wedge c \wedge d \in \Lambda^4(H/3H)$ for any $a, b, c, d \in H$. It is straightforward to check that all the relations in $D_2(H)$, are sent to 0, except maybe the IHX relation:

$$\begin{aligned}\kappa\Big(\overset{a\;\;\;d}{\underset{b\;\;\;c}{\vdash\!\!\dashv}} - \overset{a\;\;\;d}{\underset{c\;\;\;b}{\vdash\!\!\dashv}} - \overset{a\;\;\;b}{\underset{d\;\;\;c}{\vdash\!\!\dashv}}\Big) &= a \wedge b \wedge c \wedge d - a \wedge c \wedge b \wedge d - a \wedge d \wedge c \wedge b \\ &= 3(a \wedge b \wedge c \wedge d) \\ &= 0 \in \Lambda^4(H/3H).\end{aligned}$$

The map κ is then a well-defined $\mathrm{Sp}(H)$-equivariant homomorphism.

Reciprocally, we show that the map ν, sending $a \wedge b \wedge c \wedge d$ to the class of $\overset{a\;\;\;d}{\underset{b\;\;\;c}{\vdash\!\!\dashv}}$ in Q is also well-defined. It will follow that κ is an isomorphism with $\kappa^{-1} = \nu$. To prove that ν is well-defined, we need to show that the relation $a \wedge b \wedge c \wedge d + a \wedge c \wedge b \wedge d = 0$ is sent to $0 \in Q$ (the other relations in $\Lambda^4(H/3H)$ are easily verified). Hence we want to show that $\overset{a\;\;\;d}{\underset{b\;\;\;c}{\vdash\!\!\dashv}} + \overset{a\;\;\;d}{\underset{c\;\;\;b}{\vdash\!\!\dashv}}$ is equal to 0 in Q.

We first show that for any a, b in H, $a \odot b$ is equal to 0 in Q. The equality $a \odot b = b \odot a$, for any $a, b \in H$ allows us to suppose that we always have $\omega(a, b) \geq 0$. We proceed by induction on $\omega(a, b) \in \mathbb{N}$. If $\omega(a, b) = 1$, we know that there exists a subsurface F of genus 1 in Σ, with one boudary component γ, such that there are two simple closed curves α and β, inducing a and b in homology, intersecting once and inducing a symplectic basis of F. We then have by Lemma 2.7 that $\tau_2(T_\gamma) = a \odot b$, hence $a \odot b = 0 \in Q$. If $c \in H$ is such that $\omega(a, c) = 0$, we have $\omega(a, b + c) = \omega(a, b - c) = 1$, and:

$$a \odot (b + c) - a \odot b - a \odot c = \overset{a\;\;\;c}{\underset{b\;\;\;a}{\vdash\!\!\dashv}}$$

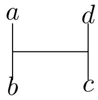
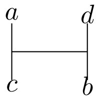
$$a \odot (b - c) - a \odot b - a \odot c = -\overset{a\;\;\;c}{\underset{b\;\;\;a}{\vdash\!\!\dashv}}$$

which implies that $2a \odot c$, and hence $a \odot c$, is trivial in Q. Now, if $\omega(a, b) = k + 1$ for some $k \geq 1$, either a is not primitive in H (and we can reduce the value of k), either there exists $b_2 \in H$ such that $\omega(a, b_2) = 1$, and we can write $b = b_1 + b_2$ with $\omega(a, b_1) = k$. Thus we have that $a \odot (b_1 + b_2) + a \odot (b_1 - b_2) = 2(a \odot b_1) + 2(a \odot b_2) \in Q$. By using the induction hypothesis for $k, k - 1$ and 1, we conclude that $a \odot b = a \odot (b_1 + b_2)$ is trivial in Q.

Furthermore, for any $a, b, c, d \in H$, we have the following, where all the terms of the left hand-sides are trivial in Q:

$$(a + c) \odot b - a \odot b - c \odot b = \overset{a\;\;\;b}{\underset{b\;\;\;c}{\vdash\!\!\dashv}}$$

$$\begin{aligned}\overset{a\;\;\;b+d}{\underset{b+d\;\;\;c}{\vdash\!\!\dashv}} - \overset{a\;\;\;b}{\underset{b\;\;\;c}{\vdash\!\!\dashv}} - \overset{a\;\;\;d}{\underset{d\;\;\;c}{\vdash\!\!\dashv}} &= \overset{a\;\;\;d}{\underset{b\;\;\;c}{\vdash\!\!\dashv}} + \overset{a\;\;\;b}{\underset{d\;\;\;c}{\vdash\!\!\dashv}} \\ &= 2\,\overset{a\;\;\;d}{\underset{b\;\;\;c}{\vdash\!\!\dashv}} - \overset{a\;\;\;d}{\underset{c\;\;\;b}{\vdash\!\!\dashv}}.\end{aligned}$$

We conclude, because we work modulo 3, that  +  is trivial in Q. $\square$

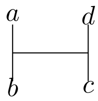
The characterization given by Proposition 2.12 might be helpful to build other elements $\psi \in J_3$ such that $d(\psi) = 1$, making use of the fact that d is trivial on $\mathcal{K}'$. Indeed, for any product of left and right Dehn twists of genus 2, such that the algebraic number of twists is -3, and whose image by τ_2 is in $\tau_2(\mathcal{K}')$, we can always multiply it by an element of $\mathcal{K}'$ such that the product is in J_3 and has Casson invariant equal to 1.

Nevertheless, this will always involve a lot of Dehn twists, as we now explain. This is related to the properties of d, the core the Casson invariant, and another invariant d' defined by Morita in [43]. As we recalled, the map d is the homomorphism from $\mathcal{K}$ to $\mathbb{Z}$ sending a twist of genus h to $4h(h-1)$. Similarly, d' can be defined as the homomorphism from $\mathcal{K}$ to $\mathbb{Z}$ sending a twist of genus h to $h(2h+1)$. The particularity of the map d' is that it factors through τ_2:

$$d' := \bar{d}' \circ \tau_2 \tag{2.5}$$

where $\bar{d}'$ is defined on $D_2(H)$ by

$$\bar{d}'(a \odot b) := 3\omega(a,b)^2$$

$\bar{d}'($  $) := 4\omega(a,b)\omega(c,d) - 2\omega(a,d)\omega(b,c) + 2\omega(a,c)\omega(b,d).$

Morita [43, Th. 5.4] showed that the $\mathcal{M}$-equivariant homomorphisms from $\mathcal{K}$ to $\mathbb{Z}$ are rational linear combinations of d and d' (which are linearly independent): thus $H^1(\mathcal{K};\mathbb{Z})^{\mathcal{M}}$ is free abelian of rank two and generated over the rational by d and d'. In fact the following lemma can be obtained by direct computation.

Lemma 2.13. *If T_1 is a twist of genus 1 and T_2 is a twist of genus 2, then:*

$$\frac{d}{8}(T_2) = \frac{4d'-5d}{12}(T_1) = 1$$
$$\frac{d}{8}(T_1) = \frac{4d'-5d}{12}(T_2) = 0.$$

Therefore we obtain that the abelian group $H^1(\mathcal{K};\mathbb{Z})^{\mathcal{M}}$ is freely generated by $\frac{4d'-5d}{12}$ and $\frac{d}{8}$. Furthermore, we have the following.

Proposition 2.14. *For any $g \geq 2$, $\mathcal{K}/[\mathcal{K},\mathcal{M}]$ is free abelian of rank 2 and canonically isomomorphic to its dual $H^1(\mathcal{K};\mathbb{Z})^{\mathcal{M}}$.*

Proof. Choose two twists of genus 1 and 2, denoted respectively T_1 and T_2. Then, the map from $H^1(\mathcal{K};\mathbb{Z})^{\mathcal{M}}$ to $\mathcal{K}/[\mathcal{K},\mathcal{M}]$ defined by sending f to $T_1^{f(T_1)}T_2^{f(T_2)}$ is surjective and canonical, as any two Dehn twists of same genus are always conjugated to each other. It is injective by Lemma 2.13. $\square$

Remark 2.15. Proposition 2.14 implies that to compute the algebraic number of Dehn twists of genus 1 and 2 involved in an element of $\mathcal{K}$, it is enough to know its values under τ_2 and d. Also, for any $\phi \in \mathrm{Ker}(d)$, we see that there is a $k \in \mathbb{Z}$ such that $d'(\phi) = 3k$. Then we get that for any BSCC γ of genus 1, $\phi(T_\gamma)^{-k}$ is in $\mathrm{Ker}(d) \cap \mathrm{Ker}(d') = [\mathcal{K},\mathcal{M}]$. This proves that in genus $g \geq 2$, $\mathrm{Ker}(d) = [\mathcal{K},\mathcal{M}]\mathcal{K}'$.

To get an element $\psi \in J_3$ such that $\lambda(\psi) = 1$, we need to have $d(\psi) = -24$, and $d'(\psi) = 0$ (as d' factorizes by τ_2). This implies that $\frac{4d'-5d}{12}(\psi) = 10$ and $\frac{d}{8}(\psi) = -3$. Thus, by the previous computations, the algebraic numbers of Dehn twists of genus 1 and 2 in ψ are respectively 10 and -3. Thus we obtain the following:

Proposition 2.16. *In genus $g \geq 2$, any element in J_3 whose Casson invariant is 1 is the composition of BSCC maps of genus 1 and 2 such that the algebraic number of BSCC maps of genus 1 is 10 and the algebraic number of BSCC maps of genus 2 is -3.*

3 Triviality of the J_4-equivalence

In this section, we prove Theorem B. We have shown in Section 2 that there exists an element $\psi \in J_4$ whose Casson invariant is equal to 1. Following [39, Rem. 6.4], we explain how this implies the triviality of the J_4-equivalence relation on the set of homology 3-spheres.

We first recall the definition of the Y_k-equivalence and the J_k-equivalence relations, and refer to [39, Section 2] for more details. For a compact oriented 3-manifold M, a submanifold $S \subset \mathrm{int}(M)$ homeomorphic to $\Sigma_{g,1}$ for some $g \geq 1$, and any φ in the mapping class group of S, we define the 3-manifold $M_{S,\varphi}$ by removing from M a neighborhood $S \times [-1,1]$ of S, and regluing it twisting by the map φ:

$$M_{S,\varphi} := (M \backslash \mathrm{int}(S \times [-1,1])) \cup_{\overline{\varphi}} (S \times [-1,1])$$

where $\overline{\varphi}$ is the map from $\partial(S \times [-1,1])$ to M defined by $(Id \times (-1)) \cup (\partial S \times Id) \cup (\varphi \times 1)$.

Whenever the map φ is in the Torelli group J_1 of S, we call the move from M to $M_{S,\varphi}$ a *Torelli surgery.* Torelli surgeries preserve the set $\mathcal{S}(3)$ of homeomorphism classes of homology 3-spheres.

Definition 3.1. *The Y_k-equivalence and J_k-equivalence relations are defined on $\mathcal{S}(3)$ as follows:*

$$M \overset{Y_k}{\sim} M' \Leftrightarrow \exists S \subset M,\ \exists \varphi \in \Gamma_k \mathcal{I}(S) \text{ such that } M' \cong M_{S,\varphi},$$

$$M \overset{J_k}{\sim} M' \Leftrightarrow \exists S \subset M,\ \exists \varphi \in J_k(S) \text{ such that } M' \cong M_{S,\varphi}.$$

These relations are equivalence relations, and Y_k-equivalence implies J_k-equivalence. It is known that J_3-equivalence is trivial among homology 3-spheres [49]. We now improve this result. We first sketch the proof of the following fact, announced by Habiro [15]:

Theorem 3.2 (Habiro)**.** *For any $M, M' \in \mathcal{S}(3)$, the following statements are equivalent:*

$$\begin{aligned} &(1)\quad M \overset{Y_3}{\sim} M' \\ &(2)\quad M \overset{Y_4}{\sim} M' \\ &(3)\quad \lambda(M) = \lambda(M') \in \mathbb{Z}. \end{aligned}$$

Sketch of proof. Habiro [15, Section 8] studied the *Y-filtration* on the monoid of homology cylinders (we refer to [16, Sections 5 and 6] for a survey). A similar study was made by Goussarov [11], with a different vocabulary. Here, we need only to use the results of Habiro in the genus 0 case, corresponding to the monoid of homology 3-spheres (where the multiplication is the connected sum operation). In this case $\mathcal{S}(3)$ is filtered by $(Y_k(\mathcal{S}(3)))_{k\geq 1}$, where $Y_k(\mathcal{S}(3))$ is the submonoid consisting of homology 3-spheres Y_k-equivalent to S^3. For any $k \geq 1$, the quotient of $\mathcal{S}(3)$ by the Y_k-equivalence is a group and the quotient of $Y_k(\mathcal{S}(3))$ by the Y_{k+1}-equivalence is an abelian group. We consider the associated graded space

$$\mathrm{Gr}^Y(\mathcal{S}(3)) := \bigoplus_{k \geq 1} Y_k(\mathcal{S}(3))/Y_{k+1}.$$

By using the techniques of clasper surgery, Habiro was able to define (in degree greater than 2) a surjective graded map from a certain space of Jacobi diagrams, namely trivalent graphs subject to the AS and IHX relations (see Figure 2.3), to $\mathrm{Gr}^Y(\mathcal{S}(3))$. In degree 2, there is only one (up to scalar) such Jacobi diagram, and in degree 3, there are none. This implies that $Y_2(\mathcal{S}(3))/Y_3 \simeq \mathbb{Z}$ (the isomorphism being given by the Casson invariant), and $Y_3(\mathcal{S}(3))/Y_4 = 0$. We deduce that two homology 3-spheres are Y_4-equivalent if and only if they are Y_3-equivalent, i.e. if and only if they have the same Casson invariant (see [15] or [39]). $\square$

Proof of Theorem B. Let S be such that the standard Heegaard surface of genus 2 in S^3 is obtained from S by capping it with a disk. Let us pick $\psi \in J_4(S)$ such that $\lambda(\psi) = 1$: the existence of such an element has been proved in Section 2. We have by definition that $P := S^3_{S,\psi}$ is a homology 3-sphere whose Casson invariant is equal to 1. Furthermore, the homology 3-sphere P is by construction J_4-equivalent to S^3. Let M and M' be two homology 3-spheres, and assume that $\lambda(M) \leq \lambda(M')$. Then, the additivity of the Casson invariant implies that $\lambda(M\#P^{(\lambda(M')-\lambda(M))}) = \lambda(M')$, hence

$$M \cong M\#(S^3)^{(\lambda(M')-\lambda(M))} \overset{J_4}{\sim} M\#P^{(\lambda(M')-\lambda(M))} \overset{Y_4}{\sim} M'$$

where the last equality follows from Theorem 3.2. By transitivity, we get that M is J_4-equivalent to M'. $\square$

Remark 3.3. Knowing the values of λ on J_5 would *a priori* not be enough to discuss the J_5-equivalence, as the Y_5-equivalence is related to higher-order finite-type invariants.

2.A SageMath computer program

In this appendix, we provide the code that we used to verify that the map ψ constructed in Section 2.3 belongs to J_3, compute $\tau_3(\psi)$ and write it as a linear combination of degree 3 tree-like Jacobi diagrams.

We divide the program in two parts. The first part of the program is functional for any genus $g \geq 1$ and allows one to compute the maps L_k^θ on commutators, for $k \in \{4, \dots, N\}$, provided we are given a symplectic expansion θ up to degree $N-2$. Here we take $N := 5$ and the expansion used in the program is a truncation of the one given in [36]. The entries of the functions L_k^θ in the program are strings describing elements in π (for example $\alpha_1\beta_2^{-1}$ is encoded as 'a1+b2-'). The output of the function L_k^θ in the code is actually an element of degree k in the free algebra on $2g$ generators, instead of an element of the tensor algebra $\widehat{T}$. The second part of the code is specific to the genus $g := 2$, and allows us to compute $\tau_3(\psi)$ by entering *barcodes* associated to the spines in Figure 2.4. A barcode is a list of non-zero integers between g and $-g$, which corresponds to a word in π through the correspondences $\pm(2i-1) \sim \alpha_i^{\pm 1}$ and $\pm 2i \sim \beta_i^{\pm 1}$. Such a barcode is then transformed in a string encoding an element of π and we use the function L_k^θ to compute $\tau_3(\psi)$. In the end, the program compares $\tau_3(\psi)$ to the expansion of trees claimed in Section 2.4.

We explain on an example how to construct a barcode encoding a curve in the surface. We consider the boundary of a neighborhood of the spine of the twist T_7 in Figure 2.4, orient it in any way, and denote it δ_7. We lift δ_7 to an element y_7 of the fundamental group in an arbitrary way, by linking it to the base point with an arc. Precisely, the based curve y_7 is obtained by first travelling from x_0 to the boundary of the spine along this arc (drawn with two opposite arrows in Figure 2.5), then going along δ_7, and finally travelling back along the arc from the spine to x_0. We can now express the curve defined by the spine as a word in the fundamental group, using the basis of π chosen in Figure 2.2. Respecting this procedure, one will obtain a commutator $[U, V] \in \pi$ where U and V are elements corresponding to the curves of the spine, lifted to elements of π using the arc γ. In our case, we get that $y_7 = [\alpha_2^{-1}\beta_2[\alpha_1, \beta_1^{-1}]\beta_1^{-1}\alpha_1^{-1}\beta_2^{-1}, \beta_2] \in \pi$, where the brackets refer to the commutator bracket in the free group π. We now use the correspondency described above and we get that the barcode corresponding to y_7 is $[-3, 4, 1, -2, -1, 2, -2, -1, -4, 4, 4, 1, 2, -2, 1, 2, -1, -4, 3, -4]$. Of course, by changing the word describing the element y_7, we would get a different barcode (for example $[-3, 4, 1, -2, -1, -1, 4, 1, 1, 2, -1, -4, 3, -4]$ is a simpler barcode for y_7), but this will not affect the result.

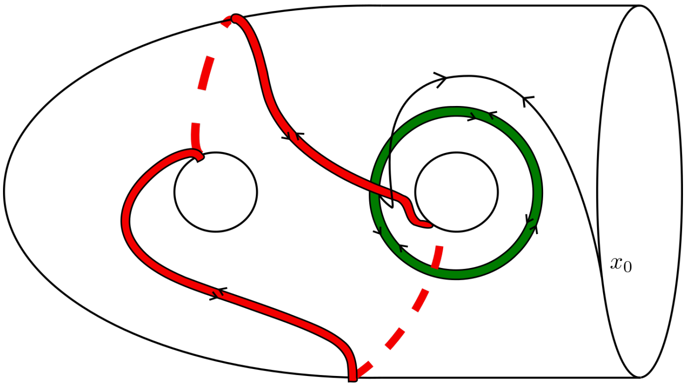


Figure 2.5: The element $y_7 \in \pi$, based at x_0

```
# Choose the genus and the nilpotency class

g=2
N=5


# The free associative algebra on 2g generators a1,...,ag,b1,...,bg

variables = ''
for i in range(g): variables = variables + 'a' + str(i+1) + ','
for i in range(g): variables = variables + 'b' + str(i+1) + ','
variables = variables[:6*g-1]

A=FreeAlgebra(QQ,2*g,variables)
a=[A(1)]+[A.gen(i) for i in range(g)]
b=[A(1)]+[A.gen(i+g) for i in range(g)]


# "Fast" operations (product and bracket) in A up to degree N

def fpr(u,v):
  res = 0
  u = A(u)
  v = A(v)
  data_u =[(w.to_word(), cf) for (w,cf) in u]
  data_v =[(w.to_word(), cf) for (w,cf) in v]
  for (wu,cfu)  in data_u:
    for (wv,cfv) in data_v:
      if len(wu) + len(wv) <= N:
        res = res + cfu*cfv*A.monomial(wu)*A.monomial(wv)
  return res

def fbr(u,v):
  return fpr(u,v)-fpr(v,u)


# Transforms a tensor into a cyclic tensor

def cyc(x):
  res = A(0)
  data = [(w.to_word(), cf) for (w,cf) in x]
  for (w, cf) in data:
    for i in range(len(w)):
      res = res + cf*A.monomial(Word(w[i:] + w[:i]))
  return res


# Extracts the degree k part

def extract(expr,k):
  data = [(w.to_word(), cf) for (w,cf) in expr if len(w)==k]
```

```
  return A.sum_of_terms(term for term in data)


# Truncates up to degree k

def truncate(expr,k):
  data = [(w.to_word(), cf) for (w, cf) in expr if len(w)<=k]
  return A.sum_of_terms(term for term in data)


# Computes the degree

def degree(expr):
  data = [len(w.to_word()) for (w,cf) in expr]
  return max(data)


# Computes, up to degree N, the exponential of a tensor without constant term

def exp(x):
  p= [A(1) for i in range(N+1)]
  for i in range(1,N+1):
    p[i] = fpr(p[i-1],x)
  res = A(0)
  for i in range(N+1):
    res = res + p[i]*(1/factorial(i))
  return truncate(res,N)


# Computes, up to degree N, the logarithm of a tensor whose constant term is
    one

def log(x):
  p = [A(1) for i in range(N+1)]
  d = x-1
  for i in range(1,N+1):
    p[i] = fpr(p[i-1],d)
  res = A(0)
  for i in range(1,N+1):
    res = res+p[i]*((-1)^(i+1))/i
  return truncate(res,N)


# Values of a symplectic expansion "theta" up to order N

logtheta_a = [A(0)] + [a[i]-(1/2)*fbr(a[i],b[i])+(1/12)*fbr(fbr(a[i],b[i]),b[i
    ])-(1/2)*fbr(sum(fbr(a[j],b[j]) for j in range(i)),a[i])
for i in range(1,g+1)]

logtheta_b = [A(0)] +  [b[i]-(1/2)*fbr(a[i],b[i])+(1/4)*fbr(fbr(a[i],b[i]),b[i
    ])+(1/12)*fbr(a[i],fbr(a[i],b[i]))+(1/2)*fbr(b[i],sum(fbr(a[j],b[j]) for j
     in range(i)))
for i in range(1,g+1)]

theta_a = [exp(logtheta_a[i]) for i in range(g+1)]

theta_b = [exp(logtheta_b[i]) for i in range(g+1)]

theta_a_inv = [exp(-logtheta_a[i]) for i in range(g+1)]

theta_b_inv = [exp(-logtheta_b[i]) for i in range(g+1)]


# Computation of theta from a string such a 'a1+b2-a1-' which encodes an
    element of the fundamental group

def theta(lis):
  res = A(1)
  for j in range(len(lis)/3):
          index = int(lis[3*j+1])
```

```
            if [lis[3*j],lis[3*j+2]]==['a','+']: res = fpr(res,theta_a[index])
            if [lis[3*j],lis[3*j+2]]==['a','-']: res = fpr(res,theta_a_inv[
    index])
            if [lis[3*j],lis[3*j+2]]==['b','+']: res = fpr(res,theta_b[index])
            if [lis[3*j],lis[3*j+2]]==['b','-']: res = fpr(res,theta_b_inv[
    index])
  return truncate(res,N)


# Checks that this expansion is symplectic up to some degree

boundary = ''
for i in range(g): boundary = boundary + 'b' + str(i+1) + '-' + 'a' + str(i+1)
     + '+'  + 'b' + str(i+1) + '+' + 'a' + str(i+1) + '-'

exp_omega_tilde = theta(boundary)
exp_omega = exp(sum( fbr(a[i],b[i]) for i in range(1,g+1)))
diff = exp_omega_tilde - exp_omega

print('Computations are done up to degree '+str(N)+'.')

d = 0
for i in range(N+1):
  if extract(diff,i)==0: d=i
print('The expansion is symplectic up to order '+str(d)+'.')


#  The map "L^theta_k" of Kawazumi & Kuno for a commutator

def Ltheta(lis,k):
  logtheta = log(theta(lis))
  ltheta = [extract(logtheta,i) for i in range(0,k-1)]
  res = A(0)
  for i in range(2,k-1):
    res = res + cyc(A(fpr(ltheta[i],ltheta[k-i])))
  return res*(1/2)
```

```
# Computation of the cyclic tensor corresponding a tree-like Jacobi diagram of
     the following form :
#
#     b   c   d
#     |   |   |
# a-----------e
#

def br(u,v):
  return u*v-v*u

def treetotens(a,b,c,d,e):
  return cyc(br(a,b)*br(c,br(d,e)))


# Here we assume that g=2

# We represent elements of the fundamental group with barcodes or strings:
# Transforms a barcode such as [1,-2,3] to a string such as 'a1+b1-a2+'

def list_to_string(x):
    res=''
    for j in range(len(x)):
        if x[j] == 1 : res = res + 'a1+'
        if x[j] == -1 : res = res + 'a1-'
        if x[j] == 2 : res = res + 'b1+'
        if x[j] == -2 : res = res + 'b1-'
        if x[j] == 3 : res = res + 'a2+'
        if x[j] == -3 : res = res + 'a2-'
        if x[j] == 4 : res = res + 'b2+'
        if x[j] == -4 : res = res + 'b2-'
    return res
```

```python
# Description of psi, by entering the expression (as barcodes) of the curves defining the twists of which psi is composed

def brack(a,b):
  return [a,b,-a,-b]

def bra(a,b):
  mira = list(reversed(a))
  mirb = list(reversed(b))
  return a+b+[-1*i for i in mira] + [-1*i for i in mirb]

gamma2 = list_to_string(brack(3,-4)+brack(1,-2))
t1 = list_to_string(bra(brack(-2,1)+[-4,1],[-2]))
t2 = list_to_string(bra([1],[-4,3,4,-2]))
t3 = list_to_string(bra([1],[-4,-3,4]+brack(1,-2)+[-2]))
t4 = list_to_string(bra([3],[-1,-4]))
t5 = list_to_string(bra([1],[-4,-3,-2]))
t6 = list_to_string(bra([3],[-2,-1,-4]))
t7 = list_to_string(bra([-3,4]+brack(1,-2)+[-2,-1,-4],[4]))
t8 = list_to_string(bra([3,4,1],[-2]))
t9 = list_to_string(bra([1],[-4,-2]))
t10 = list_to_string(bra([-4,-3,4]+brack(1,-2)+[-2],[4]))
t11 = list_to_string(bra(brack(-2,1)+[-4,3,4,1],[-2]))
t12 = list_to_string(bra([1,-4,-3,4]+brack(1,-2)+[4,1]+brack(-2,1)+[-4,3,4],[-4,-3,4]+brack(1,-2)+[-2]))
t13 = list_to_string(bra([-4,-3,4]+brack(1,-2)+[-2,-1],[1,2,4]))
s1 = list_to_string(brack(1,-2))
s2 = list_to_string(brack(3,-4))

listelem = [gamma2,t1,t2,t3,t4,t5,t6,t7,t8,t9,t10,t11,t12,t13,s1,s2]
listcoeff = [-3,-1,-1,+2,+2,+1,-1,-1,+1,-1,+1,-1,-1,+1,+7,+2]


# Computation of tau_2(psi) and tau_3(psi)

tau2_psi = sum(listcoeff[i]*Ltheta(listelem[i],4) for i in range(16))
tau3_psi = sum(listcoeff[i]*Ltheta(listelem[i],5) for i in range(16))


# Comparison with the linear combinations of tree-like Jacobi diagrams

candidate = (-treetotens(a[2],a[1],a[1],b[1],a[1])-treetotens(a[2],b[1],a[1],a[2],a[1])-treetotens(b[2],a[1],a[1],b[1],a[1])-treetotens(b[2],b[1],a[1],b[1],a[1])+treetotens(b[2],a[2],a[1],b[1],a[1])+treetotens(b[2],a[2],a[1],a[2],a[1])+treetotens(b[2],a[2],a[1],b[2],a[1])+treetotens(b[2],a[2],b[1],b[2],a[1])+3*treetotens(b[2],a[2],a[2],b[1],a[1])+treetotens(b[2],a[2],a[2],a[2],a[1])+treetotens(b[2],a[2],b[2],b[1],a[1])-treetotens(b[2],a[1],a[2],b[1],a[1])+treetotens(b[2],b[1],a[2],b[1],a[1])+treetotens(b[2],a[2],b[2],a[2],a[1])-treetotens(b[2],a[2],b[2],a[2],b[1]))

# candidate is also equal to candidate_bis = (treetotens((b[1]+a[2]),a[1],(a[1]+a[2]+b[2]),a[2],(a[1]+b[2]))+treetotens(a[2]-a[1],b[2],a[1]+b[1],a[1],b[1]+b[2])-treetotens(a[2]-a[1],b[1],b[2],a[2],b[1]+b[2])+treetotens(b[2],a[2],(2*a[2]-2*a[1]+b[2]),b[1],a[1]))

print('psi is in J_3 : ' + str(tau2_psi == 0))
print('tau3_psi == candidate : ' + str(tau3_psi == candidate))
```

Appendix A

Study of some surgery equivalence relations on 3-manifolds

This appendix can be read independently. Nevertheless, we refer to the introduction of this dissertation for more details on the motivation, the notions and the notations that are not specific to this appendix. We shall give the proofs of Propositions 1.15 and 1.19 of the introduction, that we shall reproduce below.

1 Definitions

We define an equivalence relation on the set of connected compact oriented 3-manifolds. Let us consider a surface $\Sigma := \Sigma_{g,1}$ obtained from a closed surface Σ_g by removing a small disk. Let V be a handlebody bounded by Σ_g. The inclusion of Σ in V induces projections $\pi := \pi_1(\Sigma) \to \pi' := \pi_1(V)$ and $H := H_1(\Sigma) \to H' := H_1(V)$. We denote $D_k(H)$ (resp. $D_k(H')$) the set of positive symplectic derivations of degree k of the free Lie algebra $\mathcal{L}(H)$ (resp. $\mathcal{L}(H')$). For an element f of the mapping class group $\mathcal{M}$ of Σ, $f_* \in \mathrm{Sp}(H)$ stands for the action of f on H. We still write abusively f for the action of f on the fundamental group.

Definition 1.1. *The Lagrangian Torelli group is defined by:*

$$\mathcal{I}^L := \{h \in \mathcal{M} \mid h_*(A) \subset A \text{ and } h_* \text{ is the identity on } A\}.$$

Definition 1.2. *For $k \geq 1$, the group $L_k(V)$ is defined by:*

$$L_k(V) := \Big\{h \in \mathcal{I}^L \;\Big|\; p\big(h(\mathbb{A})\big) \subset \Gamma_{k+1}\pi'\Big\}.$$

We sometimes drop the "V" in the notation $L_k(V)$ when the context is clear. Levine showed that for any $k \geq 1$, the set L_k is a group [32, Section 4]. The groups L_k define a filtration of $\mathcal{I}^L$ which is non-separating. We also recall the definition of the subgroup J_k of $\mathcal{M}$. An element of $\mathcal{M}$ belongs to J_k if it acts trivially on the quotient $\pi/\Gamma_{k+1}\pi$ (i.e. it acts trivially modulo the $(k+1)$-th commutators). Plainly, we have that $J_k(\Sigma) \subset L_k(V)$. Next proposition defines the *L_k-equivalence.*

Proposition 1.3. *Two oriented compact 3-manifolds M and M' are said to be L_k-equivalent if M' can be obtained from M by removing a handlebody V and regluing it by twisting with an element of $L_k(V)$. This is an equivalence relation.*

Similarly, one can define the J_k-equivalence by removing the thickening of a surface Σ with one boundary component and regluing it, twisting by an element of $J_k(\Sigma)$ (see Definition 1.7 of the introduction).

Proof of Proposition 1.3. The reflexivity and symmetry are straightforward, because L_k is a group. For the transitivity, remark first that (as in the case of the J_k and the Y_k-equivalences considered in [39]) the L_k-equivalence is preserved under stabilization: one can always add to the surgered handlebody trivial 1-handles that will be removed and reglued in a trivial way. We can hence always increase the genus of the surgery. Suppose now that $M \overset{L_k}{\sim} M'$ and that $M' \overset{L_k}{\sim} M''$, and that the respective corresponding surgeries are performed on a handlebody V_1 in M and a handlebody V_2 in M'. The image of V_1 in M' can be isotoped to be disjoint of V_2, as handlebodies are homotopy equivalent to wedges of circles. Hence, V_2 has a "twin" handlebody in M, say V_2', disjoint of V_1. If we "tube" V_1 and V_2' (i.e. we link these two handlebodies by a 1-handle), we get a third handlebody $V \subset M$. To get M'' from M, we need to remove V, and glue it back using homeomorphisms of $L_k(V_1)$ for the V_1 part and $L_k(V_2)$ for the V_2' parts, and the identity along the tube. It is not hard to see that this new gluing map is an element of $L_k(V)$. Hence we get that $M \overset{L_k}{\sim} M''$. □

Because $J_k \subset L_k$ as subgroups of $\mathcal{M}$, it is clear that the J_k-equivalence dominates the L_k-equivalence. It is hence natural to compare these two relations. We also hope to tackle the question of the triviality of the J_k-equivalence for homology 3-spheres via this comparison. As the J_k-equivalence is trivial among homology 3-spheres for $1 \leq k \leq 4$, so is the L_k-equivalence. We can nevertheless compare these equivalence relations in a more general context. In the next sections, we shall prove the two following propositions.

Proposition 1.4. *Any two compact oriented 3-manifolds that are L_2-equivalent are J_2-equivalent.*

Proposition 1.5. *Among all compact oriented 3-manifolds, the L_3-equivalence is strictly weaker than the J_3-equivalence.*

Specifically, we provide counter-examples in two cases: homology cylinders and closed 3-manifolds.

2 L_2-equivalence

We now show that the L_2-equivalence and the J_2-equivalence are the same among 3-manifolds. There are two proofs of this fact. The first one uses the fact that $L_2 = J_2 \cdot L_\infty$ [34] where L_∞ is the intersection of all the L_k's and is included in the subgroups of $\mathcal{M}$ of elements extending to the handlebody V [32]. This implies that a gluing element in L_2 can always be replaced by a gluing element in J_2 up to an element that does not change the homeomorphism type of the manifold. We give a second proof.

Proof of Proposition 1.4. Recall first that, plainly, J_2-equivalence implies L_2-equivalence, because $J_2 \subset L_2$, as subgroups of $\mathcal{I}$. Let us denote V a handlebody involved in some L_2-surgery. It was shown by Levine [32, Prop. 4.1] that the group $L_2(V)$ is generated by Dehn twists along curves which are null-homologous in the handlebody. By Lickorish's trick, twisting by such a Dehn twist is equivalent to doing a (± 1)-framed surgery on a knot K bounding a surface in the manifold. This is what Cochran, Gerges and Orr call a *2-surgery* relation [6], and they show that it is equivalent to the J_2-equivalence relation [6, Theorem 3.18]. Here, we just need the easy part of this implication, and we recall the argument. Let us consider the surgery on K. This knot has a Seifert surface S which can be chosen to be properly embedded in V (the curve is null homologous in V). With this choice, the framing of K allows us to apply once again Lickorish's trick, the handlebody being this time a thickening of S. Hence, the surgery is equivalent to removing the thickening of S and gluing it back, twisting along a BSCC map on its boundary. Hence the surgery is a J_2-surgery, as BSCC maps belong to J_2 [26]. Finally, by transitivity, any L_2-surgery can be decomposed into a sequence of surgeries done on generators of L_2, and we conclude that is also a J_2-surgery. □

We now turn to the case of the L_3-equivalence. Of course, it is trivial among homology 3-spheres, because it is dominated by the J_3-equivalence, but we shall show that it is not the case in general. The rest of the appendix is dedicated to the proof of Proposition 1.5.

3 L_3-equivalence

We shall exhibit two compact oriented 3-manifolds with boundary which are L_3-equivalent but *not* J_3-equivalent. We then use this counter-example to produce another one in the case of closed oriented 3-manifolds.

A *cobordism* (M, m) over a surface $\Sigma_{g,1}$ is a compact oriented 3-manifold M equipped with a homeomorphism (called the boundary parametrization) $m : \partial(\Sigma_{g,1} \times [-1,1]) \to \partial M$. The restriction of m to the top $\Sigma_{g,1} \times \{+1\}$ (resp. bottom $\Sigma_{g,1} \times \{-1\}$) part of the thickened surface is denoted $m_+ : \Sigma_{g,1} \to \partial M$ (resp. $m_- : \Sigma_{g,1} \to \partial M$). Cobordisms are considered up to oriented homeomorphisms that commute with the boundary parametrizations. A *homology cylinder* (M, m) is a homology cobordism which has the same homology type as the trivial cobordism $(\Sigma_{g,1} \times [-1,1], \mathrm{Id})$. Concretely, we mean that there is an isomorphism $h : H_*(\Sigma_{g,1} \times [-1,1]) \to H_*(M)$ such that the following diagram commutes:

$$\begin{array}{ccc} H_*(\Sigma_{g,1} \times [-1,1]) & \xrightarrow{h} & H_*(M) \\ \uparrow & & \uparrow \\ H_*(\partial(\Sigma_{g,1} \times [-1,1])) & \xrightarrow{m_*} & H_*(\partial M). \end{array}$$

We denote by $\mathcal{IC} = \mathcal{IC}(\Sigma_{g,1})$ the set of homology cylinders. Two homology cylinders can be stacked one above the other, which gives $\mathcal{IC}$ a monoid structure. Let us denote $\Sigma := \Sigma_{g,1}$ to lighten the notations. The mapping cylinder $c(f)$ of a homeomorphism $f : \Sigma \to \Sigma$ is the homology cylinder $\Sigma \times [-1,1]$ with boundary parametrization $(\mathrm{Id} \times (-1)) \cup (\partial\Sigma \times \mathrm{Id}) \cup (f \times 1)$. For an element of the Torelli group $\mathcal{I}$, the mapping cylinder construction yields a homology cylinder.

We will exhibit, first, a homology cylinder which is L_3-equivalent to the trivial homology cylinder, but not J_3-equivalent to it.

3.1 A property of the Birman-Craggs homomophism

To show that two homology cylinders are not J_3-equivalent we will use the Birman-Craggs homomorphism β [22] and its extension to homology cylinders [32, 38]. We fix a surface Σ with one boundary component. The map β is a monoid homomorphism from $\mathcal{IC}$ to the space $B_{\leq 3}$ of boolean cubic functions on $\mathrm{Spin}(\Sigma)$, the set of spin structures on the surface Σ. Recall from [23] that $\mathrm{Spin}(\Sigma)$ can be identified with the set of quadratic forms on $H \otimes \mathbb{Z}_2$ whose polar form is the intersection pairing ω mod 2. Notice that in the sense given by this identification, a spin structure can be *evaluated* at any element $x \in H$ (after tensorization by $\mathbb{Z}_2$). We denote $\mu : \mathcal{S}(3) \to \mathbb{Z}_2$ the Rokhlin invariant of homology 3-spheres, and define, for any embedding $j : \Sigma \to S^3$, the manifold $S^3(M, j)$ to be the homology 3-sphere defined by:

$$S^3(M, j) := (S^3 \backslash (j(\Sigma) \times [-1,1])) \cup_{\tilde{j} \circ m^{-1}} M \tag{3.1}$$

where $j(\Sigma) \times [-1,1]$ is a closed regular neighborhood of $j(\Sigma)$, $\tilde{j}$ is the restriction to the boundary of $j \times \mathrm{Id} : \Sigma \times [-1,1] \to j(\Sigma) \times [-1,1]$, and $m : \partial(\Sigma \times [-1,1]) \to \partial M$ the boundary parametrization of M. Then a definition of β is given, with j varying among all possible embeddings of Σ in S^3, by:

$$\begin{array}{rcccl} \beta & : & \mathcal{IC} & \longrightarrow & B_{\leq 3} \\ & & M & \longmapsto & (j^* \sigma_0 \mapsto \mu(S^3(M, j)) \end{array}$$

where σ_0 is the unique spin structure on S^3. This definition naturally extends the definition of the usual Birman-Craggs homomorphism $\beta : \mathcal{I} \to B_{\leq 3}$, in the sense that for any f in the Torelli group, we have $\beta(c(f)) = \beta(f)$.

It can be deduced from [39, Lemma 6.2 and Lemma 3.20] that for any homology cylinder M which is J_3-equivalent to the trivial homology cylinder, $\beta(M)$ is a constant map. We show this in a more direct way, using only the fact that $\beta(J_3) = B_0$ (see [25, p.178], [39, Rem. 3.21] or Remark 4.15 of Chapter 1), where $B_0 \subset B_{\leq 3}$ is the subspace of constant boolean functions.

Proposition 3.1. *If M is J_3-equivalent to the trivial homology cylinder $c(\mathrm{Id}) = (\Sigma \times [-1,1], \mathrm{Id})$ over the surface Σ, then the boolean function $\beta(M)$ is constant.*

Proof. Let j' be an embedding of some surface Σ' in $c(\mathrm{Id})$, and $f \in J_3(\Sigma')$ such that $M := (c(\mathrm{Id})\backslash(j'(\Sigma') \times [-1,1])) \cup_{\tilde{j'} \circ \tilde{f}^{-1}} (\Sigma' \times [-1,1])$. Here, $\tilde{f}$ designates the boundary parametrization of the mapping cylinder $c(f)$. Take any embedding $j : \Sigma \to S^3$, and define ι_j as an inclusion of $c(\mathrm{Id})$ in S^3 given by the choice of a closed regular neighborhood of $j(\Sigma)$ in the surgery. Then, we compute, as the boundary parametrization of M is given by the identity:

$$\begin{aligned}
\beta(M)\big(j^*\sigma_0\big) &= \mu\big(S^3(M,j)\big) \\
&= \mu\big((S^3\backslash(j(\Sigma) \times [-1,1])) \cup_{\tilde{j} \circ m^{-1}} M\big) \\
&= \mu\big(S^3\backslash((\iota_j \circ j')(\Sigma') \times [-1,1])) \cup_{\widetilde{\iota_j \circ j'} \circ \tilde{f}^{-1}} (\Sigma' \times [-1,1])\big) \\
&= \mu\big(S^3(c(f), \iota_j \circ j')\big) \\
&= \beta\big(c(f)\big)\big((\iota_j \circ j')^*\sigma_0\big) \\
&= \beta(f)\big((\iota_j \circ j')^*\sigma_0\big)
\end{aligned}$$

and $\beta(f)$, as $f \in J_3(\Sigma')$, is known to be a constant function. Hence, the value of $\beta(M)$ is independent of the spin structure on Σ' it is applied to, which concludes. $\square$

3.2 A counter-example

We now a build a homology cylinder M which is L_3-equivalent to the trivial homology cylinder, and such that $\beta(M)$ is not constant. This shows, by Proposition 3.1 above, that the L_3-equivalence is strictly weaker than the J_3-equivalence. For any simple closed curve γ on a given surface, we denote by T_γ the right Dehn twist along γ.

We consider a surface Σ of genus one, with one boundary component, and its thickening, the handlebody $V := \Sigma \times [-1,1]$. We choose two oriented simple closed curves c and d inducing a basis of $H_1(\Sigma)$, as shown in Figure A.1.

Let us keep the notation V when we consider this handlebody as the trivial homology cylinder over Σ. We will perform on V a L_3-surgery. This surgery is made on a concentric handlebody in V, that we denote $V' := \Sigma \times [-1+\epsilon, 1-\epsilon]$ for some small $\epsilon \in]0,1[$. We also chose a set of curves on $\Sigma' := \partial V' \setminus D^2$, the boundary of the handlebody V' minus a small disk as shown in Figure A.2. This induces a basis of $H_1(\Sigma')$.

Now we draw on Figures A.3 and A.4 the *spines* of two curves γ_1 and γ_2 on $\partial V'$. This means that each of these curve is the boundary of the neighborhood of the union of the two simple closed curves with one intersection point that are drawn in the respective figures. We consider the element $s := T_{\gamma_1} \circ T_{\gamma_2}^{-1} \in \mathcal{M}(\Sigma')$ and perform the surgery:

$$V \rightsquigarrow V_s := ((V - V') \cup_s V').$$

This choice of s is inspired by the fact that, as one can deduce from Theorem 2.4 of Chapter 1, the element $\tau_2(s)$ is not the image by τ_2 of an element extending to the handlebody V'. We need the following lemma.

Lemma 3.2. *The reduction of $\tau_2(s) \in D_2(H_1(\Sigma'))$ to $D_2(H_1(V'))$ is trivial.*

Proof. According to Morita's formula from [42], which is recalled in Lemma 2.3 of Chapter 1, we compute that $\tau_2(s) = (b_1 + b_2) \odot (b_2 - a_1) - b_1 \odot (b_2 - a_1)$ which reduces to 0 in $D_2(H_1(V'))$. $\square$

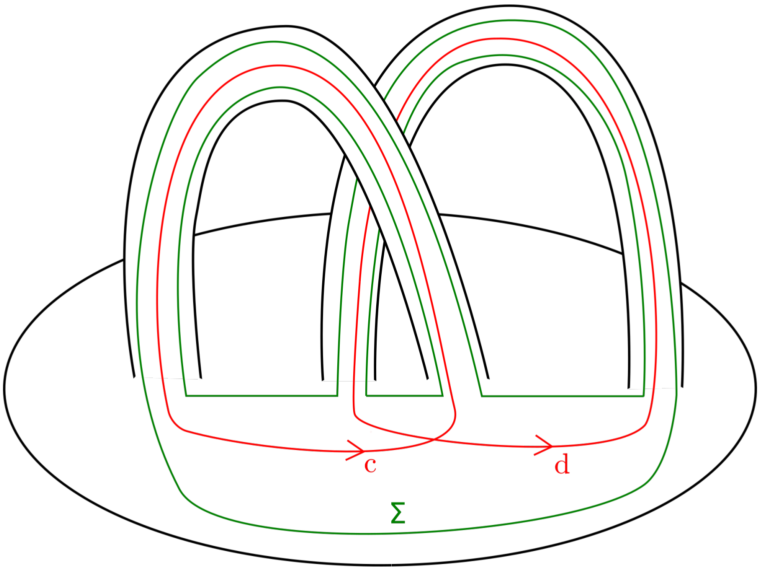


Figure A.1: A copy of the surface Σ in the handlebody V, with a choice of basis for $H_1(\Sigma)$

According to [34, Section 3], this implies that $s \in L_3(V')$ so that V_s is L_3-equivalent to V. We will prove that $\beta(V_s)$ is not a constant map, which will in turn imply, by Proposition 3.1, that V_s cannot be J_3-equivalent to the trivial homology cylinder V. To do this we use clasper calculus from [15], and [39, Lemma 3.16] reproduced below. We refer to [15] for the definition of a surgery on a Y-graph, and to [39, Appendix B] for the definition of the framing number in a homology cylinder. We denote by M_Y the homology cylinder obtained from a homology cylinder M by performing the surgery on the Y-graph $Y \subset M$.

Lemma 3.3. *Let Y be a Y-graph in a homology cylinder M over some surface S, whose leaves are ordered and oriented in an arbitrary way. Denote by $h_1, h_2, h_3 \in H_1(S)$ their homology classes and by $f_1, f_2, f_3 \in \mathbb{Z}$ their framing numbers in M. Then we have*

$$\beta(M_Y) - \beta(M) = \prod_{i=1}^{3} \left(\overline{h_i} + f_i \cdot \overline{1}\right) \in B_{\leq 3} \tag{3.2}$$

where $\overline{x}$ denotes the evaluation at x for any $x \in H_1(S)$, and $\overline{1}$ is the constant function equal to $1 \in \mathbb{Z}_2$.

In order to use Lemma 3.3, we need to convert the surgeries given by the Dehn twists on γ_1 and γ_2 into surgeries on Y-graphs. We recall in Figure A.10 how to transform a right Dehn twist along a curve bounding a surface (the curve is represented by dashes in Figure A.5) into a Y-graph with a special leaf. To deal with a left Dehn twist, one would simply need to change the signs of the framings. Lickorish's trick allows us to convert the surgery on a right Dehn twist on some surface to a (+1)-surgery on a knot bounding a surface. The first Kirby move allows us to add a trivial component with a (+11)-framing, and we get to Step 2. The second Kirby move (a "slide") yields Step 3. We then apply moves (1) and (2) from [15] to get Step 4. Move (9) from [15] gives step 4 and concludes.

The two claspers corresponding to γ_1 and γ_2 are drawn in Figures A.11 and A.12. These two claspers are drawn in two disjoint parallel layers of a thickening of Σ'.

Proposition 3.4. *V_s is not J_3-equivalent to V.*

Proof. Denote respectively Y_1 and Y_2 the Y-graphs in Figures A.11 and A.12. Recall that c and d on Figure A.1 are homology classes in $H_1(\Sigma)$, that we consider here as homology

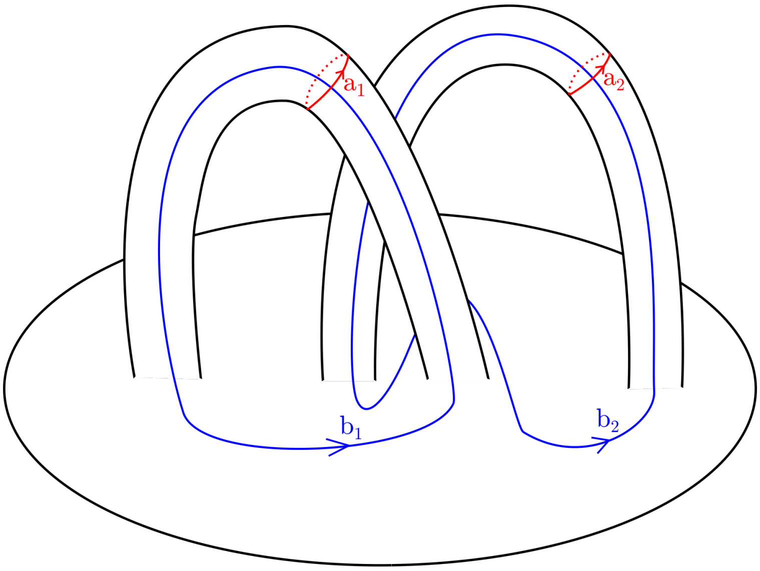


Figure A.2: The handlebody V' and a choice of basis for $H_1(\Sigma')$

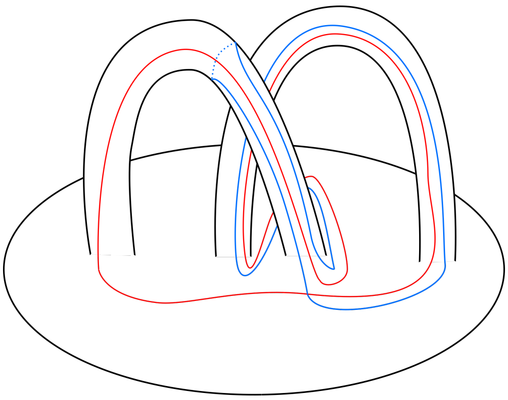


Figure A.3: Spine of γ_1

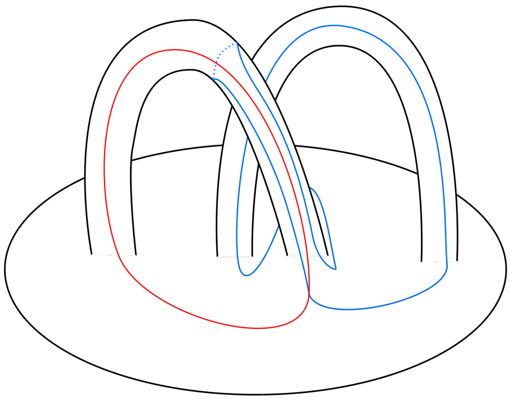


Figure A.4: Spine of γ_2

classes in $H_1(V)$. Thanks to equation (3.2) we compute:

$$\begin{aligned}\beta(V_s) - \beta(V) &= (\beta((V_{Y_2})_{Y_1}) - \beta(V_{Y_2})) + (\beta(V_{Y_2}) - \beta(V))\\ &\overset{(3.2)}{=} (\overline{c+d} + (-1)\cdot\overline{1})(\overline{d} + 0\cdot\overline{1})(\overline{0} + (-1)\cdot\overline{1}) + (\overline{c} + 0\cdot\overline{1})(\overline{d} + 0\cdot\overline{1})(\overline{0} + 1\cdot\overline{1})\\ &= (\overline{c} + \overline{d})\overline{d} + \overline{c}\overline{d}\\ &= \overline{d}^2\end{aligned}$$

where we have used [39, Lemma B.2] to compute the framings in the cylinder $V = \Sigma\times[-1,1]$. The second part of this lemma implies that to compute the framing of a curve in a thickened surface, it is enough to count algebraically the crossings in a projection of the curve to the top surface $\sigma \times \{+1\}$. Besides the framings of the leaves of Y_1 in V_{Y_2} can be computed in the homology cylinder V using the same method, because the third part of the same lemma claims that Torelli surgeries do not affect linking numbers. Notice also that we have used, for any $x, y \in H$, the equality $\overline{x+y} = \overline{x} + \overline{y} + \omega(x,y)\ mod\ 2$. Finally, as the square function is the identity in $\mathbb{Z}_2$, we have that $\overline{x}^2 = \overline{x}$ for any $x \in H$, and we obtain:

$$\beta(V_s) - \beta(V) = \overline{d} \tag{3.3}$$

This concludes together with Proposition 3.1. $\square$

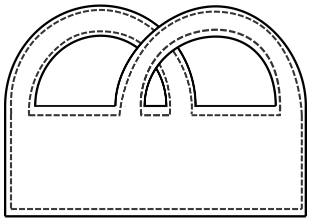


Figure A.5: A surface F into which the curve bounds.

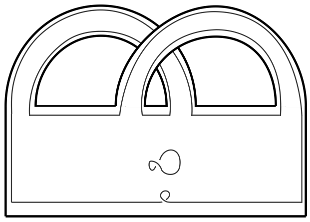


Figure A.6: Step 2

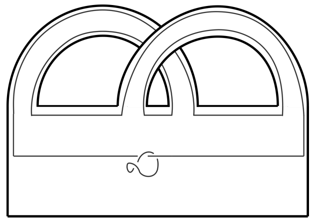


Figure A.7: Step 3

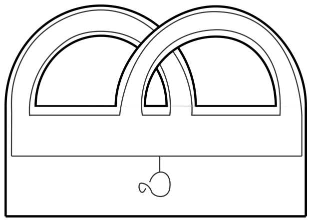


Figure A.8: Step 4

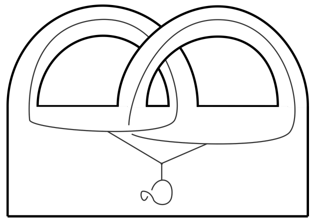


Figure A.9: Step 5

Figure A.10: Equivalent surgery descriptions of the mapping cylinder of a BSCC map of genus 1. The claspers are drawn in a thickening of the surface F.

We now conclude with the following result.

Corollary 3.5. *There exist two closed oriented 3-manifolds that are L_3-equivalent but not J_3-equivalent.*

In order to prove Corollary 3.5 we need to connect the Birman-Craggs homomorphism of a homology cylinder to the Rokhlin function of its *closure*. The closure of a homology cylinder C with boundary parametrization c is the closed oriented 3-manifold defined as $\overline{C} := (C/\sim) \cup (S^1 \times D^2)$, where $C/\sim$ denotes the quotient of C by the idenfication of the bottom of C with the top of C via $c_+ \circ c_-$. The solid torus is glued along the toroidal boundary of this quotient by sending the meridian to some circle fibering over a point of the boundary of Σ. The following is explained in [16, Section 6.3] and we refer to this survey for more details. Remind that the Rokhlin invariant of a closed oriented 3-manifold C equipped with a spin structure σ is defined as the signature of a compact connected oriented smooth 4-manifold which is bounded by C and to which σ can be extended. This invariant is well-defined (independent of the choices made) only after reduction modulo 16. Hence we get the Rokhlin function of C:

$$R_C : \mathrm{Spin}(C) \to \mathbb{Z}_{16}$$

where $\mathrm{Spin}(C)$ is the set of spin structures on C, and happens to be an affine space over $H^1(M, \mathbb{Z}_2)$. Furthermore, whenever $H_1(M, \mathbb{Z})$ is torsion-free, the Rokhlin function is trivial modulo 8. This applies for example to the closure of a homology cylinder. Hence, when dividing by 8, we get an invariant in $\mathbb{Z}_2$. Also, for any $C \in \mathcal{IC}_{g,1}$, the inclusions $c_\pm : \Sigma_{g,1} \to C$ induce an affine isomorphism $c^* : \mathrm{Spin}(\overline{C}) \to \mathrm{Spin}(\Sigma_{g,1})$. This gives sense to the following: the Birman-Craggs homomorphism $\beta : \mathcal{IC}_{g,1} \longrightarrow B_{\leq 3}$ coincides with the map

$$C \mapsto \frac{1}{8} R_{\overline{C}} \circ c^{*,-1}.$$

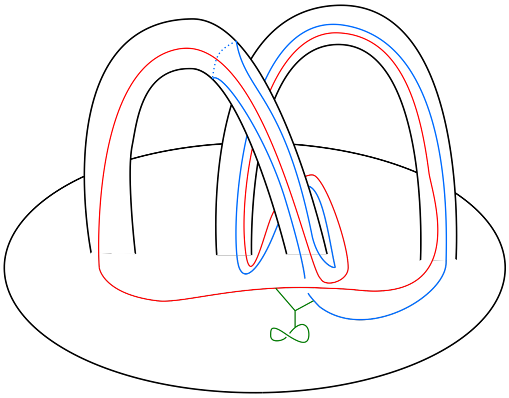


Figure A.11: Y-graph corresponding to T_{γ_1}

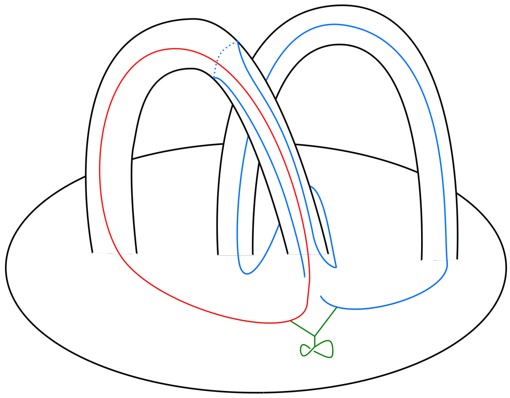


Figure A.12: Y-graph corresponding to T_{γ_2}

We shall need the following variation formula for the Rokhlin function [16, Prop. 6.13].

Lemma 3.6. *Let N be a closed connected oriented 3-manifold. Given a cobordism $C \in \mathcal{IC}_{g,1}$ and an embedding $j : \Sigma_{g,1} \to N$, consider the 3-manifold N' obtained by "cutting" N along j and by "inserting" C. Then, there is a canonical bijection $\sigma \mapsto \sigma'$ between* $\mathrm{Spin}(N)$ *and* $\mathrm{Spin}(N')$, *such that*

$$R_{N'}(\sigma') - R_N(\sigma) = 8 \cdot \beta(C)\,(j^*(\sigma)) \in \mathbb{Z}_{16}. \tag{3.4}$$

This is all we need to prove Corollary 3.5.

Proof of Corollary 3.5. Consider the closures of the homology cylinders V_s and V, whose boundary parametrization is denoted by v. Suppose they are related by a surgery on an element $f \in J_3(\Sigma)$ for some surface embedding $j : \Sigma_{g,1} \to V \subset \overline{V}$. By formula (3.4), we get for any $\sigma \in \mathrm{Spin}(V) \simeq \mathrm{Spin}(\overline{V})$ (and the associated spin structure σ_s on V_s):

$$R_{\overline{V_s}}(\sigma_s) - R_{\overline{V}}(\sigma) = 8 \cdot \beta(c(f))\,(j^*(\sigma)) \in \mathbb{Z}_{16}$$

which implies that

$$\beta(V_s)(v^*(\sigma_s)) - \beta(V)(v^*(\sigma)) = \beta(f)(j^*(\sigma)) \in \mathbb{Z}_2$$

Hence, as $f \in J_3(\Sigma_{g,1})$, and V is a trivial homology cylinder, we get that $\beta(V_s)$ is a constant map which is absurd in virtue of equation (3.3). $\square$

We have seen that the L_k-equivalence does not necessarily help to study the J_k-equivalence in full generality. Nevertheless, it might still be the case that, for homology 3-spheres, these relations coincide even though working with the L_k-equivalence provides a wider class of applications when performing surgeries. Among homology 3-spheres, the J_k-equivalence is trivial up to $k = 4$. The following questions hence remain.

Question 3.7. *For homology 3-spheres, and any $k \geq 5$, is the L_k-equivalence trivial ?*

Question 3.8. *For homology 3-spheres, and any $k \geq 5$, do the L_k-equivalence and J_k-equivalence relations coincide ?*

Bibliography

[1] S. Andreadakis. On the automorphisms of free groups and free nilpotent groups. *Proc. London Math. Soc. (3)*, 15:239–268, 1965.

[2] Mamoru Asada and Hiroaki Nakamura. On graded quotient modules of mapping class groups of surfaces. *Israel J. Math.*, 90(1-3):93–113, 1995.

[3] Joan S. Birman. On Siegel's modular group. *Math. Ann.*, 191:59–68, 1971.

[4] Joan S. Birman. On the equivalence of Heegaard splittings of closed, orientable 3-manifolds. In *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, pages 137–164. Ann. of Math. Studies, No. 84. 1975.

[5] Joan S. Birman and R. Craggs. The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold. *Trans. Amer. Math. Soc.*, 237:283–309, 1978.

[6] Tim D Cochran, Amir Gerges, and Kent Orr. Dehn surgery equivalence relations on 3-manifolds. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 131, pages 97–127. Cambridge University Press, 2001.

[7] Alexandru Dimca, Richard Hain, and Stefan Papadima. The abelianization of the Johnson kernel. *J. Eur. Math. Soc. (JEMS)*, 16(4):805–822, 2014.

[8] Quentin Faes. The handlebody group and the images of the second Johnson homomorphism. To appear in *Algebraic & Geometric Topology*, arXiv:2001.09825, preprint 2021.

[9] Quentin Faes. Triviality of the J_4-equivalence among homology 3-spheres, preprint 2021, arXiv:2105.14253.

[10] William Fulton and Joe Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

[11] Mikhail Goussarov. Finite type invariants and n-equivalence of 3-manifolds. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(6):517–522, 1999.

[12] H. B. Griffiths. Automorphisms of a 3-dimensional handlebody. *Abh. Math. Sem. Univ. Hamburg*, 26:191–210, 1963/64.

[13] M. N. Gusarov. Variations of knotted graphs. The geometric technique of n-equivalence. *Algebra i Analiz*, 12(4):79–125, 2000.

[14] Nathan Habegger and Gregor Masbaum. The Kontsevich integral and Milnor's invariants. *Topology*, 39(6):1253–1289, 2000.

[15] Kazuo Habiro. Claspers and finite type invariants of links. *Geom. Topol.*, 4:1–83, 2000.

[16] Kazuo Habiro and Gwénaël Massuyeau. From mapping class groups to monoids of homology cobordisms: a survey. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 465–529. Eur. Math. Soc., Zürich, 2012.

[17] Richard Hain. Infinitesimal presentations of the Torelli groups. *J. Amer. Math. Soc.*, 10(3):597–651, 1997.

[18] Richard Hain. Relative weight filtrations on completions of mapping class groups. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 309–368. Math. Soc. Japan, Tokyo, 2008.

[19] Sebastian Hensel. A primer on handlebody groups. In *Handbook of group actions. V*, volume 48 of *Adv. Lect. Math. (ALM)*, pages 143–177. Int. Press, Somerville, MA, 2020.

[20] Susumu Hirose. The action of the handlebody group on the first homology group of the surface. *Kyungpook Math. J.*, 46(3):399–408, 2006.

[21] Dennis Johnson. An abelian quotient of the mapping class group I. *Mathematische annalen*, 249(3):225–242, 1980.

[22] Dennis Johnson. Quadratic forms and the Birman-Craggs homomorphisms. *Trans. Amer. Math. Soc.*, 261(1):235–254, 1980.

[23] Dennis Johnson. Spin structures and quadratic forms on surfaces. *J. London Math. Soc. (2)*, 22(2):365–373, 1980.

[24] Dennis Johnson. The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$. *Ann. of Math. (2)*, 118(3):423–442, 1983.

[25] Dennis Johnson. A survey of the Torelli group. *Contemporary Math*, 20:165–179, 1983.

[26] Dennis Johnson. The structure of the Torelli group II. A characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113–126, 1985.

[27] Dennis Johnson. The structure of the Torelli group III. The abelianization of $\mathcal{I}$. *Topology*, 24(2):127–144, 1985.

[28] Dennis L. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.

[29] Jamie Bradley Jorgensen. *Surface homeomorphisms that do not extend to any handlebody and the Johnson filtration*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Rice University.

[30] Nariya Kawazumi. Cohomological aspects of magnus expansions, preprint 2005, arXiv:0505497.

[31] Nariya Kawazumi and Yusuke Kuno. The logarithms of Dehn twists. *Quantum Topol.*, 5(3):347–423, 2014.

[32] Jerome Levine. Homology cylinders : an enlargement of the mapping class group. *Algebraic & Geometric Topology*, 1(1):243–270, 2001.

[33] Jerome Levine. Labeled binary planar trees and quasi-lie algebras. *Algebraic & Geometric Topology*, 6(2):935–948, 2006.

[34] Jerome Levine. The Lagrangian filtration of the mapping class group and finite-type invariants of homology spheres. *Mathematical Proceedings of the Cambridge Philosophical Society*, 141(2):303–315, 2006.

[35] Gwénaël Massuyeau. Cohomology rings, Rochlin function, linking pairing and the Goussarov-Habiro theory of three-manifolds. *Algebr. Geom. Topol.*, 3:1139–1166, 2003.

[36] Gwénaël Massuyeau. Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant. *Bull. Soc. Math. France*, 140(1):101–161, 2012.

[37] Gwénaël Massuyeau. Quelques aspects de la théorie des invariants de type fini en topologie de dimension trois, 2012. Habilitation à diriger des recherches, Université de Strasbourg, tel-00734378v3.

[38] Gwénaël Massuyeau and Jean-Baptiste Meilhan. Characterization of Y_2-equivalence for homology cylinders. *J. Knot Theory Ramifications*, 12(4):493–522, 2003.

[39] Gwénaël Massuyeau and Jean-Baptiste Meilhan. Equivalence relations for homology cylinders and the core of the Casson invariant. *Transactions of the American Mathematical Society*, 365(10):5431–5502, 2013.

[40] Gwénaël Massuyeau and Vladimir Turaev. Fox pairings and generalized Dehn twists. *Ann. Inst. Fourier (Grenoble)*, 63(6):2403–2456, 2013.

[41] S. V. Matveev. Generalized surgeries of three-dimensional manifolds and representations of homology spheres. *Mat. Zametki*, 42(2):268–278, 345, 1987.

[42] Shigeyuki Morita. Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I. *Topology*, 28(3):305–323, 1989.

[43] Shigeyuki Morita. On the structure of the Torelli group and the Casson invariant. *Topology*, 30(4):603–621, 1991.

[44] Shigeyuki Morita. Abelian quotients of subgroups of the mapping class group of surfaces. *Duke Mathematical Journal*, 70(3):699–726, 1993.

[45] Shigeyuki Morita, Takuya Sakasai, and Masaaki Suzuki. Structure of symplectic invariant Lie subalgebras of symplectic derivation Lie algebras. *Adv. Math.*, 282:291–334, 2015.

[46] Shigeyuki Morita, Takuya Sakasai, and Masaaki Suzuki. Torelli group, Johnson kernel, and invariants of homology spheres. *Quantum Topol.*, 11(2):379–410, 2020.

[47] Yuta Nozaki, Masatoshi Sato, and Masaaki Suzuki. Abelian quotients of the Y-filtration on the homology cylinders via the LMO functor. To appear in *Geom. Topol.*, arXiv:2001.09825, preprint 2020.

[48] Genki Omori. A small normal generating set for the handlebody subgroup of the Torelli group. *Geometriae Dedicata*, 201(1):353–367, 2019.

[49] Wolfgang Pitsch. Integral homology 3-spheres and the Johnson filtration. *Transactions of the American Mathematical Society*, 360(6):2825–2847, 2008.

[50] Wolfgang Pitsch. Trivial cocycles and invariants of homology 3-spheres. *Adv. Math.*, 220(1):278–302, 2009.

[51] Jerome Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.

[52] Kurt Reidemeister. Zur dreidimensionalen topologie. *Abh. Math. Sem. Univ. Hamburg*, 9(1):189–194, 1933.

[53] Ricard Riba and Wolfgang Pitsch. Invariants of rational homology 3-spheres from the abelianization of the mod-p Torelli group, preprint 2021, arXiv:2103.15519.

[54] James Singer. Three-dimensional manifolds and their Heegaard diagrams. *Transactions of the American Mathematical Society*, 35(1):88–111, 1933.

[55] Shin'ichi Suzuki. On homeomorphisms of a 3-dimensional handlebody. *Canadian Journal of Mathematics*, 29(1):111–124, 1977.

[56] Anderson Vera. Alternative versions of the Johnson homomorphisms and the LMO functor. To appear in *Alg. Geom. Topol.*, arXiv:1902.10012, preprint 2019.

[57] Friedhelm Waldhausen. Heegaard-Zerlegungen der 3-Sphäre. *Topology*, 7:195–203, 1968.

[58] Yuji Yokomizo. An $\mathrm{Sp}(2g;\mathbb{Z}_2)$-module structure of the cokernel of the second Johnson homomorphism. *Topology and its Applications*, 120(3):385–396, 2002.

Equivalence relations among homology 3-spheres and the Johnson filtration

Abstract: The Torelli group of a surface consists of isotopy classes of homeomorphisms of this surface acting trivially at the homological level. The structure of the Torelli group can be approached by the study and the comparison of two filtrations of this group: its lower central series, and the "Johnson" filtration, given by the kernels of the natural actions on the successive nilpotent quotients of the fundamental group of the surface. It is now known that there are, via the notion of "Heegaard splittings", rich interactions between this 2-dimensional study and the study of some 3-manifolds topological invariants: we refer here precisely to the so-called "finite-type" invariants. In this PhD, we are interested, through the study of the Torelli group, to some equivalence relations on homology 3-spheres. This allows us both to state results about homology 3-spheres and their surgeries, and results about the Johnson filtration of the Torelli group. Specifically, we study first the second Johnson homomorphism (a homomorphism defined on the second term of the Johnson filtration), and its interaction with the subgroup of elements extending to a handlebody bounded by the considered surface. This allows us to give a new description of the set of homology 3-spheres. In a second part, we prove that a certain equivalence relation is trivial among homology 3-spheres. Two homology 3-spheres are always related by a surgery along an element of the fourth term of the Johnson filtration. This is shown by proving the surjectivity of the restriction of a homomorphism called "the core of the Casson invariant" to the fourth term of the Johnson filtration. In the proof, we exhibit a "non-trivial" element of the fourth term of the Johnson filtration.

Résumé : Le groupe de Torelli d'une surface est constitué des classes d'isotopie d'homéomorphismes de cette surface qui agissent trivialement sur son homologie. La structure du groupe de Torelli peut être approchée par l'étude comparée de deux filtrations sur ce groupe : d'un côté, sa suite centrale descendante et, de l'autre, la filtration dite « de Johnson », donnée par les noyaux des actions naturelles sur les quotients nilpotents successifs du groupe fondamental de la surface. On sait désormais qu'il existe (via les scindements de Heegaard) de riches interactions entre cette étude en dimension deux et l'étude de certains invariants topologiques en dimension trois : il s'agit précisément des invariants « de type fini » des 3-variétés. Dans cette thèse, nous nous intéressons, à travers l'étude du groupe de Torelli, à des relations d'équivalences sur les 3-sphères d'homologie. Cela nous permet à la fois d'énoncer des résultats sur ces variétés et leurs chirurgies, et des résultats sur la filtration de Johnson du groupe de Torelli. Spécifiquement, nous étudions d'abord le second homomorphisme de Johnson (un homomorphisme défini sur le deuxième sous-groupe de la filtration de Johnson), et son interaction avec le sous-groupe des transformations s'étendant à un corps en anses bordé par la surface considérée. Dans un deuxième temps, nous prouvons qu'une certaine relation d'équivalence est triviale sur l'ensemble des 3-sphères d'homologie. Deux 3-sphères d'homologie sont toujours reliés par une chirurgie utilisant un élément dans le quatrième terme de la filtration de Johnson. Ceci est notamment montré en montrant la surjectivité de la restriction d'un homomorphisme appelé le « coeur de l'invariant de Casson » au quatrième terme de la filtration de Johnson.