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Caractérisation topologique de tresses virtuelles
(Topological characterization of virtual braids)

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Contents

1 Introduction **2**
1.1 Finite type invariants and virtual knots 2
1.2 Braids and virtual braids 3
1.3 Results 8

2 Virtual braids and Gauss diagrams **12**
2.1 Virtual braids. 12
2.2 Braid Gauss diagrams. 14
2.3 Virtual braids as Gauss diagrams 26
2.4 Presentation of PV_n. 28

3 Abstract diagrams **31**
3.1 Diagrams on surfaces 31
3.2 Abstract diagrams and monotony 35
3.3 Abstract and virtual braids 37
3.4 Abstract string links 40

4 Minimal realization of a virtual braid **42**
4.1 Braids in thickened surfaces. 43
4.2 Minimal genus of abstract objects 46

Chapter 1

Introduction

1.1 Finite type invariants and virtual knots

Knots, braids, tangles and all kind of linked objects have always fascinated humanity for many reasons: by their shape as motives in architecture and art, by their utility for sailors, even in philosophy when we think in the Borromean rings as Lacan's topology interpretation of human subjectivity. But, how to describe knots, braids, links or tangles have always been an intricate task. Their study has led to the development of knot theory as a branch of mathematics, that nowadays extend to almost all other fields in mathematics.

A *knot* is an embedding of S^1 in $\mathbb{R}^3$. Knots are identified up to isotopy in $\mathbb{R}^3$. The set of isotopy classes of knots is denoted by $\mathcal{K}$. A *knot invariant* is a function $I : \mathcal{K} \to G$, where G is an abelian group.

Finite-type invariants of knots are special type of invariant that was defined independently by Vassiliev [55] and Goussarov [26]. It has been proved that all know polynomial invariants [29, 23] and quantum invariants [47, 53] of knots can be expressible in terms of finite-type invariants [4, 12, 10]. For more information of finite-type invariants we refer the reader to [4, 15].

Vassiliev [2, 55] discovered finite-type invariants, while he was studying the topology of discriminant sets of smooth maps $\mathcal{M} = \{S^1 \to \mathbb{R}^3\}$. The main idea of Vassiliev relies in the fact that the space of smooth maps $\mathcal{M}$ is arcwise connected. In particular any knot can be connected to the unknot. The path connecting both embeddings, the knot and the unknot, may be deformed by an isotopy, in such way that it is always in general position (either it is an embedding or it is an immersion with transversal double points). The immersion having only double points is called *singular knot*. The set of isotopy classes of singular knots is called the discriminant Σ. Consequently knots are in bijective correspondence with the connected components of $\mathcal{M} \setminus \Sigma$. A knot invariant on G corresponds to a cohomology class of $H^0(\mathcal{M} \setminus \Sigma, G)$. Finite type invariants come from a filtration of Σ by subspaces of a given number of singular points.

A combinatorial definition of finite type invariants was developed independently by Goussarov [26] and refined later in the work of Polyak and Viro [48] in terms of Gauss diagrams. Gauss diagrams are combinatorial objects that encode the information of a knot diagram in terms of its crossings. They are identified by their equivalent Reidemeister moves. In particular Goussarov, Polyak and Viro proved that all finite-type invariants could be obtained by combinatorial formulae in terms of Gauss diagrams [27, 28]. However, in order to find such formulation of finite-type invariants, it is necessary to use Gauss diagrams that do not correspond to knot diagrams. Such diagrams are called *non-realizable* Gauss diagrams.

It is worth to mention that Gauss diagrams are interesting by themselves and have lead to the definition and overall the calculation of knot invariants, not only in $\mathbb{R}^3$ but in general 3-manifolds. For more information about Gauss diagrams and their invariants we refer the reader to [20, 21, 22]. However it remains the question: how can we interpret a non-realizable Gauss diagram?

Kauffman answered this question defining *virtual knots* [32, 33]. Virtual knots are equivalence classes of knot-like diagrams with an extra type of crossing, called *virtual*. The equivalence relation is generated by the *virtual Reidemeister moves*. Virtual knots are in a bijective correspondence with Gauss diagrams.

In a first glance this description seems completely combinatorial, but it has a nice topological interpretation as knots embedded in thickened surfaces, identified up to isotopy, compatiblity and stability [14, 30]. *Compatibility* is the equivalence relation generated by orientation preserving diffeomorphisms of the thickened surface. *Stability* is the equivalence relation generated by adding or deleting thickened handles to the thickened surface supporting the knot.

It does not seem convenient that a virtual knot may be represented in different thickened surfaces. Kuperberg [39] proved that, up to compatibility and isotopy, there is a unique embedding of the virtual knot in the thickened surface of minimal genus supporting the virtual knot. In particular this implies that classical knot theory embeds in virtual knot theory. In this sense virtual knot theory is a generalization of classical knot theory. For a comprehensive discussion about virtual knot theory we refer the reader to [32, 33, 44].

1.2 Braids and virtual braids

Braid groups on n strands, $\mathcal{B}_n$, were introduced by Artin in 1925 [3] and can be defined in different ways. Each one of them admits a natural generalization and have beautiful connections with other fields of mathematics. For a complete discussion on the equivalence of the definitions we refer the reader to [11, 46, 38]. Here we present some definitions of the braid group and we discuss their relations with the virtual braid group.

Geometrically, the braid group on n strads, is defined as the embedding of n disjoint curves in the 3-cube, connecting n fixed points of one face with n fixed points of the opposite face, satisfying a monotony condition. Geometric braids are identified up to isotopy, preserving the monotony condition. The multiplication of two geometric braids is given by concatenation. As in the case of knots and links, we consider their projection on the plane and work with braid diagrams.

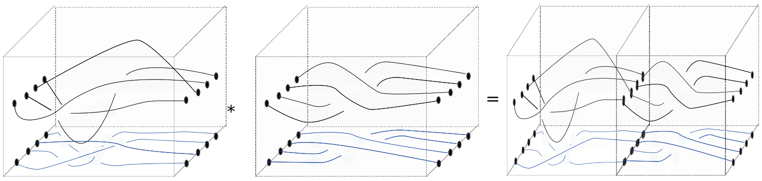

Figure 1.1: Geometric braids and its diagrams.

A braid diagram is a projection of a geometric braid in general position. That is, the image of the projection is a graph with transversal double points. We indicate in each crossing if it is an *undercrossing* or an *overcrossing*. The braid group, $\mathcal{B}_n$, is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the braid represented in Figure 1.2. Braid diagrams are identified by Reidemeister moves (Figure 1.2) and isotopy.

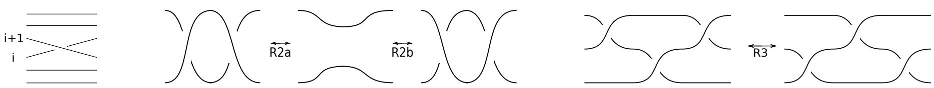

Figure 1.2: Classical generators and Reidemeister moves.

Algebraically the braid group is defined by means of its presentation, that is

$$\mathcal{B}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle .$$

If we add to this presentation the relations $\sigma_i^2 = 1$, we obtain a presentation of the symmetric group S_n. Thus, there is a natural epimorphism $\mathcal{B}_n \to S_n$. Its kernel is called the *pure braid group on n strands* and is denoted by $\mathcal{P}_n$. An element of $\mathcal{P}_n$ is a braid whose strands begin and end at the same points.

The braid group admits an interpretation as a subgroup of the automorphism of the free group on n generators. Let F_n be de free group with generators $x_1, \ldots, x_n$. Let $\varphi_n : \mathcal{B}_n \to Aut(F_n)$ be defined as follows:

$$\varphi_n(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad \text{if } j \neq i, i+1 \end{cases}$$

Artin proved that φ_n is a faithful representation of $\mathcal{B}_n$ for all $n \geq 2$. Furthermore, an automorphism of the free group, $\psi \in Aut(F_n)$, is a braid automorphism if and only if

1. there exists $\pi \in S_n$ such that $\psi(x_k)$ is conjugate in F_n to $x_{\pi(k)}$ for all $k \in \{1, \ldots, n\}$,

2. it fixes the element $x_1 \ldots x_n$, i.e. $\psi(x_1 \ldots x_n) = x_1 \ldots x_n$.

Virtual braids were defined by Kauffman in the same spirit of virtual knots [32]. They have a combinatorial definition by means of braid like diagrams. A *virtual braid diagram on n strands* is the union of n smooth curves in general position on the plane connecting points $(0, i)$ with points $(1, a_i)$. There exists $\pi \in S_n$, with $a_i = \pi(i)$, and these curves are monotonic with respect to the first coordinate. To each crossing we indicate if it is an *overcrossing, undercrossing* or a *virtual crossing.*

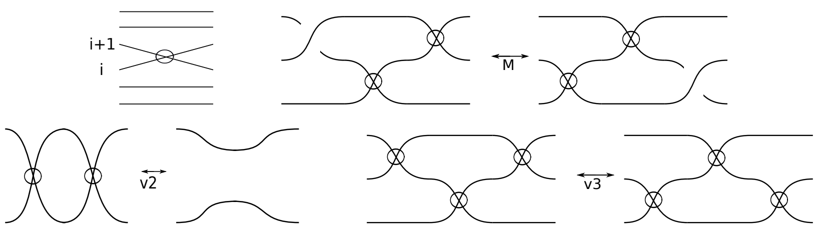

Figure 1.3: Virtual generators and virtual Reidemeister moves.

As in the classical case, the multiplication of two virtual braid diagrams is given by concatenation of the diagrams. Virtual braid diagrams are identified up to isotopy, Reidemeister moves (Figure 1.2) and virtual Reidemeister moves (Figure 1.3). The set of equivalence classes of virtual braid diagrams form a group that is generated by $\sigma_1, \ldots \sigma_{n-1}$ (classical generators, Figure 1.2) and $\tau_1, \ldots, \tau_{n-1}$ (virtual generators, Figure 1.3). We say that two virtual braid diagrams are *virtually equivalent* if they are related by isotopy and virtual Reidemeister moves.

Algebraically the virtual braid group, VB_n, admits the following presentation:

$$\left\langle \begin{array}{c} \sigma_1,\ldots,\sigma_{n-1} \\ \tau_1,\ldots,\tau_{n-1} \end{array} \middle| \begin{array}{lll} |i-j|\geq 2 & \sigma_i\sigma_j=\sigma_j\sigma_i & \tau_i\tau_j=\tau_j\tau_i \\ |i-j|=1 & \sigma_j\sigma_i\sigma_j=\sigma_i\sigma_j\sigma_i & \tau_j\tau_i\tau_j=\tau_i\tau_j\tau_i \\ & \sigma_j\tau_i\tau_j=\tau_i\tau_j\sigma_i & \\ 1\leq j\leq n-1 & \tau_j^2=1 & \end{array} \right\rangle.$$

As in the case of classical braids, there is an epimorphism $\theta_P : VB_n \to S_n$, given by $\theta_P(\sigma_i) = (i, i+1)$ and $\theta_P(\tau_i) = (i, i+1)$. The kernel of θ_P is called the *pure virtual braid group* and is denoted by PV_n, it consists in all virtual braid diagrams whose strands begin and end in the same fixed points. In the case of the virtual braid group there is another projection on the symmetric group, $\theta_K : VB_n \to S_n$, given by $\theta_K(\sigma_i) = 1$ and $\theta_K(\tau_i) = (i, i+1)$. The kernel of θ_K is called the *kure virtual braid group* and is denoted by KB_n.

In his master thesis [49], Rabenda gave a presentation of KB_n. In particular KB_n is an Artin-Tits group. On the other hand, a presentation of pure virtual braid group is given by Bardakov in [6].

A presentation of the pure virtual braid group, PV_n, is

$$\left\langle R_{i,j}\ 1\leq i\neq j\leq n \middle| \begin{array}{ll} R_{i,j}R_{k,j}R_{k,i}=R_{k,i}R_{k,j}R_{i,j} & \text{for } i,j,k \text{ distinct} \\ R_{i,j}R_{k,l}=R_{k,l}R_{i,j} & \text{for } i,j,k,l \text{ distinct} \end{array} \right\rangle.$$

A presentation of the kure virtual braid group, KB_n, is

$$\left\langle \delta_{i,j}\quad 1\leq i\neq j\leq n \middle| \begin{array}{ll} \delta_{i,j}\delta_{j,k}\delta_{i,j}=\delta_{j,k}\delta_{i,j}\delta_{j,k} & \text{for } i,j,k \text{ distinct} \\ \delta_{i,j}\delta_{k,l}=\delta_{k,l}\delta_{i,j} & \text{for } i,j,k,l \text{ distinct} \end{array} \right\rangle.$$

The relation of the pure virtual braid group on n strands with the Yang Baxter equations is studied in [9], under the name n-th quasitriangular group QTr_n. In the same paper a classifying space for the pure virtual braid group is constructed.

In the study of Artin groups of type FC, Godelle and Paris [25] constructed a classifying space for KB_n and proved that it has dimension $n-1$, with cohomological dimension $n-1$. In particular KB_n is of FP type and is torsion free. As a consequence, VB_n is virtually torsion free and its virtual cohomological dimension is $n-1$. Furthermore in the same paper, Godelle and Paris gave a solution to the word problem.

The relation between the pure and the kure virtual braid groups was studied by Bardakov and Bellingeri [7]. The group KB_n coincides with the normal closure of $\mathcal{B}_n$ in VB_n, and the intersection of PV_n and KB_n is called the extended pure virtual braid group on n strands, and is denoted by EP_n. In particular it contains the normal closure of PV_n in VB_n.

It is worth to mention that the center of the virtual braid group is trivial [19].

Other algebraic decomposition of VB_n have been studied by Kauffman and Lambropoulou. They propose a categorical model for the virtual braid group [36, 37]. In the construction of this categorical model, they deduce a presentation of VB_n with only one classical generator, and all the virtual ones. The relations are deduced from the action of the symmetric group on VB_n. By looking to the action of the symmetric group on this presentation, they deduce the presentation of PV_n given before. This categorical model propose how to use solutions of the algebraic Yang-Baxter equation to obtain representations of the pure virtual braid group.

Lebed in her Ph.D. thesis and subsequent papers [40, 41, 42] proposed another approach to the categorification of the virtual braid group. Her approach is by considering "locally" braided objects inside of a symmetric category. This is closer in spirit to the categorification of $\mathcal{B}_n$ in terms of braided categories, and it produces a convenient machine for constructing representations of VB_n. In particular she recovers the Burau representation for the virtual braid group, deduced by Vershinin [54], and extended it to the twisted virtual case.

Another categorification of the virtual braid group was proposed by Thiel [50, 51].

Recall that in the case of classical braids, Alexander's theorem [1] assures us that any link can be expressed as the closure of a braid. On the other hand, Markov's theorem [11, 45, 52] states necessary and sufficient conditions that relate two braids whose closure give isotopic links. In the case of virtual braids, the equivalent theorems relating virtual braids and virtual links were proved first by Kamada [31] and afterwards improved by Kauffman and Lambropoulou [34, 35].

A representation of the virtual braid group on the free group on $n+1$ generators has been proposed independently by Bardakov [5] and Manturov [43]. It is not known whether this representation is faithful or not. Let F_{n+1} be the free group generated by $x_1, \ldots, x_n, y$. The representation proposed by Bardakov and Manturov is given as follows.

$$\hat{\varphi}_n(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j \quad \text{if } j \neq i, i+1 \\ y \mapsto y \end{cases} \qquad \hat{\varphi}_n(\tau_i) : \begin{cases} x_i \mapsto y x_{i+1} y^{-1} \\ x_{i+1} \mapsto y^{-1} x_i y \\ x_j \mapsto x_j \quad \text{if } j \neq i, i+1 \\ y \mapsto y \end{cases}$$

The *group of a classical link* is the fundamental group of its complement in S^3. A presentation of the group of a link can be obtained by means of the Wirtinger method. There is an intrinsic relation between Artin's representation of the braid group and the group of the closure of a braid. Let L be a link in S^3 and $\beta \in \mathcal{B}_n$, such that $cl(\beta) = L$. Denote $G(L) := \pi_1(S^3 \setminus L)$. Then [56]

$$G(L) \cong \langle\, x_1, \ldots, x_n \mid \varphi_n(\beta)(x_i) = x_i \,\rangle.$$

The *group of a virtual link* is defined by an extension of the Wirtinger method [33]. However it was noted in [28] that the upper Wirtinger group of a virtual knot is not necessary isomorphic to the corresponding lower Wirtinger group. Bellingeri and Bardakov [13] proved that the representation of the virtual braid group given before induces an invariant of virtual links as follows. Let L be a virtual link, and β be a virtual braid whose closure coincides with L. Then

$$\hat{G}(L) := \langle\, x_1, \ldots, x_n \mid \hat{\varphi}_n(\beta)(x_i) = x_i \,\rangle$$

is an invariant of the virtual link. They proved that the group $\hat{G}$ remains invariant by the virtual Markov moves applied to the braid. However this group does not coincides with the group of a virtual link defined by means of the Wirtinger method. If we denote by $G(L)$ the group obtained by the Wirtinger method in the case of virtual knots, the relation is given by $G(K) \cong \hat{G}(K)/\langle\langle y \rangle\rangle$.

Other generalizations of classical braid invariants have been constructed for virtual braids. Manturov [43, 44] proposed an invariant of virtual braids. This invariant is a generalization, in some sense, of the action of $\mathcal{B}_n$ on the free group. In his article he proves that the linking number of a virtual braid is encoded by the action of VB_n on F_{n+1}. However it is still an open question if this is a complete invariant for virtual braids.

Bardakov, Vesnin and Wiest [8] defined *Dynnikov coordinates* on virtual braid groups, extending an action of $\mathcal{B}_n$ on $\mathbb{Z}^{2n}$ to VB_n. In the classical case this action is faithful [18]. In the virtual case it is proved to be faithful for $n = 2$. In the general case it is not know if it is faithful. Another question about this action is: what is the relation of this invariant with the one defined by Manturov?

Chterental [16], defined *virtual curve diagrams* as a generalization of curve diagrams [24]. As in the classical case he defined an action of VB_n on the virtual curve diagrams. In his paper he claims that such action is faithful for all n. Consequently, virtual curve diagrams are a complete invariant for virtual braids.

1.3 Results

In the second chapter we study the relation between Gauss diagrams and virtual braids. In particular, we consider *braid-Gauss diagrams*, also called horizontal Gauss diagrams.

Definition. Let $A = \{ A_{i,j} \mid 1 \leq i \neq j \leq n \}$. A *braid-Gauss diagram* on n strands is a couple (w, π), where $w \in (A^{\pm 1})^*$ and $\pi \in S_n$. We say that w is related with w' by an *isotopy* if $w = w_0 A_{i,j}^{\pm 1} A_{k,l} \pm 1 w_1$ and $w = w_0 A_{k,l} \pm 1 A_{i,j} \pm 1 w_1$. We identify braid-Gauss diagrams up to isotopy.

We can draw a braid-Gauss diagram as the union of n disjoint ordered parallel intervals, connected with perpendicular disjoint arrows labeled with a sign. As in the case of virtual knots, to each virtual braid β we can associate a braid-Gauss diagram $G(\beta)$. On the other hand, if two virtual braid diagrams are related by virtual Reidemeister moves, we say that they are *virtually equivalent*.

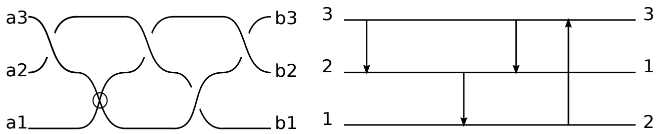

Figure 1.4: A virtual braid with its braid-Gauss diagram.

There is a bijective correspondence between braid-Gauss diagrams and virtually equivalent virtual braid diagrams.

Proposition 2.10.

1. Let g be a braid-Gauss diagram on n strands. There exists a virtual braid diagram β such that $G(\beta) = g$.

2. Let β_1 and β_2 be two virtual braids diagrams on n strands. Then $G(\beta_1) = G(\beta_2)$ if and only if β_1 is virtually equivalent to β_2.

Moreover, we can interpret the Reidemeister moves in terms of braid-Gauss diagrams as follows. Let $i, j, k \in \{1, \dots, n\}$ distincts, $\epsilon \in \{\pm 1\}$ and $w, w', w_0, w_1 \in A^*$.

1. We say that w is related with w' by an $\Omega 2$ move if
$$w = w_0 A_{i,j}^{\epsilon} A_{i,j}^{-\epsilon} w_1 \quad \text{and} \quad w' = w_0 w_1.$$

2. We say that w is related with w' by an $\Omega 3$ move if
$$w = w_0 A_{i,j}^{\epsilon} A_{i,k}^{\epsilon} A_{j,k}^{\epsilon} w_1 \quad \text{and} \quad w' = w_0 A_{j,k}^{\epsilon} A_{i,k}^{\epsilon} A_{i,j}^{\epsilon} w_1.$$

Definition. Let (w, π) and (w', π') be two braid-Gauss diagrams. We say that (w, π) is equivalent to (w', π') if $\pi = \pi'$ and w is related with w' by a finite number of Ω and isotopy moves. The set of equivalence classes of braid-Gauss diagrams is denoted by bG_n.

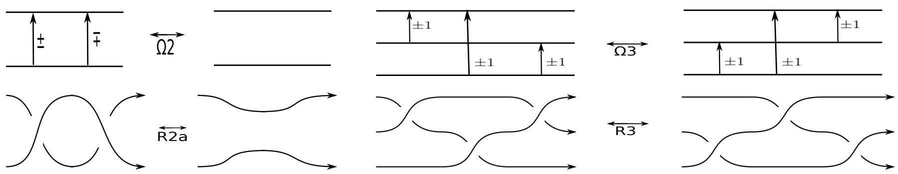

Figure 1.5: Ω moves.

Theorem 2.24. There is a bijective correspondence between the set of equivalence classes of braid Gauss diagrams and virtual braids.

As a corollary of Theorem 2.24 we recover the well-known presentation for the pure virtual braid group.

In the third chapter the main objective is to construct a topological interpretation of virtual braids that remains compatible with the topological interpretation of virtual knots and links and that preserves the combinatorial properties of virtual objects. Moreover, the topological interpretation proposed generalizes the geometric definition of classical braids.

In the same spirit of abstract links [30], we introduce an *abstract diagram* as a pair (S, β), where S is an oriented compact surface with two boundary components, and β is an immersion of n disjoint curves connecting fixed ordered points in the boundary components. We identify abstract diagrams up to isotopy, compatibility and stability. *Compatibility* is the equivalence relation generated by orientation preserving diffeomorphisms of the surface. *Stability* is the equivalence relation generated by adding or deleting handles to the surface.

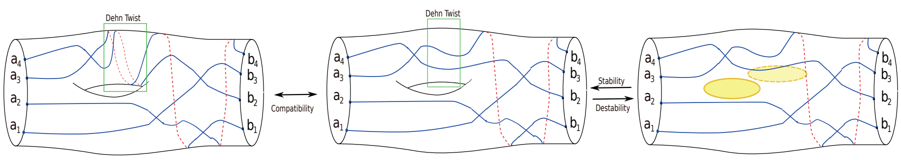

Figure 1.6: Braids in a thickened surface.

Now we state a necessary and sufficient condition to assure that the diagram on the surface accomplishes the characteristic monotonic condition of braids. An *abstract braid diagram* is a triple (S, β, ϵ), where (S, β) is an abstract diagram, such that the graph of β does not have any oriented cycle, and ϵ is a function indicating the over and undercrossings in the diagram. The isotopy, compatibility and stability equivalence are inherited from abstract diagrams. We identify abstract braid diagrams up to Reidemeister moves (Figure 1.2) that do not generate oriented cycles.

Using braid-Gauss diagrams we prove that abstract braid diagrams are a good topological representation for virtual braids.

Theorem 3.24. There exists a bijection between abstract braids on n strands and virtual braids on n strands.

Finally we introduce *abstract string links*, that can be seen as braid diagrams without the monotony condition.

In the fourth chapter, inspired by the work of Kuperberg [39] for virtual links, we prove that for any abstract braid, up to compatibility and Reidemeister moves, there exists a unique representative of minimal genus. An immediate consequence of this result is that classical braids embeds in abstract braids, consquently in virtual braids. Furthermore we proved that if an abstract string link admits a braid position, then its minimal representative also admits a braid position.

In order to accomplish this, we introduce braids in a thickened surface and identify them by isotopy, preserving the monotony condition. Then we translate the concept of destabilization and compatibility to braids in a thickened surface. We verify that we can identify braids in a thickened surface with abstract braids. Finally we prove the next theorem.

Theorem 4.13. Given a braid (sting link) in a thickened surface. Up to isotopy and compatibility, it admits a unique representative of minimal genus.

That translated in terms of abstract braids, reads as follows.

Corollary. Given an abstract braid, up to compatibility and Reidemeister equivalence, it admits a unique representative of minimal genus.

Corollary. Classical braids embeds in abstract braids.

Corollary. Given an abstract string link that admits a braid representative. Its minimal representative admits a braid position.

The last corollary suggests that abstract braids embeds in abstract string links. To prove this it would be sufficient to prove that ambient isotopy on braids in a thickened surface is equivalent to braid isotopy.

Chapter 2

Virtual braids and Gauss diagrams

We fix the next notation: set n a natural number, the interval $[0,1]$ is denoted by I, and the 2-cube is denoted by $\mathbb{D} = I \times I$. The projections on the first and second coordinate from the 2-cube to the interval, are denoted by $\pi_1 : \mathbb{D} \to I$ and $\pi_2 : \mathbb{D} \to I$, respectively. A set of planar curves is said to be in *general position* if all its multiple points are transversal double points.

2.1 Virtual braids.

Definition 2.1. A *strand diagram* on n strands is an n-tuple of curves, $\beta = (\beta_1, \ldots, \beta_n)$, where $\beta_k : I \to \mathbb{D}$ for $k = 1, \ldots, n$, such that:

1. There exists $\sigma \in S_n$ such that, for $k = 1, \ldots, n$, we have $\beta_k(0) = a_k$ and $\beta_k(1) = b_{\sigma(k)}$, where $a_k = (0, \frac{k}{n+1})$ and $b_k = (1, \frac{k}{n+1})$.

2. For $k = 1, \ldots, n$ and all $t \in I$, $(\pi_1 \circ \beta_k)(t) = t$.

3. The set of curves in β is in general position.

The curves β_k are called *strands* and the transversal double points are called *crossings*. The set of crossings is denoted by $\mathcal{C}(\beta)$.

A *virtual braid diagram* on n strands is a strand diagram on n strands β endowed with a function $\epsilon : \mathcal{C}(\beta) \to \{+1, -1, v\}$. The crossings are called *positive, negative* or *virtual* according to the value of the function ϵ. The positive and negative crossings are called *regular crossings* and the set of regular crossings is denoted by $R(\beta)$. In the image of a regular neighbourhood (homeomorphic to a disc sending the center to the crossing) we replace the image of the involved strands as in Figure 2.1, according to the crossing type.

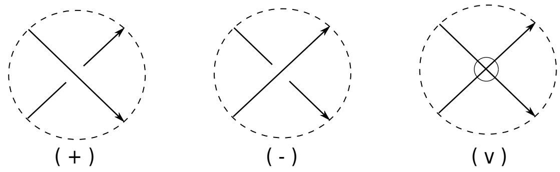

Figure 2.1: Positive, negative and virtual crossings.

Without loss of generality we draw the braid diagrams from left to right. We denote the set of virtual braid diagrams on n strands by VBD_n.

Definition 2.2. Given two virtual braid diagrams on n strands, β_1 and β_2, and a neighbourhood $V \subset \mathbb{D}$, homeomorphic to a disc, such that:

- Up to isotopy $\beta_1 \setminus V = \beta_2 \setminus V$.
- Inside V, β_1 differs from β_2 by a diagram as either in Figure 2.2, or in Figure 2.3, or in Figure 2.4.

Then we say that β_2 is obtained from β_1 by an $R2a$, $R2b$, $R3$, $V2$, $V3$, M or M' moves.

The moves $R2a$, $R2b$, $R3$ are called *Reidemeister moves*, the moves $V2$ and $V3$ are called *virtual moves*, and the moves M and M' are called *mixed moves*.

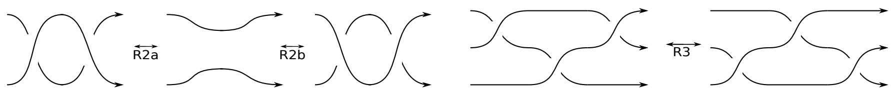

Figure 2.2: Reidemester moves.

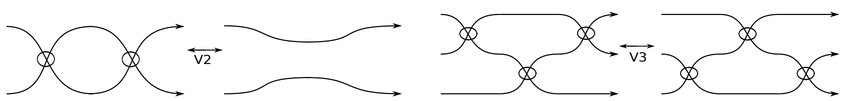

Figure 2.3: Virtual moves.

Let β and β' be two virtual braid diagrams. Note that if β can be obtained from β' by a finite series of virtual, mixed or Reidemeister moves, necessarily β and β' have the same number of strands.

If β' can be obtained from β by isotopy and a finite number of virtual, Reidemeister or mixed moves, β and β' are *virtually Reidemeister equivalent.* We denote this by $\beta \sim \beta'$. These equivalence classes are called *virtual braids on n strands*. We denote by $VB_n = VBD_n/\sim$ the set of virtual braids on n strands.

If β' can be obtained from β by isotopy and a finite number of virtual or mixed moves, β and β' are *virtually equivalent.* We denote this by $\beta \sim_{vm} \beta'$.

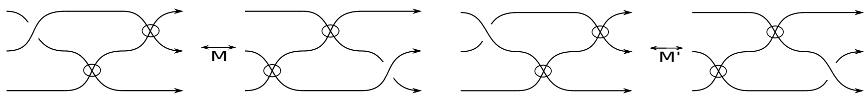

Figure 2.4: Mixed moves.

If β' can be obtained from β by isotopy and a finite number of Reidemeister moves, β and β' are *Reidemeister equivalent*. We denote this by $\beta \sim_R \beta'$.

Remark 2.3. Define the product of two virtual braids diagrams as the concatenation of the diagrams and an isotopy in the obtained diagram, to fix it in $\mathbb{D}$. With this operation the set of virtual braid diagrams has the structure of a monoid. It is not hard to see that it factorizes in a group when we consider the virtual Reidemeister equivalence classes. Thus, the set of virtual braids has the structure of a group with the product defined as the concatenation of virtual braids. The virtual braid group on n strands has the following presentation:

- Generators: $\sigma_1, \dots, \sigma_{n-1}, \tau_1, \dots, \tau_{n-1}$.
- Relations:

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \leq i \leq n-2 \\
\sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| \geq 2 \\
\tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} && \text{for } 1 \leq i \leq n-2 \\
\tau_i\tau_j &= \tau_j\tau_i && \text{if } |i-j| \geq 2 \\
\sigma_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\sigma_{i+1} && \text{for } 1 \leq i \leq n-2 \\
\tau_i\sigma_j &= \sigma_j\tau_i && \text{if } |i-j| \geq 2 \\
\tau_i^2 &= 1 && \text{for } 1 \leq i \leq n-1
\end{aligned}$$

Remark 2.4. The mixed moves can be replaced by the moves indicated in Figure 2.5.

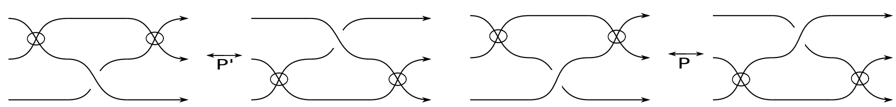

Figure 2.5: Equivalent mixed moves.

2.2 Braid Gauss diagrams.

Definition 2.5. A *Gauss diagram* on n strands G is an ordered collection of n oriented intervals $\sqcup_{k=1}^n I_k$, together with a finite number of arrows and a permutation $\sigma \in S_n$ such that:

- Each arrow connects by its ends two points in the interior of the intervals (possibly the same interval).
- Each arrow is labelled with a sign ± 1.
- The end point of the k-th interval is labelled with $\sigma(i)$.

Gauss diagrams are considered up to orientation preserving homeomorphism of the underlying intervals.

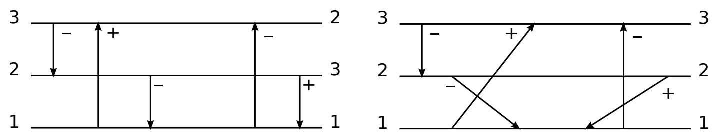

Figure 2.6: Gauss diagrams

Definition 2.6. Let β be a virtual braid diagram on n strands. The *Gauss diagram* of β, $G(\beta)$, is a Gauss diagram on n strands given by:

- Each strand of $G(\beta)$ is associated to the corresponding strand of β.
- The endpoints of the arrows of $G(\beta)$ corresponds to the preimages of the regular crossings of β.
- Arrows are pointing from the over-passing string to the under-passing string.
- The signs of the arrows are given by the signs of the crossings (their local writhe).
- The permutation of $G(\beta)$ corresponds to the permutation associated to β.

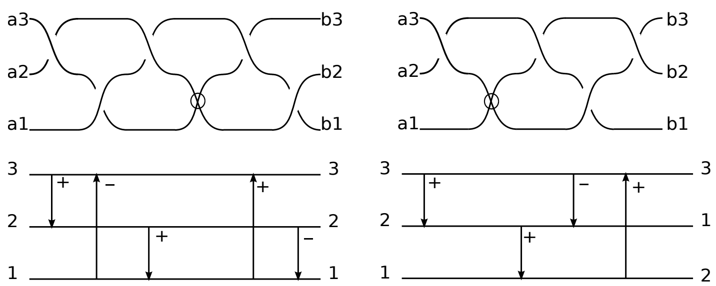

Figure 2.7: Gauss diagrams of virtual braid diagrams

Remark 2.7. The arrows of the Gauss diagram of any virtual braid diagram are pairwise disjoint and each arrow connects two different intervals. Furthermore we can draw them perpendicular to the underlying intervals, i.e. we can parametrize each interval, I_k, with respect to the standard interval $I = [0, 1]$, in such a way that the beginning and ending points of each arrow correspond to the same $t \in I$ and such that different arrows correspond to different t's in I.

Definition 2.8. Gauss diagrams that satisfies the conditions of Remark 2.7 are called *braid Gauss diagrams*. The set of braid Gauss diagrams on n strands is denoted by bGD_n.

Definition 2.9. Given a braid-Gauss diagram, G, we can associate a total order to the set of arrows in G, given by the order in which the arrows appear in the interval I, i.e. let a and b be two arrows in G, such that a appears first, then $a > b$. This order is not defined in the equivalence class of the Gauss diagram, as it may change with orientation preserving homeomorphisms of the underlying intervals.

We denote by $P(G)$ the partial order obtained as the intersection of the total orders associated to G. Given a virtual braid diagram β, let $G(\beta)$ be its Gauss diagram. Then $P(G(\beta))$ defines a partial order in the set of regular crossings, $R(\beta)$. We denote it by $P(\beta)$.

Proposition 2.10.

1. Let g be a braid-Gauss diagram on n strands. Then there exists $\beta \in VBD_n$ such that $G(\beta) = g$.
2. Let β_1 and β_2 be two virtual braids on n strands. Then $G(\beta_1) = G(\beta_2)$ if and only if $\beta_1 \sim_{vm} \beta_2$.

From now on we fix $n \in \mathbb{N}$ the number of strands on the braid-Gauss diagrams, and we say braid-Gauss diagram instead of braid-Gauss diagram on n strands. We split the proof of this proposition into some lemmas.

Lemma 2.11. Let g be a braid-Gauss diagram. Then there exists $\beta \in VBD_n$ such that $G(\beta) = g$.

Proof. Let g be a braid-Gauss diagram and $A = \{c_1, \ldots, c_k\}$ be the set of arrows of g. Set a parametrization of the intervals as described in Remark 2.7. This induces an order in A given by $c_i > c_j$ if $p_i < p_j$, where $p_i \in I$ is the corresponding endpoint of c_i. Suppose that $c_i > c_j$ if $i < j$.

Recall the notation of Definition 2.1. For $j = 1, \ldots, k$ let $d_j = (\frac{j}{k+1}, \frac{1}{2})$ and consider the disc D_j with radius $r = \frac{1}{5(k+1)}$ centered in d_j. Draw a crossing inside D_j according to the sign of c_j, and label the intersection of the crossing components with the boundary of D_j as in Figure 2.8.

Drawing the strands: let $\sigma \in S_n$ be the permutation associated to g. Fix $i \in \{1, \ldots, n\}$ and let $A_i = \{c_{i_1}, \ldots, c_{i_m}\}$ be the arrows starting or ending in the i-th interval.

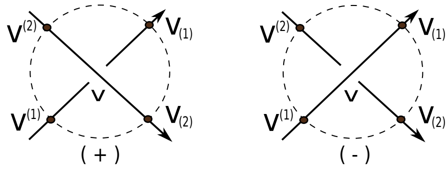

Figure 2.8: Labelled neighbourhoods of regular crossings.

For $s = 0, \ldots, m$ define o_s and t_{s+1} as follows:

1. $o_0 = a_i$ and $t_{m+1} = b_{\sigma(i)}$.
2. For $l = 1, \ldots, m$, $o_l = (d_{i_l})^{(v)}$ and $t_l = (d_{i_l})_{(v)}$ where:
   (a) If c_{i_l} is a positive arrow starting in the i-th interval or a negative arrow ending in the i-th interval then $v = 2$;
   (b) If c_{i_l} is a negative arrow starting in the i-th interval or a positive arrow ending in the i-th interval then $v = 1$.

For each $s \in \{0, \ldots, m\}$, draw a curve joining o_s to t_{s+1} such that it is strictly increasing on the first component and disjoint from the discs D_j for all $j \in \{1, \ldots, k\}$ except possibly on the points o_s and t_{s+1} defined above. In this way we have drawn a curve joining a_i with $b_\sigma(i)$ passing through the crossings $c_{i_1}, \ldots, c_{j_m}$.

For each $i \in \{1, \ldots, k\}$ we can draw a curve as described before, so that they are in general position. Consider the double points outside the discs D_j as virtual crossings. In this way we have constructed a virtual braid diagram such that its Gauss diagram coincides with g. □

Lemma 2.12. Let β_1 and β_2 be two virtual braid diagrams on n strands such that they are virtually equivalent. Then $G(\beta_1) = G(\beta_2)$.

Proof. In order to see this we only need to verify that the $V2$, $V3$, M and M' moves do not change the braid-Gauss diagram of a virtual braid diagram. In the cases of the $V2$ and $V3$ moves they involve only virtual crossings, which are not represented in the Gauss diagram, so they do not change the Gauss diagram. In the case of the M and M' moves, the Gauss diagrams of the equivalent virtual braid diagrams are equal (Figure 2.9), thus this type of move neither changes the Gauss diagram of the virtual braid diagram. □

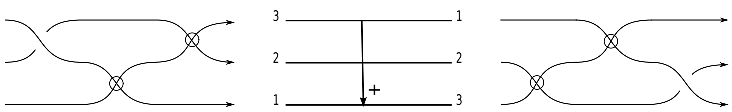

Figure 2.9: Gauss code of the mixed move.

Definition 2.13. Given $\beta \in VBD_n$ we can deform β by an isotopy, in such a way that for $c_i, c_j \in \mathcal{C}(\beta)$ with, $i \neq j$, we have that $\pi_1(c_i) \neq \pi_1(c_j)$, in this case we say that β is in *general position.* If $\beta \in VBD_n$ is in general position, let $\mathcal{D}(\beta)$ be the total order associated to $\mathcal{C}(\beta)$, given by $c_i > c_j$ if $\pi_1(c_i) < \pi_1(c_j)$. Denote by $D(\beta)$ the total order of the set of regular crossings, $R(\beta)$, induced by $\mathcal{D}(\beta)$.

Definition 2.14. A *primitive arc* of β is a segment of a strand of β which does not go through any regular crossing (but it may go through virtual ones).

Let $\beta \in VBD_n$. For $v \in R(\beta)$, set a disc D_v centered in v, with a radius small enough so that its intersection with β consists exactly in two transversal arcs as in Figure 2.8. We denote by $v^{(1)}$ and by $v^{(2)}$ the bottom and upper left intersections of β with ∂D_v, and by $v_{(2)}$ and by $v_{(1)}$ the bottom and upper right intersections of β with ∂D_v as in Figure 2.8.

Let d be a point in the diagram β, if $d \in \{b_1, \dots, b_n\}$ or $d \in \{c^{(1)}, c^{(2)}\}$ for some $c \in R(\beta)$ we denote d by d^*. Similarly if $d \in \{a_1, \dots, a_n\}$ or $d \in \{c_{(1)}, c_{(2)}\}$ for some $c \in R(\beta)$ we denote d by d_*. A *joining arc* is a primitive arc α such that there exist a_* and b^* with $\alpha(0) = a_*$ and $\alpha(1) = b^*$.

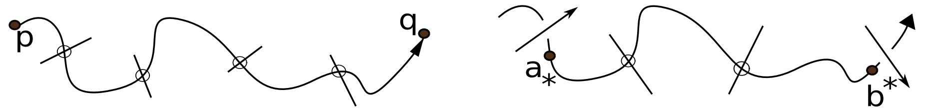

Figure 2.10: primitive and joining arcs.

As each arc is a segment of a strand $\beta_k : [0,1] \to \mathbb{D}$ we can parametrize it with respect to the projection on the first coordinate, i.e. there exists a continuous bijective map $\theta : [t_0, t_f] \to [0,1]$ with $0 < t_0 < t_f < 1$ such that $\pi_1(\alpha(\theta(t))) = t$. Without loss of generality we suppose from now on that the arcs are parametrized by the projection on the first coordinate.

Lemma 2.15. Let β be a virtual braid diagram and let α_1, α_2 be two primitive arcs of β such that:

1. The arcs α_1 and α_2 start at the same time, t_0, and end at the same time, t_f.

2. The arcs α_1 and α_2 start at the same point (a crossing which may be either virtual or regular), i.e. $\alpha_1(t_0) = \alpha_2(t_0)$.

3. The arcs α_1 and α_2 do not intersect, except at the extremes, i.e. $\alpha_1|_{(t_0,t_f)} \cap \alpha_2|_{(t_0,t_f)} = \emptyset$.

Then, there exists a virtual braid diagram β' virtually equivalent to β such that:

1. If β_1 and β_2 are the strands corresponding to α_1 and α_2 respectively, then up to isotopy they remain unchanged in β', and in their restriction to $(0, t_0) \times I$ we add only virtual crossings.

2. The diagrams β and β' coincide for $t \geq t_f$, i.e. $\beta|_{t \geq t_f} = \beta'|_{t \geq t_f}$.

3. In $(t_0, t_f) \times I$ there are only virtual crossings with α_2.

4. If $\alpha_1(t_f) = \alpha_2(t_f)$, we can choose β' such that there is no crossing in $(t_0, t_f) \times I$.

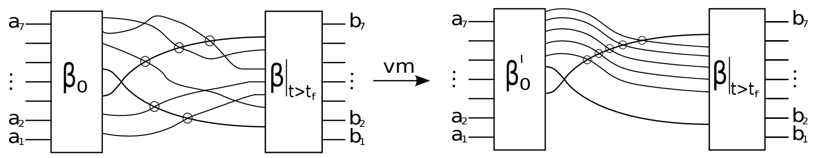

Figure 2.11: Lemma 2.15.

Proof. Suppose $\alpha_1(t_f) \neq \alpha_2(t_f)$ and that we have reduced β by all the possible $V2$ moves that may be made on it. Note that α_1, α_2 and $y = t_f$ form a triangle D.

Let C be the set of crossings in β such that their projections on the first component are in the open interval (t_0, t_f). Let m_0 be the number of crossings in C that are in the interior of D, m_1 the number of crossings in C that are on α_1, m_2 the number of crossings in C that are on α_2, and m_∞ the number of crossings in C that are outside D.

We argue by induction on $m = m_0 + m_1 + m_\infty$. Suppose $m = 1$. Then $C = \{c_1, \ldots, c_{m_2}, d\}$, where $c_1, \ldots, c_{m_2}$ are the crossings on α_2 and d is the other crossing. We have four cases:

1. The crossing d is outside D (Figure 2.12). We move d by an isotopy to the left part of $I \times [t_0, t_f)$.

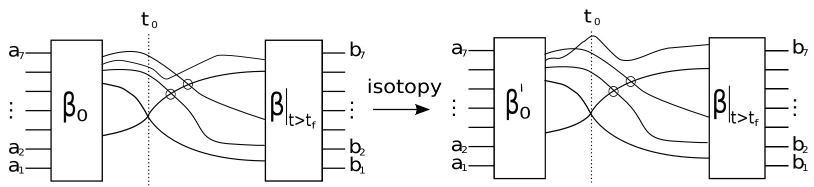

Figure 2.12: Case 1, Lemma 2.15.

2. The crossing d is inside D (Figure 2.13). There are two strands entering D that meet at the crossing d. We apply a move of type M, M' or $V3$ according to the value of the crossing d, and then apply Case 1.

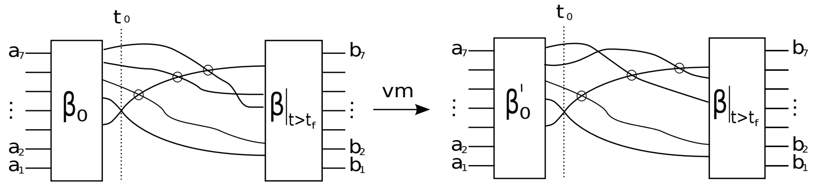

Figure 2.13: Case 2, Lemma 2.15.

3. The crossing d is on α_1 in such a way that the strand making the crossing with α_1 goes out D (Figure 2.14). Then such strand also has a crossing with α_2 and is the leftmost crossing on it. Up to isotopy we may apply a move of type M, M' or $V3$ according to the value of the crossing p. Then we are done.

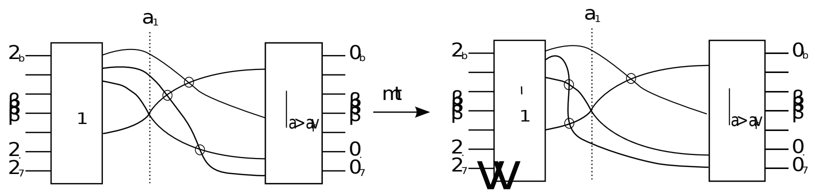

Figure 2.14: Case 3, Lemma 2.15.

4. The crossing d is on α_1 in such a way that the strand making the crossing with α_1 enters D (Figure 2.15). Let β_j be that strand. We apply a move of type $V2$ to β_j and β_2 just before the crossing p, then we apply a move of type P, P' or $V3$ according to the value of the crossing p. In this way now d is a crossing on α_2.

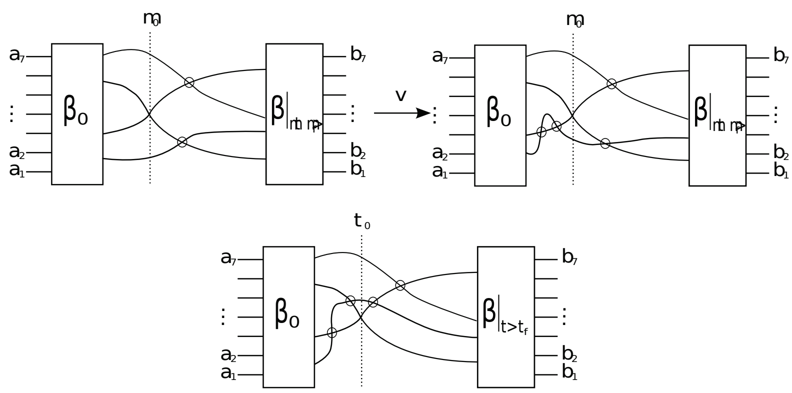

Figure 2.15: Case 4, Lemma 2.15.

Note that, in the above four cases, we have not deformed β for $t \geq t_f$.

Moreover, up to isotopy, β_1 and β_2 remain unchanged and in their restriction to $(0, t_0) \times I$ we have added only virtual crossings.

Now if $m \geq 2$ take d the leftmost crossing in $(t_0, t_f) \times I$ such that d is not on α_2. We apply the case $m = 1$ in order to get rid of this crossing and reduce the obtained diagram by all the possible $V2$ moves in it. By the induction hypothesis, we have proved (1,2,3) of the lemma.

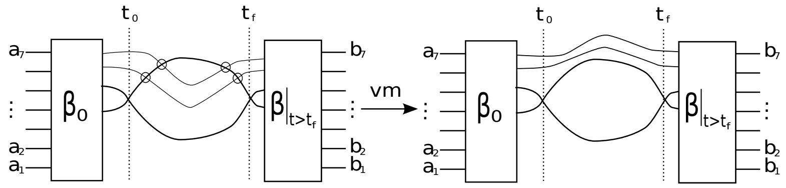

Figure 2.16: Case $\alpha_1(t_f) = \alpha_2(t_f)$, Lemma 2.15.

Now suppose $\alpha_1(t_f) = \alpha_2(t_f)$. Then α_1 and α_2 form a bigon D. We apply the same reasoning as above in order to have only crossings on α_2 (Figure 2.16). Suppose that $m_2 \neq 0$, then it necessarily is even (as each strand entering the bigon must go out by α_2). We can apply $\frac{m}{2}$ moves of type $V2$ to get rid of the crossings in α_2. But this is a contradiction as in each inductive step we are reducing the diagram by all the possible $V2$ moves in it. Therefore we can chose β' such that there are no crossings in $(t_0, t_f) \times I$. With this we complete the proof of the lemma. $\square$

Corollary 2.16. Let β be a virtual braid diagram and let α_1, α_2 be two primitive arcs of β such that:

1. The arcs α_1 and α_2 start in the same point, say p (thus a crossing, it may be virtual or regular).

2. The arcs α_1 and α_2 end at the same time, say t_f.

Then there exists a virtual braid diagram β' virtually equivalent to β such that:

1. If β_1 and β_2 are the strands corresponding to α_1 and α_2, respectively, then up to isotopy they remain unchanged in β' and in their restriction to $(0, \pi_1(p) = t_0)$ we add only virtual crossings.

2. The diagrams β and β' coincide for $t \geq t_f$, i.e. $\beta|_{t \geq t_f} = \beta'|_{t \geq t_f}$.

3. Let $\alpha_1 \cap \alpha_2 = \{p = p_1, p_2, \ldots, p_m\}$, numbered so that $\pi_1(p_i) < \pi_1(p_{i+1})$ for $1 \leq i \leq m - 1$.

    (a) If $\pi_1(p_m) = t_f$, then in $(t_0, t_f) \times I$ there are no crossings except, eventually, $p_2, \ldots, p_{m-1}$.

(b) If $\pi_1(p_m) \neq t_f$, then in $(t_0, \pi_1(p_m)) \times I$ there are no crossings except eventually $p_2, \dots, p_{m-1}$ and in $(\pi_1(p_m), t_f) \times I$ there are only virtual crossings with the corresponding upper segment of α_1 or α_2.

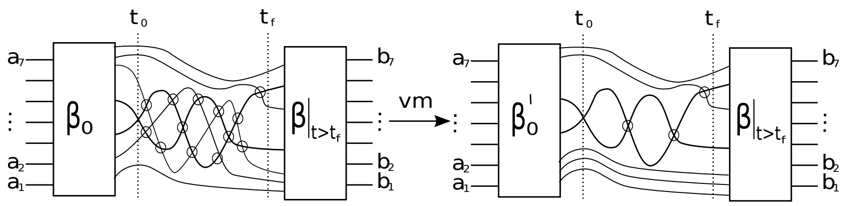

Figure 2.17: Corollary 2.16.

Proof. Suppose that $\pi_1(p_1) < \cdots < \pi_1(p_m)$. We argue by induction on m. Suppose $m = 1$. Then necessarily $p_1 = p$ and we have the hypothesis of Lemma 2.15.

Suppose $m > 1$ and that α_1 and α_2 end in the same point p_m. Consider the restrictions of α_1 and α_2 to $[\pi_1(p_{m-1}), t_f]$ and apply Lemma 2.15. We obtain a virtually equivalent diagram β' which does not have crossings neither on the restriction of α_1 nor on the restriction of α_2. Furthermore, up to isotopy the strands corresponding to α_1 and α_2 remain unchanged and their restrictions to $[t_0, \pi_1(p_{m-1})]$ go only through virtual crossings, i.e. they are primitive arcs whose intersection has $m - 1$ points. Applying induction hypothesis on them, we have proved this case.

Suppose $m > 1$ and that α_1 and α_2 do not end in the same point. Consider the restrictions of α_1 and α_2 to $[\pi_1(p_m), t_f]$ and apply Lemma 2.15. We obtain a virtually equivalent diagram β' which may have only virtual crossings with the corresponding upper segment of α_1 or α_2. Furthermore, up to isotopy, the strands corresponding to α_1 and α_2 remain unchanged and their restrictions to $[t_0, \pi_1(p_m)]$ go only through virtual crossings, i.e. they are primitive arcs whose intersection has m points and satisfies the condition of the preceding case. With this we conclude the proof. □

Corollary 2.17. Given a virtual braid diagram β in general position. Let c_1 and c_2 be two regular crossings not related in $P(\beta)$ (Definition 2.9) and such that:

1. In the total order on $R(\beta)$ (Definition 2.13), $D(\beta)$, $c_1 > c_2$.

2. There is no regular crossing between c_1 and c_2 in $D(\beta)$.

Then there exists a virtual braid diagram β' virtually equivalent to β with $c_2 > c_1$ in $D(\beta')$, and such that there is no regular crossing between them.

Furthermore, the diagrams β and β' coincide for $t > t_f$. In particular the total order on the set of elements smaller than c_2 in $D(\beta)$ is preserved in $D(\beta')$, i.e. if $c_2 > d_1 > d_2$ in $D(\beta)$, then $c_1 > d_1 > d_2$ in $D(\beta')$.

Proof. Let $t_f > \pi_1(c_2)$ such that there is no crossing in $(\pi_1(c_2), t_f) \times I$ and, let α_1 and α_2 be the primitive arcs coming from the regular crossing c_1 and finishing in t_f. Applying the last corollary to α_1 and α_2, we obtain a virtually equivalent diagram β' such that in $(\pi_1(c_1), t_f] \times I$ there are only virtual crossings and β remains unchanged for $t \geq t_f$. Thus $c_2 > c_1$ in $D(\beta')$ and if $c_2 > d_1 > d_2$ in $D(\beta)$, then $c_1 > d_1 > d_2$ in $D(\beta')$. □

Given two orders, R and R' over a set X, we say that R' is compatible with R if $R \subset R'$.

Lemma 2.18. Let β be a virtual braid diagram and let $\tilde{R}$ be a total order on $R(\beta)$ compatible with $P(\beta)$. Then there exists a virtual braid diagram β' virtually equivalent to β such that $D(\beta') = \tilde{R}$.

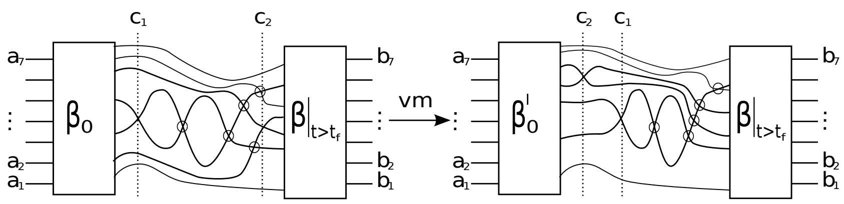

Figure 2.18: Lemma 2.18.

Proof. Suppose

$$\tilde{R} = \{c_1 > \cdots > c_m\},$$

$$D(\beta) = \{d_1 > \cdots > d_m\},$$

and that $c_l = d_l$ for $l > k$, $c_k \neq d_k$ and $c_k = d_j > d_k$ in $D(\beta)$. Note that for $k \geq l > j$, d_j is not related with d_l in $P(\beta)$. Applying the last corollary $k - j$ times we construct a virtual braid diagram β' virtually equivalent to β such that if $D(\beta') = \{d'_1 > \cdots > d'_m\}$ then $c_l = d'_l$ for $l \geq k$. Applying this procedure inductively we obtain the lemma. □

Lemma 2.19. Let β_1 and β_2 be two virtual braid diagrams on n strands, and let α_1 and α_2 be two primitive arcs of β_1 and β_2 respectively, such that:

1. The extremes of α_1 and α_2 coincide.

2. α_1 and α_2 form a bigon D.

3. $\beta_1 \setminus \alpha_1$ and $\beta_2 \setminus \alpha_2$ coincide.

4. There are no crossings in the interior of D.

Then β_1 and β_2 are virtually equivalent by isotopy and moves of type $V2$.

Proof. First note that each strand entering D must go out. Take a strand α entering D and suppose it is innermost. If it goes out by the same side, as there are no crossings in the interior of D, then we can apply a move of type $V2$ and eliminate the two virtual crossings. Therefore we can suppose that each strands entering by one side goes out by the other. Apply an isotopy following the strands crossing the bigon (if there are any) in order to identify the two primitive arcs. $\square$

Lemma 2.20. Let β_1 and β_2 be two virtual braid diagrams on n strands, and let α_1 and α_2 be two primitive arcs of β_1 and β_2 respectively, such that:

1. The extremes of α_1 and α_2 coincide.
2. α_1 and α_2 form a bigon D.
3. $\beta_1 \setminus \alpha_1$ and $\beta_2 \setminus \alpha_2$ coincide.

Then β_1 and β_2 are virtually equivalent.

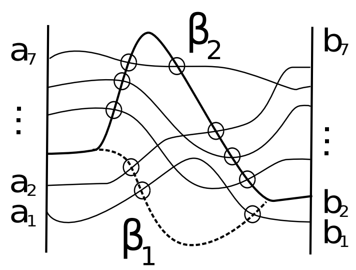

Figure 2.19: Lemma 2.20.

Proof. Call p and q the starting and ending points of α_1, and set $\beta = \beta_1 \setminus \alpha_1$. Let m be the number of crossings inside D. We argue by induction on m. If $m = 0$ we apply the last lemma.

Suppose $m \geq 1$ and let c be the leftmost crossing inside D. Choose t_0 and t_1 so that $\pi_1(p) < t_0 < t_1 < \pi_1(c)$ and so that there are no crossings in $D \cap ((t_0, \pi_1(c)) \times I)$. Draw a line a' joining $\alpha_1(t_0)$ and $\alpha_2(t_1)$. Note that a' intersects the two incoming strands that compose c. To the crossings of β with a' assign virtual crossings. Consider the following primitive arcs:

$$c_1 = \alpha_1|_{[\pi_1(p),t_0]} * a' \qquad c_3 = a' * \alpha_2|_{[t_1,\pi_1(q)]}$$
$$c_2 = \alpha_2|_{[\pi_1(p),t_1]} \qquad c_4 = \alpha_1|_{[t_0,\pi_1(q)]}.$$

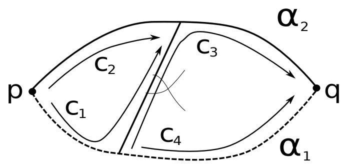

Figure 2.20: Construction in proof of Lemma 2.20.

Note that c_1 and c_2 form a bigon that has no crossing in its interior, so we apply the last lemma to $\beta_2' = (\beta_2 \cup c_1) \setminus c_2$ and β_2.

On the other hand, take the bigon D' formed by c_3 and c_4. D' has the same crossings as D, and c is still the leftmost crossing in D'. By construction of β_2' we can apply a move of type $V3$, M or M' to move a' to the other side of c. Call the obtained virtual braid diagram β_1'. Then the bigon formed between β_1' and β_1 has $m-1$ crossings in its interior. Applying the induction hypothesis to β_1 and β_1' we conclude that

$$\beta_1 \sim_{vm} \beta_1' \sim_{vm} \beta_2' \sim_{vm} \beta_2,$$

which proves the lemma. □

Corollary 2.21. Let β_1 and β_2 be two virtual braid diagrams on n strands, and let α_1 and α_2 be two primitive arcs of β_1 and β_2, respectively, such that:

1. The extremes of α_1 and α_2 coincide.
2. $\beta_1 \setminus \alpha_1$ and $\beta_2 \setminus \alpha_2$ coincide.

Then β_1 and β_2 are virtually equivalent.

Proof. Without loss of generality we can suppose that α_1 intersects transversally α_2 in a finite number of points. In this case they form a finite number of bigons. Apply the previous lemma to each one. □

Now we are able to complete the proof of Proposition 2.10. We have already proved (1) in Lemma 2.11. In Lemma 2.12 we have shown that if $\beta \sim_{vm} \beta'$ then $G(\beta) = G(\beta')$. It remains to prove that if $\beta, \beta' \in VBD_n$ are so that $G(\beta) = G(\beta')$, then $\beta \sim_{vm} \beta'$. Set $g = G(\beta) = G(\beta')$.

Let $\tilde{R}$ be a total order of $R(G)$, compatible with $P(G)$. By Lemma 2.18 there exist two virtual braid diagrams, α and α', virtually equivalent to β and β' respectively and such that $D(\alpha) = \tilde{R} = D(\alpha')$. As $D(\alpha) = D(\alpha')$ we can suppose that the regular crossings of α and α' coincide (if not, move them by an isotopy to make them coincide). In this case α and α' differ by joining arcs.

Suppose α has m joining arcs. As the regular crossings of α and α' coincide, we can suppose that the corresponding joining arcs of α and α' begin and end at the same points. Apply Corollary 2.21 m times, in order to make that each of the corresponding joining arcs coincide. We conclude that α is virtually equivalent to α' and thus β and β'.

2.3 Virtual braids as Gauss diagrams

The aim of this section is to establish a bijective correspondence between virtual braids and certain equivalence classes of braid-Gauss diagrams. We also give the group structure on the set of virtual braids in terms of Gauss diagrams, and use this to prove a presentation of the pure virtual braid group.

Definition 2.22. Let g and g' be two Gauss diagrams. A *Gauss embedding* is an embedding $\varphi : g' \to g$ which send each interval of g' into a subinterval of g, and which sends each arrow of g' to an arrow of g respecting the orientation and the sign. Note that there is no condition on the permutations associated to g' and g in the above definition. We shall say that g' is embedded in g if a Gauss embedding of g' into g is given.

Let g' be a Gauss diagram of n strands, so that it is embedded in g by sending the interval i to a subinterval of the interval k_i of g, we say that the embedding is of type $(k_1, \dots, k_n)$.

Consider the three Gauss diagrams presented in Figure 2.21. Note that g_1 is embedded in g_2 by an embedding of type $(2, 1)$, and g_3 is embedded in g_2 by an embedding of type $(1, 2, 3)$.

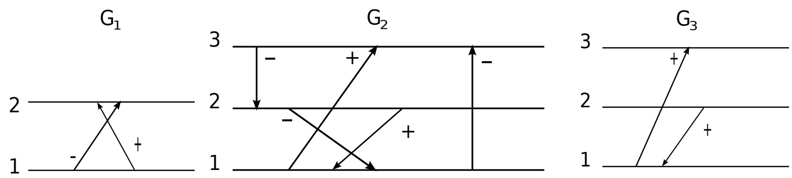

Figure 2.21: G_1 is $(1, 2)-$embedded in G_2, and G_3 is $(1, 2, 3)-$embedded in G_2.

By performing an $\Omega 3$ *move* on a braid Gauss diagram g, we mean choosing an embedding in g of the braid Gauss diagram depicted on the left hand side of Figure 2.22 (or on the right hand side of Figure 2.22), and replacing it by the braid Gauss diagram depicted on the right hand side of Figure 2.22 (resp. on the left hand side of Figure 2.22).

Let g be a Gauss diagram with n strands and $i, j, k \in \{1, \dots, n\}$ with $i < j < k$. The six different types of embeddings of the Gauss diagram in Figure 2.22 in g are illustrated in Figures 2.23, 2.24 and 2.25. According to the type of embedding the $\Omega 3$ move is called $\Omega 3$ move of type (k_1, k_2, k_3).

Similarly, by performing an $\Omega 2$ *move* on a braid Gauss diagram g, we mean choosing an embedding in g of the braid Gauss diagram depicted on

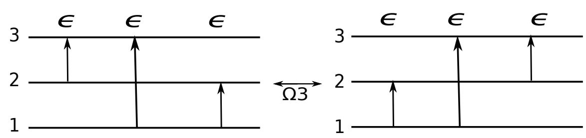

Figure 2.22: $\Omega 3$ move on Gauss diagrams, with $\epsilon \in \{\pm 1\}$.

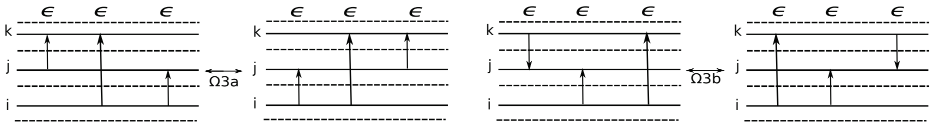

Figure 2.23: $\Omega 3$ moves of type (i, j, k) and (i, k, j).

the left hand side of Figure 2.26 (or on the right hand side of Figure 2.26), and replacing it by the braid Gauss diagram depicted on the right hand side of Figure 2.26 (resp. on the left hand side of Figure 2.26).

In this case there are only two types of embeddings. They are illustrated in Figure 2.27.

Definition 2.23. The equivalence relation generated by the $\Omega 2$ and the $\Omega 3$ moves in the set of braid Gauss diagrams is called *Reidemeister equivalence.* The set of equivalence classes of braid Gauss diagrams is denoted by bG_n.

Theorem 2.24. There is a bijective correspondence between bG_n and VB_n.

Proof. By Proposition 2.10 we know that there is a bijective correspondence between the set of virtually equivalent virtual braid diagrams and the braid Gauss diagrams. Therefore we need to prove that if two virtual braid diagrams are related by a Reidemeister move then their braid Gauss diagrams are Reidemeister equivalent, and that if two braid Gauss diagrams are related by an $\Omega 2$ or an $\Omega 3$ move then their virtual braid diagrams are virtually Reidemeister equivalent.

Let β and β' be two virtual braid diagrams that differ by a Reidemeister move. Suppose that they are related by a $R3$ move, and that the strands involved in the move are a, b, and c, with $a, b, c \in \{1, \ldots, n\}$. Then, up to isotopy we can deform the diagrams so that they coincide outside the subinterval $I_0 := [t_0, t_f] \subset I$, and in I_0 there are only the crossings involved in the $R3$ move. In I_0 the diagrams look as in Figure 2.28. Thus, their braid-Gauss diagrams coincide outside I_0 and in I_0 they differ by an $\Omega 3$ move of type (a, b, c). The case $R2$ is proved in the same way.

Now, let g and g' be two braid Gauss diagrams and let $a, b, c \in \{1, \ldots, n\}$ be pairwise different. Suppose that g and g' are related by an $\Omega 3$ move of type (a, b, c). There exists a subinterval $I_0 = [t_0, t_f] \subset I$, that contains only the three arrows involved in the $\Omega 3$ move. There exists a virtual braid diagram β, representing g, that in the subinterval I_0 it looks as the left hand

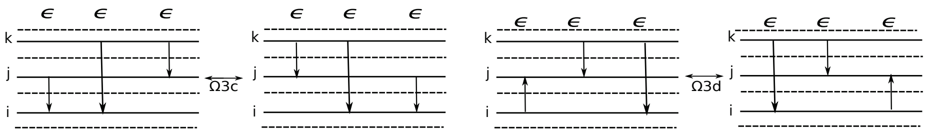

Figure 2.24: $\Omega 3$ moves of type (j, i, k) and (j, k, i).

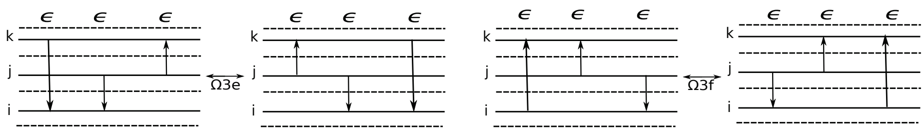

Figure 2.25: $\Omega 3$ moves of type (k, i, j) and (k, j, i).

side (or the right hand side) of Figure 2.28. By performing an $R3$ move on $\beta|_{I_0}$, we obtain a virtual braid diagram β'. Their braid Gauss diagrams coincides outside I_0 and in I_0 the differ by an $\Omega 3$ move of type (a, b, c), i.e. $G(\beta') = g'$.

□

2.4 Presentation of PV_n.

Recall that VB_n has a group structure, with the product given by the concatenation of the diagrams (Remark 2.3). By Theorem 2.24, bG_n has a group structure induced by the one on VB_n.

A presentation of the *pure virtual braid group* was given by Bardakov [6]. We present an alternative proof by means of the braid Gauss diagrams.

Recall that the symmetric group, S_n, has the next presentation:

- Generators: $t_1, \ldots, t_{n-1}$.
- Relations:

$$\begin{aligned} t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} && \text{for } 1 \leq i \leq n-2 \\ t_i t_j &= t_j t_i && \text{for } |i-j| \geq 2. \\ t_i^2 &= 1 && \text{for } 1 \leq i \leq 2. \end{aligned}$$

From the presentation of VB_n (Remark 2.3), there exists an epimorphism $\theta_P : VB_n \to S_n$, given by

$$\theta_P(\tau_i) = t_i = \theta_P(\sigma_i) \quad \text{for } 1 \leq i \leq n-1.$$

The kernel of θ_P is called the *pure virtual braid group* and is denoted by PV_n. The elements of this group correspond to the virtual braids diagrams whose strands begin and end in the same marked points, i.e. the permutation associated to its braid Gauss diagram is the identity.

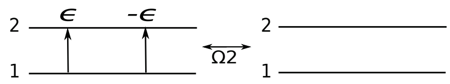

Figure 2.26: $\Omega 2$ move on Gauss diagrams.

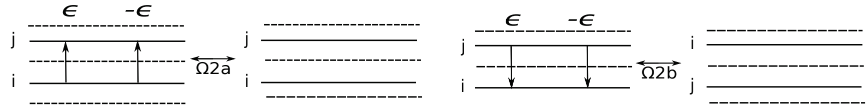

Figure 2.27: $\Omega 2$ moves of type (i, j) and (j, i).

On the other hand, a braid Gauss diagram is composed by the next elements:

1. A finite ordered set of n intervals, say $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$.
2. A finite set of arrows connecting the different intervals, so that to each arrow corresponds a different time.
3. A function assigning a sign, $\{\pm 1\}$, to each arrow.
4. A permutation, $\sigma \in S_n$, labelling the endpoint of each interval.

Denote by $X^{\epsilon}_{i,j}$ the arrow from the interval i to the interval j with sign $\epsilon \in \{\pm 1\}$. Let

$$X = \{X^{\epsilon}_{i,j} \mid 1 \leq i \neq j \leq n \ , \ \epsilon \in \{\pm 1\}\}$$

and denote by X^* the set of all words in X union the empty word, denoted by e.

Given a braid Gauss diagram, g, its arrows have a natural order induced by the parametrization of the intervals. Let $W \in X^*$ be the word given by the concatenation of the arrows in g, according to the order in which they appear, and $\sigma \in S_n$ its associated permutation. Thus any braid Gauss diagram can be expressed as $g = (W, \sigma)$. We denote $\bar{e} := (e, Id_{S_n})$.

Proposition 2.25. (Bardakov [6]) The group PV_n has the following presentation:

- Generators: $A_{i,j}$ with $1 \leq i \neq j \leq n$.
- Relations:

$$\begin{aligned} A_{i,j}A_{i,k}A_{j,k} &= A_{j,k}A_{i,k}A_{i,j} \quad \text{for } i, j, k \text{ distinct.} \\ A_{i,j}A_{k,l} &= A_{k,l}A_{i,j} \quad \text{for } i, j, k, l \text{ distinct.} \end{aligned}$$

Proof. Given a pure virtual braid diagram β, its braid Gauss diagram is given by $G(\beta) = (W, Id_{S_n})$. Thus any pure virtual braid diagram may be expressed as an element in X^*.

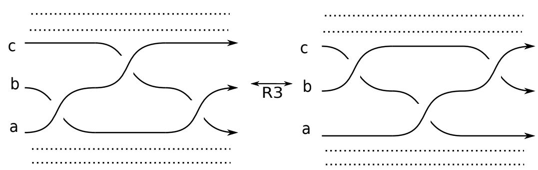

Figure 2.28: A labelled $R3$ move.

Recall that, as elements of bG_n, the braid Gauss diagrams are related by three different moves (and its inverses) on the subwords of any word in X^*:

1. Reparametrization:

$$X_{i,j}^{\epsilon_1} X_{k,l}^{\epsilon_2} = X_{k,l}^{\epsilon_2} X_{i,j}^{\epsilon_1} \text{ for } i,j,k,l \text{ distinct and } \epsilon_1, \epsilon_2 \in \{\pm 1\}.$$

2. The $\Omega 2$ move:

$$X_{i,j}^{\epsilon} X_{i,j}^{-\epsilon} = e \text{ for } i,j \text{ distinct and } \epsilon \in \{\pm 1\}.$$

3. The $\Omega 3$ move:

$$X_{i,j}^{\epsilon} X_{i,k}^{\epsilon} X_{j,k}^{\epsilon} = X_{j,k}^{\epsilon} X_{i,k}^{\epsilon} X_{i,j}^{\epsilon} \text{ for } i,j,k \text{ distinct and } \epsilon \in \{\pm 1\}.$$

Denote by PG_n the set of equivalence classes of X^*. Note that PG_n has the structure of group with the product defined as the concatenation of the words. On the other hand, $G : PV_n \to PG_n$ is an homomorphism, i.e. $G(\beta_1\beta_2) = G(\beta_1)G(\beta_2)$. By Theorem 2.24, G is a bijection. Consequently it is an isomorphism.

Let Γ be the group with presentation as stated in the proposition. Let $\Psi : \Gamma \to PG_n$ be given by

$$\Psi(A_{i,j}) = X_{i,j},$$

and let $\Phi : PG_n \to \Gamma$ be given by

$$\Phi(X_{i,j}^{\epsilon}) = A_{i,j}^{\epsilon}.$$

Note that Φ and Ψ are well-defined homomorphisms. Furthermore $\Psi \circ \Phi = Id_{PG_n}$ and $\Phi \circ \Psi = Id_{\Gamma}$. Consequently PV_n has the presentation stated in the proposition. $\square$

Chapter 3

Abstract diagrams

The aim of this section is to present a topological framework to study virtual braids and string links. In order to do this, first we introduce certain class of diagrams on surfaces, that can represent virtual braids. Then we give a necessary and sufficient condition to be braids (as diagrams) and we state some results concerning their monotony. Finally we prove the equivalence between abstract braids and virtual braids.

I am deeply grateful to Christian Bonatti for his suggestions and remarks, that have made possible to restate this results with much more simplicity and transparency.

3.1 Diagrams on surfaces

Definition 3.1. An *abstract diagram on n strands* is a pair (S, β), such that:

1. S is a connected, compact and oriented surface.

2. The boundary of S has only two connected components, i.e. $\partial S = C_0 \sqcup C_1$, with $C_0 \approx S^1 \approx C_1$. They are called *distinguished boundary components.*

3. Each boundary component of S has n marked points, say $K_0 = \{a_1, \ldots, a_n\} \subset C_0$ and $K_1 = \{b_1, \ldots, b_n\} \subset C_1$. Such that:

   (a) The elements of K_0 and K_1 are linearly ordered.

   (b) Let $\kappa_0 : S^1 \to C_0$ and $\kappa_1 : S^1 \to C_1$ be parametrizations of C_0 and C_1 compatible with the orientation of S. Up to isotopy we can put $a_k = \kappa_0(e^{\frac{2\pi i}{k}})$ and $b_k = \kappa_1(e^{-\frac{2\pi i}{k}})$ for $k \in \{1, \ldots, n\}$.

4. β is an n-tuple of curves $\beta = (\beta_1, \ldots, \beta_n)$ with

   (a) For $k = 1, \ldots, n$, $\beta_k : I \to S$.

   (b) For $k = 1, \ldots, n$, $\beta_k(0) = a_k$.

(c) There exists $\sigma \in S_n$ such that

$$\beta_k(1) = b_{\sigma(k)},$$

for all $k \in \{1, \ldots, n\}$.

(d) The n-tuple of curves β is in general position, i.e. there are only transversal double points, called crossings.

Similarly to Defintion 2.1, denote by $R(\beta)$ the set of crossings of β. As usual we fix n and we say abstract diagram instead of abstract diagram on n strands. Note that the β is an oriented graph embedded in S.

Definition 3.2. An *isotopy* of abstract diagrams is a continuous family of abstract diagrams $G = \{(S, \beta^s)\}_{s \in I}\}$, such that β^s fixes the end points. We write $(S, \beta^0) \simeq (S, \beta^1)$. The isotopy relation is an equivalent relation on the set of abstract diagrams.

Definition 3.3. Let (S, β) and (S, β') be two abstract diagrams. We say that (S, β) and (S, β') are *compatible* if there exists a diffeomorphism $\varphi : (S, \partial S) \to (S, \partial S)$, such that $(S, \varphi \circ \beta)$ is isotopy equivalent to (S, β'). We denote it by $(S, \beta) \approx (S, \beta')$.

Remark 3.4. The compatibility relation is an equivalence relation on the set of abstract diagrams, and the isotopy equivalence is included in the compatibility relation. We denote by AD_n the set of compatibility classes of abstract diagrams on n strands.

Definition 3.5. Given (S, β) and (S', β'). We say that they are related by a *stability move* if there exist two disjoint embedded discs, D_0 and D_1, in $S \setminus \beta$ and an embedding $\varphi : (S \setminus (D_0 \cup D_1), \partial S) \to (S', \partial S')$, with $S' \setminus \varphi(S \setminus (D_0 \cup D_1)) \approx S^1 \times I$, such that $(S', \varphi \circ \beta) \approx (S', \beta')$.

Definition 3.6. Given (S, β) and (S', β'). We say that they are related by a *destability move* or a *destabilization*, if there exists an essential non-separating simple curve C in $S \setminus \beta$ and an embedding $\varphi : (S \setminus C, \partial S) \to (S', \partial S')$, such that $S' \setminus \varphi(S \setminus C)$ is homeomorphic to the disjoint union of two closed discs, and $(S', \varphi \circ \beta) \approx (S', \beta')$.

Note that stability and destability moves are complementary, in the following sense. Let (S, β) and (S', β') be two abstract diagrams. If (S', β) is obtained from (S, β) by a destabilization, we can recover (S, β) from (S', β') by a stabilization, and vice-versa.

Definition 3.7. The equivalence relation on the set of abstract diagrams generated by the stability (and destability) moves is called *stability equivalence.* We denote it by $\sim_s$.

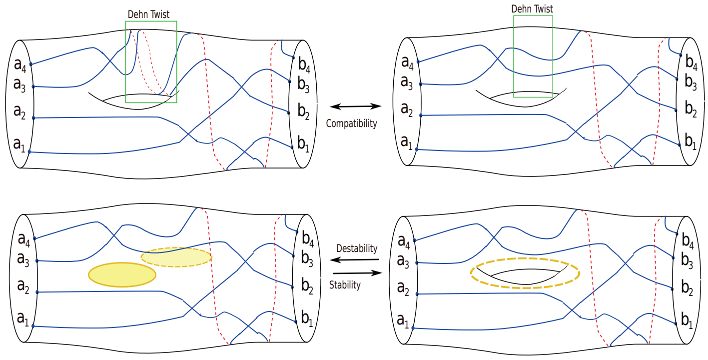

Figure 3.1: Compatibility and stability equivalence.

Definition 3.8. Let $\bar{\beta} = (S, \beta)$ be an abstract diagram, and let C be a simple closed essential curve embedded in $S \setminus \beta$. Denote by S_C the connected component of $S \setminus C$ containing β. Let S'_C be a compact, connected, oriented surface and $\varphi : S_C \to S'_C$ such that:

1. The surface S'_C has only two boundary components C'_0 and C'_1.
2. The map φ is an embedding such that $\varphi(C_0) = C'_0$ and $\varphi(C_1) = C'_1$.
3. Let k_C be the number of connected components of $S \setminus C$.
   (a) If $k_C = 1$, then $S'_C \setminus \varphi(S_C)$ is homeomorphic to a disjoint union of two discs.
   (b) If $k_C = 2$, then $S'_C \setminus \varphi(S_C)$ is homeomorphic to a disc.

Then $\bar{\beta}_C = (S'_C, \varphi \circ \beta)$ is an abstract diagram. We say that we obtain $\bar{\beta}_C$ by *destabilizing* $\bar{\beta}$ along C, and is called a *generalized destabilization*.

Proposition 3.9. Let $\bar{\beta} = (S, \beta)$ be an abstract diagram, and let C be an embedded simple closed curve in $S \setminus \beta$. Then $\bar{\beta}_C$ is stable equivalent to $\bar{\beta}$ by a finite number of destabilizations.

Proof. First note that if $S \setminus C$ has only one connected component the generalized destabilization along C coincides with the definition of destabilization. Thus $\bar{\beta} \sim_s \bar{\beta}_C$ by one destabilization.

We can assume that $S \setminus C$ has two connected components, one of which contains β (we call it S_C) and the other is a compact connected surface with one boundary component, thus it is homeomorphic to $\Sigma_{g,1}$. We will prove the proposition by induction on g.

If $g = 0$ then $\Sigma_{g,1}$ is a disc, consequently $\bar{\beta}_C \approx \bar{\beta}$.

If $g = 1$ then $\Sigma_{1,1}$ is a torus with one boundary component, which corresponds to the curve C. Let C' be a closed simple essential non separating curve in $\Sigma_{1,1}$. We claim that $\bar{\beta}_C \approx \bar{\beta}_{C'}$.

Note that $\Sigma_{1,1} \setminus C'$ is homeomorphic to a pair of pants (Figure 3.2), whose exterior boundary is the curve C and whose interior boundaries correspond to those generated by cutting S along C'.

On the other hand, consider the curve C' embedded in S. The surface $S \setminus C'$ has one connected component, and two (non distinguished) boundary components. Let S' be the surface obtained from $S \setminus C'$ by capping the boundary components corresponding to C'. There exist a disc, D', embedded in S' so that its boundary corresponds to the curve C. Thus S_C is embedded in $S_{C'}$ and $S_{C'}$ is embedded in S'. We conclude that $\bar{\beta}_C \approx \bar{\beta}_{C'}$. In particular, $\bar{\beta} \sim_s \bar{\beta}_C$ by a unique destabilization.

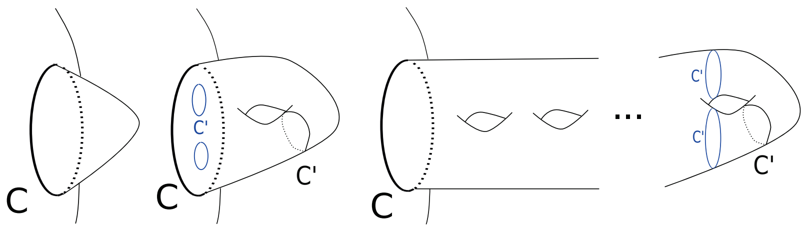

Figure 3.2: Generalized destabilization along a curve C.

Suppose that the proposition is true when the second connected component is homeomorphic to $\Sigma_{k,1}$.

Choose a simple essential closed curve C which divides S in two connected components, from which the component that does not contain β is homeomorphic to $\Sigma_{k+1,1}$. Take a simple essential closed curve C' in $\Sigma_{k+1,1}$, which is not isotopic to C in $\Sigma_{k+1,1}$. Destabilize $\bar{\beta}$ along C'. Then, by induction, $\bar{\beta}$ is stable equivalent to $\bar{\beta}_{C'}$. The curve C is still a simple closed curve in $S_{C'} \setminus \beta$, thus we can destabilize $\bar{\beta}_{C'}$ along C (Figure 3.2).

By induction hypothesis, the destabilization of $\bar{\beta}_{C'}$ along C is stable equivalent to $\bar{\beta}_{C'}$. Thus $\bar{\beta}$ is stable equivalent to $(\bar{\beta}_{C'})_C$.

Without loss of generality we can write $(\varphi_{C'})_C = \varphi_C$. Consequently $(\bar{\beta}_{C'})_C \approx \bar{\beta}_C$ and $\bar{\beta} \sim_s \bar{\beta}_C$. □

Proposition 3.10. Let (S, β). Up to compatibility and isotopy, there is a unique stable representative (S', β), with S' of minimal genus. Moreover it only depends on the oriented graph β.

Proof. Let N be a regular neighbourhood of $\beta \cup C_0 \cup C_1$. Note that N is a compact oriented surface homeomorphic $\Sigma_{g,b+2}$. In particular, N has b non distinguished boundary components. We can destabilize (S, β) along them to obtain an abstract braid diagram (S', β) that does not admit any other destabilization. Thus (S', β) is of minimal genus and is unique up to compatibility. □

Corollary 3.11. Let (S, β), (S', β') and $\varphi : C_0 \cup C_1 \cup \beta \to C_0 \cup C_1 \cup \beta'$ be an orientation preserving homeomorphism. Then (S, β) and (S', β) are stable equivalent.

Proof. Without loss of gnerality we can suppose that (S, β) and (S', β') are of minimal genus. Let $N(\beta)$ and $N(\beta')$ be regular neighbourhoods of $\beta \cup C_0 \cup C_1$ and $\beta' \cup C_0 \cup C_1$, we can extend φ to a diffeomorphism $\varphi' : N(\beta) \to N(\beta')$. By the construction of S and S', we can extend $\varphi'' : S \to S'$. Consequently $(S, \beta) \approx (S', \beta')$. □

Definition 3.12. The *genus of an abstract diagram*, (S, β), is the genus of S. We write $g(S, \beta)$. The minimal genus of (S, β) is the genus of its stable representative of minimal genus, by the previous proposition it only depends on β. We write $g(\beta)$.

3.2 Abstract diagrams and monotony

Let (S, β) be an abstract diagram. Recall that β is an oriented graph embedded in S. The vertices of β, $V(\beta)$, are given by $R(\beta) \cup K_0 \cup K_1$, where K_0 and K_1 are the initial and final points of β, respectively. The edges, $E(\beta)$, are given by the segments of β joining them.

Let e be an edge. There exists $1 \leq k \leq n$, and $[t_0, t_1] \subset I$ such that $e = \beta_k([t_0, t_k])$. Note that $\beta_k(t_0)$ and $\beta_k(t_1)$ are in $V(\beta)$. Define $s, t : E(\beta) \to V(\beta)$, by $s(e) = \beta(t_0)$ and $t(e) = \beta(t_1)$. Then $\Gamma(\beta) := (V(\beta), E(\beta), s, t)$ is an oriented graph.

A *path* in β is given by a finite sequence $p = (e_1, \ldots, e_l)$ with $s(e_{k+1}) = t(e_k)$, for $1 \leq k \leq l - 1$. An *oriented cycle* is a path, $p = (e_1, \ldots, e_l)$, so that $t(e_l) = s(e_1)$.

On the other hand, let Γ be an oriented graph without cycles. The set of vertices of Γ admits a partial order given by $v < v'$ if there is path $p = (e_1, \ldots, e_l)$, such that $s(e_1) = v$ and $t(e_l) = v'$. Is not difficult to prove that this is a well defined order. Given two orders, A and B over a set V, we say that B extends A, if $v < v'$ in A then the same is true in B. Any partial order A admits a total ordered extension.

Lemma 3.13. Let $(S, \partial S)$ be a surface with non empty boundary, and let $f : \partial S \to \mathbb{R}$ be a continuous function. There exists $F : S \to \mathbb{R}$, continuous, such that $F|_{\partial S} = f$. Furthermore, if F and F' are two extensions of f, then they are homotopy equivalent relative to the boundary.

Proof. Let $A = S^1 \times I$ be an annulus, and let $f : S^1 \times \{0\} \to \mathbb{R}$ be a continuous function. Define $F_f : S^1 \times I \to \mathbb{R}$ as $F_f(x, t) := (1 - t)f(x) + t$. Then F_f is a continuous function, with $F_f|_{S^1 \times \{0\}} = f$ and $F_f|_{S^1 \times \{1\}} = 1$.

Now, let S be an oriented surface with $\partial S = C_0 \sqcup \cdots \sqcup C_m \neq \emptyset$, let $f : \partial S \to \mathbb{R}$ be a continuous function, and let $N := N(\partial S)$ be a closed collar neighbourhood of ∂S. We can choose $N = N_0 \sqcup \cdots \sqcup N_m$, such that there exists an homeomorphism $\varphi_k : N_k \to S^1 \times I$, for all $0 \leq k \leq m$. Set $k \in \{0, 1, \ldots, m\}$ and denote $\partial_0 N_k := S^1 \times \{0\}$ and $\partial_1 N_k := S^1 \times \{1\}$. Without loss of generality, we can suppose that $C_k = \partial_0 N_k$. Let $F_k : N_k \to \mathbb{R}$ be a continuous function defined by $F_k := F_{f|_{C_k}} \circ \varphi_k$, and define $F : S \to \mathbb{R}$ as follows

$$F(x) := \begin{cases} F_k(x) & \text{if } x \in N_k, \\ 1 & \text{otherwise.} \end{cases}$$

Then F is a continuous function such that $F|_{\partial S} = f$, as required.

On the other hand, let $F, F' : S \to \mathbb{R}$ be two continuous functions such that $F|_{\partial S} = f = F|_{\partial S}$. Define $H : S \times I \to \mathbb{R}$ as follows, $H(x,t) = (1-t)F(x) + tF'(x)$. Then H is an homotopy from F to F'. Moreover, for $x \in \partial S$, $H(x,t) = (1-t)f(x) + tf(x) = f(x)$. Consequently this homotopy preserves the values on the boundary, as required. $\square$

Let (S, β) be an abstract diagram and $f : S \to I$. We say that f is a function *strictly increasing on* β if $f \circ \beta_k$ is strictly increasing for all $1 \leq k \leq n$.

Proposition 3.14. Let (S, β) be an abstract diagram. There exists a function $f : S \to I$ strictly increasing on β, if and only if β does not have any oriented cycle.

Proof. If S admits such a function, then $f|_{R(\beta)}$ is a well defined function. Suppose β admits oriented cycles. There exists $t < t' \in I$ and an oriented path, γ, in β such that $\gamma(t) = v = \gamma(t')$. But this is absurd because f is strictly increasing on β.

On the other hand, if β does not have oriented cycles. Then it induces a partial order on $R(\beta)$. Call this partial order $P(\beta)$. Extend $P(\beta)$ to a total order $D(\beta)$. Then $D(\beta)$ induces a reparametrization of β_k, for $1 \leq k \leq n$, such that if $v < v'$ in $R(\beta)$, then $\beta^{-1}(v) < \beta^{-1}(v)$. In particular it induces a function $f_\beta : \beta \cup C_0 \cup C_1 \to I$ so that $f^{-1}(0) = C_0$, $f^{-1}(1) = C_1$ and f is strictly increasing on β_k for $1 \leq k \leq n$. Let N be a regular neighbourhood of $\beta \cup C_0 \cup C_1$. By the previous lemma we can extend f to the whole surface. $\square$

Proposition 3.15. Let (S, β) be an abstract diagram, and let $f, g : S \to I$ be two functions strictly increasing on β. Then f and g are homotopy equivalent among the strictly increasing functions on β.

Proof. By the precedent proposition we know that β is an oriented graph without cycles, then it induces a partial order in $R(\beta)$. Call this order $P(\beta)$. On the other hand, if $f : S \to I$ is a strictly increasing function on β, then it induces an order on $R(\beta)$ that extends $P(\beta)$. Without loss of generality we can suppose it is a total order. Call this order $D(\beta, f)$.

Let $f, g : S \to I$ be two functions that are strictly increasing on β. Let $N := N(\beta \cup C_0 \cup C_1)$ be a regular neighbourhood of $\beta \cup C_0 \cup C_1$. By the previous lemma, we can suppose that f and g only differs inside N. As N is homotopy equivalent to β, it suffices to modify f and g along β.

The two following claims are proved by a simple reparametrization of β.

Claim 3.16. If $D(\beta, f)$ and $D(\beta, g)$ coincide, then f and g are homotopic among the functions strictly increasing on β.

Claim 3.17. Let $D(\beta, f) = \{v_1 < \cdots < v_m\}$ and suppose there exists $1 \leq k < m$ such that v_k and v_{k+1} are not related in $P(\beta)$. Then there exists f' homotopy equivalent to f among the strictly increasing functions on β, so that $D(\beta, f') = \{v_1 < \cdots < v_{k+1} < v_k < v_{k+2} < \ldots, v_m\}$.

Claim 3.18. Suppose that $D(\beta, f) = \{v_1 < \cdots < v_m\}$ and $D(\beta, g) = \{p_1 < \cdots < p_m\}$ so that $p_k = v_k$ for $1 \leq k < l$. Then there exists g' homotopy equivalent to g among the strictly increasing functions on β, so that $D(\beta, g) = \{v_1 < \cdots < v_l < p'_{l+1} < \ldots, p'_m\}$.

Proof. We have that $p_l \neq v_l$, then there exist $m_0 > 0$ so that $p_{l+m_0} = v_l$. First, note that p_l is not related to p_{l+k} in $P(\beta)$, for $1 \leq k \leq m_0$. Then we can use the last claim, m_0 times, to construct the desired function. □

Now we have sufficient elements to prove the proposition. Let $D(\beta, f)$ and $D(\beta, g)$ the orders induced by f and g respectively. Applying the last claim inductively, we can construct a function g' homotopy equivalent to g among the strictly increasing functions on β, so that $D(\beta, g') = D(\beta, f)$. Finally by the first claim we have that f is homotopic to g' among the strictly increasing functions on β, which proofs the proposition.

□

3.3 Abstract and virtual braids

Definition 3.19. An *abstract braid diagram* on n strands is a triple (S, β, ϵ) with (S, β) an abstract diagram on n strands, so that β does not have oriented cycles, and $\epsilon : R(\beta) \to \{\pm 1\}$. The set of abstract braid diagrams is denoted by ABD_n.

The isotopy, compatibility and stability equivalences are extended naturally to the class of abstract braid diagrams. That is, we consider the compatibility and stability among abstract diagrams without oriented cycles, ignoring the function ϵ. As usual we fix n, and we refer to them simply as abstract braid diagrams instead of abstract braid diagrams on n strands.

Definition 3.20. Given two abstract braid diagrams $\bar{\beta} = (S, \beta, \epsilon)$ and $\bar{\beta}' = (S, \beta', \epsilon')$, we say that they are related by a *Reidemeister move* or simply by an *R-move* if there exists a neighbourhood D in S, homeomorphic to a disc, such that $\beta\backslash D = \beta'\backslash D$, $\epsilon|_{\beta\backslash D} = \epsilon'|_{\beta'\backslash D}$, and inside D we can transform β into β' by a Reidemeister move and isotopy (Figure 2.2). The equivalence relation generated by the R-moves is called *Reidemeister equivalence* or simply *R-equivalence*. We denote it by $\bar{\beta} \sim_R \bar{\beta}'$.

Definition 3.21. Let $\sim$ be the equivalence relation on the abstract braid diagrams on n strands generated by the compatibility, stability and Reidemeister moves. The equivalence classes of abstract braid diagrams are called *abstract braids*, and the set of abstract braids is denoted by AB_n.

Remark 3.22. The definition of *braid Gauss diagram* is extended in a natural way to the set of abstract braid diagrams. The braid Gauss diagram of an abstract braid diagram is invariant under compatibility (resp. under isotopy) and stability. Thus, there is a well defined map from ABD_n to bGD_n, which associates to each abstract braid diagram its braid Gauss diagram. This map is well defined up to compatibility and stability. By abuse of notation we denote the induced map still by G.

Recall that the set of braid Gauss diagrams is in bijective correspondence with the set of virtually equivalent virtual braid diagrams. Thus, braid Gauss diagrams are a good tool to prove that abstract braids are a good geometric interpretation of virtual braids. We present an analogous of Proposition 2.10 for abstract braid diagrams.

Proposition 3.23. The map $G : ABD_n \to bGD_n$ induces a bijection between the stable equivalence classes of abstract braid diagrams and the braid Gauss diagrams.

Proof. Recall that the function is well defined from the stable and compatibility equivalence classes of Abstract braid diagrams to the braid Gauss diagrams (Remark 3.22).

Now we proof the surjectivity. Let $g \in bGD_n$. Then by Proposition 2.10 there exists a virtual braid diagram β such that $G(\beta) = g$. For each $\beta \in VBD_n$ we can construct an abstract braid diagram $\bar{\beta}$ such that $G(\beta) = G(\bar{\beta})$ as follows.

Let β be a virtual braid diagram, and let N be a regular neighbourhood of $\beta \cup (\{0\} \times I) \cup (\{1\} \times I)$ in $\mathbb{D} = I \times I$ (Figure 3.3). Note that N can be seen as the union of regular neighbourhoods of each strand and of the two extremes of the virtual braid diagram.

Now consider the standard embedding of $\mathbb{D}$ in $\mathbb{R}^3$. Around each virtual crossing perturb the regular neighbourhoods of the strands involved in the crossing, so that they do not intersect, as pictured in Figure 3.3. To the regular neighbourhood of each extreme of the diagram, attach a ribbon so that each extreme is now a cylinder, as in Figure 3.3. In this way we obtain a compact oriented surface, S', with more than the two distinguished boundary components.

Note that S' is compact, connected and oriented, then it is diffeomorphic to $\Sigma_{g,b+2}$. Cap the b non-distinguished boundary components. We obtain a surface S that has only the distinguished boundary components. From this we conclude that the function G is surjective.

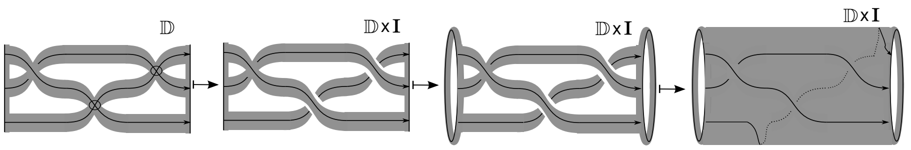

Figure 3.3: Construction of $\bar{\beta}$ from β such that $G(\bar{\beta}) = G(\beta)$.

Now to prove injectivity of the induced function, let $\bar{\beta} = (S, \beta, \epsilon)$ and $\bar{\beta}' = (S', \beta', \epsilon')$ be two abstract braid diagrams such that $G(\bar{\beta}) = G(\bar{\beta}')$. We claim that $\bar{\beta}$ is equivalent to $\bar{\beta}'$.

Note that $G(\bar{\beta}) = G(\bar{\beta}')$ implies that the graph given by $\Gamma = C_0 \cup \beta \cup C_1 \subset S$ is homeomorphic to $\Gamma' = C_0' \cup \beta' \cup C_1' \subset S'$. From Corollary 3.11 we have that (S, β) and (S', β') are stable equivalent as abstract diagrams. The signs on the crossings coincides, as this is encoded also in the Gauss diagram. Consequently (S, β, ϵ) and (S, β, ϵ') are stable equivalent. □

Theorem 3.24. There exists a bijection between the abstract braids on n strands and the virtual braids on n strands.

Proof. We need to verify that the function induced by G, from AB_n to bG_n, is well defined and that it remains injective. By abuse of notation we denote the induced map still by G.

Let (S, β, ϵ) and (S, β', ϵ') be two abstract braid diagrams related by an R-move. We need to see that $G((S, \beta, \epsilon))$ is related to $G((S, \beta', \epsilon'))$ by an $\Omega 2$ or an $\Omega 3$ move. By definition of an R-move, there exists a neighbourhood, D, diffeomorphic to a disc, such that β and β' coincide outside D, and we can obtain β' from β by a R-move and isotopy. Looking to the strands involved in the R-move in the braid Gauss diagram, we can perform the corresponding Ω move. Consequently G is well defined from AB_n to bG_n.

To prove the injectivity, let (S, β, ϵ) and (S', β', ϵ') be two abstract braids diagrams such that $G((S, \beta, \epsilon))$ and $G((S', \beta', \epsilon'))$ are related by an $\Omega 2$ move. Let N and N' be the regular neighbourhoods of $C_0 \cup C_1 \cup \beta$ and $C_0 \cup C_1 \cup \beta'$, respectively. Note that N and N' differs only in the intervals where the $\Omega 2$ move has effect. Locally they look either as in the right hand side or as in the left hand side of Figure 3.4. Deform the regular neighbourhood of the right hand side by gluing a disc in the middle, so that it looks as in the center of Figure 3.4. Then we can embed both diagrams in the same surface and relate them by a $R2$ move. The case when $G(\beta)$ and $G(\beta')$ are related by an $\Omega 3$ move is proved similarly and illustrated in Figure 3.5. With this we conclude G is injective and the theorem is true. □

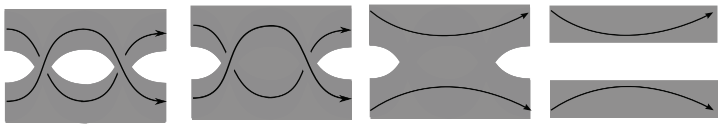

Figure 3.4: Strands involved in the $\Omega 2$ move.

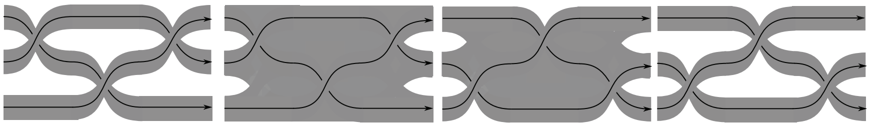

Figure 3.5: Strands involved in the $\Omega 3$ move.

Remark 3.25. Note that, in the proof of the injectivity, if we have two braid Gauss diagrams that differ only by an $\Omega 3$ move, the minimal surface constructions of the their respective diagrams coincide. In particular this implies that the $\Omega 3$ moves do not change the genus of an abstract braid.

3.4 Abstract string links

Definition 3.26. An *abstract string link* is a triple (S, β, ϵ), with (S, β) an abstract diagram and $\epsilon : R(\beta) \to \{\pm 1\}$.

In particular an abstract braid diagram is an abstract string link, whose oriented graph does not admit oriented cycles. As in the case of abstract braids, the isotopy, compatibility and stability are extended naturally to the set of abstract string links, by ignoring the function ϵ.

In order to introduce the Reidemeister equivalence on the set of abstract string links, we present the complete Reidemeister moves, illustrated in Figure 3.6. Note that they differ from the (braid) Reidemeister moves essentially in two aspects. The first one is that we do not care about the orientation of the strands. The second one is that we introduce the Reidemeister move 1.

Definition 3.27. Given two abstract string links diagrams (S, β, ϵ) and (S, β', ϵ'), we say that they are related by a *complete Reidemeister move* or simply by a *cR-move* if there exists a neighbourhood D in S, homeomorphic to a disc, such that $\beta \setminus D = \beta' \setminus D$, $\epsilon|_{\beta \setminus D} = \epsilon'|_{\beta' \setminus D}$, and inside D we can transform β into β' by a Reidemeister move and isotopy (Figure 3.6). The equivalence relation generated by the cR-moves is called *complete Reidemeister equivalence* or simply *cR-equivalence*. We denote it by $(S, \beta, \epsilon) \sim_R (S, \beta', \epsilon')$

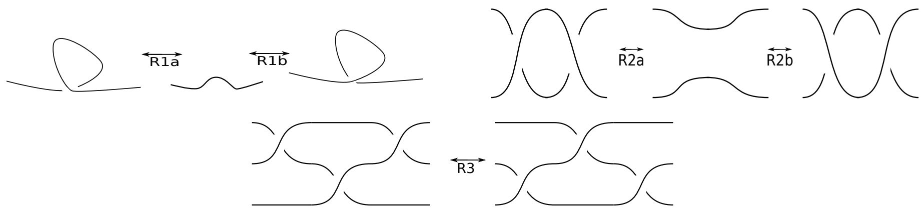

Figure 3.6: Complete Reidemeister moves.

By abuse of notation we will confuse the complete Reidemeister moves with the Reidemeister moves when there is no confusion. That is, when we are working with abstract string link diagrams, they will be identified by complete Reidemeister moves, and in the case of abstract braid diagrams, they will be identified by (braid) Reidemeister moves.

Definition 3.28. Let $\sim$ be the equivalence relation on the abstract string link diagrams on n strands generated by the compatibility, stability and Reidemeister moves. The equivalence classes of abstract string link diagrams are called *abstract string links*, and the set of abstract string links is denoted by ASl_n.

Chapter 4

Minimal realization of a virtual braid

Let (S, β, ϵ) be an abstract braid (string link) diagram. By Proposition 3.10, up to compatibility there is a unique stable equivalent representative of (S, β, ϵ) with minimal genus. Furthermore, we know that it only depends on the graph of β. In this section, we extend this result to the abstract braids (string links). That is, given an abstract braid (string link), (S, β, ϵ), up to compatibility and (complete) Reidemeister equivalence, there is a unique stable equivalent representative of (S, β, ϵ) with minimal genus.

In order to attain this, we introduce braids in a thickened surface. We must be careful to not confuse braids embedded in a thickened surface, with surface braids, which are elements of the fundamental group of the configuration space of n points in the surface. We prove that braids (string links) in a thickened surface are in bijective correspondence with abstract braids (string links), thus with virtual braids.

A braid in a thickened surface is an embedding of n strands in a thickened surface. The embedded strands are disjoint by pairs and they accomplish certain monotony condition. As in the classical case, we identify the braids by isotopy preserving the monotony condition. On the other hand, as in the case of abstract braids, we have to deal with stability and compatibility.

As a consequence of the main theorem, the minimal genus of an abstract braid is well defined. Consequently, it is an invariant of the virtual braid. Therefore, we can determine if a virtual braid is a classical one by means of its genus. On the other hand, virtual braids of genus 0 correspond to cylindrical braids, up to compatibility. We call them *virtual cylindrical braids.* We know that cylindrical braids are embedded in a classical braid group. The classical braid group has fast solutions to the word problem. Consequently, virtual cylindrical braid groups have a fast solution to the word problem.

4.1 Braids in thickened surfaces.

Definition 4.1. A *string link in a thickened surface on n strands* is a pair, (M, β), such that:

1. There exists a compact, connected and oriented surface S, such that $M = S \times I$. Write $\pi : M \to S$ the projection on the first coordinate.

2. The boundary of S has only two disjoint connected components, called *distinguished boundary components.*

3. Each boundary component of S has n marked points, say $K_0 = \{a_1, \ldots, a_n\} \subset C_0$ and $K_1 = \{b_1, \ldots, b_n\} \subset C_1$. Such that:

   (a) The elements of K_0 and K_1 are lineary ordered.

   (b) Let $\kappa_0 : S^1 \to C_0$ and $\kappa_1 : S^1 \to C_1$ be parametrizations of C_0 and C_1 compatible with the orientation of S. Up to isotopy we can put $a_k = \kappa_0(e^{\frac{2\pi i}{k}})$ and $b_k = \kappa_1(e^{-\frac{2\pi i}{k}})$ for $k \in \{1, \ldots, n\}$.

4. β is an n-tuple of curves $\beta = (\beta_1, \ldots, \beta_n)$ with:

   (a) For $k = 1, \ldots, n$, $\beta_k : I \to M_S$.

   (b) For $k = 1, \ldots, n$, $\beta_k(0) = (a_k, \frac{1}{2})$.

   (c) There exists $\sigma \in S_n$ such that for $k = 1, \ldots, n$,
   $$\beta_k(1) = (b_{\sigma(k)}, \frac{1}{2}).$$

   (d) For $i \neq j$, $\beta_i \cap \beta_j = \emptyset$.

From now on we fix $n \in \mathbb{N}$ and we say string links in a thickened surface instead of string links in a thickened surface on n strands.

Definition 4.2. An *isotopy* of string links in a thickened surface is a continuous family of string links in a thickened surface $G = \{(M, \beta^s)\}_{s \in I}$, such that for all $s \in I$, $K_0^s = K_0^0$ and $K_1^s = K_1^0$. We say that (M, β^0) and (M, β^1) are isotopic and we denote it by $G : (M, \beta^0) \simeq (M, \beta^1)$.

Let (M, β) be a string link in a thickened surface. Up to an isotopy we can suppose that $\pi \circ \beta$ is an oriented graph. In such a case, we say that (M, β) is in *general position.*

Given an isotopy of string link diagrams, $G = \{(M, \beta^s)\}_{s \in I}$, we can suppose that there is a finite number of points $s_1 < \cdots < s_m \in I$, such that (M, β^{s_k}) is not in general position. We call this points, *singular points.* The isotopy $\{(M, \beta^t)\}_{t \in (s_k, s_{k+1})}$, factorizes to an isotopy of abstract string links $\{(S, \pi \circ \beta^t, \epsilon^t)\}_{t \in (s_k, s_{k+1})}$, denote (S, β_k) this isotopy equivalence class of abstract string link diagrams. Without loss of generality, we can suppose that (S, β_k) is related with (S, β_{k+1}) by a complete Reidemeister move.

From the previous discussion we have the following proposition.

Proposition 4.3. There is a bijective correspondence between the isotopy classes of string links in a thickened surface and the abstract string link diagrams identified by complete Reidemeister moves.

Definition 4.4. A *braid in a thickened surface* is a string link in a thickened surface (M, β) in general position, such that $\pi \circ \beta$ does not admit oriented cycles.

Definition 4.5. Let $G = \{(M, \beta^s)\}_{s \in I}$ be an isotopy of string links. Recall that it splits in a finite number of string links in general position (M, β_k), with $0 \leq k \leq m$. We say that G is a *braid isotopy* if (M, β_k) is a braid in a thickened surface for $0 \leq k \leq m$.

The previous discussion applies as well for braids in a thickened surfaces and abstract braids. Thus, we have the following lemma.

Lemma 4.6. There is a bijective correspondence between braid isotopy classes of braids in thickened surfaces and Reidemeister equivalence classes of abstract braids.

Definition 4.7. Let (M, β) be a braid (string link) in a thickened surface, with $M = S \times I$. A *compatibility move* is an orientation preserving diffeomorphism $\varphi : M \to M$ such that $\pi \circ \varphi \circ \beta$ is homeomorphic to $\pi \circ \beta$. We write $\varphi : (M, \beta) \approx (M, \varphi \circ \beta)$ or simply $(M, \beta) \approx (M, \varphi \circ \beta)$. The equivalence relation generated by the compatibility moves is called *compatibility*.

Definition 4.8. Let (M, β) be a string link in a thickened surface. Given $A, B \subset M_S$ we say that *A is isotopic to B relative to ∂M_S* if there exists a continuous function $H : A \times I \to M_S$ such that:

1. $H_0 = id_A$ and $H_1(A) = B$.
2. For all $s \in I$, H_s is an embedding.
3. For all $s \in I$, $H_s(A \cap \partial M_S) \subset \partial M_S$.

In particular A is diffeomorphic to B, and H induces an isotopy of $A \cap \partial M_S$ and $B \cap \partial M_S$ in ∂M_S.

Definition 4.9. Let (M, β) be a string link in a thickened surface.

1. A *vertical annulus in* (M, β) is an annulus $A \subset M \setminus \beta$, such that A is isotopic to $C \times I \subset S \times I$ with C a simple closed curve in S.
2. A *destabilization* of (M, β) is an annulus $A \subset M \backslash \beta$ isotopic to a vertical annulus $C \times I$ relative to ∂M, with C essential and non-separating in S.

3. A *destabilization move* on (M, β) along a destabilization A, is to cut M along A and cap the two boundary components with two thickened discs. We also say to *destabilize* (M, β) along A and we denote the obtained thickened abstract braid by (M_A, β).

4. The equivalence relation generated by these moves in the set of string links in a thickened surface is called *stable equivalence.*

Note that compatibility and stability moves on string links in thickened surfaces coincide with the corresponding compatibility and stability moves on abstract string links. Following Proposition 4.3, abstract string links in a thickened surface and abstract links are in a bijective correspondence.

On the other hand, note that compatibility is well defined in the set of braids in a thickened surface. However we need to verify that the stability relation is well defined as well.

Lemma 4.10. The stability relation is well defined in the set of braids in a thickened surface.

Proof. Let (M, β) be a string link in a thickened surface. Recall that by Proposition 3.14, $(S, \pi \circ \beta)$ does not admit oriented cycles if and only if there exists a function $f : S \to I$ that is strictly increasing along β.

Let (M, β) be a braid in a thickened surface, and A be a destabilization in $M \setminus \beta$. We need to verify that (M_A, β) is a braid in a thickened surface. Let $(S, \pi \circ \beta)$ be the abstract diagram associated to (M, β). By Proposition 3.14 there exists a function $f : S \to I$ such that it is strictly increasing along β. We can extend such a function to a smooth function $F : M \to I$, so that $f = F \circ \pi$. Destabilize (M, β) along A. That is, cut the thickened surface along A and cap the obtained boundary components with two thickened discs. Denote the two thickened discs by B_0 and B_1. A thickened disc is homeomorphic to a 3-ball.

On the other hand, the function F restricted to $M \setminus A$ induces a function $F' : M_A \setminus (B_0 \cup B_1) \to I$. Since B_0 and B_1 are homeomorphic to 3-balls. We can extend F' to M_A in such a way that $F' \circ \pi$ remains strictly increasing along $\pi \circ \beta$. By Proposition 3.14 this is equivalent to say that (M_A, β) is a braid in a thickened surface. $\square$

From this lemma and the previous discussion we have the next Proposition.

Proposition 4.11. There is a bijective correspondence between abstract braids (string links) and braids (string links) in a thickened surface identified up to compatibility and stability.

4.2 Minimal genus of abstract objects

Definition 4.12. Let (M, β) be a braid (string link) in a thickened surface. A *descendant* of (M, β) is a braid (string link) in a thickened surface (M', β'), that is obtained by compatibility, isotopy and at least one destabilization move. An *irreducible descendant* of (M, β) is a descendant that does not admit any destabilization.

Let (M, β) be a braid (string link) in a thickened surface and A be a vertical annulus in $M \setminus \beta$. If ∂A is a pair of non essential curves in S, then they bound discs in their respective copies of S. Say D_0 and D_1. Note that $A \cup D_0 \cup D_1$ bounds a ball in $M \setminus \beta$. We say that A *bounds a ball*, and we refer to such ball as *the ball bounded by* A.

Let (M, β) be a braid (string link) in a thickened surface. If $M = S \times I$. We say that S is the *supporting surface* of (M, β). *The genus of* (M, β) is the genus of S, we write $g(M, \beta)$. Note that if (M', β') is a descendant of (M, β), then $g(M', \beta') < g(M, \beta)$. In particular the destabilization induces a partial order on the equivalence classes of braids (string links) in a thickened surface. An irreducible descendant is a representative that does not admit any destabilization. Note that there may be two irreducible descendants of different genus, or even, there may exist two descendants of the same genus that are not related neither by compatibility, nor by isotopy. The following theorem asserts that the irreducible descendant is unique up to compatibility and isotopy. Our proof is inspired by that of Kuperberg [39] for virtual links.

Theorem 4.13. Let (M, β) be a braid (string link) in a thickened surface. Up to isotopy and compatibility, there is a unique irreducible descendant of (M, β).

Proof. Let (M, β) be a braid (string link) in a thickened surface. Suppose that (M, β) has two irreducible descendants. In this case there is a representative in the equivalence class of (M, β), such that it is of minimal genus among the representatives admitting two different irreducible descendants. Without loss of generality we can suppose that (M, β) is such a representative. Let S be its supporting surface.

Since each destabilization reduces the genus, by minimality of the genus of S each destabilization of (M, β) has a unique irreducible descendent. Two destabilizations of (M, β) are called *descendant equivalent* if they have the same irreducible descendent.

We claim that all destabilizations in (M, β) are descendent equivalent. Suppose there exist two destabilizations A_1 and A_2 of (M, β) that are not equivalent.

Claim 4.14. The intersection of A_1 and A_2 is nonempty.

Proof. Suppose that A_1 and A_2 are disjoint. We can destabilize (M, β) along A_1 and then along A_2 and vice-versa. In both cases we obtain a common descendent, i.e. $((M_{A_1})_{A_2}, \beta) \approx ((M_{A_2})_{A_1}, \beta)$. This is a contradiction. □

Therefore, we can suppose that A_1 and A_2 intersect transversally and so that the number of curves in the intersection ($m_{1,2} \geq 1$) is minimal. Furthermore, we can choose A_1 and A_2 so that $m_{1,2}$ is minimal among inequivalent pairs of destabilizations of $\bar{\beta}$.

The intersection between two transversal surfaces is a disjoint union of 1-manifolds. A curve in $A_1 \cap A_2$ is thus either a circle or an arc. A *horizontal circle* in an annulus A is a circle that does not bound a disc in A (Figure 4.1). A *vertical arc* in an annulus A is a simple arc in A such that its extremes connect the two boundary components of A (Figure 4.1).

Given a horizontal circle C in an annulus A, it divides A in two annuli A' and A'' (Figure 4.1) such that:

$$\partial A' = (\partial A \cap (S \times \{0\}) \cup C \qquad \text{and} \qquad \partial A'' = (\partial A \cap (S \times \{1\}) \cup C.$$

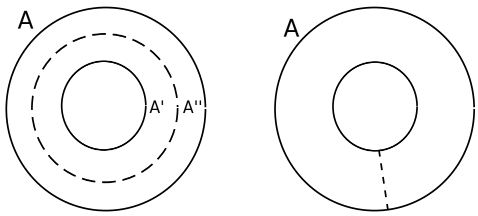

Figure 4.1: Horizontal circle and vertical arc in A.

Claim 4.15. All the 1-manifolds in $A_1 \cap A_2$ are either horizontal circles or vertical arcs in A_1 and in A_2.

Proof. Suppose there exists $C \subset A_1 \cap A_2$ such that C is a non-horizontal circle in A_1. Thus, the circle C bounds a disc D in A_1, in particular it is null-homotopic in $M_S \setminus \beta$. On the other hand, if C is horizontal in A_2 it is homotopic to an essential circle in S and so it is not null-homotopic in M_S. Therefore C is non-horizontal in A_2.

Suppose that C is innermost (i.e. $int(D) \cap A_2 = \emptyset$). Consider a regular neighbourhood of D in $M_S \setminus \beta$, $N(D)$. The boundary of $N(D)$, $\partial N(D)$, intersects A_2 in two disjoint circles C' and C''. The circle C' (resp. C'') bounds a disc D' (resp. D'') in $\partial N(D)$ (Figure 4.2). The surface $A_2 \setminus N(D)$ has two connected components that we can complete with D' and D'' in order to obtain two surfaces, say A_2' and A_2''. They can be spheres, annuli or discs in M_S.

Since C is non-horizontal in A_2 and C is innermost in A_1, necessarily, up to exchanging A_2' with A_2'', A_2' is a sphere and A_2'' is an annulus isotopic to A_2 (Figure 4.2). By construction $A_1 \cap A_2''$ has less connected components than $A_1 \cap A_2$. This is a contradiction. We conclude that all the circles in $A_1 \cap A_2$ are horizontal in A_i for $i = 1, 2$.

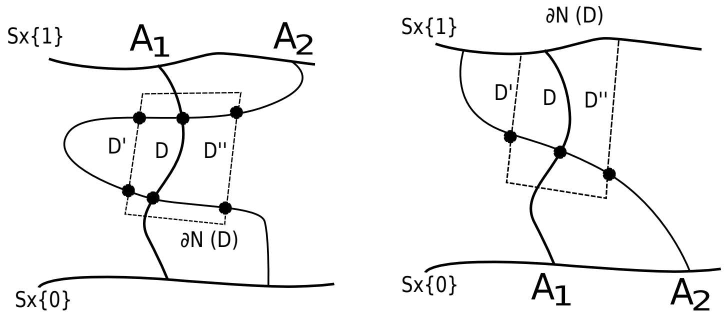

Figure 4.2: A non-horizontal circle and a non-vertical arc in A_1.

Let $C \subset A_1 \cap A_2$ be a non-vertical arc in A_1. Hence, the extremes of C are in the same component of ∂A_1. Let α be the segment of the component of ∂A_1 that joins the extremes of C so that $C \cup \alpha$ is a simple closed curve that bounds a disc D in A_1. In particular C is null-homotopic in M_S relative to ∂M_S, consequently, C is also a non vertical arc in A_2.

Suppose that C is innermost, in the sense that $A_2 \cap int(D) = \emptyset$. Let $N(D)$ be a regular neighbourhood of D in $M_S \setminus \beta$. The boundary of $N(D)$, $\partial N(D)$, intersects A_2 in two disjoint non-vertical arcs, C' and C''. With a similar construction as for C, we can find arcs α' and α'' in $\partial N(D) \cap \partial M_S$ such that $C' \cup \alpha'$ (resp. $C'' \cup \alpha''$) bounds a disc D' (resp. D'') in $\partial N(D)$. The surface $A_2 \setminus N(D)$ has two connected components that can be completed with D' and D'' in order to obtain two surfaces A_2' and A_2''.

Since C is non-vertical in A_2 and C is innermost in A_1, necessarily, up to exchanging A_2' with A_2'', A_2' is a disc and A_2'' is an annulus isotopic to A_2 (Figure 4.2). By construction $A_1 \cap A_2''$ has less connected components than $A_1 \cap A_2$ which is a contradiction. We conclude that all the arcs in $A_1 \cap A_2$ are vertical in A_i for $i = 1, 2$. $\square$

Claim 4.16. The intersection $A_1 \cap A_2$ does not contain any horizontal circle.

Proof. Let $C \subset A_1 \cap A_2$ be a horizontal circle in A_1. We have seen that necessarily it is a horizontal circle in A_2. Then C splits A_1 and A_2 in four annuli, A_1', A_1'', A_2' and A_2''. We can choose C so that it is exterior in A_1 in the sense that $int(A_1'') \cap A_2 = \emptyset$. In this case the annulus A_1'' is isotopic to A_2'' in $M_S \setminus \beta$ relative to ∂M_S. Let A_3 be the annulus $A_1'' \cup A_2'$ deformed by an isotopy in such a way that it is in general position with respect to A_1 (Figure 4.3).

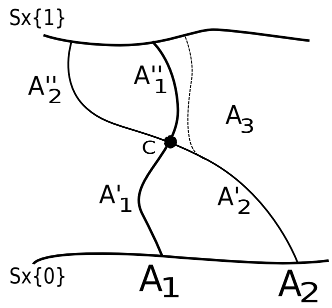

Figure 4.3: The intersection of two destabilizations along a horizontal circle.

The number of curves in $A_3 \cap A_1$ is strictly inferior than the number of curves in $A_2 \cap A_1$. Furthermore A_2 is isotopy equivalent to $A_1'' \cup A_2'$ which is isotopy equivalent to A_3 by construction. Hence A_3 is a destabilization equivalent to A_2, and $A_3 \cap A_1$ has strictly less curves than $A_2 \cap A_1$. This is a contradiction. □

Claim 4.17. The intersection $A_1 \cap A_2$ does not contain any vertical arc.

Proof. Let N be a regular neighbourhood of $A_1 \cup A_2$ in $M_S \setminus \beta$. Then ∂N is a disjoint union of m surfaces in M_S. Since there are only vertical arcs in $A_1 \cap A_2$ these surfaces are isotopic to vertical annuli, say $\partial N = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m$. Therefore, either there is a destabilization in ∂N or all the vertical annuli are non-essential.

Suppose that for some $k \in \{1, \ldots, m\}$, B_k is a destabilization, i.e. isotopic to an essential vertical annulus. Since B_k is disjoint from A_1 and A_2, it is descendent equivalent to both. This is a contradiction.

Suppose that for all $k = 1, \ldots, m$, B_k is isotopic to a non-essential vertical annulus. Let E_k be the ball bounded by B_k and $S_k = \partial E_k$.

We claim that there exists $k \in \{1, \ldots, m\}$ such that $A_1 \cup A_2 \subset E_k$. This is equivalent to say that there exists $k \in \{1, \ldots, m\}$ such that $(A_1 \cup A_2) \cap E_k \neq \emptyset$. It is clear that if $A_1 \cup A_2 \subset E_k$ then the intersection is nonempty. On the other hand, suppose there exists $k \in \{1, \ldots, m\}$, such that $(A_1 \cup A_2) \cap E_k \neq \emptyset$. Since $B_k \cap (A_1 \cup A_2) = \emptyset$ and by connectivity of $A_1 \cup A_2$ and of E_k, we have $A_1 \cup A_2 \subset E_k$.

Now, suppose there exist $j, k \in \{1, \ldots, m\}$, such that $j \neq k$ and $S_k \cap S_j \neq \emptyset$. Then, up to exchanging E_k with E_j, $E_j \subset E_k$. Note that B_j (resp. B_k) separates M_S in two connected components. Furthermore, B_k and $A_1 \cup A_2$ (resp. B_j and $A_1 \cup A_2$) are in the same connected component of $M_S \setminus B_j$ (resp. $M_S \setminus B_k$). Thus $A_1 \cup A_2$ is in the shell bounded by S_j and S_k. In particular $A_1 \cup A_2 \subset E_k$.

If $S_i \cap S_j = \emptyset$, then $E_i \cap E_j = \emptyset$. Suppose that $E_i \cap E_j \neq \emptyset$. As $B_i \cap B_j = \emptyset$, up to exchanging E_i with E_j, $E_i \subset E_j$ and $(S_i \cap \partial M_S) \subset (S_j \cap \partial M_S)$. This is a contradiction.

Suppose that for all $k = 1, \dots, m$, $(A_1 \cup A_2) \cap E_k = \emptyset$ and that $S_i \cap S_j = \emptyset$ for $i \neq j$. As $E_i \cap E_j = \emptyset$ for $i \neq j$, the connected components of $M_S \setminus (\cup_{k=1}^m B_k)$ are $int(E_1)$, $\dots$, $int(E_m)$, and $M_S \setminus (\cup_{k=1}^m E_k)$. But $(A_1 \cup A_2) \cap E_k = \emptyset$ for $k = 1, \dots, m$. Thus β and $A_1 \cup A_2$ are in the same connected component. This is a contradiction, because ∂N separates β and $A_1 \cup A_2$. We conclude that there exists $k \in \{1, \dots, m\}$ such that $A_1 \cup A_2 \subset E_k$.

For $j = 1, 2$ and $i = 0, 1$, set $\gamma_j^i = (S \times \{i\}) \cap A_j$. Since $A_1 \cup A_2 \subset E_k$, we have $\gamma_j^i \subset E_k$, thus γ_j^i is null-homotopic. This is a contradiction. We conclude that there are no vertical arcs in $A_1 \cap A_2$. □

Finally by Claim 4.14, $A_1 \cap A_2 \neq \emptyset$. On the other hand, by Claim 4.15, $A_1 \cap A_2$ has only vertical arcs or horizontal circles. But Claims 4.16 and 4.17 state that $A_1 \cap A_2$ does not have neither horizontal circles nor vertical arcs, thus $A_1 \cap A_2 = \emptyset$. This is a contradiction. We conclude that there are no descendent inequivalent destabilizations of (M, β). Thus there is a unique irreducible descendent. □

Now we state the equivalent sentence in terms of abstract braids (string links).

Corollary 4.18. Given an abstract braid (string link), up to compatibility and Reidemeister equivalence, it admits a unique representative of minimal genus.

On the other hand, if a string link in a thickened surface admits a braid position, by Lemma 4.10 a destabilization preserves the braid position. In particular its minimal representative admits a braid position.

Corollary 4.19. Given an abstract string link that admits a braid representative. Its minimal representative admits a braid position.

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