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From resurgent functions to real resummation through combinatorial Hopf algebras

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Résumé

Le problème de la resommation réelle consiste à associer à une série divergente réelle une fonction analytique qui lui est asymptotique sur un secteur du plan complexe bissecté par une des deux demi-directions réelles. Jean Ecalle a esquissé, pour le résoudre, les grandes lignes d'une théorie dite des bonnes moyennes uniformisantes. Celle-ci est basée sur plusieurs de ses découvertes : le calcul moulien simple et arborifié, les opérateurs étrangers et les fonctions résurgentes.

Nous nous proposons dans cette thèse de détailler complètement la théorie des moyennes d'Ecalle. Il s'agit de l'appliquer à la resommation de la conjuguante formelle des champs analytiques réels de type noeud-col et des difféomorphismes analytiques tangents à l'identité dans leur classe formelle la plus simple. Une partie conséquente de la thèse est consacrée à la théorie de l'arborification. C'est l'un des ingrédients majeurs de la théorie des moyennes mais pour laquelle Ecalle n'avait délivré que peu de détails.

Un chapitre de la thèse traite de géométrie o-minimale. Il s'agit de démontrer l'existence d'un « isomorphisme formel »entre les familles de germes d'ensembles semi-analytiques issus de deux classes quasi-analytiques isomorphes. Bien que ce chapitre soit disjoint de la théorie des moyennes, il est probable que cette dernière permette à l'avenir d'obtenir de nouvelles classes quasi-analytiques.

Enfin, nous proposons de faire le lien entre un procédé de resommation réelle de la conjuguante formelle du noeud-col réel élaboré par R. Schäfke et les moyennes d'Ecalle.

Remerciements.

Mes remerciements vont tout d'abord à Claudine Mitschi qui m'a incité, suite à la préparation à l'agrégation de l'université de Strasbourg, à poursuivre et débuter une thèse.

Je dois à Reinhard Schäfke mes premiers pas dans le monde de la recherche. Il m'a initié au problème de la resommation et m'a confié mon premier problème de recherche. La matinée pendant laquelle il m'expliqua la théorie des modules de Martinet-Ramis restera à jamais gravée dans ma mémoire et fut une étape déterminante dans mon apprentissage de la dynamique analytique locale. Enfin, c'est sur sa proposition que je me suis intéressé au problème de la resommation réelle et aux travaux d'Ecalle sur les bonnes moyennes.

C'est un honneur que mon travail ait été rapporté par Daniel Panazzolo et Jean-Yves Thibon. Je remercie en particulier Daniel Panazzolo pour ses suggestions qui ont permis d'améliorer le manuscrit initial de la thèse.

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Alors étudiant en deuxième année à l'université de Bourgogne, j'ai eu la chance de suivre les TDs de Jean-Philippe Rolin. Je dois dire combien son dynamisme et son enthousiasme étaient communicatifs et combien ces séances furent formatrices non seulement au niveau mathématique mais aussi pour l'enseignant que j'allais devenir. Ce fut pour moi une grande joie qu'il accepte de diriger ma thèse et je lui en suis profondément reconnaissant.

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J'ai une pensée pour mes parents, mes soeurs et leurs familles respectives ainsi que mes deux grand mères, mes beaux parents, ma belle soeur et sa famille. L'affection dont ils m'entourent est des plus précieuses. Qu'ils sachent toute l'affection qu'en retour j'ai pour eux.

Enfin, terminons sur cette phrase d'Hermann Hesse : « *A tout commencement préside un miracle, qui nous aide à vivre et nous protège* ». Le miracle de ma vie est sans doute de t'avoir rencontrée, Eloïse. Sans ce bonheur que tu m'apportes, sans ta tendre protection, je suis assez certain que je n'aurais pas eu la force de mener ce travail à bien. Que ces pages soient aussi le moment de te dire combien je t'aime.

La jeunesse, pour vivre, doit porter un secret.
Gaston Rébuffat - Étoiles et tempêtes

À Lucien, Adèle, Valentine et Joseph

Contents

Introduction

Préambule à l'introduction

En utilisant la série d'Euler comme un fil d'Ariane, nous allons expliquer la problématique de la resommation et le procédé de Borel-Laplace. Nous expliciterons alors le phénomène de Stokes et l'automorphisme associé puis la combinatoire qu'il permet de faire émerger. Nous pourrons dés lors aborder le problème plus difficile posé par la resommation réelle et nous expliquerons comment cette combinatoire permettra de le résoudre.

L'exemple de la série d'Euler

La série d'Euler est donnée par

$$\tilde{f}(x) = \sum_{n\geq 0} (-1)^n \, n! x^{n+1}.$$

C'est l'unique série formelle solution de l'équation différentielle du même nom $x^2y' + y = x$. Elle est divergente et n'admet donc pas de fonction somme au sens usuel du terme. En appliquant le théorème de résolution des équations différentielles linéaires du premier ordre et la méthode de variation de la constante, on peut néanmoins montrer que l'unique solution bornée au voisinage de $0 \in \mathbb{R}_+^*$ de cette équation est donnée par $f_0(x) = \int_0^x \frac{1}{t} e^{1/x-1/t} \mathrm{d}t$. De plus, pour tout sous-secteur fermé du demi plan $\Re(x) > 0$ de sommet l'origine, pour tout x élément de ce secteur et pour tout $N \in \mathbf{N}$, on a :

$$\left| f_0(x) - \sum_{n=0}^{N} (-1)^n ! x^{n+1} \right| \leq aN! \, |x|^{N+1} .$$

Autrement dit, f_0 est *Gevrey*−1 *asymptotique* à $\tilde{f}$ dans le secteur S bissecté par $\mathbb{R}_+$ et d'ouverture π. Le théorème de Watson nous assure du fait que f_0 est l'unique fonction satisfaisant cette propriété et f_0 est appelée la *resommée* de $\tilde{f}$ sur S.

Transformées de Borel formelle et de Laplace

Si on effectue au sein de l'intégrale précédente le changement de variable $\frac{\zeta}{x} = \frac{1}{t} - \frac{1}{x}$, il vient que $f_0(x) = \int_0^{+\infty} \frac{1}{\zeta+1} e^{-\zeta/x} \mathrm{d}\zeta$ et on reconnaît que f_0 est la transformée de Laplace $\mathcal{L}^0$ de la fonction $\zeta \mapsto \frac{1}{\zeta+1}$ dans la direction $\mathbb{R}_+$. Il nous faut encore comprendre le lien entre cette dernière fonction et la série initiale. Celui-ci est donné par la transformation de Borel formelle $\mathcal{B}$ qui est linéaire et qui à x^{n+1} associe $\zeta^n/n!$. Par application de $\mathcal{B}$ à $\tilde{f}$, on obtient la série $\left(\mathcal{B}\tilde{f}\right)(\zeta) = \sum_{n\geq 0} (-1)^n \zeta^n$ convergente dans le disque ouvert de rayon 1 centré en 0 et de somme $\hat{f}(\zeta) = 1/(1+\zeta)$. La technique consistant à appliquer successivement à une série Gevrey de niveau 1 une transformée de Borel, puis une transformée de Laplace est appelée *procédé de resommation de Borel-Laplace*.

Resommation

Il y a trois ingrédients qui additionnés autorisent l'utilisation successive de ces deux transformations.

— Le premier est que $\hat{f}(\zeta)$ est un germe analytique en 0 de $\mathbb{C}$. C'est une conséquence du fait que les coefficients a_n de la série d'Euler satisfont pour tout $n \in \mathbf{N}$, $|a_n| \leq ab^n n!$, autrement dit cette *série est Gevrey de niveau* 1.

— Le second est que pour une direction $\theta \neq \pi\ [2\pi]$, $\hat{f}(\zeta)$ est analytiquement prolongeable le long de la demi-droite d'origine O et d'angle θ.

— Le troisième est que ce prolongement suivant $\mathbb{R}_+$ est à croissance au plus exponentielle ce qui permet de prendre sa transformée de Laplace dans cette direction.

On résume le procédé de resommation dans le diagramme suivant :

$$\tilde{f}(x) = \sum_{n\geq 0} a_n x^{n+1} \in x\mathbb{C}_1[[x]] \xrightarrow{\mathcal{B}} \hat{f}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n \in \mathbb{C}\{\zeta\}$$

$$\downarrow \mathcal{L}^\theta$$

$$f(x) = \int_0^{e^{i\theta}\infty} f(\zeta)\, e^{-\zeta/x} \mathrm{d}\zeta \in \mathcal{H}_\theta$$

où

— $x\mathbb{C}_1[[x]]$ est l'algèbre des séries formelles Gevrey de niveau 1 sans terme constant. Le produit est celui usuel des séries formelles.

— Après transformée de Borel, une serie Gevrey de niveau 1 définie un germe analytique en l'origine de $\mathbb{C}$. On note $\mathbb{C}\{\zeta\}$ l'espace formé par ces germes. Quand on le muni du produit de convolution, cet espace devient une algèbre et la transformation de Borel formelle un morphisme injectif d'algèbres.

— L'espace vectoriel des fonctions analytiques dans le demi plan $\Re\left(xe^{i\theta}\right) > 0$, noté $\mathcal{H}_\theta$, muni du produit usuel est aussi une algèbre et la transformée de Laplace est elle aussi un morphisme injectif d'algèbres.

Le phénomène de Stokes

Dans l'exemple de la série d'Euler, la fonction $\hat{f}$ est analytiquement prolongeable dans toutes les directions $\mathcal{D}_\theta$ sauf évidemment celle pour $\theta = \pi\ [2\pi]$. On peut donc appliquer le procédé de Borel-Laplace dans chacune des directions $\theta \in]-\pi, \pi[$ et on obtient des fonctions f_θ définies sur des demi-plans $\Re\left(xe^{i\theta}\right) > 0$. En utilisant le théorème du prolongement analytique, on peut recoller ces fonctions en une fonction analytique sur le domaine $\arg(x) \in]-3\pi/2, 3\pi/2[$. Le pôle simple de $\hat{f}$ en -1 empêche évidemment de recoller les resommées de $\tilde{f}$ dans des directions $\pi^+ > \pi$ et $\pi^- < \pi$. Par contre on peut « mesurer la différence d'analyticité »de ces deux solutions par un simple calcul de résidu[1] :

$$\left(\mathcal{L}^{\pi^+}\hat{f}\right)(x) - \left(\mathcal{L}^{\pi^-}\hat{f}\right)(x) = 2i\pi e^{1/x},$$

c'est le *phénomène de Stokes*. L'automorphisme d'espace vectoriel $\dot{\Delta}^-$ qui à $f^-(x) := \left(\mathcal{L}^{\pi^-}\hat{f}\right)(x)$ associe $f^+(x) := \left(\mathcal{L}^{\pi^+}\hat{f}\right)(x) = f^-(x) + 2i\pi e^{1/x}$ est *l'automorphisme de Stokes*.

En résumé, au terme de cette étude, on a obtenu une fonction f Gevrey-1 asymptotique à $\tilde{f}$ sur un secteur d'ouverture 3π et de sommet O mais cette fonction est multivaluée. Plus précisément, elle est définie et analytique sur la surface de Riemann du logarithme et pas sur $\mathbb{C}$. Le théorème de Watson nous garantit que cette fonction est unique à vérifier ces propriétés et l'obstruction au fait d'obtenir pour $\tilde{f}$ une resommée analytique sur un voisinage de l'origine provient de la singularité en -1 de sa transformée de Borel, c'est-à-dire du phénomène de Stokes.

Plus loin avec le phénomène de Stokes

Les transformées de Borel de séries divergentes issues de systèmes dynamiques ([50], [51], [24], [36], [9], ...) ont la propriété d'être analytiquement prolongeables dans le plan complexe le long de chemins évitant un ensemble discret de singularités Ω. Cet ensemble possède une structure de semi-groupe additif[2]. Ce sont ces singularités, comme on l'a vu, qui empêchent d'appliquer la transformation de Laplace dans une direction intersectant Ω. De telles fonctions sont dites *résurgentes*[3] et elles forment une sous-algèbre de $\mathbb{C}\{\zeta\}$ appelé *modèle convolutif* des fonctions résurgentes. L'image par la transformée de Borel inverse de cette sous-algèbre est appelée *modèle formel* des fonctions résurgentes et son image par la transformée de Laplace dans une direction θ valide est appelée *modèle géométrique*.

Le phénomène de Stokes est une matérialisation de l'impossibilité de resommer les séries divergentes en des fonctions analytiques définies sur un disque

1. On peut aussi utiliser le fait que $\mathcal{L}^{\pi^+}\hat{f} - \mathcal{L}^{\pi^-}\hat{f}$ est solution de $xy' + y = 0$ mais cela ne permet pas de déterminer le coefficient $2i\pi$ devant l'exponentielle.
2. Dans le cas de l'équation d'Euler par exemple, on a $\Omega = -\mathbf{N}^*$. Mais seule la singularité en -1 intervient.
3. Cette appellation sera précisée et justifiée plus loin.

centré en 0. Mais il est aussi, comme nous allons l'expliquer, l'origine d'une foisonnante et féconde combinatoire qui va permettre d'analyser les systèmes considérés.

Ce qui distingue l'oeuvre d'Ecalle sur les systèmes dynamiques des autres travaux sur le sujet est l'usage systématique qu'il fait du modèle convolutif $\mathbb{C}\{\zeta\}$ et du jeu de va et vient qu'il exerce entre les différents modèles. Ce modèle est généralement sacrifié, dans les travaux non résurgents sur la dynamique complexe, au bénéfice du modèle géométrique. Pourtant, comme on la vu dans l'exemple précédent, c'est après la transformée de Borel et avant la transformée de Laplace qu'on peut bien comprendre les obstructions à la resommation d'une série divergente. Mais il y a plus. En effet, l'automorphisme de Stokes est sécable dans le modèle convolutif en une somme de sous-opérateurs élémentaires. Ces opérateurs, qui ont des propriétés algébriques remarquables, vont servir de briques pour obtenir de nouveaux opérateurs non constructibles, pour la plupart d'entre eux, dans le modèle géométrique : *les opérateurs étrangers*.

Les fonctions résurgentes

Pour une fonction résurgente $\hat{f}$, on s'intéresse, au voisinage d'une de ses singularités ω, à sa classe modulo $\mathbb{C}\{\zeta-\omega\}$, qu'on note $\mathbf{Sing}_\omega\left(\hat{f}\right)$. Dans le cas de la série d'Euler, on a au voisinage de $\omega=-1$ un pôle simple et pour ζ proche de -1, on sait que $\hat{f}(\zeta)=\frac{1}{\zeta+1}$. On convient d'identifier $\mathbf{Sing}_{-1}\left(\hat{f}\right)$ à $2i\pi\delta$ où δ représente la distribution de Dirac. Les fonctions résurgentes simples sont, par définition, au voisinage d'une de leur singularité ω, de la forme

$$\hat{f}(\zeta)=\frac{C}{2i\pi(\zeta-m)}+\frac{1}{2i\pi}\hat{a}(\zeta-m)\log(\zeta-m)+\hat{b}(\zeta)$$

où $\hat{a}$ et $\hat{b}$ sont des germes de fonctions holomorphes. La donnée de la classe de $\hat{f}$ au voisinage de ω équivaut donc à la donnée du résidu C en ω et de la différence des prolongements par la gauche et par la droite de ω de $\hat{f}$, c'est-à-dire au mineur de $\hat{f}$. On convient alors d'identifier $\mathbf{Sing}_\omega\left(\hat{f}\right)$ avec $C\delta+\hat{a}$. Comme on le démontrera dans le chapitre 1, le germe $\hat{a}$ est encore celui d'une fonction résurgente simple. Pour que l'opérateur $\mathbf{Sing}_\omega$ soit bien défini dans l'espace des fonctions résurgentes, il est nécessaire d'augmenter ce dernier par le sous-espace engendré par la distribution de Dirac. Cette dernière est de plus l'unité pour le produit de convolution et fait de l'espace des fonctions résurgentes une algèbre commutative unitaire.

L'automorphisme de Stokes et l'algèbre des opérateurs étrangers

Le premier point important est que l'automorphisme de Stokes $\dot{\Delta}^+$ peut se décrire dans le modèle convolutif explicitement grâce aux opérateurs[4] $\mathbf{Sing}_\omega$:

4. En fait grâce à ces opérateurs tensorisés par des distributions de Dirac.

$$\dot{\Delta}^+ = \sum_{m\in\Omega} \delta_\omega \star \mathbf{Sing}_\omega.$$

Nous en proposons une démonstration originale dans la proposition 1.2.4 page 27.

Le second point essentiel est que l'automorphisme de Stokes est non seulement un morphisme d'espaces vectoriels mais aussi un morphisme d'algèbres, voir la proposition 1.2.5.

Il est alors naturel de considérer l'algèbre engendrée par les composantes homogènes de l'automorphisme de Stokes et leurs compositions [5]. En munissant cette algèbre de la topologie de Krull (voir section 2.1.5) puis en la complétant, on obtient des opérateurs qui sont des sommes infinies de telles compositions. Ces opérateurs sont les fameux opérateurs étrangers d'Ecalle et leur construction n'est possible, pour l'essentiel d'entre eux, que dans le modèle convolutif. Leur rôle va être déterminant car ils vont servir de boite à outils pour manipuler les fonctions résurgentes.

Un opérateur étranger **op** est donc de la forme

$$\mathbf{op} = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}} \dot{\Delta}^+_{\underline{\omega}} \in \mathbf{ALIEN}$$

où

— $\Omega^\bullet$ est l'ensemble des mots avec lettres dans Ω auquel on ajoute le mot vide.
— $(M^{\underline{\omega}})$ est une famille scalaire indexée par $\Omega^\bullet$ appelée *moule*.
— La famille d'opérateurs $\left(\dot{\Delta}^+_{\underline{\omega}}\right)$ indexée par $\Omega^\bullet$ est appelée un *comoule* et pour tout $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ l'opérateur $\dot{\Delta}^+_{\underline{\omega}}$ est donné par la composition $\dot{\Delta}^+_{\underline{\omega}} = \dot{\Delta}^+_{\omega_r} \ldots \dot{\Delta}^+_{\omega_1}$.

De plus, l'opérateur de Stokes étant un morphisme d'algèbres, on peut munir l'algèbre des opérateurs étrangers d'un coproduit en considérant l'action des composantes homogènes du Stokes sur un produit de fonctions résurgentes. Ce coproduit fait de **ALIEN** une algèbre de Hopf. L'automorphisme de Stokes $\dot{\Delta}^+$ est naturellement un élément group-like de cette algèbre et son générateur infinitésimal $\dot{\Delta}$, appelé *dérivation étrangère standard* par Ecalle, est un élément primitif.

Les algèbres de Hopf combinatoires

On devine la richesse combinatoire de l'algèbre **ALIEN** quand on sait qu'il s'agit d'une réalisation de l'algèbre des fonctions symétriques non commutatives **Sym**. Cette dernière a été étudiée en profondeur ([26],[34],...) et est toujours le sujet d'actives recherches. En plus de servir de cadre pour obtenir

5. Ces compositions sont possibles car les opérateurs $\mathbf{Sing}_\omega$ préservent l'espace des fonctions résurgentes.

des versions non commutatives de certaines identités classiques ([26]), l'algèbre **Sym** intervient dans de nombreux domaines comme la théorie des représentations ([35]), la topologie algébrique ([1]), la géométrie des polytope ([6])... Les composantes homogènes $\dot{\Delta}_k^+$ de l'automorphisme de Stokes sont identifiables aux fonctions symétriques élémentaires Λ_k. Considérant la série génératrice $\lambda(t) = 1 + \sum_{k\geq 1} t^k \Lambda_k$, les composantes homogènes ϕ_k de $\log\left(\lambda^{-1}(-t)\right)$ sont, elles, identifiables à $k\dot{\Delta}_k$.

L'étude des fonctions résurgentes, et donc des systèmes dynamiques, est complètement liée à la théorie des algèbres de Hopf combinatoires. Dû au caractère group-like de l'automorphisme de Stokes, les moules sont identifiables à des formes linéaires sur l'algèbre de Hopf des quasi-shuffles sur Ω ([31]). Les éléments group-like et primitifs de **ALIEN** peuvent être détectés grâce au moule qui les définit, si ce moule, après identification avec les formes linéaires de l'algèbre des quasi-shuffles, est un caractère ou un caractère infinitésimal de l'algèbre des quasi-shuffles. Ces propriétés se traduisent en termes de symétries mouliennes dans la terminologie d'Ecalle. L'utilisation de ces structures combinatoire permet de gérer la complexité des calculs qu'on rencontre en dynamique locale.

En dualisant le coproduit de l'algèbre **ALIEN**, on peut munir l'espace des moules d'un produit, qui correspond au produit de convolution sur l'algèbre des quasi-shuffles. Ce produit mime la composition des opérateurs étrangers. Il est essentiel de remarquer que les opérations entre opérateurs étrangers se trouvent ainsi traduites en en une combinatoire sur les mots de $\Omega^\bullet$.

L'équation du pont

La dérivation étrangère standard joue un rôle de premier plan dans la théorie d'Ecalle. En effet, l'action de ses composantes homogènes sur une intégrale formelle du système considéré est proportionnelle, via une égalité appelée *équation du pont*, à celle d'un opérateur différentiel ordinaire agissant sur l'espace des paramètres de cette intégrale formelle. Plus précisément, si on convient de noter $\hat{f}(\zeta, u)$ cette dernière, ζ étant la variable résurgente et $u = (u_1, \ldots, u_n)$ le paramètre, l'équation du pont est de la forme, pour tout $\omega \in \Omega$:

$$\Delta_\omega \hat{f}(\zeta, u) = C_\omega \mathbb{D}_\omega \hat{f}(\zeta, u)$$

où $\mathbb{D}_\omega$ est un opérateur différentiel ordinaire sur $\mathbb{C}[[u]]$ et $C_\omega \in \mathbb{C}$.

Les coefficients de proportionnalité $C_\omega \in \mathbb{C}$ caractérisent complètement la classe analytique du système étudié. On les appelle généralement *invariants d'Ecalle*. Il sont évidemment reliés à ceux déjà connus, par exemple les *invariants de Martinet-Ramis* dans le cas des champs de vecteurs analytiques de type noeud-col ou aux coefficients de l'application corne dans le cas des difféos analytiques tangents à l'identité.

L'équation du pont permet de définir, pour chaque classe analytique du système étudié, un anti-morphisme d'algèbres de Hopf, appelé *morphisme de réduction*, entre **ALIEN** et une algèbre d'opérateurs différentiels ordinaires. Il est défini comme étant l'unique anti-morphisme d'algèbres **red** : **ALIEN** $\to$ **ENDOM** $(\mathbb{C}[[u]])$ vérifiant [6]

$$\mathbf{red}\left(\dot{\Delta}_\omega\right) = C_\omega \mathbb{D}_\omega.$$

Ce morphisme va être d'une importance capitale dans la théorie de la resommation réelle. Il va nous permettre de caractériser les opérateurs étrangers qui préservent la croissance au plus exponentielle des fonctions résurgentes. Cette propriété sera en fait traduite au niveau de l'algèbre des opérateurs différentiels correspondante grâce à ce morphisme. On verra en effet qu'un opérateur étranger préserve la croissance géométrique si et seulement si, pour toute réduction de cette opérateur, l'opérateur différentiel ordinaire obtenu préserve les germes analytiques (dés que c'est le cas pour l'automorphisme de Stokes).

L'arborification

Le morphisme de réduction nous conduit à considérer des opérateurs différentiels ordinaires agissant sur l'espace des séries formelles $\mathbb{C}[[u_1,\dots,u_n]]$ où $u_1,\dots,u_n$ sont les paramètres de l'intégrale formelle et à trouver un moyen de détecter parmi ceux-ci lesquels vont préserver $\mathbb{C}\{u_1,\dots,u_n\}$. Dans notre thèse, les intégrales formelles considérées n'ont qu'un paramètre aussi on ne travaillera qu'avec des opérateurs différentiels agissants sur $\mathbb{C}[[u]]$. Ces opérateurs, comme on le verra, s'écrivent :

$$\mathbf{F} = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbb{D}_{\underline{\omega}} \in \mathbf{ENDOM}(\mathbb{C}[[u]])$$

où

- $\mathbb{D} = \sum_{m\in\Omega}\mathbb{D}_m \in \mathbf{ENDOM}(\mathbb{C}[[u]])$ est un opérateur différentiel ordinaire sur $\mathbb{C}[[u]]$ décomposé en la somme de ses composantes homogènes.
- $\left(\mathbb{D}_{\underline{\omega}}\right)$ est un *comoule* donné pour tout $\underline{\omega} = (\omega_1,\dots,\omega_r)$ par les compositions $\mathbb{D}_{\underline{\omega}} = \mathbb{D}_{\omega_r} \ \dots \ \mathbb{D}_{\omega_1}$.

Il est malheureusement très difficile de détecter lesquels parmi ces opérateurs vont préserver $\mathbb{C}\{u\}$. Ce problème est intimement lié à la croissance [7] des compositions $\mathbb{D}_{\omega_r} \ \dots \ \mathbb{D}_{\omega_1}$. Cayley a été le premier à noter la pertinence (voir [8]) d'indexer les opérateurs intervenant dans le développement de telles compositions $\mathbb{D}_{\underline{\omega}}$ par des forêts $\underline{\omega}^<$ décorés par Ω. On construit ainsi un comoule $\left(\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}}\right)$ non plus indexé par des mots mais par des forêts. Ce comoule est le *pré-coarborifié* du comoule $\mathbb{D}_\bullet$.

La pré-coarborification est une étape importante dans le procédé de *coarborification* et elle n'en diffère que par un facteur rationnel dépendant de la forêt $\underline{\omega}^<$ indexant l'opérateur, son facteur de symétrie $s\left(\underline{\omega}^<\right)$. Si on note $\left(\mathbb{D}_{\underline{\omega}^<}\right)$ le

6. On montrera que tout opérateur étranger $\mathbf{op} = \sum M^\bullet \dot{\Delta}^+$ peut se décomposer suivant le comoule associé à la dérivation étrangère standard $\dot{\Delta}$, c'est-à-dire en une somme $\mathbf{op} = \sum N^\bullet \dot{\Delta}$. Alors $\mathbf{red}(\mathbf{op}) = \sum_{\underline{\omega}\in\Omega^\bullet} N^{\underline{\omega}}\mathbb{D}_{\mathbf{rev}(\underline{\omega})}$ où $\mathbf{rev}(\underline{\omega})$ représente le mot $\underline{\omega}$ retourné.

7. un choix ayant été fait d'une famille de semi-normes sur l'algèbre des opérateurs différentiels ordinaires considérés

coarborifié de $(\mathbb{D}_{\underline{\omega}})$, on a en fait pour toute forêt $\underline{\omega}^{<}$, $\mathbb{D}_{\underline{\omega}^{<}} = 1/s\,(\underline{\omega}^{<})\,\mathbb{D}_{\underline{\omega}}^{\mathbf{prearbo}}$. L'idée est alors de remplacer chaque opérateur $\mathbb{D}_{\underline{\omega}}$ dans la somme donnant $\mathbf{F}$ par la somme correspondante d'opérateurs coarborifiés. On n'obtient rien d'autre qu'une nouvelle organisation de la somme définissant $\mathbf{F}$:

$$\mathbf{F} = \sum_{\underline{\omega}^{<}} M^{\underline{\omega}^{<}} \mathbb{D}_{\underline{\omega}^{<}}$$

et le moule $\left(M^{\underline{\omega}^{<}}\right)$ indicié par des forêts se calcule au moyen d'une formule explicite à partir du moule $M^{\bullet}$ et des facteurs de symétries. Cette formule s'obtient en dualisant l'opération de coraborification. Le moule $M^{\bullet<}$ est *l'arborifié* du moule $M^{\bullet}$ et l'opération consistant à passer du moule $M^{\bullet}$ au moule $M^{\bullet<}$ est *l'arborification.*

Parmi ces opérateurs différentiels, on sait que ceux dont l'arborifié $(M^{\underline{\omega}^{<}})$ est à croissance géométrique en la norme de $\underline{\omega}^{<}$ [8], vont préserver les germes analytiques. On peut lire une très belle preuve de ce fait dans [43].

De la même façon qu'on traduit les opérations entre opérateurs (étrangers ou différentiels ordinaires) en une combinatoire sur les mots, on peut traduire celles des opérateurs différentielles en une combinatoire sur les forêts. Cette combinatoire, re-découverte par Ecalle, est celle de l'algèbre de Grossman-Larson ([27]). Son dual est l'algèbre de Connes-Kreimer ([12]) et le produit de cette dernière est exactement le produit des moules arborifés tel qu'Ecalle l'avait énoncé dans [15] huit ans auparavant. Un raisonnement cohomologique (voir [22]) montre qu'il existe un unique morphisme d'algèbre de Hopf entre l'algèbre des quasishuffles et celle de Connes-Kreimer. Ce morphisme correspond justement au procédé d'arborification (voir [60] et [18]).

Le problème de la resommation réelle

Revenons à la série d'Euler. Elle est réelle, tout comme l'équation différentielle qu'elle solutionne, aussi il est naturel d'en chercher des resommées réelles sur l'axe réel.

Il n'y a pas de difficulté à cela dans la direction $\mathbb{R}^{+}$, cette condition est remplie par la fonction f_0. Les choses se compliquent néanmoins dans la direction $\mathbb{R}^{-}$ à cause du phénomène de Stokes.

Il est cependant facile de contourner ici cette complication grâce à la linéarité de l'équation différentielle d'Euler. En effet, il suffit de considérer la demi-somme f des deux resommées $f^{\pi^{+}}$ et $f^{\pi^{-}}$. Ces deux fonctions étant conjuguées, f est réelle. Sa restriction à $\mathbb{R}_{-}$ est analytique et comme elle est aussi solution de l'équation d'Euler (c'est là que la linéarité intervient), elle reste Gevrey-1 asymptotique à $\tilde{f}$.

Ces considérations ne restent évidemment pas pertinentes dans le cas général car les systèmes étudiés ne sont pas toujours linéaires. La question est alors

8. C'est-à-dire la somme des décorations de $\underline{\omega}^{<}$

la suivante. Considérons une série formelle Gevrey-1 provenant d'un système dynamique analytique réel. Est-il possible de lui associer, pour chacune des deux directions réelles, une fonction réelle qui lui est Gevrey-1 asymptotique ?

Exposons une autre tentative. On choisit à cette fin une série formelle $\tilde{\phi}$ Gevrey-1 solution formelle d'une équation différentielle analytique non linéaire $f(x, y, y') = 0$ et dont la transformée de Borel $\hat{\phi}$ est résurgente avec des singularités situées (entre autres) sur $\mathbb{R}_+$. On souhaite évidemment resommer cette série dans cette direction. On suppose qu'on peut prolonger $\tilde{\phi}$ dans des directions $\theta > 0$ et $-\theta$ et que ces deux prolongements, respectivement notés ϕ^+ et ϕ^- sont à croissance au plus exponentielle [9]. La série initiale étant réelle, ces deux prolongements sont conjugués l'un de l'autre. La présence de singularités suivant $\mathbb{R}_+$ nous invite à considérer l'automorphisme de Stokes $\dot{\Delta}^-$ suivant $\mathbb{R}_+$. Il va échanger $\hat{\phi}^+$ et $\hat{\phi}^-$. L'automorphisme inverse est noté $\dot{\Delta}^+$ et il échange bien évidemment $\hat{\phi}^-$ et $\hat{\phi}^+$. De plus dans le cas d'un système différentiel réel, on montrera qu'il est unitaire, c'est-à-dire que son inverse est égal à son conjugué. On remarque que la racine carrée $\sqrt{\dot{\Delta}^-}$ de l'automorphisme de Stokes satisfait

$$\sqrt{\dot{\Delta}^-}\phi^+ = \sqrt{\dot{\Delta}^+}\phi^-.$$

Autrement dit,

$$\overline{\sqrt{\dot{\Delta}^-}\phi^+} = \sqrt{\dot{\Delta}^-}\phi^+$$

et donc $\sqrt{\dot{\Delta}^-}\phi^+$ est réel. De plus, l'opérateur de Stokes étant un morphisme pour la convolution, il en est de même pour sa racine carrée et si $\sqrt{\dot{\Delta}^-}\phi^+$ était à croissance exponentielle, on pourrait prendre sa transformée de Laplace et obtenir une fonction réelle qui serait solution de l'équation différentielle. Cette dernière serait alors asymptotique Gevrey-1 à la série initiale et on aurait trouvé une resommée réelle de cette série. Malheureusement, la racine carrée d'un opérateur analytique ne reste pas nécessairement analytique et donc la croissance de $\sqrt{\dot{\Delta}^-}\phi^+$ peut être plus qu'exponentielle.

La solution d'Ecalle

L'idée simple et élégante d'Ecalle est de considérer, pour une fonction résurgente donnée comme la transformée de Borel d'une série divergente qu'on souhaite resommer, les prolongements de cette fonction le long de tous les chemins possibles évitant les singularités de Ω (on prend évidemment un chemin par classe d'homotopie) puis de combiner le tout en affectant chaque prolongement d'un poids complexe dépendant du chemin choisi. C'est ce système de poids, quand il vérifie certaines règles appelées relations d'autocohérence, qu'Ecalle appelle une moyenne.

On obtient à l'issue de cette opération une fonction uniforme sur $\mathbb{R}_+$. L'enjeu réside bien entendu dans la possibilité de trouver un système de poids judicieux pour obtenir à la fin une fonction à la fois :

1. réelle,

9. On peut donc leur appliquer la transformée de Laplace.

2. à croissance au plus exponentielle le long de $\mathbb{R}^{+}$ [10],
3. et dont la transformée de Laplace est asymptotique à la série initiale.

Une moyenne vérifiant ces trois propriétés est appelée une bonne moyenne et le travail consiste maintenant à comprendre comment les traduire en terme de poids.

Pour ce qui est de la réalité, c'est facile, les poids affectés à des chemins conjugués doivent être conjugués. Pour les deux autres conditions, les choses sont bien moins évidentes.

Une petite avancée permettant de commencer à formaliser la troisième propriété consiste à remarquer que si la moyenne préserve la convolution alors après transformée de Laplace la fonction obtenue est solution du système différentiel considéré et est donc asymptotique à la série initiale. Mais il est difficile en l'état de faire mieux.

Avant de tenter quoique ce soit d'autre, introduisons les deux moyennes **mur** et **mul**. La première affecte du poids 1 tout chemin contournant systématiquement par la droite les singularités et du poids 0 tout autre chemin. Il en est de même pour **mul** si ce n'est que ce sont les chemins contournant les singularités par la gauche qui sont affectés du poids 1. On reconnaît en fait que $\mathbf{mul}\hat{\phi} = \phi^{+}$ et $\mathbf{mur}\hat{\phi} = \phi^{-}$. Ces deux moyennes ne sont pas des bonnes moyennes. Elles vérifient en fait les propriétés 2. et 3. mais pas la 1.. Elles sont par contre conjuguées l'une de l'autre et sont reliées par l'automorphisme de Stokes : $\mathbf{mur}\dot{\Delta}^{-}\hat{\phi} = \mathbf{mul}\hat{\phi}$.

Une propriété fondamentale des moyennes est qu'elles sont toutes reliées par un opérateur étranger à l'une des deux moyennes **mur** ou **mul** par une relation de la forme :

$$\mathbf{m} = \mathbf{mur\ rem} = \mathbf{mul\ lem}$$

où $\mathbf{rem}, \mathbf{lem} \in \mathbf{ALIEN}$.

On peut alors transcrire au niveau de ces opérateurs étrangers les propriétés caractérisant les bonnes moyennes. On montre que la moyenne $\mathbf{m} = \mathbf{mur\ rem}$ est bonne si et seulement si elle satisfait :

1. $\mathbf{rem} = \dot{\Delta}^{+}\overline{\mathbf{rem}}$ pour la réalité,
2. **rem** est group like pour le respect de la convolution
3. pour toute réduction **red**, $\mathbf{red}\left(\mathbf{rem}\right)$ est un opérateur différentiel ordinaire analytique dés que la réduction de l'automorphisme de Stokes $\mathbf{red}\left(\dot{\Delta}^{-}\right)$ est analytique [11], pour le respect de la croissance exponentielle.

Mais les opérateurs étrangers étant complètement définis dés qu'on connaît leur moule relativement à un comoule donné, ces trois conditions deviennent, pour $\mathbf{rem} = \sum M^{\bullet}\dot{\Delta}^{+}_{\bullet}$:

10. Pour pouvoir appliquer la transformée de Laplace dans cette direction
11. On dit qu'un opérateur différentiel ordinaire agissant sur $\mathbb{C}[[u]]$ est analytique s'il préserve $\mathbb{C}\{u\}$.

1. $\overline{M}^{\bullet} \circ J^{\bullet} = M^{\bullet} \times (1+I)^{\bullet}$ [12] pour la réalité,
2. $M^{\bullet}$ est symetrel [13],
3. l'arborifié $M^{\bullet <}$ de $M^{\bullet}$ est à croissance géométrique [14].

Et le problème de trouver des bonnes moyennes se ramène à celui de trouver des moules satisfaisant ces trois propriétés. La difficulté reste immense car elles ont tendance à mutuellement s'exclure et il est absolument remarquable qu'Ecalle ait réussi à exhiber des moules solutions de ce problème.

Parmi ces moules, un de ceux ci permet d'aboutir aux moyennes dites induites par diffusion. Celles-ci sont fondamentales dans le sens où toutes les moyennes connues jusqu'ici en découlent, soit par le choix du semi-groupe convolutif les définissant (moyenne de Catalan, moyenne brownienne,...), soit par un passage à la limite (moyenne organique). Bien qu'Ecalle en donne une interprétation probabiliste dans [14], il n'a malheureusement pas explicité les motivations qui l'ont amenés à les découvrir. Des travaux récents ([45],[61]) montrent néanmoins que les moyennes induites par diffusion, ainsi que la moyenne organique, peuvent être reconstruites assez naturellement en utilisant des outils développés pour la théorie quantique des champs perturbés (pQFT). Expliquons brièvement pourquoi, même si cela nous éloigne momentanément du sujet de la thèse. Un des problèmes fondamentaux en pQFT est de pouvoir renormaliser des intégrales (de Feynman) divergentes [15] afin d'en tirer des quantités finies ayant un sens physique. Les physiciens ont découvert un algorithme de renormalisation connu sous le nom de formule de Bogoliubov. Mais jusqu'aux travaux de Connes et Kreimer ([12]), cet algorithme était dépourvu de fondement mathématique. Le processus de renormalisation peut s'exprimer algébriquement, comme un problème de factorisation de caractères sur l'algèbre de Hopf de Connes-Kreimer.

Ce processus de factorisation est en fait un cas particulier du procédé de factorisation d'Atkinson dans les algèbres de Rota-Baxter. Et le problème de trouver des bonnes moyennes est justement un problème de factorisation de caractère sur l'algèbre de Hopf des quasi-shuffles. On a expliqué que les moules symetrels pouvaient s'identifier à des caractères sur l'algèbre des quasi-shuffles. Trouver des moyennes vérifiant les propriétés de réalité et de convolution revient alors, pour un bon choix d'une algèbre de Rota-Baxter, à trouver une factorisation du moule identité. Pour vérifier la propriété de bonne croissance de ces moyennes, il est nécessaire de re-effectuer cette factorisation sur l'algèbre de Connes-Kreimer. On tombe ainsi directement sur l'arborifié du moule $M^{\bullet}$. Il convient alors de majorer géométriquement cet arborifié, ce qui passe, pour les exemples connus, par l'obtention d'une formule close pour cet arborifié.

12. Les notations et opérations utilisées seront explicitées plus loin dans la thèse
13. Ce qui garantit que **rem** est group-like
14. Et donc, dans toute réduction, **rem** préserve les germes analytiques.
15. En théorie des hautes énergies, les collisions entre particules sont modélisées par des graphes, dits de Feynman. On obtient des mesures physiques de ces phénomènes en considérant des intégrales définies sur ces graphes.

Structures o-minimales

La théorie des structures o-minimales est une généralisation de la géométrie analytique réelle. Mais c'est aussi une tentative d'obtenir l'axiomatique d'une topologie « modérée » ; comme la décrivait Grothendieck dans son fameux manuscrit « Esquisse d'un programme »(voir [28]), dans laquelle les ensembles considérés ne sont pas « trop singuliers ».

La théorie des structures o-minimales peut être construite à partir de la théorie des modèles, de la notion d'extension d'un langage et plus précisément du langage du corps des réels $(\mathbb{R}, <, 0, 1, +, -, .)$. On ajoute à ce dernier une famille $\mathcal{F}$ de nouveaux symboles de fonctions réelles. On étudie ensuite les propriétés de la structure obtenue et en particulier quels sont les sous-ensembles de $\mathbb{R}^n$ qu'on peut obtenir localement comme équations ou inéquations de fonctions définissables dans la structure.

Une approche équivalente mais plus géométrique de la théorie des modèles consiste à s'intéresser à des familles de sous-ensembles de $\mathbb{R}^n$ contenant les ensembles semi-algébriques pour tout $n \in \mathbf{N}$ et stables par une famille d'opérations (réunion, intersection, prise du complémentaire, produit cartésien et projection, contenant les sous espaces $x_1 = \ldots = x_n$).

Une structure est dite o-minimale si les sous-ensembles de $\mathbb{R}$ définissables en son sein ont un nombre fini de composantes connexes. Elle est modèle complète si l'opération de prise du complémentaire dans la liste ci-dessus laisse invariante la famille.

Parmi les extensions déjà étudiées, les plus notables sont :

— $\mathbb{R}_{\mathcal{F}} = \mathbb{R}_{an}$ où $\mathcal{F}$ est l'ensemble des fonctions analytiques, voir [25]. Cette extension fourni une axiomatisation de la géométrie semi-analytique.
— $\mathbb{R}_{\mathcal{F}} = \mathbb{R}_{\exp}$ où $\mathcal{F} = \{\exp\}$, voir [63],
— l'extension $\mathbb{R}_{an,\exp}$, voir [58],
— les extensions avec une solution d'une équation Pfaffienne, voir [55]
— les extensions avec une solution d'une équation différentielle ordinaire, voir [47],

Un des objectifs est d'obtenir à la fin de nouvelles propriétés géométriques comme conséquences de la théorie des modèles appliquée à la structure. Dans [33] par exemple, les auteurs ont construit une extension du corps des réels dans laquelle, pour un champ de vecteurs analytique réel du plan avec un point singulier isolé hyperbolique non-résonnant, les applications de transition en ce point singulier sont définissables. Une conséquence est que certaines applications de retour de Poincaré sont aussi définissables dans la structure, ce qui permet d'affirmer que le champ de vecteurs a un nombre fini de cycles limites et prouve la conjecture de Dulac dans ce cas particulier.

Une voie suivie avec succès dans deux articles [48] et [54] consiste à trouver de nouvelles extensions du corps des réels en prenant pour $\mathcal{F}$ une classe de fonctions quasianalytiques [16]. Dans le premier papier, $\mathcal{F}$ est une classe de Denjoy-

16. une classe de fonctions est dite quasianalytique si l'application qui associe à une fonction

Carlemann et dans le second c'est une classe de fonctions réels construites comme sommes de séries multisommables. Dans les deux cas, la preuve de l'o-minimalité de l'extension considérée est basée sur un algorithme de désingularisation généralisant celui de Bierstone et Milman [3].

Dans ce dernier exemple, les séries considérées n'ont pas de phénomène de Stokes suivant $\mathbb{R}_+$ aussi il est possible de les resommer (que ce soit avec le procédé de Borel-Laplace ou sa généralisation par Ecalle, Martinet, Ramis, Malgrange et d'autres, la théorie de la multisommabilité, voir [40] ou [2]).

Mais si il y a un phénomène de Stokes dans une des directions réelles, alors la théorie de la multisommabilité n'est plus adaptée à trouver des resommées réelles. Même si on peut outrepasser ces difficultés dans certain cas (par exemple, dans [47] les auteurs ont prouvé l'o-minimalité d'une extension du corps des réels par les fonctions analytiques réelles et la solution non-oscillante d'une équation de type Euler ayant un phénomène de Stokes suivant $\mathbb{R}_+$. Mais l'existence de cette solution a été démontrée sans recours à la théorie de la resommation, voir [62]).

On comprend ce que la théorie des bonnes moyennes pourrait apporter à cette problématique. Elle devrait permettre de construire de nouvelles classes quasianalytiques. Même si ces considérations ont constituées le point de départ de la thèse, nous n'abordons néanmoins pas ce problème ici si ce n'est dans la conclusion où nous dresserons une liste des obstacles à surmonter pour y parvenir.

Plan de la thèse

L'objet de la thèse est de proposer une approche complètement détaillée de la resommation réelle d'Ecalle et de l'appliquer à deux systèmes dont la résurgence est maintenant parfaitement comprise, les champs analytiques de type noeud-col et les difféomorphismes analytiques tangents à l'identité dans leur classe formelle la plus simple. Elle est organisée comme suit.

Dans le chapitre 1, nous introduisons la notion de fonction résurgente simple ainsi que l'automorphisme de Stokes. Dans le chapitre 2, nous effectuons quelques rappels sur les algèbres de Hopf, nous introduisons l'algèbre de concaténation et nous en tirons deux représentations. La première, construite grâce aux composantes homogènes de l'automorphisme de Stokes, est évidemment **ALIEN**. La seconde est une sous-algèbre des endomorphismes de $\mathbb{C}[[u]]$ composée d'opérateurs différentiels ordinaires.

Dans le chapitre 3, nous établissons l'équation du pont pour les champs de type noeud-col et les difféos paraboliques. Nous introduisons les invariants analytiques d'Ecalle et nous les relions à ceux déjà existants, c'est-à-dire ceux de Martinet-Ramis pour le noeud-col et les coefficients de l'application corne pour les difféos. Dans le cas des champs de type noeud-col nous donnons aussi des formules explicites et nouvelles permettant de calculer explicitement les

son développement de Taylor en 0 est injective.

invariants d'Ecalle en fonction de ceux de Martinet-Ramis et nous démontrons leur croissance 1-Gevrey. Nous construisons alors le morphisme de réduction.

Comme on l'a expliqué précédemment, la construction de bonnes moyennes fait un usage fondamental de l'arborification. Ecalle n'a que très partiellement exposé cette théorie et il n'en a explicité que les grandes lignes ([14],[15]). Nous en proposons une description complète dans les chapitres 4 et 5. Dans le chapitre 6, nous proposons plusieurs calculs explicites d'arborifiés. L'existence de formules closes pour les arborifiés des moules intervenant dans la thèse va permettre de conclure en leur croissance géométrique.

Enfin, dans le chapitre 7, nous construisons la théorie des moyennes uniformisantes. Nous démontrons, en suivant de près le formalisme d'Ecalle, que les moyennes induites par diffusion sont de bonnes moyennes et nous appliquons la théorie à la resommation de la conjuguante du noeud-col réel et de la normalisante des difféos paraboliques réels.

Le chapitre 8 concerne la géométrie o-minimale. Dans [48], les auteurs prouvent l'o-minimalité et la modèle complétude d'extension du corps des réels par les fonctions analytiques réelles et une classe de Denjoy-Carlemann. Ils montrent, pour ce faire, la quasianalyticité d'une algèbre de fonctions construite à partir de cette classe de Denjoy-Carlemann. Les résultats de ces travaux ont été appliqués dans [47] à une classe de fonctions engendrées par les germes de fonctions analytiques et une solution réelle ϕ_1 d'une équation différentielle de type noeud-col admettant un phénomène de Stokes dans une des deux directions réelles. Si on considère la même classe mais engendrée par une autre solution ϕ_2 de la même équation différentielle, on obtient une classe isomorphe à la première. Un problème naturel est alors de comparer les germes d'ensembles analytiques obtenus pour chacune des deux classes. On prouve dans le chapitre 8 l'existence d'un isomorphisme entre ces deux familles de germes. Cet isomorphisme envoie de plus le germe du graphe de la solution ϕ_1 sur celui de la solution ϕ_2. Par contre, on montre qu'on ne peut l'étendre aux germes de sous-ensembles analytiques à moins qu'ils ne vérifient une certaine propriété de régularité en l'origine.

Le dernier chapitre 9 a pour objet de comparer un résultat de R. Schäfke avec la théorie des bonnes moyennes. On considère un champ de vecteurs analytique réel de type noeud-col et on note $y(t,u)$ une paramétrisation de la conjuguante formelle associée à ce noeud-col. On note aussi $y^+(t,u)$ la resommée (au sens de Borel-Laplace) de cette conjuguante formelle dans une direction $\theta > 0$ proche de 0. R. Schäfke a montré, dans un travail non publié, l'existence d'un germe analytique $f_{\mathcal{S}}(u)$ tel que $y^+\,(t, f(u))$ est réel analytique et asymptotique 1-Gevrey à $y(t,u)$. Ce germe analytique est solution d'une équation fonctionnelle faisant intervenir l'isotropie sectorielle du noeud-col considéré. Nous montrons que, à toute bonne moyenne d'Ecalle, on peut associer un tel germe mais que, réciproquement, pour tout germe analytique vérifiant cette équation fonctionnelle, il n'existe pas forcément de moyenne permettant de l'incarner. En nous restreignant à certaines classes analytiques de noeud-cols (ceux ayant une famille d'invariants analytiques browniens à croissance au moins géométrique), on construit par contre des moyennes correspondantes au germe $f_{\mathcal{S}}(u)$. Pour ce faire, nous utilisons la notion d'automorphisme et de dérivation appariés, déjà investiguée par F. Menous dans sa thèse ([42]). Nous mettons en évidence la nécessité de travailler avec une dérivation étrangère réelle et équianalytique à l'automorphisme de Stokes [17]. Nous explorons une première piste qui consiste à considérer la dérivation appariée à l'automorphisme de Stokes. Cette dernière n'est malheureusement pas réelle mais elle permet de retrouver une dérivation réelle déjà découverte par J. Ecalle, la dérivation organique, mais ici par un procédé original et purement algébrique. La dérivation organique n'est malheureusement pas équianalytique à l'automorphisme de Stokes et nous avons dû travailler dans une autre direction. La solution vient des dérivations induites par diffusion et plus précisément de la dérivation Brownienne. F. Menous avait déjà démontré dans sa thèse l'équianalyticité de cette dérivation avec l'automorphisme de Stokes mais suite à des calculs très complexes relatifs à la moyenne de Catalan. Nous proposons une preuve très raccourcie de ce fait et grâce à un théorème de factorisation, nous concluons alors en l'existence des moyennes recherchées.

Les résultats combinatoires des chapitres 2, 4 ou 5 auraient pu recevoir un traitement plus algébrique en considérant les moules arborifiés ou non comme des caractères, respectivement, sur l'algèbre de Hopf de Connes-Kreimer et sur celle des quasi-shuffles. Nous avons pris le parti ici de rester proche du formalisme d'Ecalle. Dans [60], nous traitons par contre l'arborification de manière purement combinatoire. Nous y re-démontrons nos résultats sur les opérations arborifiés et les appliquons à la théorie des B et P-series.

De la même façon, comme nous l'avons déjà signalé, la théorie des bonnes moyennes développée dans le chapitre 7 peut être totalement re-écrite ([61]) en utilisant les outils des algèbres de Rota-Baxter. La force de ce formalisme est qu'il permet de s'abstraire totalement de la notion de poids, ce qui conduit à grandement simplifier l'exposé. De plus, la factorisation d'Atkinson fournit un procédé naturel de construction des moyennes et permet d'aborder l'arborification sans travail technique préalable.

Les principaux résultats de la thèse sont déjà énoncés dans les travaux d'Ecalle mais avec au mieux des esquisses de leur preuve. Sauf mention du contraire, les démonstrations proposées ici et les résultats intermédiaires sont originaux.

17. Deux opérateurs étrangers sont équianalytiques si pour chaque réduction, quant l'un est analytique alors l'autre l'est aussi.

Chapitre 1

Simple resurgent functions and Stokes automorphism

Break on through to the other side
Jim Morrison

1.1 Resurgent functions

1.1.1 Resurgent functions

The formal Borel transform is a $\mathbb{C}$-linear homomorphism given by :

$$\mathcal{B} : \tilde{\phi}(z) = \sum_{n\geq 0} c_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]] \mapsto \hat{\phi}(\zeta) = \sum_{n\geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]].$$

As a consequence of Cauchy's inequalities, $\hat{\phi}$ is a convergent series and defines an entire function of exponential type if and only if $\tilde{\phi}$ is convergent. Moreover, $\tilde{\phi}$ is a Gevrey-1 series if and only if $\hat{\phi}$ is convergent in a small neighborhood of 0.

We can define a convolution product between two germs of analytic functions

$$(\phi \star \psi)(\zeta) = \int_0^\zeta \phi(\zeta_1)\, \psi(\zeta - \zeta_1)\, \mathrm{d}\zeta_1$$

for $|\zeta|$ small enough.

With this convolution product, $(\mathbb{C}\{\zeta\}, \star)$ get a structure of algebra without unit that is isomorphic via $\mathcal{B}$ to the algebra $\left(z^{-1}\mathbb{C}[[z^{-1}]]_1, \times\right)$.

By adjunction of the formal symbol $\delta := \mathcal{B}(1)$, which can be interpreted as the Dirac distribution, we obtain an isomorphism of unitary algebras :

$$\mathcal{B} : \mathbb{C}[[z^{-1}]] \mapsto \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$$

The counterpart of the differential operator $\frac{\mathrm{d}}{\mathrm{d}z}$ in $\mathbb{C}[[z^{-1}]]$ via $\mathcal{B}$ is the operator $\hat{\partial} : c\delta + \hat{\phi}(\zeta) \mapsto -\zeta\hat{\phi}(\zeta)$.

If $c\delta + \hat{\phi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ admits analytic continuation in the direction θ of the Borel plane, i.e. if there is no singularities for $\hat{\phi}$ on the half line of origin 0 and direction θ, and if this analytic continuation is of sub-exponential growth then we can perform a Laplace transform $\mathcal{L}^\theta$ to $c\delta + \hat{\phi}$:

$$\mathcal{L}^\theta \left(c\delta + \hat{\phi}\right)(z) = c + \int_0^{e^{i\theta}\infty} \hat{\phi}(\zeta)\, e^{-z\zeta}\mathrm{d}\zeta.$$

We obtain an analytic function $\mathcal{L}^\theta \left(c\delta + \hat{\phi}\right)$ defined on an half-plane $\Re\left(ze^{i\theta}\right) \geq c$ where $c \in \mathbb{R}_+^*$.

Definition 1.1.1.1 *Let Ω be a discrete (and so countable) subset of $\mathbb{C}^*$.*

- *We call convolutive model of resurgent functions the sub-algebra $\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}(\Omega)$ of $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ containing elements of the form $c\delta + \hat{\phi}$ where $c \in \mathbb{C}$ and $\hat{\phi}$ is a germ of analytic function that admits an analytic continuation along any rectifiable oriented path $\gamma : [0,1] \to \mathbb{C} \setminus \Omega$.*
- *We call formal model of resurgent functions the sub-algebra $\widetilde{\boldsymbol{RESUR}}_{\mathbb{C}}(\Omega)$ of $\mathbb{C}[[z^{-1}]]$ given by*

$$\widetilde{\boldsymbol{RESUR}}_{\mathbb{C}}(\Omega) := \mathcal{B}^{-1}\left(\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}(\Omega)\right).$$

- *We say that $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{C}}(\Omega)$ is a simple resurgent function if for any $m \in \Omega$ and for any ζ close enough m :*

$$\hat{\phi}(\zeta) = \frac{C}{2i\pi(\zeta - m)} + \frac{1}{2i\pi}\hat{a}(\zeta - m)\log(\zeta - m) + \hat{b}(\zeta - m)$$

where $C \in \mathbb{C}$, $\hat{a}, \hat{b} \in \mathbb{C}\{\zeta\}$. We will denote by $\widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ the convolutive model of simple resurgent functions and by $\widetilde{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ its counterpart in $\mathbb{C}[[z^{-1}]]$ via the inverse Borel transform.

This definition is legitimized by the fact that if $\hat{\phi}$ and $\hat{\psi}$ are resurgent functions, i.e. if they admit analytical continuation along a path avoiding $\mathbb{C}\setminus\Omega$, then so does their convolution product, see [51]. Moreover, the convolution product of two simple resurgent functions is also a simple resurgent functions.

Remark 1.1.1.1 *With the previous notations, $\hat{a}$ and $\hat{b}$ are simple resurgent functions.*

1.1.2 The spaces $\mathbb{C}//\Omega$ and $\mathbb{R}_+//\Omega_+$

In order to simplify the study of resurgent functions, it is convenient to introduce the space $\mathbb{C}//\Omega$ of all homotopy classes $[\gamma]$ of rectifiable oriented paths $\gamma : [0,1] \to \mathbb{C} \setminus \Omega$ of $\mathbb{C}$ starting from 0, avoiding Ω and without coming back.

We denote by π the projection from $\mathbb{C}//\Omega$ to $\mathbb{C} \setminus \Omega$ defined by, for all $[\gamma] \in \mathbb{C}//\Omega : \pi([\gamma]) = \gamma(1)$. If we consider π as a covering map then $\mathbb{C}//\Omega$ gets a Riemann surface structure.

Instead of working with resurgent functions as analytic functions which admits analytic continuation along paths of $\mathbb{C}\setminus\Omega$, we can also work with resurgent function as analytic functions defined on $\mathbb{C}//\Omega$. This is the viewpoint we will adopt here.

Our subject being real resummation, the considered semi-group Ω will be a subset of $\mathbb{R}_+$ or $\mathbb{R}_-$, then one can identify each point $[\gamma]$ of $\mathbb{C}//\Omega$ with its address $(\epsilon_1, \dots, \epsilon_n)$ and its endpoint ζ. Such a notation means that $\gamma : [0,1] \to \mathbb{C} \setminus \Omega$ is a rectifiable oriented paths starting from 0, ending in ζ and in addition that γ avoids each $\omega_1 < \dots < \omega_n \in \Omega$ from the right if $\epsilon_i = +$ or from the left if $\epsilon_i = -$.

Notation 1.1.2.1 *For a given $\zeta^{\epsilon_1,\dots,\epsilon_m} = [\gamma] \in \mathbb{C}//\Omega$ with $\Omega \in \mathbb{R}_+^*$ or $\Omega \in \mathbb{R}_-^*$, we will denote by*

$$\hat{\phi}\left(\zeta^{\epsilon_1,\dots,\epsilon_m}\right)$$

the analytic continuation of $\hat{\phi}$ along the path avoiding the singularities of $\omega_1 < \dots < \omega_m \in \Omega$ from the right if $\epsilon_i = +$ or from the left if $\epsilon_i = -$ and ending in $\zeta \in \mathbb{C} \setminus \Omega$. With the help of this notation, we can see resurgent functions as defined on $\mathbb{C}//\Omega$.

In the case of real summation, we have to deal with the subalgebra of $\widehat{\mathbf{RESUR}}_{\mathbb{C}}(\Omega)$ constituted of the real resurgent functions. We will denote it $\widehat{\mathbf{RESUR}}_{\mathbb{R}}(\Omega)$ and we will call it *convolutive model of real resurgent functions*. It contains functions of the form $c\delta + \hat{\phi}$ where $c \in \mathbb{R}$ and $\hat{\phi}$ is a germ of real analytic function that admits analytic continuation along any rectifiable oriented path $\gamma : [0,1] \to \mathbb{C}\setminus\Omega$. The formal model of real resurgent functions is the sub-algebra $\widetilde{\mathbf{RESUR}}_{\mathbb{R}}(\Omega)$ of $\mathbb{R}[[z^{-1}]]$ given by $\widetilde{\mathbf{RESUR}}_{\mathbb{R}}(\Omega) := \mathcal{B}^{-1}\left(\widehat{\mathbf{RESUR}}_{\mathbb{R}}(\Omega)\right)$.

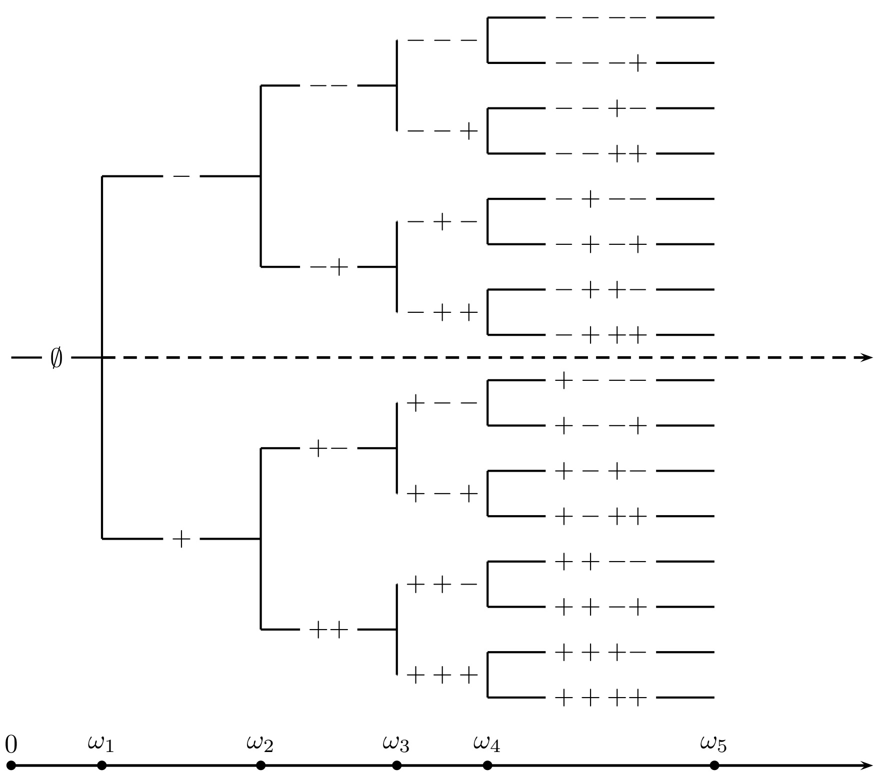
In such a case, we also considered analytic continuations of real resurgent functions along some rectifiable oriented paths $\gamma(t) = (x(t), y(t))$ starting from 0 avoiding $\Omega_+ = \Omega \cap \mathbb{R}_+^*$ (resp. $\Omega_- = \Omega \cap \mathbb{R}_-^*$) and such that for all $t \in [0,1]$, $x'(t) > 0$ (no coming back) (resp. $x'(t) < 0$).

Such construction justifies the introduction of the space $\mathbb{R}_+//\Omega_+$ called ramified line by J. Ecalle and defined as the set of all homotopy classes $[\gamma]$ of rectifiable oriented path $\gamma : [0,1] \to \mathbb{R} \setminus \Omega_+$ of $\mathbb{C}$ starting from 0, ending in $\mathbb{R}^+ \setminus \Omega_+$ $(\gamma(1) \in \mathbb{R}^+ \setminus \Omega_+^*)$, that avoids Ω_+ and such that for all $t \in [0,1]$, $x'(t) > 0$. It is a sub-topological space of $\mathbb{C}//\Omega$.

The set $\mathbb{R}_+//\Omega_+$ can be seen as constituted of :
— one segment $\left[0, \omega_1^1\right[$,
— of two segments $\left]\omega_1^1, \omega_2^i\right[$ with $i = 1, 2$,
— ...,
— of 2^k segments $\left]\omega_{k-1}^i, \omega_k^j\right[$ with $i = 1, ..., k-1$ and $j = 1, 2$,
— etc...
with for any $i \in \mathbf{N}^*$ and $j \in \{1, 2\}$, $\pi\left(\omega_i^j\right) = i$.

The set $\mathbb{R}_-//\Omega_-$ can be constructed symmetrically.

Every point ζ of $\mathbb{R}_+//\Omega_+$ can be identified with its address $(\epsilon_1, \ldots, \epsilon_n)$ which signifies that if $\zeta = [\gamma]$ where $\gamma : [0,1] \to \mathbb{R} \setminus \Omega_+$ is a rectifiable oriented path starting from 0, ending in $]\omega_n, \omega_{n+1}[$ then γ avoids each $\omega_i \in [\![1, n]\!]$ from the right if $\epsilon_i = +$ or from the left if $\epsilon_i = +$.



For a point $\zeta^{\epsilon_1,\ldots,\epsilon_n} \in \mathbb{R}_+//\Omega$ that corresponds to the homotopy class of some path γ starting from 0 and avoiding the points of Ω, we define the point $\overline{\zeta^{\epsilon_1,\ldots,\epsilon_n}} \in \mathbb{R}_+//\Omega$ corresponding to the homotopy class of the paths $\overline{\gamma}$. In fact, we have : $\overline{\zeta^{\epsilon_1,\ldots,\epsilon_n}} = \zeta^{\overline{\epsilon}_1,\ldots,\overline{\epsilon}_n}$ with $\overline{+} = -$ and $\overline{-} = +$. The conjugate of a function $\hat{\phi} : \mathbb{R}_+//\Omega \to \mathbb{C}$ is defined by :

$$\overline{\hat{\phi}} : \begin{cases} \mathbb{C}//\mathbb{Z}^* & \longrightarrow & \mathbb{C} \\ \zeta^{\epsilon_1,\ldots,\epsilon_n} & \mapsto & \overline{\hat{\phi}\left(\overline{\zeta^{\epsilon_1,\ldots,\epsilon_n}}\right)} \end{cases} .$$

Proposition 1.1.1 *If* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{R}}^{s}$ *then* $\overline{\hat{\phi}} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{R}}^{s}$*. Moreover, for any* ζ *close enough to* $m \in \Omega$*, if we get*

$$\hat{\phi}(\zeta) = \frac{C}{2i\pi(\zeta - m)} + \frac{1}{2i\pi}\hat{a}(\zeta - m)\log(\zeta - m) + \hat{b}(\zeta - m)$$

where $C \in \mathbb{R}$*,* $\hat{a}, \hat{b} \in \mathbb{R}\{\zeta\}$ *then*

$$\overline{\hat{\phi}}(\zeta) = -\frac{\overline{C}}{2i\pi(\zeta - m)} - \frac{1}{2i\pi}\overline{\hat{a}}(\zeta - m)\log(\zeta - m) + \overline{\hat{b}}(\zeta - m).$$

Proof As long as $\hat{\phi}$ is a real germ at 0, we get in a small neighborhood of 0, $\hat{\phi} - \overline{\hat{\phi}} = 0$. But the analytic continuation of the null resurgent function is the null function and so for any $\zeta^{\epsilon_1,\ldots,\epsilon_n} \in \mathbb{C}//\mathbb{Z}^*$, $\left(\overline{\hat{\phi}} - \hat{\phi}\right)(\zeta^{\epsilon_1,\ldots,\epsilon_n}) = 0$. Then one has $\overline{\hat{\phi}} = \hat{\phi}$ on $\mathbb{C}//\mathbb{Z}^*$. We then obtain the expression of $\overline{\hat{\phi}}$ in a neighborhood of m as a direct consequence. □

1.2 Stokes automorphism

1.2.1 The operators Δ^+ and Δ^-

We consider a set of singularities constituted of a semi-group $\Omega \subset \mathbb{R}_+^*$ and the associated simple resurgent functions algebra $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$. For $m \in \Omega$, we want to define two operators Δ_m^+ and Δ_m^- on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ that may measure, for a given simple resurgent function $\hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$, the behavior of $\hat{\phi}$ in the neighborhood of the singularity m. As in [50], we consider $m \in \Omega$ and a path γ starting from 0, avoiding Ω and ending in a small neighborhood of m. We consider the analytic continuation $\hat{\phi}_\gamma$ of $\hat{\phi}$ along this path. We know that in a neighborhood of m, $\hat{\phi}_\gamma$ reads

$$\hat{\phi}_\gamma(\zeta) = \frac{C}{2i\pi(\zeta - m)} + \frac{1}{2i\pi}\hat{a}(\zeta - m)\log(\zeta - m) + \hat{b}(\zeta)$$

where $C \in \mathbb{C}$, $\hat{a}, \hat{b} \in \mathbb{C}\{\zeta\}$. As $\hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$, we must have $\hat{a}, \hat{b} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ too. Moreover, we get, for ζ close enough to 0 :

$$\hat{\phi}_\gamma(m + \zeta) - \hat{\phi}_\gamma\left(m + \zeta e^{-2i\pi}\right) = \hat{a}(\zeta) \text{ and } C = \mathbf{res}_m(\phi_\gamma)$$

where $\mathbf{res}_m(\phi_\gamma)$ is the residue of ϕ_γ in m.

We then introduce the operator $\mathbf{Sing}_{m,\gamma}$ defined on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ by

$$\mathbf{Sing}_{m,\gamma}\left(\hat{\phi}\right) = C\delta + \hat{a}.$$

It is well defined on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ and it depends not on the path γ but on the homotopy class of this path.

Notation 1.2.1.1 *It will be convenient for* $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_s) \in \{\pm\}^s$ *to introduce the notation* $(\epsilon_1)^{r_1} \ldots (\epsilon_s)^{r_s}$ *meaning :*

$$(\epsilon_1)^{r_1} \ldots (\epsilon_s)^{r_s} = \underbrace{\epsilon_1, \ldots, \epsilon_1}_{r_1 \ terms}, \ldots, \underbrace{\epsilon_s, \ldots, \epsilon_s}_{r_s \ terms}.$$

Definition 1.2.1.1 *For any* $m \in \Omega$*, we define the operators* Δ_m^+ *and* Δ_m^- *by their action, called proper action, on* $\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$*. For* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$ *and* ζ *in a punctured positive small enough neighborhood of* 0 *(i.e.* $0 < |\zeta| < \omega_{m+1} - \omega_m$ *and* $0 < |\zeta| < \omega_1$*), we set :*

$$\begin{aligned}\left(\Delta_m^+\left(\hat{\phi}\right)\right)(\zeta) &= \boldsymbol{Sing}_{m,(\zeta+m)^{(+)^{m-1}}}\left(\hat{\phi}\right)(\zeta)\\ &= \hat{\phi}\left((m+\zeta)^{(+)^{m-1},+}\right) - \hat{\phi}\left((m+\zeta)^{(+)^{m-1},-}\right) + \boldsymbol{res}_0\left(\hat{\phi}\left((m+\zeta)^{(+)^{m-1}}\right)\right)\delta\end{aligned}$$

$$\begin{aligned}\left(\Delta_m^-\left(\hat{\phi}\right)\right)(\zeta) &= \boldsymbol{Sing}_{m,(\zeta+m)^{(-)^{m-1}}}\left(\hat{\phi}\right)(\zeta)\\ &= \hat{\phi}\left((m+\zeta)^{(-)^{m-1},-}\right) - \hat{\phi}\left((m+\zeta)^{(-)^{m-1},+}\right) - \boldsymbol{res}_0\left(\hat{\phi}\left((m+\zeta)^{(-)^{m-1}}\right)\right)\delta\end{aligned}$$

Remark 1.2.1.1

— *Using notations of subsection 1.1.2, one has for* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$ *and* ζ *in a punctured positive small enough neighborhood of* 0

$$\left(\Delta_m^+\left(\hat{\phi}\right)\right)(\zeta) = \hat{\phi}\left(\zeta^{(+)^{m-1},+}\right) - \hat{\phi}\left(\zeta^{(+)^{m-1},-}\right) + \boldsymbol{res}_0\left(\hat{\phi}\left(\zeta^{(+)^{m-1}}\right)\right)\delta$$

— *The operators* $\Delta_m^{\pm}$ *send* $\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$ *onto itself. It is a consequence of remark 1.1.1.1.*

— *If* ω *is not a singular point for* $\hat{\phi}$ *then* $\Delta_\omega^{\pm}\left(\hat{\phi}\right) = 0$*. The reciprocal is false, see footnote 14 of [50].*

— *Using the inverse Borel transform, we can define the action of* $\Delta_m^{\pm}$ *on* $\widetilde{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$*.*

Proposition 1.2.1 *For any* $m \in \Omega$*, we have* $\left[\Delta_m^{\pm}, \hat{\partial}\right] = -m\Delta_m^{\pm}$*.*

Proof Let us consider $\hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^s(\Omega)$ and ζ small enough. We know that :

$$\hat{\phi}\left((\zeta+m)^{(+)^{m-1}}\right) = \frac{c}{2i\pi\zeta} + \frac{1}{2i\pi}\hat{a}(\zeta)\log(\zeta) + \hat{b}(\zeta)$$

where $\hat{a}, \hat{b} \in \mathbb{C}\{\zeta\}$ and $c \in \mathbb{C}$. Then

$$\begin{aligned}(\zeta+m)^{(+)^{m-1}}\hat{\phi}\left((\zeta+m)^{(+)^{m-1}}\right) &= (\zeta+m)\frac{c}{2i\pi\zeta} + \frac{1}{2i\pi}(\zeta+m)\hat{a}(\zeta)\log(\zeta) + (\zeta+m)\hat{b}(\zeta)\\ &= \frac{mc}{2i\pi\zeta} + \frac{1}{2i\pi}(\zeta+m)\hat{a}(\zeta)\log(\zeta) + \frac{c}{2i\pi} + (\zeta+m)\hat{b}(\zeta)\end{aligned}$$

and $\Delta_m^+\left(\hat{\partial}\hat{\phi}\right)(\zeta) = -mc\delta - (\zeta+m)\,\hat{a}(\zeta)$. As long as $\hat{\partial}\left(\Delta_m^+\hat{\phi}\right)(\zeta) = -\zeta\hat{a}(\zeta)$, we obtain

$$\left(\left[\Delta_m^+, \hat{\partial}\right]\hat{\phi}\right)(\zeta) = -mc\delta - m\hat{a}(\zeta) = -m\Delta_m^{\pm}\left(\hat{\phi}\right)(\zeta)$$

□

1.2.2 Dotted operators

We will now introduce the dotted operators $\dot{\Delta}_m^{\pm}$. They act on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ but in a different way than the non-dotted ones $\Delta_m^{\pm}$.

There is at least three good reasons to define such operators :

— First, for any $m \in \Omega$, we will have $\left[\dot{\Delta}_m^{\pm}, \hat{\partial}\right] = 0$.

— Then, the **ALIEN** operators that will intervenes in the Bridge Equation in chapter 3 are the dotted ones.

— To finish, the infinite sums [1] $\sum_{m\geq 0}\dot{\Delta}_m^{\pm}$ are well defined on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$, whereas it is not the case of $\sum_{m\geq 0}\Delta_m^{\pm}$.

Before introducing them, and in order to give a meaning to the $\dot{\Delta}_{\underset{\sim}{\omega}}^{\pm}$ as internal operators, it will be necessary to consider the graded algebra $\widetilde{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)\left[\left[e^{-z}\right]\right]$ instead of $\widetilde{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ (which is obviously a sub-algebra of this algebra). As already performed in subsection 1.1.1 when introducing the formal symbol δ, we decide to denote by the symbol δ_m the Borel transform of e^{-mz}. This symbol can be identify with the Dirac distribution to the point m defined for any test function f by $\langle \delta_m, f\rangle = f(m)$ [2]. First of all, let us recall some basic facts about distributions.

We denote by $\mathcal{D}(\mathbb{C}//\Omega)$ the space of test functions on the Riemann surface $\mathbb{C}//\Omega$ fitted of its usual topology and by $\mathcal{D}'(\mathbb{C}//\Omega)$ the linear dual of this space. By definition, it is the space of distributions on $\mathcal{D}(\mathbb{C}//\Omega)$. We denote by $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}//\Omega)$ the space of locally integrable functions on $\mathbb{C}//\Omega$. To each function $\phi \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}//\Omega)$ one can associate a distribution $T_\phi \in \mathcal{D}'(\mathbb{C}//\Omega)$ given for any test function $f \in \mathcal{D}(\mathbb{C}//\Omega)$ by

$$T_\phi(f) = \int_{\mathbb{C}//\Omega} f(z)\phi(z)\mathrm{d}z.$$

The space $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+//\Omega)$ of locally integrable functions on the ramified line is usually denoted $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$. It can be identified to a sub-space of $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}//\Omega)$ by assuming locally integrable functions on $\mathbb{R}_+//\Omega$ as null on $(\mathbb{C}//\Omega)\setminus(\mathbb{R}_+//\Omega)$. One has naturally $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega) \subset \mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$.

1. where $\dot{\Delta}_0^{\pm} = \mathrm{i}d$.

2. Indeed, when applying Laplace transform $\mathcal{L}$ on the distribution δ_m, one obtains $\mathcal{L}(\delta_m) = \left\langle \delta_m, e^{-tz}\right\rangle = e^{-mz}$.

The convolution between a distribution $T \in \mathcal{D}'(\mathbb{C}//\Omega)$ and a test function $\phi \in \mathcal{D}(\mathbb{C}//\Omega)$ is defined by

$$(T \star f)(x) = T\left(\tau_x \check{f}\right)$$

where $\check{f}(x) = f(-x)$, $\tau_x(f)(.) = f(. - x)$. Note that

$$\left(\tau_x \check{f}\right)(y) = \check{f}(y - x) = f(x - y).$$

If $T = T_\phi$ with $\phi \in \mathcal{L}^1_{\text{loc}}(\mathbb{C}//\Omega)$, then one obtains for any $z' \in \mathbb{C}//\Omega$:

$$(T_\phi \star f)(z') = \int_{\mathbb{C}//\Omega} \phi(z) f(z' - z) \mathrm{d}z.$$

If U is an open subset of $\mathbb{C}//\Omega$ on which $T(\phi) = 0$ for any test function f with compact support included in U then we say that $T \in \mathcal{D}'(\mathbb{C}//\Omega)$ vanishes on U. If V is the union of all such open sets U, then the support of T is the complementary of V in $\mathbb{C}//\Omega$.

Following [49], for any distributions $S, T \in \mathcal{D}'(\mathbb{C}//\Omega)$, assuming at least one of these two distributions to be compactly supported, the convolution $S \star T$ is defined for any test function $f \in \mathcal{D}(\mathbb{C}//\Omega)$ by

$$(T \star S) \star f = T \star (S \star f).$$

The Dirac distribution δ is the unit for the convolution product. Let us observe that, for any $m \in \Omega$, the functional δ_m is not an element of $\mathcal{D}'(\mathbb{C}//\Omega)$. But for any locally integrable function ϕ, the convolution $\delta_m \star T_\phi$ is yet a distribution of $\mathcal{D}'(\mathbb{C}//\Omega)$. Indeed, for any test function $f \in \mathcal{D}(\mathbb{C}//\Omega)$, one has :

$$\begin{aligned}
((\delta_m \star T_\phi) \star f)\, x &= \delta_m \left(y \mapsto \tau_x (T_\phi \check{\star} f)(y)\right) \\
&= \delta_m \left(y \mapsto (T_\phi \star f)(x - y)\right) \\
&= \delta_m \left(y \mapsto \int_{\mathbb{C}//\Omega} \phi(t) f(x - y - t) \mathrm{d}t\right) \\
&= \int_{\mathbb{C}//\Omega} \phi(t) f(x - m - t) \mathrm{d}t \\
&= \int_{\mathbb{C}//\Omega} \phi(t - m) f(x - t) \mathrm{d}t
\end{aligned}$$

where the last equality is obtained using the change of variable $t \to m + t$. Thus one has $\delta_m \star T_\phi = T_{\tau_m(\phi)}$.

Considering ϕ a locally integrable function on $\mathbb{R}_+//\Omega$, one has, identifying T_ϕ with ϕ

$$(\delta_m \star \phi)\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right) = \begin{cases} \phi\left(\zeta^{\alpha_1,\ldots,\alpha_{n-m}}\right) & \text{if } n \geq m \\ 0 & \text{otherwise} \end{cases}. \tag{1.1}$$

because elements of $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+//\Omega)$ are null everywhere outside $\mathbb{R}_+//\Omega$.

We can sum up what precedes in here below :

$$(\delta_m \star (\alpha\delta + \phi))\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right) = \begin{cases} \phi\left(\zeta^{\alpha_1,\ldots,\alpha_{n-m}}\right) & \text{if } n \geq m \\ 0 & \text{otherwise} \end{cases}$$

and it is now possible to define the dotted operators.

Definition 1.2.2.1 *We define for any $m \in \Omega$, the operators $\dot{\Delta}^{\pm}_m$ by their action, called stationnary action, on $\widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$. For any $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ and any $\zeta^{\alpha_1,\ldots,\alpha_n} \in \mathbb{C}//\Omega$ we set*

$$\dot{\Delta}^{\pm}_m\left(\hat{\phi}\right)\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right) = \left(\delta_m \star \Delta^{\pm}_m\left(\hat{\phi}\right)\right)\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right).$$

The counterpart of the operators $\dot{\Delta}^{\pm}_{\omega} = \delta_{\omega} \star \Delta^{\pm}_{\omega}$ in the formal model are naturally the operators $\dot{\Delta}^{\pm}_{\omega} = e^{-\omega z}\Delta^{\pm}_{\omega}$.

Let us observe that the operators $\dot{\Delta}^{\pm}_{\omega}$ are perverseness graduation operators because $\dot{\Delta}^{\pm}_m e^{-nz}\widetilde{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{C}}(\Omega) \subset e^{-(m+n)z}\widetilde{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{C}}(\Omega)$.

Remark 1.2.2.1 *Using calculations of remark 1.2.1.1, we have for any $(\alpha_1,\ldots,\alpha_n) \in \{\pm\}^n$, $m \in \Omega$ and $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$*

$$\begin{aligned} &\left(\dot{\Delta}^{+}_m\hat{\phi}\right)\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right) \\ = &\begin{cases} \left(\Delta^{+}_m\hat{\phi}\right)\left(\zeta^{\alpha_1,\ldots,\alpha_{n-m}}\right) & \textit{if} \quad m \leq n \\ 0 & \textit{if} \quad m > n \end{cases} \\ = &\begin{cases} \hat{\phi}\left(\zeta^{(+)^{m-1},+,\alpha_1,\ldots,\alpha_{n-m}}\right) - \hat{\phi}\left(\zeta^{(+)^{m-1},-,\alpha_1,\ldots,\alpha_{n-m}}\right) & \textit{if} \quad m \leq n \\ 0 & \textit{if} \quad m > n \end{cases} \\ = &\begin{cases} \sum_{\epsilon_m=\pm}\epsilon_m\phi\left(\zeta^{(+)^{m-1},\epsilon_m,\alpha_1,\ldots,\alpha_{n-m}}\right) & \textit{if} \quad m \leq n \\ 0 & \textit{if} \quad m > n \end{cases}. \end{aligned}$$

In the same way, one has

$$\left(\dot{\Delta}^{-}_m\hat{\phi}\right)\left(\zeta^{\alpha_1,\ldots,\alpha_n}\right) = \begin{cases} \sum_{\epsilon_1,\ldots,\epsilon_m=\pm} -\epsilon_m\phi\left(\zeta^{(-)^{m-1},\epsilon_m,\alpha_1,\ldots,\alpha_{n-m}}\right) & \textit{if} \quad m \leq n \\ 0 & \textit{if} \quad m > n \end{cases}.$$

A direct consequence is that the infinite sum $\sum_{m\geq 0}\dot{\Delta}^{\pm}_m$ is well defined because it becomes finite when evaluating it on a resurgent function at a given point. We then introduce the formal operators $\dot{\Delta}^{\pm} = id + \sum_{m\in\Omega}\dot{\Delta}^{\pm}_m$. The operator $\dot{\Delta}^{+}$ (or $\dot{\Delta}^{-}$) is the Stokes automorphism along the directions $\mathbb{R}_+$ or $\mathbb{R}_-$, see Proposition 1.2.4.

Example 1.2.2.1 *In the case of a resurgent function $\hat{\phi}$ with only one singularity located at $m = 1$ (a situation similar to the solution of the Euler equation described in the introduction) one has :*

$$\begin{aligned}
\left(\dot{\Delta}^{+}\hat{\phi}\right)(\zeta^{--}) &= \left(\left(id + \dot{\Delta}_1^{+}\right)\hat{\phi}\right)(\zeta^{--}) \\
&= \hat{\phi}(\zeta^{--}) + \phi(\zeta^{+-}) - \phi(\zeta^{--}) \\
&= \hat{\phi}(\zeta^{+-}) \\
&= \hat{\phi}(\zeta^{++})
\end{aligned}$$

because there is no singularity at $m = 2$.

1.2.3 Properties of the Stokes automorphism

As announced before, for any $m \in \Omega$ the operators $\dot{\Delta}^{\pm}$ and ∂_z commute :

Proposition 1.2.2 *For any $m \in \Omega$, we have $\left[\dot{\Delta}_m^{\pm}, \hat{\partial}\right] = 0$.*

Proof The proof is easier in the formal model :

$$\begin{aligned}
\left[\dot{\Delta}_m^{\pm}, \partial\right] &= e^{-mz}\Delta_m^{\pm}.\partial - \partial\left(e^{-mz}\dot{\Delta}_m^{\pm}.\right) \\
&= e^{-mz}\Delta_m^{\pm}.\partial + me^{-mz}\Delta_m^{\pm} - e^{-mz}\partial.\Delta_m^{\pm} \\
&= e^{-mz}\left[\Delta_m^{\pm}, \hat{\partial}\right] + me^{-mz}\Delta_m^{\pm} \\
&= -me^{-mz}\Delta_m^{\pm} + me^{-mz}\Delta_m^{\pm} \\
&= 0
\end{aligned}$$

because of Proposition 1.2.1. □

We can define the product of the operators Δ^{+} and Δ^{-} by their action on $\widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$:

$$\forall \hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega), \quad \left(\dot{\Delta}^{+}.\dot{\Delta}^{-}\right)\hat{\phi} = \dot{\Delta}^{+}.\left(\dot{\Delta}^{-}\hat{\phi}\right).$$

Proposition 1.2.3 *We have $\dot{\Delta}^{+}.\dot{\Delta}^{-} = \dot{\Delta}^{-}.\dot{\Delta}^{+} = 1$.*

Proof For $\hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{C}}^{\mathrm{s}}(\Omega)$ and $\zeta^{\epsilon_1,\ldots,\epsilon_n} \in \mathbb{C}//\Omega$, we have :

$$\left(\dot{\Delta}^{+}.\dot{\Delta}^{-}\right)\left(\hat{\phi}\right)(\zeta^{\epsilon_1,\ldots,\epsilon_n}) = \left(1 + \sum_{m=1}^{n} \sum_{\substack{r+s=m \\ r,s \geq 0}} \left(\dot{\Delta}_r^{+}.\dot{\Delta}_s^{-}\right)\right)\left(\hat{\phi}\right)(\zeta^{\epsilon_1,\ldots,\epsilon_n}).$$

Let us take $m \in [\![1,n]\!]$. If for example $m=2$ it comes,

$$\begin{aligned}
\left(\dot{\Delta}^{+}.\dot{\Delta}^{-}\right)\left(\hat{\phi}\right)\left(\zeta^{\epsilon_1,\ldots,\epsilon_n}\right) &= \left(\dot{\Delta}_2^{+}+\dot{\Delta}_1^{+}\dot{\Delta}_1^{-}+\dot{\Delta}_2^{-}\right)\left(\hat{\phi}\right)\left(\zeta^{\epsilon_1,\ldots,\epsilon_n}\right)\\
&= \hat{\phi}\left(\zeta^{+,+,\epsilon_2,\ldots,\epsilon_n}\right)-\hat{\phi}\left(\zeta^{+,-,\ldots,\epsilon_n}\right)\\
&\quad +\hat{\phi}\left(\zeta^{-,+,\ldots,\epsilon_n}\right)-\hat{\phi}\left(\zeta^{-,-,\ldots,\epsilon_n}\right)\\
&\quad -\hat{\phi}\left(\zeta^{+,+,\ldots,\epsilon_n}\right)+\hat{\phi}\left(\zeta^{+,-,\ldots,\epsilon_n}\right)\\
&\quad +\hat{\phi}\left(\zeta^{-,-,\epsilon_2,\ldots,\epsilon_n}\right)-\hat{\phi}\left(\zeta^{-,+,\ldots,\epsilon_n}\right)\\
&= 0
\end{aligned}$$

In the general case, one has :

$$\sum_{\substack{r+s=m\\ r,s\geq 0}} \left(\dot{\Delta}_r^{+}.\dot{\Delta}_s^{-}\right)\left(\hat{\phi}\right)\left(\zeta^{\epsilon_1,\ldots,\epsilon_n}\right) =$$

$$\sum_{\substack{r+s=m\\ r,s\geq 0}} [\ \hat{\phi}\left(\zeta^{(++)^r(--)^s,\epsilon_1,\ldots,\epsilon_{n-m}}\right)-\hat{\phi}\left(\zeta^{(+-)^r(--)^s,\epsilon_1,\ldots,\epsilon_{n-m}}\right)$$

$$-\hat{\phi}\left(\zeta^{(++)^r(-+)^s,\epsilon_1,\ldots,\epsilon_{n-m}}\right)+\hat{\phi}\left(\zeta^{(+-)^r(-+)^s,\epsilon_1,\ldots,\epsilon_{n-m}}\right)\ ]$$

where $(++)^r = (\underbrace{+,\ldots,+}_{r-1 \text{ plus signs}},+)$ if $r\geq 1$ and $(+)^r$ is the empty sequence if $r=0$.
It is not difficult to see that all the terms in this sum compensate themselves and thus, that this sum is null. The result follows. □

Proposition 1.2.4 *For any $\hat{\phi}\in\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^{s}(\Omega)$ we have :*

$$\forall \zeta^{+,\ldots,+}\in\mathbb{R}_+//\Omega,\quad \left(\dot{\Delta}^{-}\hat{\phi}\right)\left(\zeta^{+,\ldots,+}\right)=\hat{\phi}\left(\zeta^{-,\ldots,-}\right).$$

In other words, Δ^{-} exchanges the analytic continuation of $\hat{\phi}$ along an half line starting from 0 and contained in the superior half plane with its symmetric compared with the real axes.

Proof We compute :

$$\begin{aligned}
\left(\dot{\Delta}^{-}\hat{\phi}\right)\left(\zeta^{(+)^m}\right) &= \left(1+\dot{\Delta}_1^{-}+\ldots+\dot{\Delta}_m^{-}\right)\left(\hat{\phi}\right)\left(\zeta^{(+)^m}\right)\\
&= \phi\left(\zeta^{(+)^m}\right)+\sum_{i=1}^{m}\left(\phi\left(\zeta^{(-)^i(+)^{m-i}}\right)-\phi\left(\zeta^{(-)^{i-1}(+)^{m-i+1}}\right)-C_{(-)^i}\right)\\
&= \hat{\phi}\left(\zeta^{(-)^m}\right)
\end{aligned}$$

□

Proposition 1.2.5 *For any* $\hat{\phi},\hat{\psi}\in\widehat{\boldsymbol{RESUR}}^{s}_{\mathbb{C}}(\Omega)$ *and any* $\omega\in\Omega$ *:*

$$\dot{\Delta}^{\pm}_{\omega}\left(\hat{\phi}\star\hat{\psi}\right)=\left(\dot{\Delta}^{\pm}_{\omega}\hat{\phi}\right)\star\hat{\psi}+\sum_{\substack{\omega_1+\omega_2=\omega\\ \omega_1,\omega_2\in\Omega}}\left(\dot{\Delta}^{\pm}_{\omega_1}\hat{\phi}\right)\star\left(\dot{\Delta}^{\pm}_{\omega_2}\hat{\psi}\right)+\hat{\phi}\star\left(\dot{\Delta}^{\pm}_{\omega}\hat{\psi}\right).$$

Proof The proof is easier in the geometrical model. For $\hat{\phi},\hat{\psi}\in\widehat{\mathbf{RESUR}}^{s}_{\mathbb{C}}(\Omega)$, one has

$$\mathcal{L}^{0^-}\left(\hat{\phi}\star\hat{\psi}\right) = \mathcal{L}^{0^+}\dot{\Delta}^{-}\left(\hat{\phi}\star\hat{\psi}\right)$$

but also

$$\begin{aligned}\mathcal{L}^{0^-}\left(\hat{\phi}\star\hat{\psi}\right) &= \mathcal{L}^{0^-}\hat{\phi}.\mathcal{L}^{0^-}\hat{\psi}\\ &= \mathcal{L}^{0^+}\dot{\Delta}^{-}\hat{\phi}.\mathcal{L}^{0^+}\dot{\Delta}^{-}\hat{\psi}\end{aligned}$$

and so, the Laplace transform being bijective, $\dot{\Delta}^{-}\left(\hat{\phi}\star\hat{\psi}\right)=\dot{\Delta}^{-}\hat{\phi}\star\dot{\Delta}^{-}\hat{\phi}$ from which follows the result.

□

Chapitre 2

Algebra reminders

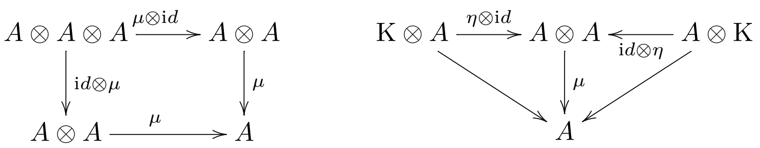
Johner : Hey, Ripley. I heard you, like, ran into these things before ?
Ripley : That's right.
Johner : Wow, man. So, like, what did you do ?
Ripley : I died.
Jean-Pierre Jeunet - Alien, Resurrection

2.1 Algebras, coalgebras, bialgebras and Hopf algebras

We begin this chapter by some reminders about algebra, coalgebra, bialgebra and Hopf algebra. We invite the reader to refer to [19] for more details and explanations.

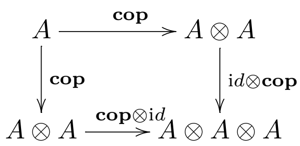
2.1.1 Algebras and coalgebras

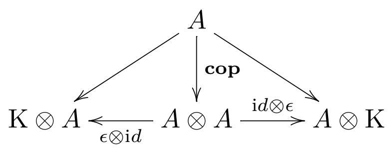
We first recall that an *algebra* A over a field K is defined as a vector space over K with a linear *multiplication* $\mu : A \otimes A \to A$ and a unit given by a linear map $\eta : K \to A$ in a such way that the following diagrams commute :



The first diagram ensures that the algebra is associative. If we set $1_A := \eta(1)$, the second brings us to $\mu(1_A \otimes x) = \mu(x \otimes 1_A) = x$ for any $x \in A$.

We now define the notion of *coalgebra* by dualising the previous diagrams. One says that the vector space A over the field K is a coalgebra if there is two linear maps, a *comultiplication* $\mathbf{cop}: A \to A \otimes A$ and a *counit* $\epsilon : A \to \mathrm{K}$ such that the two following diagrams commute :





The first diagram is a translation of the fact that the comultiplication is coassociative.

It is convenient at this point to introduce Sweedler's notations. For x an element of a coalgebra A, one writes :

$$\mathbf{cop}\,(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}.$$

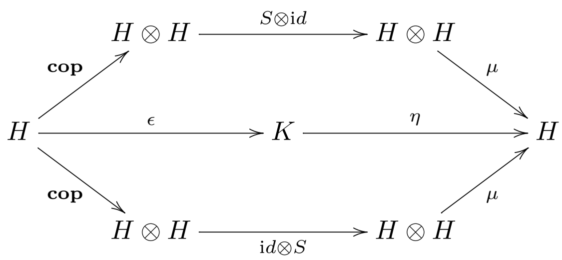
Using it, the second diagram ensures that $x = \sum_{(x)} \epsilon\left(x^{(1)}\right) x^{(2)} = \sum_{(x)} \epsilon\left(x^{(2)}\right) x^{(1)}$. The coalgebra A is said to be *cocommutative* if for any $x \in A$,

$$\mathbf{cop}\,(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} = \sum_{(x)} x^{(2)} \otimes x^{(1)}.$$

In a coalgebra A an element $a \in A$ is said to be :
— *group-like* if $\mathbf{cop}\,(a) = a \otimes a$.
— *primitive* if $\mathbf{cop}\,(a) = a \otimes 1 + 1 \otimes a$.

2.1.2 Bialgebras and Hopf algebras

A vector space that is both an algebra and a coalgebra with compatibility between both structures (see [19] for details) is said to be a *bialgebra*. Finally, an *Hopf algebra* is a bialgebra H with a linear map $S : H \to H$ called *antipode* so that the following diagram commutes :



which can be written using Sweedler's notation :

$$\sum_{(x)} S\left(x^{(1)}\right) x^{(2)} = \sum_{(x)} x^{(1)} S\left(x^{(2)}\right) = \eta \circ \epsilon\,(x)$$

for any $x \in H$.

2.1.3 Graduation

When one can split a bialgebra A into sub-spaces A_m each one with different degrees m in a such way that $A = \oplus_{m\in\mathbf{N}} A_m$ and that :
— $A_0 = K$
— $\mu\,(A_i \otimes A_j) \subset A_{i+j}$
— $\mathbf{cop}\,(A_m) \subset \oplus_{i+j=m} A_i \otimes A_j$.
then $(A, \mu, \eta, \mathbf{cop}, \epsilon)$ is called a *graded bialgebra*.

An important result concerning graded bialgebra is that if A is a graded bialgebra with A_0 one-dimensional (A is said to be *connected*) then A is in fact an Hopf algebra with antipode given by $S(1_H) = 1_H$ and inductively by one of the two formulas :

$$S(x) = -x - \sum_{(x)} S\left(x^{(1)}\right) x^{(2)}$$

$$S(x) = -x - \sum_{(x)} x^{(1)} S\left(x^{(2)}\right)$$

for $x \in \ker \epsilon$.

These definitions and properties are still right when considering any totally ordered finite or countable monoid Ω instead of $\mathbf{N}$. Such a bialgebra graded by Ω instead of $\mathbf{N}$ will be called an Ω-*graded bialgebra*. This notion was already investigated in [60] and it is the correct way to formalize things relative for example to B or P-series.

2.1.4 The concatenation algebra

For a totally ordered and commutative monoid $\Omega \subset \mathbb{C}$, we introduce the set $\Omega^\bullet$ of all words with letters are in Ω plus the empty one. For a reason that will become clearer after, we decide to write $\mathbf{e}_{\underline{\omega}} = \mathbf{e}_{\omega_r} \ldots \mathbf{e}_{\omega_1}$ the word $\underline{\omega} = (\omega_1, \ldots, \omega_r)$. We consider the linear span $< \Omega^\bullet >$ of $\Omega^\bullet$. We turn this space into an algebra by considering the product μ given by words concatenations. The unit $\eta : \mathrm{K} \to < \Omega^\bullet >$ is defined by $\eta\,(1) = \emptyset$ and we will denote it by 1.

There are two natural ways to define a coproduct on $< \Omega^\bullet >$. The first one is to consider the words of length one as primitive

$$\forall m \in \Omega, \quad \mathbf{cop}\,(\mathbf{e}_m) = \mathbf{e}_m \otimes 1 + 1 \otimes \mathbf{e}_m \tag{2.1}$$

and the second one to consider them as group-like

$$\forall m \in \Omega, \quad \mathbf{cop}\left(\mathbf{e}_m\right) = \sum_{i+j=m} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.2}$$

We consider $< \Omega^\bullet >$ with one of these two coproducts. We define a counit on $< \Omega^\bullet >$ by setting $\epsilon(\emptyset) = 1$ and $\epsilon\left(\mathbf{e}_{\underline{\omega}}\right) = 0$ if $\mathrm{l}(\underline{\omega}) \geq 1$. It is easy to verify that $(< \Omega^\bullet >, \mu, \eta, \mathbf{cop}, \epsilon)$ is a bialgebra.

The algebra $< \Omega^\bullet >$ is clearly Ω-graded by the norm $\|\ \|$ of words : if $\underline{\omega} \in \Omega^\bullet$ then

$$\|\mathbf{e}_{\underline{\omega}}\| := \|\left(\omega_1, \ldots, \omega_r\right)\| = \omega_1 + \ldots + \omega_r.$$

2.1.5 Krull topology

In fact, we need to work no only with finite sums of elements of $< \Omega^\bullet >$ but also infinite ones. It is necessary to this end to introduce a topology on $\Omega^\bullet$. Let us first recall the definition of a summable family.

Definition 2.1.5.1 *In a normed vector space $(E, |||.|||)$, a family $(u_i)_{i\in I}$ is summable of sum $u \in E$ if and only if*

$$\forall \epsilon > 0, \quad (\exists J_0 \subset I, J_0 \text{ finite}) : \quad (\forall J \subset I, J \text{ finite}, J_0 \subset J), \quad |||\sum_{j\in J} u_j - u||| \leq \epsilon.$$

When a family $(u_i)_{i\in I}$ is summable then [1]

$$\forall \epsilon > 0, \quad (\exists J_0 \subset I, J_0 \text{ finite}) : \quad (\forall J \subset I, J \text{ finite}, J \subset I \setminus J_0), \quad |||\sum_{i\in J} u_i||| \leq \epsilon.$$

A family $(u_i)_{i\in I}$ satisfying this last property is called a *Cauchy family.*

To each exhaustion $(\mathcal{I}_k)_{k\in\mathbf{N}}$ of I by finite sets one can associate a sequence $(U_k)_{k\in\mathbf{N}}$ defined for any $k \in \mathbf{N}$ by $U_k = \sum_{i\in\mathcal{I}_k} u_i$.

Proposition 2.1.1 *The family $(u_i)_{i\in I}$ is summable in E of sum u if and only if for any exhaustion $(\mathcal{I}_k)_{k\in\mathbf{N}}$ of I by finite sets, the sequence (U_k) satisfies $U_k \xrightarrow[k\to+\infty]{} u$.*

Proof Assuming the family $(u_i)_{i\in I}$ to be summable of sum u and considering $\epsilon > 0$, then there is a finite $I_0 \subset I$ such that for any finite $I_0 \subset J \subset I$, $|||\sum_{i\in J} u_i - u||| \leq \epsilon$. Consider an exhaustion $(\mathcal{I}_k)_{k\in\mathbf{N}}$ of I by finite set. There

1. Indeed, for $\epsilon > 0$, there exists $J_0 \subset I$ finite is a such way that for any finites $J_0 \subset J_1 \subset I$, $J_0 \subset J_2 \subset I$ one has $|||\sum_{j\in J_1} u_j - u||| \leq \epsilon/2$ and $|||\sum_{j\in J_2} u_j - u||| \leq \epsilon/2$. Then for a finite $J \subset I \setminus J_0$, there is some finites $J_0 \subset J_1 \subset I$ and $J_0 \subset J_2 \subset I$ such that $J = J_1 \setminus J_2$. Thus it comes $|||\sum_{j\in J} u_j||| \leq \epsilon$.

is $N \in \mathbf{N}$ such that for any $n \geq N$, $I_0 \subset \mathcal{I}_n$ and then $|||U_n - u||| \leq \epsilon$ for any $n \geq N$ as needed.

Conversely, we assume the family (u_i) to be not summable. Then for any $u \in E$, there exists $\epsilon > 0$ such that for any finite $I_0 \subset I$, there is a finite $I_0 \subset J \subset I$ such that $|||\sum_{i \in J} u_i - u||| \geq \epsilon$. In order to precise the dependencies of J relatively to I_0 we write it $J(I_0)$. We consider now an exhaustion $(\mathcal{I}_k)$ of I by finite sets and we construct from it by induction an other exhaustion $\mathcal{J}_k$ of I by finite sets as follows. We set $\mathcal{J}_0 = \mathcal{I}_0$ and we assume $\mathcal{J}_n$ being constructed. We then set $\mathcal{J}_{n+1} = J(\mathcal{I}_{n+1} \cup \mathcal{J}_n)$. We then consider the sequence (U_n) associated to this exhaustion. This sequence is by construction divergent. Indeed, for any $n \in \mathbf{N}$,

$$|||U_n - u||| = |||\sum_{i \in \mathcal{J}_n} u_i - u||| = |||\sum_{i \in J(\mathcal{I}_n \cup \mathcal{J}_{n-1})} u_i - u||| \geq \epsilon$$

which ends the proof.

□

One has the same property for Cauchy families :

Proposition 2.1.2 *The family $(u_i)_{i \in I}$ is a Cauchy family if and only if for any exhaustion $(\mathcal{I}_k)_{k \in \mathbf{N}}$ of I by finite sets, (U_k) is a Cauchy sequence.*

Proof Assuming the family $(u_i)_{i \in I}$ to be a Cauchy one and considering $\epsilon > 0$, then there is a finite $I_0 \subset I$ such that for any finite $J \subset I \setminus I_0$, $|||\sum_{i \in J} u_i||| \leq \epsilon$.

Consider an exhaustion $(\mathcal{I}_k)_{k \in \mathbf{N}}$ of I by finite set. There is $N \in \mathbf{N}$ such that for any $n \geq N$, $I_0 \subset \mathcal{I}_n$. Moreover, for any $k \geq l > N$, $J = I_k \setminus I_l$ is finite and contained in $I \setminus I_0$. Then $|||U_k - U_l||| = |||\sum_{i \in J} u_i||| \leq \epsilon$ as needed.

Conversely, if the family (u_i) is not a Cauchy one, then there exists $\epsilon > 0$ such that for any finite $I_0 \subset I$, there is a finite $J \subset I \setminus I_0$ such that $|||\sum_{i \in J} u_i||| \geq \epsilon$. As previously, in order to precise the dependence of J relatively to I_0 we denote it $J(I_0)$. For a given exhaustion $(\mathcal{I}_k)$ of I by finite sets, we set :

— $\mathcal{J}_0 = \mathcal{I}_0$,
— $\mathcal{J}_1 = \mathcal{I}_1$,

and by induction :

— $\mathcal{J}_{2n-1} = \mathcal{J}_{2n-1} \cup \mathcal{I}_n$,
— $\mathcal{J}_{2n} = \mathcal{J}_{2n-1} \cup J(\mathcal{J}_{2n-1})$.

The family $\mathcal{J}_k$ is an exhaustion of I by finite sets. We denote by (U_k) the sequence associated to this exhaustion. For any $n \in \mathbf{N}$, one has

$$|||U_{2n+1} - U_{2n}||| = |||\sum_{i \in J(\mathcal{J}_{2n})} u_i||| \geq \epsilon$$

which proves that (U_k) does not satisfy the Cauchy property.

□

We introduce now an Ω-pseudo-valuation on $<\Omega^\bullet>$, i.e a map val $:<\Omega^\bullet>\to\Omega\cup\{\infty\}$ satisfying the three following points :

1. $\mathrm{val}\,(x)=\infty$ if $x=0$.
2. $\mathrm{val}\,(x_1+x_2)\geq\min\{\mathrm{val}\,(x_1)\,,\mathrm{val}\,(x_2)\}$
3. $\mathrm{val}\,(x_1x_2)\geq\mathrm{val}\,(x_1)+\mathrm{val}\,(x_2)$.

for any $x_1,x_2\in<\Omega^\bullet>$.

For $x=\sum_{\underline{\omega}\in\mathscr{X}}M^{\underline{\omega}}\mathbf{e}_{\underline{\omega}}$ a sum of words $\mathbf{e}_{\underline{\omega}}$ weighted by scalars $M^{\underline{\omega}}$ with $\underline{\omega}$ in a finite subset $\mathscr{X}$ of $\Omega^\bullet$, we set $\mathrm{val}\,(x)=\min\{||\underline{\omega}||\mid\underline{\omega}\in\mathscr{X}\}$ on $<\Omega^\bullet>$. Let us observe that the application val takes as requested its values in the monoid Ω.

We then define a norm on $<\Omega^\bullet>$ by setting, for any $x\in<\Omega^\bullet>$:

$$|||x|||=2^{-\mathrm{val}(x)}$$

which has its meaningful because $\Omega\subset\mathbb{C}$. It turns $<\Omega^\bullet>$ into a normed vector space with an inherited topology known as *Krull topology.*

We translate Definition 2.1.5.1 in terms of pseudo-valuation :

Proposition 2.1.3 *In the normed vector space* $(<\Omega^\bullet>,|||.|||)$ *and for a family* $(u_i)_{i\in I}$ *of elements of* $<\Omega^\bullet>$*, one has an equivalence between :*

1. $(u_i)_{i\in I}$ *is summable of sum* $u\in E$*.*
2. $\forall N\in\Omega$, $(\exists J_0\subset I, J_0$ *finite* $)$: $\quad(\forall J\subset I, J$ *finite* $, J_0\subset J)\,,\quad$ *val* $\left(\sum_{j\in J}u_j-u\right)\geq N$.

In the same way, one has for Cauchy families :

Proposition 2.1.4 *In the normed vector space* $(<\Omega^\bullet>,|||.|||)$ *and for a family* $(u_i)_{i\in I}$ *of elements of* $<\Omega^\bullet>$*, one has equivalence between :*

1. $(u_i)_{i\in I}$ *is a Cauchy family of* E*.*
2. $\forall N\in\Omega$, $(\exists J_0\subset I, J_0$ *finite* $)$: $\quad(\forall J\subset I, J$ *finite* $, J\subset I\setminus J_0)\,,\quad$ *val* $\left(\sum_{j\in J}u_j\right)\geq N$.
3. $\forall N\in\Omega$ *the set* $E_N=\{i\in I\mid \mathit{val}(u_i)<N\}$ *is finite.*

Proof The equivalence between the two first assertions is easy. We assume that $(u_i)_{i\in I}$ is a Cauchy family. If there exists $N\in\Omega$ such that the set E_N is not finite, then for any finite $J_0\subset I$, one can find some finite $J\subset I\setminus J_0$ such that $J\subset E_N$. Then $\mathrm{val}\left(\sum_{i\in J}u_i\right)\geq\min\{\mathrm{val}\,(u_i)\}>N$ which contradicts the fact that $(u_i)_{i\in I}$ is a Cauchy family.

Conversely, if for all $N\in\Omega$, the set $E_N=\{i\in I\mid\mathrm{val}\,(u_i)<N\}$ is finite. For a given $N\in\Omega$, let us set $J_0=E_n$. For any finite $J\subset I\setminus J_0$, one has $\mathrm{val}\left(\sum_{i\in J}u_i\right)\geq\min\{\mathrm{val}\,(u_i)\}\geq N$ and so the family is a Cauchy one. □

We then consider the completion A of $< \Omega^\bullet >$ for the Krull topology. As a consequence of what precedes, the summable families of A are exactly the Cauchy ones and for a monoid Ω of $\mathbb{R}_+^*$ or $\mathbb{R}_-^*$, it follows that any family $\left(M^{\underline{\omega}}\mathbf{e}_{\underline{\omega}}\right)_{\underline{\omega}\in\Omega}$ is summable in A.

From now and for all the following, we assume the continuity of the product and the coproduct and we extend them to the whole of A which so becomes a graded bialgebra too and also an Hopf algebra. The elements of this space are infinite sums of the form

$$\sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbf{e}_{\underline{\omega}}.$$

It is what Ecalle call a *mould-comould expansion*. The scalar family indexed by words $(M^{\underline{\omega}})$ is a *mould* and the family $\left(\mathbf{e}_{\underline{\omega}}\right)$ is a *comould*. An important thing is that it is possible to construct a comould from any element $\mathbf{a}$ of A. Indeed, using the graduation, one can expand $\mathbf{a}$ as the sum of its homogeneous components : $\mathbf{a} = \sum_{m\in\Omega}\mathbf{a}_m$. So, for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$, one can consider the products $\mathbf{a}_{\underline{\omega}} := \mathbf{a}_{\omega_r}\ldots\mathbf{a}_{\omega_1}$ which defines a new comould associated to $\mathbf{a}$ that we will denote $\mathbf{a}_\bullet$.

It is now possible to consider mould-comould expansion of the form $\sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$. Such a mould-comould expansion is often denoted $\sum M^\bullet\mathbf{a}_\bullet$.

As a consequence of the graded structure, one has :

Proposition 2.1.5 *For* $\mathbf{b} = \sum_{m\in\Omega}\mathbf{b}_m \in A$ *:*

— $\mathbf{b}$ *is primitive if and only if for any* $m \in \Omega$, $\mathbf{b}_m$ *is primitive, i.e.* $\mathbf{b}_m$ *satisfies relation 2.1 page 31.*

— $\mathbf{b}$ *is group-like if and only if for any* $m \in \Omega$, $\mathbf{b}_m$ *is group-like, i.e.* $\mathbf{b}_m$ *satisfies relation 2.2 page 32.*

Proof We perform the proof just in the first case. It is similar in the second one.

We assume that $\mathbf{b}$ is primitive. Then one has $\mathbf{cop}(\mathbf{b}) = \mathbf{b}\otimes 1 + 1\otimes\mathbf{b}$. Writing $\mathbf{b} = \sum_{m\in\Omega}\mathbf{b}_m$ where $\mathbf{b}_m$ denotes the homogeneous component of degree m of $\mathbf{b}$, one has, using the continuity of the coproduct :

$$\sum_{m\in\Omega}\mathbf{cop}(\mathbf{b}_m) = \sum_{m\in\Omega}\left(\mathbf{b}_m\otimes 1 + 1\otimes\mathbf{b}_m\right).$$

But A, being a graded bialgebra, it comes for any $m \in \Omega$, $\mathbf{cop}(\mathbf{b}_m) = \mathbf{b}_m\otimes 1 + 1\otimes\mathbf{b}_m$. The reciprocal is easily to establish using analogous computations. □

We will denote by A_p the space of mould-comould expansions defined by assuming the words of length 1 as primitive and A_g the one defined by assuming them group-like.

Moreover, we will denote by $\mathcal{M}(\Omega)$ the set of complex valued mould indexed by words with letters in Ω.

Definition 2.1.5.2 *The comould* $\mathbf{a}_\bullet$ *is said to be :*
— *cosymmetral if* $\mathbf{a}$ *is primitive.*
— *cosymmetrel if* $\mathbf{a}$ *is group-like.*

2.2 Moulds symmetries

J. Ecalle has discovered four fundamental symmetry properties on mould that allow to detect if an element of A_p or A_g is a primitive or a group-like one. These symmetries depend of the comould part, if it is primitive or group-like. The goal of the two next sections is to explain this.

2.2.1 Alternality and symmetrality

We first work with the Hopf algebra A_p.

Notation 2.2.1.1 *For any sequence* $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$, *we denote by* $\boldsymbol{sh}\left(\underline{\omega}^1, \underline{\omega}^2\right)$ *the set of all shuffles of* $\underline{\omega}^1$ *and* $\underline{\omega}^2$ *that respect the internal order of each of the two sequences* $\underline{\omega}^1$ *and* $\underline{\omega}^2$.

Example 2.2.1.1

$$\boldsymbol{sh}\left((a,b),(\alpha,\beta)\right) = \left\{(\alpha,a,b,\beta),(\alpha,a,\beta,b),(\alpha,\beta,a,b),(a,\alpha,b,\beta),(a,\alpha,\beta,b),(a,b,\alpha,\beta)\right\}.$$

Definition 2.2.1.1 *We say that a mould* $M^\bullet$ *is*
— *alternal if for any sequence* $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$

$$\sum_{\underline{\omega} \in \boldsymbol{sh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = 0,$$

— *symmetral if for any sequence* $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$

$$\sum_{\underline{\omega} \in \boldsymbol{sh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = M^{\underline{\omega}^1} M^{\underline{\omega}^2}.$$

Let us consider $\mathbf{b} = \sum M^\bullet \mathbf{a}_\bullet \in A$. We want to understand which constraints on the mould $M^\bullet$ bring the fact that $\mathbf{b}$ is primitive or group-like .

Example 2.2.1.2 *We assume that* $\Omega = \mathbf{N}^*$ *and we begin by a small computation with the homogeneous component* $\mathbf{b}_2$ *of* $\mathbf{b}$. *One has :*

$$\begin{aligned}
\boldsymbol{cop}\left(\mathbf{b}_2\right) &= \left(\sum_{\underline{\omega}\in\Omega^\bullet, \|\underline{\omega}\|=2} M^{\underline{\omega}} \boldsymbol{cop}\left(\mathbf{a}_{\underline{\omega}}\right)\right) \\
&= M^{1,1} \boldsymbol{cop}\left(\mathbf{a}_1 . \mathbf{a}_1\right) + M^2 \boldsymbol{cop}\left(\mathbf{a}_2\right) \\
&= M^{1,1}\left(\mathbf{a}_2\mathbf{a}_1 \otimes 1 + \mathbf{a}_1 \otimes \mathbf{a}_2 + \mathbf{a}_2 \otimes \mathbf{a}_1 + 1 \otimes \mathbf{a}_2\mathbf{a}_1\right) + M^2\left(\mathbf{a}_2 \otimes 1 + 1 \otimes \mathbf{a}_2\right) \quad (\star).
\end{aligned}$$

— *If* $\mathbf{b}$ *is primitive then* $\boldsymbol{cop}\,(\mathbf{b}_2) = \mathbf{b}_2 \otimes 1 + 1 \otimes \mathbf{b}_2$*. Then it comes :*

$$\begin{aligned}\boldsymbol{cop}\,(\mathbf{b}_2) &= \mathbf{b}_2 \otimes 1 + 1 \otimes \mathbf{b}_2 \\ &= \left(M^{1,1}\mathbf{a}_1.\mathbf{a}_1 + M^2\mathbf{a}_2\right) \otimes 1 + 1 \otimes \left(M^{1,1}\mathbf{a}_1.\mathbf{a}_1 + M^2\mathbf{a}_2\right) \qquad (\star\star)\end{aligned}$$

and when we compare $(\star)$ *and* $(\star\star)$*, we obtain* $M^{1,1} = 0$*, i.e.* $\sum_{\underline{\omega}\in \boldsymbol{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} = 0$ *with* $\underline{\omega}^1 = \underline{\omega}^2 = (1)$*.*

— *If* $\mathbf{b}$ *is group-like then*

$$\begin{aligned}\boldsymbol{cop}\,(\mathbf{b}_2) &= \mathbf{b}_2 \otimes 1 + \mathbf{b}_1 \otimes \mathbf{b}_1 + \mathbf{b}_2 \otimes 1 \\ &= \left(M^{1,1}\mathbf{a}_1.\mathbf{a}_1 + M^2\mathbf{a}_2\right) \otimes 1 + \left(M^1\right)^2 \mathbf{a}_1 \otimes \mathbf{a}_2 + 1 \otimes \left(M^{1,1}\mathbf{a}_1.\mathbf{a}_1 + M^2\mathbf{a}_2\right) \qquad (\star\star\star)\end{aligned}$$

and when we compare $(\star)$ *and* $(\star\star\star)$*, we obtain* $M^{1,1} = \left(M^1\right)^2$*, i.e.* $\sum_{\underline{\omega}\in \boldsymbol{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} = M^{\underline{\omega}^1} M^{\underline{\omega}^2}$ *with* $\underline{\omega}^1 = \underline{\omega}^2 = (1)$*.*

We then observe that the fact to be group like or primitive is in relation with the symmetrality or the alternality of $M^\bullet$. In order to prove it, we need to compute $\mathbf{cop}\left(\mathbf{a}_{\underline{\omega}}\right)$ for $\underline{\omega} \in \Omega^\bullet$.

For any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$, we have :

$$\begin{aligned}&\mathbf{cop}\,(\mathbf{a}_{\omega_1}) = \mathbf{a}_{\omega_1} \otimes 1 + 1 \otimes \mathbf{a}_{\omega_1} \\ &\mathbf{cop}\,(\mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1}) = \mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1} \otimes 1 + \mathbf{a}_{\omega_1} \otimes \mathbf{a}_{\omega_2} + \mathbf{a}_{\omega_2} \otimes \mathbf{a}_{\omega_1} + 1 \otimes \mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1} \\ &\mathbf{cop}\,(\mathbf{a}_{\omega_3}\mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1}) = \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1} \otimes 1 + \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_1} \otimes \mathbf{a}_{\omega_2} + \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_2} \otimes \mathbf{a}_{\omega_1} + \mathbf{a}_{\omega_3} \otimes \mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1} \\ &\qquad + \mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1} \otimes \mathbf{a}_{\omega_3} + \mathbf{a}_{\omega_1} \otimes \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_2} + \mathbf{a}_{\omega_2} \otimes \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_1} + 1 \otimes \mathbf{a}_{\omega_3}\mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1}\end{aligned}.$$

An easy induction leads to

$$\mathbf{cop}\left(\mathbf{a}_{\underline{\omega}}\right) = \sum_{(\underline{\omega}^1,\underline{\omega}^2)\in E_{\underline{\omega}}} \mathbf{a}_{\underline{\omega}^1} \otimes \mathbf{a}_{\underline{\omega}^2}$$

where $E_{\underline{\omega}}$ is the set of pairs $\left(\underline{\omega}^1, \underline{\omega}^2\right)$ that can give $\underline{\omega}$ for a conveniently chosen shuffle.

Now, for $\mathbf{b} = \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}}\mathbf{a}_{\underline{\omega}} \in A$, using the continuity of $\mathbf{cop}$ for the Krull topology, it comes :

$$\begin{aligned}\mathbf{cop}\,(\mathbf{b}) &= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}}\mathbf{cop}\,(\mathbf{a}_\omega) \\ &= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}} \sum_{(\underline{\omega}^1,\underline{\omega}^2)\in E_{\underline{\omega}}} \mathbf{a}_{\underline{\omega}^1} \otimes \mathbf{a}_{\underline{\omega}^2} \\ &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet} \left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} \right) \mathbf{a}_{\underline{\omega}^1} \otimes \mathbf{a}_{\underline{\omega}^2}.\end{aligned}$$

Thus we establish the following important theorem :

Theorem 2.2.1

— $\mathbf{b}$ *is primitive if and only if* $M^\bullet$ *is alternal.*
— $\mathbf{b}$ *is group like if and only if* $M^\bullet$ *is symmetral.*

Proof

— if $\mathbf{b}$ is primitive, then $\mathbf{cop}(\mathbf{b}) = \mathbf{b}\otimes 1 + 1\otimes \mathbf{b}$ and using the previous formula, we get :

$$\begin{aligned}\mathbf{cop}(\mathbf{b}) &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}\\ &= \mathbf{b}\otimes 1 + 1\otimes\mathbf{b} + \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet,\underline{\omega}^1,\underline{\omega}^2\neq\emptyset}\left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}.\end{aligned}$$

Then for any non empty sequences $\underline{\omega}^1,\underline{\omega}^2$, we must have $\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} = 0$ and $M^\bullet$ is alternal. The reciprocal is easy.

— if $\mathbf{b}$ is group like, then $\mathbf{cop}(\mathbf{b}) = \mathbf{b}\otimes\mathbf{b}$ and as previously

$$\begin{aligned}\mathbf{cop}(\mathbf{b}) &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}\\ &= \mathbf{b}\otimes\mathbf{b} + \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(M^{\underline{\omega}^1}M^{\underline{\omega}^2} - \left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}\end{aligned}$$

Then for any non empty sequences $\underline{\omega}^1,\underline{\omega}^2$, we must have

$$\sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(M^{\underline{\omega}^1}M^{\underline{\omega}^2} - \left(\sum_{\underline{\omega}\in\mathbf{sh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\right) = 0$$

and the symmetrality of $M^\bullet$ follows. The reciprocal is easy.

□

2.2.2 Alternelity and symmetrelity

We now work with the Hopf algebra A_g.

Notation 2.2.2.1 *For any sequences* $\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet$*, we denote by* $\boldsymbol{csh}\left(\underline{\omega}^1,\underline{\omega}^2\right)$ *the set of all possible shuffles* $\underline{\omega}$ *of* $\underline{\omega}^1$*,*$\underline{\omega}^2$ *following by possible sum contractions of several pair* (a,b) *of consecutive elements of* $\underline{\omega}$ *in the case where* $a\in\underline{\omega}^1$ *and* $b\in\underline{\omega}^2$ *or* $a\in\underline{\omega}^2$ *and* $b\in\underline{\omega}^1$*.*

Example 2.2.2.1

$$\boldsymbol{csh}((a,b),(\alpha,\beta)) = \boldsymbol{sh}((a,b),(\alpha,\beta))\cup\{\,(\alpha+a,b,\beta),(\alpha,a,b+\beta),(\alpha+a,\beta,b),$$
$$(\alpha+a,\beta+b),(\alpha,a,\beta+b),(\alpha,\beta+a,b),(a+\alpha,b,\beta),(a+\alpha,b+\beta),(a,b+\alpha,\beta)\,\}.$$

Definition 2.2.2.1 *We say that a mould $M^\bullet$ is*

— *alternel if for any sequences $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$*

$$\sum_{\underline{\omega} \in \boldsymbol{csh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = 0,$$

— *symmetrel if for any sequences $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$*

$$\sum_{\underline{\omega} \in \boldsymbol{csh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = M^{\underline{\omega}^1} M^{\underline{\omega}^2}.$$

As previously, we compute for any $\underline{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$:

$$\mathbf{cop}\,(\mathbf{a}_{\omega_1}) = \sum_{m_1+m_2=m, m_1, m_2 \geq 0} \mathbf{a}_{m_1} \otimes \mathbf{a}_{m_2}$$

$$\mathbf{cop}\,(\mathbf{a}_{\omega_2}\mathbf{a}_{\omega_1}) = \sum_{\substack{m_1^1 + m_2^1 = \omega_1, m_1^1, m_2^1 \geq 0 \\ m_1^2 + m_2^2 = \omega_2, m_1^2, m_2^2 \geq 0}} \mathbf{a}_{m_1^2}\mathbf{a}_{m_1^1} \otimes \mathbf{a}_{m_2^2}\mathbf{a}_{m_2^1}$$

and more generally one has :

$$\mathbf{cop}\left(\mathbf{a}_{\underline{\omega}}\right) = \sum_{\substack{m_1^i + m_2^i = \omega_i \\ m_1^i, m_2^i \geq 0 \\ i \in [\![1, r]\!]}} \mathbf{a}_{m_1^r} \dots \mathbf{a}_{m_1^1} \otimes \mathbf{a}_{m_2^r} \dots \mathbf{a}_{m_2^1} \quad (\star)\,.$$

For a pair $\left(\underline{\mathrm{x}}, \underline{\mathrm{y}}\right) \in \Omega^{\bullet 2}$, we consider the set $T_{\underline{\mathrm{x}},\underline{\mathrm{y}}}$ of all couples $(\underline{\mathrm{X}}, \underline{\mathrm{Y}}) \in \Omega^{\bullet 2}$ such that :

— $r := \mathrm{l}(\underline{\mathrm{X}}) = \mathrm{l}(\underline{\mathrm{Y}})$.

— $\underline{\mathrm{X}}$ (resp. $\underline{\mathrm{Y}}$) is built from $\underline{\mathrm{x}}$ (resp. $\underline{\mathrm{y}}$) by inserting some 0 before or after terms of $\underline{\mathrm{x}}$ (resp. $\underline{\mathrm{y}}$).

— We can not have a zero at the same position in $\underline{\mathrm{X}}$ and $\underline{\mathrm{Y}}$. More precisely, if $\underline{\mathrm{X}} = (X_1, \dots, X_r)$ and $\underline{\mathrm{Y}} = (Y_1, \dots, Y_r)$, for any $i \in [\![1, r]\!]$, we can not have $X_i = 0$ if $Y_i = 0$ or $Y_i = 0$ if $X_i = 0$.

We will denote by $R_{\underline{\mathrm{x}},\underline{\mathrm{y}}}$ the set

$$R_{\underline{\mathrm{x}},\underline{\mathrm{y}}} = \left\{ \underline{\mathrm{X}} + \underline{\mathrm{Y}} \mid \ (\underline{\mathrm{X}}, \underline{\mathrm{Y}}) \in T_{\underline{\mathrm{x}},\underline{\mathrm{y}}} \right\}.$$

And $(\star)$ can be written

$$\mathbf{cop}\left(\mathbf{a}_{\underline{\omega}}\right) = \sum_{\underline{\mathrm{x}},\underline{\mathrm{y}} \in \Omega^\bullet \ \mid \ \underline{\omega} \in R_{\underline{\mathrm{x}},\underline{\mathrm{y}}}} \mathbf{a}_{\underline{\mathrm{x}}} \otimes \mathbf{a}_{\underline{\mathrm{y}}}.$$

Lemma 2.2.2 *We have $R_{\underline{x},\underline{y}} = \boldsymbol{csh}\left(\underline{x}, \underline{y}\right)$.*

Proof The direct inclusion is easy. For the reciprocal one, we will perform an induction on the integer $r = \mathrm{l}(\underline{\mathrm{x}}) + \mathrm{l}\left(\underline{\mathrm{y}}\right)$. The property is clearly true if $r = 1$.

We assume that it is true for a given $r \in \mathbf{N}^*$ and for any $\underline{x}, \underline{y} \in \Omega^\bullet$ such that $\mathrm{l}(\underline{x}) + \mathrm{l}(\underline{y}) = r$. We will prove it for two sequences $\underline{x}, \underline{y} \in \Omega^\bullet$ such that $s+t = r+1$ with $s = \mathrm{l}(\underline{x})$ and $t = \mathrm{l}(\underline{y})$. Let us consider $\underline{\omega} = (\underline{\omega}', \omega_r) \in \mathbf{csh}(\underline{x}, \underline{y})$. We then have three possibilities :

— The first one is that $\omega_r = x_s$. Using the induction hypothesis, we have $\underline{\omega}' \in \mathbf{csh}(\underline{x}', \underline{y}) = R_{\underline{x}', \underline{y}}$ and then there exists $(\underline{X}', \underline{Y}) \in T_{\underline{x}', \underline{y}}$ such that $\underline{\omega}' = \underline{X}' + \underline{Y}$. It comes that $\underline{\omega} = (\underline{X}', x_s) + (\underline{Y}, 0)$.

— The second one is that $\omega_r = y_t$. We conclude as previously.

— The third one is that $\omega_r = x_s + y_t$. We then have $\omega' \in \mathbf{csh}(\underline{x}', \underline{y}') = R_{\underline{x}', \underline{y}'}$. There exists $(\underline{X}', \underline{Y}') \in T_{\underline{x}', \underline{y}'}$ such that $\underline{\omega}' = \underline{X}' + \underline{Y}'$ and the result follows from the fact that $\underline{\omega} = (\underline{X}', x_s) + (\underline{Y}', y_t)$.

In the three cases, $\underline{\omega} \in R_{\underline{x}, \underline{y}}$. □

For an element $\mathbf{b} = \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}} \mathbf{a}_\omega \in A$, using the continuity of **cop**, we obtain here

$$
\begin{aligned}
\mathbf{cop}(\mathbf{b}) &= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}} \mathbf{cop}(a_{\underline{\omega}}) \\
&= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}} \sum_{\underline{x},\underline{y}\in\Omega^\bullet \ | \ \underline{\omega}\in\mathbf{csh}(\underline{x},\underline{y})} \mathbf{a}_{\underline{x}} \otimes \mathbf{a}_{\underline{y}} \\
&= \sum_{\underline{x},\underline{y}\in\Omega^\bullet} \left(\sum_{\underline{\omega}\in\mathbf{csh}(\underline{x},\underline{y})} M^{\underline{\omega}} \right) \mathbf{a}_{\underline{x}} \otimes \mathbf{a}_{\underline{y}}.
\end{aligned}
$$

Then we have the counterpart of Theorem 2.2.1 in this second case :

Theorem 2.2.3

— $\mathbf{b}$ *is primitive if and only if* $M^\bullet$ *is alternel.*

— $\mathbf{b}$ *is group like if and only if* $M^\bullet$ *is symmetrel.*

Proof

— if $\mathbf{b}$ is primitive, then $\mathbf{cop}(\mathbf{b}) = \mathbf{b} \otimes 1 + 1 \otimes \mathbf{b}$ and using the previous formula we get :

$$
\mathbf{cop}(\mathbf{b}) = \mathbf{b} \otimes 1 + 1 \otimes \mathbf{b} + \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet, \underline{\omega}^1,\underline{\omega}^2\neq\emptyset} \left(\sum_{\underline{\omega}\in\mathbf{csh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} \right) \mathbf{a}_{\underline{\omega}^1} \otimes \mathbf{a}_{\underline{\omega}^2}
$$

Then for any non empty sequences $\underline{\omega}^1, \underline{\omega}^2$, we must have $\sum_{\underline{\omega}\in\mathbf{csh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}} = 0$ and $M^\bullet$ is alternel. The reciprocal is easy.

— if $\mathbf{b}$ is group like, then $\mathbf{cop}(\mathbf{b}) = \mathbf{b}\otimes\mathbf{b}$ in the same way than previously

$$\begin{aligned}\mathbf{cop}(\mathbf{b}) &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(\sum_{\underline{\omega}\in\mathbf{csh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}\\ &= \mathbf{b}\otimes\mathbf{b}+\sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(M^{\underline{\omega}^1}M^{\underline{\omega}^2}-\left(\sum_{\underline{\omega}\in\mathbf{csh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\right)\mathbf{a}_{\underline{\omega}^1}\otimes\mathbf{a}_{\underline{\omega}^2}\end{aligned}$$

Then for any non empty sequences $\underline{\omega}^1, \underline{\omega}^2$, we must have

$$\sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet}\left(M^{\underline{\omega}^1}M^{\underline{\omega}^2}-\left(\sum_{\underline{\omega}\in\mathbf{csh}(\underline{\omega}^1,\underline{\omega}^2)} M^{\underline{\omega}}\right)\right)=0$$

and the symmetrelity of $M^\bullet$ follows. The reciprocal is easy.

□

2.3 Mould calculus

The graded Hopf algebra structure on A_p or A_g induces an algebra structure on the set of complex valued moulds $\mathcal{M}(\Omega)$ that we will describe now. We will indifferently denote by A one of the two Hopf algebra A_p or A_g when there is no need to precise the coproduct.

2.3.1 Mould product

We will now give an explicit formula for mould product.

Proposition 2.3.1 (Mould product) *If* $\mathbf{b} = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$, $\mathbf{c} = \sum_{\underline{\omega}\in\Omega^\bullet} N^{\underline{\omega}}\mathbf{a}_{\underline{\omega}} \in A$ *and if* $\mathbf{b}.\mathbf{c} = \sum_{\underline{\omega}\in\Omega^\bullet} P^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$ *then for all* $\underline{\omega}\in\Omega^\bullet$ *:*

$$P^{\underline{\omega}} = \sum_{\underline{\omega}_1\cdot\underline{\omega}_2=\underline{\omega}} N^{\underline{\omega}_1}M^{\underline{\omega}_2}$$

where the sum ranges over all possible concatenations of the two sequences $\underline{\omega}_1$ *and* $\underline{\omega}_2$ *into* $\underline{\omega}$ *including the empty one. This operation is called mould product and we note* $P^\bullet = M^\bullet \times N^\bullet$.

Proof The family $\left(P^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}\right)$ is evidently a summable one. One has to prove that its sum is given by $\mathbf{b}.\mathbf{c}$. Using the continuity of the product, one can write.

$$\begin{aligned}\mathbf{b}.\mathbf{c} &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet} M^{\underline{\omega}^1}N^{\underline{\omega}^2}\mathbf{a}_{\underline{\omega}^1}.\mathbf{a}_{\underline{\omega}^2}\\ &= \sum_{\underline{\omega}^1,\underline{\omega}^2\in\Omega^\bullet} M^{\underline{\omega}^1}N^{\underline{\omega}^2}\mathbf{a}_{\underline{\omega}^2\cdot\underline{\omega}^1}\\ &= \sum_{\underline{\omega}\in\Omega^\bullet}\left(\sum_{\substack{\left(\underline{\omega}^1,\underline{\omega}^2\right)\in\Omega^{\bullet 2}\\ \underline{\omega}^1\cdot\underline{\omega}^2=\underline{\omega}}} N^{\underline{\omega}^1}M^{\underline{\omega}^2}\right)\mathbf{a}_{\underline{\omega}}\end{aligned}$$

where the internal sum ranges over all the possible pairs of sequences $\left(\underline{\omega}^1,\underline{\omega}^2\right)\in\Omega^{\bullet 2}$ such that the concatenation of $\underline{\omega}^1$ with $\underline{\omega}^2$ is equal to $\underline{\omega}$ and where $\underline{\omega}^1$ or $\underline{\omega}^2$ may be empty.

□

The unit for mould product is the mould $1^\bullet$ given by $1^{\underline{\omega}}=1$ if $\underline{\omega}=\emptyset$ and $1^{\underline{\omega}}=0$ elsewhere. This mould is an element of the *constant type moulds family* because its values $M^{\underline{\omega}}$ depend only of the length of the sequence $\underline{\omega}$.

We are now going to show three other fundamental examples of constant type mould : $I^\bullet$, $\log^\bullet$ and $\exp_a^\bullet$ where $a\in\mathbb{C}$. They are defined for any $\underline{\omega}\in\Omega^\bullet$ by :

$$I^{\underline{\omega}}=\begin{cases}1 & \text{if } r=1\\ 0 & \text{otherwise}\end{cases},\quad \log^{\underline{\omega}}=\begin{cases}\dfrac{(-1)^{r+1}}{r} & \text{if } r\neq 0\\ 0 & \text{otherwise}\end{cases},\quad \exp_a^{\underline{\omega}}=\frac{a^r}{r!}$$

with $r=\mathrm{l}(\underline{\omega})$.

One easily verify that a mould $M^\bullet$ is invertible for mould product if and only if $M^\emptyset\neq 0$. We will give a formula for mould inverse in subsection 2.3.3.

2.3.2 Mould composition

We consider now three elements $\mathbf{a},\mathbf{b},\mathbf{c}\in A$. We assume that there exists three moulds $M^\bullet, N^\bullet$ and $P^\bullet$ such that $\mathbf{b}=\sum_{\underline{\omega}\in\Omega^\bullet}N^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$, $\mathbf{c}=\sum_{\underline{\omega}\in\Omega^\bullet}M^{\underline{\omega}}\mathbf{b}_{\underline{\omega}}=\sum_{\underline{\omega}\in\Omega^\bullet}P^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$. We want to compute the mould $P^\bullet$ in terms of the moulds $M^\bullet$ and $N^\bullet$. Such an operation, allowing to obtain $P^\bullet$ in terms of $M^\bullet$ and $N^\bullet$ is known as mould composition.

Proposition 2.3.2 (Mould composition) *If* $\mathbf{b}=\sum_{\underline{\omega}\in\Omega^\bullet}N^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$, $\mathbf{c}=\sum_{\underline{\omega}\in\Omega^\bullet}M^{\underline{\omega}}\mathbf{b}_{\underline{\omega}}\in\mathbb{A}^\Omega$ *and if* $\mathbf{c}=\sum_{\underline{\omega}\in\Omega^\bullet}P^{\underline{\omega}}\mathbf{a}_{\underline{\omega}}$ *then for any* $\underline{\omega}\in\Omega^\bullet$ *:*

$$P^\emptyset=M^\emptyset \text{ and } P^{\underline{\omega}}=\sum_{\underline{\omega}_1\cdot\ldots\cdot\underline{\omega}_r=\underline{\omega}} M^{\|\underline{\omega}_1\|,\ldots,\|\underline{\omega}_r\|}N^{\underline{\omega}_1}\ldots N^{\underline{\omega}_r} \text{ if } \underline{\omega}\neq\emptyset$$

where the sum ranges over all the possible concatenations of $\underline{\omega}$ into non empty sub-sequences $\underline{\omega}_1, \ldots, \underline{\omega}_r$. The obtained mould $P^\bullet$ is called the composition of the moulds $M^\bullet$ and $N^\bullet$ and will be denoted $P^\bullet = M^\bullet \circ N^\bullet$.

Proof Because of the graduation, we have for any $m \in \Omega$, $\mathbf{b}_m = \sum_{\underline{\omega}\in\Omega^\bullet, \|\underline{\omega}\|=m} N^{\underline{\omega}} \mathbf{a}_{\underline{\omega}}$ and for any $\underline{m} = (m_1, \ldots, m_r) \in \Omega^\bullet$:

$$\begin{aligned} \mathbf{b}_{\underline{m}} &= \mathbf{b}_{m_r} \ldots \mathbf{b}_{m_1} \\ &= \sum_{\substack{\underline{\omega}^1, \ldots, \underline{\omega}^r \in \Omega^\bullet \\ \|\underline{\omega}^1\| = m_1, \ldots, \|\underline{\omega}^r\| = m_r}} N^{\underline{\omega}^1} \ldots N^{\underline{\omega}^r} \mathbf{a}_{\underline{\omega}^r} \ldots . \mathbf{a}_{\underline{\omega}^1}. \end{aligned}$$

But

$$\begin{aligned} \mathbf{c} &= \sum P^\bullet a_\bullet \\ &= \sum M^\bullet \mathbf{b}_\bullet \\ &= M^\emptyset \mathbf{b}_\emptyset + \sum_{\underline{m}\in\Omega^\bullet, \underline{m}\neq\emptyset} M^{\underline{m}} \sum_{\substack{\underline{\omega}^1, \ldots, \underline{\omega}^r \in \Omega^\bullet \\ \|\underline{\omega}^1\| = m_1, \ldots, \|\underline{\omega}^r\| = m_r}} N^{\underline{\omega}^1} \ldots N^{\underline{\omega}^r} \mathbf{a}_{\underline{\omega}^r} \ldots . \mathbf{a}_{\underline{\omega}^1} \\ &= M^\emptyset \mathbf{a}_\emptyset + \sum_{\underline{\omega}\in\Omega^\bullet, \underline{m}\neq\emptyset} \sum_{\underline{\omega}_1 \cdot \ldots \cdot \underline{\omega}_r = \underline{\omega}} M^{\|\underline{\omega}_1\|, \ldots, \|\underline{\omega}_r\|} N^{\underline{\omega}_1} \ldots N^{\underline{\omega}_r} \mathbf{a}_{\underline{\omega}}. \end{aligned}$$

□

For a mould $M^\bullet$, it is easy to verify that $M^\bullet \circ I^\bullet = M^\bullet$ but one has $I^\bullet \circ M^\bullet = M^\bullet$ only if $M^\emptyset = 0$. In order to avoid this problem, we assume that the composition reading $M^\bullet \circ N^\bullet$ is in fact the composition $M^\bullet \circ \left(N^\bullet - N^\emptyset 1^\bullet\right)$. In particular, if $N^\bullet$ is symmetrel, then $M^\bullet \circ N^\bullet := M^\bullet \circ (N^\bullet - 1^\bullet)$

2.3.3 Antipode

As explained before, A being a graded bialgebra, it has an antipode we will denote s.

Proposition 2.3.3 *If* $\mathbf{b} = \sum M^\bullet \mathbf{e}_\bullet \in A$ *then*

$$s(\mathbf{b}) = \begin{cases} \sum (-1)^{l(\bullet)} \boldsymbol{rev}(M)^\bullet \mathbf{e}_\bullet & \text{if } A = A_p \\ \sum \left(\boldsymbol{rev}(M) \circ \left((1+I)^{-1} - 1\right)\right)^\bullet \mathbf{e}_\bullet & \text{if } A = A_g \end{cases}.$$

Proof Working with $A = A_p$, we know that $\mathbf{e}_\omega$ is primitive for any $\omega \in \Omega$ and then $s(\mathbf{e}_\omega) = -\mathbf{e}_\omega$. Thus the formula is a direct consequence of the fact that the antipode is an antimorphism.

Otherwise, if $A = A_g$, then one can write $\mathbf{e} = \sum (1+I)^\bullet \mathbf{e}_\bullet$. The symmetrel mould $(1+I)^\bullet$ is invertible for mould product and one easily finds

$\left((1+I)^{-1}\right)^{\underline{\omega}} = (-1)^{\mathrm{l}(\underline{\omega})}$. Then we obtain, using the graduation, for any $m \in \Omega^\bullet$:

$$s(\mathbf{e}_m) = \sum_{\underline{\omega}\in\Omega;\|\underline{\omega}\|=m} (-1)^{\mathrm{l}(\underline{\omega})} \mathbf{e}_{\underline{\omega}}.$$

and s being an antimorphism of A, it comes :

$$\begin{aligned} s(\mathbf{b}) &= \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}} s\left(\mathbf{e}_{\underline{\omega}}\right) \\ &= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} M^{\underline{\omega}} s(\mathbf{e}_{\omega_r}) \cdot \ldots \cdot s(\mathbf{e}_{\omega_1}) \\ &= \sum_{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet} \sum_{\substack{\underline{W}^1,\ldots,\underline{W}^r \in \Omega^\bullet \\ \|\underline{W}^1\|=\omega_1,\ldots,\|\underline{W}^r\|=\omega_r}} M^{\mathbf{rev}(\underline{\omega})} \times \\ &\qquad (-1)^{\mathrm{l}\left(\underline{W}^1\right)} \ldots (-1)^{\mathrm{l}(\underline{W}^r)} \mathbf{e}_{\underline{W}^1} \cdot \ldots \cdot \mathbf{e}_{\underline{W}^r} \end{aligned}$$

and the formula follows.

□

Replacing $\mathbf{e}$ by an element $\mathbf{a} \in A$, one obtain identical formulas than previously when assuming $\mathbf{a}$ to be primitive or group-like.

The action of the antipode S can be read on the mould space $\mathcal{M}(\Omega)$.

Corollary 2.3.4 *If $b = \sum M^\bullet e_\bullet$ and if $s(b) = \sum N^\bullet e_\bullet$ then for any $\underline{\omega} \in \Omega^\bullet$:*

$$N^{\underline{\omega}} = \begin{cases} (-1)^{l(\underline{\omega})} M^{rev(\underline{\omega})} & \text{if } A = A_p \\ (-1)^{l(\underline{\omega})} \sum_{\substack{\underline{W}^1,\ldots,\underline{W}^r \in \Omega^\bullet \\ \underline{W}^1 \cdot \ldots \cdot \underline{W}^r = \underline{\omega} \\ \|\underline{W}^1\|=m_1,\ldots,\|\underline{W}^r\|=m_r}} M^{\|\underline{W}^r\|,\ldots,\|\underline{W}^1\|} & \text{if } A = A_g \end{cases}$$

As a direct corollary, these last formulas allow to compute the product inverse of a symmetral or a symmetrel mould :

Corollary 2.3.5 *The product inverse of a mould $M^\bullet$ is the mould $N^\bullet$ given for any $\underline{\omega} \in \Omega^\bullet$ by :*

$$N^{\underline{\omega}} = \begin{cases} (-1)^{l(\underline{\omega})} M^{rev(\underline{\omega})} & \text{if } M^\bullet \text{ is symmetral} \\ (-1)^{l(\underline{\omega})} \sum_{\substack{\underline{W}^1,\ldots,\underline{W}^r \in \Omega^\bullet \\ \underline{W}^1 \cdot \ldots \cdot \underline{W}^r = \underline{\omega} \\ \|\underline{W}^1\|=m_1,\ldots,\|\underline{W}^r\|=m_r}} M^{\|\underline{W}^r\|,\ldots,\|\underline{W}^1\|} & \text{if } M^\bullet \text{ is symmetrel} \end{cases}$$

Proof It is a direct consequence of the fact that for a group-like element $\mathbf{a}$ in an Hopf algebra with antipode s, one has $s(\mathbf{a}) = \mathbf{a}^{-1}$. □

2.4 Two fundamental examples of Hopf algebra

We introduce now two fundamental Hopf algebras in Ecalle's work as realizations of the spaces A_p and A_g.

2.4.1 The Hopf algebra of ALIEN operators

We replace in the definition of A the symbols $\mathbf{e}_m$ by $\dot{\Delta}^+_m$ for any $m \in \Omega$. For $\underline{\omega} = (\omega_1, \dots, \omega_r)$, the product $\mathbf{e}_{\underline{\omega}} = \mathbf{e}_{\omega_r} \dots \mathbf{e}_{\omega_1} = \dot{\Delta}^+_{\omega_r} \dots \dot{\Delta}^+_{\omega_1}$ corresponds now to the composition of r operators. Proposition 1.2.5 allows to affirm that the $\dot{\Delta}^+_m$ are group-like.

This realization of the Hopf algebra A_g is the one of what Ecalle call **ALIEN** operators. It is denoted **ALIEN** (Ω). As proved in [50], the comould $\dot{\Delta}^+_\bullet$ defines a basis of this algebra. We have explained in sub-section 1.2.2 how the homogeneous components of the Stokes automorphism $\dot{\Delta}^+$ act on $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{C}}(\Omega)[[\delta_1]]$ which permits to define the action [2] of an **ALIEN** operator on $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{C}}(\Omega)[[\delta_1]]$. The **ALIEN** operators act on $\widetilde{\mathbf{RESUR}}^{s}_{\mathbb{C}}(\Omega)[[e^t]]$ by inverse Borel transform).

We are now going to introduce a fundamental **ALIEN** operator, the standard **ALIEN** derivation.

Proposition 2.4.1 *We define the standard* ***ALIEN*** *derivation* $\dot{\Delta}$ *by :*

$$\dot{\Delta} := \log\left(\dot{\Delta}^+\right) := \sum \log^\bullet \dot{\Delta}^+_\bullet = \sum_{\underline{\omega}\in\Omega^\bullet} \frac{(-1)^{l(\underline{\omega})+1}}{l(\underline{\omega})} \dot{\Delta}^+_{\underline{\omega}}.$$

Moreover,

$$\dot{\Delta}^+ = \exp\left(\dot{\Delta}\right) := \sum \exp^\bullet \dot{\Delta}_\bullet = \sum_{\underline{\omega}\in\Omega^\bullet} \frac{(-1)^{l(\underline{\omega})}}{l(\underline{\omega})!} \dot{\Delta}_{\underline{\omega}}.$$

Proof The fact that $\dot{\Delta}$ is a derivation on $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{C}}(\Omega)[[\delta_1]]$ is a direct consequence of the alternelity of $\log^\bullet$. The formula for $\dot{\Delta}^+$ follows from a direct computation. □

Remark 2.4.1.1

- *We have* $\dot{\Delta} = -\log\left(\dot{\Delta}^-\right)$ *because* $0 = \log(1) = \log\left(\dot{\Delta}^+\dot{\Delta}^-\right) = \log\left(\dot{\Delta}^+\right) + \log\left(\dot{\Delta}^-\right)$.
- *As long as* $\dot{\Delta}$ *is a derivation and* $\dot{\Delta}^\pm$ *is a convolution automorphism, the mould* $\exp^\bullet$ *is symmetral.*
- *A consequence of proposition 1.2.2 is that for any* $m \in \Omega$, $\left[\dot{\Delta}_m, \partial\right] = 0$.

2. This action makes sense because when an **ALIEN** operator $\mathbf{op} = \sum M^\bullet \dot{\Delta}^+_\bullet$ is evaluated on a resurgent function to a point $\zeta \in \mathbb{C}//\Omega$, the infinite sum defining it becomes a finite one as explained in remark 1.2.2.1.

Let us translate in **ALIEN** (Ω) what the meaning, for an operator, of being group-like or primitive.

Definition 2.4.1.1 *An* ***ALIEN*** *operator* $\boldsymbol{op} \in \boldsymbol{ALIEN}(\Omega)$ *is :*
— *a* ***convolution automorphism*** *if*

$$\forall \hat{\phi}, \hat{\psi} \in \widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)[[\delta_1]], \quad \boldsymbol{op}\left(\hat{\phi} \star \hat{\psi}\right) = \boldsymbol{op}\hat{\phi} \star \boldsymbol{op}\hat{\psi}.$$

— *a* ***derivation*** *if*

$$\forall \hat{\phi}, \hat{\psi} \in \widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)[[\delta_1]], \quad \boldsymbol{op}\left(\hat{\phi} \star \hat{\psi}\right) = \left(\boldsymbol{op}\hat{\phi}\right) \star \hat{\psi} + \hat{\phi} \star \left(\boldsymbol{op}\hat{\psi}\right).$$

Then we easily verify that :
— a primitive element of **ALIEN** (Ω) is a derivation of **ALIEN** (Ω).
— a group-like element of **ALIEN** (Ω) is a convolution automorphism of **ALIEN** (Ω).
Let us mention an important corollary of Proposition 1.2.2 :

Corollary 2.4.2 *For any* $m \in \Omega$*, we have* $\left[\boldsymbol{op}_m, \hat{\partial}\right] = 0$ *where* $\boldsymbol{op}_m$ *is the homogeneous component of order* m *of* $\boldsymbol{op} \in \boldsymbol{ALIEN}(\Omega)$

Proof It is a direct consequence of the mould-comould expansion of **op** with the comould $\dot{\Delta}^{\pm}$ and of the bilinearity of the Lie bracket. □

We conclude this paragraph by an important result already mentioned in [50] and proved in [53] (Theorem 30.9) about composition of simple resurgent functions and the standard **ALIEN** derivation.

Proposition 2.4.3 *If* $\tilde{\phi}_1, \tilde{\phi}_2 \in \widetilde{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ *then* $\tilde{\phi}_1 \circ \left(id + \tilde{\phi}_2\right) \in \widetilde{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ *and for any* $\omega \in \Omega$*,*

$$\dot{\Delta}_\omega\left(\tilde{\phi}_1 \circ \left(id + \tilde{\phi}_2\right)\right) = \left(\dot{\Delta}_\omega \tilde{\phi}_1\right) \circ \left(id + \tilde{\phi}_2\right) + \left(\left(\partial \tilde{\phi}_1\right) \circ \left(id + \tilde{\phi}_2\right)\right) \dot{\Delta}_\omega \tilde{\phi}_2.$$

2.4.2 A sub-algebra of differential operators

In this section, we set $\Omega = \mathbf{N}^*$ but all the construction is still valid when considering for Ω a totally ordered semi-group of $\mathbb{C}$.

We denote by **ENDOM** $(\mathbb{C}[[u]])$ the algebra of endomorphisms on $\mathbb{C}[[u]]$ and we introduce a family of derivation $\mathbb{B} = (\mathbb{B}_m)_{m \in \Omega} \subset$ **ENDOM** $(\mathbb{C}[[u]])$. We then consider the realization of A_p with the $\mathbb{B}_m$ instead of the $\mathbf{e}_m$ and we denote it by $\mathcal{D}(\mathbb{B})$. In the next, with $m \in \mathbf{N}^*$, we will take $\mathbb{B}_m = u^{m+1}\partial_u$ or $\mathbb{B}_m = e^{-mu}\partial_u$.

Contrary to what happens with the **ALIEN** algebra and the comould defined by the Stokes automorphism, the comould $\mathbb{B}_\bullet$ does not define a basis of $\mathcal{D}(\mathbb{B})$ because it does not define a free family.

As previously, we will translate in $\mathcal{D}(\mathbb{B})$ what means for an operator to be group like or primitive. To do this, we need to put a topology on $\mathbb{C}[[u]]$. We do as in section 2.1.5, we define a pseudo-valuation

$$\mathrm{val}: \begin{cases} \mathbb{C}[[u]] & \longrightarrow & \mathbf{N}\cup\{\infty\} \\ \sum_{n\in\mathbf{N}} f_n u^n & \mapsto & \begin{cases} \min\{n\in\mathbf{N} \mid f_n \neq 0\} & \text{if } f\neq 0 \\ \infty & \text{otherwise} \end{cases} \end{cases}$$

and we consider $\mathbb{C}[[u]]$ as a metric space with metric given by $d(f(u),g(u)) = 2^{-\mathrm{val}(f-g)}$ for any $f(u), g(u) \in \mathbb{C}[[u]]$.

Proposition 2.4.4 *A primitive operator $\mathbb{D}$ on $\mathbb{C}[[u]]$ is a derivation on $\mathbb{C}[[u]]$ of the form $\mathbb{D} = d(u)\partial_u$ where $d(u) := \mathbb{D}.u \in \mathbb{C}[[u]]$.*

Moreover, if $\mathbb{D} = \sum M^\bullet \mathbb{B}_\bullet \in \mathcal{D}(\mathbb{B})$ with $M^\bullet$ alternal then $\mathbb{D} = d(u)\partial_u$ with $d(u) := \sum_{m\geq 1} d_m u^{m+1} \in u\mathbb{C}[[u]]$ and where for all $m \geq 1$,

$$d_m = \sum_{\underline{\omega}\in\Omega^\bullet, \|\underline{\omega}\|=m} \beta_{\underline{\omega}} M^{\underline{\omega}}$$

with for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$

$$\beta_{\underline{\omega}} = \begin{cases} (\check{\omega}_1+1)(\check{\omega}_2+1)\ldots(\check{\omega}_{r-1}+1) & \text{if } r\geq 2 \\ 1 & \text{otherwise} \end{cases}.$$

Finally $\mathbb{D}$ is continuous if $d(u) \in u\mathbb{C}[[u]]$.

Proof If $\mathbb{D}$ is primitive then it satisfies the Leibniz rule and it is a derivation on $\mathbb{C}[[u]]$. But the set of derivations on $\mathbb{C}[[u]]$ is a module of dimension one over the ring $\mathbb{C}[[u]]$. Then $\mathbb{D} = d(u)\partial_u$ with $d(u) := \mathbb{D}.u \in \mathbb{C}[[u]]$.

We easily verify for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ that

$$\mathbb{B}_{\underline{\omega}} = \mathbb{B}_{\omega_r}.\ldots.\mathbb{B}_{\omega_1}.u = \left(u^{\omega_r+1}\partial.\left(\ldots\left(u^{\omega_1+1}\partial.u\right)\right)\right) = \beta_{\underline{\omega}} u^{\|\underline{\omega}\|+1}$$

then for any $m \in \Omega$,

$$\mathbb{D}_m.u = \left(\sum_{\underline{\omega}\in\Omega^\bullet, \|\underline{\omega}\|=m} M^\bullet\mathbb{B}_\bullet\right).u = \left(\sum_{\underline{\omega}\in\Omega^\bullet, \|\underline{\omega}\|=m} M^{\underline{\omega}}\beta_{\underline{\omega}}\right) u^{m+1} = d_m u^{m+1}.$$

As a consequence of the alternality of M, it comes $M^\emptyset = 0$ and so $d(u) \in u\mathbb{C}[[u]]$.

We consider now the space $\mathcal{D}(\mathbb{B}).u$. It is a sub-space of $\mathbb{C}[[u]]$ with the same pseudo-valuation. In this sub-space, the family $(M^{\underline{\omega}}\mathbb{B}_w.u)_{\underline{\omega}\in\Omega^\bullet\setminus\{\emptyset\}}$ is summable of sum $d(u)$. Indeed, for $\omega \in \Omega$ and $I_0 = \{\underline{\omega}\in\Omega^\bullet \mid \|\underline{\omega}\| \leq \omega\}$, one has for any finite $I_0 \subset J \subset \Omega^\bullet$

$$\mathrm{val}\left(\sum_{\underline{\omega}\in J} M^{\underline{\omega}}\mathbb{B}_{\underline{\omega}}.u - d(u)\right) \geq \mathrm{val}\left(\sum_{\underline{\omega}\in I_0} M^{\underline{\omega}}\mathbb{B}_{\underline{\omega}}.u - d(u)\right) = \mathrm{val}\left(\sum_{m>\omega} d_m u^{m+1}\right) > \omega.$$

Finally, for $\omega \in \Omega$, and for $f(u) \in \mathbb{C}[[u]]$ of pseudo-valuation at less ω, one has $\mathrm{val}\,(\mathbb{D}f(u)) = \mathrm{val}\,(d(u)f'(u)) \geq \omega$ if $d(u) \in u\mathbb{C}[[u]]$. In this case, $\mathbb{D}$ is continuous.

□

Proposition 2.4.5 *A continuous group-like element $\mathbf{F} \in \mathcal{D}(\mathbb{B})$ is a substitution automorphism*

$$\mathbf{F} = \mathbf{F}_f : \left\{ \begin{array}{ccc} \mathbb{C}[[u]] & \longrightarrow & \mathbb{C}[[u]] \\ h(u) & \mapsto & h(f(u)) \end{array} \right.$$

where $f(u) := \mathbf{F}.u \in \mathbb{C}[[u]]$. If $\mathbf{F}_m$ denotes the homogeneous component of order m of $\mathbf{F}$, then one has also, if $f(u) \in u + u^2\mathbb{C}[[u]]$,

$$\forall m \geq 1, \quad \mathbf{F}_m = \sum_{\substack{\omega_1, \ldots, \omega_r \geq 1 \\ \omega_1 + \ldots + \omega_r = m}} \frac{f_{\underline{\omega}}}{r!} u^{m+r} \partial_u^r$$

with for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$

$$f_{\underline{\omega}} = \begin{cases} f_{\omega_1} \times \ldots \times f_{\omega_r} & \text{if } r \geq 1 \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, if $F = \sum M^\bullet \mathbb{B}_\bullet$ with $M^\bullet$ symmetral then $f(u) = M^\emptyset u + \sum_{m\geq 1} f_m u^{m+1} \in u\mathbb{C}[[u]]$ with for any $m \geq 1$

$$f_m = \sum_{\underline{\omega} \in \Omega^\bullet, \|\underline{\omega}\| = m} \beta_{\underline{\omega}} M^{\underline{\omega}}$$

and where $M^\emptyset = 0$ or 1.

Conversely, if $\mathbf{F}$ is a substitution automorphism

$$\mathbf{F} = \mathbf{F}_f : \left\{ \begin{array}{ccc} \mathbb{C}[[u]] & \longrightarrow & \mathbb{C}[[u]] \\ h(u) & \mapsto & h(f(u)) \end{array} \right.$$

then $\mathbf{F}$ is a group-like element of $\mathcal{B}$ and it is continuous if $f(u) \in u\mathbb{C}[[u]]$.

Proof We assume that $\mathbf{F}$ is a continuous group-like element of $\mathcal{D}(\mathbb{B})$. Then for any $n \in \Omega$, one has $\mathbf{F}(u^\omega) = (\mathbf{F}.u)^\omega = (f(u))^\omega$. And so, for the Krull topology, it comes, for any $h(u) \in \mathbb{C}[[u]]$,

$$\mathbf{F}.h(u) = \mathbf{F}.\left(\lim_{n\to\infty} h^{[n]}(u)\right) = \lim_{n\to\infty} \mathbf{F}.\left(h^{[n]}\right)(u) = \lim_{n\to\infty} h^{[n]}(f(u)) = h(f(u))$$

where $h^{[n]}(u) := \sum_{k\geq 0}^{n} h_k u^k$ if $h(u) = \sum_{k\geq 0} h_k u^k$.

Let us assume that $f(u) = u + \sum_{n\geq 1} f_n u^{n+1} \in \mathbb{C}[[u]]$. Using a Taylor expansion, we obtain :

$$\begin{aligned}\mathbf{F}.h(u) &= h(f(u)) \\ &= h\left(u+\sum_{n\geq 1} f_n u^n\right) \\ &= \sum_{r\in\mathbf{N}} \frac{1}{r!}\left(\sum_{n\geq 1} f_n u^{n+1}\right)^r \partial_u^r h(u) \\ &= \sum_{m\in\mathbf{N}} \sum_{\substack{\omega_1,\ldots,\omega_r\geq 1\\ \omega_1+\ldots+\omega_r=m}} \frac{f_{\omega_1}\times\ldots\times f_{\omega_r}}{r!} u^{m+r}\partial_u^r h(u) \\ &= \sum_{m\in\mathbf{N}} \mathbf{F}_m\end{aligned}$$

with for any $m\in\mathbf{N}^*$,

$$\mathbf{F}_m = \sum_{\substack{\omega_1,\ldots,\omega_r\geq 1\\ \omega_1+\ldots+\omega_r=m}} \frac{f_{\omega_1}\times\ldots\times f_{\omega_r}}{r!} u^{m+r}\partial_u^r.$$

We assume from now that $\mathbf{F}=\sum M^\bullet\mathbb{B}_\bullet$ with $M^\bullet$ symmetral (and so $\mathbf{F}$ is group-like). Then one has $\left(M^\emptyset\right)^2=M^\emptyset$ and so $M^\emptyset=0$ or 1. We easily verify as in the previous proposition that $\left(M^{\underline{\omega}}\mathbb{B}_{\underline{\omega}}.u\right)_{\underline{\omega}\in\Omega^\bullet}$ is summable in $\mathbb{C}[[u]]$ of sum $f(u)=M^\emptyset u+\sum_{m\geq 1} f_m u^{m+1}\in u\mathbb{C}[[u]]$ and that for any $m\geq 1$

$$f_m=\sum_{\underline{\omega}\in\Omega^\bullet,\|\underline{\omega}\|=m} \beta_{\underline{\omega}} M^{\underline{\omega}}.$$

We prove that $\mathbf{F}$ is continuous. Let us observe that $\mathbb{C}[[u]]$ is the completion for the Krull topology of the space $\mathbb{C}[u]$. And $\mathbf{F}_{|\mathbb{C}[[u]]}:\mathbb{C}[u]\to\mathbb{C}[[u]]$ is an uniformly continuous map. Indeed, if $P(u),Q(u)\in\mathbb{C}[[u]]$, then one has $\operatorname{val}\left(\mathbf{F}_{|\mathbb{C}[[u]]}.P(u)-\mathbf{F}_{|\mathbb{C}[[u]]}.Q(u)\right)=\operatorname{val}\left(P(f(u))-Q(f(u)\right)\geq\operatorname{val}\left(P(u)-Q(u)\right)$ because $f(u)\in u\mathbb{C}[[u]]$. Then the continuation $\mathbf{F}$ of $\mathbf{F}_{|\mathbb{C}[[u]]}$ on $\mathbb{C}[[u]]$ is continuous on the completion $\mathbb{C}[[u]]$ of $\mathbb{C}[u]$. Applying what precedes, we can affirm that $F=\mathbf{F}_f$.

Finally, if $\mathbf{F}=\mathbf{F}_f\in\mathcal{D}(\mathbb{B})$ is a a substitution automorphim then one verify easily it is group-like and that it is continuous if $f(u)\in u\mathbb{C}[[u]]$. □

Remark 2.4.2.1 *With the previous notations, for any $n,m\in\mathbf{N}$, it comes :*

$$\mathbf{F}_m.u^n = \sum_{\substack{\omega_1,\ldots,\omega_r\geq 1\\ \omega_1+\ldots+\omega_r=m}} f_{\omega_1}\times\ldots\times f_{\omega_r}\binom{n}{r}u^{n+m}$$

and for $m_1,\ldots,m_s\in\mathbf{N}$, we get :

$$\mathbf{F}_{m_s}.\ldots.\mathbf{F}_{m_1}.u^n = \sum_{\|\omega^1\|=m_1,\ldots,\|\omega^s\|=m_s} f_{\underline{\omega}^1}\times\ldots\times f_{\underline{\omega}^s}\binom{n}{r_1}\binom{n+m_1}{r_2}\binom{n+\check{m}_2}{r_3}\times\ldots\times\binom{n+\check{m}_{s-1}}{r_s}u^{n+m_1+\ldots+m_s}$$

with for any $i \in [\![1, s]\!]$, $\underline{\omega}^i = \left(\omega_1^i, \ldots, \omega_{r_i}^i\right) \in \Omega^\bullet$, $r_i = l\left(\underline{\omega}^i\right)$ *and* $f_{\underline{\omega}^i} = f_{\omega_1^i} \times \ldots \times f_{\omega_k^i}$.

Chapitre 3

ALIEN operators, bridge equation and reduction

Le malheur peut être un pont vers le bonheur
Proverbe japonais

3.1 Introduction

The bridge equation is a fundamental tool in Ecalle's work. It allows to translate **ALIEN** operators into ordinary differential operators.

More precisely, for a given dynamical system, the bridge equation enables the construction of a continuous [1] anti-morphism called *reduction* and denoted **red** between the algebras of **ALIEN** operators and a sub-algebra of **ENDOM** $(\mathbb{C}[[u_1,\ldots,u_n]])$ that is the set of all endomorphisms on the space of formal power series in n variables $\mathbb{C}[[u_1,\ldots,u_n]]$ [2].

In this chapter, we will give two fundamental examples. The first one consists of the saddle-node problem, and the second one concerns analytic germs tangent to identity. In both cases, the space of parameters will just count one variable and we will focus ourselves in the thesis to reduction from **ALIEN** into **ENDOM** $(\mathbb{C}[[u]])$.

As a consequence of the fact that **red** will be an antimorphism of Hopf algebras and of section 2.4, we have :

Proposition 3.1.1 *The reduction of*
— *a derivation D of* ***ALIEN*** *is a derivation $\mathbb{D}_d \in$* ***ENDOM*** $(\mathbb{C}[[u]])$ *where $d(u) =$* ***red*** $(D).u$.

1. for the topology inherited by the graduation.
2. The number n of variables depends on the studied dynamical system.

— *a convolution automorphism* F *of* $\boldsymbol{ALIEN}$ *is a substitution automorphism* $\mathbf{F}_f \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *where* $f(u) = \boldsymbol{red}(F).u$.

We will now give the two aforementioned explicit examples.

3.2 The saddle-node problem

3.2.1 Introduction

Consider a germ of a complex analytic 2-dimensional vector field :

$$X = x^2 \frac{\partial}{\partial_x} + A(x,y) \frac{\partial}{\partial_y} \qquad A(x,y) \in \mathbb{C}\{x,y\} \tag{3.1}$$

for which we assume :

$$A(0,y) = y, \qquad \frac{\partial^2 A}{\partial x \partial y}(0,0) = 0.$$

It is well known that there exists a unique formal diffeomorphism θ of the form $\theta(x,y) = (x, \varphi(x,y))$ (with $\varphi(x,y) = y + \sum_{n \in \mathbf{N}} \varphi_n(x) y^n$ and where $\forall n \in \mathbf{N}, \quad \varphi_n(x) \in x\mathbb{C}[[x]]$) that conjugates the vector field X to the vector field

$$X_0 = x^2 \frac{\partial}{\partial_x} + y \frac{\partial}{\partial_y}.$$

Observe that we can parametrize the flow of X_0 by :

$$\begin{cases} x &= -1/t \\ y &= ue^t \end{cases}, \quad u \in \mathbb{R}, \quad t \in \mathbb{R}^*$$

and then

$$\begin{cases} x &= -1/t \\ y &= ue^t + \sum_{n \in \mathbf{N}} \varphi_n(-1/t) u^n e^{nt} \end{cases}, \quad u \in \mathbb{C}, \quad t \in \mathbb{C}^*$$

is a formal parametrization of the flow of X.

A classical result is that for all $n \in \mathbf{N}$, the functions $\tilde{\varphi}_n(t) := \varphi_n(-1/t)$ are simple resurgent ones and their Borel transform $\hat{\varphi}_n(\zeta)$ admit singularities belonging to $n - \mathbf{N}^*$ (see [50]).

Moreover, the analytic continuation of the $\hat{\varphi}_n$ along a path starting from 0 and cutting $\mathbb{R} \setminus \Omega$ a finite number of times is of exponential growth (see [50]).

We can then perform the Laplace transform of such analytic continuation and we obtain a sum of the formal conjugant.

In particular, for all $n \in \mathbf{N}$ we can perform a Laplace transform of $\hat{\varphi}_n$ along the half line of origin 0 and direction $\theta \in]0,\pi[$ or $\theta \in]\pi, 2\pi[$ in the Borel plane. In addition, it exists $R, \epsilon > 0$ such that

— $y^+(t,u) = ue^t + \sum_{n\in\mathbf{N}} \varphi_n^+(t)\, u^n e^{nt}$ that is defined and analytic for

$$t \in S^+(R,\epsilon) = \{t \in \mathbb{C} \mid -\pi/2 + \epsilon \le \arg t \le 3\pi/2 - \epsilon, |t| \ge R\}$$

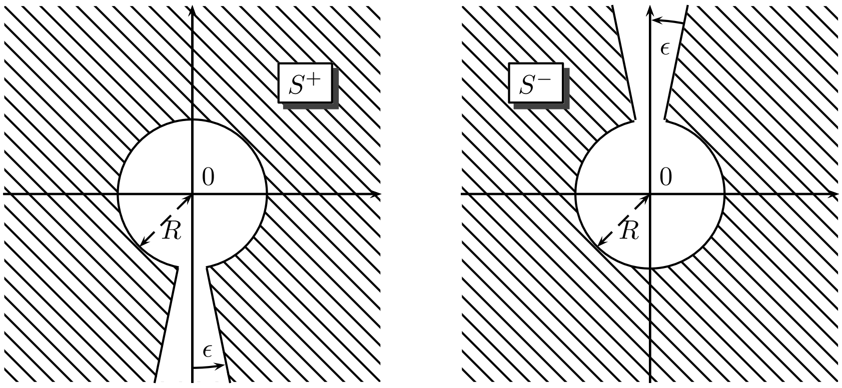
and u small enough

— $y^-(t,u) = ue^t + \sum_{n\in\mathbf{N}} \varphi_n^-(t)\, u^n e^{nt}$ that is defined and analytic for

$$t \in S^-(R,\epsilon) = \{t \in \mathbb{C} \mid -3\pi/2 + \epsilon \le \arg t \le \pi/2 - \epsilon, |t| \ge R\}$$

and u small enough

are two sums of the formal conjugant.



J. Martinet and J. P. Ramis (see [39]) have proved that there exists a unique pair

$$\left(\xi_{-1}, \xi(t) = t + \sum_{m\in\mathbf{N}^*} \xi_m t^{m+1}\right) \in \mathbb{C} \times u\mathbb{C}\{u\}$$

such that :

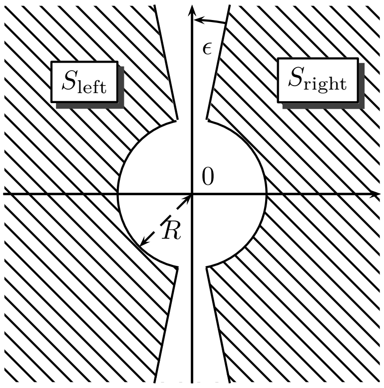
$$\forall t \in S_{\text{left}}, \quad y^-(t,u) = y^+(t, u + \xi_{-1})$$

$$\forall t \in S_{\text{right}}, \quad y^-(t,u) = y^+(t, \xi(u))$$

where

$$S_{\text{left}} = S^+ \cap S^- \cap \{z \in \mathbb{C} \mid \Re(z) < 0\} \text{ and } S_{\text{right}} = S^+ \cap S^- \cap \{z \in \mathbb{C} \mid \Re(z) > 0\}.$$

Moreover, this pair characterizes completely the analytic class of the studied saddle-node.



3.2.2 The bridge equation for the saddle-node

Consider a given complex analytic saddle-node X as in the previous section and a formal parameterization of its flow $\tilde{\varphi}(t,u) = ue^t + \sum_{n\in\mathbf{N}} \tilde{\varphi}_n(t)\, u^n e^{nt}$. The formal series $\tilde{\varphi}(t,u)$ satisfies the differential equation :

$$\partial_t y = \tilde{A}(t,y)$$

where $\tilde{A}(t,y) = A(-1/t, y)$. For all $n \in \mathbf{N}$, $\tilde{\varphi}_n(t) = \varphi_n(-1/t) \in \widetilde{\mathbf{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)$, and thus $\tilde{\varphi}(t,u) \in \widetilde{\mathbf{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)[[ue^t]]$. We extend the action of $\mathbf{ALIEN}(\mathbb{Z}^*)$ on $\widetilde{\mathbf{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)[[ue^t]]$ by setting for all $\phi(t,u) = \sum_{n\geq 0} \tilde{\phi}_n(t)\, u^n e^{nt} \in \widetilde{\mathbf{RESUR}}_{\mathbb{C}}[[ue^t]]$ and any $\mathbf{op} \in \mathbf{ALIEN}(\mathbb{Z}^*)$:

$$\mathbf{op}\phi(t,u) := \sum_{n\geq 0} \left(\mathbf{op}\tilde{\phi}_n(t)\right) u^n e^{nt}.$$

Theorem 3.2.1 *There exists $(C_m)_{m\in\{-1\}\cup\mathbf{N}^*}$ a complex sequence such that :*

$$\forall m \in \{-1\} \cup \mathbf{N}^*, \quad \boxed{e^{-mt}\Delta_m \tilde{\varphi}(t,u) = C_m u^{m+1} \partial_u \tilde{\varphi}(t,u)}. \tag{3.2}$$

This equality is ***called Bridge Equation for the saddle-node*** *and the terms of the complex sequence $(C_m)_{m\in\{-1\}\cup\mathbf{N}^*}$ are called Ecalle's invariants. Moreover $e^{-mt}\Delta_m \tilde{\varphi}(t,u) = 0$ if $m \leq -2$.*

Proof Let us consider $m \in \mathbf{N}^*$. As a consequence of the commutation between $\dot{\Delta}$ and ∂_t (see corollary 2.4.2), we know that the operators $\dot{\Delta}_m$ and ∂_z commute. The same occurs for the operators ∂_t and ∂_u. We apply the two operators $\dot{\Delta}_m$ and ∂_u to the equality $\partial_t \tilde{\varphi}(t,u) = \tilde{A}(t, \varphi(t,u))$ and we obtain the two relations :

$$\begin{aligned} \partial_t \left(\dot{\Delta}_m \tilde{\varphi}(t,u)\right) &= \dot{\Delta}_m \tilde{\varphi}(t,u)\, \partial_y \tilde{A}(t, \tilde{\varphi}(t,u)) \\ \partial_t \left(\partial_u \tilde{\varphi}(t,u)\right) &= \partial_u \tilde{\varphi}(t,u)\, \partial_y \tilde{A}(t, \tilde{\varphi}(t,u)). \end{aligned}$$

Consequently, $\partial_u \tilde{\varphi}(t,u)$ and $\dot{\Delta}_m \tilde{\varphi}(t,u)$ are formal solutions of a same first order linear equation :

$$\partial_t Y = Y \partial_y \tilde{A}(t, \tilde{\varphi}(t,u))$$

and so are proportional :

$$\dot{\Delta}_m \tilde{\varphi}(t,u) = \alpha \partial_u \tilde{\varphi}(t,u) \quad (\clubsuit)$$

where α is a complex number that may depend on m, u but in no way on t. We are now going to determine α. If $m \geq 1$, on one hand one has :

$$\begin{aligned}
\dot{\Delta}_m \tilde{\varphi}(t,u) &= \sum_{k\geq 0} \dot{\Delta}_m \tilde{\varphi}_k(t) u^k e^{kt} \\
&= \sum_{k\geq m} \dot{\Delta}_m \tilde{\varphi}_k(t) u^k e^{kt} \text{ because } \dot{\Delta}_m \tilde{\varphi}_k(t) = 0 \text{ if } k < m \\
&= \sum_{k\geq m} \Delta_m \tilde{\varphi}_k(t) u^k e^{(k-m)t} \\
&= \sum_{k\geq 0} \Delta_m \tilde{\varphi}_{k+m}(t) u^{k+m} e^{kt} \\
&= u^{m+1} \sum_{k\geq 0} \Delta_m \tilde{\varphi}_{k+m}(t) u^{k-1} e^{kt} \ (\spadesuit)
\end{aligned}$$

and on the other one :

$$\partial_u \tilde{\varphi}(t,u) = \sum_{k\geq 0} \tilde{\varphi}_k(t) k u^{k-1} e^{kt} \quad (\diamondsuit).$$

The equality $\dot{\Delta}_m \tilde{\varphi}_k(t) = 0$ if $k < m$ is an obvious consequence of the aforementioned property of the $\hat{\varphi}_n(\zeta)$ which admit singularities belonging only to $n - \mathbf{N}^*$ (see [50]).

Using the equalities ($\spadesuit$) and ($\diamondsuit$) into the relation ($\clubsuit$), we obtain $\alpha = C_m u^{m+1}$ with $C_m \in \mathbb{C}$.

If $m = -1$ then $\dot{\Delta}_{-1}\tilde{\varphi}(t,u) = \sum_{k\geq 0} \Delta_{-1}\tilde{\varphi}_k(t) u^k e^{(k+1)t}$ and $\partial_u \tilde{\varphi}(t,u) = \sum_{k\geq 0} k\tilde{\varphi}_k(t) u^{k-1} e^{kt} = \sum_{k\geq 1} (k+1) \tilde{\varphi}_{k+1}(t) u^k e^{(k+1)t}$. These two vectors are collinear if and only if there exists a constant $C_{-1} \in \mathbb{C}$ in a such way that $\dot{\Delta}_{-1}\tilde{\varphi}_k = C_{-1}\tilde{\varphi}_{k+1}$.

If $m \leq -2$ then $\dot{\Delta}_m \tilde{\varphi}(t,u) = \sum_{k\geq 0} \dot{\Delta}_{-1}\tilde{\varphi}_k(t) u^k e^{(k+m)t}$ that can not be proportional to $\partial_u \tilde{\varphi}(t,u)$.

Moreover, let us remark that we have the relations :

$$\forall n \in \mathbf{N}^*, \quad \Delta_m \tilde{\varphi}_n = \begin{cases} (n-m) C_m \tilde{\varphi}_{n-m} & \text{if } n > m \\ 0 & \text{otherwise} \end{cases}.$$

□

The Martinet-Ramis invariants characterize completely the analytic class of the studied saddle-node. As a consequence of the formulas of the next section which express the Ecalle's invariant's in terms of those of Martinet-Ramis, the same occurs for the Ecalle's invariants.

Remark 3.2.2.1 *Let us observe that the only property used about $\dot{\Delta}$ in the previous proof is the fact that it is a derivation. Thus, we can write a bridge equation for any derivation of **ALIEN** from which one a new family of analytical invariants is obtained. We will come back to this important point at the end of this chapter.*

We set $\mathbb{B}_m := u^{m+1}\partial_u$. As a consequence of the commutation between $\dot{\Delta}_m$ and ∂, we can write :

$$\dot{\Delta}_m \dot{\Delta}_n \tilde{\varphi}(t,u) = C_m C_n \mathbb{B}_n . \mathbb{B}_m . \tilde{\varphi}(t,u) .$$

Let us remark the reversal of the indexes. For $\Omega = \mathbf{N}^*$ (or $\Omega = -\mathbf{N}^*$), let us define **red** on the Hopf algebra $\langle \dot{\Delta} \rangle$ spanned by the homogeneous components of the standard **ALIEN** derivation and their compositions to the Hopf algebra $\mathcal{D}(\mathbb{B})$ by assuming it to be an Hopf algebra anti-morphism and by setting $\mathbf{red}\left(\dot{\Delta}_m\right) = C_m \mathbb{B}_m$. One easily prove that **red** is continuous for the topology inherited by the gradation on this two algebras and so one can extend it on the completion $\mathbf{ALIEN}(\Omega)$ of $\langle \dot{\Delta} \rangle$ into a continuous Hopf algebra anti-morphism $\mathbf{red} : \mathbf{ALIEN}(\Omega) \to \mathcal{D}(\mathbb{B})$. For an **ALIEN** operator $\mathbf{op} = \sum M^{\bullet} \dot{\Delta}_{\bullet} \in \mathbf{ALIEN}(\Omega)$, we can write :

$$\mathbf{op}\,\tilde{\varphi}(t,u) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} \dot{\Delta}_{\underline{\omega}} \tilde{\varphi}(t,u) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\omega)} \tilde{\varphi}(t,u) = \mathbf{red}(\mathbf{op}) . \tilde{\varphi}(t,u) \tag{3.3}$$

with

$$\mathbf{red}(\mathbf{op}) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\underline{\omega})}$$

and where $C_{\underline{\omega}} = C_{\omega_1} \times \ldots \times C_{\omega_r}$ if $\underline{\omega} = (\omega_1, \ldots, \omega_r)$.

The equality 3.3 becomes in the convolutive model

$$\mathbf{op}\,\hat{\varphi}(\zeta,u) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} \dot{\Delta}_{\underline{\omega}} \hat{\varphi}(\zeta,u) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\omega)} \hat{\varphi}(\zeta,u) = \mathbf{red}(\mathbf{op}) . \tilde{\varphi}(\zeta,u) . \tag{3.4}$$

Unexpectedly, when evaluating them at a point $\zeta \in \mathbb{R}_+//\Omega$, the sums contained in this equality are finite ones. Let us assume that $\zeta \in]M, M+1[$ with $M \in \mathbf{N}$. In the left side of the equality, the finiteness of the sum is a consequence of remark 1.2.2.1 page 25 and one has

$$\mathbf{op}\,\hat{\varphi}(\zeta,u) = \sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} \dot{\Delta}_{\underline{\omega}} \hat{\varphi}(\zeta,u) = \mathbf{op}^{[M]} \hat{\varphi}(\zeta,u)$$

where $\mathbf{op}^{[M]} = \sum_{k \leq M} \mathbf{op}_k$.

For the right side, one has, according to relation[3] 1.1 page 25,

$$\begin{aligned}
&\sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\omega)} \hat{\varphi}(\zeta,u)\\
=& \sum_{n\geq 0}\left(\hat{\varphi}_n(\zeta)\sum_{\underline{\omega}\in\Omega^{\bullet}} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\omega)}.u^n\right)\star\delta_{-n}\\
=& \sum_{n\geq 0}\sum_{m\geq 0}\left(\sum_{\underline{\omega}\in\Omega^{\bullet},\|\underline{\omega}\|=m} M^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega},n} u^{n+m}\hat{\varphi}_n(\zeta)\right)\star\delta_{-n}\\
=& \sum_{n\geq 0}\sum_{m\geq 0}\left(\sum_{\underline{\omega}\in\Omega^{\bullet},\|\underline{\omega}\|=m} M^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega},n} (\hat{\varphi}_n\star\delta_m)(\zeta)\right)u^{n+m}\star\delta_{-n-m}\\
=& \mathbf{F}^{[M]}\hat{\varphi}(\zeta,u)
\end{aligned}$$

where $\mathbf{F}^{[M]}=\sum_{k\leq M}\mathbf{F}_k$ and where for any $\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^{\bullet}$

$$\beta_{\underline{\omega},n}=\begin{cases}(\check{\omega}_1+n)(\check{\omega}_2+n)\ldots(\check{\omega}_{r-1}+n) & \text{if } r\geq 2\\ n & \text{otherwise}\end{cases}.$$

We have well defined a morphism of graded algebras $\mathbf{red}:\mathbf{ALIEN}(\Omega)\to\mathbf{ENDOM}\left(\widetilde{\mathbf{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)[[ue^t]]\right)$. As an application of Proposition 2.4.5 page 48, one has :

Corollary 3.2.2 *For a convolution automorphism* $\boldsymbol{op}\in\boldsymbol{ALIEN}(Z^*)$ *and for a given reduction*

$$\boldsymbol{red}:\boldsymbol{ALIEN}(\Omega)\to\boldsymbol{ENDOM}\left(\widetilde{\boldsymbol{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)[[ue^t]]\right)$$

one has

$$\boldsymbol{op}\tilde{\varphi}(t,u)=\tilde{\varphi}(t,f(u))$$

where $f(u)=\boldsymbol{red}(\boldsymbol{op}).u$.

Proof Indeed, $\mathbf{red}(\mathbf{op})$ is a substitution automorphism $\mathbf{F}_f$ of $\widetilde{\mathbf{RESUR}}_{\mathbb{C}}(\mathbb{Z}^*)[[ue^t]]$ with $f(u)=\mathbf{red}(\mathbf{op}).u$. □

When computing the reduction of $\dot{\Delta}^+$, using the fact that it is a convolution automorphism and also the results of section 2.4, we know that $\mathbf{red}\left(\dot{\Delta}^+\right)$ is a substitution automorphism $\mathbf{F}_f$ of $\mathbf{ENDOM}(\mathbb{C}[[u]])$ with $f(u):=\mathbf{red}\left(\dot{\Delta}^+\right).u=u+\sum_{m\geq 1}f_m u^{m+1}$ if $\Omega=\mathbf{N}^+$. Then $\dot{\Delta}^+\tilde{\varphi}(\zeta,u)=\varphi(\zeta,f(u))$. But we know too that for any $\zeta^{\epsilon_1,\ldots,\epsilon_m}\in\mathbb{R}_+//\mathbf{N}^*$, $\dot{\Delta}^+\hat{\varphi}(\zeta^{\epsilon_1,\ldots,\epsilon_m},u)=\hat{\varphi}\left(\zeta^{\overline{\epsilon_1},\ldots,\overline{\epsilon_m}}\right)$ and then $y^-(t,u)=y^+(t,f(u))$ for any $t\in S_{\text{right}}$. We recognize that f is the Martinet-Ramis sectoral isotropy : $\mathbf{red}\left(\dot{\Delta}^+\right).u$ is thus an analytic germ.

3. which delivers here $\hat{\varphi}_n\star\delta_m=0$ while $m>M$.

3.2.3 Link between Martinet-Ramis invariants and Ecalle's invariants

Moreover, if $\Omega = \mathbf{N}^*$, we have the following expression for $\xi(u)$:

$$\begin{aligned}
\xi(u) &= \mathbf{red}\left(\dot{\Delta}^+\right).u \\
&= \mathbf{red}\left(\sum_{\underline{\omega}} < \dot{\Delta}^+, \Delta >^{\underline{\omega}} \Delta_{\underline{\omega}}\right).u \\
&= \mathbf{red}\left(\sum_{\underline{\omega}} \frac{(-1)^{\mathrm{l}(\underline{\omega})}}{\mathrm{l}(\underline{\omega})!}\Delta_{\underline{\omega}}\right).u \\
&= u + \sum_{m\in\mathbf{N}} \sum_{\substack{\|\underline{\omega}\| = m \\ \underline{\omega} \in \Omega^\bullet}} \frac{(-1)^{\mathrm{l}(\underline{\omega})}}{\mathrm{l}(\underline{\omega})!} C_{\underline{\omega}}\beta_{\underline{\omega}}u^{m+1}
\end{aligned}$$

and then if $\xi(u) = u + \sum_{m\in\mathbf{N}} \xi_m u^{m+1}$ we have, for all $m \in \mathbf{N}$:

$$\xi_m = \sum_{\substack{\|\underline{\omega}\| = m \\ \underline{\omega} \in \Omega^\bullet}} \frac{(-1)^{\mathrm{l}(\underline{\omega})}}{\mathrm{l}(\underline{\omega})!} C_{\underline{\omega}}\beta_{\underline{\omega}}$$

and we find back the expression of ξ given in [50]. If $\Omega = -\mathbf{N}^*$, the same computation leads to $\xi(u) = u - C_{-1}$.

We will now compute the Ecalle's invariants $(C_m)_{m\geq 1}$ in terms of those of Martinet-Ramis $(\xi_m)_{m\geq 1}$. We will deduce from our calculation that the sequence $(C_m)_{m\geq 1}$ is $1-$Gevrey. The same computation can be made to establish that the infinitesimal generator of an analytic diffeomorphism tangent to identity is a 1-Gevrey vector field.

Proposition 3.2.3 *For any $m \in \mathbf{N}^*$, we have :*

$$C_m = \sum_{\|\underline{m}\|=m} \frac{(-1)^{s+1}}{s} \sum_{\|\underline{\omega}^1\|=m_1,\ldots,\|\underline{\omega}^s\|=m_s} \xi_{\underline{\omega}^1}\times\ldots\times\xi_{\underline{\omega}^s}\binom{\check{m}_{s-1}}{r_s}\times\ldots\times\binom{\check{m}_1}{r_2}\binom{1}{r_1}$$

with for any $i \in [\![1, s]\!]$, $\underline{\omega}^i = \left(\omega^i_1, \ldots, \omega^i_{r_i}\right) \in \Omega^\bullet$, $r_i = l\left(\underline{\omega}^i\right)$ and $\xi_{\underline{\omega}^i} = \xi_{\omega^i_1} \times \ldots \times \xi_{\omega^i_k}$ and $s = l(\underline{m})$, $\underline{m} = (m_1, \ldots, m_s)$.

Proof We know that the reduction of the Stokes automorphism $\dot{\boldsymbol{\Delta}}^+ = \mathbf{red}\left(\dot{\Delta}^+\right)$ is the substitution automorphism $\mathbf{F}_\xi$ of $\mathbb{C}[[u]]$ associated to the sectoral isotropy of Martinet-Ramis $\xi(u) = u + \sum_{m\geq 1} \xi_m u^{m+1}$. We know too that $\dot{\boldsymbol{\Delta}} =$

$\sum_{m\geq 1} C_m u^{m+1}\partial_u$ is the infinitesimal generator of $\dot{\boldsymbol{\Delta}}^{+}$: $\dot{\boldsymbol{\Delta}} = \log \dot{\boldsymbol{\Delta}}^{+}$. Then using mould calculus, we get :

$$\dot{\boldsymbol{\Delta}} = \sum_{\underline{m}=(m_1,\ldots,m_s)\in\Omega^\bullet} \frac{(-1)^{\mathrm{l}(\underline{m})+1}}{\mathrm{l}(\underline{m})} \dot{\boldsymbol{\Delta}}^{+}_{m_1}\ldots\dot{\boldsymbol{\Delta}}^{+}_{m_s}$$

and it comes, using remark 2.4.2.1 and with $\dot{\boldsymbol{\Delta}}_m = C_m u^{m+1}\partial_u$:

$$\dot{\boldsymbol{\Delta}}_m.u = \sum_{\substack{\underline{m}=(m_1,\ldots,m_s)\in\Omega^\bullet\\ \|\underline{m}\|=m}} \frac{(-1)^{\mathrm{l}(\underline{m})+1}}{\mathrm{l}(\underline{m})} \sum_{\|\omega^1\|=m_1,\ldots,\|\omega^s\|=m_s} \xi_{\underline{\omega}^1}\times\ldots$$
$$\ldots\times\xi_{\underline{\omega}^s}\binom{\check{m}_{s-1}}{r_s}\times\ldots\times\binom{\check{m}_1}{r_2}\binom{1}{r_1}u^{m+1}$$

and the result follows.

□

Lemma 3.2.4 *We have the classical bounds for the binomial coefficients :*

$$\forall n\in\mathbf{N},\quad \forall k\in[\![1,n]\!],\quad \left(\frac{n}{k}\right)^k\leq\binom{n}{k}\leq\frac{n^k}{k!}\leq\left(\frac{ne}{k}\right)^k$$

Corollary 3.2.5 *The sequence $(C_m)_{m\geq1}$ of Ecalle's invariants of a given saddle-node is $1-$Gevrey.*

Proof We use the same notations than in the proof of Proposition 3.2.3.

$$\begin{aligned}\binom{1}{r_1}\binom{\check{m}_1}{r_2}\times\ldots\times\binom{\check{m}_{s-1}}{r_s} &\leq \frac{1}{r_1!}\left(\frac{\check{m}_1}{r_2}\right)^{r_2}\times\ldots\times\left(\frac{\check{m}_{s-1}}{r_s}\right)^{r_s}e^{r_1+\ldots+r_s}\\ &\leq \frac{m^{r_1+\ldots+r_s}}{r_1^{r_1}\ldots r_s^{r_s}}e^{r_1+\ldots+r_s}\\ &\leq (me)^m\\ &\leq \frac{(m)^m}{m!}e^m m!\\ &\leq e^{2m}m!.\end{aligned}$$

Indeed with an easy inductive proof we get $\forall m\in\mathbf{N},\ \dfrac{m^m}{m!}\leq e^m$. As the sectoral isotropy ξ is an analytic germ, there exists $a,b\in\mathbb{R}_+^*$ such that : $\forall m\in$

$\mathbf{N}^*$, $|\xi_m| \le ab^m$. Then using formulas of Proposition 3.2.3, we obtain :

$$\begin{aligned}
|C_m| &\le \sum_{\|\underline{m}\|=m} \frac{1}{s} \sum_{\|\omega^1\|=m_1,\ldots,\|\omega^s\|=m_s} \left|\xi_{\underline{\omega}^1}\right| \times \ldots \times \left|\xi_{\underline{\omega}^s}\right| \binom{\check{m}_{s-1}}{r_s} \times \ldots \times \binom{\check{m}_1}{r_2}\binom{1}{r_1} \\
&\le \sum_{\|\underline{m}\|=m} \frac{1}{s} \sum_{\|\omega^1\|=m_1,\ldots,\|\omega^s\|=m_s} a^{l(\underline{\omega}_1)} \ldots a^{l(\underline{\omega}_s)} b^{\|\underline{\omega}_1\|} \ldots b^{\|\underline{\omega}_s\|} \\
&\le \sum_{\|\underline{m}\|=m} \frac{(2ab)^m e^{2m} m!}{s} \\
&\le \left(4abe^2\right)^m m!
\end{aligned}$$

□

3.3 Tangent to identity holomorphic germs

3.3.1 Introduction

Let us consider a non degenerate parabolic germ at infinity, i.e. any transformation f of the form [4] $f(z) = z + 2i\pi + a(z)$ where $a(z) = \sum_{n\ge2} a_n z^{-n} \in z^{-1}\mathbb{C}\left\{z^{-1}\right\}$ is a convergent power series. Let us observe that we have assumed the coefficient a_1 of z^{-1} in $a(z)$ to be null. The opposite of this coefficient ρ is called iterative residue in Ecalle's work (resiter for short) and we limit ourselves to the case where $\rho = 0$. In this subsection, we interest in the classification of such analytic germs.

A classical result (see [51]) is that for a given non degenerate parabolic germ at infinity with vanishing resiter there exists a unique formal transformation $\tilde{U}(z) = z + \tilde{u}(z) \in z + z^{-1}\mathbb{C}\left[\left[z^{-1}\right]\right]$ such that

$$\tilde{V} \circ f \circ \tilde{U}(z) = z + 2i\pi$$

where

$$\tilde{V} = \tilde{U}^{-1} = z + \tilde{v}(z), \quad \tilde{v}(z) \in z^{-1}\mathbb{C}\left[\left[z^{-1}\right]\right]$$

is the formal normalizing.

Instead of working with a sub-algebra of $\mathbb{C}[[z]][[ue^t]]$ as in the previous section, here we have to deal with a sub-algebra of $\mathbb{C}[z][[z^{-1}]]$.

Let us observe that the automorphism of substitution of $\mathbf{ENDOM}\left(\mathbb{C}[z][[z^{-1}]]\right)$ given by $\tilde{\phi}(z) \mapsto \tilde{\phi}(z+2i\pi)$ is $\exp(2i\pi\partial) = \sum_{r\ge0} \frac{(2i\pi)^r}{r!}\partial^r$ because this operator is group-like [5] and because one has $\exp(2i\pi\partial).z = z + 2i\pi$. An other expression of the conjugation equation for tangent to identity holomorphic germs is :

$$\exp(2i\pi\partial).\tilde{U}(z) = f \circ \tilde{U}(z).$$

4. The choice of the constant $2i\pi$ will induce that the singularities appearing here must be all located on $\mathbb{Z}^*$

5. It is also a consequence of the Taylor formula.

It is established in [51] that $\tilde{u}(z), \tilde{v}(z)$ are simple resurgent functions and their Borel transform $\hat{u}(\zeta), \hat{v}(\zeta)$ admit singularities belonging to $\mathbb{Z}^*$. The analytic continuation of $\hat{u}$ and $\hat{v}$ along a path starting from 0 and cutting $\mathbb{C} \setminus \mathbb{Z}^*$ a finite number of times is of exponential growth (see [51]). Then we can perform the Laplace transform of $\hat{u}$ and $\hat{v}$ along half line of origin 0 and direction $\theta \in]0, \pi[$ or $\theta \in]\pi, 2\pi[$ in the Borel plane. It exists $R, \epsilon > 0$ such that

— U^+ and V^+ defined and analytic in

$$S^+(R,\epsilon) = \{z \in \mathbb{C} \mid -\pi/2 + \epsilon \leq \arg z \leq 3\pi/2 - \epsilon, |z| \geq R\}$$

— U^- and V^- defined and analytic in

$$S^-(R,\epsilon) = \{z \in \mathbb{C} \mid -3\pi/2 + \epsilon \leq \arg z \leq \pi/2 - \epsilon, |z| \geq R\}$$

are two respective sums of the formal normalizing maps $\tilde{U}$ and $\tilde{V}$. We set

$$S_{\text{left}} = S^+ \cap S^- \cap \{z \in \mathbb{C} \mid \Re(z) < 0\} \text{ and } S_{\text{right}} = S^+ \cap S^- \cap \{z \in \mathbb{C} \mid \Re(z) > 0\}.$$

3.3.2 The bridge equation for tangent to identity holomorphic germs

Before writing the bridge equation for tangent to identity holomorphic germs, one notes that $\tilde{U}(z) = z + \tilde{u}(z)$ is not a resurgent function. One can define the Borel transform of z as being δ'. Indeed, using an integration by parts, one has

$$\mathcal{L}^\theta(\delta') = \langle \delta', e^{z\zeta} \rangle \, d\zeta = \langle \delta, -ze^{z\zeta} \rangle \, d\zeta = -z.$$

But the identity $Id : z \mapsto z$ is an entire function and so one can assume that $\dot{\Delta}_m \delta' = 0$ which permits to write the following Theorem.

Theorem 3.3.1 *The bridge equation for tangent to identity holomorphic germs is given in the formal model for any* $m \in \mathbb{Z}^*$ *by :*

$$\boxed{\dot{\Delta}_m \tilde{U}(z) = A_m e^{-mz} \partial_z \tilde{U}(z)}$$

where $(A_n)_{n\in\mathbb{Z}^*}$ *is a complex sequence.*

Proof As a consequence of the commutation between $\dot{\Delta}$ and ∂_z (see corollary 2.4.2), we know that the operators $\dot{\Delta}_m$ and ∂_z commute. Then the same occurs for the operators $\dot{\Delta}_m$ and $\exp(2i\pi\partial_z)$: $[\dot{\Delta}_m, \exp(2i\pi\partial_z)] = 0$. In the same way, we easily verify that $[\partial_z, \exp(2i\pi\partial_z)] = 0$. We consider now the conjugation relation $\exp(2i\pi\partial_z).\tilde{U}(z) = f \circ \tilde{U}(z)$ and we apply to it the operators $\dot{\Delta}_m$ and ∂_z. Because of the aforementioned commutations and with the help of Proposition 2.4.3, we obtain :

$$\begin{aligned} \exp(2i\pi\partial_z).\left(\dot{\Delta}_m \tilde{U}(z)\right) &= \dot{\Delta}_m \tilde{U}(z) f' \circ \tilde{U}(z) \\ \exp(2i\pi\partial_z).\left(\partial_z \tilde{U}(z)\right) &= \partial_z \tilde{U}(z) f' \circ \tilde{U}(z). \end{aligned}$$

Then $\dot{\Delta}_m \tilde{U}(z)$ and $\partial_z \tilde{U}(z)$ are formal solutions of the same linear equation :

$$\exp(2i\pi\partial_z).F(z) = F(z) f' \circ \tilde{U}(z).$$

But as long as $\partial_z\tilde{U}(z) = 1 + \partial_z\tilde{u}(z)$, $\partial_z\tilde{U}(z)$ is an invertible series and $\dot{\Delta}_m\tilde{U}(z)/\partial_z\tilde{U}(z)$ is a formal solution of the differences equation

$$\tilde{g}(z+2i\pi) - \tilde{g}(z) = 0.$$

Using a Taylor expansion, this equation becomes $\sum_{r\geq 1}(2i\pi)^r/r!\partial^r\tilde{g}(z) = 0$ and taking its Borel transform, we obtain $\sum_{r\geq 1}(-2i\pi\zeta)^r/r!\hat{g}(\zeta) = 0$ which can be written

$$\left(e^{-2i\pi\zeta} - 1\right)\hat{g}(\zeta) = 0.$$

The solutions of this equation are, in the convolutive model, the functions $\hat{g}(z) = \sum_{n\in\mathbb{Z}} A_n\delta_n(\zeta)$ where $(A_m)_{m\in\mathbb{Z}}$ is a complex sequence. These solutions become in the formal model $\tilde{g}(z) = \sum_{n\in\mathbb{Z}} A_n e^{-nz}$. Then there exists a complex sequence $(A_n)_{n\in\mathbb{Z}}$ such that $\dot{\Delta}_m\tilde{U}(z)/\partial_z\tilde{U}(z) = \sum_{n\in\mathbb{Z}} A_n e^{-nz}$. Using the definition of $\dot{\Delta}_m$, it comes

$$\frac{\Delta_m\tilde{U}(z)}{\partial_z\tilde{U}(z)} = \sum_{n\in\mathbb{Z}} A_n e^{-(n+m)z}.$$

But $\Delta_m\tilde{U}(z) \in \mathbb{C}[[z^{-1}]]$ and the same occurs for $\Delta_m\tilde{U}(z)/\partial_z\tilde{U}(z)$. Then we necessarily have $A_n = 0$ if $n+m \neq 0$ and $\Delta_m\tilde{U}(z)/\partial_z\tilde{U}(z) = A_{-m}$. We have proved that $\dot{\Delta}_m\tilde{U}(z) = A_{-m}e^{-mz}\partial_z\tilde{U}(z)$.

□

The complex sequence $(A_m)_{m\in\mathbb{Z}^*}$ characterizes completely the analytic class of f and its terms are called Ecalle's invariants for tangent to identity holomorphic germs.

We set $\mathbb{B}_m := A_m e^{-mz}\partial_z$. Because of the commutation between $\dot{\Delta}_m$ and ∂, we can write :

$$\dot{\Delta}_m\dot{\Delta}_n\tilde{U}(z) = \mathbb{B}_n.\mathbb{B}_m.\tilde{U}(z).$$

Observe the reversal of the indexes. As in the previous section, this relation permits to define a continuous anti-morphism of Hopf algebras **red**. For $\Omega = \mathbf{N}^*$ or $\Omega = -\mathbf{N}^*$ and for an **ALIEN** operator $\mathbf{op} = \sum M^\bullet\dot{\Delta}_\bullet \in \mathbf{ALIEN}(\Omega)$, we have[6] :

$$\mathbf{op}\,\tilde{U}(z) = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\dot{\Delta}_{\underline{\omega}}\tilde{U}(z) = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbb{B}_{\mathbf{rev}(\underline{\omega})}\tilde{U}(z) = \mathbf{red}(\mathbf{op}).\tilde{U}(z).$$

But we know that $\dot{\Delta}^+$ is a convolution automorphism of ENDOM $\left(\mathbb{C}[z][[z^{-1}]]\right)$ then with $\Omega = \mathbf{N}^*$

$$\dot{\Delta}^+\tilde{U}(z) = \mathbf{red}\left(\dot{\Delta}^+\right)\tilde{U}(z) = \tilde{U}(H_+(z))$$

with

$$H_+(z) := \mathbf{red}\left(\dot{\Delta}^+\right).z = z + \sum_{m\geq 1}\left(\sum_{\substack{\underline{\omega} = (\omega_1,\ldots,\omega_r)\\ \|\underline{\omega}\| = m}} \frac{(-1)^r}{r!}A_{\underline{\omega}}\beta_{\underline{\omega}}\right)e^{-mz}$$

6. As explained in the previous sections, these sums are in fact finite one.

if $\Re(z) > \tau_0$ and where

$$\beta_{\underline{\omega}} = \begin{cases} 1 & \text{if } \mathrm{l}(\underline{\omega}) = 1 \\ \omega_1(\omega_1+\omega_2)\ldots(\omega_1+\ldots+\omega_{r-1}) & \text{if } \underline{\omega} = (\omega_1,\ldots,\omega_r),\ r \geq 2 \end{cases}.$$

If $\Omega = -\mathbf{N}^*$, then we obtain

$$\dot{\Delta}^+\tilde{U}(z) = \mathbf{red}\left(\dot{\Delta}^+\right)\tilde{U}(z) = \tilde{U}(H_-(z))$$

with

$$H_-(z) := \mathbf{red}\left(\dot{\Delta}^+\right).z = z + \sum_{m\geq 1}\left(\sum_{\substack{\underline{\omega}=(\omega_1,\ldots,\omega_r)\\ \|\underline{\omega}\|=m}} \frac{(-1)^r}{r!}A_{\underline{\omega}}\beta_{\underline{\omega}}\right)e^{mz}$$

if $\Re(z) < -\tau_0$. The functions H_- and H_+ are the well known horn maps. They define analytic germs respectively in $\Re(z) > \tau_0$ and $\Re(z) < -\tau_0$. Their coefficients, as for the Ecalle's invariants, determine completely the analytic class of the studied non degenerate parabolic germ.

Moreover, according to Proposition 1.2.4, for $\theta > 0$ small enough and for $z \in S_{\text{right}}$, one has :

$$\left(\mathcal{L}_\theta.\mathcal{B}.\dot{\Delta}^+.\tilde{U}\right)(z) = \left(\mathcal{L}_\theta.\dot{\Delta}^+.\hat{U}\right)(z) = \left(\mathcal{L}_{-\theta}\hat{U}\right)(z) = U^-(z)$$

and, according to the previous calculus, one has two :

$$\left(\mathcal{L}_\theta.\mathcal{B}.\dot{\Delta}^+.\tilde{U}\right)(z) = \left(\mathcal{L}_\theta.\mathcal{B}.\tilde{U}\circ H\right)(z) = U^+(H(z)).$$

Thus we have verified that $U^-(z) = U^+(H(z))$.

3.4 Analytical invariants and ALIEN derivations

As mentioned in remark 3.2.2.1 page 56, considering a real dynamical system like those studied in the two previous sections, one can build a bridge equation for any **ALIEN** derivation D as we has perform it for the standard one. In the case of the saddle-node for example, one obtains a bridge equation like

$$e^{-mt}D_m\tilde{\varphi}(t,u) = A_m\mathbb{B}_m\tilde{\varphi}(t,u)$$

where D_m denotes the homogeneous component of order m of D, where A_m is a complex scalar and where $\mathbb{B}_m = u^{m+1}\partial_u$.

We will now explain that under certain conditions, the obtained scalar family (A_m) characterizes completely the analytical class of the studied saddle-node.

Let us observe that if $D = \sum N^\bullet\Delta_\bullet$ with $N^\bullet$ satisfying $N^{\underline{\omega}} \neq 0$ for any sequence $\underline{\omega} \in \Omega^\bullet$ of length 1, then the mould $N^\bullet$ is invertible for mould composition. Thus there exists an alternal mould $M^\bullet$ such that $\Delta = \sum M^\bullet D_\bullet$. In other words, the comould $D_\bullet$ defines a basis of the **ALIEN** algebra.

Considering from now an analytic class of saddle-nodes and the associated reduction $\mathbf{red} : \mathbf{ALIEN} \to \mathbf{ENDOM}\left(\mathbb{C}\left[[u]\right]\right)$.

We have define the reduction morphism $\mathbf{red} : \mathbf{ALIEN} \to \mathbf{ENDOM}\left(\mathbb{C}\left[[u]\right]\right)$ using the bridge equation for the standard **ALIEN** derivation but it is possible to perform the same construction with the bridge equation related to D. Let us denote by $\mathbf{red}'$ the obtained morphism. We naturally have $\mathbf{red} = \mathbf{red}'$. Indeed, one must have for any homogeneous component D_m of D :

$$\mathbf{red}\left(D_m\right) = \mathcal{A}_m \mathbb{B}_m \text{ and } \mathbf{red}'\left(D_m\right) = A_m \mathbb{B}_m$$

with a complex scalar $\mathcal{A}_m$. But these two equalities are the translation of

$$D_m \tilde{\varphi}(t,u) = A_m \partial_u \tilde{\varphi}(t,u),$$

and so necessarily $\mathcal{A}_m = A_m$ from what follows $\mathbf{red}' = \mathbf{red}$. In particular, one has $\mathbf{red}\left(\Delta\right) = \sum_{m\geq 1} C_m \mathbb{B}_m$ and $\mathbf{red}\left(D\right) = \sum_{m\geq 1} A_m \mathbb{B}_m$. The family (C_m) is the one already detailed in section 3.2 of Ecalle's analytical invariants.

It is then very easy to compute the family (A_m) starting from the Ecalle's invariants (C_m) and conversely. For example, one has :

$$\begin{aligned} \mathbf{red}\left(D\right) &= \sum_{\underline{\omega}} N^{\underline{\omega}} \mathbf{red}\left(\Delta_{\underline{\omega}}\right) \\ &= \sum_{\underline{\omega}} N^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\underline{\omega})} \end{aligned}$$

and so :

$$\begin{aligned} \sum_{m\geq 1} A_m u^{m+1} &= \mathbf{red}\left(D\right).u \\ &= \sum_{\underline{\omega}} N^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\underline{\omega})}.u \\ &= \sum_{m\geq 1} \left(\sum_{\underline{\omega}\in\Omega^{\bullet}, \|\underline{\omega}\|=m} N^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}} \right) u^{m+1}. \end{aligned}$$

Finally, it comes $A_m = \sum_{\underline{\omega}\in\Omega^{\bullet}, \|\underline{\omega}\|=m} N^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}}$. In the same way, one finds $C_m = \sum_{\underline{\omega}\in\Omega^{\bullet}, \|\underline{\omega}\|=m} M^{\underline{\omega}} A_{\underline{\omega}} \beta_{\underline{\omega}}$. Thus there is a biunivocal correspondence between Ecalle's families of analytical invariants and families (A_m) appearing in the bridge equation relative to D. That is why we call such families of coefficients (A_m) *families of invariants associated to* D. We abusively use this terminology in the following of the thesis to call the scalars appearing in the bridge equation on an **ALIEN** derivation $D = \sum M^{\bullet} \Delta_{\bullet}$, even if $M^{\underline{\omega}}$ may be null for words $\underline{\omega}$ of length 1.

Chapitre 4

Pre-coarborification

Je suis un être greffé. Je me suis fait à moi même plusieurs greffes. Greffer des mathématiques sur de la poésie, de la rigueur sur des images libres. des idées claires sur un tronc superstitieux.
Paul Valéry - Cahiers

4.1 Introduction

An essential concept in Mould theory is *arborification*, where the objects are indexed by trees ; in fact arborification goes along with the dual concept of *coarborification*. In the present chapter and the next one, we give an overall description of arborification/coarborification, remaining as close as possible to the presentation given by Ecalle, yet giving all the detailed proofs, in an elementary way, of the numerous combinatorial and algebraic properties involved. All the constructions are due to Ecalle, but for many of the intermediate results, no proof was to be found in the literature. We also include an original theorem on the "inverse of composition for arborescent moulds", and give several applications of this operation of composition, at the arborified level. We indicate at the relevant places the connections of Ecalle's theory with some well known Hopf algebras on trees (Connes-Kreimer, Grossman-Larson, etc) but we make no use of the language nor of results on Hopf algebras ; we shall only need to use the notion of coproduct.

Coarborification has to do with the composition of ordinary differential operators ; it has been known since Cayley (see [8]) that trees are the relevant combinatorial objects in this context and we introduce the subject by going right away into this. Already, when we compose two operators $\mathbb{B}_{m_1} = u^{m_1}\partial$ and $\mathbb{B}_{m_2} = u^{m_2}\partial$ where ∂ is the derivation with respect to the variable u (we choose to denote by the symbol $\bullet$ this composition), then, for a test function $f(u) \in \mathbb{C}[[u]]$ (we sometimes denote with a dot the action of an operator on a test function), we obtain :

$$\mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} f(u) = u^{m_2}\partial.\,(u^{m_1}\partial f(u)) = u^{m_2}(\partial u^{m_1})\partial f(u) + u^{m_2+m_1}\partial^2 f(u)\,.$$

We observe that $\mathbb{B}_{m_2}$ acts on the test function $f(u)$ but also on $\mathbb{B}_{m_1}$. The first action can be represented by the graph $\begin{smallmatrix} m_2 \\ | \\ \bullet_{m_1}\end{smallmatrix}$ (the node m_2 is placed to the end of an edge starting from the node m_1) and the second by the graph $\bullet_{m_1}\bullet_{m_2}$ (the node m_2 is placed at the same level than the node m_1). Consequently, it is suitable to index the operators that appear in the composition by graphs :

$$\mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} f(u) = \mathbb{B}_{\begin{smallmatrix} m_2 \\ | \\ \bullet_{m_1}\end{smallmatrix}} f(u) + \mathbb{B}_{\bullet_{m_1}\bullet_{m_2}} f(u).$$

For the composition of three such operators, we can write :

$$\begin{aligned}
\mathbb{B}_{m_3} \bullet \mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} f(u) &= u^{m_3}\partial.\left(u^{m_2}(\partial u^{m_1})\partial f(u) + u^{m_2+m_1}\partial^2 f(u)\right) \\
&= u^{m_3}(\partial u^{m_2})(\partial u^{m_1})\partial f(u) + u^{m_3+m_2}\left(\partial^2 u^{m_1}\right)\partial f(u) \\
&\quad + u^{m_3+m_2}(\partial u^{m_1})\partial^2 f(u) + u^{m_3+m_1}(\partial u^{m_2})\partial^2 f(u) \\
&\quad + u^{m_3+m_2}(\partial u^{m_1})\partial^2 f(u) + u^{m_3+m_2+m_1}\partial^3 f(u).
\end{aligned}$$

And using the previous encoding, we obtain :

$$\begin{aligned}
\mathbb{B}_{m_3} \bullet \mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} f(u) &= \mathbb{B}_{m_3} \bullet \left(\mathbb{B}_{\begin{smallmatrix} m_2 \\ | \\ \bullet_{m_1}\end{smallmatrix}} f(u) + \mathbb{B}_{\bullet_{m_1}\bullet_{m_2}} f(u)\right) \\
&= \mathbb{B}_{\begin{smallmatrix} m_3 \\ | \\ m_2 \\ | \\ \bullet_{m_1}\end{smallmatrix}} f(u) + \mathbb{B}_{\begin{smallmatrix} m_2 \;\; m_3 \\ \diagdown\diagup \\ \bullet_{m_1}\end{smallmatrix}} f(u) + \mathbb{B}_{\begin{smallmatrix} m_2 & \\ | & \\ \bullet_{m_1} & \bullet_{m_3}\end{smallmatrix}} f(u) + \\
&\quad \mathbb{B}_{\begin{smallmatrix} & m_3 \\ & | \\ \bullet_{m_1} & \bullet_{m_2}\end{smallmatrix}} f(u) + \mathbb{B}_{\begin{smallmatrix} & m_2 \\ & | \\ \bullet_{m_1} & \bullet_{m_3}\end{smallmatrix}} f(u) + \mathbb{B}_{\bullet_{m_1}\bullet_{m_2}\bullet_{m_3}} f(u).
\end{aligned}$$

The tree like structure of the graphs we use is becoming apparent. Before continuing, let us recall briefly some properties about trees and forests. We will be more precise in the next sections. For a set Ω, a rooted Ω-decorated tree is either the empty set, either a finite connected oriented graph, with elements of Ω located at each vertex, and with at most one incoming edge to each vertex. There is just one vertex without incoming edge, *the root* of the tree. An Ω-decorated forest is a collection of rooted Ω-decorated trees.

Let us observe that, in the previous expansions, all the operators intervene are of the form $P(u)\partial^k$ where $P(u) \in \mathbb{C}[u]$ and $k \in \mathbf{N}^*$. We say that $P(u)$ is the **polynomial part** of the operator $P(u)\partial^k$ and that ∂^k is its **derivative part**.

The rules for coding operators with rooted decorated forests are the following ones. For a forest T :

— If $\mathbb{B}_m$ acts on the derivative part of $\mathbb{B}_T$ then we write $\mathbb{B}_{\bullet_T \ \bullet_m}$ the resulting operator (where $\bullet_T \ \bullet_m$ is the forest obtained when juxtaposing the tree constituted of the single node m to the forest T).

— If $\mathbb{B}_m$ acts on the polynomial part of $\mathbb{B}_T$ and more precisely on the factor $u^{m'}$ (then m' is a node of T), we write $\mathbb{B}_{\substack{m \\ | \\ \bullet_T}}$ the resulting operator (where $\substack{m \\ | \\ \bullet_T}$ represents the forest obtained by inserting a node just behind the node m').

Now, the collection of differential operators $\mathbb{B}_T$ constructed by the previous rules will be called **the precoarborified** of the family $(\mathbb{B}_m)$. This notion doesn't appear as such in Ecalle, it will only be used as an *intermediate step* to define arborification, which is the true important concept indeed. We have decided to give a name to this intermediate construct to insist on the crucial role played by the *symmetry factors* ; these are in fact "hidden" in Ecalle's presentation but they are *essential* both for the algebraic structures and the eventual applications to analysis (growth estimates, see [14] or [15]) and we have considered it a necessity to explicit and highlight the way they appear in the calculations.

The point is that *the algebraic properties of the composition of operators are lifted to the tree level.* Indeed, if we consider two operators $\mathbb{B}_{\substack{m_2 \ m_3 \\ \vee \\ \bullet_{m_1}}} = u^{m_2+m_3}\left(\partial u^{m_1}\right)\partial$ and $\mathbb{B}_{\substack{m_5 \\ | \\ \bullet_{m_4}}} = u^{m_5}\left(\partial u^{m_4}\right)\partial$ then we can compute the composition of the first by the second one and our encoding allows us to write the expansion :

$$\mathbb{B}_{\substack{m_5 \\ | \\ \bullet_{m_4}}} \bullet \mathbb{B}_{\substack{m_2 \ m_3 \\ \vee \\ \bullet_{m_1}}} = \mathbb{B}_{\substack{m_5 \\ | \\ m_4 \\ | \\ m_2 \ m_3 \\ \vee \\ \bullet_{m_1}}} + \mathbb{B}_{\substack{m_5 \\ | \\ m_4 \\ | \\ m_2 \ m_3 \\ \vee \\ \bullet_{m_1}}} + \mathbb{B}_{\substack{m_5 \\ | \\ m_2 \ m_3 \ m_4 \\ \vee \\ \bullet_{m_1}}} + \mathbb{B}_{\substack{m_2 \ m_3 \\ \vee \\ \bullet_{m_1}} \ \substack{m_5 \\ | \\ \bullet_{m_4}}} .$$

As a consequence, one can understand the composition of two precoarborified operators as the product of two elements of the space spanned by decorated forests.

Grossman and Larson have given in the 80's a description of this in terms of Hopf algebras, see [27].

In numerical analysis, there is the foundational work of Butcher that has essentially uncovered the Hopf-algebraic structures underlying Runge-Kutta processes. Recent developments are the concepts of B-series ([11],[23], [37]) or P-series ([60]).

The interactions (isomorphisms, dualities of connected-graded Hopf algebras, coactions, ...) between the Hopf algebras involved are described in several papers ([30],[64],[46]).

Precoarborification is an important step in the process of coarborification and it differs from it just by a rational number depending of the tree that indexes the operator. That is what we will explain now.

The vector space spanned by decorated forests endowed with the product already mentioned and induced by the composition of differential operators becomes a bigebra with the coproduct given by :

$$\mathbf{cop}\,(T) = \sum_{T_1 \oplus T_2 = T} T_1 \otimes T_2$$

where the sum ranges over all the decompositions of T into two sub-forests. One recognize the Grossman-Larson coproduct (see [27]) but defined on forests instead of trees. For example :

$$\mathbf{cop}\left(\bullet 2 \quad \bullet 2\right) = \bullet 2 \quad \bullet 2 \otimes \emptyset + \bullet 2 \otimes \bullet 2 + \emptyset \otimes \bullet 2 \quad \bullet 2\,.$$

For the ordinary family of operators $(\mathbb{B}_m)$ with $\mathbb{B}_m = u^m \partial$ (which are the homogeneous components of a derivation, i.e. a primitive element of $\mathbf{ENDOM}\,(\mathbb{C}\,[[u]])$), and more precisely for the corresponding precoarborified family, one can write too :

$$\mathbf{cop}\left(\mathbb{B}_{\bullet_2\ \bullet_2}\right) = \mathbb{B}_{\bullet_2\ \bullet_2} \otimes 1 + 2\mathbb{B}_{\bullet_2} \otimes \mathbb{B}_{\bullet_2} + 1 \otimes \mathbb{B}_{\bullet_2\ \bullet_2} \qquad (✠)$$

which means that :

$$\begin{aligned} \mathbb{B}_{\bullet_2\ \bullet_2}(fg)\,(u) &= \left(u^4\partial^2\right)(f.g)\,(u) \\ &= u^4\partial^2 f\,(u)\,.g\,(u) + 2u^4\partial f\,(u)\,.\partial g\,(u) + f\,(u)\,.u^4\partial^2 g\,(u)\,. \end{aligned}$$

We observe, by contrast with what happens for the product, the coproduct of ordinary differential operators is not compatible with the one of decorated forests : we should have $\mathbb{B}_{\bullet_2\ \bullet_2} f\,(u)\,.g\,(u) + \mathbb{B}_{\bullet_2} f\,(u)\,.\mathbb{B}_{\bullet_2} g\,(u) + f\,(u)\,.\mathbb{B}_{\bullet_2\ \bullet_2} g\,(u)$ instead of (✠).

In our example, it is very easy to correct it. Indeed, instead of considering the operator $\mathbb{B}_{\bullet_2\ \bullet_2}$, we have to deal with the operator $\frac{1}{2}\mathbb{B}_{\bullet_2\ \bullet_2}$. We easily verify that

$$\frac{1}{2}\mathbb{B}_{\bullet_2\ \bullet_2}(fg)\,(u) = \frac{1}{2}\mathbb{B}_{\bullet_2\ \bullet_2} f\,(u)\,.g\,(u) + \mathbb{B}_{\bullet_2} f\,(u)\,.\mathbb{B}_{\bullet_2} g\,(u) + f\,(u)\,.\frac{1}{2}\mathbb{B}_{\bullet_2\ \bullet_2} g\,(u)\,.$$

In order to generalize this operation, we have to understand and to interpret the coefficient $\frac{1}{2}$ that we have plugged in the previous example. For a given pre-coarborified operator $\mathbb{D}_T$, we will prove that the inverse of this coefficient is the well known by algebraists **internal symmetry factor** $s\,(T)$ of the forest T. This integer measures the number of symmetries which preserve T. We will be more precise in Section 5.1. We just mention that this number is defined recursively and we give some examples, m, m', m'' being three distinct elements of Ω :

$$s\left(\bullet m\right) = 1!, \quad s\left(\bullet m \ \bullet m\right) = 2!, \quad s\left(\bullet m \ \bullet m'\right) = 1!,$$

$$s\left(\begin{array}{c} m \\ | \\ \bullet m \end{array}\right) = s\left(\bullet m\right) = 1!, \quad s\left(\begin{array}{c} m' \\ | \\ \bullet m \end{array}\right) = s\left(\bullet m'\right) = 1!,$$

$$s\left(\begin{array}{c} m' \quad m' \\ \diagdown\!\diagup \\ \bullet m \end{array}\right) = s\left(\bullet m' \, \bullet m'\right) = 2!, \quad s\left(\begin{array}{c} m' \quad m'' \\ \diagdown\!\diagup \\ \bullet m \end{array}\right) = s\left(\bullet m' \, \bullet m''\right) = 1!,$$

$$s\left(\begin{array}{c} m \\ | \\ m \\ | \\ \bullet m \end{array}\right) = s\left(\begin{array}{c} m \\ | \\ \bullet m \end{array}\right) = 1!, \quad s\left(\begin{array}{cc} m & \\ | & \\ \bullet m & \bullet m \end{array}\right) = s\left(\begin{array}{c} m \\ | \\ \bullet m \end{array}\right) \times s\left(\bullet m\right) = 1!,$$

$$s\left(\begin{array}{cc} m' \quad m' & m' \quad m' \\ \diagdown\!\diagup & \diagdown\!\diagup \\ \bullet m & \bullet m \end{array}\right) = 2! \times \left(s\left(\begin{array}{c} m' \quad m' \\ \diagdown\!\diagup \\ \bullet m \end{array}\right)\right)^2 = 2! \times (2!)^2.$$

We can then construct for a given family of ordinary differential operator $(\mathbb{B}_m)_{m\in\mathbf{N}}$ a family of operators $\left(\tilde{\mathbb{B}}_T\right)$ indexed by decorated forests with product and coproduct compatible with those of the bigebra of decorated forests.

The family of operators so obtained was called **the coarborified** of the family $(\mathbb{B}_m)_{m\in\mathbf{N}}$ by its discoverer, J. Ecalle.

Most other works involving trees for taming the complexity of the composition of differential operators are not really concerned by the *size* of the coefficients in the reexpansions obtained by an "indexation by trees". This crucial point, for the applications to difficult questions on analytic dynamical systems at singularities is evoked at the end of chapter 5 ; it is precisely this necessity of obtaining some form of *geometrical growth* for the coefficients that has lead Ecalle to the very rich algebraic structures of arborescent moulds and the like. Indeed, he was confronted to the problem of the convergence of normalizing transformations for differential equations. These transformations act on elements of $\mathbb{C}[[u]]$ and are constituted of infinite sum of composed operators like

$$\mathbf{F} = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\mathbb{D}_{\underline{\omega}} \in \mathbf{ENDOM}\left(\mathbb{C}\left[[u]\right]\right)$$

where as explained in chapter 2

— $\Omega \subset \mathbb{R}_+^*$ is a semi-group (i.e. a non empty set together with an associative binary operation) of $\mathbb{R}_+^*$. In this paper, we can consider that $\Omega = \mathbf{N}^*$.
— $\Omega^\bullet$ is the set of all finite sequences elements of Ω including the empty one.
— $(M^{\underline{\omega}})$ is an $\Omega^\bullet$ indexed family of scalars, scalar functions, etc... called **mould**.

— $\mathbb{D} = \sum_{m\in\Omega} \mathbb{D}_m \in \mathbf{ENDOM}\,(\mathbb{C}\,[[u]])$ is an ordinary differential operator on $\mathbb{C}[[u]]$ and the composed operator $\left(\mathbb{D}_{\underline{\omega}}\right)$, called **comould**, is given for any $\underline{\omega} = (\omega_1, \dots, \omega_r)$ by the product $\mathbb{D}_{\underline{\omega}} = \mathbb{D}_{\omega_r} \bullet\ \dots\ \bullet \mathbb{D}_{\omega_1}$ defined by its action on $\mathbb{C}[[u]]$:

$$\forall f(u) \in \mathbb{C}\,[[u]]\,, \quad (\mathbb{D}_{\omega_r} \bullet\ \dots\ \bullet \mathbb{D}_{\omega_1})\,.f(u) = (\mathbb{D}_{\omega_r}.\ (\dots\ .(\mathbb{D}_{\omega_1} f(u))))\,.$$

Generally, such an operator fails to preserve $\mathbb{C}\,\{u\}$.

Roughly speaking, it is sometimes possible to restore its convergence while rearranging its terms using coarborification. The idea is to replace each operator $\mathbb{D}_{\underline{\omega}}$ by the corresponding sum of coarborified operators. We then obtain a new organization of the previous sum :

$$\mathbf{F} = \sum_{\underline{\omega}^<} M^{\underline{\omega}^<} \mathbb{D}_{\underline{\omega}^<}$$

and the coefficients $M^{\underline{\omega}^<}$ that appears before the coarborified operators $\mathbb{D}_{\underline{\omega}^<}$ are the elements of a tree indexed family $(M^{\bullet^<})$ called **arborification** of the mould $(M^{\bullet})$. For numerous moulds and particulary for most of those intervenes naturally in dynamical system, the point is that their arborification have a geometrical growth and, provided that $\mathbb{D}$ preserves $\mathbb{C}\,\{u\}$, the same occurs for $\mathbf{F}$. One can read a very nice proof of this fact in [43].

The following is organized as follow.

In the beginning of the first section of this chapter, we explain the process of precoarborification, i.e. how one can encode the comould associated to a derivation with arborified sequences. The second part of the first section is devoted to the study of the product of two precoarborified comoulds that will reflect the tree product and in the third part, we will describe the composition of precoarborified comoulds. If one can expand a precoarborified operator in terms of a second, and this second in terms of a third, the problem is to find the relation between the first and the third. All the results of this section are new.

In the next chapter, we make explicitly the link between pre-coarborification and Ecalle's coarborification. Before establishing this link, we need to prove several technical combinatorial lemmae. The internal symmetry factor connecting the precoarborified form and the coarborified form of a differential operator appears as a direct consequence of Lemma 5.3.1. All these results are new excepted the theorem of existence and uniqueness of the coarborified that is announced in the articles already cited of J. Ecalle with a sketch of proof. But the crucial role of the symmetry factor is not apparent in these articles. We then study mould arborification and we transpose mould product and composition in arborified terms. One can find mentions of this two operations in [15], [41] and [56] but without proof and explanation. We prove moreover that these transformations preserve the geometrical growth.

We then study three applications of arborified composition. The first consists in the anti-arborification process already introduced by J. Ecalle and B. Vallet in [16]. We obtain as a corollary of our work that the geometrical growth of the antiarborified of a given mould is equivalent to the one of its arborified. In the second and the third application, we prove formulae for the arborification of the product inverse and the composition inverse of a mould. The first was already given by J. Ecalle in [15] but the second is new.

Finally, in chapter 6, we gives some recipes for mould's arborification and we illustrate this by several explicit examples.

Let us mention that a presentation of arborification in the language of Hopf algebras can be found in [43] and [60].

4.2 Notations

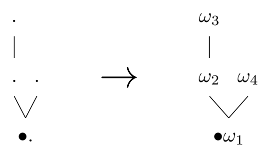
Let us consider a semi-group Ω, i.e. a set Ω with an associative binary operation. In this chapter and the next one, we can consider that $\Omega = \mathbf{N}^*$ with addition, but all the constructions go through for a "locally finite $\mathbf{N}$-graded semi-group", see [31].

Definition 4.2.0.1 *A family of ordinary differential operators* $(\mathbb{B}_m)_{m\in\Omega} \in \boldsymbol{ENDOM}(\mathbb{C}\,[[u]])$ *is an* ***admissible family of ordinary differential operators*** *if for all* $m \in \Omega$*, there exists* $b_m \in \mathbb{C}$ *such that* $\mathbb{B}_m = b_m u^{m+1}\partial$*.*[1]

For a given $\underline{m} = (m_1, \ldots, m_r) \in \Omega^\bullet$, we want to make explicit the effect of the Leibniz rules when we expand the product $\mathbb{B}_{\underline{m}} = \mathbb{B}_{m_r} \bullet \ldots \bullet \mathbb{B}_{m_1}$. More precisely, we will explain that it will be very convenient to index the resulting operators with trees in the manner of A. Cayley (see [8]).

4.3 Decorated trees

A rooted forest is a graph with connected components that are rooted trees. An Ω-decorated forest is given by a couple $(\mathcal{F}, f)$ where f is a map from the vertices of $\mathcal{F}$ to Ω. We naturally represent a decorated forest by placing on each vertex of $\mathcal{F}$ its image by f. By example :

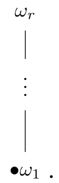


A decorated vertex of a decorated forest $(\mathcal{F}, f)$ will be called a node of $(\mathcal{F}, f)$. We will denote by $\Omega^{\bullet<}$ the set of all Ω-decorated forest. In Ecalle's terminology, an Ω-decorated forest is called an arborified sequence and is denoted $\underline{\omega}^< = (\omega_1, \ldots, \omega_r)^<$ where $\omega_1, \ldots, \omega_r \in \Omega$ are the nodes of $\underline{\omega}^<$.

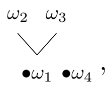
We can put a partial order between the elements of an arborified sequence. For two elements ω_i, ω_j of an arborified sequence $\underline{\omega}^< = (\omega_1, \ldots, \omega_r)^<$, we will

1. In other words, the terms of $(\mathbb{B}_m)$ are the homogeneous components of a one dimensional vector field.

assume that the notation $\omega_i < \omega_j$ means that ω_i is anterior to ω_j for the internal order of $\underline{\omega}^<$. A sequence $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ can be seen as a totally ordered decorated forest :



For example, in the arborified sequence



we get : $\omega_1 < \omega_2$ and $\omega_1 < \omega_3$. Moreover, ω_2 and ω_3 are not comparable and ω_4 is not comparable with ω_1, ω_2 and ω_3.

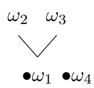
In order to simplify the explanation, we assume that all the arborified sequences $\underline{\omega}^<$ considered here contain the $\emptyset$ element and that this element is anterior to all nodes of $\underline{\omega}^<$. The element $\emptyset$ will be called the root of the arborified sequence. We admit of never writing it when we represent a tree. We say that

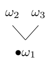
— a node of $\underline{\omega}^<$ without successor is a summit node or a *leaf* of $\underline{\omega}^<$.

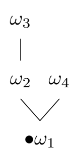
— a node of $\underline{\omega}^<$ that admits only $\emptyset$ as predecessor is a root of the tree $\underline{\omega}^<$.

The length and the norm of an arborified sequence are defined in the same way as for a totally ordered sequence of $\Omega^\bullet$.

An arborified sequence with only one root is said to be irreducible. If $\underline{\omega}^<$ admits several roots ω_{i_1}, …, ω_{i_s} then the tree $\underline{\omega}^<$ is the disjoint union of s irreducible trees $\underline{\omega}^{<i_1}$, …, $\underline{\omega}^{<i_s}$ with respective roots ω_{i_1}, …, ω_{i_s}. We then write $\underline{\omega}^< = \underline{\omega}^{<i_1} \oplus \ldots \oplus \underline{\omega}^{<i_s}$ and we will denote by $\deg(\underline{\omega}^<)$ the number of irreducible components of $\underline{\omega}^<$. This notation is the one of J. Ecalle and it is different of the common one which is $\underline{\omega}^< = \underline{\omega}^{<i_1} \ldots \underline{\omega}^{<i_s}$. Now, we give examples

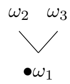
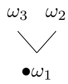
$\underline{\omega}^< =$  .

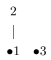
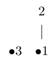
In this tree, we count three summit nodes $\omega_2, \omega_3, \omega_4$ and two roots : ω_1, ω_4. The predecessor of ω_2 or ω_3 is ω_1 (and $\emptyset$). Moreover, $\underline{\omega}^< = \underline{\omega}^{1<} \oplus \underline{\omega}^{2<}$ with $\underline{\omega}^{1<} =$  and $\underline{\omega}^{2<} = \bullet\omega_4$, so $\deg(\underline{\omega}) = 2$. In the irreducible tree

$\underline{\omega}^< =$ 

ω_3 get two predecessors, ω_2 and ω_1 (and $\emptyset$). Moreover $\deg(\underline{\omega}^<) = 1$.

Observe that the arborified sequences

 and 

are identical here. Our trees are *non planar* (see for example [20] for more precisions). In the same way, the forests  and  are identical from our point of view. We will now define a product on trees. Let us fix two trees $\underline{m}^< = (m_1, \ldots, m_r)^<$, $\underline{n}^< = (n_1, \ldots, n_s)^<$ and a node $m = m_k$ or $m = \emptyset$ of $\underline{m}^<$. We assume that $\underline{n}^<$ is irreducible. Our product consist in grafting the tree $\underline{n}^<$ to the tree $\underline{m}^<$ at the level of the node m while connecting the root of $n^<$ to the tree $\underline{m}^<$ directly after the node m. The obtained tree is denoted $\underline{m}^< \times_m \underline{n}^<$ and is characterized by the following :

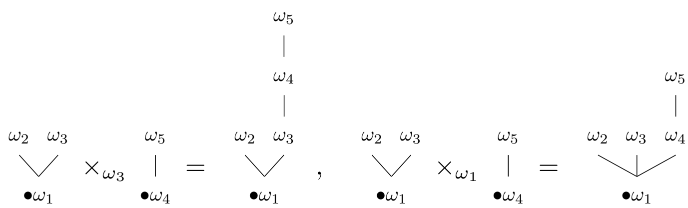
— The nodes of $\underline{m}^< \times_m \underline{n}^<$ are $m_1, \ldots, m_r$, $n_1, \ldots, n_s$.

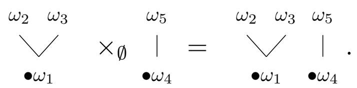
— If ω_i, ω_j are two nodes of $\underline{m}^< \times_m \underline{n}^<$ then we get

$$\omega_i < \omega_j \iff \begin{cases} \omega_i < \omega_j \text{ for the order of } \underline{m}^< & \text{if } \omega_i, \omega_j \in \underline{m}^< \\ \omega_i < \omega_j \text{ for the order of } \underline{n}^< & \text{if } \omega_i, \omega_j \in \underline{n}^< \\ \omega_i \in \underline{m}^< \text{ and } \omega_j \in \underline{n}^< \end{cases}$$

Let us observe that this product is compatible with the planar representation of the forests.

Example 4.3.0.1





If $\underline{n}^< = \underline{n}^{1<} \oplus \ldots \oplus \underline{n}^{p<}$ is not irreducible and if $n_{j_1}, \ldots n_{j_p}$ are the roots of $\underline{n}^<$ (n_{j_i} is the roots of $\underline{n}^{i<}$) then we define the product

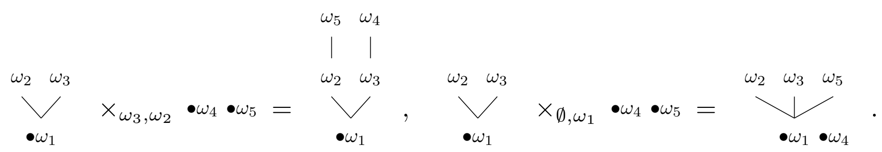
$$\underline{m}^< \times_{m_{i_1}, \ldots, m_{i_p}} \underline{n}^<$$

as the arborified sequence

$$\Big(\big((\underline{m}^< \times_{m_{i_1}} \underline{n}^{1<}\big) \times_{m_{i_2}} \underline{n}^{2<}\big) \ldots\Big) \times_{m_{i_p}} \underline{n}^{p<}.$$

Observe that there could be repetitions in the nodes $m_{i_1}, \ldots, m_{i_p}$

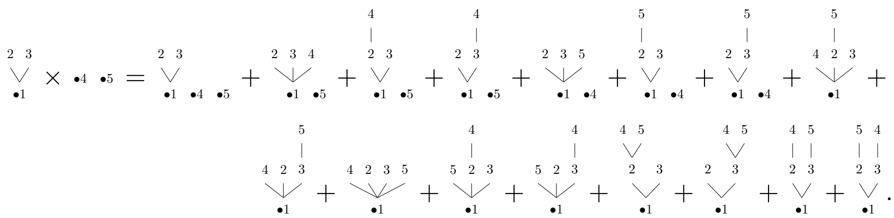
Example 4.3.0.2

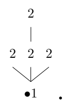


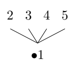
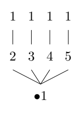
For the same two trees $\underline{m}^{<} = (m_1, \ldots, m_r)^{<}$ and $\underline{n}^{<} = \underline{n}^{1<} \oplus \ldots \oplus \underline{n}^{p<}$ than before, we finally define the product $\underline{m}^{<} \times \underline{n}^{<}$ as

$$\underline{m}^{<} \times \underline{n}^{<} = \sum_{\left(m_{i_1},\ldots,m_{i_p}\right) \in (\emptyset, m_1, \ldots, m_r)^p} \left(\left(\left(\underline{m}^{<} \times_{m_{i_1}} \underline{n}^{1<}\right) \times_{m_{i_2}} \underline{n}^{2<}\right) \ldots\right) \times_{m_{i_p}} \underline{n}^{p<}.$$

Example 4.3.0.3



We will then denote by **preconcat** $(m^{<}, n^{<})$ the family of all the rooted trees that appear in this sum. It is the family of all possible concatenations of $\underline{m}^{<}$ and $\underline{n}^{<}$, i.e the family of all possible products between $\underline{m}^{<}$ and $\underline{n}^{<}$. There may be several repetitions of a given rooted tree in **preconcat** $(m^{<}, n^{<})$. To convince oneself of this fact, one can expand the product  $\times \bullet_2 \bullet_2$. We then find for example four repetitions of the tree  .

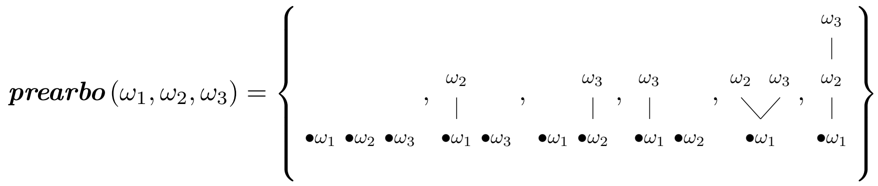
Remark 4.3.0.1 *Let us observe the important following fact : the number of repetitions of a tree in the tree product can be of factorial type. In the product*  $\times \bullet_1 \bullet_1 \bullet_1 \bullet_1$, *we obtain* $4!$ *repetitions of the tree*  .

We will denote by **concat** $(m^{<}, n^{<})$ the set of elements of the family **preconcat** $(m^{<}, n^{<})$. It contains the same elements than **preconcat** $(m^{<}, n^{<})$ but without repetition.

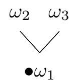
For $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, we will denote by **prearbo** $(\underline{\omega})$ the family of all arborified sequences with nodes ω_1,...,ω_r and with an internal order that respect the internal order of $\underline{\omega}$ counted with possibly repetitions.

The set of all $\underline{\omega}^{<} \in$ **prearbo** $(\underline{\omega})$ (with no repetition) is denoted **arbo** $(\underline{\omega})$.

Example 4.3.0.4



Notation 4.3.0.1 *We write $\omega_1.'\underline{\omega}^<$ the irreducible sequence $\underline{\omega}^<$ with root ω_1. The arborescent sequence $'\underline{\omega}^<$ corresponds to $\underline{\omega}^<$ truncated of ω_1. For example,*

if $\underline{\omega}^< =$  *then $\underline{\omega}^< = \omega_1.'\underline{\omega}^<$ with $'\underline{\omega}^< = \bullet\omega_2 \bullet\omega_3$.*

4.4 Pre-coarborification

As explained in the introduction, we consider an admissible family of ordinary differential operators $(\mathbb{B}_m)_{m\in\Omega} \subset \mathbf{ENDOM}(\mathbb{C}[[u]])$ where for all $m \in \Omega$, $\mathbb{B}_m = b_m u^{m+1}\partial$ with $b_m \in \mathbb{C}$. The scalars b_m are not important in the process of pre-coarborification so we assume that they are all equal to 1. We want to expand the composition $\mathbb{B}_{\underline{m}} = \mathbb{B}_{m_r} \bullet \ldots \bullet \mathbb{B}_{m_1}$ using Leibniz rule. Let us begin with some examples.

For two integers $m_1, m_2 \in \mathbf{N}^*$, we have :

$$\mathbb{B}_{m_1,m_2} = \mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} = u^{m_2+1}\partial \bullet u^{m_1+1}\partial = \underbrace{u^{m_2+1}(\partial u^{m_1+1})\partial}_{=:B_{m_2}\bullet_{m_1}\mathbb{B}_{m_1}} + \underbrace{u^{m_2+m_1+2}\partial^2}_{=:\mathbb{B}_{m_2}\bullet_{\emptyset}\mathbb{B}_{m_1}}$$

As announced, we will use trees to describe the two operators contained in this sum.

We set :
— $\mathbb{B}_{m_2} \bullet_{m_1} \mathbb{B}_{m_1} =: \mathbb{B}_{m_1 \bullet_{m_1} m_2} = \mathbb{B}_{\substack{m_2 \\ | \\ \bullet m_1}}$.
— $\mathbb{B}_{m_2} \bullet_{\emptyset} \mathbb{B}_{m_1} =: \mathbb{B}_{m_1 \bullet_{\emptyset} m_2} = \mathbb{B}_{\bullet m_1 \bullet m_2}$ . .
With these new notations, we obtain :

$$\mathbb{B}_{m_1,m_2} = \mathbb{B}_{m_1 \bullet_{\emptyset} m_2} + \mathbb{B}_{m_1 \bullet_{m_1} m_2} = \mathbb{B}_{\substack{m_2 \\ | \\ \bullet m_1}} + \mathbb{B}_{\bullet m_1 \bullet m_2} .$$

We will then study what happens for the composition of three such operators.

$$\begin{aligned}
\mathbb{B}_{m_1,m_2,m_3} &= \mathbb{B}_{m_3} \bullet (\mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1}) \\
&= u^{m_3+1}\partial \bullet \left(u^{m_2+1}(\partial u^{m_1+1})\partial + u^{m_2+m_1+2}\partial\right) \\
&= u^{m_3+1}\left(\partial u^{m_2+1}\right)\left(\partial u^{m_1+1}\right)\partial + u^{m_3+m_2+2}\left(\partial u^{m_1+1}\right)\partial \\
&+ u^{m_3+m_2+2}\left(\partial u^{m_1+1}\right)\partial + u^{m_3+m_1+2}\left(\partial u^{m_2+1}\right)\partial \\
&+ u^{m_3+m_2}\left(\partial u^{m_1}\right)\partial + u^{m_3+m_2+m_1+3}\partial
\end{aligned}$$

With the same notations than previously, we are then able to write :

$$\begin{aligned}
\mathbb{B}_{m_1,m_2,m_3} &= \mathbb{B}_{m_3} \bullet (\mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1}) \\
&= \mathbb{B}_{m_3} \bullet (\mathbb{B}_{m_2} \bullet_{m_1} \mathbb{B}_{m_1} + \mathbb{B}_{m_2} \bullet_{\emptyset} \mathbb{B}_{m_1}) \\
&= \mathbb{B}_{m_3} \bullet_{m_2} (\mathbb{B}_{m_2} \bullet_{m_1} \mathbb{B}_{m_1}) + \mathbb{B}_{m_3} \bullet_{m_1} (\mathbb{B}_{m_2} \bullet_{m_1} \mathbb{B}_{m_1}) + \\
&\mathbb{B}_{m_3} \bullet_{\emptyset} (\mathbb{B}_{m_2} \bullet_{m_1} \mathbb{B}_{m_1}) + \mathbb{B}_{m_3} \bullet_{m_2} (\mathbb{B}_{m_2} \bullet_{\emptyset} \mathbb{B}_{m_1}) + \\
&\mathbb{B}_{m_3} \bullet_{m_1} (\mathbb{B}_{m_2} \bullet_{\emptyset} \mathbb{B}_{m_1}) + \mathbb{B}_{m_3} \bullet_{\emptyset} (\mathbb{B}_{m_2} \bullet_{\emptyset} \mathbb{B}_{m_1}) \\
&:= \mathbb{B}_{m_1 \times_{m_1} m_2 \times_{m_2} m_3} + \mathbb{B}_{m_1 \times_{m_1} m_2 \times_{m_1} m_3} + \mathbb{B}_{m_1 \times_{m_1} m_2 \times_{\emptyset} m_3} + \\
&\mathbb{B}_{m_1 \times_{\emptyset} m_2 \times_{m_2} m_3} + \mathbb{B}_{m_1 \times_{\emptyset} m_2 \times_{m_1} m_3} + \mathbb{B}_{m_1 \times_{\emptyset} m_2 \times_{\emptyset} m_3} \\
&= \mathbb{B}_{\substack{m_3 \\ | \\ m_2 \\ | \\ \bullet_{m_1}}} + \mathbb{B}_{\substack{m_2 \;\; m_3 \\ \diagdown\diagup \\ \bullet_{m_1}}} + \mathbb{B}_{\substack{m_2 \\ | \\ \bullet_{m_1} \bullet_{m_3}}} + \mathbb{B}_{\substack{m_3 \\ | \\ \bullet_{m_1} \bullet_{m_2}}} + \mathbb{B}_{\substack{m_2 \\ | \\ \bullet_{m_1} \bullet_{m_3}}} + \mathbb{B}_{\bullet_{m_1} \bullet_{m_2} \bullet_{m_3}} .
\end{aligned}$$

Let us remark two important facts :

— The first is that the operator $\mathbb{B}_{m_3} \bullet \mathbb{B}_{\substack{m_2 \\ | \\ \bullet_{m_1}}}$ is the sum of all operators $B_{\underline{\omega}^<}$ where $\underline{\omega}^<$ ranges over all trees obtained by the product of the trees $\substack{m_2 \\ | \\ \bullet_{m_1}}$ and $\bullet_{m_3}$. The same occurs for $\mathbb{B}_{m_3} \bullet \mathbb{B}_{\bullet_{m_1} \bullet_{m_2}}$.

— The decomposition of $\mathbb{B}_{m_1,m_2,m_3}$ into tree-indexed operators $B_{\underline{\omega}^<}$ ranges over all trees that could be constructed from (m_1, m_2, m_3), the order of which respects the order of the sequence (m_1, m_2, m_3). In other words, we get :

$$\mathbb{B}_{m_3} \bullet \mathbb{B}_{m_2} \bullet \mathbb{B}_{m_1} = \sum_{\underline{\omega}^< \in \mathbf{prearbo}(m_1,m_2,m_3)} \mathbb{B}_{\underline{\omega}^<} .$$

We will then establish the following :

Theorem 4.4.1 *For any sequence $\underline{m} = (m_1, \ldots, m_r) \in \Omega^\bullet$ and for any admissible family of ordinary differential operators $(\mathbb{B}_m) \in$ **ENDOM**$(\mathbb{C}[[u]])$, we get :*

$$\mathbb{B}_{\underline{m}} = \mathbb{B}_{m_r} \bullet \ldots \bullet \mathbb{B}_{m_1} \quad = \sum_{\underline{\omega}^< \in \boldsymbol{prearbo}(\underline{m})} \mathbb{B}_{\underline{\omega}^<} \qquad (P_1)$$

where the indexed by forests operators family $\mathbb{B}_{\bullet^<}$ is inductively defined by :

1. *if* $\underline{\omega}^< = \omega_1.'\underline{\omega}^<$ *then*

$$\mathbb{B}_{\underline{\omega}^<} = \mathbb{B}_{'\underline{\omega}^<}.\left(\mathbb{B}_{\omega_1}.u\right)\partial,$$

2. *if* $\deg\left(\underline{\omega}^<\right) = d$ *, i.e. if* $\underline{\omega}^< = \left(\underline{\omega}^{1<}\right) \oplus \ldots \oplus \left(\underline{\omega}^{d<}\right)$ *where* $\forall i \in [\![1, d]\!]$, $\underline{\omega}^{i<} \in \Omega^{\bullet<}_{irred}$ *then* $\mathbb{B}_{\underline{\omega}^<}$ *can be written*

$$\mathbb{B}_{\underline{\omega}^<} = P_1(u)\ldots P_d(u)\partial^d$$

where for all $i \in [\![1, d]\!]$, $P_i(u) = \mathbb{B}_{\underline{\omega}^{i<}}.u$.

The tree indexed comould $\mathbb{B}_{\bullet^<}$ *is called the* ***pre-coarborification*** *of the admissible family of ordinary differential operators* $(\mathbb{B}_m) \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$.

Before proving the theorem, we need the following lemma :

Lemma 4.4.2

$$\forall m \in \Omega, \quad \forall \underline{\omega}^< \in \Omega^{\bullet<}, \quad \mathbb{B}_m \bullet \mathbb{B}_{\underline{\omega}^<} = \sum_{\underline{W}^< \in \boldsymbol{preconcat}\left(\underline{\omega}^<,\ \bullet m\right)} \mathbb{B}_{\underline{W}^<}.$$

Proof The proof is by induction on the length r of $\underline{\omega}^<$. If $r = 0$, one has trivially $\mathbb{B}_m \bullet \mathbb{B}_\emptyset = \mathbb{B}_m = \mathbb{B}_{\bullet m} = \sum_{\underline{W}^< \in \mathbf{preconcat}\left(\emptyset,\ \bullet m\right)} \mathbb{B}_{\underline{W}^<}$. We assume the lemma true for any arborescent sequences of length r and we prove it for an arborescent sequence $\underline{\omega}^<$ of length $r+1$.

If $\underline{\omega}^< = \underline{\omega}^{1<} \oplus \ldots \oplus \underline{\omega}^{d<}$ with $\underline{\omega}^{i<} \in \Omega^{\bullet<}_{\mathrm{irred}}$ and $d \geq 2$ then using Leibniz rule and the induction hypothesis

$$\begin{aligned}
\mathbb{B}_m \bullet \mathbb{B}_{\underline{\omega}^<} &= \mathbb{B}_m \bullet \left(\left(\mathbb{B}_{\underline{\omega}^{1<}}.u\right)\ldots.\left(\mathbb{B}_{\underline{\omega}^{d<}}.u\right)\partial^d\right) \\
&= \sum_{i=1}^{d} \left(\mathbb{B}_{\underline{\omega}^{1<}}.u\right)\ldots\left(\mathbb{B}_m \bullet \mathbb{B}_{\underline{\omega}^{i<}}\right).u\ldots\left(\mathbb{B}_{\underline{\omega}^{d<}}.u\right)\partial^d \\
&\qquad\qquad + \left(\mathbb{B}_{\underline{\omega}^{1<}}.u\right)\ldots\left(\mathbb{B}_{\underline{\omega}^{d<}}.u\right)\mathbb{B}_m \bullet \partial^d \\
&= \sum_{i=1}^{d} \sum_{\underline{W}^{i<} \in \mathbf{preconcat}^*\left(\underline{\omega}^{i<},\ \bullet m\right)} \mathbb{B}_{\bullet\underline{\omega}^{1<}\bullet\ldots\bullet\underline{W}^{i<}\bullet\ldots\bullet\underline{\omega}^{d}} + \mathbb{B}_{\bullet\underline{\omega}^{1<}\bullet\ldots\bullet\underline{\omega}^{d<}\bullet m} \\
&= \sum_{\underline{W}^< \in \mathbf{preconcat}\left(\underline{\omega}^<,\ \bullet m\right)} \mathbb{B}_{\underline{W}^<}
\end{aligned}$$

where $\mathbf{preconcat}^*\left(\underline{\omega}^{i<},\ \bullet m\right)$ is the family of all grafts of $\bullet m$ on $\underline{\omega}^{i<}$ excepted on the root of $\underline{\omega}^{i<}$.

If $\underline{\omega}^<$ is irreducible then $\underline{\omega}^< = \omega_1.'\underline{\omega}^< = \begin{array}{c}\underline{\mathrm{X}}^{1<}\ \ldots\ \underline{\mathrm{X}}^{d<} \\ \diagdown\ |\ \diagup \\ \bullet\omega_1\end{array} \in \Omega^{\bullet<}_{\mathrm{irred}}$ with $'\underline{\omega}^< = \underline{\mathrm{X}}^{1<} \oplus \ldots \oplus \underline{\mathrm{X}}^{d<}$ and where $\underline{\mathrm{X}}^{i<} \in \Omega^{\bullet<}_{\mathrm{irred}}$. Then using Leibniz rule and the induction hypothesis, one obtains :

$$\begin{aligned}
\mathbb{B}_m \bullet \mathbb{B}_{\underline{\omega}^<} &= \mathbb{B}_m \bullet \left(\mathbb{B}_{'\underline{\omega}^<} . \left(\mathbb{B}_{\omega_1} .u\right) \partial\right) \\
&= \left(\mathbb{B}_m \bullet \mathbb{B}_{'\underline{\omega}^<}\right) . \left(\mathbb{B}_{\omega_1} .u\right) \partial + \mathbb{B}_{'\underline{\omega}^<} . \left(\mathbb{B}_{\omega_1} .u\right) \mathbb{B}_m \bullet \partial \\
&= \sum_{\underline{W}^< \in \mathbf{preconcat}\left('\underline{\omega}^<, \bullet m\right)} \mathbb{B}_{\underline{W}^<} . \left(\mathbb{B}_{\omega_1} .u\right) \partial + \mathbb{B}_{\bullet \underline{\omega}^< \bullet m} \\
&= \sum_{\underline{\mathrm{X}}^< \in \mathbf{preconcat}\left(\underline{\omega}^<, \bullet m\right)} \mathbb{B}_{\underline{\mathrm{X}}^<}
\end{aligned}$$

from what follows the lemma. □

We then prove the theorem :

Proof As previously, we will perform an induction on the length r of $\underline{m}$. We have already proved the theorem in the previous examples for small values of r. We assume it is true for any sequence $\underline{m}$ of length less than r and we prove it for a sequence $\underline{m} = (m_1, \dots, m_{r+1})$ of length $r+1$. Using the induction hypothesis, the previous lemma and with $\underline{n} =' \underline{m}$, we have :

$$\begin{aligned}
\mathbb{B}_{\underline{m}} &= \mathbb{B}_{m_{r+1}} \bullet \mathbb{B}_{\underline{n}} \\
&= \mathbb{B}_{m_{r+1}} \bullet \left(\sum_{\underline{n}^< \in \mathbf{prearbo}(\underline{n})} \mathbb{B}_{\underline{n}^<} \right) \\
&= \sum_{\underline{n}^< \in \mathbf{prearbo}(\underline{n})} \sum_{\underline{W}^< \in \mathbf{preconcat}\left(\underline{n}^<, \bullet m_{r+1}\right)} \mathbb{B}_{\underline{W}^<} \\
&= \sum_{\underline{\omega}^< \in \mathbf{prearbo}(\underline{m})} \mathbb{B}_{\underline{\omega}^<}
\end{aligned}$$

□

4.5 Precoarborified product

We will now study how we can encode a product of two operators $\mathbb{B}_{\underline{n}^<}$ and $\mathbb{B}_{\underline{m}^<}$.

If $\underline{n}^<$ is an irreducible arborescent sequence, then $\mathbb{B}_{\underline{n}^<}$ is of the form $P(u)\,\partial$ with $P(u) \in \mathbb{C}[u]$ and it acts on $\mathbb{B}_{\underline{m}^<}$ exactly like the operator $\mathbb{B}_m$ on the operator $\mathbb{B}_{\underline{\omega}^<}$ in lemma 4.4.2, but instead of grafting the node m on the arborescent sequence $\underline{\omega}^<$ in the indexation of the obtained operators, one has to graft the tree $\underline{n}^<$ on the forest $\underline{m}^<$. We then obtain that

$$\mathbb{B}_{\underline{n}^<} \bullet \mathbb{B}_{\underline{m}^<} = \sum_{\underline{\omega}^< \in \mathbf{preconcat}(\underline{m}^<, \underline{n}^<)} \mathbb{B}_{\underline{\omega}^<} .$$

Otherwise, if $\underline{n}^< = \underline{n}^{1<} \oplus \ldots \oplus \underline{n}^{p<}$ is not irreducible (with $\underline{n}^{i<} \in \Omega^{\bullet<}_{\text{irred}}$ for all $i \in [\![1,p]\!]$) then the action of $\mathbb{B}_{\underline{n}^<}$ on $\mathbb{B}_{\underline{m}^<}$ can be seen as the action of p operators of the form $P_i(u)\partial$ (with $P_i(u) \in \mathbb{C}[u]$) on the operator $\mathbb{B}_{\underline{m}^<}$ and as previously, we obtain that

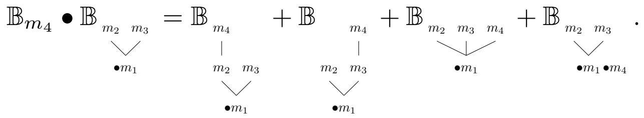
$$\mathbb{B}_{\underline{n}^<} \bullet \mathbb{B}_{\underline{m}^<} \quad = \sum_{\underline{\omega}^< \in \mathbf{preconcat}(\underline{m}^<, \underline{n}^<)} \mathbb{B}_{\underline{\omega}^<}.$$

We then have :

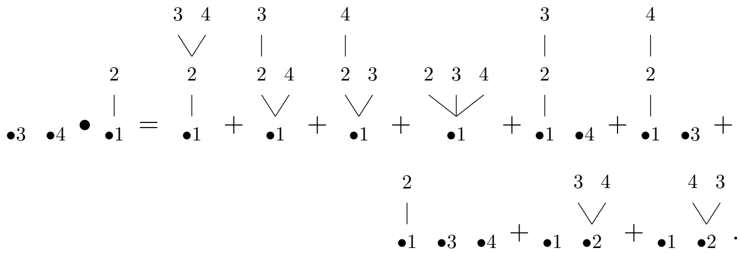
Proposition 4.5.1 *For any arborified sequence* $\underline{m}^<$ *and* $\underline{n}^<$ *and for any admissible family of ordinary differential operators* $(\mathbb{B}_m)_{m\in\Omega} \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$*, we get :*

$$\mathbb{B}_{\underline{n}^<} \bullet \mathbb{B}_{\underline{m}^<} \quad = \sum_{\underline{\omega}^< \in \boldsymbol{preconcat}(\underline{m}^<, \underline{n}^<)} \mathbb{B}_{\underline{\omega}^<}.$$

Example 4.5.0.5 *Let us give a first example :*



For the second example, we omit the symbols $\mathbb{B}$ *and* m *to focus on trees :*



4.6 Precoarborified composition

We consider now two admissible families of ordinary differential operators $(\mathbf{F}_m)_{m\in\Omega}$ and $(\mathbf{G}_m)_{m\in\Omega}$ of $\mathbf{ENDOM}(\mathbb{C}[[u]])$ such that :

$$\forall m \in \Omega, \quad \mathbf{G}_m = g_m u^{m+1}\partial \text{ and } \mathbf{F}_m = f_m u^{m+1}\partial$$

where $(f_m), (g_m)$ are complex sequences. We assume that there exists a family of complex scalars $\left(N^{\underline{\omega}^<}\right)$ indexed by $\Omega^{\bullet<}$ such that for any $m \in \Omega$:

$$\mathbf{G}_m = \sum_{\|\underline{\omega}^<\| = m} N^{\underline{\omega}^<} \mathbf{F}_{\underline{\omega}^<}.$$

The operator $\mathbf{G}_m$ is a first order derivation, so we shall keep in this sum only operators $\mathbf{F}_{\underline{\omega}^<}$ indexed by irreducible trees.

The aim of this section is to understand what will happen if we replace the operators $\mathbf{G}_{m_1}, \ldots, \mathbf{G}_{m_r}$ in $\mathbf{G}_{(m_1,\ldots,m_r)^<}$ with operators $\mathbf{F}_{\underline{\omega}^<}$.

Consider first the operator $\mathbf{G}_{\substack{m_2 \\ | \\ \bullet m_1}} = \mathbf{G}_{m_2} \bullet_{m_1} \mathbf{G}_{m_1} = g_{m_2} u^{m_2} \left(\partial \left(g_{m_1} u^{m_1}\right)\right) \partial.$

Because $\mathbf{G}_m$ is a first order derivation, we know that for $i = 1, 2$,

$$\begin{aligned} \mathbf{G}_{m_i} &= \sum_{\|\underline{\omega}^<\| = m_i, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} \mathbf{F}_{\underline{\omega}^<} \\ &= \sum_{\|\underline{\omega}^<\| = m_i, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} P_{\underline{\omega}^<}(u) \partial \end{aligned}$$

where $P_{\underline{\omega}^<}(u) \in \mathbb{C}[u]$.

In $\mathbf{G}_{\substack{m_2 \\ | \\ \bullet m_1}}$, the operator $\mathbf{G}_{m_2} = \sum_{\|\underline{\omega}^<\| = m_2, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} P_{\underline{\omega}^<}(u) \partial$ acts only on the polynomial part of the operators in the sum giving $\mathbf{G}_{m_1} = \sum_{\|\underline{\omega}^<\| = m_1, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} P_{\underline{\omega}^<}(u) \partial$. When we replace $\mathbf{G}_{m_i}$ by $\mathbf{F}_{\underline{\omega}^i}$ for $i = 1, 2$ in $\mathbf{G}_{\substack{m_2 \\ | \\ \bullet m_1}}$, the obtained operators are those that are encoded as the graft of the irreducible tree $\underline{\omega}^{2<}$ on one of the nodes of $\underline{\omega}^{1<}$. More precisely :

$$\mathbf{G}_{\substack{m_2 \\ | \\ \bullet m_1}} = \sum_{\|\underline{\omega}^{1<}\| = m_1, \|\underline{\omega}^{2<}\| = m_2} N^{\underline{\omega}^{1<}} N^{\underline{\omega}^{2<}} \sum_{\omega \in \underline{\omega}^{1<}, \omega \neq \emptyset} \mathbf{F}_{\underline{\omega}^{1<} \times_{\omega} \underline{\omega}^{2<}}.$$

In $\mathbf{G}_{\bullet m_1 \bullet m_2} = g_{m_1} g_{m_2} u^{m_2 + m_1} \partial$, the operator $\mathbf{G}_{m_2} = \sum_{\|\underline{\omega}^<\| = m_2, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} P_{\underline{\omega}^<}(u) \partial$ acts only on the rooted part of $\mathbf{G}_{m_1} = \sum_{\|\underline{\omega}^<\| = m_1, \underline{\omega}^< \text{ irred.}} N^{\underline{\omega}^<} u^{\cdots} P_{\underline{\omega}^<}(u) \partial$. When we perform the same replacements as before, the operators obtained are the ones that are encoded as the graft of the irreducible tree $\underline{\omega}^2$ on the root $\emptyset$ of $\underline{\omega}^1$:

$$\mathbf{G}_{\bullet m_1 \bullet m_2} = \sum_{\|\underline{\omega}^{1<}\| = m_1, \|\underline{\omega}^{2<}\| = m_2} N^{\underline{\omega}^{1<}} N^{\underline{\omega}^{2<}} \mathbf{F}_{\underline{\omega}^{1<} \times_{\emptyset} \underline{\omega}^{2<}}.$$

For a tree $\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet <}$ and $\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{\text{irred}}$ in such a way that for any $i \in [\![1, s]\!]$, $\|\underline{\omega}^{i<}\| = m_i$, we will denote by **precompo** $\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ the list of all the arborified sequences constructed in the following way :

— we replace each node m_i of $\underline{m}^<$ by the tree $\underline{\omega}^{i<}$.

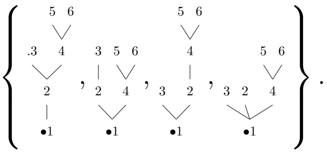
— If $m_{i'}$ is the predecessor of $m_i \in \underline{m}^<$ in $\underline{m}^<$ then we connect one of the node of $\underline{\omega}^{i'<}$ with the root of $\underline{\omega}^{i<}$.

Moreover, we will denote by **compo** $\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ the same list but without repetition.

Example 4.6.0.6

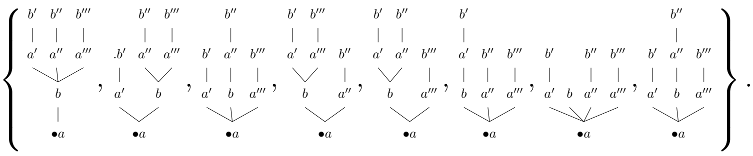
— *If* $\underline{m}^{<} = \overset{3\ \ 15}{\bigvee}_{\bullet 3}$ *and* $\underline{\omega}^{1<} = \overset{2}{|}_{\bullet 1}$, $\underline{\omega}^{2<} = \bullet_3$ *and* $\underline{\omega}^{3<} = \overset{5\ \ 6}{\bigvee}_{\bullet 4}$ *we obtain :*

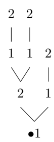
$$\textit{\textbf{precompo}}\left(\underline{m}^{<};\underline{\omega}^{1<},\underline{\omega}^{2<},\underline{\omega}^{3<}\right) =$$



— *Let us consider* $a, a', a'', a''', b, b', b'', b''' \in \Omega$ *and* $c = a + b$, $c' = a' + b'$, $c'' = a'' + b''$, $c''' = a''' + b'''$. *If* $\underline{m}^{<} = \overset{c'\ \ c''\ \ c'''}{\bigvee}_{\bullet c}$, $\underline{\omega}^{1<} = \overset{b}{|}_{\bullet a}$, $\underline{\omega}^{2<} = \overset{b'}{|}_{\bullet a'}$, $\underline{\omega}^{3<} = \overset{b''}{|}_{\bullet a''}$ *and* $\underline{\omega}^{4<} = \overset{b'''}{|}_{\bullet a'''}$, *then :*

$$\textit{\textbf{precompo}}\left(\underline{m}^{<};\underline{\omega}^{1<},\underline{\omega}^{2<},\underline{\omega}^{3<},\underline{\omega}^{4<}\right) =$$



We observe that if $a = a' = a'' = a''' = 1$ *and if* $b = b' = b'' = b''' = 2$ *then we count three repetitions of the tree*  *in the set* $\textit{\textbf{precompo}}(\underline{m}^{<};\underline{\omega}^{1<}, \underline{\omega}^{2<},\underline{\omega}^{3<},\underline{\omega}^{4<})$.

We want to prove that if we replace the operators $\mathbf{G}_{m_1}, \dots, \mathbf{G}_{m_r}$ in $\mathbf{G}_{(m_1,\dots,m_r)^<}$ with operators $\mathbf{F}_{\underline{\omega}^{i<}}$ then we obtain the expansion :

$$\mathbf{G}_{(m_1,\dots,m_r)^<} = \sum_{\underline{\omega}^{<} \in \mathbf{precompo}(\underline{m}^{<};\underline{\omega}^{1<},\dots,\underline{\omega}^{s<})} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}} \mathbf{F}_{\underline{\omega}^{<}}.$$

We have already verified that this property is true if $\mathrm{l}\left(\underline{m}^{<}\right) = 2$. We assume that it is true if $\mathrm{l}\left(\underline{m}^{<}\right) = r \geq 2$ and we will prove it if $\mathrm{l}\left(\underline{m}^{<}\right) = r + 1$. Assume that the predecessor of m_{r+1} in $\underline{m}^{<} = (m_1, \dots, m_r, m_{r+1})^{<}$ is m_i. If we replace the operators $\mathbf{G}_{m_1}, \dots, \mathbf{G}_{m_r}$ in $\mathbf{G}_{(m_1,\dots,m_r,m_{r+1})^<}$ with operators $\mathbf{F}_{\underline{\omega}^{i<}}$ then we obtain the sum

$$\sum_{\underline{\omega}^{<} = (\underline{\omega}'^{<}, m_r)^{<}, \underline{\omega}'^{<} \in \mathbf{precompo}(\underline{m}'^{<};\underline{\omega}^{1<},\dots,\underline{\omega}^{r<})} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}} \mathbf{G}_{m_{r+1}} \bullet_{m_i} F_{\underline{\omega}^{<}}$$

where the operator $\mathbf{G}_{m_{r+1}}$ and so $\mathbf{F}_{\underline{\omega}^{r+1<}}$ with $\|\underline{\omega}^{r+1<}\| = m_{r+1}$ acts only on the operator $F_{\underline{\omega}^{i<}}$ and where $\underline{m}'^{<}$ denotes the tree $\underline{m}^{<}$ where we have removed the summital node m_{r+1}. But $\mathbf{F}_{\underline{\omega}^{r+1<}} \bullet \mathbf{F}_{\underline{\omega}^{i<}} = \sum_{\underline{\omega}^{<} \in \mathbf{preconcat}(\underline{\omega}^{i<},\underline{\omega}^{r+1<})} \mathbf{F}_{\underline{\omega}^{<}}$ and the formula follows.

We have thus proved the following :

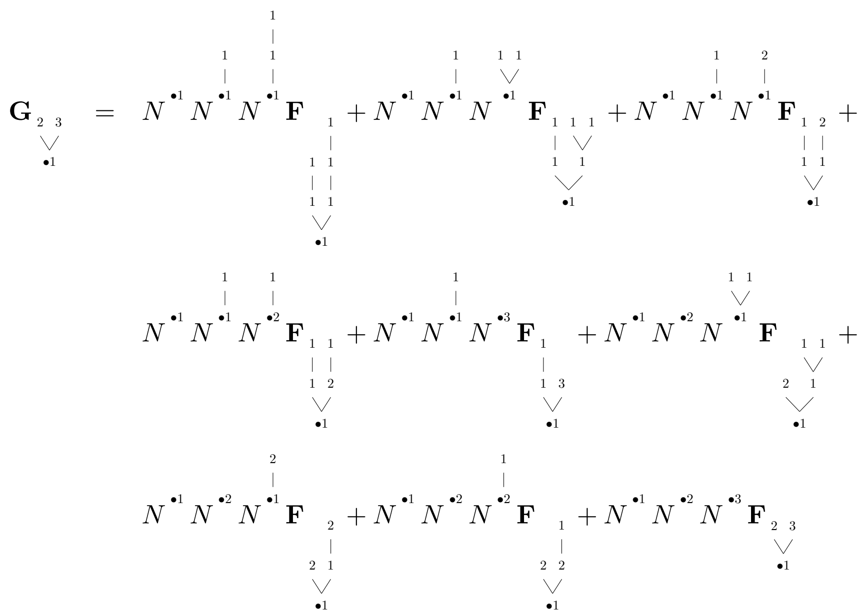
Proposition 4.6.1 *Let us consider two families of ordinary differential operators* $(\mathbf{F}_m)_{m\in\Omega}$ *and* $(\mathbf{G}_m)_{m\in\Omega}$ *of* $\boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *such that for any* $m \in \Omega$ *:*

$$\mathbf{G}_m = \sum_{\substack{\|\underline{\omega}^{<}\| = m \\ \underline{\omega}^{<} \in \Omega^{\bullet<}_{irred}}} N^{\underline{\omega}^{<}} \mathbf{F}_{\underline{\omega}^{<}}.$$

Then for any $\underline{m}^{<} = (m_1, \ldots, m_r)^{<} \in \Omega^{\bullet<}$.

$$\mathbf{G}_{(m_1,\ldots,m_r)^{<}} = \sum_{\underline{\omega}^{<} \in \boldsymbol{precompo}(\underline{m}^{<};\underline{\omega}^{1<},\ldots,\underline{\omega}^{s<})} N^{\underline{\omega}^{1<}} \ldots N^{\underline{\omega}^{s<}} \mathbf{F}_{\underline{\omega}^{<}}.$$

Example 4.6.0.7



Chapitre 5

Simple and contracting (co-)arborification

On juge l'arbre à ses fruits.
Saint Matthieu - La Bible

5.1 Some definitions and lemmae

We first introduce two notations :

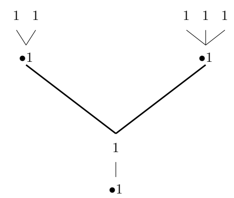
Notation 5.1.0.2

— *For two arborified sequences* $\underline{m}^<, \underline{n}^< \in \Omega^{\bullet<}$*, we will denote by* $\underline{m}^< \bullet \underline{n}^<$ *the couple* $(\underline{m}^<, \underline{n}^<) \in (\Omega^{\bullet<})^2$ *with the partial order inherited of these of* $\underline{m}^<$ *and* $\underline{n}^<$ *and where each node of* $\underline{m}^<$ *is considered as anterior to each node of* $\underline{n}^<$*.*

— *For* $\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet<}$ *and* $\underline{\omega}^{i<} \in \Omega^{\bullet<}_{irred}$*, we will denote by* $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *the s-uple* $\left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *with the partial order inherited of each* $\underline{\omega}^{i<}$ *for* $i \in [\![1, s]\!]$ *and where each node of* $\underline{\omega}^{i_0<}$ *is considered as anterior to each node of* $\underline{\omega}^{j_0<}$ *if* m_{i_0} *is anterior to* m_{j_0} *for the arborescent order of* $\underline{m}^<$ *and for any* $i_0, j_0 \in [\![1, s]\!]$ *such that* $i_0 \neq j_0$*.*

Remark 5.1.0.2 *For* $\underline{m}^< = \underset{\bullet 2}{\overset{3\ \ 4}{\vee}}$, $\underline{\omega}^{1<} = \underset{\bullet 1}{\overset{1}{|}}$, $\underline{\omega}^{2<} = \underset{\bullet 1}{\overset{1\ \ 1}{\vee}}$ *and* $\underline{\omega}^{3<} = \underset{\bullet 1}{\overset{1\ \ 1\ \ 1}{\vee\!\!\!|}}$, *the*

3-*uple* $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \underline{\omega}^{2<}, \underline{\omega}^{3<}\right)$ *can be seen as the "tree of trees" :*



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Definition 5.1.0.2 *Let us consider* $\underline{\omega} \in \Omega^{\bullet}$, $\underline{\omega}^{<} \in \textbf{\textit{arbo}}\left(\underline{\omega}\right)$ *and* $r = l(\underline{\omega})$. *We say that*

1. $\sigma : [\![1, r]\!] \mapsto [\![1, r]\!]$ *is a* ***monotonous permutation*** *of* $\underline{\omega}^{<}$ *if* σ *is a permutation of* $[\![1, r]\!]$ *and if* $\omega_{i_1} < \omega_{i_2}$ *for the arborescent order of* $\underline{\omega}^{<}$ *then* $\omega_{\sigma(i_1)} < \omega_{\sigma(i_2)}$ *for the arborescent order of* $\underline{\omega}^{<}$.
2. $\sigma : [\![1, r]\!] \mapsto [\![1, r]\!]$ *is a* ***monotonous projection*** *from* $\underline{\omega}^{<}$ *into* $\underline{\omega}$ *if* σ *is a permutation of* $[\![1, r]\!]$ *such that for any* $i \in [\![1, r]\!]$, *if* $\omega_{i_1} < \omega_{i_2}$ *for the arborescent order of* $\underline{\omega}^{<}$ *then* $\omega_{\sigma(i_1)} < \omega_{\sigma(i_2)}$ *for the total order of* $\underline{\omega}$.
3. $\phi : \underline{\omega}^{<} \to \underline{m}^{<} \bullet \underline{n}^{<}$ *is a* ***monotonous injection from*** $\underline{\omega}^{<}$ ***into*** $\underline{m}^{<} \bullet \underline{n}^{<}$, *if* ϕ *verifies for any* $\omega_i, \omega_j \in \underline{\omega}^{<}$, $\omega_i < \omega_j$ *for the arborescent order of* $\underline{\omega}^{<}$ *if and only if*

$$\begin{cases} \phi\left(\omega_i\right) < \phi\left(\omega_j\right) \text{ for the arborescent order of } \underline{m}^{<} & \text{if } \phi\left(\omega_i\right), \phi\left(\omega_j\right) \in \underline{m}^{<} \\ \phi\left(\omega_i\right) < \phi\left(\omega_j\right) \text{ for the arborescent order of } \underline{n}^{<} & \text{if } \phi\left(\omega_i\right), \phi\left(\omega_j\right) \in \underline{n}^{<} \\ \phi\left(\omega_i\right) \in \underline{m}^{<}, \quad \phi\left(\omega_j\right) \in \underline{n}^{<} & \end{cases} .$$

4. $\phi : \underline{\omega}^{<} \to \underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *is a* ***monotonous injection from*** $\underline{\omega}^{<}$ ***into*** $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *where* $\underline{m}^{<} = (m_1, \ldots, m_s)^{<} \in \Omega^{\bullet <}$ *and* $\underline{\omega}^{i<} \in \Omega^{\bullet <}_{irred}$ *if and only if* ϕ *is an injection from* $\underline{\omega}^{<}$ *in* $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *which satisfies for any* $\omega_i, \omega_j \in \underline{\omega}^{<}$, $w_i < w_j$ *for the arborescent order of* $\underline{\omega}^{<}$ *if and only if*

$$\begin{cases} \phi\left(\omega_i\right) < \phi\left(\omega_j\right) \text{ for the arborescent order of } \underline{\omega}^{i_0<} & \text{if } \phi\left(\omega_i\right), \phi\left(\omega_j\right) \in \underline{\omega}^{i_0<} \\ m_{i_0} < m_{j_0} \text{ for the arborescent order of } \underline{m}^{<} & \text{if } \phi\left(\omega_i\right) \in \underline{\omega}^{i_0<}, \\ & \phi\left(\omega_j\right) \in \underline{\omega}^{j_0<},\ i_0 \neq j_0 \end{cases}$$

Remark 5.1.0.3 *With the same notations as in the definition, it is easy to understand the following*

1. *a* ***monotonous permutation*** σ *of* $\underline{\omega}^{<}$ *as an application* $\phi : \underline{\omega}^{<} \mapsto \underline{\omega}^{<}$ *such that, for any* $i_1, i_2 \in [\![1, r]\!]$ *and for the arborescent order of* $\underline{\omega}^{<}$, *if* $\omega_{i_1} < \omega_{i_2}$ *then* $\phi\left(\omega_{i_1}\right) < \phi\left(\omega_{i_2}\right)$.
2. *a* ***monotonous projection*** σ *from* $\underline{\omega}^{<}$ *into* $\underline{\omega}$ *as an application* $\phi : \underline{\omega}^{<} \mapsto \underline{\omega}$ *such that for any* $i_1, i_2 \in [\![1, r]\!]$, *if* $\omega_{i_1} < \omega_{i_2}$ *for the arborescent order of* $\underline{\omega}^{<}$ *then* $\phi\left(\omega_{i_1}\right) < \phi\left(\omega_{i_2}\right)$ *for the total order of* $\underline{\omega}$. *Let us observe that this notion can be formulate in terms of linear extension of special posets, see [21], p.12.*

3. *a* ***monotonous injection*** *from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$ *as a partition of* $\underline{\omega}^<$ *into the two subtrees* $\underline{m}^<$ *and* $\underline{n}^<$ *which respect the arborescent order of* $\underline{\omega}^<$*.*
4. *a* ***monotonous injection*** *from* $\underline{\omega}^<$ *into* $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ *as a partition of* $\underline{\omega}^<$ *in s irreducible subtrees which respects the arborescent order of* $\underline{\omega}^<$*.*

Before explaining these new objects with examples, it will be convenient to introduce the following notations :

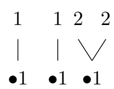
Notation 5.1.0.3 *For* $\underline{\omega}^<, \underline{m}^<, \underline{n}^< \in \Omega^{\bullet<}$ *and* $\underline{\omega}$ *in* $\Omega^\bullet$*, we will denote by :*
— ***bij*** $(\underline{\omega}^<)$ *the number of monotonous permutations of* $\underline{\omega}^<$*.*
— ***proj***$\left(\frac{\underline{\omega}^<}{\underline{\omega}}\right)$ *the number of monotonous projections from* $\underline{\omega}^<$ *into* $\underline{\omega}$*. from* $\underline{\omega}^<$ *into* $\underline{\omega}$*.*
— ***proj***$\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right)$ *the number of monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$*.*
— ***proj***$\left(\frac{\underline{\omega}^<}{\underline{m}^< \circ (\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<})}\right)$ *the number of monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$*.*
— $K\left(\frac{\underline{\omega}^<}{\underline{\omega}}\right)$ *the number of repetitions of* $\underline{\omega}^<$ *in* ***prearbo*** $(\underline{\omega})$*.*
— $K\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right)$ *the number of repetitions of* $\underline{\omega}^<$ *in* ***preconcat*** $(\underline{m}, \underline{n})$*.*
— $K\left(\frac{\underline{\omega}^<}{\underline{m}^< \circ (\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<})}\right)$ *the number of repetitions of* $\underline{\omega}^<$ *in* ***precompo***$(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<})$*.*

Notation 5.1.0.4 *For an arborescent sequence* $\underline{\omega}^< \in \Omega^{\bullet<}$*, we define inductively the internal symmetry factor* $s\left(\underline{\omega}^<\right)$ *of* $\underline{\omega}^<$ *by*

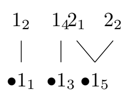
$$s\left(\underline{\omega}^<\right) = \begin{cases} 1 & \text{if } \underline{\omega}^< = \emptyset \\ d_1! \ldots d_s! \left(s\left(\underline{\omega}^{1<}\right)\right)^{d_1} \ldots \left(s\left(\underline{\omega}^{s<}\right)\right)^{d_s} & \text{if } \underline{\omega}^< = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s} \\ s\left('\underline{\omega}^<\right) & \text{if } \underline{\omega}^< = \omega_1.'\underline{\omega}^< \end{cases}$$

where $\underline{\omega}^{i<} \neq \underline{\omega}^{j<}$ *if* $i \neq j$*. The appellation of this integer will be legitimated in the first point of Lemma 5.1.1.*

Example 5.1.0.8 *We first give two examples of monotonous permutations. Let us consider the arborified sequence*

$\underline{\omega}^< =$  .

We easily compute that $s\left(\underline{\omega}^<\right) = 2!.2! = 4$*. In order to identify the monotonous permutations of* $\underline{\omega}^<$*, we index its nodes in the following way :*

$\underline{\omega}^< =$  .

Then, the only monotonous permutations ϕ of $\underline{\omega}^{<}$ are those given by the following tables :

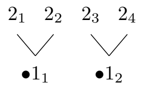
ω	1_1	1_2	1_3	1_4	1_5	2_1	2_2
$\phi(\omega)$	1_1	1_2	1_3	1_4	1_5	2_1	2_2

ω	1_1	1_2	1_3	1_4	1_5	2_1	2_2
$\phi(\omega)$	1_3	1_4	1_1	1_2	1_5	2_1	2_2

ω	1_1	1_2	1_3	1_4	1_5	2_1	2_2
$\phi(\omega)$	1_1	1_2	1_3	1_4	1_5	2_2	2_1

ω	1_1	1_2	1_3	1_4	1_5	2_1	2_2
$\phi(\omega)$	1_3	1_4	1_1	1_2	1_5	2_2	2_1

and we verify that $\boldsymbol{bij}(\underline{\omega}^{<}) = s(\underline{\omega}^{<})$*. For the second example, let us take*

$\underline{\omega}^{<} =$ 

then $s(\underline{\omega}^{<}) = 2!\,(2!)^2 = 8$ *and the different monotonous permutations of* $\underline{\omega}^{<}$ *are given by :*

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_1	2_1	2_2	1_2	2_3	2_4

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_1	2_1	2_2	1_2	2_4	2_3

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_1	2_2	2_1	1_2	2_3	2_4

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_1	2_2	2_1	1_2	2_4	2_3

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_2	2_3	2_4	1_1	2_1	2_2

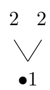
ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_2	2_3	2_4	1_1	2_2	2_1

ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_2	2_4	2_3	1_1	2_1	2_2

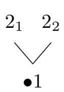
ω	1_1	2_1	2_2	1_2	2_3	2_4
$\phi(\omega)$	1_2	2_4	2_3	1_1	2_2	2_1

and we find again $\boldsymbol{bij}(\underline{\omega}^{<}) = s(\underline{\omega}^{<})$.

Example 5.1.0.9 *We will now give two examples of monotonous projections. We consider first*

$\underline{\omega}^{<} =$  *and* $\underline{\omega} = (1, 2, 2)$.

In order to distinguish the elements of $\underline{\omega}$ *and* $\underline{\omega}^{<}$*, we write*

$\underline{\omega}^{<} =$  *and* $\underline{\omega} = (1', 2'_1, 2'_2)$.

We then write the only two monotonous projections from $\underline{\omega}^{<}$ *into* $\underline{\omega}$ *:*

ω	1	2_1	2_2
$\phi(\omega)$	$1'$	$2'_1$	$2'_2$

ω	1	2_1	2_2
$\phi(\omega)$	$1'$	$2'_2$	$2'_1$

We verify that $1 < 2_1$ *and* $1 < 2_2$ *in* $\underline{\omega}^<$ *and we still have* $\phi(1) < \phi(2_1)$ *and* $\phi(1) < \phi(2_2)$ *in* $\underline{\omega}$*.*

We consider now

$$\underline{\omega}^< = \begin{array}{cc} 1\ 1 & 1 \\ \vee & | \\ \bullet 1 & \bullet 1 \end{array} \text{ and } \underline{\omega} = (1,1,1,1,1).$$

As previously, in order be able to distinguish the elements of $\underline{\omega}$*, we index them :* $\underline{\omega}^< = (1_1, 1_2, 1_3, 1_4, 1_5)$*. We first compute all the repetitions of* $\underline{\omega}^<$ *in* ***prearbo***$(\underline{\omega})$ *:*

$$\begin{array}{cc} 1_2\,1_3 & 1_5 \\ \vee & | \\ \bullet 1_1 & \bullet 1_4 \end{array}, \begin{array}{cc} 1_3\,1_4 & 1_5 \\ \vee & | \\ \bullet 1_1 & \bullet 1_2 \end{array}, \begin{array}{cc} 1_4\,1_5 & 1_3 \\ \vee & | \\ \bullet 1_1 & \bullet 1_2 \end{array}, \begin{array}{cc} 1_3\,1_5 & 1_4 \\ \vee & | \\ \bullet 1_1 & \bullet 1_2 \end{array}, \begin{array}{cc} 1_2\,1_4 & 1_5 \\ \vee & | \\ \bullet 1_1 & \bullet 1_3 \end{array},$$
$$\begin{array}{cc} 1_2\,1_5 & 1_4 \\ \vee & | \\ \bullet 1_1 & \bullet 1_3 \end{array}, \begin{array}{cc} 1_3\,1_4 & 1_5 \\ \vee & | \\ \bullet 1_2 & \bullet 1_1 \end{array}, \begin{array}{cc} 1_3\,1_5 & 1_4 \\ \vee & | \\ \bullet 1_2 & \bullet 1_1 \end{array}, \begin{array}{cc} 1_4\,1_5 & 1_3 \\ \vee & | \\ \bullet 1_2 & \bullet 1_1 \end{array}, \begin{array}{cc} 1_4\,1_5 & 1_2 \\ \vee & | \\ \bullet 1_3 & \bullet 1_1 \end{array}$$

and $K\binom{\underline{\omega}^<}{\underline{\omega}} = 10$*. Each of these repetitions corresponds to an indexation of the nodes of* $\underline{\omega}^<$ *which respects the total order of* $\underline{\omega}$*.*

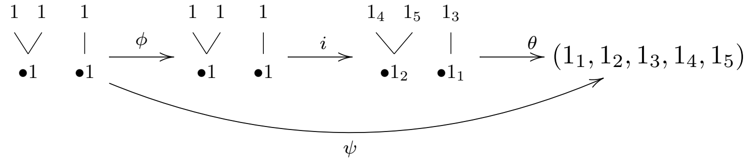
To construct a monotonous projection ψ *from* $\underline{\omega}^<$ *to* $\underline{\omega}$*, we choose an indexation of the nodes of* $\underline{\omega}^<$ *among the ten repetitions of* $\underline{\omega}^<$ *in* ***prearbo***$(\underline{\omega})$*. When we perform such an operation, we associate to each node of* $\underline{\omega}^<$ *an index* $i_k \in [\![1,5]\!]$ *:*

$$\underline{\omega}^< = \begin{array}{cc} 1_{i_2}\,1_{i_3} & 1_{i_5} \\ \vee & | \\ \bullet 1_{i_1} & \bullet 1_{i_4} \end{array}$$

and then we set

$$\theta : \left\{ \begin{array}{ccc} \underline{\omega}^< = \begin{array}{cc} 1_{i_2}\,1_{i_3} & 1_{i_5} \\ \vee & | \\ \bullet 1_{i_1} & \bullet 1_{i_4} \end{array} & \longrightarrow & \underline{\omega} = (1_1, 1_2, 1_3, 1_4, 1_5) \\ 1_{i_k} & \mapsto & 1_{i_k} \end{array} \right. .$$

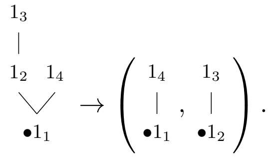
For example :



Let us remark that if we compose this application on the right by a monotonous permutation ϕ *of* $\underline{\omega}^<$ *(there are* $s(\underline{\omega}^<) = 2$ *such permutations), we don't alter the arborified sequence* $\underline{\omega}^<$ *and we then find another monotonous projection of* $\underline{\omega}^<$ *into* $\underline{\omega}$*. Moreover, we find that* ***proj***$\binom{\underline{\omega}^<}{\underline{\omega}} = 2 \times 10 = s(\underline{\omega}^<)\,K\binom{\underline{\omega}^<}{\underline{\omega}}$*.*

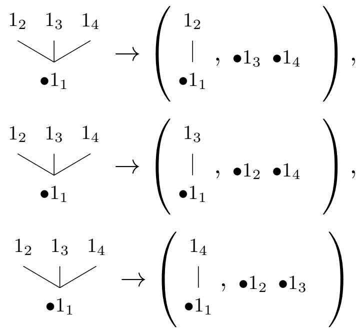
Example 5.1.0.10 *We give four examples of monotonous injections from $\underline{\omega}^<$ into $\underline{m}^< \bullet \underline{n}^<$.*

1. *We first consider* $\underline{\omega}^< = \overset{\overset{1}{|}}{1}\ 1 \bigvee_{\bullet 1}$, $\underline{m}^< = \overset{1}{\underset{\bullet 1}{|}}$ *and* $\underline{n}^< = \overset{1}{\underset{\bullet 1}{|}}$. *Then* $s(\underline{\omega}^<) = s(\underline{m}^<) = s(\underline{n}^<) = 1$. *There is just a repetition of* $\underline{\omega}^<$ *in* ***preconcat***$(\underline{m}^<, \underline{n}^<)$ *and* $K\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right) = 1$ *and we find just a monotonous injection from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$ *:*



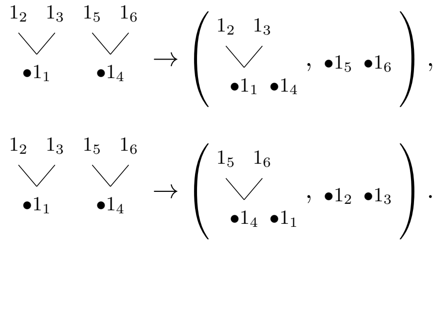
We observe that $\textbf{\textit{proj}}\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right) = \dfrac{s(\underline{\omega}^<)}{s(\underline{m}^<)\, s(\underline{n}^<)} K\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right)$.

2. *Let us now consider* $\underline{\omega}^< = {}^{1\ 1\ 1}\bigvee_{\bullet 1}$ *and* $\underline{m}^< = \overset{1}{\underset{\bullet 1}{|}}$, $\underline{n}^< = \bullet 1\ \bullet 1$. *Then* $s(\underline{\omega}^<) = 3!$, $s(\underline{m}^<) = 1$ *and* $s(\underline{n}^<) = 2!$ *We just find one occurrence of* $\underline{\omega}^<$ *in* ***preconcat***$(\underline{m}^<, \underline{n}^<)$ *and* $K\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right) = 1$. *But we find* 3 *monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$ *:*

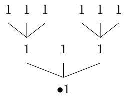
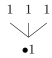


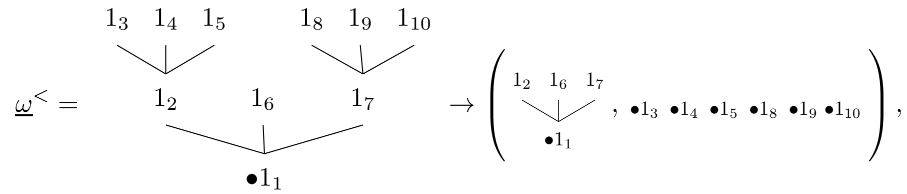
and we have yet $\textbf{\textit{proj}}\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right) = \dfrac{s(\underline{\omega}^<)}{s(\underline{m}^<)\, s(\underline{n}^<)} K\left(\begin{smallmatrix}\underline{\omega}^<\\ \underline{m}^< \bullet \underline{n}^<\end{smallmatrix}\right)$.

3. *For the next example, we take* $\underline{\omega}^< = {}^{1\ 1}\bigvee_{\bullet 1}\ {}^{1\ 1}\bigvee_{\bullet 1}$, $\underline{m}^< = {}^{1\ 1}\bigvee_{\bullet 1}\ \bullet 1$ *and* $\underline{n}^< = \bullet 1\ \bullet 1$. *We find* $s(\underline{\omega}^<) = 2!.(2!)^2$, $s(\underline{m}^<) = 2!$ *and* $s(\underline{n}^<) = 2!$. *There is just one occurrence of* $\underline{\omega}^<$ *in* ***preconcat***$(\underline{m}^<, \underline{n}^<)$ *and* 2 *monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$ *:*



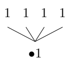
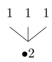
We verify that $\textbf{\textit{proj}}\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right) = \dfrac{s\left(\underline{\omega}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} K\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right)$.

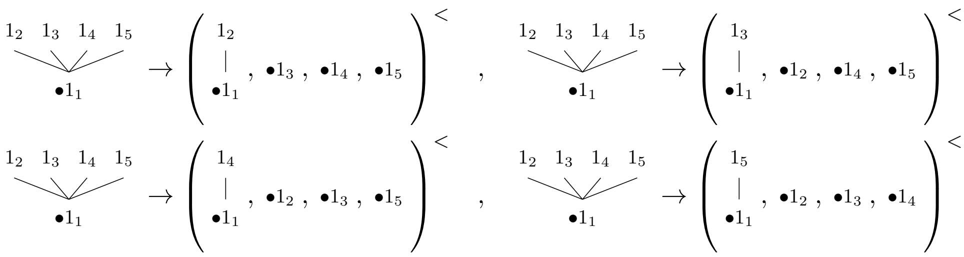
4. *For the last example, we take* $\underline{\omega}^< =$  , $\underline{m}^< =$  *and* $\underline{n}^< =$ $\bullet_1\ \bullet_1\ \bullet_1\ \bullet_1\ \bullet_1\ \bullet_1$. *We find* $s\left(\underline{\omega}^<\right) = 2!.(3!)^2$, $s\left(\underline{m}^<\right) = 3!$ *and* $s\left(\underline{n}^<\right) = 6!$ *We find too* $3!\binom{6}{3}/2!$ *repetitions of* $\underline{\omega}^<$ *in* ***preconcat*** $\left(\underline{m}^<, \underline{n}^<\right)$ *(indeed to choose such a repetition, one has to firstly choose two nodes between the three summital ones of* $\underline{m}^<$*, then we have to choose for this two nodes, the three ones between the six of* $\underline{n}^<$ *that will be grafted on it, there is* $\binom{6}{3}$ *possibilities. The division by 2! is due to the symmetry of* $\underline{\omega}^<$*) and* 1 *monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \bullet \underline{n}^<$ *:*

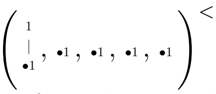


Once again, $\textbf{\textit{proj}}\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right) = \dfrac{s\left(\underline{\omega}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} K\left(\frac{\underline{\omega}^<}{\underline{m}^< \bullet \underline{n}^<}\right)$.

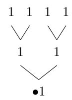
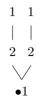
Example 5.1.0.11 *We give three examples of monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$.

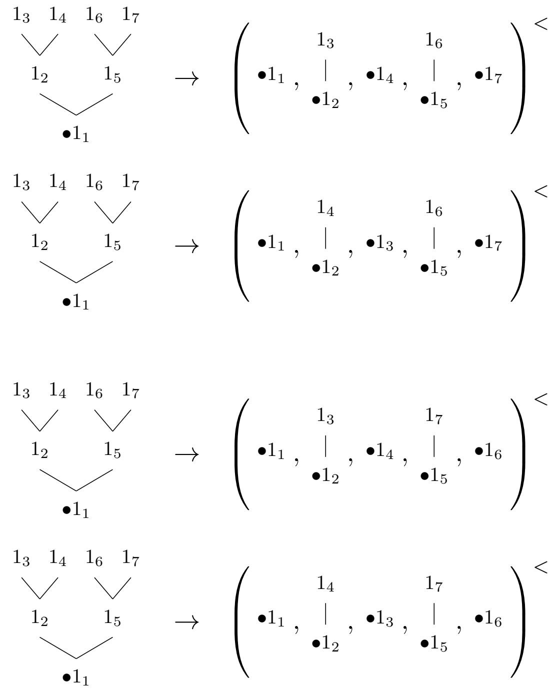
1. *For the first example, we set* $\underline{\omega}^< =$  , $\underline{m}^< =$  , $\underline{\omega}^{1<} =$  *and* $\underline{\omega}^{2<} = \underline{\omega}^{3<} = \underline{\omega}^{4<} = \bullet_1$. *Then* $s\left(\underline{\omega}^<\right) = 4!$, $s\left(\underline{m}^<\right) = 3!$ *and* $\forall i \in [\![1,4]\!]$, $s\left(\underline{\omega}^{i<}\right) = 1$. *We count* 4 *monotonous injections from* $\underline{\omega}^<$ *into* $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{4<}\right)$ *:*

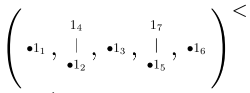


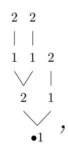
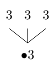
where  *is a sequence of trees with the partial order inherited from the partial order of* $\underline{m}^<$.
Moreover, there is just a repetition of $\underline{\omega}^<$ *in* ***precompo*** $\left(\underline{m}^<; \underline{\omega}^{1<}, \underline{\omega}^{2<}, \underline{\omega}^{3<}, \underline{\omega}^{4<}\right)$ *and we verify that*

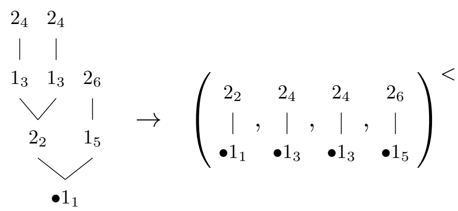
$$\textbf{\textit{proj}}\left(\frac{\underline{\omega}^<}{\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)}\right) = \frac{s\left(\underline{\omega}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{\omega}^{1<}\right) \ldots s\left(\underline{\omega}^{s<}\right)} K\left(\frac{\underline{\omega}^<}{\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)}\right) \quad (\star).$$

2. *We consider now* $\underline{\omega}^{<} =$  , $\underline{m}^{<} =$  , $\underline{\omega}^{1<} = \underline{\omega}^{2<} = \underline{\omega}^{4<} = \bullet_1$ *and* $\underline{\omega}^{3<} = \underline{\omega}^{5<} =$ . *Then* $s\left(\underline{\omega}^{<}\right) = 2.(2!)^2$, $s\left(\underline{m}^{<}\right) = 2!$ *and* $\forall i \in [\![1,5]\!]$, $s\left(\underline{\omega}^{i<}\right) = 1$. *We count* 4 *monotonous injections from* $\underline{\omega}^{<}$ *into* $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{4<}\right)$ *:*



where  *is a tree sequence with the partial order inherited of* $\underline{m}^{<}$. *There is just one repetition of* $\underline{\omega}$ *in* ***precompo***$(\underline{m}^{<}; \underline{\omega}^{1<}, \underline{\omega}^{2<}, \underline{\omega}^{3<}, \underline{\omega}^{4<}, \underline{\omega}^{5<})$ *and the relation* $(\star)$ *is already verified.*

3. *If* $\underline{\omega}^{<} =$  , $\underline{m}^{<} =$  *and* $\forall i \in [\![1,4]\!]$, $\underline{\omega}^{i<} =$ , *then we find just one monotonous injection from* $\underline{\omega}^{<}$ *into* $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{4<}\right)$ *:*



where $\left(\begin{smallmatrix}2\\|\\\bullet 1\end{smallmatrix}, \begin{smallmatrix}2\\|\\\bullet 1\end{smallmatrix}, \begin{smallmatrix}2\\|\\\bullet 1\end{smallmatrix}, \begin{smallmatrix}2\\|\\\bullet 1\end{smallmatrix}\right)^{<}$ *is a tree sequence with the partial order inherited of this of* $\underline{m}^{<}$. *Moreover,* $s\left(\underline{\omega}^{<}\right) = 2!$, $s\left(\underline{m}^{<}\right) = 3!$, $s\left(\underline{\omega}^{i<}\right) = 1$ *for* $i \in [\![1,4]\!]$ *and as explained in example 4.6.0.6,* $K\left(^{\underline{\omega}^{<}}_{\underline{m}^{<}\circ(\underline{\omega}^{1<},\ldots,\underline{\omega}^{4<})}\right) = 3$. *One more time, we verify that* $(\star)$ *is true.*

Lemma 5.1.1 *For any arborescent sequences* $\underline{\omega}^{<}, \underline{m}^{<}, \underline{n}^{<} \in \Omega^{\bullet <}$ *and any sequence* $\underline{\omega} \in \Omega^{\bullet}$ *we have :*

1. $\boldsymbol{bij}\left(\underline{\omega}^{<}\right) = s\left(\underline{\omega}^{<}\right)$
2. $\boldsymbol{proj}\left(^{\underline{\omega}^{<}}_{\underline{\omega}}\right) = s\left(\underline{\omega}^{<}\right) K\left(^{\underline{\omega}^{<}}_{\underline{\omega}}\right)$
3. $\boldsymbol{proj}\left(^{\underline{\omega}^{<}}_{\underline{m}^{<}\bullet\underline{n}^{<}}\right) = \dfrac{s\left(\underline{\omega}^{<}\right)}{s\left(\underline{m}^{<}\right) s\left(\underline{n}^{<}\right)} K\left(^{\underline{\omega}^{<}}_{\underline{m}^{<}\bullet\underline{n}^{<}}\right)$ [1]
4. $\boldsymbol{proj}\left(^{\underline{\omega}^{<}}_{\underline{m}^{<}\circ(\underline{\omega}^{1<},\ldots,\underline{\omega}^{s<})}\right) = \dfrac{s\left(\underline{\omega}^{<}\right)}{s\left(\underline{m}^{<}\right) s\left(\underline{\omega}^{1<}\right)\ldots s\left(\underline{\omega}^{s<}\right)} K\left(^{\underline{\omega}^{<}}_{\underline{m}^{<}\circ(\underline{\omega}^{1<},\ldots,\underline{\omega}^{s<})}\right)$

Proof

1. We prove the first formula by an induction on $r = l(\underline{\omega}^{<})$. The result is clearly true if $r = 0$ or $r = 1$. We assume it is true for any arborified sequence $\underline{\omega}^{<}$ of length $\leq r$ and we consider an arborified sequence $\underline{\omega}^{<}$ of length $r+1$.
   - We first consider the case where $\underline{\omega}^{<} = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s}$ is not irreducible. For any $i \in [\![1,s]\!]$, a monotonous bijection $\phi : \underline{\omega}^{<} \to \underline{\omega}^{<}$ sent each irreducible component $\underline{\omega}^{i<}$ of $\left(\underline{\omega}^{i<}\right)^{d_i}$ into an irreducible component $\underline{\omega}^{i<}$ of $\left(\underline{\omega}^{i<}\right)^{d_i}$. It is a consequence of the fact that ϕ preserves the arborescent order of $\underline{\omega}^{<}$. There are $d_i!$ possible choices for the image by ϕ of a factor $\underline{\omega}^{i<}$ of $\left(\underline{\omega}^{i<}\right)^{d_i}$ and, using the induction hypothesis, there are for each factor $\underline{\omega}^{i<}$, $s\left(\underline{\omega}^{i<}\right)$ possible choices for a monotonous permutation $\phi : \underline{\omega}^{i<} \to \underline{\omega}^{i}$. Then there are $d_i! s\left(\underline{\omega}^{i<}\right)^{d_i}$ possible choices for a monotonous permutation $\phi : \left(\underline{\omega}^{i<}\right)^{d_i} \to \left(\underline{\omega}^{i<}\right)^{d_i}$. When i ranges over $[\![1,s]\!]$, that makes
   $$s\left(\underline{\omega}^{<}\right) = d_1! \ldots d_s! \left(s\left(\underline{\omega}^{1<}\right)\right)^{d_1} \ldots \left(s\left(\underline{\omega}^{s<}\right)\right)^{d_s}$$
   possible choices for ϕ.
   - If $\underline{\omega}^{<} = \omega_1.'\underline{\omega}^{<}$ is irreducible (see Notation 4.3.0.1) then a monotonous bijection $\phi : \underline{\omega}^{<} \to \underline{\omega}^{<}$ sends ω_1 and $'\underline{\omega}^{<}$ into themselves. Then we get $s\left(\underline{\omega}^{<}\right) = s\left('\underline{\omega}^{<}\right)$ possible choices for ϕ.
2. Choosing an occurrence of $\underline{\omega}^{<} = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s}$ in **prearbo** $(\underline{\omega})$ is equivalent to decomposing $\underline{\omega}^{<}$ into $d = d_1 + \ldots + d_s$ totally ordered sequences $\underline{W}^1_1, \ldots, \underline{W}^1_{d_1}$, …, $\underline{W}^s_1, \ldots, \underline{W}^s_{d_s}$ with an order inherited by that of $\underline{\omega}$ and such that $\underline{\omega}^{i<} \in$ **prearbo** $\left(\underline{W}^i_j\right)$ for any $j \in [\![1, d_i]\!]$. By definition, there are $K\left(^{\underline{\omega}^{<}}_{\underline{\omega}}\right)$ such decompositions of $\underline{\omega}$.

1. This formula was already mentioned and proved in [32] but the proof purposed here is original.

Let us consider a monotonous projection ϕ of $\underline{\omega}^<$ into $\underline{\omega}$. As ϕ shall preserve the arborescent order of $\underline{\omega}^<$, for each $i \in [\![1, s]\!]$, it shall send each irreducible component $\underline{\omega}^{i<}$ of $\underline{\omega}^<$ into a totally ordered sequence $\underline{W}_k^i$ where $k \in [\![1, d_i]\!]$ and where $\underline{\omega}^{i<}$ in **prearbo** $\left(\underline{W}_k^i\right)$ and $\underline{W}_k^i$ is a subsequence of $\underline{\omega}$ totally ordered by the one which is inherited from $\underline{\omega}$. As explained just before, the sequences $\underline{W}_1^1, \ldots, \underline{W}_{d_1}^1$, ..., $\underline{W}_1^s, \ldots, \underline{W}_{d_s}^s$ are in correspondence with the choice of an occurrence of $\underline{\omega}^<$ in **prearbo** $(\underline{\omega})$. If ψ is another monotonous projection of $\underline{\omega}^<$ into $\underline{\omega}$ that sends each $\underline{\omega}^{i<}$ in the same W_k^i that ϕ, then $\phi \circ \psi^{-1}$ is a monotonous projection of $\underline{\omega}^<$. The result follows.

3. Let us consider a monotonous injection ϕ from $\underline{\omega}^<$ into $\underline{m}^< \bullet \underline{n}^<$ (we assume that $\underline{\omega}^< \in$ **preconcat** $(\underline{m}^<, \underline{n}^<)$). We know that there exist two subtrees $\underline{W}^{1<}$ and $\underline{W}^{2<}$ of $\underline{\omega}^<$ such that $\phi^{-1}(\underline{m}^<) = \underline{W}^{1<}$ and $\phi^{-1}(\underline{n}^<) = \underline{W}^{2<}$. Moreover, each element of $\underline{W}^{1<}$ is anterior to each element of $\underline{W}^{2<}$ for the arborescent order of $\underline{\omega}^<$. Then when we fix a monotonous injection from $\underline{\omega}^<$ into $\underline{m}^< \bullet \underline{n}^<$, we fix an occurrence of $\underline{\omega}^<$ into **preconcat** $(\underline{m}^<, \underline{n}^<)$. If ψ is an other monotonous injection from $\underline{\omega}^<$ into $\underline{m}^< \bullet \underline{n}^<$ then $\phi \circ \psi^{-1}$ is a monotonous permutation of $\underline{\omega}^<$. If θ is a monotonous permutation of $\underline{m}^<$ and $\underline{n}^<$ then $\theta \circ \phi = \phi$. The result follows.
4. We finally consider a monotonous injection ϕ from $\underline{\omega}^<$ into $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ and we assume that $\underline{\omega}^< \in$ **precompo** $\left(\underline{m}^<; \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)\right)$. As previously, we know there exist s subtrees $\underline{W}^{1<}$,...,$\underline{W}^{s<}$ of $\underline{\omega}^<$ in such a way that $\phi^{-1}\left(\underline{\omega}^{i<}\right) = \underline{W}^{i<}$ for $i \in [\![1, s]\!]$. When we fix a monotonous injection of ϕ from $\underline{\omega}^<$ into $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$, we then fix an occurrence of $\underline{\omega}^<$ in **precompo** $\left(\underline{m}^<; \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)\right)$. If ψ is an other monotonous injection from $\underline{\omega}^<$ into $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ then $\phi \circ \psi^{-1}$ is a monotonous permutation of $\underline{\omega}^<$. If θ is a monotonous permutation of $\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ that preserves each subtree $\underline{\omega}^{i<}$ for $i \in [\![1, s]\!]$ then $\theta \circ \phi = \theta$. As we count $s(\underline{m}^<)\, s\left(\underline{\omega}^{1<}\right) \ldots s\left(\underline{\omega}^{s<}\right)$ such permutations θ, the result follows.

□

Remark 5.1.0.4

— *Let us consider two monotonous injections ϕ and ψ from $\underline{\omega}^<$ to $\underline{\omega}^{1<} \bullet \underline{\omega}^{2<}$. For $i = 1, 2$, we call $\underline{W}^{i<} = \phi^{-1}\left(\underline{\omega}^{i<}\right)$ and $\underline{W}'^{i<} = \psi^{-1}\left(\underline{\omega}^{i<}\right)$. The arborified sequences $\underline{W}^{i<}$ and $\underline{W}'^{i<}$ for $i = 1, 2$ are subtrees of $\underline{\omega}^<$. The application ϕ and ψ are equal if and only if for $i = 1, 2$, $W^{i<} = W'^{i<}$. The number of monotonous injection from $\underline{\omega}^<$ into $\underline{\omega}^{1<} \bullet \underline{\omega}^{2<}$ is less than the number of partitions of $\underline{\omega}$ into the two subsets constituted of the elements of $\underline{\omega}^{1<}$ and the elements of $\underline{\omega}^{2<}$. We can then affirm that*

$$\boldsymbol{proj}\binom{\underline{\omega}^<}{\underline{\omega}^{1<} \bullet \underline{\omega}^{2<}} \leq 2^{l(\underline{\omega}^<)}.$$

— *As explained in Remark 5.1.0.3, there is a one to one correspondence between monotonous partitions of $\underline{\omega}^<$ into the irreducible subtrees $\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}$ with respect of the order given by $\underline{m}^<$ and monotonous injections from*

$\underline{\omega}^{<}$ to $\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$. In order to bound the number of such injections, one can observe the following. We fix such a partition of $\underline{\omega}^{<}$ when we decide of the position of the roots of the trees $\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}$ into $\underline{\omega}^{<}$. There are at most $\binom{l(\underline{\omega}^{<})}{s}$ such possible choices and then

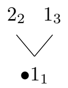
$$\boldsymbol{proj}\binom{\underline{\omega}^{<}}{\underline{m}^{<} \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)} \leq 2^{l(\underline{\omega}^{<})}.$$

We add a definition.

Definition 5.1.0.3 *We say that $\sigma : [\![1, r]\!] \mapsto [\![1, s]\!]$ is a* ***monotonous contraction*** *from $\underline{\omega}^{<}$ into $\underline{\upsilon}$ where $\underline{\upsilon} = (\upsilon_1, \ldots, \upsilon_s) \in \Omega^{\bullet}$ and $s \leq r$ if and only if σ is a surjection that verifies :*

- *if $\omega_i < \omega_j$ for the arborescent order of $\underline{\omega}^{<}$ then $\upsilon_{\sigma(i)} < \upsilon_{\sigma(j)}$ for the total order of $\underline{\upsilon}$*
- *for any $j \in [\![1, s]\!]$, $\upsilon_j = \sum_{j=\sigma(i)} \omega_i$.*

Notation 5.1.0.5 *For any $\underline{\omega}^{<} \in \Omega^{\bullet<}$ and $\underline{\omega} \in \Omega^{\bullet}$, we will denote by $\boldsymbol{cont}\binom{\underline{\omega}^{<}}{\underline{\omega}}$ the number of monotonous contractions.*

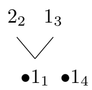
Example 5.1.0.12 *We will now give several examples of monotonous contractions. Let us consider $\underline{\omega}^{<} =$* 

.

- *if $\underline{\upsilon} = (1_1, 3_2)$ then the only monotonous contraction from $\underline{\omega}^{<}$ into $\underline{\upsilon}$ is given by the following table :*

i	1	2	3
$\sigma(i)$	1	2	2

which means that $\upsilon_1 = 1 = \omega_{\sigma(1)}$ and that $\upsilon_2 = 3 = \omega_{\sigma(2)} + \omega_{\sigma(3)}$. Moreover, we verify that in one side, we have $1_1 < 2_2$ and $1_1 < 1_3$ for the arborescent order of $\underline{\omega}^{<}$ and in the other side, we have $1_1 = \upsilon_{\sigma(1)} < \upsilon_{\sigma(2)} = 3_2$ and $1_1 = \upsilon_{\sigma(1)} < \upsilon_{\sigma(3)} = 3_2$ for the total order of $\underline{\upsilon}$.
- *If $\underline{\upsilon} = (2_1, 3_2)$ then there is no monotonous contraction from $\underline{\omega}^{<}$ into $\underline{\upsilon}$.*

We consider now $\underline{\omega}^{<} =$ 

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- *If $\underline{\upsilon} = (2_1, 3_2)$ then the only monotonous contractions from $\underline{\omega}^{<}$ into $\underline{\upsilon}$ are given by the following table :*

i	1	2	3	4
$\sigma(i)$	1	2	2	1

- *If $\underline{\upsilon} = (1_1, 4_2)$ then the only monotonous contractions from $\underline{\omega}^{<}$ into $\underline{\upsilon}$ are given by the following table :*

i	1	2	3	4
$\sigma(i)$	1	2	2	2

— *If* $\underline{\upsilon} = (1_1, 1_2, 3_3)$ *then the only monotonous contractions from* $\underline{\omega}^<$ *into* $\underline{\upsilon}$ *is given by the following table :*

i	1	2	3	4
$\sigma(i)$	1	3	3	2

i	1	2	3	4
$\sigma(i)$	1	3	2	3

5.2 Simple or contracted arborification of a mould

Definition 5.2.0.4 *We call*

— ***arborification*** *of a given mould* $M^\bullet$ *the tree indexed mould* $M^{\bullet<}$ *given for any* $\underline{\omega}^< \in \Omega^{\bullet<}$ *by :*

$$M^{\underline{\omega}^<} = \sum_{\underline{\omega} \in \Omega^\bullet} \boldsymbol{proj}\binom{\underline{\omega}^<}{\underline{\omega}} M^{\underline{\omega}}.$$

This sum is finite because we have $\boldsymbol{proj}\binom{\underline{\omega}^<}{\underline{\omega}} = 0$ *if there is no monotonous projection from* $\underline{\omega}^<$ *to* $\underline{\omega}$.

— ***contracting arborification*** *of a given mould* $M^\bullet$ *the tree indexed mould* $M^{\bullet\ll}$ *given for any* $\underline{\omega}^< \in \Omega^{\bullet<}$ *by :*

$$M^{\underline{\omega}\ll} = \sum_{\underline{\omega} \in \Omega^\bullet} \boldsymbol{cont}\binom{\underline{\omega}^<}{\underline{\omega}} M^{\underline{\omega}}.$$

For the same reason than previously, this sum is finite.

Example 5.2.0.13 *We give some examples of mould arborification :*

$$\begin{aligned}
M^{\begin{smallmatrix} 3 \\ | \\ 2 \\ | \\ \bullet 1 \end{smallmatrix}} &= M^{1,2,3} \\
M^{\bullet 1 \;\; \bullet 2 \;\; \bullet 3} &= M^{1,2,3} + M^{1,3,2} + M^{2,1,3} + M^{3,1,2} + M^{2,3,1} + M^{3,2,1} \\
M^{\begin{smallmatrix} 3 \;\; 4 \\ \vee \\ 2 \\ | \\ \bullet 1 \end{smallmatrix}} &= M^{1,2,3,4} + M^{1,2,4,3} \\
M^{\bullet 1 \;\; \begin{smallmatrix} 3 \;\; 4 \\ \vee \\ \bullet 2 \end{smallmatrix}} &= M^{1,2,3,4} + M^{2,1,3,4} + M^{2,3,1,4} + M^{2,3,4,1} + \\
&\quad M^{1,2,4,3} + M^{2,1,4,3} + M^{2,4,1,3} + M^{2,4,3,1} \\
M^{\bullet 1 \;\; \bullet 1} &= 2M^{1,1} \\
M^{\begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix} \begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix} \begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix}} &= 6M^{1,2,1,2,1,2} + 36M^{1,1,1,2,2,2} + 12M^{1,1,2,2,1,2} + 12M^{1,2,1,1,2,2}
\end{aligned}$$

and of mould contracted arborification :

$$
\begin{aligned}
M^{\bullet_1\ \bullet_1} &= 2M^{1,1}+M^{2}\\
M^{\overset{2\ \ 3}{\overset{\vee}{\bullet_1}}} &= M^{1,2,3}+M^{1,3,2}+M^{1,5}\\
M^{\bullet_1\ \bullet_2\ \bullet_3} &= M^{1,2,3}+M^{1,3,2}+M^{2,1,3}+M^{3,1,2}+M^{2,3,1}\\
&\quad +M^{3,2,1}+M^{1,5}+M^{5,1}+2M^{3,3}+M^{4,2}+M^{2,4}+M^{6}\\
M^{\overset{3}{\overset{|}{\overset{2}{\overset{|}{\bullet_1}}}}} &= M^{1,2,3}
\end{aligned}
$$

Definition 5.2.0.5 *We call*

— ***coarborified*** *of a cosymmetral comould $\mathbb{B}_\bullet$ a tree indexed comould $\mathbb{B}_{\bullet<}$ that satisfies*

$$\mathbb{B}_{\underline{\omega}} = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \boldsymbol{proj}\binom{\underline{\omega}^<}{\underline{\omega}} \mathbb{B}_{\underline{\omega}^<}$$

— ***coarborified*** *of a cosymmetrel comould $\mathbb{B}_\bullet$ a tree indexed comould $\mathbb{B}_{\bullet<}$ that satisfies*

$$\mathbb{B}_{\underline{\omega}} = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \boldsymbol{cont}\binom{\underline{\omega}^<}{\underline{\omega}} \mathbb{B}_{\underline{\omega}^<}$$

For the same reasons as in the previous definition, these two sums are finite.

5.3 The main theorem

Notation 5.3.0.6 *For a sequence $\mathbf{d}=(d_1,\ldots,d_s)\in\mathbf{N}^s$, we write $\mathbf{d}! = \pi_{i=1}^{s} d_i!$.*

Lemma 5.3.1 *Let us consider $\mathbf{d}=(d_1,\ldots,d_s)\in(\mathbf{N}^*)^s$ then*

$$\boldsymbol{cop}\left(\frac{1}{\mathbf{d}!}\partial^{\|\mathbf{d}\|}\right) = \sum_{m_i,n_i\geq 0, m_i+n_i=d_i, i\in[\![1,s]\!]} \frac{1}{\underline{m}!}\partial^{\|\underline{m}\|}\otimes\frac{1}{\underline{n}!}\partial^{\|\underline{n}\|}$$

Proof The lemma is clearly true for a sequence $\mathbf{d}$ of length 1. Assume it is true for a sequence of length $s\in\mathbf{N}^*$ and let us consider a sequence $\mathbf{d}=(d_1,\ldots,d_s,d_{s+1})$. We set $A_{\mathbf{d}}=\left\{(\underline{m},\underline{n})\in(\Omega^\bullet)^2 \mid \mathrm{l}(\underline{m})=\mathrm{l}(\underline{n})=s,\ m_i+n_i=d_i,\ m_i,n_i\geq 0\right\}$. We consider two test functions ϕ and ψ. With the help of the induction hypothesis and Leibniz formula, we get :

$$\begin{aligned}\frac{1}{\mathbf{d}!}\partial^{\|\mathbf{d}\|}.(\phi\psi) &= \frac{1}{\mathbf{d}'!d_{s+1}!}\partial^{\|\mathbf{d}'\|+d_{s+1}}.(\phi\psi)\\ &= \frac{1}{d_{s+1}!}\partial^{d_{s+1}}.\left(\sum_{(\underline{m},\underline{n})\in A_{\mathbf{d}'}}\frac{1}{\underline{m}!}\partial^{\|\underline{m}\|}\phi.\frac{1}{\underline{n}!}\partial^{\|\underline{n}\|}\psi\right)\\ &= \frac{1}{d_{s+1}!}\sum_{m_{s+1}+n_{s+1}=d_{s+1}}\frac{d_{s+1}!}{m_{s+1}!n_{s+1}!}\\ &\qquad\times\sum_{(\underline{m},\underline{n})\in A_{\mathbf{d}'}}\frac{1}{\underline{m}!}\partial^{\|\underline{m}\|+m_{s+1}}\phi.\frac{1}{\underline{n}!}\partial^{\|\underline{n}\|+n_{s+1}}\psi\\ &= \sum_{(\underline{m},\underline{n})\in A_{\mathbf{d}}}\frac{1}{\underline{m}!}\partial^{\|\underline{m}\|}\phi.\frac{1}{\underline{n}!}\partial^{\|\underline{n}\|}\psi\end{aligned}$$

and the equality is proved by induction. □

Definition 5.3.0.6 *We call* ***precoarborification of an operator*** $\mathbb{D}=\sum_{m\geq 0}\mathbb{D}_m\in$ ***ENDOM***$(\mathbb{C}[[u]])$ *the pre-coarborification in the sense of Section 4.4 of the admissible ordinary differential operators family* $\left(\tilde{\mathbb{D}}_m\right)_{m\in\Omega}$ *given by*

$$\forall m\in\Omega,\quad \tilde{\mathbb{D}}_m=(\mathbb{D}_m.u)\,\partial.$$

It will be denoted $\mathbb{D}^{\boldsymbol{prearbo}}$.

We can then expose the main theorem of this chapter. It was expressed for the first time in [15] by its discoverer J. Ecalle but with only a sketch of proof. We propose here a version of this theorem connected to what we have called pre-coarborification. This notion will make it easier to prove the existence of the coarborified of a given operator. We postpone to the end of this chapter an other proof which is more in the spirit of J. Ecalle's.

Theorem 5.3.2 *Let us consider a family of ordinary differential operators* $(\mathbb{D}_m)_{m\in\Omega}$. *There exists an unique arborescent comould* $\mathbb{D}_{\bullet^<}$ *that satisfies the three following properties :*

P₁ : $\mathbb{D}_{\bullet^<}$ *is coseparative :*

$$\mathbb{D}_\emptyset=1 \text{ and } \boldsymbol{cop}\left(\mathbb{D}_{\underline{\omega}^<}\right)=\sum_{\underline{m}^<\oplus\underline{n}^<=\underline{\omega}^<}\mathbb{D}_{\underline{m}^<}\otimes\mathbb{D}_{\underline{n}^<}$$

where the sequences $\underline{m}^<$, $\underline{n}^<$ *may be empty.*

P₂ : *If* $\deg(\underline{\omega}^<)=d$, *i.e.* $\underline{\omega}^<=\left(\underline{\omega}^{1<}\right)\oplus\ldots\oplus\left(\underline{\omega}^{d<}\right)$ *where* $\forall i\in[\![1,d]\!]$, $\underline{\omega}^{i<}\in\Omega^{\bullet<}_{irred}$ *then* $\mathbb{D}_{\underline{\omega}^<}$ *can be written*

$$\mathbb{D}_{\underline{\omega}^<}=P(u)\partial^d.$$

P₃ : *If* $\underline{\omega}^<=\omega_1.{}'\underline{\omega}^<$ *then*

$$\mathbb{D}_{\underline{\omega}^<}.u=\mathbb{D}_{'\underline{\omega}^<}.\mathbb{D}_{\omega_1}.u.$$

This arborescent comould is given by :

$$\forall \underline{\omega}^{<} \in \Omega^{\bullet <}, \quad \mathbb{D}_{\underline{\omega}^{<}} = \frac{1}{s\left(\underline{\omega}^{<}\right)} \mathbb{D}^{\textbf{\textit{prearbo}}}_{\underline{\omega}^{<}}.$$

Moreover :

— *if* $\mathbb{D}_{\bullet}$ *is the comould associated to a primitive element of* ***ENDOM***$(\mathbb{C}[[u]])$ *then* $\mathbb{D}_{\underline{\omega}^{<}}$ *is a coarborified of* $\mathbb{D}_{\bullet}$*.*

— *if* $\mathbb{D}_{\bullet}$ *is the comould associated to a group like element of* ***ENDOM***$(\mathbb{C}[[u]])$*, then* $\mathbb{D}_{\underline{\omega}^{<}}$ *is a contracted coarborified of* $\mathbb{D}_{\bullet}$*.*

In these two cases, $\mathbb{D}_{\bullet^{<}}$ *is called* ***the*** ***coarborified of*** $\mathbb{D}_{\bullet}$ ***homogeneous in*** ∂ *.*

Proof

Existence By construction, the comould $\mathbb{D}_{\bullet^{<}} = \dfrac{1}{s\left(\bullet^{<}\right)} \mathbb{D}^{\mathbf{prearbo}}_{\bullet^{<}}$ satisfies $\mathbf{P_2}$ and $\mathbf{P_3}$, see Proposition 4.4.1. Using Lemma 5.3.1, we easily verify that it satisfies $\mathbf{P_1}$ too. For $\underline{\omega}^{<} = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s} \in \Omega^{\bullet <}$, we know that

$$\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^{<}} = \left(P_1(u)\right)^{d_1} \ldots \left(P_s(u)\right)^{d_s} \partial^{d} =$$

where $d = d_1 + \ldots + d_s$ and $P_1(u) = \mathbb{D}_{\underline{\omega}^{1<}}.u, \ldots, P_s(u) = \mathbb{D}_{\underline{\omega}^{s<}}.u \in \mathbb{C}[u]$. Then for $P(u) = \left(P_1(u)\right)^{d_1} \ldots \left(P_s(u)\right)^{d_s}$ and for two test functions ϕ, ψ we have :

$$\begin{aligned}
& \mathbb{D}_{\underline{\omega}^{<}}(\phi.\psi) \\
= \; & \frac{1}{s\left(\underline{\omega}^{<}\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^{<}}(\phi.\psi) \\
= \; & \frac{1}{\mathbf{d}!\, s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} P(u)\, \partial^{d_1+\ldots+d_s}(\phi.\psi) \\
= \; & \frac{P(u)}{s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} \frac{1}{\mathbf{d}!} \partial^{\|\mathbf{d}\|}(\phi.\psi) \\
= \; & \frac{P(u)}{s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} \sum_{m_i, n_i \geq 0, m_i + n_i = d_i, i \in [\![1,s]\!]} \frac{1}{\underline{m}!} \partial^{\|\underline{m}\|} \phi . \frac{1}{\underline{n}!} \partial^{\|\underline{n}\|} \psi \\
= \; & \sum_{\underline{m}^{<} \oplus \underline{n}^{<} = \underline{\omega}^{<}} \frac{\left(P_1(u)\right)^{m_1} \ldots \left(P_s(u)\right)^{m_s}}{\underline{m}!\, s\left(\underline{\omega}^{1<}\right)^{m_1} \ldots s\left(\underline{\omega}^{s<}\right)^{m_s}} \mathbb{D}_{\underline{m}^{<}} \phi . \frac{\left(P_1(u)\right)^{n_1} \ldots \left(P_s(u)\right)^{n_s}}{\underline{n}!\, s\left(\underline{\omega}^{1<}\right)^{n_1} \ldots s\left(\underline{\omega}^{s<}\right)^{n_s}} \mathbb{D}_{\underline{n}^{<}} \psi \\
= \; & \sum_{\underline{m}^{<} \oplus \underline{n}^{<} = \underline{\omega}^{<}} \frac{1}{s\left(\underline{m}^{<}\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{m}^{<}} \phi . \frac{1}{s\left(\underline{n}^{<}\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{n}^{<}} \psi \\
= \; & \sum_{\underline{m}^{<} \oplus \underline{n}^{<} = \underline{\omega}^{<}} \mathbb{D}_{\underline{m}^{<}} \phi . \mathbb{D}_{\underline{n}^{<}} \psi
\end{aligned}$$

Uniqueness Conversely, if an arborescent comould $\tilde{\mathbb{D}}_{\bullet^{<}}$ satisfies $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$, we have to prove that $\tilde{\mathbb{D}}_{\bullet^{<}} = \dfrac{1}{s\left(\bullet^{<}\right)} \mathbb{D}^{\mathbf{prearbo}}_{\bullet^{<}}$. We shall reason inductively on the length r of the arborescent sequence $\underline{\omega}^{<}$. If $r = 1$ then using $\mathbf{P_2}$, we know that $\tilde{\mathbb{D}}_{\underline{\omega}^<} = \tilde{P}(u)\partial$ and $\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<} = P(u)\partial$ where $P(u), \tilde{P}(u) \in \mathbb{C}[u]$ and using $\mathbf{P_3}$, we have :

$$\tilde{P}(u) = \tilde{\mathbb{D}}_{\underline{\omega}^<}.u = \mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<}.u = P(u)$$

then $\tilde{\mathbb{D}}_{\underline{\omega}^<} = \mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<}$. Assume that $\tilde{\mathbb{D}}_{\underline{\omega}^<} = 1/s\left(\underline{\omega}^<\right)\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<}$ for any arborescent sequence of length $\leq r$ and consider $\underline{\omega}^< \in \Omega^{\bullet<}$ of length $r+1$.

— In the case where $\underline{\omega}^< = \omega_1.'\underline{\omega}^< \in \Omega^{\bullet<}_{\text{irred}}$ is an irreducible sequence and where $'\underline{\omega}^< \in \Omega^{\bullet<}$ is of length r, then using the induction hypothesis, we have :

$$\tilde{\mathbb{D}}_{'\underline{\omega}^<} = \frac{1}{s\left('\underline{\omega}^<\right)}\mathbb{D}^{\mathbf{prearbo}}_{'\underline{\omega}^<}.$$

Moreover, $\mathbf{P_2}$ gives the existence of $P(u), \tilde{P}(u) \in \mathbb{C}[u]$ in a such way that $\tilde{\mathbb{D}}_{\underline{\omega}^<} = \tilde{P}(u)\partial$ and $\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<} = P(u)\partial$. With $\mathbf{P_3}$, we have :

$$\tilde{P}(u) = \mathbb{D}_{\underline{\omega}^<}.u = \tilde{\mathbb{D}}_{'\underline{\omega}^<}.\mathbb{D}_{\omega_1}.u = \frac{1}{s\left('\underline{\omega}^<\right)}\mathbb{D}^{\mathbf{prearbo}}_{'\underline{\omega}^<}.\mathbb{D}_{\omega_1}.u = P(u)$$

then $\tilde{\mathbb{D}}_{\underline{\omega}^<} = 1/s\left(\underline{\omega}^<\right)\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<}$.

— In the case where $\underline{\omega}^< = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s}$ and where $\forall i \in [\![1, s]\!]$, $\underline{\omega}^{i<} \in \Omega^{\bullet<}_{\text{irred}}$, then once again, $\mathbf{P_2}$ gives the existence of $P(u), \tilde{P}(u) \in \mathbb{C}[u]$ in a such way that $\tilde{\mathbb{D}}_{\underline{\omega}^<} = \tilde{P}(u)\partial^{\|\mathbf{d}\|}$ and $\mathbb{D}_{\underline{\omega}^<} = P(u)\partial^{\|\mathbf{d}\|}$. Then using Lemma 5.3.1 :

$$\begin{aligned}
\mathbf{cop}\left(\tilde{\mathbb{D}}_{\underline{\omega}^<}\right) &= \mathbf{cop}\left(\tilde{P}(u)\partial^{\|\mathbf{d}\|}\right) \\
&= d!\tilde{P}(u)\mathbf{cop}\left(\frac{1}{\mathbf{d}!}\partial^{\|\mathbf{d}\|}\right) \\
&= \mathbf{d}!\tilde{P}(u) \sum_{\substack{m_i + n_i = d_i \\ m_i, n_i \geq 0 \\ i \in [\![1, s]\!]}} \frac{1}{\underline{m}!}\partial^{\|\underline{m}\|} \otimes \frac{1}{\underline{n}!}\partial^{\|\underline{n}\|} \qquad (\maltese)
\end{aligned}$$

But using $\mathbf{P_1}$, the induction hypothesis and Proposition 4.4.1, it comes :

$$\mathbf{cop}\left(\tilde{\mathbb{D}}_{\underline{\omega}^<}\right) = \tilde{P}(u)\partial^s \oplus 1 + 1 \oplus \tilde{P}(u)\partial^s + \sum_{\substack{\underline{m}^< \oplus \underline{n}^< = \underline{\omega}^< \\ \underline{m}^<, \underline{n}^< \neq \emptyset}} \tilde{\mathbb{D}}_{\underline{m}^<} \otimes \tilde{\mathbb{D}}_{\underline{n}^<}$$

but

$$\begin{aligned}
&\sum_{\substack{\underline{m}^< \oplus \underline{n}^< = \underline{\omega}^< \\ \underline{m}^<, \underline{n}^< \neq \emptyset}} \tilde{\mathbb{D}}_{\underline{m}^<} \otimes \tilde{\mathbb{D}}_{\underline{n}^<} \\
= & \sum_{\substack{\underline{m}^< \oplus \underline{n}^< = \underline{\omega}^< \\ \underline{m}^<, \underline{n}^< \neq \emptyset}} \frac{1}{s\left(\underline{m}^<\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{m}^<} \otimes \frac{1}{s\left(\underline{n}^<\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{n}^<} \\
= & \sum_{\substack{m_i + n_i = d_i \\ m_i, n_i \geq 0 \\ i \in [\![1, s]\!]}} \frac{\mathbb{D}^{\mathbf{prearbo}}_{(\underline{\omega}^{1<})^{m_1} \oplus \ldots \oplus (\underline{\omega}^{s<})^{m_s}}}{\underline{m}! s\left(\underline{\omega}^{1<}\right)^{m_1} \ldots s\left(\underline{\omega}^{s<}\right)^{m_s}} \otimes \frac{\mathbb{D}^{\mathbf{prearbo}}_{(\underline{\omega}^{1<})^{n_1} \oplus \ldots \oplus (\underline{\omega}^{s<})^{n_s}}}{\underline{n}! s\left(\underline{\omega}^{1<}\right)^{n_1} \ldots s\left(\underline{\omega}^{s<}\right)^{n_s}} \\
= & \frac{1}{s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} \sum_{\substack{m_i + n_i = d_i \\ m_i, n_i \geq 0 \\ i \in [\![1, s]\!]}} (P_1(u))^{m_1} \ldots (P_s(u))^{m_s} \frac{1}{\underline{m}!} \partial^{\|\underline{m}\|} \otimes \\
& \qquad (P_1(u))^{n_1} \ldots (P_s(u))^{n_s} \frac{1}{\underline{n}!} \partial^{\|\underline{n}\|} \\
= & \frac{(P_1(u))^{d_1} \ldots (P_s(u))^{d_s}}{s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} \sum_{\substack{m_i + n_i = d_i \\ m_i, n_i \geq 0 \\ i \in [\![1, s]\!]}} \frac{1}{\underline{m}!} \partial^{\|\underline{m}\|} \otimes \frac{1}{\underline{n}!} \partial^{\|\underline{n}\|}
\end{aligned}$$

and when one identifies (✠) with this last equality, we obtain :

$$\tilde{P}(u) = \frac{(P_1(u))^{d_1} \ldots (P_s(u))^{d_s}}{\mathbf{d}! s\left(\underline{\omega}^{1<}\right)^{d_1} \ldots s\left(\underline{\omega}^{s<}\right)^{d_s}} = \frac{(P_1(u))^{d_1} \ldots (P_s(u))^{d_s}}{s\left(\underline{\omega}\right)}$$

and then $\tilde{\mathbb{D}}_{\underline{\omega}^<} = \dfrac{1}{s\left(\underline{\omega}^<\right)} \mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<}$.

We have thus proved the uniqueness of the coarborified which is homogeneous in the degree of $(\mathbb{D}_m)$.

We have yet to prove that if $\mathbb{D}_\bullet$ is cosymmetral (resp. cosymmetrel) then $\mathbb{D}_{\bullet^<}$ is the (resp. contracted) coarborified of $\mathbb{D}_\bullet$.

— If $\mathbb{D}_\bullet$ is cosymmetral, we naturally get $\mathbb{D}^{\mathbf{prearbo}}_m = \mathbb{D}_m$ for any $m \in \Omega$ and according to Proposition 4.4.1 and Lemma 5.1.1, we have for any $\underline{\omega} \in \Omega^\bullet$:

$$\begin{aligned}
\mathbb{D}_{\underline{\omega}} &= \sum_{\underline{\omega}^< \in \mathbf{prearbo}(\underline{\omega})} \mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^<} \\
&= \sum_{\underline{\omega}^< \in \mathbf{arbo}(\underline{\omega})} K\binom{\underline{\omega}^<}{\underline{\omega}} s\left(\underline{\omega}^<\right) \mathbb{D}_{\underline{\omega}^<} \\
&= \sum_{\underline{\omega}^< \in \mathbf{arbo}(\underline{\omega})} \mathbf{proj}\binom{\underline{\omega}^<}{\underline{\omega}} \mathbb{D}_{\underline{\omega}^<}
\end{aligned}$$

and the result is proved.

— The case where $\mathbb{D}_\bullet$ is cosymmetrel will be a consequence of the next proposition.

□

Proposition 5.3.3 *Let us consider the coarborified $\mathbb{D}_{\bullet^<}$ of a given comould $\mathbb{D}_\bullet$ as constructed in the previous theorem. Then for any arborified sequences $\underline{m}^<, \underline{n}^< \in \Omega^{\bullet<}$ we have :*

$$\mathbb{D}_{\underline{n}^<}.\mathbb{D}_{\underline{m}^<} = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \boldsymbol{proj}\begin{pmatrix} \underline{\omega}^< \\ \underline{m}^< \bullet \underline{n}^< \end{pmatrix} \mathbb{D}_{\underline{\omega}^<}.$$

Proof According to Proposition 4.5.1 and Lemma 5.1.1, we have :

$$\begin{aligned}
\mathbb{D}_{\underline{n}^<} \times \mathbb{D}_{\underline{m}^<} &= \frac{1}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} \tilde{\mathbb{D}}^{\mathbf{prearbo}}_{\underline{n}^<} \times \tilde{\mathbb{D}}^{\mathbf{prearbo}}_{\underline{m}^<} \\
&= \frac{1}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} \sum_{\underline{\omega}^< \in \mathbf{preconcat}(\underline{m}^<, \underline{n}^<)} \tilde{\mathbb{D}}^{\mathbf{prearbo}}_{\underline{\omega}^<} \\
&= \sum_{\underline{\omega}^< \in \mathbf{concat}(\underline{m}^<, \underline{n}^<)} \frac{s\left(\underline{\omega}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} K\begin{pmatrix} \underline{\omega}^< \\ \underline{m}^< \bullet \underline{n}^< \end{pmatrix} \mathbb{D}_{\underline{\omega}^<} \\
&= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \mathbf{proj}\begin{pmatrix} \underline{\omega}^< \\ \underline{m}^< \bullet \underline{n}^< \end{pmatrix} \mathbb{D}^<_{\underline{\omega}}
\end{aligned}$$

□

Remark 5.3.0.5 *If B_+ is the operator on $\Omega^{\bullet<}$ associating to an arborified sequence $\underline{\omega}^< = \underline{\omega}^{1<} \oplus \ldots \oplus \underline{\omega}^{s<}$ the irreducible sequence obtained by grafting the root to each irreducible sequence $\underline{\omega}^{i<}$ on a same empty node, then we recognize the Grossmann-Larson product for rooted trees :*

$$B_+\left(\underline{n}^<\right) \times B_+\left(\underline{m}^<\right) = \sum_{\underline{\omega}^< \in \boldsymbol{concat}(\underline{m}^<, \underline{n}^<)} \frac{s\left(\underline{\omega}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{n}^<\right)} K\begin{pmatrix} \underline{\omega}^< \\ \underline{m}^< \bullet \underline{n}^< \end{pmatrix} B_+\left(\underline{\omega}^<\right).$$

Before completing the proof of the main Theorem 5.3.2 and in order to improve our explanations, we will make explicit the expansion $\mathbb{D}_m = \sum_{\|\underline{\omega}\|=m} \frac{f_\omega}{r!} u^{m+r} \partial^r$

in terms of coarborified operators for $m = 1, 2, 3, 4$:

$$\mathbb{D}_1 = f_1 u^2 \partial = (\mathbb{D}_1.u)\,\partial = \mathbb{D}_{\bullet_1}$$

$$\mathbb{D}_2 = \frac{f_1^2}{2!} u^4 \partial^2 + f_2 u^3 \partial = \frac{1}{2!}(\mathbb{D}_1.u)(\mathbb{D}_1.u)\,\partial^2 + (\mathbb{D}_2.u)\,\partial = \mathbb{D}_{\bullet_1\ \bullet_1} + \mathbb{D}_{\bullet_2}$$

$$\mathbb{D}_3 = \frac{f_1^3}{3!} u^6 \partial^3 + 2\frac{f_1 f_2}{2} u^5 \partial^2 + f_3 u^4 \partial = \frac{1}{3!}(\mathbb{D}_1.u)^3\,\partial^3 + (\mathbb{D}_1.u)(\mathbb{D}_2.u)\,\partial^2 + (\mathbb{D}_3.u)\,\partial$$

$$= \mathbb{D}_{\bullet_1\ \bullet_1\ \bullet_1} + \mathbb{D}_{\bullet_1\ \bullet_2} + \mathbb{D}_{\bullet_3}$$

$$\mathbb{D}_4 = \frac{f_1^4}{4!} u^8 \partial^4 + 2\frac{f_1 f_3}{2} u^6 \partial^2 + \frac{f_2^2}{2} u^6 \partial^2 + 3\frac{f_1^2 f_2}{3!} \partial^3 + f_4 u^5 \partial$$

$$= \frac{1}{4!}(\mathbb{D}_1.u)^4\,\partial^4 + \frac{1}{2!}(\mathbb{D}_1.u)^2(\mathbb{D}_2.u)\,\partial^3 + 2\frac{1}{2}(\mathbb{D}_1.u)(\mathbb{D}_3.u)\,\partial^2 + \frac{1}{2}(\mathbb{D}_2.u)^2\,\partial^2 + (\mathbb{D}_4.u)\,\partial$$

$$= \mathbb{D}_{\bullet_1\ \bullet_1\ \bullet_1\ \bullet_1} + \mathbb{D}_{\bullet_1\ \bullet_1\ \bullet_2} + \mathbb{D}_{\bullet_1\ \bullet_3} + \mathbb{D}_{\bullet_2\ \bullet_2} + \mathbb{D}_{\bullet_4}$$

Corollary 5.3.4 *The coarborified* $\mathbb{D}_{\bullet^<}$ *of a cosymmetrel comould* $\mathbb{D}_\bullet$ *constructed in Theorem 5.3.2 is the contracted coarborified of* $\mathbb{D}_\bullet$.

Proof As $\mathbb{D}_\bullet$ is cosymmetrel, according to the fact that $\mathbb{D}$ is a substitution automorphism and using Taylor formula, we know that there exists a sequence (f_m) such that for any $m \in \Omega$,

$$\mathbb{D}_m = \sum_{\|\underline{\omega}\|=m} \frac{f_{\underline{\omega}}}{r!} u^{m+r} \partial^r$$

where $r = \mathrm{l}(\underline{\omega})$. Then, if $k\left(\omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s}\right)$ denotes the number of sequences $\underline{\omega}$ containing d_1 terms ω_1, ..., d_s terms ω_s then [2]

$$\mathbb{D}_m \quad = \sum_{\substack{\underline{\omega}^< = \omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s} \\ \|\underline{\omega}^<\| = m}} \frac{k\left(\underline{\omega}^<\right)}{(d_1 + \ldots + d_s)!} f_{\omega_1}^{d_1} \ldots f_{\omega_s}^{d_s} u^{m+d_1+\ldots+d_s} \partial^{d_1+\ldots+d_s}$$

But $k\left(\omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s}\right) = \binom{d_1+\ldots+d_s}{d_1,\ldots,d_s}$ [3] and as long as $\mathbb{D}_m^{\mathbf{prearbo}} = \mathbb{D}_m.u\partial =$

2. This expression of $\mathbb{D}_m$ is already present in [43].
3. where $\binom{d_1+\ldots+d_s}{d_1,\ldots,d_s}$ is the multinomial coefficient.

$f_m u^{m+1}\partial$ for all $m \in \Omega$, we get :

$$\begin{aligned}
\mathbb{D}_m &= \sum_{\substack{\underline{\omega}^< = \omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s} \\ \|\underline{\omega}^<\| = m}} \frac{1}{d_1! \ldots d_s!} f_{\omega_1}^{d_1} \ldots f_{\omega_s}^{d_s} u^{m+d_1+\ldots+d_s} \partial^{d_1+\ldots+d_s} \\
&= \sum_{\substack{\underline{\omega}^< = \omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s} \\ \|\underline{\omega}^<\| = m}} \frac{1}{d_1! \ldots d_s!} \mathbb{D}^{\mathbf{prearbo}}_{\omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s}} \\
&= \sum_{\substack{\underline{\omega}^< = \omega_1^{d_1} \oplus \ldots \oplus \omega_s^{d_s} \\ \|\underline{\omega}^<\| = m}} \mathbb{D}_{\underline{\omega}^<} \\
&= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^<}{m} \mathbb{D}_{\underline{\omega}^<}
\end{aligned}$$

Then for $\underline{m} = (m_1, m_2) \in \Omega^\bullet$, we get :

$$\begin{aligned}
\mathbb{D}_{\underline{m}} &= \mathbb{D}_{m_2}.\mathbb{D}_{m_1} \\
&= \sum_{\substack{\underline{\omega}^{1<}, \underline{\omega}^{2<} \in \Omega^{\bullet<} \\ \|\underline{\omega}^{1<}\| = m_1, \|\underline{\omega}^{2<}\| = m_2}} \mathbb{D}_{\underline{\omega}^{2<}}.\mathbb{D}_{\underline{\omega}^{1<}} \\
&= \sum_{\substack{\underline{\omega}^{1<}, \underline{\omega}^{2<} \in \Omega^{\bullet<} \\ \|\underline{\omega}^{1<}\| = m_1, \|\underline{\omega}^{2<}\| = m_2}} \mathbf{proj}\binom{\underline{\omega}^<}{\underline{\omega}^{1<} \bullet \underline{\omega}^{2<}} \mathbb{D}^<_{\underline{\omega}} \\
&= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^<}{\underline{m}} \mathbb{D}^<_{\underline{\omega}}
\end{aligned}$$

By induction, we assume that we have $\mathbb{D}_{\underline{m}} = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^<}{\underline{m}} \mathbb{D}^<_{\underline{\omega}}$ for any sequence $\underline{m}$ of length r. Let us consider a sequence $\underline{m}$ of length $r+1$, then we get :

$$\begin{aligned}
\mathbb{D}_{\underline{m}} &= \mathbb{D}_{m_{r+1}}.\mathbb{D}_{\underline{m'}} \\
&= \sum_{\underline{\omega}^{1<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{1<}}{m}\mathbb{D}_{\underline{\omega}^{1<}}.\sum_{\underline{\omega}^{2<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{2<}}{\underline{m'}}\mathbb{D}_{\underline{\omega}^{2<}} \\
&= \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{1<}}{m}\mathbf{cont}\binom{\underline{\omega}^{2<}}{\underline{m'}}\mathbb{D}_{\underline{\omega}^{1<}}.\mathbb{D}_{\underline{\omega}^{2<}} \\
&= \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{1<}}{m}\mathbf{cont}\binom{\underline{\omega}^{2<}}{\underline{m'}}\sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}}\mathbf{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}}\mathbb{D}_{\underline{\omega}^{<}} \\
&= \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}}\left(\sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{1<}}{m}\mathbf{cont}\binom{\underline{\omega}^{2<}}{\underline{m'}}\mathbf{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}}\right)\mathbb{D}_{\underline{\omega}^{<}} \\
&= \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} \mathbf{cont}\binom{\underline{\omega}^{<}}{\underline{m}}\mathbb{D}^{<}_{\underline{\omega}}
\end{aligned}$$

The equality $\sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}}\mathbf{cont}\binom{\omega^{1<}}{m}\mathbf{cont}\binom{\underline{\omega}^{2<}}{\underline{m'}}\mathbf{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}} = \mathbf{cont}\binom{\underline{\omega}}{\underline{m}}$ is a consequence of the following fact. Let us consider a monotonous contraction ϕ from $\underline{\omega}^{<}$ into $\underline{m}$. We can decompose $\underline{\omega}^{<}$ into two subtrees $\underline{\omega}^{1<}$ and $\underline{\omega}^{2<}$ that preserve the internal order of $\underline{\omega}^{<}$ and in such a way that $\phi\left(\underline{\omega}^{2<}\right) = \underline{m'}$ and $\phi\left(\underline{\omega}^{1<}\right) = m_{r+1}$. Choosing a contraction from $\underline{\omega}^{<}$ into $\underline{m}^{<}$ is equivalent to firstly choosing a monotonous injection from $\underline{\omega}^{<}$ into $\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}$ and to secondly choosing a contraction from $w^{1<}$ into m_{r+1} and a contraction from $w^{2<}$ into $\underline{m'}$. □

Notation 5.3.0.7 *In order to simplify the notations, we shall always denote $M^{\bullet<}$ (with only one exponentiate $<$) the arborification $M^{\bullet<}$ and the contracted arborification $M^{\bullet\ll}$ of a given mould $M^{\bullet}$. If the comould part of the studied operator is cosymmetral, then we are performing simple arborification and if it is cosymmetrel, then we are dealing with contracted arborification.*

5.4 Atomicity and separativity

One of the strengths of arborification lies in the fact that the symmetries of the differential operators (group like or primitive) are translated into symmetries of moulds. If $\mathbb{A} = \sum M^{\bullet}\mathbb{D}_{\bullet}$ is a primitive element of $\mathbf{ENDOM}(\mathbb{C}[[u]])$ with $\mathbb{D}_{\bullet}$ a cosymmetral comould then the mould $M^{\bullet}$ is alternal, which implies for example that $M^{\bullet_1\ \bullet_2} = M^{12} + M^{21} = 0$. When we compute the arborified mould-comould expansion we then obtain :

$$\begin{aligned} M^{1,2}\mathbb{D}_{1,2}+M^{2,1}\mathbb{D}_{2,1} &= M^{1,2}\left(\mathbb{D}_{\substack{2\\|\\\bullet_1}}+\mathbb{D}_{\bullet_1\ \bullet_2}\right)+M^{2,1}\left(\mathbb{D}_{\substack{1\\|\\\bullet_2}}+\mathbb{D}_{\bullet_1\ \bullet_2}\right)\\ &= M^{1,2}\mathbb{D}_{\substack{2\\|\\\bullet_1}}+M^{2,1}\mathbb{D}_{\substack{1\\|\\\bullet_2}}+\underbrace{\left(M^{1,2}+M^{2,1}\right)}_{=0}\mathbb{D}_{\bullet_1\ \bullet_2} \end{aligned}$$

The aim of this section is to explain which simplifications one can obtain with arborification.

Definition 5.4.0.7 *We say (see [15]) that a (contracted or not) arborified mould $M^\bullet$ is :*

— ***atomic***[4] *if for any non irreducible arborified sequence $\underline{\omega}^<$, we have $M^{\underline{\omega}^<}=0$.*

— ***separative*** *if for any arborified sequence $\underline{\omega}^<=\left(w^{1<}\right)^{d_1}\oplus\ldots\oplus(w^{s<})^{d_s}$, we have $M^{\underline{\omega}^<}=\left(M^{w^{1<}}\right)^{d_1}\ldots\left(M^{w^{s<}}\right)^{d_s}$.*

Proposition 5.4.1 *We assume that $\mathbb{D}_{\bullet^<}$ is the coarborified of a given cosymmetral or cosymmetrel comould $\mathbb{D}_\bullet$ and we consider an operator $\mathbb{A}=\sum M^\bullet\mathbb{D}_\bullet$. Then :*

— *If $M^{\bullet<}$ is atomic then $\mathbb{A}$ is primitive.*

— *If $M^{\bullet<}$ is separative then $\mathbb{A}$ is group like.*

Proof For any $m\in\Omega$, we have :

$$\begin{aligned} \mathbf{cop}\left(\mathbb{A}_m\right) &= \sum_{\substack{\underline{\omega}^<\in\Omega^{\bullet<}\\ \|\underline{\omega}^<\|=m}} M^{\underline{\omega}^<}\mathbf{cop}\left(\mathbb{D}_{\underline{\omega}^<}\right)\\ &= \sum_{\substack{\underline{\omega}^<\in\Omega^{\bullet<}\\ \|\underline{\omega}^<\|=m}}\ \sum_{\underline{\omega}^{1<}\oplus\underline{\omega}^{2<}=\underline{\omega}^<} M^{\underline{\omega}^<}\mathbb{D}_{\underline{\omega}^{1<}}\otimes\mathbb{D}_{\underline{\omega}^{2<}} \end{aligned}$$

— If $M^{\bullet<}$ is atomic then $M^{\underline{\omega}^{1<}\oplus\underline{\omega}^{2<}}=0$ while $\underline{\omega}^{1<}\neq\emptyset$ and $\underline{\omega}^{2<}\neq\emptyset$. Then

$$\begin{aligned} \mathbf{cop}\left(\mathbb{A}_m\right) &= \sum_{\substack{\underline{\omega}^<\in\Omega^{\bullet<}\\ \|\underline{\omega}^<\|=m}} M^{\underline{\omega}^<}\mathbb{D}_{\underline{\omega}^<}\otimes 1+\sum_{\substack{\underline{\omega}^<\in\Omega^{\bullet<}\\ \|\underline{\omega}^<\|=m}} M^{\underline{\omega}^<}1\otimes\mathbb{D}_{\underline{\omega}^<}\\ &= \mathbb{A}_m\otimes 1+1\otimes\mathbb{A}_m. \end{aligned}$$

and $\mathbb{A}$ is primitive.

4. called antiseparative in [15]

— If $M^{\bullet<}$ is separative then $M^{\underline{\omega}^{1<}\oplus\underline{\omega}^{2<}} = M^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}$ then

$$\begin{aligned}
\mathbf{cop}\,(\mathbb{A}_m) &= \sum_{\substack{\underline{\omega}^{<}\in\Omega^{\bullet<}\\ \|\underline{\omega}^{<}\|=m}} \sum_{\underline{\omega}^{1<}\oplus\underline{\omega}^{2<}=\underline{\omega}^{<}} M^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}\mathbb{D}_{\underline{\omega}^{1<}}\otimes\mathbb{D}_{\underline{\omega}^{2<}}\\
&= \sum_{i+j=m,\,i,j\geq 0} \sum_{\substack{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}\\ \|\underline{\omega}^{1<}\|=i,\,\|\underline{\omega}^{2<}\|=j}} M^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}\mathbb{D}_{\underline{\omega}^{1<}}\otimes\mathbb{D}_{\underline{\omega}^{2<}}\\
&= \sum_{i+j=m,\,i,j\geq 0}\left(\sum_{\substack{\underline{\omega}^{1<}\in\Omega^{\bullet<}\\ \|\underline{\omega}^{1<}\|=i}} M^{\underline{\omega}^{1<}}\mathbb{D}_{\underline{\omega}^{1<}}\right)\otimes\left(\sum_{\substack{\underline{\omega}^{2<}\in\Omega^{\bullet<}\\ \|\underline{\omega}^{2<}\|=j}} M^{\underline{\omega}^{2<}}\mathbb{D}_{\underline{\omega}^{2<}}\right)\\
&= \sum_{i+j=m,\,i,j\geq 0}\mathbb{A}_i\otimes\mathbb{A}_j
\end{aligned}$$

and $\mathbb{A}$ is group like.

□

5.5 Simple or contracted arborified product

Proposition 5.5.1 *We consider two operators* $\mathbb{A} = \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} M^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}$ *and* $\mathbb{B} = \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} N^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}$. *Then* $\mathbb{A}\bullet\mathbb{B} = \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} P^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}$ *with for any* $\underline{\omega}^{<}\in\Omega^{\bullet<}$,

$$P^{\underline{\omega}^{<}} = \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \boldsymbol{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}} N^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}$$

Proof We compute :

$$\begin{aligned}
\mathbb{A}\bullet\mathbb{B} &= \left(\sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} M^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}\right)\left(\sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} N^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}\right)\\
&= \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} M^{\underline{\omega}^{2<}}N^{\underline{\omega}^{1<}}\mathbb{D}_{\underline{\omega}^{2<}}\times\mathbb{D}_{\underline{\omega}^{1<}}\\
&= \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} N^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}\sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}}\mathbf{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}}\mathbb{D}_{\underline{\omega}^{<}}\\
&= \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}}\left(\sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} N^{\underline{\omega}^{1<}}M^{\underline{\omega}^{2<}}\mathbf{proj}\binom{\underline{\omega}^{<}}{\underline{\omega}^{1<}\bullet\underline{\omega}^{2<}}\right)\mathbb{D}_{\underline{\omega}^{<}}\\
&= \sum_{\underline{\omega}^{<}\in\Omega^{\bullet<}} P^{\underline{\omega}^{<}}\mathbb{D}_{\underline{\omega}^{<}}.
\end{aligned}$$

We then find that

$$P^{\underline{\omega}^<} = \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \mathbf{proj}\begin{pmatrix} \underline{\omega}^< \\ \underline{\omega}^{1<} \bullet \underline{\omega}^{2<} \end{pmatrix} N^{\underline{\omega}^{1<}} M^{\underline{\omega}^{2<}}.$$

□

Remark 5.5.0.6 *Using Remark 5.1.0.3, one can interpret the previous sum*

$$P^{\underline{\omega}^<} = \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \boldsymbol{proj}\begin{pmatrix} \underline{\omega}^< \\ \underline{\omega}^{1<} \bullet \underline{\omega}^{2<} \end{pmatrix} N^{\underline{\omega}^{1<}} M^{\underline{\omega}^{2<}}$$

as the sum

$$P^{\underline{\omega}^<} = \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} N^{\underline{\omega}^{1<}} M^{\underline{\omega}^{2<}}$$

where $\left(\underline{\omega}^{1<},\underline{\omega}^{2<}\right)$ *range over all the partitions of* $\underline{\omega}^<$ *into two subtrees which respect the arborescent order of* $\underline{\omega}^<$.

Remark 5.5.0.7 *A (scalar) separative arborescent mould can be seen as a morphism of the algebra of forests into* $\mathbb{C}$ *([43]), namely a character. In this way, the product formula for separative arborescent moulds is no other than the convolution of characters, using the coproduct of Connes-Kreimer ([12]) algebra, which can be written with our notations, for any* $\underline{\omega}^< \in \Omega^{\bullet<}$, *by*

$$\boldsymbol{cop}\left(\underline{\omega}^<\right) = \sum_{\underline{\omega}^{1<},\underline{\omega}^{2<}\in\Omega^{\bullet<}} \boldsymbol{proj}\begin{pmatrix} \underline{\omega}^< \\ \underline{\omega}^{1<} \bullet \underline{\omega}^{2<} \end{pmatrix} \underline{\omega}^{1<} \otimes \underline{\omega}^{2<}.$$

Remark 5.5.0.8 *This sum is finite. Indeed, let us recall that* $\boldsymbol{proj}\left(\begin{smallmatrix} \underline{\omega}^< \\ \underline{\omega}^{1<} \bullet \underline{\omega}^{2<} \end{smallmatrix}\right) = 0$ *if* $\underline{\omega}^< \notin \boldsymbol{concat}\left(\underline{\omega}^{1<},\underline{\omega}^{2<}\right)$. *More precisely, the number of couples* $\left(\underline{\omega}^{1<},\underline{\omega}^{2<}\right)$ *such that* $\underline{\omega}^< \in \boldsymbol{concat}\left(\underline{\omega}^{1<},\underline{\omega}^{2<}\right)$ *is less than the number of partitions of* $\underline{\omega}^<$ *into two subsets. Thus the previous sum contains at most* $2^{l\left(\underline{\omega}^<\right)}$ *terms.*

Example 5.5.0.14

$$\begin{aligned}(M\times N)^{\overset{3\,4}{\overset{\vee}{\overset{2}{\overset{|}{\bullet 1}}}}} &= M^{\overset{3\,4}{\overset{\vee}{\overset{2}{\overset{|}{\bullet 1}}}}} N^{\bullet\emptyset} + M^{\overset{3}{\overset{|}{\overset{2}{\overset{|}{\bullet 1}}}}} N^{\bullet 4} + M^{\overset{4}{\overset{|}{\overset{2}{\overset{|}{\bullet 1}}}}} N^{\bullet 3} + \\ &\quad M^{\overset{2}{\overset{|}{\bullet 1}}} N^{\bullet 3\ \bullet 4} + M^{\bullet 1} N^{\overset{3\,4}{\overset{\vee}{\bullet 2}}} + M^{\bullet\emptyset} N^{\overset{3\,4}{\overset{\vee}{\overset{2}{\overset{|}{\bullet 1}}}}}\end{aligned}$$

$$(M\times N)^{\bullet 1\ \overset{3}{\overset{|}{\bullet 2}}} = M^{\bullet 1\ \overset{3}{\overset{|}{\bullet 2}}} N^{\emptyset} + M^{\bullet 1\ \bullet 2} N^{\bullet 3} + M^{\bullet 1} N^{\overset{3}{\overset{|}{\bullet 2}}} + M^{\bullet 2} N^{\bullet 1\ \bullet 3} + M^{\overset{3}{\overset{|}{\bullet 2}}} N^{\bullet 1} + M^{\emptyset} N^{\bullet 1\ \overset{3}{\overset{|}{\bullet 2}}}$$

Proposition 5.5.2 *The arborified product preserves geometrical growth.*

More precisely, we consider two operators $\mathbb{A} = \sum_{\underline{\omega}^< \in \Omega^{\bullet <}} M^{\underline{\omega}^<} \mathbb{D}_{\underline{\omega}^<}$ *and* $\mathbb{B} = \sum_{\underline{\omega}^< \in \Omega^{\bullet <}} N^{\underline{\omega}^<} \mathbb{D}_{\underline{\omega}^<}$*. We assume that* $M^\bullet$ *and* $N^\bullet$ *are both symmetra(e)l*[5] *and that there exist* $a, b, c, d \in \mathbb{R}_+^*$ *that verify*

$$\forall \underline{\omega}^< \in \Omega^{\bullet <}, \quad \left| M^{\underline{\omega}^<} \right| \leq a b^{\|\underline{\omega}^<\|} \text{ and } \left| N^{\underline{\omega}^<} \right| \leq c d^{\|\underline{\omega}^<\|}.$$

$\mathbb{A} \bullet \mathbb{B} = \sum P^{\underline{\omega}^<} \mathbb{D}_{\underline{\omega}^<}$ *then there exist* $e, f \in \mathbb{R}_+^*$ *such that for any* $\underline{\omega}^< \in \Omega^{\bullet <}$

$$\left| P^{\underline{\omega}^<} \right| \leq e f^{\|\underline{\omega}^<\|}.$$

Proof Let us consider $\underline{\omega}^< \in \Omega^{\bullet <}$. The result is a direct consequence of the previous formula :

$$P^{\underline{\omega}^<} = \sum_{\underline{\omega}^{1<}, \underline{\omega}^{2<} \in \Omega^{\bullet <}} \mathbf{proj} \begin{pmatrix} \underline{\omega}^< \\ \underline{\omega}^{1<} \bullet \underline{\omega}^{2<} \end{pmatrix} N^{\underline{\omega}^{1<}} M^{\underline{\omega}^{2<}}$$

and of the Remarks 5.1.0.4 and 5.5.0.8.

□

5.6 Simple or contracted arborified composition

We recall that for a tree $\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet <}$ and $\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{\text{irred}}$, the set **precompo** $\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ contains all the arborified sequences constructed in the following way :

— $\|\underline{\omega}^{i<}\| = m_i$

— we replace each node m_i of $\underline{m}^<$ by the tree $\underline{\omega}^{i<}$.

— If $m_{i'}$ is the predecessor of $m_i \in \underline{m}^<$ in $\underline{m}^<$ then we connect one of the (non empty) node of $\underline{\omega}^{i'<}$ with the root of $\underline{\omega}^{i<}$.

The set **compo** $\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ is the same set but without repetition.

Notation 5.6.0.8 *For a mould-comould expansion* $\mathbf{G} = \sum M^\bullet \mathbf{F}_\bullet$*, we write* $< \mathbf{G}, \mathbf{F} >^\bullet$ *the mould* $M^\bullet$*.*

Proposition 5.6.1 *We consider three operators* $\mathbf{F}, \mathbf{G}, \mathbb{H} \in$ ***ENDOM***$(\mathbb{C}[[u]])$ *such that :*

— $\mathbf{F}$*,* $\mathbf{G}$ *and* $\mathbb{H}$ *are both primitive or group like.*

— $N^\bullet :=< \mathbf{G}, \mathbf{F} >^\bullet$*,* $M^\bullet :=< \mathbb{H}, \mathbf{G} >^\bullet$ *and* $P^\bullet :=< \mathbb{H}, \mathbf{F} >^\bullet$*.*

Then for any $\underline{W}^< \in \Omega^{\bullet <}$*,*

$$P^{\underline{W}^<} = \sum_{\substack{\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet <} \\ \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{irred} \\ W^< \in \textbf{\textit{compo}}\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)}} \textbf{\textit{proj}} \begin{pmatrix} \underline{W}^< \\ \underline{m}^< \circ (\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}) \end{pmatrix} M^{\underline{m}^<} N^{\underline{\omega}^{1<}} \ldots N^{\underline{\omega}^{s<}}.$$

5. Then the mould product $(M \times N)^\bullet$ is symmetra(e)l too

Proof We know that for any $m \in \Omega$:

$$\mathbf{G}_m = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}, \|\underline{\omega}^<\| = m} N^{\underline{\omega}^<} \mathbf{F}_{\underline{\omega}^<} = \sum_{\underline{\omega}^< \in \Omega^{\bullet<}, \|\underline{\omega}^<\| = m} \frac{1}{s(\underline{\omega})} N^{\underline{\omega}^<} \mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}$$

where $\left(N^{\underline{\omega}^<}\right)$ is the arborified or the contracted arborified of the mould $(N^{\underline{\omega}})$. We then compute the pre-coarborification of the operator $\mathbf{G}$ as in Definition 5.3.0.6 :

$$\begin{aligned}
\mathbf{G}^{\mathbf{prearbo}}_m &= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}, \|\underline{\omega}^<\| = m} \frac{1}{s(\underline{\omega}^<)} N^{\underline{\omega}^<} \left(\mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}.u\right) \partial \\
&= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}_{\text{irred}}, \|\underline{\omega}^<\| = m} \frac{1}{s(\underline{\omega}^<)} N^{\underline{\omega}^<} \left(\mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}.u\right) \partial \\
&= \sum_{\underline{\omega}^< \in \Omega^{\bullet<}_{\text{irred}}, \|\underline{\omega}^<\| = m} \frac{1}{s(\underline{\omega}^<)} N^{\underline{\omega}^<} \mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}
\end{aligned}$$

because if $\underline{\omega}^<$ is not irreducible then $\mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}.u = 0$ and if $\underline{\omega}^<$ is irreducible then $\left(\mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}.u\right) \partial = \mathbf{F}^{\mathbf{prearbo}}_{\underline{\omega}^<}$.

Using Proposition 4.6.1, we know that for any $\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet<}$,

$$\begin{aligned}
\mathbf{G}_{\underline{m}^<} &= \frac{1}{s(\underline{m})} \mathbf{G}^{\mathbf{prearbo}}_{\underline{m}^<} \\
&= \sum_{\substack{\|\underline{\omega}^{1<}\| = m_1 \ldots \|\underline{\omega}^{s<}\| = m_s \\ \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet<}_{\text{irred}}}} \sum_{\underline{W}^< \in \mathbf{precompo}(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<})} N^{\underline{\omega}^{1<}} \ldots N^{\underline{\omega}^{s<}} \frac{1}{s(\underline{m}^<)\, s(\underline{\omega}^{1<}) \ldots s(\underline{\omega}^{s<})} \mathbf{F}^{\mathbf{prearbo}}_{\underline{W}^<} \\
&= \sum_{\substack{\|\underline{\omega}^{1<}\| = m_1 \ldots \|\underline{\omega}^{s<}\| = m_s \\ \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet<}_{\text{irred}}}} \sum_{\underline{W}^< \in \mathbf{precompo}(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<})} N^{\underline{\omega}^{1<}} \ldots N^{\underline{\omega}^{s<}} \frac{s\left(\underline{W}^<\right)}{s(\underline{m}^<)\, s(\underline{\omega}^{1<}) \ldots s(\underline{\omega}^{s<})} \mathbf{F}_{\underline{W}^<}
\end{aligned}$$

And then :

$$\begin{aligned}
\mathbb{H} &= \sum_{\underline{m}^< \in \Omega^{\bullet <}} M^{\underline{m}^<} \mathbf{G}_{\underline{m}^<} \\
&= \sum_{\underline{m}^< \in \Omega^{\bullet <}} M^{\underline{m}^<} \sum_{\substack{\|\underline{\omega}^{1<}\| = m_1 \dots \|\underline{\omega}^{s<}\| = m_s \\ \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{\text{irred}}}} \sum_{\underline{W}^< \in \mathbf{precompo}(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<})} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}} \\
&\qquad \times \frac{s\left(\underline{W}^<\right)}{s\left(\underline{m}^<\right) s\left(\underline{\omega}^{1<}\right) \dots s\left(\underline{\omega}^{s<}\right)} \mathbf{F}_{\underline{W}^<} \\
&= \sum_{\underline{W}^< \in \Omega^{\bullet <}} \sum_{\substack{\underline{m}^< = (m_1, \dots, m_s)^< \in \Omega^{\bullet <} \\ \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{\text{irred}} \\ W^< \in \mathbf{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)}} M^{\underline{m}^<} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}} \\
&\qquad \times \frac{s\left(\underline{W}^<\right)!}{s\left(\underline{m}^<\right) s\left(\underline{\omega}^{1<}\right) \dots s\left(\underline{\omega}^{s<}\right)} K\left({\underline{W}^< \atop \underline{m}^< \circ \left(\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)} \right) \mathbf{F}_{\underline{W}^<} \\
&= \sum_{\underline{W}^< \in \Omega^{\bullet <}} \sum_{\substack{\underline{m}^< = (m_1, \dots, m_s)^< \in \Omega^{\bullet <} \\ \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{\text{irred}} \\ W^< \in \mathbf{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)}} \mathbf{proj}\left({\underline{W}^< \atop \underline{m}^< \circ \left(\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)} \right) M^{\underline{m}^<} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}}
\end{aligned}$$

and the result follows. □

Remark 5.6.0.9 *Using Remark 5.1.0.3, one can interpret the previous sum*

$$P^{\underline{W}^<} = \sum_{\substack{\underline{m}^< = (m_1, \dots, m_s)^< \in \Omega^{\bullet <} \\ \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{irred} \\ W^< \in \boldsymbol{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)}} \boldsymbol{proj}\left({\underline{W}^< \atop \underline{m}^< \circ \left(\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)} \right) M^{\underline{m}^<} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}}$$

as the sum

$$P^{\underline{W}^<} = \sum_{\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<} \in \Omega^{\bullet <}_{irred}} M^{\underline{m}^<} N^{\underline{\omega}^{1<}} \dots N^{\underline{\omega}^{s<}}$$

where $\left(\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)$ ranges over all the partitions of $\underline{\omega}^<$ into s irreducible subtrees which respect the arborescent order of $\underline{\omega}^<$ and where s range over $[\![1, l\left(\underline{W}^<\right)]\!]$.

Remark 5.6.0.10 *The sum giving $P^{\underline{W}^<}$ in the previous theorem is finite. Indeed, we have $\boldsymbol{proj}\left({\underline{W}^< \atop \underline{m}^< \circ (\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<})} \right) = 0$ if $\underline{\omega}^< \notin \boldsymbol{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)$. More precisely, as a consequence of the second point of Remark 5.1.0.4, the number of irreducible arborified sequences $\underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}$ such that there exists $\underline{m}^< \in \Omega^{\bullet <}$ which verifies $\underline{\omega}^< \in \boldsymbol{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \dots, \underline{\omega}^{s<}\right)$ is less than $2^{l\left(\underline{\omega}^<\right)}$. Let us give another method to bound the number of monotonous partitions of $\underline{\omega}^<$. Let us denote by k_i the number of outgoing edges of the node $\omega_i \in \underline{\omega}^<$ for any $i \in [\![1, r]\!]$ where $r = l(\underline{\omega}^<)$. In order to choose a monotonous partition of $\underline{\omega}^<$, we consider the nodes of $\underline{\omega}^<$ from the last to the first ones. For each $i \in [\![1, r]\!]$, one have to choose if we group ω_i with one of its successors or if we leave it alone.*

There are k_i+1 possible choices. Then there are at most $(k_1+1)\ldots(k_r+1)$ possible monotonous partitions of $\underline{\omega}^<$. But :

$$\sup_{\sum k_i \le r} \left((k_1+1)\ldots(k_r+1)\right)^{1/(2r)} \le e^{1/e}$$

and we finally obtain that the number of monotonic partitions of $\underline{\omega}^<$ is less than $\left(e^{2/e}\right)^{l\left(\underline{\omega}^<\right)}$.

Example 5.6.0.15

$$\begin{aligned}
P^{\bullet 1} &= M^{\bullet 1} N^{\bullet 1}\\
P^{\overset{\overset{2}{|}}{\bullet 1}} &= M^{\overset{\overset{2}{|}}{\bullet 1}} N^{\bullet 1} N^{\bullet 2} + M^{\bullet 1+2} N^{\overset{\overset{2}{|}}{\bullet 1}}\\
P^{\bullet 1\ \bullet 2} &= M^{\bullet 1\ \bullet 2} N^{\bullet 1} N^{\bullet 2}\\
P^{\overset{2\ 3}{\overset{\vee}{\bullet 1}}} &= M^{\overset{2\ 3}{\overset{\vee}{\bullet 1}}} N^{\bullet 1} N^{\bullet 2} N^{\bullet 3} + M^{\overset{\overset{3}{|}}{\bullet 1+2}} N^{\overset{\overset{2}{|}}{\bullet 1}} N^{\bullet 3}\\
&\quad + M^{\overset{\overset{2}{|}}{\bullet 1+3}} N^{\overset{\overset{3}{|}}{\bullet 1}} N^{\bullet 2} + M^{\bullet 1+2+3} N^{\overset{2\ 3}{\overset{\vee}{\bullet 1}}}\\
P^{\overset{\overset{2}{|}}{\bullet 1}\ \bullet 3} &= M^{\overset{\overset{2}{|}}{\bullet 1}\ \bullet 3} N^{\bullet 1} N^{\bullet 2} N^{\bullet 3} + M^{\bullet 1+2 \bullet 3} N^{\overset{\overset{2}{|}}{\bullet 1}} N^{\bullet 3} + M^{\bullet 1+2+3} N^{\overset{\overset{2}{|}}{\bullet 1}\ \bullet 3}\\
P^{\bullet 1\ \bullet 2\ \bullet 3} &= M^{\bullet 1\ \bullet 2\ \bullet 3} N^{\bullet 1} N^{\bullet 2} N^{\bullet 3}
\end{aligned}$$

Remark 5.6.0.11 *One recognizes the coproduct already mentioned in the non decorated case in [7] :*

$$\boldsymbol{cop}\left(\underline{W}^<\right) = \sum_{\substack{\underline{m}^< = (m_1,\ldots,m_s)^< \in \Omega^{\bullet <}\\ \underline{\omega}^{1<},\ldots,\underline{\omega}^{s<} \in \Omega^{\bullet <}_{irred}\\ W^< \in \boldsymbol{compo}\left(\underline{m}^<;\underline{\omega}^{1<},\ldots,\underline{\omega}^{s<}\right)}} \boldsymbol{proj}\binom{W^<}{\underline{m}^< \circ \left(\underline{\omega}^{1<},\ldots,\underline{\omega}^{s<}\right)} \left(\underline{\omega}^{1<} \oplus \ldots \oplus \underline{\omega}^{s<}\right) \otimes \underline{m}^<.$$

Proposition 5.6.2 *The arborified composition preserves geometrical growth.*

More precisely, we consider three operators both primitive or group like $\mathbf{F},\mathbf{G},\mathbb{H} \in \boldsymbol{ENDOM}\left(\mathbb{C}\left[[u]\right]\right)$ such that $\mathbf{G} = \sum_{\underline{\omega}^< \in \Omega^{\bullet <}} N^{\underline{\omega}^<} \mathbf{F}_{\underline{\omega}^<}$ and $\mathbb{H} = \sum_{\underline{\omega}^< \in \Omega^{\bullet <}} M^{\underline{\omega}^<} \mathbf{G}_{\underline{\omega}^<} = \sum_{\underline{\omega}^< \in \Omega^{\bullet}} P^{\underline{\omega}^<} \mathbf{F}_{\underline{\omega}^<}$ and in such a way there exist $a,b,c,d \in \mathbb{R}_+^*$ that verify

$$\forall \underline{\omega}^< \in \Omega^{\bullet <}, \quad \left|M^{\underline{\omega}^<}\right| \le a b^{\|\underline{\omega}^<\|} \text{ and } \left|N^{\underline{\omega}^<}\right| \le c d^{\|\underline{\omega}^<\|}.$$

If then there exist $A,B \in \mathbb{R}_+^*$ such that for any $\underline{\omega}^< \in \Omega^{\bullet <}$

$$\left|P^{\underline{\omega}^<}\right| \le A B^{\|\underline{\omega}^<\|}.$$

More precisely, one has $A=a$ and $B = 4bcd$[6]*.*

6. We do not know the optimal bounds.

Proof The proposition is a direct consequence of the formula for arborified mould composition and of the Remarks 5.1.0.4 and 5.6.0.10.

□

5.7 Three applications of arborified composition

5.7.1 Arborified product inverse of a mould

As explained in the introduction, we work with operators elements of a subalgebra $\tilde{\mathcal{B}}$ of **ENDOM** $(\mathbb{C}[[u]])$ spanned by the elements of a family of differential operators $(\mathbf{F}_m)$ and completed for the Krull topology. We know that a given mould $M^\bullet$ is invertible for mould product if and only if $M^\emptyset \neq 0$. If $\mathbb{A}, \mathbb{B} \in \mathcal{B}$ are two differential operators such that $\mathbb{A} = \sum M^\bullet \mathbf{F}_\bullet$, $\mathbb{B} = \sum N^\bullet \mathbf{F}_\bullet$ and if $\mathbb{B}$ is the inverse of the operator $\mathbb{A}$ then we have : $\mathbb{A} \times \mathbb{B} = 1$ and $< \mathbb{B}, \mathbb{A} >^\bullet = (1+I)^{-1}$ where $\left((1+I)^{-1}\right)^{\underline{\omega}} = (-1)^{l(\underline{\omega})}$ if $\underline{\omega} \neq \emptyset$ and $\left((1+I)^{-1}\right)^{\underline{\omega}} = 1$ if $\underline{\omega} = \emptyset$. Then $< \mathbb{B}, \mathbf{F} >^\bullet = \left((1+I)^{-1} \circ < \mathbb{A}, \mathbf{F} >\right)^\bullet$. Using the result of Section 5.6, we obtain :

Proposition 5.7.1 *Let us consider a symmetra(e)l mould $M^\bullet$ such that $M^\emptyset \neq 0$. Then the mould $M^{\bullet<}$ is a separative one and for any $\underline{W}^< \in \Omega^{\bullet<}$,*

$$\left(M^{-1}\right)^{\underline{W}^<} = \sum_{\substack{\underline{m}^< = (m_1, \ldots, m_s)^< \in \Omega^{\bullet<} \\ \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet<}_{irred} \\ W^< \in \boldsymbol{compo}\left(\underline{m}^<; \underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)}} \boldsymbol{proj}\left(\underline{m}^< \circ \left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)\right)^{\underline{W}^<} (-1)^s M^{\underline{\omega}^{1<}} \ldots M^{\underline{\omega}^{s<}}.$$

Moreover if the arborified mould $M^{\bullet<}$ has a geometrical growth then the same occur for its arborified inverse.

Remark 5.7.1.1 *One can interpret the previous sum as follows :*

$$\left(< \mathbb{A}, \mathbf{F} >^{\underline{\omega}^<}\right)^{-1} = \sum_{\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<} \in \Omega^{\bullet<}_{irred}} (-1)^s M^{\underline{\omega}^{1<}} \ldots M^{\underline{\omega}^{s<}}.$$

where $\left(\underline{\omega}^{1<}, \ldots, \underline{\omega}^{s<}\right)$ range over all the partitions of $\underline{\omega}^<$ into s irreducible subtrees which respect the arborescent order of $\underline{\omega}^<$ and where s range over $[\![1, l\left(\underline{W}^<\right)]\!]$.

Example 5.7.1.1

$$\left(M^{-1}\right)^{\bullet_1} = -M^{\bullet_1}$$

$$\left(M^{-1}\right)^{\begin{smallmatrix}2\\|\\\bullet_1\end{smallmatrix}} = -M^{\begin{smallmatrix}2\\|\\\bullet_1\end{smallmatrix}} + M^{\bullet_1}M^{\bullet_2}$$

$$\left(M^{-1}\right)^{\begin{smallmatrix}2\ 3\\\vee\\\bullet_1\end{smallmatrix}} = -M^{\begin{smallmatrix}2\ 3\\\vee\\\bullet_1\end{smallmatrix}} + M^{\begin{smallmatrix}2\\|\\\bullet_1\end{smallmatrix}}M^{\bullet_3} + M^{\begin{smallmatrix}3\\|\\\bullet_1\end{smallmatrix}}M^{\bullet_2} - M^{\bullet_1}M^{\bullet_2}M^{\bullet_3}$$

$$\left(M^{-1}\right)^{\begin{smallmatrix}3\ \\|\ \\2\ 4\\\vee\\\bullet_1\end{smallmatrix}} = -M^{\begin{smallmatrix}3\ \\|\ \\2\ 4\\\vee\\\bullet_1\end{smallmatrix}} + M^{\begin{smallmatrix}2\ 4\\\vee\\\bullet_1\end{smallmatrix}}M^{\bullet_3} + M^{\begin{smallmatrix}4\\|\\\bullet_1\end{smallmatrix}}M^{\begin{smallmatrix}3\\|\\\bullet_2\end{smallmatrix}} + M^{\begin{smallmatrix}3\\|\\2\\|\\\bullet_1\end{smallmatrix}}M^{\bullet_4}$$

$$- M^{\begin{smallmatrix}4\\|\\\bullet_1\end{smallmatrix}}M^{\bullet_2}M^{\bullet_3} - M^{\begin{smallmatrix}2\\|\\\bullet_1\end{smallmatrix}}M^{\bullet_3}M^{\bullet_4} - M^{\bullet_1}M^{\begin{smallmatrix}3\\|\\\bullet_2\end{smallmatrix}}M^{\bullet_4} + M^{\bullet_1}M^{\bullet_2}M^{\bullet_3}M^{\bullet_4}$$

Remark 5.7.1.2 *As a direct consequence of our formula, the arborified inverse of the mould* $(1+I)^\bullet$ *is given by* $\left((1+I)^{-1}\right)^{\underline{\omega}^<} = (-1)^{l(\underline{\omega}^<)}$.

5.7.2 Backwards arborification

We define the application **rev** : $\left\{\begin{array}{ccc}\Omega^\bullet & \longrightarrow & \Omega^\bullet\\ (\omega_1,\ldots,\omega_r) & \mapsto & (\omega_r,\ldots,\omega_1)\end{array}\right.$. It induces an application from the set of mould on Ω into itself, called **rev** too, and defined on a mould $M^\bullet$ by

$$\mathbf{rev}\left(M^\bullet\right) = M^{\mathbf{rev}(\bullet)}.$$

With the help of this operator, we will define the backwards arborified [7], contracted or not, of a mould.

Definition 5.7.2.1 *We define the* ***backwards arborification****, contracted or not, of a mould* $M^\bullet$ *on* Ω *as the arborified, contracted or not, of the mould* $M^{\mathbf{rev}(\bullet)}$. *We will denote* $M^{\bullet>}$ *the simple backwards arborification of* $M^\bullet$ *and* $M^{\bullet\gg}$ *its contracted backwards arborification.*

Let us remark that in backwards arborification, the coefficients of the moulds are indexed by antiarborescent sequences, i.e. sequences with an antiarborescent order. An arborescent sequence is a sequence provided with an order such that each element of the sequence has at most one predecessor. In an antiarborescent sequence, each element has at most one successor.

7. This notion is from J. Ecalle, see [14]

The reason for backwards arborification lies in the fact that one cannot always obtains a closed formula for the arborified of a given mould (and so it is possibly very difficult to verify that this arborified mould has a geometrical growth) although it might be the case for its backwards arborified. There are many examples of this fact. One observes this phenomenon for the moulds $\mathbf{sofo}^{\bullet}_{\pm}$, $\mathbf{lefo}^{\bullet}_{\pm}$ or the organic one $\mathbf{romo}^{\bullet}$ (see chapter 6, 7 or the paper [16] for the definition of these moulds) or the mould S, see Subsection 6.3. The two following propositions legitimize our comment. The formulae given in the first one was already mentioned in [12].

Proposition 5.7.2 *Let us consider a mould $M^{\bullet}$ on Ω.*
— *If $M^{\bullet}$ is alternel then*

$$\boldsymbol{rev}\,(M^{\bullet}) \circ \left((1+I)^{-1} - 1\right)^{\bullet} = -M^{\bullet}.$$

— *If $M^{\bullet}$ is symmetrel then*

$$\left((1+I)^{-1} - 1\right)^{\bullet} \circ \boldsymbol{rev}\,(M^{\bullet}) \circ \left((1+I)^{-1} - 1\right)^{\bullet} = M^{\bullet}.$$

Proof Using results of 2, we know that $\mathcal{B}$[8] has a structure of Hopf algebra with antipode given by :

$$S : \begin{cases} \tilde{\mathcal{B}} & \longrightarrow & \tilde{\mathcal{B}} \\ \mathbb{A} = \sum M^{\bullet}\mathbf{F}_{\bullet} & \mapsto & s\,(\mathbb{A}) = \sum \mathbf{rev}\,(M)^{\bullet} \circ \left((1+I)^{-1} - 1\right)^{\bullet} \mathbf{F}_{\bullet} \end{cases}$$

where $\mathbf{F}$ is a given substitution automorphism of $\tilde{\mathcal{B}}$.
— If $\mathbb{A}$ is a derivation, i.e. a primitive element of $\tilde{\mathcal{B}}$ then we get $S\,(\mathbb{A}) = -\mathbb{A}$ and the first identity follows.
— If $\mathbb{A}$ is group like then $S\,(\mathbb{A}) = \mathbb{A}^{-1}$. Moreover

$$< \mathbb{A}^{-1}, \mathbf{F} >^{\bullet} = < \mathbb{A}^{-1}, \mathbb{A} >^{\bullet} \circ < \mathbb{A}, \mathbf{F} >^{\bullet} = \left((1+I)^{-1} - 1\right)^{\bullet} \circ M^{\bullet}.$$

Then

$$\left((1+I)^{-1}\right)^{\bullet} \circ M^{\bullet} = \mathbf{rev}\,(M)^{\bullet} \circ \left((1+I)^{-1} - 1\right)^{\bullet}$$

and it comes

$$\left((1+I)^{-1} - 1\right)^{\bullet} \circ \mathbf{rev}\,(M^{\bullet}) \circ \left((1+I)^{-1} - 1\right)^{\bullet} = M^{\bullet}$$

because the composition inverse of $\left((1+I)^{-1} - 1\right)^{\bullet}$ is itself.

□

Corollary 5.7.3 *Let us consider an alternel or symmetrel mould $M^{\bullet}$ on Ω. The two following facts are equivalent :*
— *The contracted arborified of the mould $M^{\bullet}$ is of geometrical growth.*

8. see the previous Section 5.7.1 for notations

— *The contracted backwards arborified of the mould $M^\bullet$ is of geometrical growth.*

Proof As long as geometrical growth is preserved by contracted arborified composition, and as a consequence of the previous proposition, it is enough to prove that the contracted arborified of the mould $(1+I)^\bullet$ is of geometrical growth, which is trivially the case because of the form of $(1+I)^\bullet$. □

5.7.3 Arborified composition inverse of a mould

Proposition 5.7.4 *Let us consider a mould $N^\bullet$ in such that for any sequence $\underline{\omega} \in \Omega^\bullet$ of length 1, $N^{\underline{\omega}} \neq 0$. Then $N^\bullet$ is invertible for the mould composition and if $M^\bullet$ denotes this inverse then $M^{\bullet<}$ is given for any $r \in \mathbf{N}^*$ and $\underline{m}^< \in \Omega^{\bullet<}_{irred}$ of length r by :*

$$M^{\bullet m_1} = \frac{1}{N^{\bullet m_1}} \text{ if } \underline{m}^< = \bullet m_1$$

$$M^{\substack{m_2\\|\\\bullet m_1}} = -\frac{N^{\substack{m_2\\|\\\bullet m_1}}}{N^{\bullet m_1+m_2} N^{\bullet m_1} N^{\bullet m_2}} \text{ if } \underline{m}^< = \substack{m_2\\|\\\bullet m_1}$$

and if $l(\underline{m}^<) \geq 2$,

$$M^{\underline{m}^<} = \frac{1}{N^{m_1^r} \dots N^{m_r^r}} \sum_{k=1}^{r-2} (-1)^{k-1} \sum_{\substack{1 \leq s_{r-k} \leq 2 < s_{r-k+1} < \dots < s_{r-1} < r \\ \underline{m}_i^< \in \left[\Omega^{\bullet<}\right]_{s_i} \\ \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<} \in \Omega^{\bullet<}_{irred} \\ \underline{m}^{i+1<} \in \boldsymbol{compo}\left(\underline{m}^{i<}; \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<}\right) \\ i \in [\![r-k, r-1]\!]}}$$
$$(-1)^{s_{r-k}} \left(\prod_{l=r-k}^{r-1} \boldsymbol{proj}\binom{\underline{m}^{l+1<}}{\underline{m}^{l<} \circ \left(\underline{\omega}_1^{l<}, \dots, \underline{\omega}_{s_l}^{l<}\right)} \frac{N^{\underline{\omega}_1^{l<}} \dots N^{\underline{\omega}_{s_l}^{l<}}}{N^{\|\underline{\omega}_1^{l<}\|} \dots N^{\|\underline{\omega}_{s_l}^{l<}\|}} \right) \frac{N^{\underline{m}^{r-k<}}}{N^{\|\underline{m}^{r-k<}\|}}$$

where

— $\underline{m}^{r<} := \underline{m}^< = (m_1^r, \dots, m_r^r)^<$
— $[\Omega^{\bullet<}]_k$ *is the subset of $\Omega^{\bullet<}$ of arborified sequences of length k*
— $\underline{m}^{k<} = \left(m_1^k, \dots, m_{s_k}^k\right)^<$.

Proof We easily verify the formula for an arborified sequence $\underline{m}^<$ of length 1 or 2. We assume that the formula is true for any sequence $\underline{m}^<$ of length $r \geq 2$ and we will prove it for a sequence $\underline{m}^{r+1<}$ of length $r+1$. Using the formula for arborified mould composition, we know that :

$$M^{\underline{m}^{r+1<}} = -\frac{1}{N^{m_1^{r+1}}\dots N^{m_{r+1}^{r+1}}} \sum_{s_r=1}^{r} \sum_{\substack{\underline{m}^{r<}\in\left[\Omega^{\bullet<}\right]_{s_r}\\ \underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\in\Omega^{\bullet<}_{\mathrm{irred}}\\ \underline{m}^{r+1<}\in\mathbf{compo}\left(\underline{m}^{r<};\underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\right)}} \mathbf{proj}\left(\begin{matrix}\underline{m}^{r+1<}\\ \underline{m}^{r<}\circ\left(\underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\right)\end{matrix}\right) M^{\underline{m}^{r<}} N^{\underline{\omega}_1^{r<}}\dots N^{\underline{\omega}_{s_r}^{r<}}$$

and using the induction hypothesis, we have :

$$M^{\underline{m}^{r+1<}} = \frac{1}{N^{m_1^{r+1}}\dots N^{m_{r+1}^{r+1}}} \Bigg[\sum_{s_r=3}^{r} \sum_{\substack{\underline{m}^{r<}\in\left[\Omega^{\bullet<}\right]_{s_r}\\ \underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\in\Omega^{\bullet<}_{\mathrm{irred}}\\ \underline{m}^{r+1<}\in\mathbf{compo}\left(\underline{m}^{r<};\underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\right)}} -\mathbf{proj}\left(\begin{matrix}\underline{m}^{r+1<}\\ \underline{m}^{r<}\circ\left(\underline{\omega}_1^{r<},\dots,\underline{\omega}_{s_r}^{r<}\right)\end{matrix}\right) \frac{N^{\underline{\omega}_1^{r<}}\dots N^{\underline{\omega}_{s_r}^{r<}}}{N^{m_1^r}\dots N^{m_{s_r}^r}} \Bigg[$$

$$\sum_{k=1}^{s_r-2}(-1)^{k-1} \sum_{\substack{2=s_{r-k}<s_{r-k+1}<\dots<s_{r-1}<s_r\\ \underline{m}_i^{<}\in\left[\Omega^{\bullet<}\right]_{s_i}\\ \underline{\omega}_1^{i<},\dots,\underline{\omega}_{s_i}^{i<}\in\Omega^{\bullet<}_{\mathrm{irred}}\\ \underline{m}^{i+1<}\in\mathbf{compo}\left(\underline{m}^{i<};\underline{\omega}_1^{i<},\dots,\underline{\omega}_{s_i}^{i<}\right)\\ i\in[\![s_r-k,s_r-1]\!]}} \left(\prod_{l=s_r-k}^{s_r-1}\mathbf{proj}\left(\begin{matrix}\underline{m}^{l+1<}\\ \underline{m}^{l<}\circ\left(\underline{\omega}_1^{l<},\dots,\underline{\omega}_{s_l}^{l<}\right)\end{matrix}\right)\frac{N^{\underline{\omega}_1^{l<}}\dots N^{\underline{\omega}_{s_l}^{l<}}}{N^{\|\underline{\omega}_1^{l<}\|}\dots N^{\|\underline{\omega}_{s_l}^{l<}\|}}\right)\frac{N^{\underline{m}^{s_r-k<}}}{N^{\|\underline{m}^{s_r-k<}\|}}$$

$$+\sum_{k=1}^{s_r-2}(-1)^{k} \sum_{\substack{1=s_{r-k}<2<s_{r-k+1}<\dots<s_{r-1}<s_r\\ \underline{m}_i^{<}\in\left[\Omega^{\bullet<}\right]_{s_i}\\ \underline{\omega}_1^{i<},\dots,\underline{\omega}_{s_i}^{i<}\in\Omega^{\bullet<}_{\mathrm{irred}}\\ \underline{m}^{i+1<}\in\mathbf{compo}\left(\underline{m}^{i<};\underline{\omega}_1^{i<},\dots,\underline{\omega}_{s_i}^{i<}\right)\\ i\in[\![s_r-k,s_r-1]\!]}} \left(\prod_{l=s_r-k}^{s_r-1}\mathbf{proj}\left(\begin{matrix}\underline{m}^{l+1<}\\ \underline{m}^{l<}\circ\left(\underline{\omega}_1^{l<},\dots,\underline{\omega}_{s_l}^{l<}\right)\end{matrix}\right)\frac{N^{\underline{\omega}_1^{l<}}\dots N^{\underline{\omega}_{s_l}^{l<}}}{N^{\|\underline{\omega}_1^{l<}\|}\dots N^{\|\underline{\omega}_{s_l}^{l<}\|}}\right) \Bigg]$$

$$+ \sum_{\substack{\underline{m}_r^{<}\in\left[\Omega^{\bullet<}\right]_2\\ \underline{\omega}_1^{r<},\underline{\omega}_2^{r<}\in\Omega^{\bullet<}_{\mathrm{irred}}\\ \underline{m}^{r+1<}\in\mathbf{compo}\left(\underline{m}^{r<};\underline{\omega}_1^{r<},\underline{\omega}_2^{r<}\right)}} \mathbf{proj}\left(\begin{matrix}\underline{m}^{r+1<}\\ \underline{m}^{r<}\circ\left(\underline{\omega}_1^{r<},\underline{\omega}_2^{r<}\right)\end{matrix}\right)\frac{N^{\underline{\omega}_1^{r<}}N^{\underline{\omega}_2^{r<}}}{N^{\|\underline{\omega}_1^{r<}\|}N^{\|\underline{\omega}_2^{r<}\|}}\frac{N^{\underline{m}^{r<}}}{N^{\|\underline{m}^{r<}\|}}$$

$$+\frac{N^{\underline{m}^{r+1<}}}{N^{\|\underline{m}^{r+1<}\|}} \Bigg]$$

and as long as for any $i \in [\![1, s_r]\!]$, $\|\underline{\omega}_i^{r<}\| = m_i^{r<}$, we obtain :

$$M^{\underline{m}^{r+1<}} = \frac{1}{N^{m_1^{r+1}} \dots N^{m_r^{r+1}}} \Bigg[$$

$$\sum_{k=1}^{r-1} (-1)^{k-1} \sum_{\substack{2 = s_{r-k} < s_{r-k+1} < \dots < s_{r-1} < s_r < r+1 \\ \underline{m}_i^{<} \in \left[\Omega^{\bullet <}\right]_{s_i} \\ \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<} \in \Omega_{\mathrm{irred}}^{\bullet <} \\ \underline{m}^{i+1<} \in \mathbf{compo}\left(\underline{m}^{i<}; \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<}\right) \\ i \in [\![r+1-k, r+1-1]\!]}}$$

$$\left(\prod_{l=1}^{k} \mathbf{proj}\left(\begin{array}{c} \underline{m}^{l+1<} \\ \underline{m}^{l<} \circ \left(\underline{\omega}_1^{l<}, \dots, \underline{\omega}_{s_l}^{l<}\right) \end{array}\right) \frac{N^{\underline{\omega}_1^{l<}} \dots N^{\underline{\omega}_{s_l}^{l<}}}{N^{\|\underline{\omega}_1^{l<}\|} \dots N^{\|\underline{\omega}_{s_l}^{l<}\|}}\right) \frac{N^{\underline{m}^{k<}}}{N^{\|\underline{m}^{k<}\|}} +$$

$$\sum_{k=1}^{r-1} (-1)^{k} \sum_{\substack{1 = s_{r-k} < 2 < s_{r-k+1} < \dots < s_{r-1} < s_r < r+1 \\ \underline{m}_i^{<} \in \left[\Omega^{\bullet <}\right]_{s_i} \\ \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<} \in \Omega_{\mathrm{irred}}^{\bullet <} \\ \underline{m}^{i-1<} \in \mathbf{compo}\left(\underline{m}^{i<}; \underline{\omega}_1^{i<}, \dots, \underline{\omega}_{s_i}^{i<}\right) \\ i \in [\![r+1-k, r+1-1]\!]}}$$

$$\left(\prod_{l=1}^{k} \mathbf{proj}\left(\begin{array}{c} \underline{m}^{l+1<} \\ \underline{m}^{l<} \circ \left(\underline{\omega}_1^{l<}, \dots, \underline{\omega}_{s_l}^{l<}\right) \end{array}\right) \frac{N^{\underline{\omega}_1^{l<}} \dots N^{\underline{\omega}_{s_l}^{l<}}}{N^{\|\underline{\omega}_1^{l<}\|} \dots N^{\|\underline{\omega}_{s_l}^{l<}\|}}\right) \qquad \Bigg]$$

□

Remark 5.7.3.1

— *In the formula given arborified composition inverse of the mould, we have* $m_i^k = \|\underline{\omega}_i^{k<}\|$.

— *In the case where the arborified mould* $N^\bullet$ *has a geometrical growth, this formula not permits to conclude to the geometrical growth for the arborified composition inverse* $N^\bullet$. *This property is generally false. A counter example is given by the mould* $\exp^\bullet$ *(see Subsection 6.4 for a definition) and its composition inverse* $\log^\bullet$. *We give us a one dimensional vector field* $X(u) = \sum_{m \geq 0} X_m u^{m+1}$. *When seeing* $X(u)$ *as a derivation on* $\mathbb{C}[[u]]$, *we define a group like element of* $\boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *(because the comould associated to* X *is cosymmetral and* $\exp^\bullet$ *is symmetral) by considering the mould-comould expansion* $\mathbf{F} = \sum \exp^\bullet \mathbb{B}_\bullet$ *where for any* $m \in \mathbf{N}$, $\mathbb{B}_m = X_m u^{m+1} \partial$. *The obtained operator is then the substitution automorphism by* $f(u) = \mathbf{F}u$ *which is a tangent to identity formal diffeomorphism. When starting from a tangent to identity diffeomorphism* $f(u)$ *and the corresponding substitution automorphism* $\mathbf{F}$, *we can by reversing this process, associated to it a one dimensional vector field. The mould that relies* X *to* $\mathbf{F}$ *is the composition inverse of* $\exp^\bullet$, *i.e.* $\log^\bullet$ *and* X *is said to be the infinitesimal generator of the diffeomorphism* $f(u)$. *The mould* $\exp^{\bullet <}$ *has a geometrical growth but it is not the case of the mould* $\log^{\bullet <}$. *Indeed, the analyticity of a tangent to identity analytic diffeomorphism does not implies this of its infinitesimal generator. Generally, the infinitesimal generator* X *is* 1*-Gevrey and not analytic, which means that* *its coefficients satisfy the following : there exist $a, b \in \mathbb{R}_+^*$ such that for all $m \in \mathbf{N}$, $|X_m| \leq ab^m m!$. See [59] and [60].*

Example 5.7.3.1

$$M^{\bullet 1} = \frac{1}{N^{\bullet 1}}$$

$$M^{\begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix}} = -\frac{N^{\begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix}}}{N^{\bullet 1+2} N^{\bullet 1} N^{\bullet 2}}$$

$$M^{\bullet 1 \ \bullet 2} = \frac{1}{N^{\bullet 1} N^{\bullet 2}}$$

$$M^{\begin{smallmatrix} 2 & & 3 \\ & \vee & \\ & \bullet 1 & \end{smallmatrix}} = \frac{1}{N^{\bullet 1} N^{\bullet 2} N^{\bullet 3}} \left(\frac{N^{\begin{smallmatrix} 3 \\ | \\ \bullet 1+2 \end{smallmatrix}} N^{\begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix}} N^{\bullet 3}}{N^{\bullet 1+2+3} N^{\bullet 1+2} N^{\bullet 3}} + \frac{N^{\begin{smallmatrix} 2 \\ | \\ \bullet 1+3 \end{smallmatrix}} N^{\begin{smallmatrix} 3 \\ | \\ \bullet 1 \end{smallmatrix}} N^{\bullet 2}}{N^{\bullet 1+2+3} N^{\bullet 1+3} N^{\bullet 2}} - \frac{N^{\begin{smallmatrix} 2 & & 3 \\ & \vee & \\ & \bullet 1 & \end{smallmatrix}}}{N^{\bullet 1+2+3}} \right)$$

$$M^{\begin{smallmatrix} 3 \\ | \\ 2 \\ | \\ \bullet 1 \end{smallmatrix}} = \frac{1}{N^{\bullet 1} N^{\bullet 2} N^{\bullet 3}} \left(\frac{N^{\begin{smallmatrix} 3 \\ | \\ \bullet 1+2 \end{smallmatrix}} N^{\begin{smallmatrix} 2 \\ | \\ \bullet 1 \end{smallmatrix}} N^{\bullet 3}}{N^{\bullet 1+2+3} N^{\bullet 1+2} N^{\bullet 3}} + \frac{N^{\begin{smallmatrix} 2+3 \\ | \\ \bullet 1 \end{smallmatrix}} N^{\begin{smallmatrix} 3 \\ | \\ \bullet 2 \end{smallmatrix}} N^{\bullet 1}}{N^{\bullet 1+2+3} N^{\bullet 2+3} N^{\bullet 1}} - \frac{N^{\begin{smallmatrix} 3 \\ | \\ 2 \\ | \\ \bullet 1 \end{smallmatrix}}}{N^{\bullet 1+2+3}} \right)$$

We will now compute $M^{\begin{smallmatrix} 4 \\ | \\ 3 \\ | \\ 2 \\ | \\ \bullet 1 \end{smallmatrix}}$. In order to alleviate the notations, we omit the denominators. We resume the variations of the indexes of the first and the second sum of the formula for mould arborified composition inverse in the two following tables :

k	$r-k$	s_i
1	3	$2 = s_3 < 4$
2	2	$2 = s_2 < s_3 < 4$

First sum

k	$r-k$	s_i
1	3	$1 = s_3 < 4$
2	2	$1 = s_2 < 2 < s_3 < 4$

Second sum

In the following expression :

- *the first bracket is relative to the first sum, $k = 1$ and $s_3 = 2$.*
- *the second bracket is relative to the first sum, $k = 2$, $s_2 = 2$ and $s_3 = 3$.*
- *the third bracket is relative to the second sum, $k = 1$ and $s_3 = 1$.*

— *the fourth bracket is relative to the second sum,* $k = 2$*,* $s_2 = 1$ *and* $s_3 = 3$*.*

$$M^{\substack{4\\|\\3\\|\\2\\|\\\bullet 1}} =$$

$$\Big[\; N^{\substack{3\\|\\2\\|\\\bullet 1}} N^{\bullet 4} N^{\substack{4\\|\\\bullet 1+2+3}} + N^{\substack{2\\|\\\bullet 1}} N^{\substack{4\\|\\\bullet 3}} N^{\substack{3+4\\|\\\bullet 1+2}} + N^{\bullet 1} N^{\substack{4\\|\\3\\|\\\bullet 2}} N^{\substack{2+3+4\\|\\\bullet 1}} \;\Big]$$

$$-\Big[\; N^{\substack{2\\|\\\bullet 1}} N^{\bullet 3} N^{\bullet 4} N^{\substack{3\\|\\\bullet 1+2}} N^{\bullet 4} N^{\substack{4\\|\\\bullet 1+2+3}} + N^{\substack{2\\|\\\bullet 1}} N^{\bullet 3} N^{\bullet 4} N^{\bullet 1+2} N^{\substack{4\\|\\\bullet 3}} N^{\substack{3+4\\|\\\bullet 1+2}}$$

$$+ N^{\bullet 1} N^{\substack{3\\|\\\bullet 2}} N^{\bullet 4} N^{\substack{2+3\\|\\\bullet 1}} N^{\bullet 4} N^{\substack{4\\|\\\bullet 1+2+3}} + N^{\bullet 1} N^{\substack{3\\|\\\bullet 2}} N^{\bullet 4} N^{\bullet 1} N^{\substack{4\\|\\\bullet 2+3}} N^{\substack{2+3+4\\|\\\bullet 1}}$$

$$+ N^{\bullet 1} N^{\bullet 2} N^{\substack{4\\|\\\bullet 3}} N^{\substack{2\\|\\\bullet 1}} N^{\bullet 3+4} N^{\substack{3+4\\|\\\bullet 1+2}} + N^{\bullet 1} N^{\bullet 2} N^{\substack{4\\|\\\bullet 3}} N^{\bullet 1} N^{\substack{3+4\\|\\\bullet 2}} N^{\substack{2+3+4\\|\\\bullet 1}} \;\Big]$$

$$-\Big[\; N^{\substack{4\\|\\3\\|\\2\\|\\\bullet 1}} \;\Big] + \Big[\; N^{\bullet 1} N^{\substack{3\\|\\\bullet 2}} N^{\bullet 4} N^{\substack{4\\|\\2+3\\|\\\bullet 1}} + N^{\bullet 1} N^{\bullet 2} N^{\substack{4\\|\\\bullet 3}} N^{\substack{3+4\\|\\2\\|\\\bullet 1}}$$

$$+ N^{\substack{2\\|\\\bullet 1}} N^{\bullet 3} N^{\bullet 4} N^{\substack{4\\|\\3\\|\\\bullet 1+2}} \;\Big]$$

These examples are original but when we specialize them to the non decorated case, they correspond to the ones already computed in [7]. Indeed, using the label 1 *to denote the absence of decoration and assuming that* $N^{\underline{\omega}^<} = 1$ *if* $l(\underline{\omega}^<) = 1$ *then we obtain for example :*

$$
\begin{aligned}
M^{\substack{1\\|\\1\\|\\1\\|\\\bullet 1}} &= N^{\substack{1\\|\\1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} + N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} + N^{\substack{1\\|\\1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} \\
&- N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} - N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} - N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} \\
&- N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} - N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} - N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} \\
&- N^{\substack{1\\|\\1\\|\\1\\|\\\bullet 1}} + N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\1\\|\\\bullet 1}} + N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\1\\|\\\bullet 1}} + N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\1\\|\\\bullet 1}} \\
&= -N^{\substack{1\\|\\1\\|\\1\\|\\\bullet 1}} + 5 N^{\substack{1\\|\\1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} - 5 N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}} N^{\substack{1\\|\\\bullet 1}}
\end{aligned}
$$

which is consistent with the calculations in [7] page 22.

5.8 Another proof of the main theorem

We give in this section an other proof of the main Theorem 5.3.2 in the spirit of the proof sketched by J. Ecalle in [15]

Theorem 5.8.1 *Let us consider a family of ordinary differential operators* $(\mathbb{D}_m)_{m\in\Omega}$. *There exists an unique arborescent comould* $\mathbb{D}_{\bullet^<}$ *that satisfies the three following properties :*

$\mathbf{P_1}$: $\mathbb{D}_{\bullet^<}$ *is coseparative :*

$$\mathbb{D}_\emptyset = 1 \quad \textit{and} \quad \boldsymbol{cop}\left(\mathbb{D}_{\underline{\omega}^<}\right) = \sum_{\underline{\omega}'^< \oplus \underline{\omega}''^< = \underline{\omega}^<} \mathbb{D}_{\underline{\omega}'^<} \otimes \mathbb{D}_{\underline{\omega}''^<}$$

where the sequences $\underline{\omega}'^<$, $\underline{\omega}''^<$ *may be empty.*

$\mathbf{P_2}$: *if* $\deg(\underline{\omega}^<) = d$, *i.e.* $\underline{\omega}^< = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s}$ *then*

$$\mathbb{D}_{\underline{\omega}^<} = \frac{1}{d_1! \ldots d_s!} \left(P_1(u)\right)^{d_1} \ldots \left(P_s(u)\right)^{d_s} \partial^d = \frac{1}{s\left(\underline{\omega}^<\right)} \mathbb{D}^{\boldsymbol{prearbo}}_{\underline{\omega}^<}$$

where
— $d = d_1 + \ldots + d_s$.

— $P_1(u) = \mathbb{D}_{\underline{\omega}^{1<}}.u, \ldots, P_s(u) = \mathbb{D}_{\underline{\omega}^{s<}}.u \in \mathbb{C}[u]$.

$\mathbf{P_3}$: *If* $\underline{\omega}^< = \omega_1.{}'\underline{\omega}^<$ *then*

$$\mathbb{D}_{\underline{\omega}^<}.u = \mathbb{D}_{'\underline{\omega}^<}.\mathbb{D}_{\omega_1}.u.$$

Moreover :

— *if* $\mathbb{D}_\bullet$ *is the comould associated to a primitive element of* ***ENDOM*** $(\mathbb{C}[[u]])$ *then* $\mathbb{D}_{\underline{\omega}^<}$ *is a coarborified of* $\mathbb{D}_\bullet$*.*

— *if* $\mathbb{D}_\bullet$ *is the comould associated to a group like element of* ***ENDOM*** $(\mathbb{C}[[u]])$*, then* $\mathbb{D}_{\underline{\omega}^<}$ *is a contracted coarborified of* $\mathbb{D}_\bullet$*.*

In this two cases, we say that $\mathbb{D}_{\bullet^<}$ *is* ***the coarborified of*** $\mathbb{D}_\bullet$ ***homogeneous in*** ∂ *.*

Proof We will perform an induction on $r = l(\underline{\omega}^<)$.

— If $r = 0$ then using $\mathbf{P_1}$ we get $\mathbb{D}_{\underline{\omega}^<} = 1$.

— if $r = 1$ then $\underline{\omega}^< = \omega_1$ and as a consequence of $\mathbf{P_2}$, we get $\mathbb{D}_{\underline{\omega}^<} = P_1(u)\,\partial$. But using $\mathbf{P_3}$, we obtain $\mathbb{D}_{\underline{\omega}^<}.u = P_1(u)\,\partial.u = \mathbb{D}_{\omega_1}.u = P_1(u)$ and so $P_1(u) = \mathbb{D}_{\omega_1}.u = \mathbb{D}^{\mathbf{prearbo}}_{\omega_1}.u$. Moreover, as $P_1(u)\,\partial$ is a first order derivation, it satisfies the leibniz rule and then $\mathbb{D}_{\omega_1} = (\mathbb{D}_{\omega_1}.u)\,\partial$ satisfies $\mathbf{P_1}$.

— We then assume by induction that for any $\underline{\omega}^<$ of length $\leq r$ we get a comould $\mathbb{D}_{\underline{\omega}^<}$ that satisfies $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$. Let us consider an arborified sequence $\underline{\omega}^< \in \Omega^{\bullet<}$ of length $r+1$. We will prove that there exists a comould $\mathbb{D}_{\underline{\omega}^<}$ that satisfy $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$.

— If $\underline{\omega}^< = \left(\underline{\omega}^{1<}\right)^{d_1} \oplus \ldots \oplus \left(\underline{\omega}^{s<}\right)^{d_s}$ is not irreducible, using $\mathbf{P_2}$, we search $\mathbb{D}_{\underline{\omega}^<}$ under the form

$$\mathbb{D}_{\underline{\omega}^<} = P(u)\,\partial^d$$

where $P(u) \in \mathbb{C}[u]$. Using $\mathbf{P_1}$ and the induction hypothesis, we shall have with $\forall i \in [\![1,s]\!],\ P_i(u) = \mathbb{D}_{\underline{\omega}^{i<}}.u$:

$$\begin{aligned}
&\mathbf{cop}\left(P(u)\,\partial^{\|\mathbf{d}\|}\right) \\
= &\sum_{\underline{\omega}'^{<}\oplus\underline{\omega}''^{<}=\underline{\omega}^{<}} \mathbb{D}_{\underline{\omega}'^{<}} \otimes \mathbb{D}_{\underline{\omega}''^{<}} \\
= &\sum_{n_i+m_i=d_i,n_i,m_i\geq 0,i\in[\![1,s]\!]} \mathbb{D}_{(\underline{\omega}^{1<})^{m_1}\oplus\ldots\oplus(\underline{\omega}^{s<})^{m_s}} \otimes \mathbb{D}_{(\underline{\omega}^{1<})^{n_1}\oplus\ldots\oplus(\underline{\omega}^{s<})^{n_s}} \\
= &\ P(u)\,\partial^{\|\mathbf{d}\|}\otimes 1 + 1\otimes P(u)\,\partial^{\|\mathbf{d}\|} \\
&+ \sum_{\substack{n_i+m_i=d_i\\ n_i,m_i\geq 0\\ i\in[\![1,s]\!]\\ \|\underline{m}\|\neq 0,\|\underline{n}\|\neq 0}} \frac{1}{\underline{m}!}(P_1(u))^{m_1}\ldots(P_s(u))^{m_s}\,\partial^{m_1+\ldots+m_s} \\
&\qquad\qquad \otimes\frac{1}{\underline{n}!}(P_1(u))^{n_1}\ldots(P_s(u))^{n_s}\,\partial^{n_1+\ldots+n_s} \\
= &\ P(u)\,\partial^{\|\mathbf{d}\|}\otimes 1 + 1\otimes P(u)\,\partial^{\|\mathbf{d}\|} + \\
&(P_1(u))^{d_1}\ldots(P_s(u))^{d_s} \sum_{\substack{n_i+m_i=d_i\\ n_i,m_i\geq 0\\ i\in[\![1,s]\!]\\ \|\underline{m}\|\neq 0,\|\underline{n}\|\neq 0}} \frac{1}{\underline{m}!}\partial^{m_1+\ldots+m_s}\otimes\frac{1}{\underline{n}!}\partial^{n_1+\ldots+n_s} \\
= &\ P(u)\,\partial^{\|\mathbf{d}\|}\otimes 1 + 1\otimes P(u)\,\partial^{\|\mathbf{d}\|} + \\
&(P_1(u))^{d_1}\ldots(P_s(u))^{d_s}\left(\frac{1}{\mathbf{d}!}\mathbf{cop}\left(\partial^{\|\mathbf{d}\|}\right) - \frac{1}{\mathbf{d}!}\partial^{\|\mathbf{d}\|}\otimes 1 - 1\otimes\frac{1}{\mathbf{d}!}\partial^{\|\mathbf{d}\|}\right)
\end{aligned}$$

and thus $P(u) = \frac{1}{\mathbf{d}!}(P_1(u))^{d_1}\ldots(P_s(u))^{d_s}$. Moreover, for any $i \in [\![1,s]\!]$, $P_i(u) = \mathbb{D}_{\underline{\omega}^{i<}}.u = \frac{1}{s\left(\underline{\omega}^{i<}\right)}\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^{i<}}.u$ and then as long as $\underline{\omega}^{i<}$ is irreducible for any $i \in [\![1,s]\!]$,

$$(P_i(u))^{d_i}\,\partial^{d_i} = \left(\mathbb{D}_{\underline{\omega}^{i<}}.u\right)^{d_i}\partial^{d_i} = \frac{1}{\left(s\left(\underline{\omega}^{i<}\right)\right)^{d_i}}\mathbb{D}^{\mathbf{prearbo}}_{(\underline{\omega}^{i<})^{d_i}}$$

and

$$\begin{aligned}
P(u) &= \frac{1}{\mathbf{d}!}(P_1(u))^{d_1}\ldots(P_s(u))^{d_s}\,\partial^{d_1+\ldots+d_s} \\
&= \frac{1}{\left(s\left(\underline{\omega}^{i<}\right)\right)^{d_1}\ldots\left(d\left(\underline{\omega}^{s<}\right)!\right)^{d_s}}\mathbb{D}^{\mathbf{prearbo}}_{(\underline{\omega}^{1<})^{d_1}\oplus\ldots\oplus(\underline{\omega}^{s<})^{d_s}}.
\end{aligned}$$

— If $\underline{\omega}^{<} = \omega_1.'\underline{\omega}^{<}$ is an irreducible arborified sequence then $\mathbf{P_2}$ entails that $\mathbb{D}_{\underline{\omega}^{<}} = P(u)\,\partial$ where $P(u) \in \mathbb{C}[u]$ and

$$P(u) = \mathbb{D}_{\underline{\omega}^{<}}.u = \mathbb{D}_{'\underline{\omega}^{<}}.\mathbb{D}_{\omega_1}.u = \frac{1}{s\left('\underline{\omega}^{<}\right)}\mathbb{D}^{\mathbf{prearbo}}_{'\underline{\omega}^{<}}.\mathbb{D}^{\mathbf{prearbo}}_{\omega_1}.u = \frac{1}{s\left(\underline{\omega}^{<}\right)}\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^{<}}.u$$

and then $\mathbb{D}_{\underline{\omega}^{<}} = \frac{1}{s\left(\underline{\omega}^{<}\right)}\mathbb{D}^{\mathbf{prearbo}}_{\underline{\omega}^{<}}$.

Conversely, we verify that with $P(u) = \frac{1}{\mathbf{d}!} (P_1(u))^{d_1} \dots (P_s(u))^{d_s}$ then the comould

$$\mathbb{D}_{\underline{\omega}^<} = P(u)\, \partial^d$$

satisfies $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$. We have then proved by induction the existence and the uniqueness of a coarborified comould $\mathbb{D}_{\bullet^<}$ that satisfy $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$.

We can then finish the proof as in Section 5.3.

□

Chapitre 6

Examples of mould arborification

On cesse de s'étonner devant un miracle constant.
André Gide

As explained in the previous chapter, the mechanism of arborification / co-arborification consists to replace in a differential operator like $\mathbf{F} = \sum M^\bullet \mathbb{D}_\bullet$ the composition $\mathbb{D}_{\underline{\omega}} = \mathbb{D}_{\omega_r} \bullet \ldots \bullet \mathbb{D}_{\omega_1}$ by $r!$ differential operators. The counterpart of this operation is to replace the mould $M^\bullet$ by an arborescent mould $M^{\bullet <}$ which is the sum, when it is indexed by a forest of length r, of $r!$ terms. So arborification/coarborification seems to be just a transferring of terms. But the miracle of this operation lies in the fact that, for almost any mould encountered in dynamical system, one has closed formula for their arborified and so one can easily bound them.

The aim of this chapter is to give several examples of mould arborification or antiarborification, whether simple or contracted.

From now, and in order to simplify the exposure, we will use the same notation $M^{\bullet <}$ for the arborified $M^{\bullet <}$ and the contracting arborified $M^{\bullet \ll}$ of a mould.

We will use the definitive following notations :

Notation 6.0.0.1 *For any sequence $\underline{\omega}^< \in \Omega^{\bullet <}$ and $\underline{\omega} \in \Omega^\bullet$, we will denote by*

— $\boldsymbol{sh}\left(\underline{\omega}^<, \underline{\omega}\right) = \boldsymbol{proj}\binom{\underline{\omega}^<}{\underline{\omega}}$

— $\boldsymbol{csh}\left(\underline{\omega}^\ll, \underline{\omega}\right) = \boldsymbol{cont}\binom{\underline{\omega}^<}{\underline{\omega}}$

— $\boldsymbol{sh}\left(\underline{\omega}^>, \underline{\omega}\right) = \boldsymbol{proj}\binom{rev(\underline{\omega}^<)}{rev(\underline{\omega})}$

— $\boldsymbol{csh}\left(\underline{\omega}^\gg, \underline{\omega}\right) = \boldsymbol{cont}\binom{rev(\underline{\omega}^<)}{rev(\underline{\omega})}$.

6.1 A detailed example with the organic mould

We give now some recipes to arborifiate a mould $M^\bullet$. Firstly, this is one of the strength of arborification, the four symmetries alterna(e)l and symmetra(e)l give rise to two very simple symmetries on its arborified. We recall that an arborified mould $M^{\bullet<}$ is said to be

— **primitive** if and only if $M^{\underline{\omega}^<}$ is null if $\underline{\omega}^<$ is not irreducible.

— **separatif** if and only if $M^{\underline{\omega}^<} = M^{\underline{\omega}^{1<}} \dots M^{\underline{\omega}^{s<}}$ if $\underline{\omega}^< = \underline{\omega}^{1<} \oplus \dots \oplus \underline{\omega}^{s<}$ is not irreducible.

One has the following :

$$\begin{array}{rcl} M^\bullet \text{ alterna(e)l} & \Rightarrow & M^{\bullet<} \text{ primitive ,} \\ M^\bullet \text{ symmetra(e)l} & \Rightarrow & M^{\bullet<} \text{ separatif .} \end{array}$$

A nice implication is that it suffices to compute the mould $M^{\bullet<}$ for irreducible arborescent sequences. The computation is generally based on an induction on the length r of $\underline{\omega}^<$, on the fact that for an irreducible arborescent sequence $\underline{\omega}^<$, one has $\mathbf{(c)sh}\,(\underline{\omega}^<, \underline{\omega}) = \mathbf{(c)sh}\,({}'\underline{\omega}^<, {}'\underline{\omega})$ and on the possibly multiplicative structure of the mould $M^\bullet$ (i.e. when $M^{\omega_1,\dots,\omega_r} = \alpha . M^{\omega_2,\dots,\omega_r}$ where α is a scalar which may be dependent on $\omega_1, \dots, \omega_r$). We then write [1]

$$\begin{array}{rcl} M^{\underline{\omega}^<} & = & \sum \mathbf{(c)sh}\left(\underline{\omega}^<, \underline{\omega}\right) M^{\underline{\omega}} \\ & = & \alpha \sum \mathbf{(c)sh}\left({}'\underline{\omega}^<, {}'\underline{\omega}\right) M^{'\underline{\omega}} \\ & = & \alpha M^{'\underline{\omega}^<} \end{array}$$

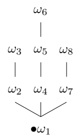
and we end the computation using the induction hypothesis.

For example, if we consider the symmetrel mould $M^{\underline{\omega}} = \prod_{i=1}^r \left(\dfrac{\frac{\omega_i}{2}}{\hat{\omega}_i} - 1 \right)$ then its contracting arborified is given by

$$M^{\underline{\omega}\ll} = \prod_{i=1}^r \left(\frac{\frac{\omega_i}{2}}{\hat{\omega}_i} - 1 \right)$$

where the sums $\hat{\omega}_i$ ranges over all the ω_i that are posterior to ω_i for the order of $\underline{\omega}^<$.

1. If $\underline{\omega}^<$ is an irreducible arborescent sequence then we write $\underline{\omega}^< = \omega_1 . {}'\underline{\omega}^<$ with root ω_1. The arborescent sequence ${}'\underline{\omega}^<$ corresponds to $\underline{\omega}^<$ truncated of ω_1. For example, if $\underline{\omega}^< =$  then $\underline{\omega}^< = \omega_1 . {}'\underline{\omega}^<$ with ${}'\underline{\omega}^< = \bullet_{\omega_2} \bullet_{\omega_3}$. We recall moreover that for a given sequence $(\omega_1, \dots, \omega_r) \in \Omega^\bullet$, ${}'\underline{\omega} = (\omega_2, \dots, \omega_r)$.

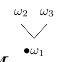
For example, for the tree , one has :

$$\begin{aligned}\hat{\omega}_1 &= \omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7+\omega_8\\ \hat{\omega}_2 &= \omega_2+\omega_3\\ \hat{\omega}_4 &= \omega_4+\omega_5+\omega_6.\end{aligned}$$

In particular, one has[2] :

M 
$$= \left(\frac{\frac{\omega_1}{2}}{\omega_1+\omega_2}-1\right)\left(\frac{\frac{\omega_2}{2}}{\omega_2}-1\right)$$

$$M^{\bullet_{\omega_1}\bullet_{\omega_2}} = -\frac{1}{4}$$

M 
$$= \left(\frac{\frac{\omega_1}{2}}{\omega_1+\omega_2+\omega_3}-1\right)\left(\frac{\frac{\omega_2}{2}}{\omega_2}-1\right)\left(\frac{\frac{\omega_3}{2}}{\omega_3}-1\right) = -\frac{(\omega_1+2\omega_2+2\omega_3)}{8(\omega_1+\omega_2+\omega_3)}$$

$$M^{\bullet_{\omega_1}\bullet_{\omega_2}\bullet_{\omega_3}} = \frac{1}{8}$$

It is a spectacular thing to verify these relations using example 5.2.0.13 and a symbolic computation program[3]

2. the last one example must be compare with example computed in next footnote.
3. the following program allows to construct the mould $M^\bullet$ with Maple :

```
>#To troncate a list of its first element
>tronque:=proc(L)
local n;
  n:=nops(L);
  [seq(op(i,L),i=2..n)];
end;

>#The mould M
>M:=proc(L)
local a,k,n,i,M;
  a:=1;
  n:=nops(L);
  M:=L;
    for i from 1 to n do
    a:=a*(M[1]/(2*convert(M,'+' ))-1);
    M:=tronque(M);
    od;
  a;
end;

>#Example of utilisation of the program
```

We apply our recipe to prove it. The formula is clearly true for an arborescent sequence $\underline{\omega}^<$ of length 1. We assume it is true for any arborescent sequence of length $\leq r$ and we consider an arborescent sequence $\underline{\omega}^<$ of length $r+1$. If $\underline{\omega}^<$ is irreducible then one has :

$$\begin{aligned} M^{\underline{\omega}^{\ll}} &= \sum \mathbf{csh}\left(\underline{\omega}^<, \underline{\omega}\right) M^{\underline{\omega}} \\ &= \sum \mathbf{csh}\left(\underline{\omega}^<, \underline{\omega}\right) \left(\frac{\frac{\omega_1}{2}}{\hat{\omega}_i} - 1\right) M'^{\underline{\omega}} \\ &= \left(\frac{\frac{\omega_1}{2}}{\hat{\omega_1}} - 1\right) \sum \mathbf{csh}\left('\underline{\omega}^<, '\underline{\omega}\right) \left(\frac{\frac{\omega_1}{2}}{\hat{\omega}_i} - 1\right) M'^{\underline{\omega}} \\ &= \prod_{i=1}^{r} \left(\frac{\frac{\omega_i}{2}}{\hat{\omega}_i} - 1\right) \end{aligned}$$

using the induction hypothesis and where the sums $\hat{\omega}_i$ range over all the ω_i that are posterior to ω_i for the order of $\underline{\omega}^<$. If $\underline{\omega}^< = \underline{\omega}^{1<} \oplus \ldots \oplus \underline{\omega}^{s<}$ is not irreducible(where for all $i \in [\![1, s]\!]$, $\underline{\omega}^{i<} \in \Omega^{\bullet<}_{\text{irred}}$), then using the induction hypothesis and the separativity of $M^{\bullet\ll}$, one easily find the attempted formula.

It is not every times possible to obtain a closed formula for the mould $M^{\bullet<}$. For example, if one consider the mould $N^\bullet$, known as the organic mould (and generally denoted **romo**$^\bullet$), given for all $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ by $N^{\underline{\omega}} = \prod_{i=1}^{r} \left(\frac{\frac{\omega_i}{2}}{\check{\omega}_i} - 1\right)$ [4], and if we compute the mould $N^{\underline{\omega}^{\ll}}$ for arborescent sequence $\underline{\omega}^<$ of small length, no simple formula appears to exprimate it [5].

```
>M([w[1],w[2],w[3]])+M([w[1],w[3],w[2]])+M([w[1],w[2]+w[3]]);

-1/2*(1/2*w[1]/(w[1]+w[2]+w[3])-1)*(1/2*w[2]/(w[2]+w[3])-1)
-1/2*(1/2*w[1]/(w[1]+w[2]+w[3])-1)*(1/2*w[3]/(w[2]+w[3])-1)
-1/4*w[1]/(w[1]+w[2]+w[3])+1/2

>simplify(%);

-1/8*(w[1]+2*w[2]+2*w[3])/(w[1]+w[2]+w[3])
```

4. For the notations, see subsection 1.1.1
5. We have for example :

```
> tronqued:=proc(L)
  local n;
    n:=nops(L);
    [seq(op(i,L),i=1..n-1)];
 end;


 > N:=proc(L)
local a,k,n,i,M;
a:=1;n:=nops(L);
M:=L;
for i from 1 to n do
a:=a*(M[n-i+1]/(2*convert(M,'+' ))-1);
M:=tronqued(M);
```

But in certain cases, and more precisely in every cases encountered when studying dynamical systems, it is possible to correct this problem using backward (or anti-)arborification, contracting or not.

Let us first introduce the application **rev** : $\left\{ \begin{array}{ccc} \Omega^\bullet & \longrightarrow & \Omega^\bullet \\ (\omega_1, \dots, \omega_r) & \mapsto & (\omega_r, \dots, \omega_1) \end{array} \right.$.
It induces an application from the set of moulds on Ω into itself, called **rev** too, and defined on a mould $M^\bullet$ by

$$\mathbf{rev}\,(M^\bullet) = M^{\mathbf{rev}(\bullet)}.$$

With the help of this operator, we will define the backwards arborified, contracted or not, of a mould.

Definition 6.1.0.1 *We define the* ***backwards arborification****, contracted or not, of a mould $M^\bullet$ on Ω as the arborified, contracted or not, of the mould $M^{rev(\bullet)}$. We will denote $M^{\bullet>}$ the simple backwards arborification of $M^\bullet$ and $M^{\bullet\gg}$ its contracted backwards arborification.*

Let us remark that in backwards arborification, the coefficients of the moulds are indexed by antiarborescent sequences, i.e. sequences with an antiarborescent order. An arborescent sequence is a sequence provided with an order such that each element of the sequence has at most one predecessor. In an antiarborescent sequence, each element get at most one successor. When computing backward arborification, instead of coming back to arborification using the involution **rev**, it is easiest to work directly with antiarborescent sequence. We so introduce the following notations.

Notation 6.1.0.2 *For any sequence $\underline{\omega}^< \in \Omega^{\bullet<}$ and $\underline{\omega} \in \Omega^\bullet$, we will denote by*
— $\boldsymbol{sh}\,(\underline{\omega}^>, \underline{\omega}) = \boldsymbol{proj}\binom{rev(\underline{\omega}^<)}{rev(\underline{\omega})}$
— $\boldsymbol{csh}\,(\underline{\omega}^\gg, \underline{\omega}) = \boldsymbol{cont}\binom{rev(\underline{\omega}^<)}{rev(\underline{\omega})}$.

It then comes for any $\underline{\omega}^< \in \Omega^{\bullet<}$:

$$\begin{aligned} M^{\underline{\omega}^<} &= \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^>, \underline{\omega}\right) M^{\underline{\omega}}, \\ M^{\underline{\omega}^\gg} &= \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{csh}\left(\underline{\omega}^\gg, \underline{\omega}\right) M^{\underline{\omega}}. \end{aligned}$$

```
od;
a;
end;

> simplify(N([w[1],w[2],w[3]])+N([w[1],w[3],w[2]])+N([w[1],w[2]+w[3]]));

-1/8*(w[3]^2*w[2]+6*w[3]*w[1]^2+2*w[3]^2*w[1]+
6*w[1]^2*w[2]+2*w[2]^2*w[1]+w[2]^2*w[3]+4*w[1]^3+
6*w[3]*w[2]*w[1])/((w[1]+w[2]+w[3])*(w[1]+w[2])*(w[1]+w[3]))
```

Example 6.1.0.2 *We give some examples of simple backward arborified :*

$$
\begin{aligned}
M^{\begin{smallmatrix}\bullet 1\\ | \\ 2\end{smallmatrix}} &= M^{2,1}\\
M^{\begin{smallmatrix}\bullet 1\\ \wedge \\ 3\;\;2\end{smallmatrix}} &= M^{2,3,1}+M^{3,2,1}\\
M^{\begin{smallmatrix}\bullet 1\\ | \\ 2\\ | \\ 3\end{smallmatrix}} &= M^{3,2,1}\\
M^{\begin{smallmatrix}\bullet 1\\ \diagup | \diagdown \\ 4\;\;3\;\;2\end{smallmatrix}} &= M^{4,3,2,1}+M^{3,4,2,1}+M^{2,3,4,1}+M^{2,4,3,1}+M^{4,2,3,1}+M^{3,2,4,1}
\end{aligned}
$$

and of some contracted backward arborified.

$$
\begin{aligned}
M^{\begin{smallmatrix}\bullet 1\\ | \\ 2\end{smallmatrix}} &= M^{2,1}\\
M^{\begin{smallmatrix}\bullet 1\\ \wedge \\ 3\;\;2\end{smallmatrix}} &= M^{2,3,1}+M^{3,2,1}+M^{2+3,1}\\
M^{\begin{smallmatrix}\bullet 1\\ | \\ 2\\ | \\ 3\end{smallmatrix}} &= M^{3,2,1}\\
M^{\begin{smallmatrix}\bullet 1\\ \diagup | \diagdown \\ 4\;\;3\;\;2\end{smallmatrix}} &= M^{4,3,2,1}+M^{3,4,2,1}+M^{2,3,4,1}+M^{2,4,3,1}+M^{4,2,3,1}+M^{3,2,4,1}\\
&\quad +M^{3+4,2,1}+M^{2+3,4,1}+M^{2+4,3,1}+M^{2+3+4,1}.
\end{aligned}
$$

Let us observe that one has the same symetries "atomic" and "separative" than in the arborified case.

When considering the previous mould $N^{\bullet}$ and when computing its backwards contracting arborified, one obtains for any antiarborescent sequence $\underline{\omega}^{>}$:

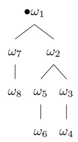
$$
N^{\underline{\omega}\gg} = \prod_{i=1}^{r} \left(\frac{\frac{\omega_i}{2}}{\check{\omega}_i} - 1 \right)
$$

where the sums $\check{\omega}_i$ ranges over all the ω_i that are anterior to ω_i for the antiarborescent order of $\underline{\omega}^{>}$ [6].

6. A small maple computation with the previous program gives :

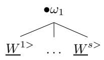
```
> simplify(N([w[2],w[3],w[1]])+N([w[3],w[2],w[1]])+N([w[2]+w[3],w[1]]));

-1/8*(w[1]+2*w[2]+2*w[3])/(w[1]+w[2]+w[3])
```

which is of the attempted form.

For example, for the tree 

, one has :

$$\begin{aligned}\check{\omega}_1 &= \omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7+\omega_8\\ \check{\omega}_2 &= \omega_2+\omega_3+\omega_4+\omega_5+\omega_6\\ \check{\omega}_7 &= \omega_7+\omega_8.\end{aligned}$$

We verify easily the formula for an anti-arborescent sequence $\underline{\omega}^>$ of length 1 and we assume it is true for any anti-arborescent sequence of length $\geq r$. We then consider an anti-arborescent sequence $\underline{\omega}^>$ of length $r+1$. If $\underline{\omega}^> = \underline{\omega}^{>\prime}.\omega_1 =$ 

with $\underline{\omega}^{>\prime} = \underline{W}^{1>} \oplus \ldots \oplus \underline{W}^{s>}$ is irreducible then one has :

$$\begin{aligned}M^{\underline{\omega}^{\gg}} &= \sum \mathbf{csh}\left(\underline{\omega}^{\gg},\underline{\omega}\right) N^{\underline{\omega}}\\ &= \sum \mathbf{csh}\left(\underline{\omega}^{\gg},\underline{\omega}\right)\left(\frac{\frac{\omega_1}{2}}{\check{\omega}_i}-1\right) N^{\underline{\omega}'}\\ &= \left(\frac{\frac{\omega_1}{2}}{\check{\omega}_1}-1\right)\sum \mathbf{csh}\left(\underline{\omega}^{\gg},\underline{\omega}'\right)\left(\frac{\frac{\omega_1}{2}}{\check{\omega}_i}-1\right) M^{\underline{\omega}'}\\ &= \prod_{i=1}^{r}\left(\frac{\frac{\omega_i}{2}}{\check{\omega}_i}-1\right)\end{aligned}$$

using the induction hypothesis and where the sums $\check{\omega}_i$ ranges over all the ω_i that are anterior to ω_i for the order of $\underline{\omega}^{\gg}$. If $\underline{\omega}^> = \underline{\omega}^{1>} \oplus \ldots \oplus \underline{\omega}^{s>}$ is not irreducible and where for all $i \in [\![1,s]\!]$, $\underline{\omega}^{i>} \in \Omega^{\bullet<}_{\text{irred}}$, then using the induction hypothesis and the separativity of $M^{\bullet\ll}$, one find easily the attempted formula.

6.2 The symmetrel mould $\not{S}^\bullet$

Proposition 6.2.1 *We consider the symmetrel mould $\not{S}^\bullet$ defined, for any $\underline{\omega} \in \Omega^\bullet$ by :*

$$\not{S}^{\underline{\omega}} = \begin{cases} 1 & \textit{if } l(\underline{\omega}) = 0\\ \dfrac{1}{(e^{\hat{\omega}_1}-1)\ldots(e^{\hat{\omega}_r}-1)} & \textit{if } l(\underline{\omega}) = r > 0\end{cases}$$

Its contracted arborified is given for any $\underline{\omega}^< = (\omega_1,\ldots,\omega_r)^< \in \Omega^{\bullet<}_{irred}$ by :

$$\not{S}^{\underline{\omega}^{\ll}} = \begin{cases} 1 & \textit{if } l(\underline{\omega}) = 0\\ \dfrac{1}{(e^{\hat{\omega}_1}-1)\ldots(e^{\hat{\omega}_r}-1)} & \textit{if } l(\underline{\omega}) = r > 0\end{cases}$$

where the sums $\hat{\omega}_i$ range over all the ω_i that are superior to ω_i for the order of $\underline{\omega}^<$.

Proof We will reason inductively on the length r of the arborified sequence $\underline{x}^{\ll}$. If $r = 1$ then the result is obvious. We assume that the result is true for any sequence of length $r > 0$ and we will consider a sequence $\underline{x}^{\ll}$ of length $r + 1$.

— if $\underline{x}^{\ll} = \underline{y}^{\ll} \oplus \underline{z}^{\ll}$ then as long as $\mathbf{S}^{\bullet\ll}$ is separative (because $\mathbf{S}^{\bullet}$ is symmetrel) we get, using our induction hypothesis

$$\cancel{\mathbf{S}}^{\underline{x}^{\ll}} = \cancel{\mathbf{S}}^{\underline{y}^{\ll}} \cancel{\mathbf{S}}^{\underline{z}^{\ll}} = \frac{1}{(e^{\hat{y}_1} - 1) \ldots \left(e^{\hat{y}_{r_1}} - 1\right)} \frac{1}{(e^{\hat{z}_1} - 1) \ldots \left(e^{\hat{z}_{r_2}} - 1\right)}$$

where
— $r_1 = \mathrm{l}\left(\underline{y}^{\ll}\right)$, $r_2 = \mathrm{l}\left(\underline{z}^{\ll}\right)$ and $r_1 + r_2 = r + 1$.
— the sums $\hat{y}_i$ and $\hat{z}_j$ are related to the arborescent order.
Then $\cancel{\mathbf{S}}^{\underline{x}^{\ll}}$ get the attempted form.

— If $\underline{x}^{\ll}$ is an irreducible sequence, then it has a root x_1 and using the definition of the contracting arborification of a mould, we obtain :

$$\begin{aligned}
\cancel{\mathbf{S}}^{\underline{x}^{\ll}} &= \sum_{\underline{x} \in \Omega^{\bullet}} \mathbf{cts}\left(\underline{x}^{\ll}, \underline{x}\right) \cancel{\mathbf{S}}^{\underline{x}} \\
&= \sum_{\underline{x} \in \Omega^{\bullet}} \mathbf{cts}\left(\underline{x}^{\ll}, \underline{x}\right) \cancel{\mathbf{S}}^{x_1, \underline{x}'} \\
&= \sum_{\underline{x}' \in \Omega^{\bullet}} \mathbf{cts}\left(\underline{x}'^{\ll}, \underline{x}'\right) \frac{1}{e^{\hat{x}_1} - 1} \cancel{\mathbf{S}}^{\underline{x}'} \\
&= \frac{1}{e^{\hat{x}_1} - 1} \cancel{\mathbf{S}}^{\underline{x}'^{\ll}}
\end{aligned}$$

which is of the attempted form.

□

6.3 The symmetrel mould $S^{\bullet}$

Proposition 6.3.1 *We consider the symmetrel mould $S^{\bullet}$ defined, for any $\underline{\omega} \in \Omega^{\bullet}$ by :*

$$S^{\underline{\omega}} = \begin{cases} 1 & \text{if } l\left(\underline{\omega}\right) = 0 \\ \dfrac{(-1)^r \, e^{\|\underline{\omega}\|}}{(1 - e^{\check{\omega}_1}) \ldots (1 - e^{\check{\omega}_r})} & \text{if } l\left(\underline{\omega}\right) = r > 0 \end{cases}$$

Its contracted anti-arborified is given for any $\underline{\omega}^{<} \in \Omega^{\bullet <}_{irred}$ by :

$$S^{\underline{\omega}^{\gg}} = \begin{cases} 1 & \text{if } l\left(\underline{\omega}\right) = 0 \\ \dfrac{(-1)^r \, e^{\|\underline{\omega}\|}}{(1 - e^{\check{\omega}_1}) \ldots (1 - e^{\check{\omega}_r})} & \text{if } l\left(\underline{\omega}\right) = r > 0 \end{cases}$$

where the sums $\check{\omega}_i$ range over all the ω_i that are anterior to ω_i for the order of $\underline{\omega}^{<}$.

Proof We will reason inductively on the length r of $\underline{x}^{\gg}$. If $r = 0$ or $r = 1$ then the result is obvious. We assume that the result is true for any sequence of length $r > 0$ and we will consider a sequence $\underline{x}^{\gg}$ of length $r + 1$.

— if $\underline{x}^{\gg} = \underline{y}^{\gg} \oplus \underline{z}^{\gg}$ then as long as $S^{\bullet\gg}$ is separative (because $S^\bullet$ is symmetrel) we get, using our induction hypothesis

$$S^{\underline{x}^{\gg}} = S^{\underline{y}^{\gg}} S^{\underline{z}^{\gg}} = \frac{(-1)^{r+1} e^{\|\underline{x}^{\gg}\|}}{(1-e^{\check{y}_1})\dots\left(1-e^{\check{y}_{r_1}}\right)(1-e^{\check{y}_1})\dots\left(1-e^{\check{y}_{r_2}}\right)}$$

where

— $r_1 = \mathrm{l}\left(\underline{y}^{\gg}\right)$, $r_2 = \mathrm{l}\left(\underline{z}^{\gg}\right)$ and $r_1 + r_2 = r+1$.

— the sums $\check{y}_i$ and $\check{z}_j$ are related to the anti-arborescent order.

Then $S^{\underline{x}^{\gg}}$ get the attempted form.

— If $\underline{x}^{\gg}$ is an irreducible sequence, then it has a root x_{r+1} and, using the definition of the anti-arborification of a mould, we have :

$$\begin{aligned}
S_+^{\underline{x}^{\gg}} &= \sum_{\underline{x}\in\Omega^\bullet} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) S^{\underline{x}} \\
&= \sum_{\underline{x}\in\Omega^\bullet} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) S^{\underline{x}', x_{r+1}} \\
&= \sum_{\underline{x}'\in\Omega^\bullet} \mathbf{cts}\left(\underline{x}'^{\gg}, \underline{x}'\right) S^{\underline{x}'} \frac{-e^{x_{r+1}}}{1-e^{-\check{x}_{r+1}}} \\
&= S^{\underline{x}'^{\gg}} \frac{-e^{x_{r+1}}}{1-e^{-\check{x}_{r+1}}}
\end{aligned}$$

which is of the announced form.

□

6.4 The symmetral exponential mould $\exp^\bullet$

The symmetral mould $\exp^\bullet$ is characterized by the following property. Let us consider $M^\bullet = t^{\mathrm{l}(\bullet)} N^\bullet$ where $N^\bullet$ is not depending on t. Then $\exp^\bullet$ is the only mould $N^\bullet$ such that $\mathbf{N}^\emptyset = 1$ and $\partial_t M^\bullet = 1^\bullet \times M^\bullet$. A small calculation permits naturally to find $\exp^\bullet = 1/\mathrm{l}(\bullet)!$. Its arborified $\exp^{\bullet<}$ shall verify the same algebraic relation. If $M^{\bullet<} = t^{\mathrm{l}(\bullet)} \exp^{\bullet<}$ then we shall have :

$$\partial_t M^{\bullet<} = 1^{\bullet<} \times M^{\bullet<}.$$

Then for an irreducible sequence $\underline{z}^< = \begin{smallmatrix} \underline{z}^{1<} & \underline{z}^{2<} & \vdots\ \vdots & \underline{z}^{s<} \\ & \diagdown\ \diagup & & \\ & \bullet z_1 & & \end{smallmatrix}$ where $z_1 \in \Omega$ and $\underline{z}^{1<}, \dots, \underline{z}^{s<} \in \Omega^{\bullet<}_{\mathrm{irred}}$, as a consequence of the separativity of $\exp^{\bullet<}$, we shall have :

$$\mathrm{l}\left(\underline{z}^<\right) t^{\mathrm{l}\left(\underline{z}^<\right)-1} \exp^{\underline{z}^<} = \exp'^{\underline{z}^<} = \exp^{\underline{z}^{1<}} \dots \exp^{\underline{z}^{s<}} .$$

We have then proved the following.

Proposition 6.4.1 *The separative arborescent mould* $\exp^{\bullet<}$ *is characterized by the inductive relation :*

$$\exp^{\emptyset<} = 1 \text{ and } \forall \underline{z}^< = \begin{smallmatrix} \underline{z}^{1<} & \underline{z}^{2<} & \vdots\ \vdots & \underline{z}^{s<} \\ & \diagdown\ \diagup & & \\ & \bullet z_1 & & \end{smallmatrix} \in \Omega^{\bullet<}_{irred}, \quad \exp^{\underline{z}^<} = \frac{1}{l\left(\underline{z}^<\right)} \exp^{\underline{z}^{1<}} \dots \exp^{\underline{z}^{s<}} .$$

Let us observe that this arborescent mould appears in several recent papers concerned with Hopf algebraic construction in the numerical analysis methods for differential equations. Indeed, it gives the coefficients of the B-series or the P-series solution of the studied differential equation, see [60], [7] page 8 or [11] page 1 . It is related too to the Connes-Moscovici coefficients, see [7] page 4.

6.5 The alternal mould $T^\bullet$

Proposition 6.5.1 *We consider the alternal mould $T^\bullet$ defined, for any $\underline{\omega} \in \Omega^\bullet$, such that two consecutive elements of $\underline{\omega}$ are not equal, by :*

$$T^{\underline{\omega}} = \begin{cases} \dfrac{1}{(\omega_1 - \omega_2)(\omega_2 - \omega_3)\dots(\omega_{r-1} - \omega_r)} & \text{if } l(\underline{\omega}) > 1 \\ 0 & \text{elsewhere} \end{cases}$$

Its simple arborified is given for any $\underline{\omega}^< \in \Omega^{\bullet <}_{irred}$ by :

$$T^{\underline{\omega}^<} = \frac{1}{\prod(\omega_i - \omega_{i+})}$$

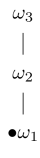
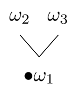
where the product ranges over pairs (ω_i, ω_{i+}) in such a way that ω_i and ω_{i+} are successive in $\underline{\omega}^<$. Moreover, $T^{\underline{\omega}^<} = 0$ for any non irreducible arborified sequence $\underline{\omega}^<$.

Proof We will reason inductively on the length r of the arborified sequence $\underline{\omega}^<$. Because $T^\bullet$ is alternal, $T^{\bullet<}$ is primitive and we will then only work with irreducible arborified sequences $\underline{\omega}^<$.

— If $r = 0$ or $r = 1$ then the result is obvious.

— If $r = 2$ then $\underline{\omega}^< =$  and

$$T^{\underline{\omega}^<} = \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) T^{\underline{\omega}} = T^{\omega_1, \omega_2} = \frac{1}{\omega_1 - \omega_2}$$

— If $r = 3$ then $\underline{\omega}^< =$  or $\underline{\omega}^< =$  . In the first case, we obtain :

$$T^{\underline{\omega}^<} = \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) T^{\underline{\omega}} = T^{\omega_1, \omega_2, \omega_3} = \frac{1}{(\omega_1 - \omega_2)(\omega_2 - \omega_3)}$$

and in the second :

$$T^{\underline{\omega}^<} = \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) T^{\underline{\omega}} = T^{\omega_1, \omega_2, \omega_3} + T^{\omega_1, \omega_3, \omega_2} = \frac{1}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)}$$

We assume that the formula for $T^{\underline{\omega}^<}$ is true for any sequence of length $r \in \mathbf{N}$ and we will now prove it for a sequence $\underline{\omega}^<$ of length $r+1$. We denote $\omega_{2_1}, \ldots, \omega_{2_s}$ the s successors of ω_1 in $\underline{\omega}^<$. Using the induction hypothesis, we have :

$$
\begin{aligned}
T^{\underline{\omega}^<} &= \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) T^{\underline{\omega}} \\
&= \sum_{\underline{\omega} \in \Omega^\bullet} \mathbf{sh}\left(\underset{\bullet\omega_1}{\underbrace{\omega_{2_1}.\underline{W}^{2_1<} \quad \omega_{2_2}.\underline{W}^{2_2<} \ \vdots \ \omega_{2_s}.\underline{W}^{2_s<}}}, \underline{\omega} \right) T^{\underline{\omega}} \\
&= \sum_{i=1}^{s} \sum_{\underline{W} \in \Omega^\bullet} \mathbf{sh}\left(\bullet\omega_{2_1}.\underline{W}^{2_1<} \vdots \bullet\underline{W}^{2_i<} \vdots \bullet\omega_{2_s}.\underline{W}^{2_s<}, \underline{W} \right) T^{\omega_1, \omega_{2_i}, \underline{W}} \\
&= T^{\underset{\bullet\omega_1}{\underset{|}{\underset{\omega_{2_1}}{\underbrace{\underline{W}^{1<} \quad \omega_{2_2}.\underline{W}^{2<} \ \vdots \ \omega_{2_s}.\underline{W}^{s<}}}}}} + T^{\underset{\bullet\omega_1}{\underset{|}{\underset{\omega_{2_2}}{\underbrace{\omega_{2_1}.\underline{W}^{1<} \quad \underline{W}^{2<} \ \vdots \ \omega_{2_s}.\underline{W}^{s<}}}}}} + \ldots + T^{\underset{\bullet\omega_1}{\underset{|}{\underset{\omega_{2_s}}{\underbrace{\omega_{2_1}.\underline{W}^{1<} \quad \omega_{2_2}.\underline{W}^{2<} \ \vdots \ \underline{W}^{s<}}}}}} \\
&= \sum_{i=1}^{s} \left(\frac{1}{\omega_1 - \omega_{2_i}} \prod_{j=1\ldots s, j\neq i} \frac{1}{\omega_{2_i} - \omega_{2_j}} \prod_{\omega_{j+} > \omega_{2_1}, \ldots, \omega_{2_s}} \frac{1}{(\omega_j - \omega_{j+})} \right) \\
&= \left(\sum_{i=1}^{s} \frac{1}{\omega_1 - \omega_{2_i}} \prod_{j=1\ldots s, j\neq i} \frac{1}{\omega_{2_i} - \omega_{2_j}} \right) \prod_{\omega_{j+} > \omega_{2_1}, \ldots, \omega_{2_s}} \frac{1}{(\omega_j - \omega_{j+})}.
\end{aligned}
$$

where the notation $\omega_{j+} > \omega_{2_1}, \ldots, \omega_{2_s}$ means that we perform the product on every nodes $\underline{\omega}_{j+}$ of $\underline{\omega}^<$ that are posterior to $\omega_{2_1}, \ldots, \omega_{2_s}$ for the arborescent order of $\underline{\omega}^<$. We have then to prove that

$$
\sum_{i=1}^{s} \frac{1}{\omega_1 - \omega_{2_i}} \prod_{j=1\ldots s, j\neq i} \frac{1}{\omega_{2_i} - \omega_{2_j}} = \prod_{i=1}^{s} \frac{1}{\omega_1 - \omega_{2_i}}.
$$

But

$$
\begin{aligned}
&\sum_{i=1}^{s} \frac{1}{\omega_1 - \omega_{2_i}} \prod_{\substack{j = 1 \ldots s \\ j \neq i}} \frac{1}{\omega_{2_i} - \omega_{2_j}} \\
=\ & \frac{\sum_{i=1}^{s} \prod_{\substack{j = 1 \ldots s \\ j \neq i}} \left(\omega_1 - \omega_{2_j}\right) \prod_{\substack{j, k = 1 \ldots s \\ k \neq i, k \neq j}} \left(\omega_{2_k} - \omega_{2_j}\right)}{\prod_{i=1\ldots s} \left(\omega_1 - \omega_{2_i}\right) \prod_{\substack{i, j = 1 \ldots s \\ i \neq j}} \left(\omega_{2_i} - \omega_{2_j}\right)}
\end{aligned}
$$

and

$$\sum_{i=1}^{s} \prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_1 - \omega_{2_j}\right) \prod_{\substack{j,k=1\dots s \\ k\neq i, k\neq j}} \left(\omega_{2_k} - \omega_{2_j}\right)$$

$$= \prod_{\substack{j,k=1\dots s \\ k\neq j}} \left(\omega_{2_k} - \omega_{2_j}\right) \sum_{i=1}^{s} \frac{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_1 - \omega_{2_j}\right)}{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_{2_i} - \omega_{2_j}\right)}.$$

So it remains to prove that

$$\sum_{i=1}^{s} \frac{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_1 - \omega_{2_j}\right)}{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_{2_i} - \omega_{2_j}\right)} = 1.$$

Let us consider the polynomial in z of degree $\leq s-1$:

$$P(z) = \sum_{i=1}^{s} \frac{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(z - \omega_{2_j}\right)}{\prod_{\substack{j=1\dots s \\ j\neq i}} \left(\omega_{2_i} - \omega_{2_j}\right)} - 1.$$

For any $k \in [\![1, s]\!]$, $P(\omega_k) = 0$. Thus, P has at least s roots, which is possible only if $P = 0$. □

Chapitre 7

Well behaved averages and derivations

Être, c'est être la somme de tout ce qu'on a été.
Victor Hugo - Choses vues

7.1 Averages

7.1.1 Introduction

We now have enough material to deal with averages. They will be given as families of weights indexed by paths avoiding the singularities of the considered dynamical system. Such families of weights must naturally satisfy different properties to deserve the name of average.

In particular, we need to take into account the fact that two resurgent functions may have two sets of singularities strictly included one into the other. Thus the weights defining an average must fulfill some relations of coherence called autocoherence relations.

We have also to explain how averages act on resurgent functions. The fact that averages respect the realness can easily be understood and defined directly on their weights. This is unfortunately not the case for the respect of the convolution product and of the at most-exponential growth.

To be able to define these properties, we will need to make the link between averages and **ALIEN** operators. We will explain that there is an **ALIEN** operator **rem** which connects an average **m** and the homogeneous components of the Stokes automorphism $\dot{\Delta}^+$. The operator of passage **rem** will be constructed using moulds and **ALIEN** calculus and to achieve this, we will need to re-define **ALIEN** operators with weights.

It will then be easy to define what is a well behaved average because we will translate on its lateral operator **rem** what should be its good properties. More precisely :

— **m** preserves the realness if and only if **rem** satisfies the **ALIEN** relation $\overline{\mathbf{rem}} = \dot{\Delta}^{+}\mathbf{rem}$.

— **m** preserves the convolution if and only if **rem** is a convolution automorphism of **ALIEN**..

— **m** preserves the at most-exponential growth if for any reductions **red**, the ordinary differential operator **red**(**rem**) preserves the analytical germs when it is the case of **red** $\left(\dot{\Delta}^{+}\right)$. Such an **ALIEN** operator is said to be analytic.

There will be a natural translation of these conditions in terms of moulds. We will consider the mould $M^{\bullet} =< \mathbf{rem}, \dot{\Delta}^{+} >^{\bullet}$ given by the components of **rem** on the basis constructed with composition of homogeneous components of the Stokes automorphism. Then

— The realness of **rem** is equivalent to the equality

$$< \mathbf{rem}, \dot{\Delta}^{+} >^{\bullet} \times \left(< \overline{\mathbf{rem}}, \dot{\Delta}^{+} >^{\bullet}\right)^{-1} = (1+I)^{\bullet}$$

— The convolution preservation of **m** is equivalent to the symmetrelity of the mould $< \mathbf{rem}, \dot{\Delta}^{+} >^{\bullet}$.

— The analyticity of **red** (**rem**) is implied by geometrical bounds for the mould $< \mathbf{rem}, \dot{\Delta}^{+} >^{\bullet}$ and more precisely of its contracted antiarborified $< \mathbf{rem}, \dot{\Delta}^{+} >^{\bullet\gg}$.

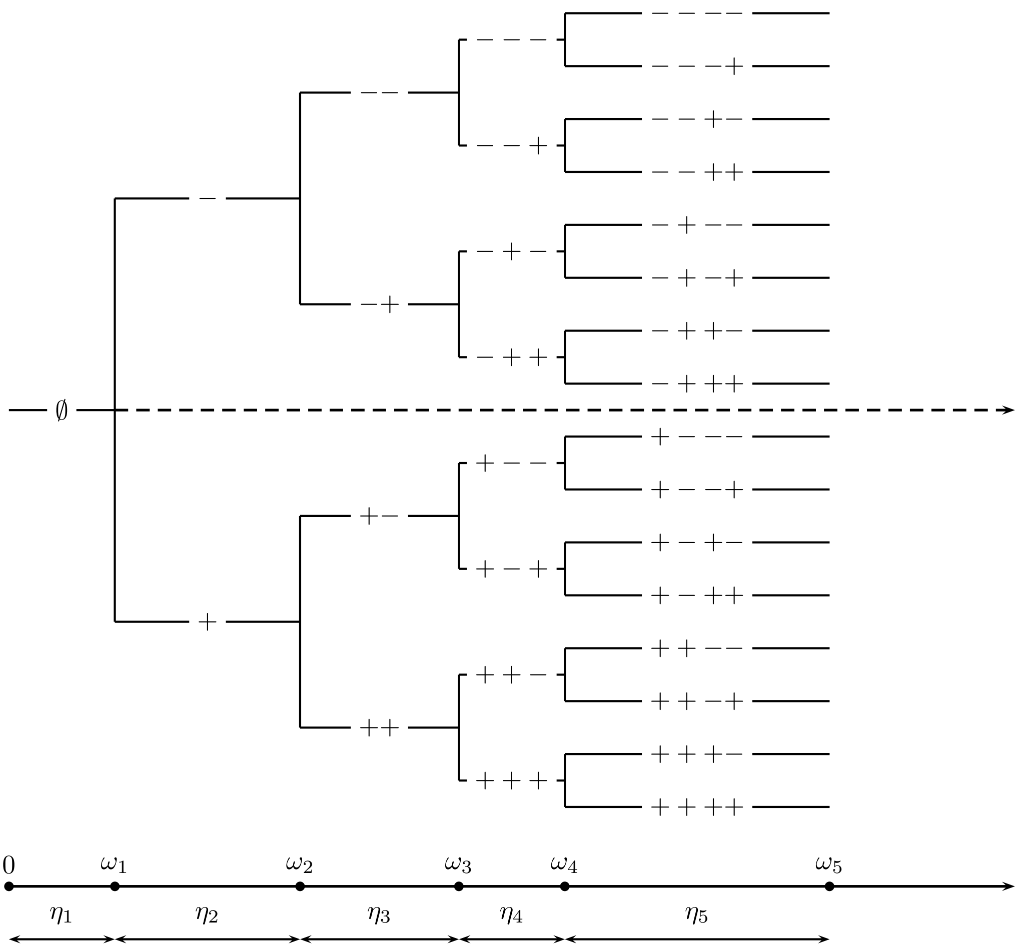
Assuming the existence of well behaved averages, we will apply the theory to the real summation of the conjugant of real saddle-nodes and the conjugant of tangent to identity real analytic germs in their simplest formal class.

In parallel to our construction of well behaved averages, we will define what must be well behaved **ALIEN** derivation. An **ALIEN** derivation is an **ALIEN** operator acting as a derivation on the space of resurgent functions. The first to be considered is the standard one, which is the infinitesimal generator of the Stokes automorphism. It particularly appears in the bridge equation and not all of its reductions are differential operators preserving $\mathbb{C}\{u\}$, namely it is not an *analytic* ***ALIEN*** *derivation*. This fact is totally related to the 1-Gevrey growth of Ecalle's invariants. In some problems, like the synthesis problem ([14]) or like the problem studied in chapter 9, it will be very convenient to have at one's disposal an **ALIEN** operator D such that :

1. D **respects the Leibniz rule** in order to be a derivation.
2. D **respects the realness :** As explained in section 3.4 page 63, considering a real dynamical system like those studied in chapter 3 and an **ALIEN** derivation D, one can construct the so called analytical invariants (A_m) associated to D. We want these coefficients A_m to be real when working with respecting realness derivation.
3. D **respects the analyticity :** the families of invariants associated to D must have a geometrical growth.

We finally end this chapter with an example of a well behaved averages family discovered by Jean Ecalle : the diffusion induced one. This family is linked to a family of well behaved **ALIEN** derivation that we will present too. We will propose a complete and self contained proof of the fact that these two families are well-behaved.

7.1.2 Averages



Let us start by fixing some notations. We consider $\Omega = \{\omega_1 < \ldots < \omega_r < \ldots\}$ an ordered semi-group of $\mathbb{R}_+^*$ or $\mathbb{R}_-^*$. We associate to Ω the set $\mathscr{I}$ of its increments $\mathscr{I} = \{0, \eta_1 = \omega_1, \eta_2 = \omega_2 - \omega_1, \ldots, \eta_r = \omega_r - \omega_{r-1}, ...\}$ and we finally consider the semi-group $\Omega^{\mathbf{inc}}$ generated by $\mathscr{I}$. We call it **the generated semi-group by increments associated to** Ω. The set Ω is clearly a sub-semi-group of $\Omega^{\mathbf{inc}}$.

An element $\zeta^{\epsilon_1,\ldots,\epsilon_n} \in \mathbb{R}_+//\Omega$ can be denoted

$$\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_n \\ \epsilon_1 & \ldots & \epsilon_n \end{pmatrix}}$$

to make clear the used increments $\eta_1, \ldots, \eta_n$.

The introduction of the thesis gives the way to follow to get a good definition of a **well behaved average m**.

For a given resurgent function $\hat{\phi} \in \widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}(\Omega)$:

— $\mathbf{m}\,\hat{\phi}$ must be real.
— $\mathbf{m}\,\hat{\phi}$ must be of at most-exponential growth if it is the case for the left and right analytic continuations of $\hat{\phi}$[1].
— $\mathbf{m}$ must respect the convolution product on $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}(\Omega)$

Moreover, in order to be able to perform the Laplace transform of $\mathbf{m}\hat{\phi}$ on $\mathbb{R}_+$ (or $\mathbb{R}_-$), we want $\mathbf{m}\hat{\phi}$ to be a uniform function on $\mathbb{R}_+//\Omega$.

An average $\mathbf{m}$ is given by its weights :

$$\mathbf{m} = \left\{ \mathbf{m}^{\begin{pmatrix} \eta_1 & \dots & \eta_r \\ \epsilon_1 & \dots & \epsilon_r \end{pmatrix}} \in \mathbb{C} \mid (\eta_1, \dots \eta_r) \in \mathscr{I}^r, (\epsilon_1, \dots, \epsilon_r) \in \{+,-\}^r, r \in \mathbf{N} \right\}.$$

Its action on the algebra of resurgent functions is given for $\zeta \in]\check{\eta}_n, \check{\eta}_{n+1}[\in \pi(\mathbb{R}_+//\Omega)$ by :

$$\left(\mathbf{m}\,\hat{\phi}\right)(\zeta) = \sum_{\substack{(\eta_1, \dots \eta_r) \in \mathscr{I}^r \\ (\epsilon_1, \dots, \epsilon_r) \in \{+,-\}^r}} \mathbf{m}^{\begin{pmatrix} \eta_1 & \dots & \eta_r \\ \epsilon_1 & \dots & \epsilon_r \end{pmatrix}} \hat{\phi}\left(\zeta^{\begin{pmatrix} \eta_1 & \dots & \eta_r \\ \epsilon_1 & \dots & \epsilon_r \end{pmatrix}}\right).$$

Let us remark that the function $\mathbf{m}\hat{\phi}$ is not necessarily a resurgent function because one can not every time realize it as the analytic continuation of an analytic germ at 0. This point motivates the introduction of two new spaces :

Definition 7.1.2.1

— *We call* $\boldsymbol{RAMIF}(\mathbb{R}_+//\Omega, int)$ *the space of locally integrable functions on* $\mathbb{R}_+//\Omega$.
— *We define* $\widehat{\boldsymbol{UNIF}}(\Omega)$ *as the linear sub-space of* $\boldsymbol{RAMIF}(\mathbb{R}_+//\Omega, int)$ *of uniform functions on* $\mathbb{R}_+//\Omega$ *i.e. :*

$$\hat{\phi} \in \widehat{\boldsymbol{UNIF}}(\Omega) \Longleftrightarrow \begin{cases} \hat{\phi} \in \boldsymbol{RAMIF}(\mathbb{R}_+//\Omega, int) \\ \forall \zeta^{\epsilon_1,\dots,\epsilon_n}, \zeta^{\tilde{\epsilon}_1,\dots,\tilde{\epsilon}_n} \in \mathbb{R}_+//\Omega, \quad \hat{\phi}\left(\zeta^{\epsilon_1,\dots,\epsilon_n}\right) = \hat{\phi}\left(\zeta^{\tilde{\epsilon}_1,\dots,\tilde{\epsilon}_n}\right) \end{cases}.$$

An average $\mathbf{m}$ is a linear operator from $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}$ into $\widehat{\mathbf{UNIF}}(\Omega)$.

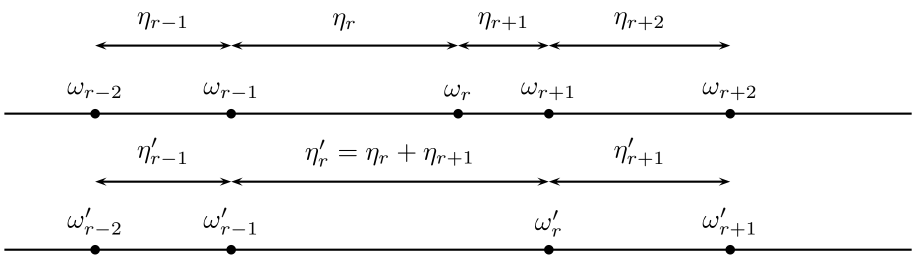
Let us observe that $\widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}(\Omega)$ is a sub-space of $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$. J. Ecalle has defined a convolution product on $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$ that extends

1. namely the analytic continuations of $\hat{\phi}$ along two half-lines on both sides of the singular direction

the convolution product on $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega)$. For the topology of uniform convergence on compact sets, the space $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega)$ is dense in $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$ and so, we can extend the action of $\mathbf{ALIEN}(\Omega)$ on $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$. We have not deeply studied these points and we invite the reader to consult the Annex C of [42] for more details.

Let us consider two sets of singularities Ω and Ω' in a such way that $\Omega' \subset \Omega$. Let us consider too the associated sets of increments $\mathscr{I}$ and $\mathscr{I}'$. We naturally get $\mathscr{I}' \subset \mathscr{I}$. Let us assume, for a given $r \in \mathbf{N}^*$, the point ω_r to be an element of Ω but not of Ω'. For a given resurgent function $\hat{\phi} \in \widehat{\mathbf{RESUR}}_{\mathbb{R}}(\Omega')$, as a consequence of the inclusion $\widehat{\mathbf{RESUR}}_{\mathbb{R}}(\Omega') \subset \widehat{\mathbf{RESUR}}_{\mathbb{R}}(\Omega)$, for $\zeta \in]\omega_r, \omega_{r+1}[$, we must have :

$$\begin{aligned}
\mathbf{m}\hat{\phi}(\zeta) &= \sum_{\substack{(\eta_1,\dots\eta_r) \in \mathscr{I}^r \\ (\epsilon_1,\dots,\epsilon_r)\in\{+,-\}^r}} \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}} \hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}}\right) \\
&= \sum_{\substack{(\eta_1,\dots\eta_{r-1}) \in \mathscr{I}^{r-1} \\ (\epsilon_1,\dots,\epsilon_{r-1})\in\{+,-\}^{r-1}}} \left(\mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & +\end{pmatrix}} + \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & -\end{pmatrix}}\right) \hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}}\right)
\end{aligned}$$



We want this quantity to be equal to :

$$\sum_{\substack{(\eta_1,\dots\eta_{r-1}) \in \left(\mathscr{I}'\right)^{r-1} \\ (\epsilon_1,\dots,\epsilon_{r-1})\in\{+,-\}^{r-1}}} \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1}\\ \epsilon_1 & \dots & \epsilon_{r-1}\end{pmatrix}} \hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1}\\ \epsilon_1 & \dots & \epsilon_{r-1}\end{pmatrix}}\right).$$

The average $\mathbf{m}$ must verify the relation (*autocoI*) :

$$\mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & +\end{pmatrix}} + \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_r\\ \epsilon_1 & \dots & -\end{pmatrix}} = \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1}\\ \epsilon_1 & \dots & \epsilon_{r-1}\end{pmatrix}}.$$

In the same way, if $\eta_n > \eta_r$ then for any $\zeta \in]\omega_n, \omega_{n+1}[$, we have :

$$\begin{aligned}
\mathbf{m}\hat{\phi}(\zeta) &= \sum_{\substack{(\eta_1,\dots\eta_n)\in\mathscr{I}^n\\(\epsilon_1,\dots,\epsilon_n)\in\{+,-\}^n}} \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_n\end{pmatrix}}\hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_n\end{pmatrix}}\right)\\
&= \sum_{(\epsilon_1,\dots,\epsilon_{r_1},\epsilon_{r+1},\dots,\epsilon_n)\in\{+,-\}^{n-1}} (\mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r & \eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & + & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}} +\\
&\quad \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r & \eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & - & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}})\hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r & \eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & + & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}}\right).
\end{aligned}$$

We want this quantity to be equal to :

$$\sum_{(\epsilon_1,\dots,\epsilon_{r-1},\epsilon_{r+1},\dots,\epsilon_n)\in\{+,-\}^{n-1}} \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r+\eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}} \times \hat{\phi}\left(\zeta^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r+\eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}}\right)$$

and $\mathbf{m}$ must verify the relation $(autocoII)$:

$$\mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r & \eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & + & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}} + \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r & \eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & - & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}} = \mathbf{m}^{\begin{pmatrix}\eta_1 & \dots & \eta_{r-1} & \eta_r+\eta_{r+1} & \dots & \eta_n\\ \epsilon_1 & \dots & \epsilon_{r-1} & \epsilon_{r+1} & \dots & \epsilon_n\end{pmatrix}}.$$

Definition 7.1.2.2 *An average $\boldsymbol{m}$ on $\widehat{\boldsymbol{RESUR}}^s_{\mathbb{R}}(\Omega)$ is given by its weights :*

$$\boldsymbol{m} = \left\{\boldsymbol{m}^{\begin{pmatrix}\omega_1 & \dots & \omega_r\\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}} \in \mathbb{C} \mid r\in\mathbf{N}, \epsilon_1,\dots,\epsilon_r\in\{+,-\}, \omega_1,\dots,\omega_r\in\Omega^{\boldsymbol{inc}}\right\}$$

which satisfy the autocoherence relations :

1. $\boldsymbol{m}^{\emptyset} = 1$
2. $(autocoI)$
3. $(autocoII)$

We denote $\boldsymbol{AVER}(\Omega)$ the linear space of averages on $\widehat{\boldsymbol{RESUR}}^s_{\mathbb{R}}(\Omega)$.

In the case of the examples concerned in chapter 3 and for the positive direction, since $\Omega = \mathbf{N}^*$, we have $\mathscr{I} = \{1,\dots,1,\dots\}$ and $\Omega^{\mathbf{inc}} = \mathbf{N}^*$. In this case and in order to simplify the notations, we will write $\mathbf{m}^{\epsilon_1,\dots,\epsilon_r}$ in place of $\mathbf{m}^{\begin{pmatrix}1 & \dots & 1\\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}}$. For this family of weights, the autocoherence relations become :

$$\begin{cases}\mathbf{m}^{\emptyset} & = 1\\ \sum_{\epsilon_n=\pm}\mathbf{m}^{\epsilon_1,\dots,\epsilon_n} & = \mathbf{m}^{\epsilon_1,\dots,\epsilon_{n-1}} \text{ for all } n\in\mathbf{N}^* \text{ and } \epsilon_1,\dots,\epsilon_{n-1}\in\{+,-\}^{n-1}\end{cases}$$

and the action of an average $\mathbf{m}$ on $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)$ becomes :

$$\forall \phi \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*), \quad \forall m \in \mathbf{N}^*, \quad \forall \zeta \in]m, m+1[,$$
$$(\mathbf{m}\phi)(\zeta) = \sum_{\epsilon_1,\ldots,\epsilon_m = \pm} \mathbf{m}^{\epsilon_1,\ldots,\epsilon_m} \phi\left(\zeta^{\epsilon_1,\ldots,\epsilon_m}\right).$$

We give now two first and fundamental examples of averages :

Example 7.1.2.1 *We define the lateral averages* ***mul*** *and* ***mur*** *with their weights by, for any* $\omega_1, \ldots, \omega_r \in \Omega^{\boldsymbol{inc}}$ *:*

$$\boldsymbol{mur}^{\begin{pmatrix} \omega_1 & \ldots & \omega_r \\ \epsilon_1 & \ldots & \epsilon_r \end{pmatrix}} = \begin{cases} 1 & \text{if } \epsilon_1 = \ldots = \epsilon_r = + \\ 0 & \text{otherwise} \end{cases}$$

$$\boldsymbol{mul}^{\begin{pmatrix} \omega_1 & \ldots & \omega_r \\ \epsilon_1 & \ldots & \epsilon_r \end{pmatrix}} = \begin{cases} 1 & \text{if } \epsilon_1 = \ldots = \epsilon_r = - \\ 0 & \text{otherwise} \end{cases}$$

The analytic continuation $\boldsymbol{mur}\hat{\phi}$ *corresponds to the analytic continuation of* $\hat{\phi}$ *along an half-line* $\mathcal{D}_\theta$ *with* $\theta < 0$ *small enough. The analytic continuation* $\boldsymbol{mul}\,\hat{\phi}$ *corresponds to the analytic continuation of* $\hat{\phi}$ *along an half-line* $\mathcal{D}_\theta$ *with* $\theta > 0$ *small enough.*

Remark 7.1.2.1 *Let us consider* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}^{s}_{\mathbb{R}}(\Omega)$ *and* $\boldsymbol{m} \in \boldsymbol{AVER}(\Omega)$ *then* $\boldsymbol{m}\,\hat{\phi} = 0$ *if and only if* $\hat{\phi} = 0$*. Indeed, if* $\zeta \in [0, \eta_1[$ *then :*

$$0 = \left(\boldsymbol{m}\,\hat{\phi}\right)(\zeta) = \hat{\phi}(\zeta)$$

and the result follows, $\hat{\phi}$ *is the germ of the null function.*

In order to get a well behaved average, the average $\mathbf{m}$

— must preserve the realness. It can be easily expressed in terms of weights. For any $n \in \mathbf{N}$, $\underline{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega^{\mathbf{inc}\bullet}$ and $(\epsilon_1, \ldots, \epsilon_n) \in \{+, -\}^n$, we must have :
$$\overline{\mathbf{m}^{\begin{pmatrix} \omega_1 & \ldots & \omega_n \\ \epsilon_1 & \ldots & \epsilon_n \end{pmatrix}}} = \mathbf{m}^{\begin{pmatrix} \omega_1 & \ldots & \omega_n \\ \overline{\epsilon_1} & \ldots & \overline{\epsilon_n} \end{pmatrix}}$$
where $\overline{+} = -$ and $\overline{-} = +$.

— must respect the convolution product. For any $\hat{\phi}, \hat{\psi} \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}$, we must have $\mathbf{m}\left(\hat{\phi} \star \hat{\psi}\right) = \left(\mathbf{m}\hat{\phi}\right) \star \left(\mathbf{m}\hat{\psi}\right)$. Take note that the first convolution product in this equality is this of $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}$ and the second one this of $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$.

— must respect the at most-exponential growth which is the most difficult point to define. We need to this end to connect **ALIEN** operators and averages. This is the object of the two next sections.

7.1.3 ALIEN operators given by weights

In order to connect averages and **ALIEN** operators, we need a definition of **ALIEN** operators in terms of weights.

Definition 7.1.3.1 *For a given monoid $\Omega = (\omega_1, \ldots, \omega_r, \ldots)$ of $\mathbb{R}_+^*$ or $\mathbb{R}_-^*$ and for the associated generated semi-group by increments $\Omega^{\boldsymbol{inc}}$, we define an $\boldsymbol{ALIEN}_w$ **operator*** $\boldsymbol{Op}$ *by its weights :*

$$\boldsymbol{Op} = \left\{ \boldsymbol{Op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_n \\ \epsilon_1 & \ldots & \epsilon_n \end{pmatrix}} \in \mathbb{C} \mid (\epsilon_1, \ldots, \epsilon_n) \in \bigcup_{n \in \mathbf{N}} \{+,-\}^n, (\omega_1, \ldots, \omega_n) \in \Omega^{\boldsymbol{inc}\bullet} \right\}$$

that satisfy the autocoherence relations :

1. $\forall n \in \mathbf{N}^*, \quad \forall r \in [\![1,n]\!], \quad \forall (\epsilon_1, \ldots, \epsilon_{r-1}, \epsilon_{r+1}, \ldots, \epsilon_n) \in \{+,-\}^n, \quad \forall (\omega_1, \ldots, \omega_n) \in \Omega^{\boldsymbol{inc}\bullet}$,

$$\sum_{\epsilon_r = \pm} \boldsymbol{Op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_{r-1} & \omega_r & \omega_{r+1} & \ldots & \omega_n \\ \epsilon_1 & \ldots & \epsilon_{r-1} & \epsilon_r & \epsilon_{r+1} & \ldots & \epsilon_n \end{pmatrix}} = \boldsymbol{Op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_{r-1} & \omega_r + \omega_{r+1} & \ldots & \omega_n \\ \epsilon_1 & \ldots & \epsilon_{r-1} & \epsilon_{r+1} & \ldots & \epsilon_n \end{pmatrix}}.$$

2. $\forall n \in \mathbf{N}^*, \quad \forall (\epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^n, \quad \forall (\omega_1, \ldots, \omega_n) \in \Omega^{\boldsymbol{inc}\bullet}$,

$$\sum_{\epsilon_n = \pm} \boldsymbol{Op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_{n-1} & \omega_n \\ \epsilon_1 & \ldots & \epsilon_{n-1} & \epsilon_n \end{pmatrix}} = 0.$$

We will denote by $\boldsymbol{ALIEN}_w(\Omega)$ the space of all $\boldsymbol{ALIEN}_w$ operators.

Definition 7.1.3.2 *For a monoid $\Omega = \{\omega_1, \omega_2, \ldots\} \subset \mathbb{R}_+^*$ given in increasing order and the associated set of increments $\mathscr{I} = (\eta_1 = \omega_1, \eta_2 = \omega_2 - \omega_1, \ldots,)$ We define the homogeneous component $\boldsymbol{Op}_{\omega_m}$ of order $\omega_m = \check{\eta}_m \in \Omega$ of an $\boldsymbol{ALIEN}_w$ operator $\boldsymbol{Op}$ by its weights : $\forall n \in \mathbf{N}, \quad \forall (\epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^n, \quad \forall (\eta_1, \ldots, \eta_n) \in \Omega^\bullet$,*

$$\boldsymbol{Op}_{\omega_m}^{\begin{pmatrix} \eta_1 & \ldots & \eta_n \\ \epsilon_1 & \ldots & \epsilon_n \end{pmatrix}} = \begin{cases} \boldsymbol{Op}^{\begin{pmatrix} \eta_1 & \ldots & \eta_m \\ \epsilon_1 & \ldots & \epsilon_m \end{pmatrix}} & \textit{if } n = m \\ 0 & \textit{otherwise} \end{cases}.$$

We will prove in this section that $\mathbf{ALIEN}_w(\Omega) = \mathbf{ALIEN}(\Omega)$, see Corollary 7.1.4. Indeed, when we know the weights of an $\mathbf{ALIEN}_w$ operator **op**, we know the mould $< \mathbf{op}, \Delta >^\bullet$. Conversely, the knowledge of this mould induces the one of the weights of **op**.

Definition 7.1.3.3

— *We define the proper action of an homogeneous operator $\boldsymbol{Op}_{\omega_m} \in \boldsymbol{ALIEN}_w(\Omega)$ of order $\omega_m = \eta_1 + \ldots + \eta_m \in \Omega$ on $\widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s(\Omega)$ by :*

$\forall \hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{C}}^s, \quad \forall \zeta$ *small enough* ,

$$\left(\boldsymbol{Op}_{\omega_m} \phi\right)(\zeta) = \sum_{\epsilon_1, \ldots, \epsilon_m = \pm} \boldsymbol{Op}^{\begin{pmatrix} \eta_1 & \ldots & \eta_m \\ \epsilon_1 & \ldots & \epsilon_m \end{pmatrix}} \phi\left((\omega_m + \zeta)^{\epsilon_1, \ldots, \epsilon_m}\right).$$

We recall that $(\omega_m+\zeta)^{\epsilon_1,\ldots,\epsilon_m}$ *represents a rectifiable oriented path from* 0 *to* $\omega_m+\zeta$ *in the complex plane that avoids the singular points* η_1,$\eta_1+\eta_2$,..., $\eta_1+\ldots+\eta_m=\omega_m$ *from the right (if* $\epsilon_i=+$*) or from the left (if* $\epsilon_i=-$*).*

— *In the same way than in section 1.2.2, we define the stationary action of* $\boldsymbol{op}_{\omega_m}\in\boldsymbol{ALIEN}_w(\Omega)$ *for all* $\hat{\phi}\in\widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$ *and* $\xi^{\begin{pmatrix}\eta_1 & \cdots & \eta_n\\ \epsilon_1 & \cdots & \epsilon_n\end{pmatrix}}\in\mathbb{R}_+//\Omega^2$ *by* $\boldsymbol{op}_{\omega_m}:=\delta_{\omega_m}\star\boldsymbol{Op}_{\omega_m}$ *which means that*

$$\left(\boldsymbol{op}_{\omega_m}\hat{\phi}\right)\left(\xi^{\begin{pmatrix}\eta_1 & \cdots & \eta_n\\ \epsilon_1 & \cdots & \epsilon_n\end{pmatrix}}\right)=$$
$$\begin{cases}\sum_{\sigma_1,\ldots,\sigma_m=\pm}\boldsymbol{Op}^{\begin{pmatrix}\eta_1 & \cdots & \eta_m\\ \sigma_1 & \cdots & \sigma_m\end{pmatrix}}\hat{\phi}\left(\xi^{\begin{pmatrix}\eta_1 & \cdots & \eta_m & \eta_{m+1} & \cdots & \eta_n\\ \sigma_1 & \cdots & \sigma_m & \epsilon_1 & \cdots & \epsilon_{n-m}\end{pmatrix}}\right) & \text{if } n\geq m\\ 0 & \text{if } n<m\end{cases}$$

To distinguish if an $\boldsymbol{ALIEN}_w$ *operator acts stationary or properly on* $\widehat{\boldsymbol{RESUR}}^s_{\mathbb{C}}(\Omega)$*, we denote it, following Ecalle, with a lowercase (**op**) if it acts stationary and with an uppercase (**Op**) if it acts properly.*

In the case of the saddle-node and for the positive direction, as long as $\Omega=\mathbf{N}^*$, we get $\Omega^{\mathbf{inc}}=\mathbf{N}^*$. To simplify the notations, we will write $\mathbf{Op}^{\epsilon_1,\ldots,\epsilon_r}$ in place of $\mathbf{Op}^{\begin{pmatrix}1 & \cdots & 1\\ \epsilon_1 & \cdots & \epsilon_n\end{pmatrix}}$. For this family of weights, the autocoherence relations become :

$$\forall n\in\mathbf{N}^*,\forall(\epsilon_1,\ldots,\epsilon_{n-1})\in\{+,-\}^{n-1}\sum_{\epsilon_n=\pm}\mathbf{Op}^{\epsilon_1,\ldots,\epsilon_n}=0$$

and the proper action of an homogeneous operator $\mathbf{Op}_m\in\mathbf{ALIEN}(\mathbf{N}^*)$ on $\mathbf{RESUR}(\mathbf{N}^*)$ becomes :

$$\forall\hat{\phi}\in\mathbf{RESUR}(\mathbf{N}^*),\quad\forall m\in\mathbf{N}^*,\quad\forall\zeta\in]m,m+1[,$$
$$\left(\mathbf{Op}_m\hat{\phi}\right)(\zeta)=\sum_{\epsilon_1,\ldots,\epsilon_m=\pm}\mathbf{Op}^{\epsilon_1,\ldots,\epsilon_m}\hat{\phi}\left((m+\zeta)^{\epsilon_1,\ldots,\epsilon_m}\right)$$

Example 7.1.3.1 *We define the Alien operators **rul**, **lur** with their weights :*

$$\boldsymbol{lur}^{\emptyset}=1\text{ and }\boldsymbol{lur}^{\begin{pmatrix}\omega_1 & \cdots & \omega_n\\ \epsilon_1 & \cdots & \epsilon_n\end{pmatrix}}=\begin{cases}\epsilon_n & \text{if }\epsilon_1=\ldots=\epsilon_n=+\\ 0 & \text{otherwise}\end{cases}$$

$$\boldsymbol{rul}^{\emptyset}=1\text{ and }\boldsymbol{rul}^{\begin{pmatrix}\omega_1 & \cdots & \omega_n\\ \epsilon_1 & \cdots & \epsilon_n\end{pmatrix}}=\begin{cases}1 & \text{if }\epsilon_1=\ldots=\epsilon_n=-\\ 0 & \text{otherwise}\end{cases}$$

We evidently recognize the operators Δ^+ *(**lur**) and* Δ^- *(**rul**).*

2. We make clear the used increments in the writing of the path.

Lemma 7.1.1 *For any* $\boldsymbol{ALIEN}_w$ *operator* $\boldsymbol{Op}$ *and any* $m_1,\ldots,m_r \in \Omega^{\boldsymbol{inc}}$*, we have*

$$\boldsymbol{Op}\begin{pmatrix} m_1 & \ldots & m_r \\ \epsilon_1 & \ldots & \epsilon_r \end{pmatrix} = \sum_{\underline{\epsilon}^1=\pm,\ldots,\underline{\epsilon}^r=\pm} \boldsymbol{Op}\begin{pmatrix} \underline{\omega}^1 & \omega_1 & \underline{\omega}^2 & \omega_2 & \ldots & \underline{\omega}^r & \omega_r \\ \underline{\epsilon}^1 & \epsilon_1 & \underline{\epsilon}^2 & \epsilon_2 & \ldots & \underline{\epsilon}^r & \epsilon_r \end{pmatrix}$$

where for all $i \in [\![1,r]\!]$, $m_i = \|\underline{\omega}^i\| + \omega_i$, $\underline{\omega}^i \in \Omega^{\boldsymbol{inc}\bullet}$, $\omega_i \in \Omega^{\boldsymbol{inc}}$, $\underline{\epsilon}^i \in \{\pm\}^{l(\underline{\omega}^i)}$, $\epsilon_i \in \{\pm\}$.

Proof We make use of the autocoherence relations, if $\underline{\omega}^r = \left(\underline{\omega}^{r\prime}, \omega\right)$ and $\underline{\epsilon}^r = \left(\underline{\epsilon}^{r\prime}, \epsilon\right)$:

$$\begin{aligned}
&\sum_{\underline{\epsilon}^1=\pm,\ldots,\underline{\epsilon}^r=\pm} \mathbf{Op}\begin{pmatrix} \underline{\omega}^1 & \omega_1 & \underline{\omega}^2 & \omega_2 & \ldots & \underline{\omega}^r & \omega_r \\ \underline{\epsilon}^1 & \epsilon_1 & \underline{\epsilon}^2 & \epsilon_2 & \ldots & \underline{\epsilon}^r & \epsilon_r \end{pmatrix} \\
&= \sum_{\epsilon=\pm} \sum_{\underline{\epsilon}^1=\pm,\ldots,\underline{\epsilon}'^r=\pm} \mathbf{Op}\begin{pmatrix} \underline{\omega}^1 & \omega_1 & \underline{\omega}^2 & \omega_2 & \ldots & \underline{\omega}^{r\prime} & \omega & \omega_r \\ \underline{\epsilon}^1 & \epsilon_1 & \underline{\epsilon}^2 & \epsilon_2 & \ldots & \underline{\epsilon}^{r\prime} & \epsilon & \epsilon_r \end{pmatrix} \\
&= \sum_{\underline{\epsilon}^1=\pm,\ldots,\underline{\epsilon}'^r=\pm} \mathbf{Op}\begin{pmatrix} \underline{\omega}^1 & \omega_1 & \underline{\omega}^2 & \omega_2 & \ldots & \underline{\omega}^{r\prime} & \omega+\omega_r \\ \underline{\epsilon}^1 & \epsilon_1 & \underline{\epsilon}^2 & \epsilon_2 & \ldots & \underline{\epsilon}^{r\prime} & \epsilon_r \end{pmatrix}
\end{aligned}$$

and the proof follows by induction. □

Lemma 7.1.2 *For any* $\underline{m} = (m_1,\ldots,m_r) \in \Omega^\bullet$ *and* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}^s_{\mathbb{R}}(\Omega)$*, we have :*

$$\left(\Delta^+_{\underline{m}}\hat{\phi}\right)(\zeta) =$$

$$\sum_{\epsilon_1=\pm,\ldots,\epsilon_r=\pm} \epsilon_1\ldots\epsilon_r\hat{\phi}\left((\zeta+m)^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} & \ldots & \eta_{i_{r-1}} & \eta_{i_{r-1}+1} & \ldots & \eta_{i_r-1} & \eta_{i_r} \\ - & \ldots & - & \epsilon_1 & \ldots & \epsilon_{r-1} & - & \ldots & - & \epsilon_r \end{pmatrix}}\right)$$

where $m = \|\underline{m}\|$ *and where for all* $i \in [\![1,r]\!]$, $m_i = \check{\eta}_i$ *(Be careful that the operator* Δ^+ *is not pointed here, we are working with the proper action).*

Proof Using the autocoherence relations, we compute :

$$\begin{aligned}
\left(\Delta^+_{m_1}\hat{\phi}\right)(\zeta) &= \sum_{\epsilon_1=\pm,\ldots,\epsilon_{i_1-1}=\pm,\epsilon_{i_1}=\pm} \mathbf{Lur}^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} \\ \epsilon_1 & \ldots & \epsilon_{i_1-1} & \epsilon_{i_1} \end{pmatrix}} \hat{\phi}\left((\zeta+m_1)^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} \\ \epsilon_1 & \ldots & \epsilon_{i_1-1} & \epsilon_{i_1} \end{pmatrix}}\right) \\
&= \sum_{\epsilon_1=\pm} \epsilon_1\hat{\phi}\left((\zeta+m_1)^{\begin{pmatrix} \underline{\omega}^1 & \omega_1 \\ (-)^{m_1-1} & \epsilon_1 \end{pmatrix}}\right)
\end{aligned}$$

and the result follows by induction. □

Theorem 7.1.3 *If* $\boldsymbol{op} = \sum M^\bullet \Delta^+_\bullet \in \boldsymbol{ALIEN}(\Omega)$ *then the weights of* $\boldsymbol{op}$ *can be computed with one of the two next relations and with the autocoherence relations I and II. For any* $\underline{\omega} = (w_1, \ldots, w_r) \in \Omega^\bullet$

$$< \boldsymbol{op}, \dot{\Delta}^+ >^{\underline{\omega}} = (-1)^r \, \boldsymbol{op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_r \\ - & \ldots & - \end{pmatrix}}$$

$$< \boldsymbol{op}, \dot{\Delta}^- >^{\underline{\omega}} = (-1)^r \, \boldsymbol{op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_r \\ + & \ldots & + \end{pmatrix}}.$$

Proof Let us consider $m \in \Omega$ and $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ such that $\|\underline{\omega}\| = m$. We consider too

$$0 < \eta_1 < \ldots < \eta_{i_1} < \eta_{i_1+1} < \ldots < \eta_{i_2} < \ldots < \eta_{i_r}$$

the $i_1 + i_2 + \ldots + i_r$ first increments of $\mathscr{I}$ where $\omega_1 = \eta_1 + \ldots + \eta_{i_1} = \check{\eta}_{i_1}$, $\omega_1 + \omega_2 = \eta_1 + \ldots + \eta_{i_1} + \eta_{i_1+1} + \ldots + \eta_{i_2} = \check{\eta}_{i_2}$, ..., $\check{m}_r = \check{\eta}_{i_r}$. We consider too the locally integrable functions $\phi_{\underline{\omega}}$ on $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$ defined on $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$ by :

— Above the $i_1 - 1$ first increments, we set for each $i \in [\![1, i_1 - 1]\!]$ and each $\epsilon_1, \ldots, \epsilon_i \in \{\pm\}$:

$$\phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \eta_2 & \ldots & \eta_i \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_i \end{pmatrix}}\right) = 1$$

— Above η_{i_1}, we set for any $\epsilon_1, \ldots, \epsilon_{i_1-1} \in \{\pm\}$,

$$\phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \eta_2 & \ldots & \eta_{i_1-1} & \eta_{i_1} \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_{i_1-1} & + \end{pmatrix}}\right) = 0 \text{ and } \phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \eta_2 & \ldots & \eta_{i_1-1} & \eta_{i_1} \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_{i_1-1} & - \end{pmatrix}}\right) = 1$$

— We then assume that $\phi_{\underline{\omega}}$ is null on the whole part of $\mathbf{RAMIF}(\mathbb{R}_+//\Omega, \mathrm{int})$ that does not contain a branch beginning with the address

$$\begin{pmatrix} \eta_1 & \eta_2 & \ldots & \eta_{i_1-1} & \eta_{i_1} \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_{i_1-1} & - \end{pmatrix}$$

with $\epsilon_1, \ldots, \epsilon_{i_1-1} \in \{\pm\}$.

— We then define $\phi_{\underline{\omega}}$ on the $i_1 + i_2 - 1$ first increments, we set for each $\epsilon_1, \ldots, \epsilon_{i_1-1} \in \{\pm\}$, for each $i \in [\![i_1 + 1, i_1 + i_2 - 1]\!]$ and each $\epsilon_{i_1+1}, \ldots, \epsilon_i \in \{\pm\}$:

$$\phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} & \eta_{i_1+1} & \ldots & \eta_{i-1} \\ \epsilon_1 & \ldots & \epsilon_{i_1-1} & - & \epsilon_{i_1+1} & \ldots & \epsilon_{i-1} \end{pmatrix}}\right) = 1.$$

— Above $\eta_{i_1+i_2}$, we set for any $\epsilon_1, \ldots, \epsilon_{i_1-1} \in \{\pm\}$ and any $\epsilon_{i_1+1}, \ldots, \epsilon_{i_1+i_2-1} \in \{\pm\}$

$$\phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} & \eta_{i_1+1} & \ldots & \eta_{i_2-1} & \eta_{i_2} \\ \epsilon_1 & \ldots & \epsilon_{i_1-1} & - & \epsilon_{i_1+1} & \ldots & \epsilon_{i_2-1} & + \end{pmatrix}}\right) = 0$$

$$\text{and } \phi_{\underline{\omega}}\left(\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_{i_1-1} & \eta_{i_1} & \eta_{i_1+1} & \ldots & \eta_{i_2-1} & \eta_{i_2} \\ \epsilon_1 & \ldots & \epsilon_{i_1-1} & - & \epsilon_{i_1+1} & \ldots & \epsilon_{i_2-1} & - \end{pmatrix}}\right) = 1.$$

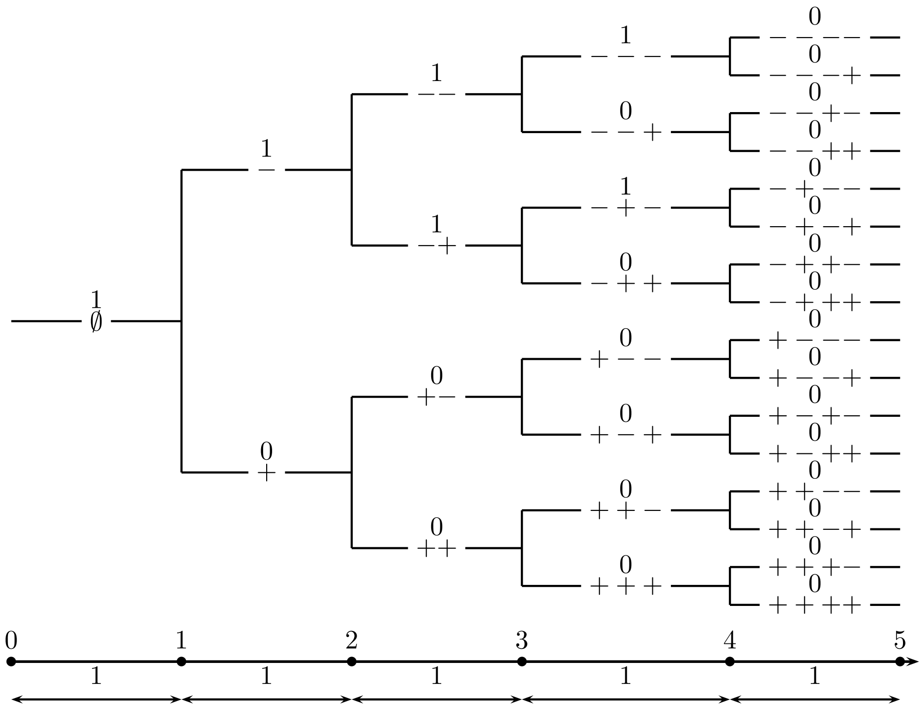
— We then assume that $\phi_{\underline{\omega}}$ is null on the whole part of **RAMIF** $(\mathbb{R}_+//\Omega, \mathrm{int})$ that does not contain a branch which begins with the address

$$\begin{pmatrix} \eta_1 & \dots & \eta_{i_1-1} & \eta_{i_1} & \eta_{i_1+1} & \dots & \eta_{i_2-1} & \eta_{i_2} \\ \epsilon_1 & \dots & \epsilon_{i_1-1} & - & \epsilon_{i_1+1} & \dots & \epsilon_{i_2-1} & - \end{pmatrix}$$

with $\epsilon_1, \dots, \epsilon_{i_1-1}, \epsilon_{i_1+1}, \dots, \epsilon_{i_2-1} \in \{\pm\}$.

— we proceed so for each increments until the increment i_r.

Here, an example with $\underline{\omega} = (1, 2)$:



We then easily verify that for ζ small enough $\left(\Delta^+_{\underline{\omega}}\left(\phi_{\underline{\omega}}\right)\right)(\zeta) = (-1)^r$. If $\underline{\upsilon} = (\upsilon_1, \dots, \upsilon_s) \in \Omega^\bullet$ is another sequence not equal to $\underline{m}$ then we easily verify that

$$\Delta^+_{\underline{\upsilon}}\left(\phi_{\underline{\omega}}\right) = 0.$$

Indeed, since $\underline{m} \neq \underline{n}$ then :

— If $\upsilon_1 < \omega_1$, we have

$$\left(\Delta^+_{\upsilon_1}\phi_{\underline{\omega}}\right)(\zeta) = \phi_{\underline{\omega}}\left((\upsilon_1+\zeta)^{(+)^{\upsilon_1-1}+}\right) - \phi_{\underline{\omega}}\left((\upsilon_1+\zeta)^{(+)^{\upsilon_1-1}-}\right) = 1-1 = 0$$

— If $\upsilon_1 > \omega_1$, we have

$$\left(\Delta^+_{\upsilon_1}\phi_{\underline{\omega}}\right)(\zeta) = \phi_{\underline{\omega}}\left((n_1+\zeta)^{(+)^{n_1-1}+}\right) - \phi_{\underline{\omega}}\left((\upsilon_1+\zeta)^{(+)^{\upsilon_1-1}-}\right) = 0-0 = 0$$

— If $\upsilon_1 = \omega_1$, then we consider the first index i in a such way that $\upsilon_i \neq \omega_i$ and we follow the previous proof.

If $\mathbf{op} = \sum M^\bullet \dot{\Delta}^+_\bullet \in \mathbf{ALIEN}(\Omega)$ we have :

$$\begin{aligned}\left(\mathbf{Op}_m \hat{\phi}_{\underline{\omega}}\right)(\zeta) &= \sum_{\underline{\omega}\in\mathscr{I}^\bullet, \|\underline{\omega}\|=m} M^{\underline{\omega}}\left(\Delta^+_{\underline{\omega}}\hat{\phi}_{\underline{\omega}}\right)(\zeta) \\ &= (-1)^r M^{\underline{\omega}}\end{aligned}$$

We assume that **op** $\in \mathbf{ALIEN}_w(\Omega)$. Then we have too :

$$\begin{aligned}
&\left(\mathbf{Op}_m \hat{\phi}_{\underline{\omega}}\right)(\zeta) \\
= & \sum_{\epsilon_i = \pm; i \in \{1,\ldots,i_r\}} \mathbf{Op}^{\begin{pmatrix} \eta_1 & \cdots & \eta_{i_1-1} & \eta_{i_1} & \cdots & \eta_{i_{r-1}+1} & \cdots & \eta_{i_r-1} & \eta_{i_r} \\ \epsilon_1 & \cdots & \epsilon_{i_1-1} & \epsilon_{i_1} & \cdots & \epsilon_{i_{r-1}+1} & \cdots & \epsilon_{i_r-1} & \epsilon_{i_r} \end{pmatrix}} \times \\
& \hat{\phi}_{\underline{\omega}}\left((\zeta+m)^{\begin{pmatrix} \eta_1 & \cdots & \eta_{i_1-1} & \eta_{i_1} & \cdots & \eta_{i_{r-1}+1} & \cdots & \eta_{i_r-1} & \eta_{i_r} \\ \epsilon_1 & \cdots & \epsilon_{i_1-1} & \epsilon_{i_1} & \cdots & \epsilon_{i_{r-1}+1} & \cdots & \epsilon_{i_r-1} & \epsilon_{i_r} \end{pmatrix}}\right) \\
= & \sum_{\substack{\epsilon_i = \pm;\, i \in \{1,\ldots,i_r\} \setminus \{i_1, i_2, \ldots, i_r\} \\ \epsilon_{i_1} = \ldots = \epsilon_{i_r} = -}} \mathbf{Op}^{\begin{pmatrix} \eta_1 & \cdots & \eta_{i_1-1} & \eta_{i_1} & \cdots & \eta_{i_{r-1}+1} & \cdots & \eta_{i_r-1} & \eta_{i_r} \\ \epsilon_1 & \cdots & \epsilon_{i_1-1} & - & \cdots & \epsilon_{i_{r-1}+1} & \cdots & \epsilon_{i_r-1} & - \end{pmatrix}} \times \\
& \hat{\phi}_{\underline{\omega}}\left((\zeta+m)^{\begin{pmatrix} \eta_1 & \cdots & \eta_{i_1-1} & \eta_{i_1} & \cdots & \eta_{i_{r-1}+1} & \cdots & \eta_{i_r-1} & \eta_{i_r} \\ \epsilon_1 & \cdots & \epsilon_{i_1-1} & \epsilon_{i_1} & \cdots & \epsilon_{i_{r-1}+1} & \cdots & \epsilon_{i_r-1} & \epsilon_{i_r} \end{pmatrix}}\right)
\end{aligned}$$

Thus using Lemma 7.1.1, we obtain for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ such that $\|\underline{\omega}\| = m$:

$$\begin{aligned}
(-1)^r M^{\underline{\omega}} &= \mathbf{Op}^{\begin{pmatrix} \eta_1 & \cdots & \eta_{i_1-1} & \eta_{i_1} & \cdots & \eta_{i_{r-1}+1} & \cdots & \eta_{i_r-1} & \eta_{i_r} \\ \epsilon_1 & \cdots & \epsilon_{i_1-1} & - & \cdots & \epsilon_{i_{r-1}+1} & \cdots & \epsilon_{i_r-1} & - \end{pmatrix}} \\
&= \mathbf{Op}^{\begin{pmatrix} \omega_1 & \cdots & \omega_r \\ - & \cdots & - \end{pmatrix}}
\end{aligned}$$

where for all $k \in [\![1, r]\!]$, $\omega_k = \check{\eta}_{i_k}$. The proof for the second relation is identical.

□

Corollary 7.1.4 *One has*

$$\boldsymbol{ALIEN}_w(\Omega) = \boldsymbol{ALIEN}(\Omega)\,.$$

Proof Using Theorem 7.1.3, we know the components of an $\mathbf{ALIEN}_w$ operator in the basis given by the comould associated to the Stokes automorphism and so it is an **ALIEN** operator. Conversely, using one times more Theorem 7.1.3 and autocoherence relations, knowing the components of an **ALIEN** operator in the previous basis permits to construct its family of weights. □

7.1.4 Post-composition of an operator by an average

We have now in position to connect averages and **ALIEN** operators. More precisely, if we consider two averages $\mathbf{m}, \mathbf{m}' \in \mathbf{AVER}$, we look for a stationary **ALIEN** operator **op** such that the following diagram commutes :

$$\begin{array}{ccc} \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega) & \xrightarrow{\mathbf{op}} & \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega)\,[[\delta]] \\ & \searrow^{\mathbf{m}'} & \downarrow \mathbf{m} \\ & & \mathbf{UNIF} \end{array}$$

We assume that such an operator **op** exists. For a monoid $\Omega = (\omega_1, \omega_2, \ldots)$ given in increasing order and m in $\mathbf{N}^*$, we introduce the monoid $\Omega' = \{\check{\omega}_m, \omega_{m+1}, \omega_{m+2}, \ldots\}$. The set $\mathscr{I}$ of increments associated to Ω is given by $\{\eta_1, \eta_2, \ldots\}$ and this $\mathscr{I}'$ associated to Ω' is given by $\{\check{\eta}_m, \eta_{m+1}, \ldots\}$. We recall that if $\hat{\phi} \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega)$ then $\mathbf{op}_{\omega_m}\hat{\phi} \in \delta_{\omega_m} \star \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega) = \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\Omega')$. This is why for any $\omega_n \in \Omega$ such that $\omega_n \geq \omega_m$ and for any $\zeta \in]\omega_n, \omega_{n+1}[$, we must have :

$$\begin{aligned} & \left(\left(\mathbf{mop}_{\omega_m}\right)\left(\hat{\phi}\right)\right)(\zeta) \\ = & \sum_{\mu_{m+1}=\pm,\ldots,\mu_n=\pm} \mathbf{m}^{\begin{pmatrix} \eta_{m+1} & \ldots & \eta_n \\ \mu_{m+1} & \ldots & \mu_n \end{pmatrix}} \sum_{\epsilon_1,\ldots,\epsilon_m=\pm} \mathbf{op}^{\begin{pmatrix} \eta_1 & \ldots & \eta_m \\ \epsilon_1 & \ldots & \epsilon_m \end{pmatrix}} \times \\ & \hat{\phi}\left(\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_m & \eta_{m+1} & \ldots & \eta_n \\ \epsilon_1 & \ldots & \epsilon_m & \mu_{m+1} & \ldots & \mu_n \end{pmatrix}}\right) \\ = & \sum_{\epsilon_1=\pm,\ldots,\epsilon_m=\pm,\epsilon_{m+1}=\pm,\ldots,\epsilon_n=\pm} \mathbf{m}^{\begin{pmatrix} \eta_{m+1} & \ldots & \eta_n \\ \epsilon_{m+1} & \ldots & \epsilon_n \end{pmatrix}} \mathbf{op}^{\begin{pmatrix} \eta_1 & \ldots & \eta_m \\ \epsilon_1 & \ldots & \epsilon_m \end{pmatrix}} \times \\ & \hat{\phi}\left(\zeta^{\begin{pmatrix} \eta_1 & \ldots & \eta_m & \eta_{m+1} & \ldots & \eta_n \\ \epsilon_1 & \ldots & \epsilon_m & \epsilon_{m+1}1 & \ldots & \epsilon_n \end{pmatrix}}\right). \end{aligned}$$

It then comes :

$$\begin{aligned} & \left(\left(\mathbf{mop}\right)\left(\hat{\phi}\right)\right)(\zeta) \\ = & \sum_{m=0}^{n} \left(\left(\mathbf{mop}_m\right)\left(\hat{\phi}\right)\right)(\zeta) \\ = & \sum_{m=0}^{n} \sum_{\epsilon_1=\pm,\ldots,\epsilon_m=\pm,\epsilon_{m+1}=\pm,\ldots,\epsilon_n=\pm} \mathbf{m}^{\begin{pmatrix} \omega_{m+1} & \ldots & \omega_n \\ \epsilon_{m+1} & \ldots & \epsilon_n \end{pmatrix}} \mathbf{op}^{\begin{pmatrix} \omega_1 & \ldots & \omega_m \\ \epsilon_1 & \ldots & \epsilon_m \end{pmatrix}} \times \\ & \hat{\phi}\left(\zeta^{\begin{pmatrix} \omega_1 & \ldots & \omega_m & \omega_{m+1} & \ldots & \omega_n \\ \epsilon_1 & \ldots & \epsilon_m & \eta_1 & \ldots & \eta_{n-m} \end{pmatrix}}\right). \end{aligned}$$

This equality suggests us to define the post-composition of a stationary operator by an average as following :

Definition 7.1.4.1 *Let* $\boldsymbol{m} \in \boldsymbol{AVER}$ *and* $\boldsymbol{op} \in \boldsymbol{ALIEN}$. *We define the average*

$\boldsymbol{m.op}$ *with the relations :* $(\boldsymbol{m.op})^{\emptyset} = 1$ *and* $\forall \underline{\epsilon} \in \bigcup_{n\in\mathbf{N}} \{+,-\}^n$, $\forall \underline{\eta} \in \mathscr{I}^\bullet$

$$(\boldsymbol{m.op})^{\binom{\underline{\eta}}{\underline{\epsilon}}} = \sum_{\underline{\epsilon}^1.\underline{\epsilon}^2=\underline{\epsilon},\underline{\eta}^1.\underline{\eta}^2=\underline{\eta}} \boldsymbol{op}^{\binom{\underline{\eta}^1}{\underline{\epsilon}^1}} \boldsymbol{m}^{\binom{\underline{\eta}^2}{\underline{\epsilon}^2}}$$

Theorem 7.1.5 *For a given average* $\mathbf{m} \in \boldsymbol{AVER}$*, there exists a unique operator denoted* $\mathbf{lem} \in \boldsymbol{ALIEN}$ *and called* ***left lateral mould associated to m*** *which satisfies the equality* $\mathbf{m} = \mathbf{mul.lem}$*. It is given by :*

$$\forall (\omega_1, \ldots, \omega_r) \in \Omega^\bullet, \quad \boldsymbol{lem}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ - & \ldots & -\end{pmatrix}} = \boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ - & \ldots & -\end{pmatrix}}$$

and the autocoherence relations. In the same way, there exists a unique operator denoted $\mathbf{rem} \in \boldsymbol{ALIEN}$ *and called* ***right lateral mould associated to m*** *which satisfies the equality* $\mathbf{m} = \mathbf{mur.rem}$*. It is given by :*

$$\forall (\omega_1, \ldots, \omega_r) \in \Omega^\bullet, \quad \boldsymbol{rem}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ + & \ldots & +\end{pmatrix}} = \boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ + & \ldots & +\end{pmatrix}}$$

and the autocoherence relations.

Proof Let us consider $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$. Then according to the formula of postcomposition of an an **ALIEN** operator by an average, we have for any $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$:

$$\mathbf{m}^{\binom{\underline{\omega}}{\underline{\epsilon}}} = (\mathbf{mul.lem})^{\binom{\underline{\omega}}{\underline{\epsilon}}} = \sum_{\underline{\epsilon}^1.\underline{\epsilon}^2=\epsilon,\underline{\omega}^1.\underline{\omega}^2=\underline{\omega}} \mathbf{lem}^{\binom{\underline{\omega}^1}{\underline{\epsilon}^1}} \mathbf{mul}^{\binom{\underline{\omega}^2}{\underline{\epsilon}^2}}$$

and if we put $\epsilon_1 = \ldots = \epsilon_r = -$ then we obtain the attempted formula for $\mathbf{lem}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ - & \ldots & -\end{pmatrix}}$. □

Remark 7.1.4.1 *We have* $\boldsymbol{mur} = \boldsymbol{mul.lur}$ *and* $\boldsymbol{mul} = \boldsymbol{mur.rul}$ *which is another writing of relation in proposition 1.2.4 page 27.*

Remark 7.1.4.2 *The knowledge of* **rem** *(or* **lem***) implies the one of* **m** *and conversely. For example, if we know* **rem** *then the average* **m** *is given by :*

$$\boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ - & \ldots & -\end{pmatrix}} := (-1)^r < \boldsymbol{op}, \boldsymbol{rul} >^{\omega_1,\ldots,\omega_r}$$

and by the auto-coherence relations. Conversely, if we know **m** *then for any* $r \in \mathbf{N}$ *and* $(\omega_1, \ldots, \omega_r) \in \{+,-\}^r$ *:*

$$\boldsymbol{rem}^{\epsilon_1,\ldots,\epsilon_r} = (-1)^r \boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ + & \ldots & +\end{pmatrix}}$$

and

$$\boldsymbol{lem}^{\omega_1,\ldots,\omega_r} = (-1)^r \boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ - & \ldots & -\end{pmatrix}}.$$

7.2 ALIEN operators

7.2.1 Analytic operators

Definition 7.2.1.1 *We say that a differential operator* $\mathbf{F} \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *is analytic if it preserves* $\mathbb{C}\{u\}$.

Remark 7.2.1.1 *Using the results of paragraph 3.1,*

— *a derivation* $\mathbb{D} \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *is analytic if and only if* $\mathbb{D}.u$ *is an analytic germ.*

— *a substitution automorphism* $\mathbf{F} \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *is analytic if and only if* $\mathbf{F}.u$ *is an analytic germ.*

Using for $\mathbb{D}$ or $\mathbf{F}$ a mould-comould expansion like $\sum M^{\underline{\omega}}\mathbb{B}_w$ with of a comould $\mathbb{B}_\bullet$ constructed from an analytic operator[3] $\mathbb{B}$, it is very difficult to prove its analyticity unless having for $M^\bullet$ majorations like :

$$\forall \underline{\omega} \in \Omega^\bullet, |M^{\underline{\omega}}| \leq \frac{ab^{\|\underline{\omega}\|}}{\mathrm{l}(\underline{\omega})!}.$$

Unfortunately, this is generally not the case.

It is here that arborification (simple or contracted, direct or backward) step in.

The following theorem, for which F. Menous has established a very nice proof in section 5 of [43] (and which is generalized in [18], Theorem 10), is the achievement of arborification theory. .

Theorem 7.2.1 *Let us consider an analytic group-like or primitive element* $\mathbb{B}$ *of* $\boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *and a group-like or primitive operator* $\mathbf{F} = \sum M^\bullet \mathbb{B}_\bullet \in \boldsymbol{ENDOM}(\mathbb{C}[[u]])$. *If the arborified*[4] $M^{\bullet <}$ *of* $M^\bullet$ *has a geometrical growth :*

$$\exists a, b \in \mathbb{R}_+^*, \quad \forall \underline{\omega}^< \in \Omega^{\bullet <}, \quad \left| M^{\underline{\omega}^<} \right| \leq ab^{\|\underline{\omega}^<\|}$$

then the operator $\mathbf{F} = \sum M^{\bullet <} \mathbb{B}_{\bullet <}$ *is analytic.*

Let us recall that one of the main results of chapter 5 is that the geometrical growth for the arborified of the mould $M^\bullet$ is equivalent to the one of its anti-arborified.

We end this subsection by a definition.

Definition 7.2.1.2 *One says that an* $\boldsymbol{ALIEN}$ *operator* $\boldsymbol{op} \in \boldsymbol{ALIEN}$ *is analytic if for any reductions* $\boldsymbol{red} : \boldsymbol{ALIEN} \to \boldsymbol{ENDOM}(\mathbb{C}[[u]])$, *if* $\boldsymbol{red}(\dot{\Delta}^+)$ *is analytic, then the same is also true for* $\boldsymbol{red}(\boldsymbol{op})$.

3. like a reduction of the Stokes automorphism
4. simple if $\mathbb{B}_\bullet$ is cosymmetral and contracted if $\mathbb{B}_\bullet$ is cosymmetrel

7.2.2 Preserving realness operators

We will now define the conjugate of an **ALIEN** operator.

Definition 7.2.2.1 *For any* $\boldsymbol{op} \in \boldsymbol{ALIEN}$*, we define its conjugate* $\overline{\boldsymbol{op}}$ *by its action on* $\widehat{\boldsymbol{RESUR}}_{\mathbb{R}}(\Omega)$ *:*

$$\forall \hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{R}}(\Omega), \quad \overline{\boldsymbol{op}}\hat{\phi} = \overline{\boldsymbol{op}\overline{\hat{\phi}}}.$$

As a direct corollary of Proposition 1.2.4 and of this definition, we obtain :

Corollary 7.2.2 *Considering their restriction to* $\widehat{\boldsymbol{RESUR}}_{\mathbb{R}}(\Omega)$*, one has for the Stokes automorphism and its reciprocal :*

$$\overline{\dot{\Delta}^{+}} = \dot{\Delta}^{-} \quad \text{and} \quad \overline{\dot{\Delta}^{-}} = \dot{\Delta}^{+}.$$

Corollary 7.2.3 *With the same hypothesis than before, we have* $\overline{\dot{\Delta}} = -\dot{\Delta}$*.*

Proof It is a direct consequence of the previous proposition and of the fact that $\dot{\Delta} =: \log\left(\dot{\Delta}^{+}\right) = -\log\left(\dot{\Delta}^{-}\right)$. □

As an important corollary, we get :

Corollary 7.2.4 *The family of Ecalle's invariant* (C_m) *associated to* $\dot{\Delta}$ *in the case of the real classification of analytic saddle-node or tangent to identity analytic germs is purely imaginary.*

Proof One can verify this property directly at the level of the bridge equation. One performs the proof for the saddle-node, it is identical for tangent to identity analytic germs. According to Corollary 7.2.3, we obtain :

$$\overline{C_m}u^{m+1}\partial u = \overline{C_m u^{m+1}\partial u} = \overline{e^{-mt}\Delta_m} = -e^{-mt}\Delta_m = -C_m u^{m+1}\partial u$$

and it follows : $\overline{C}_m = -\overline{C}_m$ and $C_m \in i\mathbb{R}$ □

An important consequence of this section is :

Proposition 7.2.5 *Let us consider an* **ALIEN** *operator* $\boldsymbol{op} = \sum M^{\bullet}\Delta_{\bullet} \in \boldsymbol{ALIEN}$ *and a reduction* $\boldsymbol{red}$ *of* **ALIEN** *into* $\boldsymbol{ENDOM}(\mathbb{C}[[u]])$ *associated to a real saddle-node or a tangent to identity diffeomorphism like those studied in chapter 3. We set* $\psi(u) = \boldsymbol{red}(\boldsymbol{op}).u$*.*

Then $\boldsymbol{red}(\overline{\boldsymbol{op}}).u = \overline{\psi}(u)$*.*

Proof We perform the proof in the case of the saddle-node. It is identical in the case of tangent to identity diffeomorphism. Using the continuity of **red** for the Krull topology, one has :

$$\begin{aligned}
\mathbf{red}\left(\overline{\mathbf{op}}\right).u &= \mathbf{red}\left(\sum_{m\in\mathbf{N}}\sum_{\|\underline{\omega}\|=m}\overline{<\mathbf{op},\Delta>^{\underline{\omega}}}\;\overline{\Delta_{\underline{\underline{\omega}}}}\right).u \\
&= \mathbf{red}\left(\sum_{m\in\mathbf{N}}\sum_{\|\underline{\omega}\|=m}\overline{<\mathbf{op},\Delta>^{\underline{\omega}}}\,(-1)^{\mathrm{l}(\underline{\omega})}\;\Delta_{\underline{\underline{\omega}}}\right).u \\
&= \sum_{m\in\mathbf{N}}\sum_{\|\underline{\omega}\|=m}\overline{<\mathbf{op},\Delta>^{\underline{\omega}}}\,(-1)^{\mathrm{l}(\underline{\omega})}\;\mathbf{red}\left(\Delta_{\underline{\underline{\omega}}}\right).u \\
&= \sum_{m\in\mathbf{N}}\left(\sum_{\|\underline{\omega}\|=m}\overline{<\mathbf{op},\Delta>^{\underline{\omega}}}\,(-1)^{\mathrm{l}(\underline{\omega})}\;\beta_{\underline{\omega}}C_{\underline{\omega}}\right)u^{m+1}
\end{aligned}$$

From another side, one has :

$$\begin{aligned}
\psi(u) &= \mathbf{red}\left(\mathbf{op}\right).u \\
&= \sum_{m\in\mathbf{N}}\left(\sum_{\|\underline{\omega}\|=m}<\mathbf{op},\Delta>^{\underline{\omega}}\,\beta_{\underline{\omega}}C_{\underline{\omega}}\right)u^{m+1}
\end{aligned}$$

and so, as for all $\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^{\bullet}$, we have $C_{\underline{\omega}}=C_{\omega_1}\times\ldots\times C_{\omega_r}$ with $C_{\omega_1},\ldots,C_{\omega_r}\in i\mathbb{R}$, then $\overline{C_{\underline{\omega}}}=(-1)^{\mathrm{l}(\underline{\omega})}\,C_{\underline{\omega}}$ and :

$$\begin{aligned}
\overline{\psi}(u) &= \sum_{m\in\mathbf{N}}\left(\sum_{\|\underline{\omega}\|=m}\overline{<\mathbf{op},\Delta>}^{\underline{\omega}}\;(-1)^{\mathrm{l}(\underline{\omega})}\,\beta_{\underline{\omega}}C_{\underline{\omega}}\right)u^{m+1} \\
&= \mathbf{red}\left(\overline{\mathbf{op}}\right).u
\end{aligned}$$

□

7.2.3 Definition of well-behaved ALIEN derivations

Definition 7.2.3.1 *An **ALIEN** operator D is said to be a well-behaved **ALIEN** derivation if :*

1. *D is an **ALIEN** derivation.*
2. *D preserves the realness, i.e.* $D=\overline{D}$
3. *D is analytic.*

Let us remark that the standard derivation $\dot{\Delta}$ is not well behaved because it is not analytic. See Corollary 3.2.5. One of the main interests of well-behaved derivations lays in the fact that they produce analytical invariants (see section 3.4 page 63) with a geometrical growth instead of a Gevrey one.

7.2.4 Characterization in terms of lateral operators of well-behaved averages

Preserving convolution averages

Definition 7.2.4.1 *We say that an average* $\boldsymbol{m} \in \boldsymbol{AVER}$ *preserves the convolution if for any* $\hat{\phi}, \hat{\psi} \in \widehat{\boldsymbol{RESUR}}^{s}_{\mathbb{R}}\Omega$*, we have :*

$$\boldsymbol{m}\left(\hat{\phi} \star \hat{\psi}\right) = \boldsymbol{m}\hat{\phi} \star \boldsymbol{m}\hat{\psi}.$$

We will admit that the averages **mur** and **mul** preserve the convolution. See [42], Annexe C for more details.

Proposition 7.2.6 *An average* $\boldsymbol{m} \in \boldsymbol{AVER}$ *preserves the convolution if and only if* ***rem*** *(or* ***lem****) is a convolution automorphism of* ***ALIEN****.*

Proof We have the equivalences :

$$\begin{aligned}
& \mathbf{m} \text{ preserves the convolution} \\
\Longleftrightarrow\quad & \forall \hat{\phi}, \hat{\psi} \in \widehat{\mathbf{RESUR}}_{\mathbb{R}}, \quad \mathbf{m}\left(\hat{\phi} \star \hat{\psi}\right) = \mathbf{m}\hat{\phi} \star \mathbf{m}\hat{\psi} \\
\Longleftrightarrow\quad & \forall \hat{\phi}, \hat{\psi} \in \widehat{\mathbf{RESUR}}_{\mathbb{R}}, \quad \mathbf{mur}\,\mathbf{rem}\left(\hat{\phi} \star \hat{\psi}\right) = \mathbf{mur}\,\mathbf{rem}\hat{\phi} \star \mathbf{mur}\,\mathbf{rem}\hat{\psi} \\
\Longleftrightarrow\quad & \forall \hat{\phi}, \hat{\psi} \in \widehat{\mathbf{RESUR}}_{\mathbb{R}}, \quad \mathbf{mur}\,\mathbf{rem}\left(\hat{\phi} \star \hat{\psi}\right) = \mathbf{mur}\left(\mathbf{rem}\hat{\phi} \star \mathbf{rem}\hat{\psi}\right) \\
\Longleftrightarrow\quad & \forall \hat{\phi}, \hat{\psi} \in \widehat{\mathbf{RESUR}}_{\mathbb{R}}, \quad \mathbf{rem}\left(\hat{\phi} \star \hat{\psi}\right) = \mathbf{rem}\hat{\phi} \star \mathbf{rem}\hat{\psi} \\
& \qquad \text{because of remark 7.1.2.1} \\
\Longleftrightarrow\quad & \mathbf{rem} \text{ is a convolution automorphism of } \widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}[[\delta]]
\end{aligned}$$

□

Remark 7.2.4.1 *This property can be extended to* $\boldsymbol{RAMIF}(\mathbb{R}_+//\Omega, int)$ *using the density of* $\widehat{\boldsymbol{RESUR}}_{\mathbb{R}}(\Omega)$ *into* $\boldsymbol{RAMIF}(\mathbb{R}_+//\Omega, int)$ *and the continuity of the convolution product.*

Preserving realness averages

Let us recall the following definition.

Definition 7.2.4.2 *An average* $\boldsymbol{m} \in \boldsymbol{AVER}$ *preserves the realness if for any* $\hat{\phi} \in \widehat{\boldsymbol{RESUR}}_{\mathbb{R}}\Omega$*,*

$$\overline{\boldsymbol{m}\phi} = \boldsymbol{m}\phi.$$

Theorem 7.2.7 *Let* $\boldsymbol{m}$ *be a « respecting »convolution average and let* $\boldsymbol{rem}$ *be its right lateral operator. Then* $\boldsymbol{m}$ *preserves the realness if and only if* $\boldsymbol{rem}$ *satisfies the conjugation equation :*

$$\boldsymbol{rem} = \boldsymbol{rul}\, \overline{\boldsymbol{rem}} \tag{7.1}$$

Proof We have the equivalences :

$$\begin{aligned}
\mathbf{m} = \overline{\mathbf{m}} &\iff \mathbf{mur\ rem} = \overline{\mathbf{mur}}\ \overline{\mathbf{rem}} \\
&\iff \mathbf{mur\ rem} = \overline{\mathbf{mul}}\ \overline{\mathbf{lur}}\ \overline{\mathbf{rem}} \\
&\iff \mathbf{mur\ rem} = \mathbf{mur\ rul}\ \overline{\mathbf{rem}} \\
&\overset{(\clubsuit)}{\iff} \mathbf{rem} = \mathbf{rul}\ \overline{\mathbf{rem}}
\end{aligned}$$

as long as $(\clubsuit)$ is a consequence of remark 7.1.2.1. □

Preserving exponential growth averages

Let us consider a real formal series $\tilde{\phi}$ associated to a real analytical dynamical problem[5]. Then we get a reduction **red** from **ALIEN** (Ω) to the space **ENDOM** $(\mathbb{C}[[u_1, \ldots, u_n]])$ where $u_1, \ldots, u_n$ are the parameters of our problem. Applying an average **m** on $\hat{\phi}$ corresponds :

- to apply first the left lateral operator **lem** associated to **m** to $\tilde{\phi}$. By the Bridge equation, it is equivalent to apply the operator of substitution $\mathbf{F} = \mathbf{red}\,(\mathbf{lem})$ to $\tilde{\phi}$
- to compute the Borel Transform of $\tilde{\phi}$
- and then to compose the resulting series by the average **mul**. We know that $\mathbf{mul}\hat{\phi}$ is of at most-exponential growth. The same occurs for $\mathbf{m}\hat{\phi}$ if and only if **F** is an analytic operator of **ENDOM** $(\mathbb{C}[[u_1, \ldots, u_n]])$.

Definition 7.2.4.3 *One says that an average* $\boldsymbol{m}$ *preserves the exponential growth if its left lateral operator (or conversely its right lateral operator) is analytic.*

Well behaved averages

Finally, one has :

Definition 7.2.4.4 *An average* $\boldsymbol{m} \in \boldsymbol{AVER}$ *is said to be well behaved if :*

1. *it preserves the convolution.*
2. *it preserves the realness.*
3. *it preserves the exponential growth.*

5. namely a resurgent problem for which one there is a bridge equation, i.e. some linear relations permitting to translate the action of the homogeneous components of the standard **ALIEN** derivation on $\tilde{\phi}$ into the action of the homogeneous components of an ordinary derivation on $\tilde{\phi}$. For example $\tilde{\phi}$ is the formal conjuguant of a real saddle-node, or the conjuguant of a real tangent to identity holomorphic germs, ...

7.3 Two examples of real summation

7.3.1 The saddle-node problem

We consider a real saddle-node and its formal solution $\hat{\varphi}(t,u) = \sum_{n\geq 0} \tilde{\varphi}_n(t)\, u^n e^{nt}$ which is a real formal series of $t^{-1}\mathbb{R}[[t^{-1}]][[ue^t]]$. We can not use Borel-Laplace summation in one of the two real directions to obtain a real sum of $\tilde{\varphi}(t,u)$ because, as explained in section 3.2, the resurgent functions $\hat{\varphi}_n$ have singularities located on $\mathbb{R}$. We must use Ecalle's averages theory. To this end, we have to extend the action of averages and **ALIEN** operators to the space $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u \star \delta_{-1}(\zeta)]]$.

We define the proper action of an **ALIEN** operator $\mathbf{op} \in \mathbf{ALIEN}(\mathbf{N}^*)$, of a differential operator $\mathbf{F} \in \mathbf{ENDOM}(\mathbb{C}[[u]])$ and the action of an average $\mathbf{m} \in \mathbf{AVER}$ on $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u \star \delta_{-1}]]$ by, for all $f = \sum \tilde{f}_n(\zeta)u^n\delta_{-n} \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u \star \delta_{-1}]]$

$$\mathbf{op}f := \sum_n \left(\mathbf{op}\hat{f}_n\right)(\zeta)u^n \star \delta_{-n}$$

$$\mathbf{F}f := \hat{f}_n(\zeta)\,(\mathbf{F}.u^n) \star \delta_{-n}$$

$$\mathbf{m}f := \sum_n \left(\mathbf{m}\hat{f}_n\right)(\zeta)u^n \star \delta_{-n}$$

We clearly have $\mathbf{op}f \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u\star\delta_{-1}]]$ and $\mathbf{m}f \in \mathbf{UNIF}(\mathbf{N}^*)[[u\star\delta_{-1}]]$. As long as $\mathbf{F}$ is a preserving graduation operator of $\mathbf{ENDOM}(\mathbb{C}[[u]])$, we have too $\mathbf{F}.f \in \widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u\star\delta_{-1}]]$. Moreover, we easily verify that on $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)[[u \star \delta_{-1}]]$, $[\mathbf{m},\mathbf{F}] = 0$ and $[\mathbf{op},\mathbf{F}] = 0$.

Since the Borel transform defines an isomorphism between the algebras $\widetilde{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)$ and $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}(\mathbf{N}^*)$, we get an isomorphism between $\widetilde{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}[[ue^t]]$ and $\widehat{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}[[u \star \delta_{-1}]]$. Then we can define the action of an operator $\mathbf{op} \in \mathbf{ALIEN}$ or of a differential operator $\mathbf{F} \in \mathbf{ENDOM}(\mathbb{C}[[u]])$ on $\widetilde{\mathbf{RESUR}}^{\mathrm{s}}_{\mathbb{R}}[[ue^t]]$ by pulling back this action via the inverse formal Borel Transform.

If $\mathbf{m} = \mathbf{mur\ rem}$ is an average which preserves the convolution product, then the action of $\mathbf{m}$ on $\hat{\varphi}(t,u)$ can be computed in the following way :

— We apply the operator **rem** on $\tilde{\varphi}(t,u)$.

— As $\mathbf{m}$ preserves the convolution product, **rem** is a convolution automorphism and $\mathbf{F} = \mathbf{red}(\mathbf{rem})$ is a substitution automorphism then

$$\mathbf{rem}\ \tilde{\varphi}(t,u) = \tilde{\varphi}(t,\mathbf{F}.u) = \tilde{\varphi}(t,f(u))$$

where $f(u) = \mathbf{F}.u$ is a formal series of $u\mathbb{C}[[u]]$.

— We then apply the formal Borel transform to **rem** $\tilde{\varphi}(t,u)$ and we obtain the series :

$$\sum_{n\geq 0} \hat{\varphi}_n(\zeta)(f(u))^n \star \delta_{-n}.$$

— We finally apply the average **mur** :

$$\sum_{n\geq 0} \mathbf{mur}\ \hat{\varphi}_n(\zeta)\, f(u)^n \star \delta_{-n}.$$

In order to sum it, we have to extend the action of the Laplace transform, when it is possible to $\mathbf{UNIF}[[u \star \delta_{-1}]]$.

We consider the set

$$\begin{gathered} \mathbf{UNIF}\{u \star \delta_{-1}\} := \{\sum \hat{f}_n(\zeta) u^n \star \delta_{-n} \in \mathbf{UNIF}[[u\star\delta_{-1}]] \mid \\ \exists A,B,C \in \mathbb{R}_+^*, \quad \forall \zeta \in \mathbb{C}//\mathbb{Z}^*, \quad \forall n \in \mathbf{N}, |f_n(\zeta)| \leq AB^n e^{c|\zeta|}\} \end{gathered}$$

We define the Laplace transform for $f(\zeta,u) = \sum \hat{f}_n(\zeta)u^n \star \delta_{-n} \in \mathbf{UNIF}\{u \star \delta_{-1}\}$ in the following way :

$$(\mathcal{L}f)(t,u) := \sum \mathcal{L}\left(\hat{f}_n\right)(t)\, u^n e^{nt}$$

with, for any $\hat{\varphi} \in \mathbf{UNIF}$ which is of exponential growth

$$\mathcal{L}(\hat{\varphi})(t) = \int_0^{+\infty} \hat{\varphi}(\zeta)\, e^{-t\zeta} \mathrm{d}\zeta.$$

The function $f(t,u)$ is defined for u in a small neighborhood of $0 \in \mathbb{C}$ and t in a neighborhood of $-\infty$.

Let us remark that for uniform functions, the path of integration in the previous integral does not depend on the choice of the branch of $\mathbb{R}_+//\mathbf{N}^*$.

Theorem 7.3.1 *If* $\boldsymbol{m} = \boldsymbol{mur\ rem}$ *is a well-behaved average then* $\boldsymbol{m}\,\varphi(\zeta,u) \in \boldsymbol{UNIF}\{u \star \delta_{-1}\}$.

Proof Let us consider $\mathbf{F} = \mathbf{red}(\mathbf{rem}) = 1 + \sum_{m\geq 1} \mathbf{F}_m$ and $f(u) = \mathbf{F}.u \in \mathbb{C}[[u]]$. As $\mathbf{F}$ is a substitution automorphism of $\mathbf{ENDOM}(\mathbb{C}[[u]])$, for any $m,k \in \mathbf{N}$, there exists $\gamma_{m,k} \in \mathbb{C}$ such that :

$$\mathbf{F}_m.u^k = \sum_{\underline{\omega}=m} M^{\underline{\omega}} C_{\underline{\omega}} \mathbb{B}_{\underline{\omega}}.u^k = \gamma_{m,k} u^{k+m}.$$

But

$$\mathbf{F}.u^k = (f(u))^k = \sum_{m\geq 0} \sum_{\substack{\|\underline{\omega}\| = m \\ \mathrm{l}(\underline{\omega}) = k}} f_{\omega_1} \dots f_{\omega_k} u^{m+k}$$

where $f(u)=\sum_{m\geq 1} f_m u^m$. Then

$$\gamma_{m,k}=\sum_{\substack{\|\underline{\omega}\|=m\\ \mathrm{l}(\underline{\omega})=k}} f_{\omega_1}\dots f_{\omega_k}$$

As $\mathbf{m}$ is a well behaved averages, f is an analytic function and the sequence (f_m) has a geometrical growth. There exists $a,b\in\mathbb{R}_+^*$ such that $\forall m\in\mathbf{N}, |f_m|\leq ab^m$. Moreover, since $\hat{\varphi}(\zeta,u)=\sum_{n\geq 0}\hat{\varphi}_n(\zeta)\,u^n\star\delta_{-n}$ is the Borel transform of the formal solution of the considered saddle-node, for all $n\in\mathbf{N}$, the resurgent functions $\hat{\varphi}_n$ are of at most-exponential growth along some paths that cross the critical direction $\mathbb{R}_+$ a finite number of times and there exists $A,B,C\in\mathbb{R}_+^*$ independent of n (see [50]) such that for all $\zeta\in\mathbb{R}_+//\mathbf{N}^*$, $|\mathbf{mur}\hat{\varphi}_n(\zeta)|\leq AB^n e^{C\zeta}$. For $\zeta\in\mathbb{R}_+//\mathbf{N}^*$, we have :

$$\begin{aligned}
\mathbf{m}\hat{\varphi}(\zeta,u) &= \mathbf{mur\ rem}\hat{\varphi}(\zeta,u)\\
&= \sum_{n\geq 0}\sum_{m\geq 0}\mathbf{mur}\hat{\varphi}_n(\zeta)\,\mathbf{F}_m.u^n\star\delta_{-n}\\
&= \sum_{n\geq 0}\sum_{m\geq 0}\mathbf{mur}\hat{\varphi}_n(\zeta)\,\gamma_{m,n}u^{n+m}\star\delta_{-n}\\
&= \sum_{n\geq 0}\sum_{m\geq 0}\left[(\mathbf{mur}\hat{\varphi}_n\star\delta_m)(\zeta)\sum_{\substack{\|\underline{\omega}\|=m\\ \mathrm{l}(\underline{\omega})=n}} f_{\omega_1}\dots f_{\omega_n}\right]u^{n+m}\star\delta_{-n-m}.\\
&= \sum_{N\geq 0}\sum_{m+n=N,\,m,n\geq 0}\left[(\mathbf{mur}\hat{\varphi}_n\star\delta_m)(\zeta)\sum_{\|\underline{\omega}\|=m} f_{\omega_1}\dots f_{\omega_n}\right]u^{N}\star\delta_{-N}. \qquad (\star)
\end{aligned}$$

But :

$$\begin{aligned}
\sum_{m+n=N,\,m,n\geq 0}\left|(\mathbf{mur}\hat{\varphi}_n\star\delta_{-n})(\zeta)\sum_{\|\underline{\omega}\|=m} f_{\omega_1}\dots f_{\omega_n}\right| &\leq \sum_{m+n=N,\,m,n\geq 0} AB^n2^m a^n b^m 2^m e^{C\zeta}\\
&\leq NA\gamma^{n+m}e^{C|\zeta|}2^m\alpha^{n+m}\beta^{n+m}\\
&\leq A\,(2e\alpha\beta\gamma)^N\,e^{C|\zeta|}
\end{aligned}$$

with $\alpha=\max(a,1)$, $\beta=\max(b,1)$ and $\gamma=\max(B,1)$. Then for each term of the sum $(\star)$, we can perform Laplace transform and the resulting series is normally convergent for u and t small enough.

□

In conclusion, we can apply the Laplace transform to $\mathbf{m}\hat{\varphi}(\zeta,u)$ and the obtained series $\varphi(t,u)$, defined for $t\in\mathbb{C}$ such that $|t|\geq R$ $(R>0)$, $\arg(t)\in\,]-\pi,\pi[$ and u small enough is the real sum of $\tilde{\varphi}(t,u)$ in the sector of ∞ of opening π and bisected by $\mathbb{R}_+^*$.

Theorem 7.3.2 *When applying Ecalle's averages theory to the formal conjuguant $\tilde{\varphi}(t,u)$ of the real saddle-node in its simplest formal class, one obtain a real analytic conjuguant $\varphi(t,f(u))$ defined and asymptotic to $\tilde{\varphi}(t,u)$ in an half-plane $\Re(z)>c$ for $c>0$ big enough.*

Example 7.3.1.1 *We apply Ecalle's averages theory in the simple case of the Euler equation $x^2y'=x+y$. As usually, we perform the change of variables $x=-1/t$ and the equation becomes $\tilde{y}'(t)-\tilde{y}(t)=-1/t$ where $\tilde{y}(t)=y(-1/t)$. Solutions are $\tilde{y}(t)=ue^t+\tilde{\varphi}_0(t)$ where $u\in\mathbb{R}$ and $\tilde{\varphi}_0(t)=\left(\int e^{-t}/t\right)e^t$. But one has $\tilde{\varphi}_0'(t)=t+\tilde{\varphi}_0(t)$ and so using the formal Borel transform, it comes $\tilde{\varphi}_0(t)=-\mathcal{B}^{-1}\left(\zeta\mapsto\dfrac{1}{1+\zeta}\right)=\sum_{n\geq 1}(-1)^n(n-1)!/t^n$ which is evidently divergent and 1-Gevrey.*

The formal conjuguant can then be written $\tilde{\varphi}(t,u)=\tilde{\varphi}_0(t)+ue^{-1/t}$. The only singularity is located at $m=-1$ and so $\Delta_m\tilde{\phi}(t,u)=0$ for $m\geq 1$. One has $e^t\Delta_{-1}\tilde{\phi}(t,u)=C_{-1}\phi(t,u)$ and a simple computation gives $C_{-1}=-2i\pi$.

We will now compute a real sum for this formal conjuguant in the direction $\mathbb{R}_-$. In this case, one has $\Omega=-\mathbf{N}^*$.

We consider a well behaved average $\boldsymbol{m}=\boldsymbol{mur}\ \boldsymbol{rem}$. The reduction of $\boldsymbol{rem}=\sum M^\bullet\dot{\Delta}_\bullet$ is the substitution operator $\mathbf{F}$ by the germ $\boldsymbol{red}(\boldsymbol{rem}).u=u-2i\pi M^{-1}$. A real sum of the formal conjuguant is then

$$\varphi(t,u):=\mathcal{L}^{\pi^-}(\boldsymbol{m}\mathcal{B}(\varphi))(t,u)=\mathcal{L}^{\pi^-}(\hat{\varphi}_0)(t)+\mathbf{F}.ue^{-1/t}=\mathcal{L}^{\pi^-}(\hat{\varphi}_0)(t)+\left(u-2i\pi M^{-1}\right)e^{-1/t}$$

which is convergent for t big enough and which is asymptotic to the formal conjuguant. Let us verify it is real. One has

$$\overline{\varphi(t,u)}=\mathcal{L}^{\pi^+}(\mathcal{B}(\varphi))(t)+\left(u+2i\pi\overline{M^{-1}}\right)e^{-1/t}.$$

The residues formula yealds

$$\mathcal{L}^{\pi^-}(\mathcal{B}(\varphi_0))(t)-\mathcal{L}^{\pi^+}(\mathcal{B}(\varphi_0))(t)=2i\pi e^{-1/t}$$

and so

$$\overline{\varphi(t,u)}=\varphi(t,u)+2i\pi\left(\overline{M^{-1}}+M^{-1}-1\right)e^{-1/t}.$$

But the relation $\overline{\boldsymbol{rem}}=\boldsymbol{lur}\ \boldsymbol{rem}$ (see Theorem 7.2.7) gives

$$(-1)^{l(\bullet)}\overline{<\boldsymbol{rem},\Delta>^\bullet}=<\boldsymbol{rem},\Delta>^\bullet\times<\Delta^+,\Delta>^\bullet$$

and using mould calculus, it comes $-\overline{M}^{-1}=M^{-1}-1$. We have well $\overline{\varphi(t,u)}=\varphi(t,u)$.

7.3.2 Tangent to identity holomorphic germs

We use the same notations than in section 3.3. We consider a non degenerate parabolic germ at infinity with null resiter $f(z)=z+2i\pi+a(z)$ where $a(z)\in$

$z^{-2}\mathbb{C}\left\{z^{-1}\right\}$ is a convergent powers series. As reminded in section 3.3, there exists a formal series $\tilde{U}(z) = z + \tilde{u}(z)$ with $\tilde{u}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ such that

$$f \circ \tilde{U}(z) = \tilde{U}(z + 2i\pi).$$

Moreover, we assume that f satisfies the equality $f \circ \overline{f} = \overline{f} \circ f = id$. Then we can easily prove that $\tilde{U}(z) \in z + z^{-1}\mathbb{R}[[z^{-1}]]$. Indeed, we obtain from the conjugation relation the two equalities

$$\tilde{U}^{-1} \circ f^{-1}(z) = \tilde{U}^{-1}(z) - 2i\pi \text{ and } \overline{f} \circ \overline{\tilde{U}}(z) = \overline{\tilde{U}}(z - 2i\pi).$$

Since $f^{-1} = \overline{f}$, the first becomes $\overline{f}(\tilde{U}(z)) = \tilde{U}(z - 2i\pi)$. Then we obtain two formal conjuguants $\tilde{U}$ and $\overline{\tilde{U}}$ for the non degenerate parabolic germ $\overline{f}$. But this conjuguant is unique then $\overline{\tilde{U}} = \tilde{U}$.

We can not compute a real sum of $\tilde{U}$ using Borel-Laplace summation, because as mentioned in section 3.3, all the singular points of $\hat{U}$ are located on $\mathbb{R}$. We have to use Ecalle's averages to this end.

We recall that the effect of an average $\mathbf{m} = \mathbf{mur}\ \mathbf{rem}$ where $\mathbf{rem} \in \mathbf{ALIEN}(\mathbf{N}^*)$ on a resurgent function $\hat{\phi} \in \widehat{\mathbf{RESUR}}^{s}_{\mathbb{R}}(\mathbf{N}^*)$ is equivalent to the action of the operator $\mathbf{rem}$ on $\tilde{\phi}$ in $\widetilde{\mathbf{RESUR}}^{s}_{\mathbb{R}}(\mathbf{N}^*)$ followed by the Borel transform and of the action of $\mathbf{mur}$. Then $\mathbf{m}\ \tilde{\phi}$ is summable if and only if $\mathcal{B}(\mathbf{rem}\ \tilde{\phi})$ has a at most-exponential growth.

In the case of a well behaved average $\mathbf{m}$, we know that $\mathbf{rem} = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}}\dot{\Delta}_{\underline{\omega}} \in \mathbf{ALIEN}(\mathbf{N}^*)$ is a real analytic convolution automorphism, and then $\mathbf{red}(\mathbf{rem})$ is a real analytic substitution automorphism of $\mathbf{ENDOM}(\mathbb{C}[z][[z^{-1}]])$. We then have

$$\mathbf{rem}\tilde{U}(z) = \mathbf{red}(\mathbf{rem})\,\tilde{U}(z) = \tilde{U}(\phi(z))$$

where

— $\mathbf{red}(\mathbf{rem}) = \sum_{\underline{\omega}\in\Omega^\bullet} M^{\underline{\omega}} A_{\underline{\omega}} \mathbb{B}_{\mathbf{rev}(\underline{\omega})}$ with $\mathbb{B}_m = e^{-mz}\partial_z$

— $\phi(z) = \mathbf{red}(\mathbf{rem}).z = z + \sum_{m\in\mathbf{N}^*} \gamma_m e^{-mz}$ with

$$\gamma_m = \sum_{\substack{\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet \\ \|\underline{\omega}\| = m}} M^{\underline{\omega}} A_{\underline{\omega}} \beta_{\underline{\omega}}.$$

We want to compute the Borel-Laplace transform of $\tilde{U}(\phi(z)) = \phi(z) + \tilde{u}(\phi(z))$: the one of $\phi(z)$ is obtain easily because it is an analytical germ of $\mathbb{C}[z]\left\{z^{-1}\right\}$. We then focus on $\tilde{u}(\phi(z))$. As $\phi(z) \in \mathbb{C}[z]\left\{z^{-1}\right\}$, the sequence (γ_m) has a geometrical growth [6] and using a Taylor expansion :

$$\begin{aligned}
\tilde{u}(\phi(z)) &= \tilde{u}(z+\sum_{m\in\mathbf{N}^*}\gamma_m e^{-mz}) \\
&= \sum_{r\geq 0}\frac{\left(\sum_{m\in\mathbf{N}^*}\gamma_m e^{-mz}\right)^r}{r!}\partial^r\tilde{u}(z) \\
&= \sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet\\ \mathrm{l}(\underline{\omega})=r}}\gamma_{\omega_1}\ldots\gamma_{\omega_r}e^{-\|\underline{\omega}\|z}\partial^r\tilde{u}(z) \\
&= \sum_{m\geq 1}\left(\sum_{\|\underline{\omega}\|=m}\frac{\gamma_{\underline{\omega}}}{\mathrm{l}(\underline{\omega})!}\partial^{\mathrm{l}(\underline{\omega})}\tilde{u}(z)\right)e^{-mz}
\end{aligned}$$

and its Borel transform is for $\zeta\in\mathbb{R}_+//\mathbf{N}^*$

$$\begin{aligned}
\widehat{u\circ\phi}(\zeta) &= \sum_{m\geq 1}\left(\sum_{\|\underline{\omega}\|=m}\frac{\gamma_{\underline{\omega}}}{\mathrm{l}(\underline{\omega})!}\left((-\zeta)^{\mathrm{l}(\underline{\omega})}\hat{u}(\zeta)\right)\right)\star\delta_m \\
&= \sum_{1\leq m\leq|\zeta|}\sum_{\|\underline{\omega}\|=m}\frac{\gamma_{\underline{\omega}}}{\mathrm{l}(\underline{\omega})!}(-\zeta+m)^{\mathrm{l}(\underline{\omega})}\hat{u}(\zeta-m)
\end{aligned}$$

Let us observe that $\widehat{u\circ\phi}$ is not an element of $\widehat{\mathbf{RESUR}}_{\mathbb{R}}(\mathbf{N}^*)$ but of $\widehat{\mathbf{RESUR}}_{\mathbb{R}}(\mathbf{N}^*)[[\delta]]$. We extend naturally to this space the action of averages. Since $\mathbf{mur}\ \hat{u}(\zeta)$ is of at most-exponential growth and since (γ_m) got a geometrical growth there exists $a,b,A,B\in\mathbb{R}_+^*$ such that for all $m\in\mathbf{N}^*$ and $\zeta\in\mathbb{R}_+//\mathbf{N}^*$ such that

$$|\gamma_m|\leq AB^m \quad\text{and}\quad |\mathbf{mur}\ \hat{u}(\zeta)|\leq ae^{b\zeta}.$$

It then comes :

$$\begin{aligned}
\left|\mathbf{mur}\ \widehat{u\circ\phi}(\zeta)\right| &= \left|\sum_{1\leq m\leq|\zeta|}\sum_{\|\underline{\omega}\|=m}\frac{\gamma_{\underline{\omega}}}{\mathrm{l}(\underline{\omega})!}(-\zeta+m)^{\mathrm{l}(\underline{\omega})}\mathbf{mur}\ \hat{u}(\zeta-m)\right| \\
&\leq \left(\sum_{1\leq m\leq|\zeta|}a2^mA^mB^m|\zeta|^m\right)e^{b\zeta}
\end{aligned}$$

which is yet of at most-exponential growth.

Let us introduce the space

$$\begin{aligned}
\mathbf{UNIF}\{\delta\} := \{&\sum\hat{f}_n(\zeta)\star\delta_n(\zeta)\in\mathbf{UNIF}[[\delta]]\mid\exists A,B,C\in\mathbb{R}_+^*, \\
&\forall\zeta\in\mathbb{C}//\mathbf{N}^*,\quad\forall n\in\mathbf{N},|f_n(\zeta)|\leq AB^ne^{C|\zeta|}\}.
\end{aligned}$$

We have then proved the following theorem

6. Indeed, if $f(z)=\sum_{m\geq 1}\gamma_m e^{-mz}$ is an analytical germ at infinity, then there is a closed neighborhood $|z|\geq R$ with $R>0$ of infinity on which one f is defined (we denote by the same symbol f the germ and an element of the class it defined) and analytic. Then $\int_R^{R+2\pi}f(z)e^{mz}\mathrm{d}z=2\pi\gamma_m$ and if $M=\sup_{[R,R+2\pi]}|f|$ then $|\gamma_m|\leq M\left(e^R\right)^m$.

Theorem 7.3.3 *If $\boldsymbol{m} = \boldsymbol{mur\ rem}$ is a well-behaved average then $\boldsymbol{m}\,\hat{U} \in$ $\boldsymbol{UNIF}\{\delta\}$.*

We can then perform the Laplace transform of $\widehat{u \circ \phi}$ and we obtain a real analytical germ $u \circ \phi(z)$, defined for $z \in \mathbb{C}$ in a sectoral region of ∞ such that $|z| \geq R$ $(R > 0)$, $\arg(z) \in]-\pi, \pi[$ which is the sum of $\tilde{u} \circ \phi(z)$. The germ $U(z) := \phi(z) + u \circ \phi(z)$ is then the sum of $\tilde{U}$ in this sectoral region.

Theorem 7.3.4 *When applying Ecalle's averages theory to the formal conjuguant $z + \tilde{u}(z)$ of a tangent to identity analytic germ of null resiter verifying $f \circ \overline{f} = \overline{f} \circ f = id$, one obtain a real analytic conjuguant $\phi(z) + u(\phi(z))$ defined and asymptotic to $z + \tilde{u}(z)$ in an half-place $\Re(z) > c$ for $c > 0$ big enough.*

7.4 The diffusion induced families of averages and derivations

This section is a presentation of the work of Jean Ecalle about diffusion-induced averages and derivations. We will follow the same framework than in [14] and the same notations will be used. This family of averages (and derivations) is of utmost importance because all the known examples of well behaved averages are specializations of this family or are obtained using a limit process applied to this family.

We first consider an analytic function $\gamma(y)$ on $\mathbb{R}$. For any $\omega \in \Omega$, we consider $g_\omega(y) = e^{-\omega\gamma(y)}$ and its Fourier transform $f_\omega(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g_\omega(y)\, e^{ixy}\mathrm{d}y$. We obtain easily that :

$$g_{\omega_1}(y)\, g_{\omega_2}(y) = g_{\omega_1+\omega_2}(y) \text{ and } \left(f_{\underline{\omega}_1} \star f_{\underline{\omega}_1}\right)(x) = \int_{\mathbb{R}} f_{\underline{\omega}_1}(x)\, f_{\underline{\omega}_2}(x - x_1)\,\mathrm{d}x_1 = f_{\underline{\omega}_1+\underline{\omega}_2}(x)$$

We assume that γ is chosen in a such way that for any $\omega \in \Omega$, $\int_{\mathbb{R}} |f_\omega(x)|\,\mathrm{d}x = 1$.

7.4.1 An example of well behaved average : the diffusion induced one

Definition 7.4.1.1 *We consider the family of weights $\boldsymbol{m}$ given by $\boldsymbol{m}^{\emptyset} = 1$ and for any $r \in \mathbf{N}^*$, $(\omega_1, \ldots \omega_r) \in \Omega^r, (\epsilon_1, \ldots, \epsilon_r) \in \{+,-\}^r$:*

$$\boldsymbol{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ \epsilon_1 & \ldots & \epsilon_r\end{pmatrix}} = \int_{\mathbb{R}^r} f_{\omega_1}(x_1) \ldots f_{\omega_r}(x_r)\, \sigma_{\epsilon_1}(\check{x}_1) \ldots \sigma_{\epsilon_r}(\check{x}_r)\, dx_1 \ldots dx_r$$

where

$$\sigma_+(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad \sigma_-(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}.$$

Proposition 7.4.1 *$\mathbf{m}$ satisfies the auto-coherence relations (autocoI) and (autocoII) of paragraph 7.1.1 and then is an average called diffusion-induced average.*

Proof We will prove that $\mathbf{m}$ verifies the auto-coherence relation (*autocoI*). The proof is the same for the second auto-coherence relation. Let us consider $r \in \mathbf{N}^*$ and $i \in [\![1, n]\!]$. Then for any $(\omega_1, \dots \omega_r) \in \Omega^r, (\epsilon_1, \dots, \epsilon_r) \in \{+, -\}^r$, we get :

$$\begin{aligned} & \mathbf{m}\begin{pmatrix} \omega_1 & \dots & \omega_{i-1} & \omega_i & \omega_{i+1} & \dots & \omega_r \\ \epsilon_1 & \dots & \epsilon_{i-1} & + & \epsilon_{i+1} & \dots & \epsilon_r \end{pmatrix} + \mathbf{m}\begin{pmatrix} \omega_1 & \dots & \omega_{i-1} & \omega_i & \omega_{i+1} & \dots & \omega_r \\ \epsilon_1 & \dots & \epsilon_{i-1} & - & \epsilon_{i+1} & \dots & \epsilon_r \end{pmatrix} \\ = & \int_{\mathbb{R}^r} f_{\omega_1}(x_1) \dots f_{\omega_{i-1}}(x_{i-1}) f_{\omega_i}(x_i) f_{\omega_{i+1}}(x_{i+1}) \dots f_{\omega_r}(x_r) \sigma_{\epsilon_1}(\check{x}_1) \dots \\ & \sigma_{\epsilon_{i-1}}(\check{x}_{i-1}) \left(\underbrace{\sigma_+(\check{x}_i) + \sigma_-(\check{x}_i)}_{=1} \right) \sigma_{\epsilon_{i+1}}(\check{x}_{i+1}) \dots \sigma_{\epsilon_r}(\check{x}_r) \mathrm{d}x_1 \dots \mathrm{d}x_r \end{aligned}$$

We perform the change of variables :$X_1 = x_1, \dots, X_{i-1} = x_{i-1}, X_i = x_i + x_{i+1}, X_{i+1} = x_{i+2}, \dots, X_{r-1} = x_r$. Using Fubini Theorem, the previous integral becomes :

$$\begin{aligned} & \int_{\mathbb{R}^r} f_{\omega_1}(X_1) \dots f_{\omega_{i-1}}(X_{i-1}) f_{\omega_i}(x_i) f_{\omega_{i+1}}(X_i - x_i) f_{\omega_{i+2}}(X_{i+1}) \dots \\ & f_{\omega_r}(X_{r-1}) \sigma_{\epsilon_1}\left(\check{X}_1\right) \dots \sigma_{\epsilon_{i-1}}\left(\check{X}_{i-1}\right) \sigma_{\epsilon_{i+1}}\left(\check{X}_{i+1}\right) \dots \sigma_{\epsilon_r}\left(\check{X}_{r-1}\right) \mathrm{d}X_1 \dots \mathrm{d}x_i \dots \mathrm{d}X_r \\ = & \int_{\mathbb{R}^{r-1}} f_{\omega_1}(X_1) \dots f_{\omega_{i-1}}(X_{i-1}) \left(\int_{\mathbb{R}} f_{\omega_i}(x_i) f_{\omega_{i+1}}(X_i - x_i) \mathrm{d}x_i \right) f_{\omega_{i+2}}(X_{i+1}) \dots \\ & f_{\omega_r}(X_{r-1}) \sigma_{\epsilon_1}\left(\check{X}_1\right) \dots \sigma_{\epsilon_{i-1}}\left(\check{X}_{i-1}\right) \sigma_{\epsilon_{i+1}}\left(\check{X}_{i+1}\right) \dots \sigma_{\epsilon_r}\left(\check{X}_{r-1}\right) \mathrm{d}X_1 \dots \mathrm{d}X_r \\ = & \int_{\mathbb{R}^{r-1}} f_{\omega_1}(X_1) \dots f_{\omega_{i-1}}(X_{i-1}) f_{\omega_i + \omega_{i+1}}(X_i) f_{\omega_{i+2}}(X_{i+1}) \dots \\ & f_{\omega_r}(X_{r-1}) \sigma_{\epsilon_1}\left(\check{X}_1\right) \dots \sigma_{\epsilon_{i-1}}\left(\check{X}_{i-1}\right) \sigma_{\epsilon_{i+1}}\left(\check{X}_{i+1}\right) \dots \sigma_{\epsilon_r}\left(\check{X}_{r-1}\right) \mathrm{d}X_1 \dots \mathrm{d}X_r \\ = & \mathbf{m}\begin{pmatrix} \omega_1 & \dots & \omega_{i-1} & \omega_i + \omega_{i+1} & \dots & \omega_r \\ \epsilon_1 & \dots & \epsilon_{i-1} & \epsilon_{i+1} & \dots & \epsilon_r \end{pmatrix} \end{aligned}$$

and (*autocoI*) is proved.

□

Proposition 7.4.2 *$\mathbf{m}$ respects the realness if and only if γ is even.*

Proof — We assume that γ is even then so are f_ω. Indeed for any $\omega \in \Omega$, $f_\omega(-x) = \overline{f}_\omega(x) = \int_{\mathbb{R}} e^{-\omega\gamma(y)} e^{-ixy} \mathrm{d}y = \int_{\mathbb{R}} e^{-\omega\gamma(-y)} e^{ixy} \mathrm{d}y = f_\omega(x)$.

Then for any $r \in \mathbf{N}^*$, $(\omega_1, \ldots \omega_r) \in \Omega^r, (\epsilon_1, \ldots, \epsilon_r) \in \{+,-\}^r$:

$$\begin{aligned}\overline{\mathbf{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ \epsilon_1 & \ldots & \epsilon_r\end{pmatrix}}} &= \int_{\mathbb{R}^r} \overline{f}_{\omega_1}(x_1)\ldots \overline{f}_{\omega_r}(x_r)\,\sigma_{\epsilon_1}(\check{x}_1)\ldots\sigma_{\epsilon_r}(\check{x}_r)\,\mathrm{d}x_1\ldots \mathrm{d}x_r\\ &= \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\,\sigma_{\overline{\epsilon_1}}(\check{x}_1)\ldots\sigma_{\overline{\epsilon_r}}(\check{x}_r)\,\mathrm{d}x_1\ldots \mathrm{d}x_r\\ &= \mathbf{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ \overline{\epsilon_1} & \ldots & \overline{\epsilon_r}\end{pmatrix}}\end{aligned}$$

and then $\mathbf{m}$ preserves the realness [7].

— We assume that $\mathbf{m}$ preserves the realness. Then for any $\omega \in \Omega$, $\overline{\mathbf{m}^{\binom{\omega}{+}}} = \mathbf{m}^{\binom{\omega}{-}}$ whence $\int_{\mathbb{R}} \overline{f_\omega(x)}\sigma_+(x)\,\mathrm{d}x = \int_{\mathbb{R}} f_\omega(x)\sigma_-(x)\,\mathrm{d}x$. Thus $\int_{\mathbb{R}}\left(\overline{f_\omega(x)} - f_\omega(-x)\right)\sigma_+(x)\,\mathrm{d}x = 0$ and then $\overline{f_\omega(x)} = f_\omega(-x)$ which is possible only if γ is odd.

□

Proposition 7.4.3 *The right and left lateral moulds associated to* $\boldsymbol{m}$ *are given by :*

$$\boldsymbol{rem}^{\omega_1,\ldots,\omega_r} = \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\,\boldsymbol{sofo}_+^{x_1,\ldots,x_r}\,dx_1\ldots dx_r$$

$$\boldsymbol{lem}^{\omega_1,\ldots,\omega_r} = \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\,\boldsymbol{sofo}_-^{x_1,\ldots,x_r}\,dx_1\ldots dx_r$$

where $\boldsymbol{sofo}_\pm = (-1)^r\,\sigma_\pm(\check{x}_1)\ldots\sigma_\pm(\check{x}_r)$

Proof This follows directly from the definition of $\mathbf{rem}^\bullet$ and $\mathbf{lem}^\bullet$. See remark 7.1.4.2. □

Proposition 7.4.4 *The moulds* $\boldsymbol{sofo}_\pm^\bullet$ *,* $\boldsymbol{rem}^\bullet$ *and* $\boldsymbol{lem}^\bullet$ *are symmetrel. So* $\boldsymbol{m}$ *respects the convolution.*

Proof Let us consider two sequences $\underline{x}$ and $\underline{y} \in \Omega^\bullet$ of respective lengths r and s. We must prove that

$$\mathbf{sofo}_+^{\underline{x}}\mathbf{sofo}_+^{\underline{y}} = \sum_{\underline{\omega}\in\mathbf{csh}(\underline{x},\underline{y})} \mathbf{sofo}_+^{\underline{\omega}} \quad (\star)\,.$$

Let us use the following fact. The set $\mathbf{csh}(\underline{x},\underline{y})$ can be decomposed into the disjoint unions

$$\mathbf{csh}(\underline{x},\underline{y}) = \left(\mathbf{csh}(\underline{x}`,\underline{y}), x_r\right)\bigsqcup\left(\mathbf{csh}(\underline{x},\underline{y}'), y_s\right)\bigsqcup\left(\mathbf{csh}(\underline{x}',\underline{y}'), (x_r+y_s)\right).$$

7. One has too $\overline{\mathbf{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ \epsilon_1 & \ldots & \epsilon_r\end{pmatrix}}} = \mathbf{m}^{\begin{pmatrix}\omega_1 & \ldots & \omega_r\\ \epsilon_1 & \ldots & \epsilon_r\end{pmatrix}}$ and so the weights of a diffusion induced average associated to an odd function γ are real ones.

One performs an induction on the integer $t = r + s$. If $t = 0$ or $t = 1$ then the result is obvious. If $t = 2$, then there are two possibilities. The first is that one of the two integers r or s is null and the second is 2 and we clearly obtain $(\star)$. If $r = s = 1$ then $\underline{x} = x \in \Omega$, $\underline{y} = y \in \Omega$ and using the definition of σ_+, we easily verify that

$$\sigma_+ (x)\, \sigma_+ (x + y) + \sigma_+ (y)\, \sigma_+ (x + y) = \sigma_+ (x)\, \sigma_+ (y) + \sigma_+ (x + y) \qquad (\star\star)$$

that can be written $\mathbf{sofo}_+^{x}\mathbf{sofo}_+^{y} = \mathbf{sofo}_+^{x,y} + \mathbf{sofo}_+^{y,x} + \mathbf{sofo}_+^{x+y}$ and then $(\star)$ is true in this case. Let us consider $t \in \mathbf{N}$. We assume that $(\star)$ is true for any sequences $\underline{x}$ and $\underline{y} \in \Omega^\bullet$ of respective lengths r and s such that $r + s \leq t$. We consider two sequences $\underline{x}$ and $\underline{y} \in \Omega^\bullet$ of respective lengths r and s such that $r + s = t + 1$. If one of the two integers r, s is null then the result is obvious. We assume that it is not the case. Then we compute :

$$\sum_{\underline{\omega} \in \mathbf{csh}(\underline{x},\underline{y})} \mathbf{sofo}_+^{\underline{\omega}} = \left(\sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x}',\underline{y}),x_r)} + \sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x},\underline{y}'),y_s)} + \sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x}',\underline{y}'),(x_r+y_s))} \right) \mathbf{sofo}_+^{\underline{\omega}}.$$

But using the induction hypothesis

$$\begin{aligned} \sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x}',\underline{y}),x_r)} \mathbf{sofo}_+^{\underline{\omega}} &= \sum_{\underline{\omega} \in \mathbf{csh}((\underline{x}',\underline{y}))} \mathbf{sofo}_+^{\underline{\omega}} \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right) \\ &= \mathbf{sofo}_+^{\underline{x}'} \mathbf{sofo}_+^{\underline{y}} \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right) \end{aligned}$$

We obtain in the same way :

$$\sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x},\underline{y}'),y_s)} \mathbf{sofo}_+^{\underline{\omega}} = \mathbf{sofo}_+^{\underline{x}} \mathbf{sofo}_+^{\underline{y}'} \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right)$$

$$\sum_{\underline{\omega} \in (\mathbf{csh}(\underline{x}',\underline{y}'),(x_r+y_s))} \mathbf{sofo}_+^{\underline{\omega}} = \mathbf{sofo}_+^{\underline{x}'} \mathbf{sofo}_+^{\underline{y}'} \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right).$$

Then using equality $(\star\star)$

$$\begin{aligned} & \sum_{\underline{\omega} \in \mathbf{csh}(\underline{x},\underline{y})} \mathbf{sofo}_+^{\underline{\omega}} \\ = \; & (-1)^{t-1} \sigma_+ (\check{x}_1) \ldots \sigma_+ (\check{x}_{r-1})\, \sigma_+ (\check{y}_1) \ldots \sigma_+ (\check{y}_{s-1}) \left[\sigma_+ (\check{y}_s)\, \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right) \right. \\ & \left. + \sigma_+ (\check{x}_r)\, \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right) - \sigma_+ \left(\|\underline{x}\| + \|\underline{y}\| \right) \right] \\ = \; & (-1)^{t+1} \sigma_+ (\check{x}_1) \ldots \sigma_+ (\check{x}_{r-1})\, \sigma_+ (\check{y}_1) \ldots \sigma_+ (\check{y}_{s-1})\, \sigma_+ (\check{x}_r)\, \sigma_+ (\check{y}_t) \\ = \; & \mathbf{sofo}_+^{\underline{x}} \mathbf{sofo}_+^{\underline{y}} \end{aligned}$$

and thus the symmetrelity of $\mathbf{sofo}_+^\bullet$ is proved. The symmetrelity of $\mathbf{rem}^\bullet$ and $\mathbf{lem}^\bullet$ follows easily from this of $\mathbf{sofo}_+^\bullet$. □

Proposition 7.4.5 *The moulds $\boldsymbol{sofo}_\pm^\bullet$ retain their form under contracting anti-arborification :*

$$\boldsymbol{sofo}_\pm^{(x_1,\ldots,x_r)\gg} = \sigma_\pm (\check{x}_1) \ldots \sigma_\pm (\check{x}_r)$$

but the sums $\check{x}_i$ are related to the anti-arborescent order. So $\boldsymbol{m}$ is analytic[8].

Proof Let us perform the proof for the mould $\mathbf{sofo}_+$, the proof for $\mathbf{sofo}_-$ being identical. We will reason inductively on the length r of $\underline{x}^{\gg}$. If $r = 0$ or $r = 1$ then the result is obvious. We assume that the result is true for any sequences of length $r > 0$ and we will consider a sequence $\underline{x}^{\gg}$ of length $r + 1$.

— if $\underline{x}^{\gg} = \underline{y}^{\gg} \oplus \underline{z}^{\gg}$ then since $\mathbf{sofo}_+^{\bullet\gg}$ is separatif (because $\mathbf{sofo}_+^{\bullet}$ is symmetrel) we get, using our induction hypothesis

$$\mathbf{sofo}_+^{\underline{x}^{\gg}} = \mathbf{sofo}_+^{\underline{y}^{\gg}} \mathbf{sofo}_+^{\underline{z}^{\gg}} = (-1)^{r+1} \sigma(\check{y}_1) \dots \sigma(\check{y}_{r_1}) \sigma(\check{z}_1) \dots \sigma(\check{z}_{r_2})$$

where
— $r_1 = \mathrm{l}\left(\underline{y}^{\gg}\right)$, $r_2 = \mathrm{l}\left(\underline{z}^{\gg}\right)$ and $r_1 + r_2 = r + 1$.
— the sums $\check{y}_i$ and $\check{z}_j$ are related to the anti-arborescent order.
Then $\mathbf{sofo}_+^{\underline{x}^{\gg}}$ get the expected form.

— If $\underline{x}^{\gg}$ is an irreductible sequence, then it get a root x_{r+1} and using the definition of the anti-arborification of a mould we obtain :

$$\begin{aligned}
\mathbf{sofo}_+^{\underline{x}^{\gg}} &= \sum_{\underline{x} \in \mathbf{ct}(\underline{x}^{\gg})} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) \mathbf{sofo}_+^{\underline{x}} \\
&= \sum_{\underline{x} \in \mathbf{ct}(\underline{x}^{\gg})} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) \mathbf{sofo}_+^{\underline{x}', x_{r+1}} \\
&= \sum_{\underline{x}' \in \mathbf{ct}(\underline{x}'^{\gg})} \mathbf{cts}\left(\underline{x}'^{\gg}, \underline{x}'\right) \mathbf{sofo}_+^{\underline{x}'} \sigma_+\left(\|\underline{x}'\| + x_{r+1}\right) \\
&= \mathbf{sofo}_+^{\underline{x}'^{\gg}} \sigma_+\left(\|\underline{x}'\| + x_{r+1}\right)
\end{aligned}$$

which is of the announced form.

□

7.4.2 An example of well behaved derivation : the diffusion induced one

Similarly to the previous section, we will present here the work of J. Ecalle about derivation induced by diffusion, keeping the same notations than in the previous paragraph.

Definition 7.4.2.1 *We consider the family of weights Δ_{diff} given by $\Delta_{diff}^{\emptyset} = 0$ and for any $r \in \mathbf{N}^*$, $(\omega_1, \dots \omega_r) \in \Omega^r, (\epsilon_1, \dots, \epsilon_r) \in \{+, -\}^r$:*

$$\Delta_{diff}^{\begin{pmatrix} \omega_1 & \dots & \omega_r \\ \epsilon_1 & \dots & \epsilon_r \end{pmatrix}} = \frac{\epsilon_r}{2i\pi} \int_{\mathbb{R}^r} f_{\omega_1}(x_1) \dots f_{\omega_r}(x_r) \sigma_{\epsilon_1}(\check{x}_1) \dots \sigma_{\epsilon_{r-1}}(\check{x}_{r-1}) \delta(\check{x}_r)\, dx_1 \dots dx_r.$$

Proposition 7.4.6 *This family of weights satisfies the auto-coherences relations and then it defines an **ALIEN** operator Δ_{diff}.*

8. Indeed, $\left|\mathbf{sofo}^{(x_1, \dots, x_r)\gg}\right| \leq 1$ and one can thus apply Theorem 7.2.1

Proof The computation that allows to prove that Δ_{diff} satisfies $(autocoII)$ is the same than in Proposition 7.4.1. In order to prove the auto-coherence relations $(autocoI)$, we consider $r \in \mathbf{N}^*$ and $i \in [\![1, n]\!]$. Then for any $(\omega_1, \dots \omega_r) \in \Omega^r, (\epsilon_1, \dots, \epsilon_{r-1}) \in \{+, -\}^{r-1}$, we have :

$$\Delta_{\text{diff}}^{\begin{pmatrix}\omega_1 & \dots & \omega_{r-1} & \omega_r \\ \epsilon_1 & \dots & \epsilon_{r-1} & +\end{pmatrix}} + \Delta_{\text{diff}}^{\begin{pmatrix}\omega_1 & \dots & \omega_{r-1} & \omega_r \\ \epsilon_1 & \dots & \epsilon_{r-1} & -\end{pmatrix}}$$
$$= \frac{1}{2i\pi}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\,\sigma_{\epsilon_1}(\check{x}_1)\dots\sigma_{\epsilon_{r-1}}(\check{x}_{r-1})\,\delta(\check{x}_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r$$
$$-\frac{1}{2i\pi}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\,\sigma_{\epsilon_1}(\check{x}_1)\dots\sigma_{\epsilon_{r-1}}(\check{x}_{r-1})\,\delta(\check{x}_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r = 0$$

and then $(autocoI)$ is satisfied. □

Proposition 7.4.7 *The operator Δ_{diff} preserves the realness.*

Proof Indeed, for any $r \in \mathbf{N}^*$, $(\omega_1, \dots \omega_r) \in \Omega^r, (\epsilon_1, \dots, \epsilon_r) \in \{+, -\}^r$:

$$\begin{aligned}
\overline{\Delta_{\text{diff}}^{\begin{pmatrix}\omega_1 & \dots & \omega_r \\ \epsilon_1 & \dots & \epsilon_r\end{pmatrix}}} &= -\frac{\epsilon_r}{2i\pi}\int_{\mathbb{R}^r} \overline{f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)}\sigma_{\epsilon_1}(\check{x}_1)\dots\sigma_{\epsilon_{r-1}}(\check{x}_{r-1})\,\delta(\check{x}_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r \\
&= \frac{\overline{\epsilon_r}}{2i\pi}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\,\sigma_{\overline{\epsilon_1}}(\check{x}_1)\dots\sigma_{\overline{\epsilon_{r-1}}}(\check{x}_{r-1})\,\delta(\check{x}_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r \\
&= \Delta_{\text{diff}}^{\begin{pmatrix}\omega_1 & \dots & \omega_r \\ \overline{\epsilon_1} & \dots & \overline{\epsilon_r}\end{pmatrix}} \qquad \text{Q.e.d}
\end{aligned}$$

□

Let us remark that the right lateral mould of Δ_{diff} is given by :

$$< \Delta_{\text{diff}}, \Delta^+ >^{\omega_1,\dots,\omega_r} := \mathbf{red}^{\omega_1,\dots,\omega_r} = \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\,\mathbf{lefo}_+^{x_1,\dots,x_r}\mathrm{d}x_1\dots\mathrm{d}x_r$$

where $\mathbf{lefo}_\pm^{\omega_1,\dots,\omega_r} = (-1)^r\,\sigma_\pm(\check{x}_1)\dots\sigma_\pm(\check{x}_{r-1})\,\delta(\check{x}_r)$.

Proposition 7.4.8 *The moulds $\mathbf{lefo}_\pm^\bullet$, $\mathbf{red}^\bullet$ and $\mathbf{led}^\bullet$ are alternel. So Δ_{diff} is a derivation called **diffusion-induced derivation**.*

Proof Let us consider two sequences $\underline{x}$ and $\underline{y} \in \Omega^\bullet$ of respective lengths r and s. We must prove that

$$\sum_{\underline{\omega}\in\mathbf{cts}(\underline{x},\underline{y})} \mathbf{sofo}_+^{\underline{\omega}} = 0 \quad (\star)\,.$$

As for the symmetrelity of $\mathbf{sofo}_\pm^\bullet$, we will use the fact that the set $\mathbf{csh}\left(\underline{x}, \underline{y}\right)$ can be decomposed into the disjoint union :

$$\left(\mathbf{csh}\left(\underline{x}‘, \underline{y}\right), x_r\right) \bigsqcup \left(\mathbf{csh}\left(\underline{x}, \underline{y}'\right), y_s\right) \bigsqcup \left(\mathbf{csh}\left(\underline{x}', \underline{y}'\right), (x_r + y_s)\right).$$

Using the symmetrelity of $\mathbf{sofo}_{\pm}^{\bullet}$, we have :

$$\begin{aligned}
\sum_{\underline{\omega}\in\mathbf{csh}\left(\underline{x},\underline{y}\right)} \mathbf{lefo}_{+}^{\omega} &= \left(\sum_{\underline{\omega}\in\left(\mathbf{csh}\left(\underline{x}',\underline{y}\right),x_r\right)} + \sum_{\underline{\omega}\in\left(\mathbf{csh}\left(\underline{x},\underline{y}'\right),y_s\right)} + \sum_{\underline{\omega}\in\left(\mathbf{csh}\left(\underline{x}',\underline{y}'\right),(x_r+y_s)\right)} \right) \mathbf{lefo}_{+}^{\omega} \\
&= \left(\sum_{\underline{\omega}\in\mathbf{csh}\left(\underline{x}',\underline{y}\right)} + \sum_{\underline{\omega}\in\mathbf{csh}\left(\underline{x},\underline{y}'\right)} + \sum_{\underline{\omega}\in\mathbf{csh}\left(\underline{x}',\underline{y}'\right)} \right) \mathbf{sofo}_{+}^{\omega} \delta\left(\|\underline{x}\| + \|\underline{y}\|\right) \\
&= \left(\mathbf{sofo}_{+}^{\mathrm{x}'} \mathbf{sofo}_{+}^{\mathrm{y}} + \mathbf{sofo}_{+}^{\mathrm{x}} \mathbf{sofo}_{+}^{\mathrm{y}'} - \mathbf{sofo}_{+}^{\mathrm{x}'} \mathbf{sofo}_{+}^{\mathrm{y}'} \right) \delta\left(\|\underline{x}\| + \|\underline{y}\|\right) \\
&= \left(\sigma\left(\check{y}_s\right) + \sigma\left(\check{x}_s\right) - 1\right) \mathbf{sofo}_{+}^{\mathrm{x}'} \mathbf{sofo}_{+}^{\mathrm{y}'} \delta\left(\|\underline{x}\| + \|\underline{y}\|\right) \\
&= 0
\end{aligned}$$

that proves the alternelity of $\mathbf{lefo}_{+}^{\bullet}$. The alternelity of $\mathbf{red}^{\bullet}$ and $\mathbf{led}^{\bullet}$ follows easily from this of $\mathbf{lefo}_{+}^{\bullet}$. □

Proposition 7.4.9 *The moulds $\boldsymbol{lefo}_{\pm}^{\bullet}$ retain their form under contracting anti-arborification. More precisely, for any $\underline{x} \gg = (x_1, \ldots, x_r) \gg$:*

$$\boldsymbol{lefo}_{\pm}^{x\gg} = \sigma_{\pm}\left(\check{x}_1\right) \ldots \sigma_{\pm}\left(\check{x}_{r-1}\right) \delta\left(\check{x}_r\right)$$

where the sums $\check{x}_i$ are related to the anti-arborescent order. So Δ_{diff} is analytic[9]*.*

Proof The proof will be performed for the mould $\mathbf{lefo}_{+}$. The proof for the mould $\mathbf{lefo}_{-}$ being identical. As for contracting anti-arborification of the mould $\mathbf{sofo}_{\pm}$, we will reason inductively on the length r of the anti-arborified sequence $\underline{x}^{\gg}$. If $r = 1$ then the result is obvious. We assume that the result is true for any sequences of length $r > 0$ and we will consider a sequence $\underline{x}^{\gg}$ of length $r + 1$. Since $\mathbf{lefo}_{+}^{\bullet}$ is alternal, $\mathbf{lefo}_{+}^{\bullet\gg}$ is atomic and we must assume that $x^{\gg}$ is irreductible. Then there exists an antiarborified sequence $\underline{x}'^{\gg}$ and $x_{r+1} \in \Omega$ such that $x^{\gg} = x'^{\gg}.x_{r+1}$. Using the definition of the anti-arborification of a mould we get :

$$\begin{aligned}
\mathbf{lefo}_{+}^{\mathrm{x}\gg} &= \sum_{\underline{x}\in\mathbf{ct}\left(\underline{x}^{\gg}\right)} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) \mathbf{lefo}_{+}^{\mathrm{x}} \\
&= \sum_{\underline{x}\in\mathbf{ct}\left(\underline{x}^{\gg}\right)} \mathbf{cts}\left(\underline{x}^{\gg}, \underline{x}\right) \mathbf{lefo}_{+}^{\mathrm{x}',x_{r+1}} \\
&= -\sum_{\underline{x}'\in\mathbf{ct}\left(\underline{x}'^{\gg}\right)} \mathbf{cts}\left(\underline{x}'^{\gg}, \underline{x}'\right) \mathbf{sofo}_{+}^{\mathrm{x}'} \delta\left(\|\underline{x}'\| + x_{r+1}\right) \\
&= -\mathbf{sofo}_{+}^{\mathrm{x}'\gg} \delta\left(\|\underline{x}'\| + x_{r+1}\right) \\
&= \sigma_{\pm}\left(\check{x}_1\right) \ldots \sigma_{\pm}\left(\check{x}_r\right) \delta\left(\check{x_{r+1}}\right)
\end{aligned}$$

where the sums $\check{x}_i$ are related to the anti-arborescent order.

□

9. Indeed, $\left|\mathbf{lefo}^{(x_1,\ldots,x_r)\gg}\right| \leq 1$ and one can thus apply Theorem 7.2.1

Chapitre 8

Un isomorphisme entre deux familles de germes d'ensembles $\mathcal{C}$-semi-analytiques

On doit être un logicien ou un grammairien rigoureux, et être en même temps plein de fantaisie et de musique.
Hermann Hesse - Le jeu des perles de verre

8.1 Introduction

Nous nous intéressons dans ce chapitre à des sous-ensembles de $\mathbb{R}^n$ obtenus (localement ou globalement) comme solutions d'un nombre fini d'équations ou d'inéquations faisant intervenir des fonctions $f : \mathbb{R}^n \to \mathbb{R}$ prises dans une famille donnée $\mathcal{C}$. De tels ensembles sont dits $\mathcal{C}$-semi-analytiques. Un premier exemple historique est celui où les sous-ensembles sont solutions globales d'équations ou d'inéquations polynomiales réelles. On a donc $\mathcal{C} = \cup_{n\in\mathbf{N}}\mathbb{R}[X_1, \ldots, X_n]$. Les ensembles $\mathcal{C}$-semi-analytiques sont alors appelés *ensembles semi-algébriques réels.* Un second exemple historique consiste à considérer pour $\mathcal{C}$ l'ensemble des fonctions $f : \mathbb{R}^n \to \mathbb{R}$ dont la restriction à $[-1, 1]^n$ est analytique réelle (et nulle ailleurs). Les ensembles $\mathcal{C}$-semi-analytiques sont ici donnés localement et sont dits semi-analytiques (on omet le préfixe $\mathcal{C}$-). Dans ces deux cas, les ensembles $\mathcal{C}$-semi-analytiques jouissent de propriétés remarquables. Ils admettent des divisions en cellules, sont triangulables et n'ont qu'un nombre fini de composantes connexes (dès qu'ils sont bornés). Parmi ces propriétés, indiquons que :

— Le complémentaire d'un ensemble $\mathcal{C}$-semi-analytique est encore $\mathcal{C}$-semi-analytique.
— Toute projection d'un ensemble $\mathcal{C}$-semi-analytique est encore $\mathcal{C}$-semi-

analytique.

La preuve de ces deux points revient à Tarski (et Seidenberg pour la propriété du complémentaire) pour les ensembles semi-algébriques et à Gabrielov pour les ensembles semi-analytiques.

Au début des années 1980, Van Den Dries remarque que la plupart des propriétés des ensembles semi-algébriques peuvent être dérivées des axiomes définissant une structure o-minimale. En théorie des modèles, une *structure* $\mathcal{M}$ consiste en la donnée d'un *langage* $\mathcal{L}$ et d'un univers M sur lequel s'interprètent les éléments du langage. Par exemple, les axiomes définissant le corps des réels muni de sa relation d'ordre canonique consiste en la structure $(\mathbb{R}, +, -, ., <, 0, 1)$. En combinant suivant des règles précises les éléments d'un langage, on construit des *formules*. Les sous-ensembles de M^n satisfaisant certaines parmi celles-ci sont dit *définissables* pour cette structure. Nous renvoyons à [38] pour des définitions précises. Les ensembles définissables dans la structure $(\mathbb{R}, +, -, ., <, 0, 1)$ sont exactement les ensembles semi-algébriques. On a de plus les correspondances suivantes :

— Le fait que les projections d'ensembles semi-algébriques sont encore semi-algébriques est équivalent au fait que la *théorie* associée à la structure $(\mathbb{R}, +, -, ., <, 0, 1)$ est *libre de quantificateur*, ce qui signifie que toute formule avec quantificateurs est équivalente à une formule sans quantificateur.
— La finitude du nombre de composantes connexes des ensembles semi-algébriques correspond à l'*o-minimalité* de la structure $(\mathbb{R}, +, -, ., <, 0, 1)$, c'est-à-dire que les sous-ensembles de $\mathbb{R}$ définissables dans la structure ont un nombre fini de composantes connexes.
— Enfin, le fait que le complémentaire d'un ensemble semi-algébrique est encore semi-algébrique est équivalent à la *modèle complétude* de la théorie considérée.

On peut ainsi reformuler le résultat de Gabrielov en écrivant que la structure $\mathbb{R}_{\mathrm{an}} = (\mathbb{R}, +, -, ., <, 0, 1, \mathcal{C})$ [1] est *o*-minimale, modèle complète et libre de quantificateur. Ajoutons qu'une structure obtenue à partir d'une autre en lui adjoignant de nouveaux symboles est appelée une *extension* de cette première structure.

Depuis les travaux de Van Den Dries, de nombreuses extensions du corps des réels ont été investiguées. Nous en avons dressé une liste non exhaustive dans l'introduction de la thèse. Celle qui nous intéresse ici a été étudiée au début des années 2000 par Rolin, Speisseger et Wilkie ([48]). C'est une extension du corps des réels par des symboles de fonctions issues de certaines classes quasi-analytiques, que nous appellerons *classes RSW-quasianalytiques*, et dont un exemple est donné par certaines classes de Denjoy-Carleman. Les auteurs prouvent dans [48] l'o-minimalité et la modèle complétude de cette extension. Rolin, Sanz et Schäfke appliquent dans [47] ce résultat à l'extension $\mathbb{R}_{\mathrm{an,H}}$ où $H :]0, \epsilon] \to \mathbb{R}$ est une solution arbitraire d'une équation différentielle ordinaire

1. où $\mathcal{C}$ représente bien entendu l'ensemble des fonctions $f : \mathbb{R}^n \to \mathbb{R}$ dont la restriction à $[-1, 1]^n$ est analytique réelle et nulle ailleurs pour tout $n \in \mathbf{N}$.

réelle de type noeud-col [2]

$$x^2y' = A(x,y)$$

avec A analytique réelle dans un voisinage de 0 dans $\mathbb{R}^2$ et vérifiant les mêmes hypothèses que dans la section 3.2. Si $\overset{\circ}{H}$ est la série formelle solution de cette équation différentielle alors on sait qu'elle présente un phénomène de Stokes non trivial suivant les deux demi-directions réelles et donc qu'elle est divergente. La solution H est donc Gevrey-1 asymptotique à $\overset{\circ}{H}$ et n'est pas analytique au voisinage de 0. Les auteurs construisent à partir de cette solution une classe RSW-analytique que nous ne détaillerons pas, nous renvoyons le lecteur à [47]. L'extension $\mathbb{R}_{\text{an,H}}$ est o-minimale et modèle complète et les ensembles définissables dans cette structure sont les *sous-ensembles $\mathcal{C}$-sous-analytiques globaux* [3].

Il n'y a pas unicité de la solution H aussi on peut considérer une autre solution $\tilde{H}$ de cette équation différentielle ainsi que la classe RSW $\tilde{\mathcal{C}}$ et l'extension associées. Notons $\theta : \mathcal{C} \to \cup_{n\in\mathbf{N}}\mathbb{R}[[X_1,\ldots,X_n]]$ ou $\tilde{\theta} : \tilde{\mathcal{C}} \to \cup_{n\in\mathbf{N}}\mathbb{R}[[X_1,\ldots,X_n]]$ les applications qui à une fonction de $\mathcal{C}$ ou $\tilde{\mathcal{C}}$ associent son développement de Taylor en 0. Les deux classes étant quasi-analytiques, ces deux applications sont injectives. De plus, par construction, elles ont le même ensemble image. On peut alors considérer l'application $\tilde{\theta}^{-1} \circ \theta : \mathcal{C} \to \tilde{\mathcal{C}}$. Cette bijection respecte la structure des deux classes RSW $\mathcal{C}$ et $\tilde{\mathcal{C}}$ aussi on dira qu'elle constitue un *RSW isomorphisme*.

Une question naturelle est alors de savoir comment se traduit cet isomorphisme sur les sous-ensembles semi-analytiques issus de chacune de ces deux classes et en particulier de savoir si il existe une application permettant de faire correspondre les $\mathcal{C}$-semi-analytiques de l'une avec les $\tilde{\mathcal{C}}$-semi-analytiques de l'autre. On souhaiterait de plus que cette application préserve les ensembles semi-analytiques et qu'elle envoie le graphe de H sur celui de $\tilde{H}$.

Mais comme on va le vérifier immédiatement, il n'y a pas de telle correspondance. Ainsi, si dans le cas par exemple de l'équation d'Euler $x^2y' = y - x$, on considère deux solutions une de signe constant H et l'autre non $\tilde{H}$ sur un voisinage de 0^+, alors l'ensemble $\mathcal{C}$-semi-analytique d'équation $H(x) = 0$ est vide dans le premier cas et réduit à un point dans le second. Et si une telle correspondance existait, elle devrait envoyer un singleton sur le même singleton et l'ensemble vide sur l'ensemble vide.

Aussi faut-il chercher cette correspondance au niveau des germes en 0 d'ensembles. L'objet de ce chapitre est de construire une telle application Θ qui à tout germe d'ensemble $\mathcal{C}$-semi analytique fait correspondre un germe d' ensemble $\tilde{\mathcal{C}}$-semi analytique. Cet isomorphisme préservera bien les germes d'ensembles semi-analytiques et enverra le germe du graphe de H sur celui de $\tilde{H}$. On vérifiera de plus que c'est un morphisme d'algèbre booléennes.

2. dans [47], les auteurs considèrent en fait le problème plus général d'un système différentiel de la forme $x^{p+1}y' = A(x,y)$ admettant un point singulier-irrégulier, où A est analytique réelle dans un voisinage de $0 \in \mathbb{R}^{r+1}$, $x \in \mathbb{R}$, $y \in \mathbb{R}^r$, où les r valeurs propres de $\frac{\partial A}{\partial y}(0,0)$ sont non nulles et d'argument distincts, et enfin où la solution formelle $\overset{\circ}{H}$ admet un phénomène de Stokes non-trivial dans au moins une direction. Pour simplifier notre exposé, nous nous bornons au cas des noeud-cols déjà abordés dans la thèse.

3. Voir la définition 8.2.1.3 page 174.

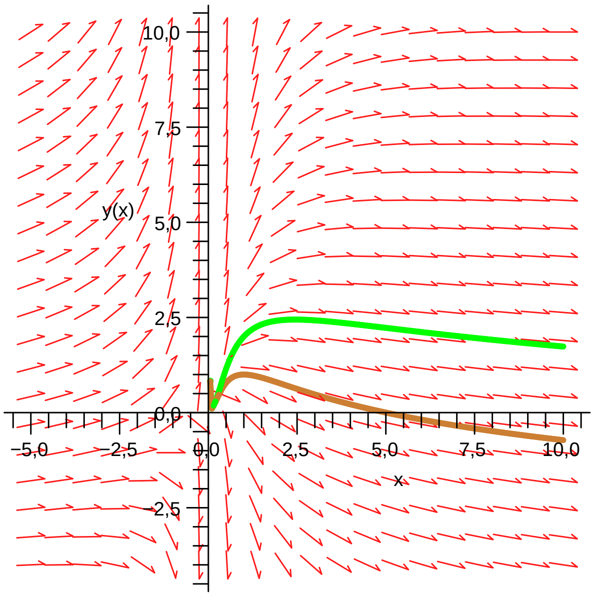


FIGURE 8.1 – Deux solutions de l'équation d'Euler de signes distincts sur un voisinage de 0^+

Nous aurons besoin pour démontrer l'existence de ce morphisme de deux lemmes :

— Le premier classique est une forme du lemme des petits chemins adaptée au ensembles $\mathcal{C}$-sous-analytiques.

— Dans le second, nous considérons un chemin $\varphi : [0, \epsilon] \to \mathbb{R}^n$ élément de $\mathcal{C}$ tel que $\varphi(0) = 0$ et tel que $\varphi\left(]0, \epsilon]\right)$ est inclus dans une réunion finie de sous-ensembles $\mathcal{C}$-sous-analytiques admettant 0 dans son adhérence. Nous montrerons que, quitte à diminuer ϵ, ce chemin est nécessairement tout entier inclus dans un de ces sous-ensembles.

Ces deux lemmes nous permettront alors de démontrer que l'application Θ est bien définie. Nous conclurons en expliquant pourquoi nous n'avons pu étendre cet isomorphisme aux germes de sous-ensembles.

8.2 Définitions et notations

8.2.1 Classes RSW et géométrie semi-analytique

Commençons par rappeler ce qu'est une classe RSW (voir aussi [48]).

Définition 8.2.1.1 *On appelle* ***classe RSW*** *la donnée pour tout* $n \in \mathbf{N}$, *pour toute boite compacte* $B = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$ *(avec* $a_i < b_i$ *pour tout* $i \in [\![1, n]\!]$*) d'une* $\mathbb{R}$*-algèbre* $\mathcal{C}_B$ *de fonctions* $f : B \to \mathbb{R}$ *telles que :*

C_1 $\mathcal{C}_B$ *contient les* ***projections*** *et pour tout* $f \in \mathcal{C}_B$, ***la restriction de*** f ***à*** $int(B)$ ***est*** $\mathcal{C}^\infty$.

C_2 ***Stabilité par composition :***
Pour toute autre boite compacte $B' \subset \mathbb{R}^m$ et si $g_1, \ldots, g_m \in \mathcal{C}_{B'}$ sont telles que $g(B') \subset B$ avec $g = (g_1, \ldots, g_m)$ alors pour tout $f \in \mathcal{C}_B$, on a $y \mapsto f(g_1(y), \ldots, g_m(y)) \in \mathcal{C}_{B'}$.

C_3 ***Stabilité par restriction :***
Si $B' \subset B$ est une boite compacte, alors pour tout $f \in \mathcal{C}_B$, $f_{|B'} \in \mathcal{C}_{B'}$ et pour tout $f \in \mathcal{C}_B$, il existe une boite compacte $B' \subset \mathbb{R}^n$ et $g \in \mathcal{C}_{B'}$ telles que $B \subset int(B')$ et $g|B = f$.

C_4 ***Stabilité par dérivation partielle :***
$\frac{\partial f}{\partial x_i} \in \mathcal{C}_B$ pour tout $f \in \mathcal{C}_B$ et tout $i \in [\![1, n]\!]$.

On introduit maintenant quelques notations.

Notation 8.2.1.1 *Soit $\mathcal{C}$ une classe RSW :*

- *Soit $r = (r_1, \ldots, r_n) \in (\mathbb{R}_+^*)^n$. On notera I_r le polydisque de $\mathbb{R}^n$ centré en 0 de polyrayon r : $I_r =]-r_1, r_1[\times \ldots \times]-r_n, r_n[$. Le polydisque fermé correspondant sera noté $\overline{I_r} (= Adh(I_r))$.*
- *On notera $\mathcal{V}_n(0)$ (ou lorsque aucune confusion n'est à craindre $\mathcal{V}(0)$) l'ensemble des voisinages de 0 dans $\mathbb{R}^n$.*
- *Si $B = \overline{I_r} = [-r_1, r_1] \times \ldots \times [-r_n, r_n]$, on notera $\mathcal{C}_{n,r} = \mathcal{C}_B$.*
- *Soient $r \in (\mathbb{R}_+^*)^n$ un polyrayon et I_r le polydisque de $\mathbb{R}^n$ centré en 0 correspondant. Soient $\mu \in \mathbf{N}^*$, $f \in \mathcal{C}_{n,r}^{\mu}$ [4], $W \subset I_r$ un voisinage de 0 dans $\mathbb{R}^n$ et $\sigma \in \{-1, 0, 1\}^{\mu}$ une condition de signe. On notera :*

$$B_W(f, \sigma) = \{x = (x_1, ..., x_n) \in W \mid signe f_1(x) = \sigma_1, ..., signe f_{\mu}(x) = \sigma_{\mu}\}$$

On définit alors les ensembles suivants :

Définition 8.2.1.2

- *Soient $r \in (\mathbb{R}_+^*)^n$ un polyrayon et I_r le polydisque de $\mathbb{R}^n$ centré en 0 correspondant. Un sous-ensemble A de I_r est dit **$\mathcal{C}$-basique**[5] si et seulement si il existe $f \in \mathcal{C}_{n,r}^{\mu}$, $W \subset I_r$ un voisinage de 0 dans $\mathbb{R}^n$ et $\sigma \in \{-1, 0, 1\}^{\mu}$ une condition de signe tels que :*

$$A = B_W(f, \sigma).$$

- *Une union finie d'ensembles $\mathcal{C}$-basiques est appelée un **$\mathcal{C}$-ensemble**.*
- *Soit $A \subset \mathbb{R}^n$. On dit que A est un **ensemble $\mathcal{C}$-semi-analytique** si pour tout $a \in \mathbb{R}^n$, il existe un polyrayon r de $\mathbb{R}^n$ tel que $(A - a) \cap I_r$ soit un $\mathcal{C}$-ensemble.*
- *Soient n, m des entiers tels que $m \leq n$ et soit $B \subset \mathbb{R}^m$. On dit que B est un **ensemble $\mathcal{C}$-sous-analytique** si il existe un ensemble $\mathcal{C}$-semi analytique $A \subset \mathbb{R}^n$ tel que $B = \Pi_m(A)$.*

4. où $\mathcal{C}_{n,r}^{\mu} = (\mathcal{C}_{n,r})^{\mu}$.
5. Attention, ce n'est pas la même définition que dans [48]. Celle donnée ici est un peu plus générale. En effet dans [48], un $\mathcal{C}$-basique est supposé de la forme $\{x \in W \mid f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}$.

On ajoute deux définitions qui ne sont pas dans [48].

Définition 8.2.1.3

- *Un ensemble de* $\mathbb{R}^n$ *est dit* ***$\mathcal{C}$-semi-analytique global*** *si son image par l'application* $\Phi_n : (x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto \left(x_1/\sqrt{1+x_1^2}, \ldots, x_n/\sqrt{1+x_n^2}\right)$ *est* $\mathcal{C}$*-semi-analytique.*
- *Un ensemble de* $\mathbb{R}^m$ *est dit* ***$\mathcal{C}$-sous-analytique global*** *si il est l'image par projection d'un* $\mathcal{C}$*-semi-analytique global*[6].

8.2.2 Classes RSW-quasianalytiques

Notation 8.2.2.1 *Soit* $\mathcal{C}$ *une classe RSW.*

- *Posons* $\mathcal{C}_n := \bigcup_{r \in]0,\infty[^n} \mathcal{C}_{n,r}$.
- *Nous noterons* $\theta_{\mathcal{C}}$ *la collection des morphismes*

$$\theta_{\mathcal{C}_n} : \mathcal{C}_n \to \mathbb{R}[[x_1, \ldots, x_n]]$$

qui à un germe de $\mathcal{C}_n$ *associe son développement de Taylor en* 0.

- *Enfin, nous noterons* $\hat{\mathcal{C}}_n$ *l'image de* $\mathcal{C}_n$ *par* $\theta_{\mathcal{C}}$ *et* $\hat{\mathcal{C}}$ *la collection des* $\hat{\mathcal{C}}_n$.

Définition 8.2.2.1 *Soit* $\mathcal{C}$ *une classe RSW. La classe* $\mathcal{C}$ *est dite* ***RSW-quasianalytique*** *(et notée RSW-QA) si elle vérifie :*

C_5 ***Quasi-analycité :***
Pour tout $n \in \mathbf{N}$*, les morphismes* $\theta_{\mathcal{C}_n}$ *sont injectifs.*

C_6 ***Fonctions implicites :***
Si $n > 1$ *et si* $f \in \mathcal{C}_n$ *est telle que* $f(0) = 0$ *et telle que* $\frac{\partial f}{\partial x_n}(0) \neq 0$ *alors il existe* $\alpha \in \mathcal{C}_{n-1}$ *telle que* $\alpha(0) = 0$ *et tel que* $f(x', \alpha(x')) = 0$ *où* $x' = (x_1, \ldots, x_{n-1})$ *si* $x = (x_1, \ldots, x_n)$.

C_7 ***Division monomiale :***
Si $f \in \mathcal{C}_n$ *et si* $i \leq n$ *sont tels que* $\widehat{f}(X) = X_i G(X)$ *pour* $G \in \mathbb{R}[X_1, ..., X_n]$ *alors il existe* $g \in \mathcal{C}_n$ *tel que* $\widehat{g} = G$ *et tel que* $f = x_i g$.

Nous rappelons le résultat principal de [48] :

Théorème 8.2.1 *Soit* $\mathcal{C}$ *une classe RSW-QA. Alors l'extension* $R_{\mathcal{D}}$ *du corps des réels par* $D = \cup_{n \in \mathbf{N}} \mathcal{C}_{[0,1]^n}$ *est o-minimale et modèle complète.*

De plus les ensembles définissables sans quantificateur dans $R_{\mathcal{D}}$ *sont exactement les ensembles* $\mathcal{C}$*-semi-analytiques globaux et les ensembles définissables sont les ensembles* $\mathcal{C}$*-sous-analytiques globaux.*

Remarque 8.2.2.1 *Il est important de comprendre que les ensembles* $\mathcal{C}$*-semi-analytiques et à fortiori les ensembles* $\mathcal{C}$*-sous-analytiques ne sont pas tous définissables dans* $\mathbb{R}_{\mathcal{D}}$*. En effet, l'ensemble* $\mathbb{Z}$ *est* $\mathcal{C}$*-semi-analytique, il est donné*

6. Remarquons que l'image par Φ d'un sous-analytique global est un sous-analytique.

au voisinage de $a \in \mathbb{Z}$ par la condition $x - a = 0$ et pour tout $x \in \mathbb{R} \setminus \mathbb{Z}$ par la condition $1 = 0$. Comme il possède une infinité de composantes connexes, il n'est pas définissable dans $\mathbb{R}_{\mathcal{D}}$ qui est o-minimale.

8.2.3 Germes d'ensembles

Rappelons la définition suivante :

Définition 8.2.3.1 *On appelle **germe associé à (ou induit par) l'ensemble** $A \subset \mathbb{R}^n$ en $a \in \mathbb{R}^n$ la classe, notée $[A]$, donnée par*

$$[A] = \{A' \subset \mathbb{R}^n \mid \exists V \in \mathscr{V}_n(a) : \quad A' \cap V = A \cap V\}.$$

Remarque 8.2.3.1 *Si on considère un germe $\mathscr{G}$ d'ensemble $\mathcal{C}$-semi-analytique en 0 et $A \subset \mathbb{R}^n$ un ensemble $\mathcal{C}$-semi-analytique qui induit ce germe : $\mathscr{G} = [A]$, par définition des ensembles $\mathcal{C}$-semi-analytiques, il existe $V \in \mathscr{V}_n(0)$ tel que $A \cap V$ est un $\mathcal{C}$-ensemble. Les ensembles A et $A \cap V$ induisent tous deux le germe $\mathscr{G}$. Tout germe d'ensemble $\mathcal{C}$-semi-analytique est donc induit par un $\mathcal{C}$-ensemble.*

Afin de pouvoir expliciter rigoureusement notre problématique, il est opportun de donner une formulation dans le cadre de la théorie des modèles de la notion de germe. C'est ce dont à quoi nous nous employons maintenant.

On considère $\mathcal{C}$ une classe RSW-QA. On introduit alors le langage $\mathcal{L}_{\hat{\mathcal{F}}} = \left(0, 1, +, \times, <, \hat{\mathcal{C}}\right)$.

Remarquons que les termes du langage ne sont pas bien définis car la composition de deux séries ne fait sens que si la série de droite a un terme constant nul. On choisit, comme dans [48], d'étendre comme étant 0 les fonctions là où elles ne sont pas définies.

Notons $\Delta_{\mathcal{C}}$ l'application qui a une formule $\phi = \bigvee_i \bigwedge_j \left(\hat{f}_{ij} \sigma_{ij} 0\right)$ [7] associe, le germe d'ensemble

$$\Delta_{\mathcal{C}}(\phi) = \bigcup_i \bigcap_j \left(\theta_{\mathcal{C}}^{-1}(\hat{f}_{ij}) \sigma_{ij} 0\right).$$

De manière réminiscente à la section 8.2.1, on a les définitions suivantes :

Définition 8.2.3.2
- *L'image par $\Delta_{\mathcal{C}}$ d'une formule de $\mathcal{L}_{\hat{\mathcal{C}}}$ sans quantificateur ni disjonction est appelée un $\mathcal{C}$-**germe basique**.*
- *L'image par $\Delta_{\mathcal{C}}$ d'une formule de $\mathcal{L}_{\hat{\mathcal{C}}}$ sans quantificateur est appelée un $\mathcal{C}$-**germe**.*

Ces notions sont liées à celles de la section 8.2.1 de la façon suivante :

7. Toute formule sans quantificateur de $\mathcal{L}_{\hat{\mathcal{F}}}$ est équivalente à une formule de cette forme.

Proposition 8.2.1

1. *Les $\mathcal{C}$-germes basiques sont exactement les germes à l'origine d'ensembles $\mathcal{C}$-basiques.*
2. *Les $\mathcal{C}$-germes sont exactement les germes à l'origine de $\mathcal{C}$-ensembles.*

Démonstration Soit $\mathscr{A}$ un $\mathcal{C}$-germe basique alors, il existe $m \in \mathbf{N}$ et $\hat{f}_j \in \hat{\mathcal{C}}$, $\sigma_j \in \{>, <, =\}$ pour $j \in [\![1, n]\!]$ tels que $\mathscr{A} = \Delta_{\mathcal{C}}(\phi)$ pour $\phi = \bigwedge_j \left(\hat{f}_j \sigma_j 0\right)$. Un représentant de ce germe d'ensemble est donné par

$$\bigcap_{j \in [\![1,n]\!]} \{x \in W \mid f_j(x)\sigma_j 0\}$$

où W est l'intersection des domaines de définition des f_j. Le germe $\mathscr{A}$ est donc bien le germe d'un ensemble $\mathcal{C}$-basique.

Réciproquement, si $\mathscr{A}$ est un germe d'ensemble $\mathcal{C}$-basique alors, par définition, un représentant de ce germe est donné par $\bigcap_{j \in [\![1,n]\!]} \{x \in W \mid f_j(x)\sigma_j 0\}$ où $f_j \in \mathcal{C}$ pour tout $j \in [\![1, n]\!]$ et donc $\mathscr{A} = \Delta_{\mathcal{C}}(\phi)$ avec $\phi = \bigwedge_j \left(\hat{f}_j \sigma_j 0\right)$. Il vient bien que $\mathscr{A}$ est un $\mathcal{C}$-germe basique.

La démonstration dans le cas des $\mathcal{C}$-germes est identique. □

Une conséquence du théorème 8.2.1 et des notions précédentes est que les ensembles définissables dans l'extension $\mathbb{R}_{\mathcal{D}}$ ne dépendent que de l'algèbre des germes à l'origine et pas des fonctions de la classe RSW. En effet, si on convient d'appeler germe $\mathcal{C}$-semi-analytique en $a \in \mathbb{R}^n$ un germe dont le translaté en a est un $\mathcal{C}$-germe alors on a :

Proposition 8.2.2 *Pour $\mathcal{D} = \cup_{n \in \mathbf{N}} \mathcal{C}_{[0,1]^n}$, les ensembles définissables sans quantificateur dans $R_{\mathcal{D}}$ sont exactement l'image par Φ^{-1} des ensembles A de $[-1, 1]^n$ tels que quelque soit $a \in [-1, 1]^n$, le germe en 0 de $A - a$ est un $\mathcal{C}$-germe.*

Démonstration Soit B un ensemble définissable sans quantificateur dans $R_{\mathcal{D}}$. Alors B est un $\mathcal{C}$-semi-analytique global. Il existe donc un $\mathcal{C}$-semi-analytique $A \subset [-1, 1]^n$ tel que $\phi^{-1}(A) = B$. Si $a \in [-1, 1]^n$, alors il existe un voisinage V de 0 tel que $(A - a) \cap V$ est un $\mathcal{C}$-ensemble, autrement dit le germe de $A - a$ en 0 est un $\mathcal{C}$-germe. □

8.2.4 Exposé du problème

Définition 8.2.4.1 *Considérons $\mathcal{C}$ et $\tilde{\mathcal{C}}$ deux algèbres RSW-QA et $\theta_{\mathcal{C}}$, $\theta_{\tilde{\mathcal{C}}}$ les collections de morphismes injectifs d'algèbres correspondants. Supposons que pour tout n, $\theta_{\mathcal{C}_n}$ et $\theta_{\tilde{\mathcal{C}}_n}$ ont la même image dans $\mathbb{R}[[x_1, ..., x_n]]$. On peut alors considérer l'application $\theta_n = \theta_{\tilde{\mathcal{C}}_n}^{-1} \circ \theta_{\mathcal{C}_n}$ qui est un isomorphisme entre $\mathcal{C}_n$ et $\tilde{\mathcal{C}}_n$ respectant la division monomiale, les fonctions implicites et la composition des fonctions. Un tel isomorphisme, quand il existe, sera appelé **RSW-isomorphisme** et les deux RSW-algèbres $\mathcal{C}$ et $\tilde{\mathcal{C}}$ seront dites **RSW-isomorphes**.*

La question naturelle est alors de comprendre comment un RSW-isomorphisme se transcrit au niveau des germes d'ensembles des deux algèbres. La réponse est donnée par le théorème suivant :

Théorème 8.2.2 *Soient $\mathcal{C}$ et $\tilde{\mathcal{C}}$ deux algèbres RSW-QA qui sont RSW isomorphes. Alors il existe un bijection Θ entre les $\mathcal{C}$-germes et les $\tilde{\mathcal{C}}$-germes vérifiant les propriétés suivantes :*

1. *Θ est un morphisme d'algèbres booléennes : pour tous $\mathcal{C}$-germes A et B,*
   (a) $\Theta(A \cup B) = \Theta(A) \cup \Theta(B)$,
   (b) $\Theta(A \cap B) = \Theta(A) \cap \Theta(B)$,
   (c) $\Theta({}^cA) = {}^c\Theta(A)$.
2. *Θ préserve le produit cartésien : pour tous $\mathcal{C}$-germes A et B, $\Theta(A \times B) = \Theta(A) \times \Theta(B)$.*
3. *Si $\mathcal{F} := \mathcal{C} \cap \tilde{\mathcal{C}}$[8] alors pour tout $\mathcal{F}$-germe A, $\Theta(A) = A$.*

La démonstration de ce théorème repose entièrement sur le théorème suivant. Avant de l'énoncer, introduisons pour les formules du langage $\mathcal{L}_{\hat{\mathcal{C}}}$ la relation d'équivalence suivante. Deux formules ϕ et ψ sont dites $\mathcal{C}$-équivalentes si et seulement si $\Delta_{\mathcal{C}}(\phi) = \Delta_{\mathcal{C}}(\psi)$. On vérifie facilement qu'il s'agit bien d'une relation d'équivalence. On a alors :

Théorème 8.2.3 *Soient $\mathcal{C}$ et $\tilde{\mathcal{C}}$ deux algèbres RSW-QA qui sont RSW isomorphes. Notons $\mathcal{L}$ le langage $\mathcal{L} := \mathcal{L}_{\hat{\mathcal{C}}} = \mathcal{L}_{\hat{\tilde{\mathcal{C}}}}$. Alors deux $\mathcal{L}$-formules sans quantificateur sont $\mathcal{C}$-équivalentes si et seulement si elles sont $\tilde{\mathcal{C}}$-équivalentes.*

La démonstration de ce théorème est l'objet des sections suivantes. Démontrons le théorème 8.2.2.

Démonstration L'application Θ se construit ainsi. Considérons un $\mathcal{C}$-germe A. Par définition, $A = \Delta_{\mathcal{C}}(\phi)$ où $\phi = \bigvee_i \bigwedge_j \left(\hat{f}_j \sigma_j 0\right)$. On veut que Θ lui fasse correspondre le $\tilde{\mathcal{C}}$-germe $\tilde{A}$ donné par $\tilde{A} := \Delta_{\tilde{\mathcal{C}}}(\phi)$. Mais Θ n'est bien définie qu'à la condition suivante. Si ψ est une formule $\mathcal{C}$-équivalente à ϕ, a-t'on encore $\tilde{A} := \Delta_{\tilde{\mathcal{C}}}(\psi)$, autrement dit ψ est-elle aussi $\tilde{\mathcal{C}}$-équivalente à ϕ ? On sait d'après le théorème 8.2.3 que c'est le cas et donc Θ est bien définie.

Il est alors simple de montrer que Θ est bijective. Il suffit en effet de considérer l'application Ξ qui à un germe $\tilde{A} = \Delta_{\mathcal{C}}(\phi)$ de $\tilde{\mathcal{C}}$-ensemble lui associe le $\mathcal{C}$-germe A donné par $\tilde{A} := \Delta_{\tilde{\mathcal{C}}}(\phi)$. Cette application est, pour la même raison, elle aussi bien définie et vérifie $\Xi \circ \Theta = \mathrm{id}$ et $\Theta_\circ \Xi = \mathrm{id}$.

Montrons maintenant que Θ est un morphisme d'algèbres booléennes. Considérons $A = \Delta_{\mathcal{C}}(\phi)$ et $B = \Delta_{\mathcal{C}}(\psi)$ deux $\mathcal{C}$-germes. Alors $\Theta(A \cup B) = \Delta_{\tilde{\mathcal{C}}}(\phi \vee \psi) = \Delta_{\tilde{\mathcal{C}}}(\phi) \cup \Delta_{\tilde{\mathcal{C}}}(\psi) = \Theta(A) \cup \Theta(B)$. On montre de même (b) et (c).

8. qui est non vide car $\mathcal{C}$ et $\tilde{\mathcal{C}}$ contiennent toutes deux les germes de polynômes réels

Montrons que Θ préserve le produit cartésien. Soient $A = \Delta_{\mathcal{C}}(\phi)$ et $B = \Delta_{\mathcal{C}}(\psi)$. Alors $A \times B = \Delta_{\mathcal{C}}(\phi) \times \Delta_{\mathcal{C}}(\psi) = \Delta_{\mathcal{C}}(\varphi)$ où φ est la formule : $\phi(x) \wedge \psi(y)$. Alors $\Theta(A \times B) = \Delta_{\tilde{\mathcal{C}}}(\varphi) = \Delta_{\tilde{\mathcal{C}}}(\phi) \times \Delta_{\tilde{\mathcal{C}}}(\psi) = \Theta(A) \times \Theta(B)$.

Enfin, si A est un $\mathcal{F}$-germe alors il existe $\phi \in \mathcal{L}_{\mathcal{F}}$ telle que $A = \Delta_{\mathcal{F}}(\phi)$. Donc, comme $\mathcal{F} = \mathcal{C} \cap \tilde{\mathcal{C}}$, il vient : $\Delta_{\mathcal{C}}(\phi) = \Delta_{\mathcal{F}}(\phi) = \Delta_{\tilde{\mathcal{C}}}(\phi)$ d'où le résultat. □

8.3 Théorème de désingularisation pour un ensemble $\mathcal{C}$-sous-analytique

Notation 8.3.0.1 *Pour une permutation λ de $\{1, ..., n\}$, on note π_λ l'application :*

$$\pi_\lambda : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (x_1, \ldots, x_n) & \mapsto & \left(x_{\lambda(1)}, \ldots, x_{\lambda(n)}\right) \end{array} \right. .$$

Définition 8.3.0.2 *Soit r un polyrayon de $\mathbb{R}^n$. On dit qu'un **ensemble** $M \subset I_r$ **est $\mathcal{C}$-trivial** si et seulement si :*

- *Soit c'est un "quadrant" : $M = B_{I_r}((x_1, ..., x_n), \sigma)$ pour une condition de signe σ.*
- *Soit il existe λ une permutation de $\{1, ..., n\}$, un ensemble $\mathcal{C}$-trivial $N \subset I_s$ où $s = (r_{\lambda(1)}, ..., r_{\lambda(n-1)})$, une application $g \in \mathcal{C}_{n-1,s}$ vérifiant $g(I_s) \subset (-r_{\lambda(n)}, r_{\lambda(n)})$ tels que la projection $\Pi_\lambda(M)$ soit égale au graphe de $g|N$.*

Remarque 8.3.0.1 *La construction d'un ensemble $\mathcal{C}$-trivial $A \subset \mathbb{R}^n$ se fait donc par récurrence en p étapes :*

- *À l'étape 0, A_0 est un quadrant de $\mathbb{R}^{n-p}$.*
- *Pour tout $i \in [\![1, p]\!]$, on construit $A_i \subset \mathbb{R}^{n-p+i}$ en fonction de $A_{i-1} \subset \mathbb{R}^{n-p+i-1}$ en considérant une injection $\lambda_i : [\![1, n-p+i]\!] \to [\![1, n]\!]$, $s_i = \left(r_{\lambda(1)}, \ldots, r_{\lambda(n-p+i-1)}\right) \in \left(\mathbb{R}_+^*\right)^{n-p+i}$, une application $g_i \in \mathcal{C}_{n-p+i-1,s_i}$ vérifiant $g_i(I_{s_i}) \subset (-r_{\lambda(n-p+i)}, r_{\lambda(n-p+i)})$ et telles que la projection $\Pi_\lambda(A_i)$ soit égale au graphe de $g_{i|A_{i-1}}$.*

À l'issue de la $p^{\text{ème}}$ étape, on obtient $A = A_p$.

Définition 8.3.0.3

1. *Un sous-ensemble M de $\mathbb{R}^n$ est une **$\mathcal{C}$-variété** si et seulement si :*
   - *M est un ensemble $\mathcal{C}$-basique contenu dans un polydisque I_r de $\mathbb{R}^n$.*
   - *M est une sous-variété de I_r.*
   - *Il existe $(f_1, ..., f_k) \in \mathcal{C}_{n,r}^k$ tel que :*
     - *f_1,...f_k s'annulent sur M.*
     - *∇f_1,...∇f_k sont linéairement indépendants.*

2. *Soit $M \subset \mathbb{R}^n$. On dit que M est une* ***variété $\mathcal{C}$-semi-analytique triviale*** *si et seulement si il existe une variété $\mathcal{C}$-triviale $N \subset \mathbb{R}^n$ et $a \in \mathbb{R}^n$ tels que $M = a + N$.*

Nous rappelons maintenant le théorème de désingularisation des ensembles $\mathcal{C}$- sous-analytiques. Il nous permettra de démontrer le lemme du petit chemin 8.4.1. Ce résultat joue un rôle fondamental pour la démonstration du théorème 8.2.3.

Théorème 8.3.1 Théorème de désingularisation pour les ensembles $\mathcal{C}$-sous-analytiques *Soit $A \subset \mathbb{R}^n$ un ensemble $\mathcal{C}$-semi-analytique borné et soit $k \leq n$. Il existe J un ensemble fini, des variétés $\mathcal{C}$-semi-analytiques triviales $N_i \subset \mathbb{R}^{n_i}$ avec $n_i \geq n$ et $i \in J$ tels que*

$$\Pi_k(A) = \bigcup_{i \in J} \Pi_k(N_i)$$

et tels que pour chaque i :

- *$d := dim(N_i) \leq k$.*
- *Il existe une fonction strictement croissante $\varphi :\ \{1, ..., d\} \to \{1, ..., k\}$ telle que $\Pi_{\varphi|N_i} :\ N_i \to \mathbb{R}^d$ est une immersion.*

8.4 Deux lemmes sur les chemins et un lemme technique

8.4.1 Lemme du petit chemin

Nous démontrons le lemme suivant, qui provient de la remarque 3.5.(2) de [48].

Lemme 8.4.1 *Soit I_r un polydisque centré en 0 de $\mathbb{R}^n$. Soit $M \subset I_r$ un ensemble $\mathcal{C}$-trivial et non réduit à un point. Soit $z \in Adh(M)$. Il existe $\varphi = (\varphi_1, ..., \varphi_n) \in \mathcal{C}^n_{1,1}$ telle que :*

1. *$\varphi : [0, 1] \to \mathbb{R}^n$.*
2. *$\varphi(]0, 1]) \subset M$*
3. *$\varphi(0) = z$*
4. *φ n'est pas constante.*

Remarque 8.4.1.1 *Si $z \in Fr(M)$,* 4. *est conséquence des points précédents.*

Démonstration Remarquons qu'un ensemble $\mathcal{C}$-trivial est toujours non vide.

— Si M est un quadrant (non réduit à un point), $M = B_{I_r}((x_1, ..., x_n), \sigma)$: M, ainsi que son adhérence, sont convexes. En considérant $z \in \text{Adh}(M)$. Il existe $m \in int(M)$ tel que $m \neq z$ ($int(M)$ est l'intérieur de M avec M muni de la topologie induite de $\mathbb{R}^n$). En effet, si un tel m n'existait pas, alors M serait réduit au point z (et d'ailleurs dans ce cas, on aurait nécessairement $z = 0$). Si $c \in [z, m]$ et si $c \neq z$ alors $c \in int(M)$. L'application $\varphi : [0,1] \to \mathbb{R}^n, \quad t \mapsto tm + (1-t)z$ correspond bien à celle recherchée.
— Supposons le lemme démontré pour tous les ensembles $\mathcal{C}$-triviaux obtenus par $n-1$ applications du procédé de construction d'un ensemble $\mathcal{C}$-trivial et démontrons le à l'étape suivante.
— Soient N un ensemble $\mathcal{C}$-trivial de $I_s \subset \mathbb{R}^{n-1}$ où $s = (r_{\lambda(1)}, ..., r_{\lambda(n-1)})$, $g \in \mathcal{C}_{n-1,s}$ telle que $g(I_s) \subset (-r_{\lambda(n)}, r_{\lambda(n)})$ et M l'ensemble $\mathcal{C}$- trivial tel que la projection $\Pi_\lambda(M)$ soit égale au graphe de $g|N$.
  — Considérons $z \in \text{Adh}(M) = \text{Adh}(\Pi_\lambda^{-1}(gr(g_{|N})))$. Il existe donc une suite $(z_k) \subset M$ telle que $z_k \xrightarrow[k \to +\infty]{} z$
  — Pour tout $k \in \mathbf{N}$, posons $u_k = \Pi_\lambda(z_k)$. Posons aussi $u = \Pi_\lambda(z)$. On a : $(u_k) \subset gr(g_{|N})$ et $u \in \text{Adh}(gr(g_{|N}))$.
  — De plus :

$$\begin{cases} u_k \xrightarrow[k \to +\infty]{} u \\ \forall k \in \mathbf{N}, \quad \Pi_{n-1}(u_k) \in N \\ \Pi_{n-1}(u_k) \xrightarrow[k \to +\infty]{} \Pi_{n-1}(u) \in \text{Adh}(N) \end{cases}$$

  et

$$\forall k \in \mathbf{N}, \quad u_k^n = g(\Pi_{n-1}(u_k))$$

  où u_k^n désigne la $n^{\text{ème}}$ coordonnée de u_k.
  — Par application de l'hypothèse de récurrence, il existe $\psi \in \mathcal{C}_{1,1}^{n-1}$ telle que
    1. $\psi : [0,1] \to \mathbb{R}^{n-1}$.
    2. $\psi(]0,1]) \subset N$
    3. $\psi(0) = \Pi_{n-1}(u)$
    4. ψ n'est pas constante.
  — Posons $\gamma = (\psi, g \circ \psi)$. γ vérifie :
    1. $\gamma : [0,1] \to \mathbb{R}^n$.
    2. $\gamma(]0,1]) \subset gr(g_{|N})$
    3. $\gamma(0) = u$
    4. γ n'est pas constante.
  — Finalement, en posant $\varphi = \Pi_\lambda^{-1} \circ \gamma$, on obtient l'application recherchée.

La propriété est alors démontrée par récurrence.

□

Remarque 8.4.1.2 *Le lemme 8.4.1 est encore valable pour une variété $\mathcal{C}$-semi-analytique triviale car de telles variétés sont des translatés d'un ensemble $\mathcal{C}$-trivial.*

Nous sommes en mesure de démontrer le lemme du petit chemin pour un ensemble $\mathcal{C}$-sous-analytique. C'est une conséquence directe du lemme 8.4.1, du théorème de désingularisation 8.3.1. Ce lemme joue un rôle crucial dans le travail réalisé ici.

Théorème 8.4.1 ***Lemme du petit chemin***
Soit $B \subset \mathbb{R}^m$ un ensemble $\mathcal{C}$-sous-analytique de $\mathbb{R}^m$ et soit $z \in Adh(B)$. Il existe $\varphi = (\varphi_1, ..., \varphi_k) \in \mathcal{C}^k_{1,1}$ telle que :

1. *$\varphi : [0,1] \to \mathbb{R}^m$.*
2. *$\varphi(]0,1]) \subset B$*
3. *$\varphi(0) = z$*
4. *φ n'est pas constante.*

Démonstration Il existe un ensemble $\mathcal{C}$-semi-analytique $A \subset \mathbb{R}^n$ $(n \geq m)$ tel que $B = \Pi_m(A)$ et en appliquant le théorème de désingularisation pour un ensemble $\mathcal{C}$-sous-analytique 8.3.1, il existe des variétés $\mathcal{C}$-semi-analytiques triviales et donc bornées $N_i \subset \mathbb{R}^{n_i}$ avec $n_i \geq n$ pour tout $i = 1, ..., J$, telles que

$$\Pi_m(A) = \Pi_m(N_1) \cup ... \cup \Pi_m(N_J)$$

Comme z est adhérent à B, z est adhérent à un des $\Pi_m(N_i)$, supposons que ce soit le $i_0^{\text{ème}}$. Comme N_{i_0} est bornée, il existe $z' \in \mathbb{R}^{n_{i_0}-m}$ tel que $(z, z') \in \text{Adh}(N_{i_0})$. Par application du lemme 8.4.1 et de la remarque 8.4.1.2, on est assuré de l'existence d'un chemin $\overline{\varphi} = (\overline{\varphi}_1, ..., \overline{\varphi}_{n_{i_0}}) \in \mathcal{C}^{n_{i_0}}_{1,1}$ tel que :

1. $\overline{\varphi} : [0,1] \to \mathbb{R}^{n_{i_0}}$.
2. $\overline{\varphi}(]0,1]) \subset B$
3. $\overline{\varphi}(0) = (z, z')$
4. $\overline{\varphi}$ n'est pas constante.

Le chemin $\varphi = \Pi_m \circ \overline{\varphi}$ est celui recherché. □

8.4.2 Lemme de choix d'un ensemble $\mathcal{C}$-sous-analytique

Proposition 8.4.2 ***Lemme de choix d'un ensemble $\mathcal{C}$-sous-analytique***

Soit B un ensemble $\mathcal{C}$-sous-analytique de $\mathbb{R}^m$ donné comme la réunion

$$B = B_1 \cup \ldots \cup B_l$$

où les B_i sont des ensembles $\mathcal{C}$-sous-analytiques inclus dans $\mathbb{R}^m$.

Supposons que $0 \in Adh(B)$ et que $\varphi \in \mathcal{C}^m_{1,1}$ soit un chemin tel que $\varphi(]0,1]) \subset B$, $\varphi(0) = 0$ et tel que $[\varphi]$ ne soit pas le germe de la fonction nulle (c'est-à-dire φ n'est pas identiquement nulle dans un voisinage de 0). Alors il existe $i_0 \in \{1, ..., l\}$ et $0 < r \leq 1$ tel que $\varphi(]0,r]) \subset B_{i_0}$.

Démontrons tout d'abord ce lemme :

Lemme 8.4.3 *Soit $\varphi \in \mathcal{C}^m_{1,1}$ un chemin non identiquement nul tel que $\varphi\,(]0,1]) \subset B$ et tel que $\varphi\,(0) = 0$. Il existe une suite $(t_n) \subset\,]0,1]$ telle que*

1. *(t_n) est strictement décroissante.*
2. *$t_n \xrightarrow[n \to +\infty]{} 0$.*
3. *$\forall n \in \mathbf{N}^*, \quad \|\varphi\,(t_n)\| = \frac{1}{n}$ où $\|\cdot\|$ représente la norme euclidienne de $\mathbb{R}^m$.*

Démonstration On va construire (t_n) par récurrence. Comme φ est non nulle sur $[0,1]$, on a $\sup \|\varphi\,(t)\| \neq 0$ et on peut supposer que $\sup \|\varphi\,(t)\| = 1$ afin de simplifier la construction qui suit.

1. $n = 1$: Comme $[0,1]$ est compact, φ atteint son sup en un point de cet intervalle et l'ensemble
$$\mathcal{E} = \{t \in [0,1] \quad | \quad |\varphi(t)| = 1\}$$
est non vide. De plus, comme $\varphi\,(0) = 0$ et que φ est continue, la borne inférieure t_1 de $\mathcal{E}$ (qui existe car $\mathcal{E}$ est un sous-ensemble minoré et non vide de $\mathbb{R}$) est strictement supérieure à 0.
2. Supposons construits les n premiers termes de la suite : $t_1 > t_2 > ... > t_n > 0$ tels que $\forall i \in [\![1,n]\!], \quad \varphi(t_i) = \frac{1}{i}$.
3. Construisons t_{n+1} : En utilisant le théorème des valeurs intermédiaires, comme $\varphi : \left\{ \begin{array}{ccc} [0,t_n] & \longrightarrow & [0,1] \\ t & \mapsto & \|\varphi\,(t)\| \end{array} \right.$ est continue, on peut affirmer que l'ensemble
$$\left\{ t \in \,]0,t_n[\quad | \quad \|\varphi(t)\| = \frac{1}{n+1} \right\}$$
est non vide. Il est par ailleurs minoré et possède donc une borne inférieure $t_{n+1} > 0$ qui vérifie, elle aussi, $\|\varphi\,(t_{n+1})\| = \frac{1}{n+1}$.

La suite (t_n) est ainsi construite par récurrence. Cette suite est bien strictement décroissante. Reste à montrer qu'elle converge vers 0. Supposons que ce ne soit pas le cas. Comme (t_n) est décroissante et minorée, elle converge vers une limite $l > 0$. Par ailleurs, par continuité de φ et par construction de (t_n), $\varphi\,(l) = 0$. Comme φ n'est pas identiquement nulle sur $[0,l]$, il existe $t \in [0,l]$ tel que $\|\varphi\,(t)\| = \alpha > 0$. Pour n assez grand, $\frac{1}{n} < \alpha$. Par le théorème des valeurs intermédiaires, il existe $T \in]0,l[$ tel que $\varphi\,(T) = \frac{1}{n}$. Mais $t_n > T$, ce qui est en contradiction avec le fait que t_n est la borne inférieure de $\left\{ t \in]0,1[\quad | \quad \|\varphi\,(t)\| = \frac{1}{n+1} \right\}$. Donc $t_n \xrightarrow[n \to +\infty]{} 0$. □

Démontrons la proposition :

Démonstration Soit (t_n) la suite construite dans le lemme précédent 8.4.3. l'ensemble

$$\Gamma = \{\varphi(t_n) \quad | \quad n \in \mathbf{N}^*\}$$

est donc de cardinal infini. Comme

$$\Gamma \subset \varphi(]0,1]) \subset B = \bigcup_{i \in \{1,\dots,l\}} B_i,$$

il existe $i_0 \in \{1, ..., l\}$ tel que B_{i_0} contienne une infinité d'éléments de Γ.

Remarquons au passage que comme $\varphi(t_n) \xrightarrow[n \to +\infty]{} 0$, il en est de même de toute suite extraite de $(\varphi(t_n))$ et donc que $0_{\mathbb{R}^m} \in \mathrm{Adh}(B_{i_0})$.

L'ensemble B_{i_0} est $\mathcal{C}$-sous-analytique donc $\mathscr{B}_{i_0} := \Phi^{-1}(B_{i_0})$ est définissable. Il existe donc une division en cellules de $\mathbb{R}^m$ adaptée à $\mathscr{B}_{i_0}$. Ceci signifie qu'il existe $J \subset \mathbf{N}$ fini et des cellules $\mathscr{C}_j$, $j \in J$ telles que

$$\mathscr{B}_{i_0} = \bigcup_{j \in J} \mathscr{C}_j$$

Ces cellules étant en nombre fini, l'une d'entre elles contient une infinité de termes de Γ. Notons $\mathscr{C}$ cette dernière cellule. De la même façon que précédemment, on peut affirmer que $0_{\mathbb{R}^m} \in \mathrm{Adh}(\mathscr{C})$.

Montrons qu'il existe $1 \geq r > 0$ tel que $(\Phi^{-1} \circ \varphi)(]0, r[) \subset \mathscr{C}$. Pour ce faire, nous allons effectuer un raisonnement par l'absurde. Posons $\psi := \Phi^{-1} \circ \varphi$ et rappelons au préalable le principe de construction de $\mathscr{C}$ (voir [57]). Celle ci se fait en m étapes (pour une cellule de $\mathbb{R}^m$). Pour tout $i \in [\![1, m]\!]$, notons $\mathscr{C}_i$ la cellule de $\mathbb{R}^i$ obtenue à l'étape i. On a $\mathscr{C}_m = \mathscr{C}$ et :

- **Étape** 1 **:** $\mathscr{C}_1$ est soit un point de $\mathbb{R}$ (on dit alors que $\mathscr{C}_1$ est de type $\mathcal{G}$), soit un intervalle ouvert de $\mathbb{R}$ (dans ce cas, on dit que $\mathscr{C}_1$ est de type $\mathcal{B}$).
- **Étape** $i+1$ $(i < m)$ **:** Supposons $\mathscr{C}_i$ construite et construisons $\mathscr{C}_{i+1}$: Il y a deux possibilités. Soit il existe $\xi_{i+1} : \mathscr{C}_i \to \mathbb{R}$ une fonction définissable telle que :

$$\mathscr{C}_{i+1} = gr(\xi_{i+1} | \mathscr{C}_i)$$

On dit alors que $\mathscr{C}_{i+1}$ est de type $\mathcal{G}$. Soit il existe $\xi^1_{i+1}, \xi^2_{i+1} : \mathscr{C}_i \to \mathbb{R}$ deux fonctions définissables telles que :

$$\mathscr{C}_{i+1} = \left\{(x', x) \in \mathbb{R}^i \times \mathbb{R} \mid \xi^1_{i+1}(x') < x < \xi^2_{i+1}(x') \text{ et } x' \in \mathscr{C}_i\right\}$$

On dit alors que $\mathscr{C}_{i+1}$ est de type $\mathcal{B}$.

Un élément $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ est élément de $\mathscr{C}$ si et seulement si pour tout $k \in [\![1, m]\!]$, les coordonnées de x vérifient les relations $(\mathcal{R}_k)_{k \in [\![1,m]\!]}$ suivantes :

- **Si** $k = 1$**, relation** $\mathcal{R}_1$ **:** Si $\mathscr{C}_1$ est de type $\mathscr{G}$ alors $x_1 = 0$ sinon, si $\mathscr{C}_1$ est de type $\mathscr{B}$ alors $x_1 \in]a, b[$ où $\mathscr{C}_1 =]a, b[$.

- Si $k > 1$**, relation** $\mathcal{R}_k$ **:** Si $\mathscr{C}_k$ est de type $\mathscr{G}$ alors $x_k = \xi_k(x_1, \ldots, x_{k-1})$ et $(x_1, \ldots, x_{k-1}) \in \mathscr{C}_{k-1}$ sinon, si $\mathscr{C}_k$ est de type $\mathscr{B}$ alors $\xi^1_{k-1}(x_1, \ldots, x_{k-1}) < x_k < \xi^2_{k-1}(x_1, \ldots, x_{k-1})$ et $(x_1, \ldots, x_{k-1}) \in \mathscr{C}_{k-1}$.

Revenons à la démonstration et supposons qu'il n'existe pas $1 \geq r > 0$ tel que $\psi(]0, r[) \subset \mathscr{C}$. Alors, pour tout $n \in \mathbf{N}^*$, il existe $T_n \in]0, \frac{1}{n}[$ tel que $\psi(T_n) \notin \mathscr{C}$. Remarquons que $T_n \xrightarrow[n\to+\infty]{} 0$ et donc, ψ étant continue, on a $\psi(T_n) \xrightarrow[n\to+\infty]{} 0$. Comme pour tout $n \in \mathbf{N}^*$, $\psi(T_n) \notin \mathscr{C}$, les coordonnées des termes de la suite $(\psi(T_n))$ ne vérifient pas les relations $(\mathcal{R}_k)_{k \in [\![1,m]\!]}$ pour une infinité de $n \in \mathbf{N}^*$. Ces relations étant en nombre finie, une, au moins, d'entre elles n'est pas vérifiée par une infinité de termes de la suite. Soit k_0 le plus petit entier dans l'ensemble $[\![1, m]\!]$ tel que les termes de la suites $(\psi(T_n))$ ne vérifient pas $\mathcal{R}_{k_0}$ pour une infinité de n. Quitte à re-indexer la suite (T_n), on suppose dorénavant que pour tout $n \in \mathbf{N}^*$, $\psi(T_n)$ ne vérifie pas $\mathcal{R}_{k_0}$ mais vérifie $\mathcal{R}_i$ pour tout $i \in [\![1, k_0 - 1]\!]$.

- si $k_0 = 1$ **:** $\mathscr{C}_1$ est soit de type $\mathcal{G}$ soit de type $\mathcal{B}$.

-Type $\mathcal{G}$ **:** Dans ce cas, $\mathscr{C}_1 = \{0\}$ et donc :

$$\forall n \in \mathbf{N}^*, \quad \Pi_1(\psi(t_n)) = 0 \text{ et } \Pi_1(\psi(T_n)) \neq 0$$

Mais $T_n \xrightarrow[n\to+\infty]{} 0$ et $t_n \xrightarrow[n\to+\infty]{} 0$ donc $\Pi_1 \circ \psi$, qui est une fonction définissable est non identiquement nulle et s'annule une infinité de fois dans un voisinage de 0. Ceci est impossible.

-Type $\mathcal{B}$ **:** On a alors $\mathscr{C}_1 =]a, b[$ où $a, b \in \mathbb{R}$ et $a < b$. Rappelons de plus que 0 est adhérent à $\mathscr{C}_1$.

Il y a alors deux cas possibles pour $\mathscr{C}_1$: soit $0 \in Int(\mathscr{C})$, soit $0 \in Fr(\mathscr{C})$.

-1er cas, $0 \in Int(\mathscr{C})$ Soit $l = \inf\{|a|, |b|\} > 0$. On a :

$$\forall n \in \mathbf{N}^*, \quad |\Pi_1(\psi(T_n))| \geq l > 0$$

On ne peut donc avoir $\psi(T_n) \xrightarrow[n\to+\infty]{} 0$.

-2nd cas, $0 \in Fr(\mathscr{C})$. Alors $\mathscr{C}_1$ est toute entière dans $\mathbb{R}_-$ ou dans $\mathbb{R}_+$. Supposons que $\mathscr{C}_1$ est dans R_-. Comme on a

$$\pi_1(\psi(T_n)) \xrightarrow[n\to+\infty]{} 0 \text{ et } \forall n \in \mathbf{N}^*, \quad \Pi_1(\psi(T_n)) \notin \mathscr{C}_1$$

la suite $(\Pi_1(\psi(T_n)))$ est positive et la suite $\Pi_1(\psi(t_n))$ est négative. On a donc deux suites $(\Pi_1(\psi(T_n)))$ et $(\Pi_1(\psi(t_n)))$ telles que

$$\forall n \in \mathbf{N}^*, \quad \Pi_1(\psi(T_n)) \leq 0 \text{ et } \Pi_1(\psi(t_n)) > 0$$

(cette dernière inégalité étant conséquence du fait que $\forall n \in \mathbf{N}, \quad \psi(t_n) \in \mathscr{C}$). Comme $t_n \xrightarrow[n\to+\infty]{} 0$ et $T_n \xrightarrow[n\to+\infty]{} 0$, on peut affirmer que $\Pi_1 \circ \psi$, qui est une fonction définissable et non identiquement nulle, s'annule une infinité de fois dans un voisinage de 0. Ce qui est, comme précédemment, impossible.

- si $k_0 > 1$ **:** $\mathscr{C}_{k_0}$ est soit de type $\mathcal{G}$ soit de type $\mathcal{B}$.

-Type $\mathcal{G}$ **:** Dans ce cas : $\mathscr{C}_{k_0} = gr(\xi_{k_0} | \mathscr{C}_{k_0 - 1})$. Comme pour tout $n \in \mathbf{N}^*$, $\psi(T_n)$ ne vérifie pas $\mathcal{R}_{k_0}$ mais vérifie $\mathcal{R}_i$ pour tout $i \in [\![1, k_0 - 1]\!]$, on a :

$$\psi_{k_0}(T_n) - \xi_{k_0}(\Pi_{k_0-1}(\psi(T_n))) \neq 0 \text{ et } \psi_{k_0}(t_n) - \xi_{k_0}(\Pi_{k_0-1}(\psi(t_n))) = 0$$

où ψ_{k_0} désigne la k_0^{eme} application coordonnée de ψ. La fonction définissable $\psi_{k_0} - \xi_{k_0} \circ \Pi_{k_0-1} \circ \psi$ est donc non constante et s'annule une infinité de fois, ce qui est, là encore impossible.

-Type $\mathcal{B}$: On a donc :

$$\mathscr{C}_{k_0} = \left\{(x, y) \in \mathbb{R}^{k_0-1} \times \mathbb{R} \mid \xi^1_{k-1}(x) < y < \xi^2_{k-1}(x) \text{ et } x \in \mathscr{C}_{k-1}\right\}$$

Comme pour tout $n \in \mathbf{N}^*$, $\psi(T_n)$ ne vérifie pas $\mathcal{R}_{k_0}$ mais vérifie $\mathcal{R}_i$ pour tout $i \in [\![1, k_0 - 1]\!]$, une des deux inégalités :

$$\psi_{k_0}(T_n) - \xi^1_{k_0}(\Pi_{k_0-1}(\psi(T_n))) \leq 0 \quad \text{ou} \quad \psi_{k_0}(T_n) - \xi^2_{k_0}(\Pi_{k_0-1}(\psi(T_n))) \geq 0$$

est vraie pour une infinité de n. Supposons que ce soit la première. On a par ailleurs

$$\psi_{k_0}(t_n) - \xi^1_{k_0}(\Pi_{k_0-1}(\psi(t_n))) > 0$$

L'application $\psi_{k_0} - \xi^1_{k_0} \circ \Pi_{k_0-1} \circ \psi$ est définissable, non constante et s'annule elle aussi une infinité de fois, ce qui est là aussi impossible.

Le lemme est prouvé par l'absurde et donc il existe $0 < r \leq 1$ tel que $\varphi(]0, r[) \subset \mathscr{C} \subset \Pi_m(N_{i_0})$. □

8.4.3 Un lemme technique

Lemme 8.4.4 ***Un lemme technique*** *Soient A et B deux sous-ensembles de $\mathbb{R}^n$. On suppose que*

1. $\forall W \in \mathscr{V}_n(0), \quad A \cap W \neq \emptyset$
2. $\forall W \in \mathscr{V}_n(0), \quad B \cap W \neq \emptyset$
3. $\forall W \in \mathscr{V}_n(0), \quad A \cap W \neq B \cap W$

Alors la disjonction suivante est toujours vraie :

$$[0 \in Adh(A \setminus B) \quad ou \quad 0 \in Adh(B \setminus A)]$$

Démonstration Supposons que

$$[0 \notin \text{Adh}(A \setminus B) \text{ et } 0 \notin \text{Adh}(B \setminus A)]$$

alors

$$\exists W \in V_n(0) : \quad W \cap (A \setminus B) = \emptyset \text{ et } W \cap (B \setminus A) = \emptyset$$

et donc, comme

$$A \cap W = ((A \setminus B) \cup (A \cap B)) \cap W$$
$$B \cap W = ((B \setminus A) \cup (A \cap B)) \cap W$$

on a nécessairement $A \cap W = B \cap W$.

□

8.5 Démonstration du théorème 8.2.3

Nous démontrons maintenant le théorème 8.2.2.

Théorème 8.5.1 *Soient $\mathcal{C}$ et $\tilde{\mathcal{C}}$ deux algèbres RSW-QA qui sont RSW isomorphes. Notons $\mathcal{L}$ le langage $\mathcal{L} := \mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\tilde{\mathcal{C}}}$. Alors deux $\mathcal{L}$-formules sans quantificateur sont $\mathcal{C}$-équivalentes si et seulement si elles sont $\tilde{\mathcal{C}}$-équivalentes.*

Démonstration Considérons deux $\mathcal{L}$-formules $\phi : \bigvee_i \bigwedge_j \left(\hat{f}_{ij}\sigma_{ij}0\right)$ et $\psi : \bigvee_i \bigwedge_j \left(\hat{\bar{f}}_{ij}\bar{\sigma}_{ij}0\right)$ qui sont $\mathcal{C}$-équivalentes. Supposons qu'elles ne sont pas $\tilde{\mathcal{C}}$-équivalentes. Par définition :

— un représentant du germe $\Delta_{\mathcal{C}}(\phi)$ est $A = \bigcup_{i=1}^{N} B_{W_i^1}\left(f^i, \sigma^i\right)$,
— un représentant du germe $\Delta_{\tilde{\mathcal{C}}}(\phi)$ est $\tilde{A} = \bigcup_{i=1}^{N} B_{W_i^2}\left(\tilde{f}^i, \sigma^i\right)$,
— un représentant du germe $\Delta_{\mathcal{C}}(\psi)$ est $\bar{A} = \bigcup_{i=1}^{M} B_{W_i^3}\left(\bar{f}^i, \bar{\sigma}^i\right)$,
— un représentant du germe $\Delta_{\tilde{\mathcal{C}}}(\psi)$ est $\tilde{\bar{A}} = \bigcup_{i=1}^{M} B_{W_i^4}\left(\tilde{\bar{f}}^i, \bar{\sigma}^i\right)$

où les W_i^k sont des voisinages de 0 sur lesquels les fonctions f^i sont définies. Comme les formules ϕ et ψ sont supposées non $\tilde{\mathcal{C}}$-équivalentes, on a :

$$\forall V \in \mathscr{V}_n(0), \quad \tilde{A} \cap V \neq \tilde{\bar{A}} \cap V$$

et alors, en appliquant le lemme technique 8.4.4, on a :

$$0 \in \operatorname{Adh}\left(\tilde{A} \setminus \tilde{\bar{A}}\right) \quad \text{ou} \quad 0 \in \operatorname{Adh}\left(\tilde{\bar{A}} \setminus \tilde{A}\right)$$

Supposons que $0 \in \operatorname{Adh}\left(\tilde{A} \setminus \tilde{\bar{A}}\right)$.

D'après le lemme du petit chemin 8.4.1, il existe $\tilde{\varphi} \in \tilde{\mathcal{C}}_{1,1}^n$ tel que :

$$\begin{cases} \tilde{\varphi}(]0,1]) \subset \tilde{A} \setminus \tilde{\bar{A}} \\ \tilde{\varphi}(0) = 0 \end{cases}$$

Il existe $\varepsilon_1 > 0$ tel que $\tilde{\varphi}(]0,\varepsilon_1]) \subset \tilde{A}$. Pour simplifier les notations, on suppose que $\varepsilon_1 = 1$. On a donc :

$$\tilde{\varphi}(]0,1]) \subset \bigcup_{i=1}^{N} B_{W_i}\left(\tilde{f}^i, \sigma^i\right)$$

D'après le lemme de choix d'un ensemble $\mathcal{C}$-sous-analytique 8.4.2, nous pouvons affirmer qu'il existe $i_0 \in \{1, \ldots, N\}$ et $\varepsilon_2 > 0$ tel que :

$$\tilde{\varphi}(]0,\varepsilon_1]) \subset B_{W_{i_0}}\left(\tilde{f^{i_0}}, \sigma^{i_0}\right)$$

On peut à nouveau supposer, sans que cela n'influe sur la suite de la démonstration, que $\varepsilon_2 = 1$.

On a donc :
$$\forall t \in]0,1], \quad \text{signe}\left(\tilde{f}^{i_0} \circ \tilde{\varphi}(t)\right) = \sigma^{i_0}$$

Mais
$$(\widehat{f^{i_0} \circ \varphi}) = (\widehat{\tilde{f}^{i_0} \circ \tilde{\varphi}})$$
donc dans un voisinage de $0 \in \mathbb{R}$, ces deux fonctions, ayant le même développement de Taylor en 0, sont de même signe (celui du signe du premier terme non nul du développement de Taylor en 0). Il existe alors $\varepsilon_3 > 0$ (qu'on peut supposer égal à 1) tel que

$$\varphi\left(]0, \varepsilon_3[\right) \subset B_{W^1_{i_0}}(f^{i_0}, \sigma^{i_0}).$$

Mais comme $[A] = [\bar{A}]$, il existe un voisinage V' de 0 dans $\mathbb{R}^n$ tel que $A \cap V' = \bar{A} \cap V'$. Quitte à diminuer à droite l'intervalle $]0,1]$, on a : $\varphi\left(]0,1]\right) \subset \bar{A}$.

Il existe par ailleurs $\varepsilon_4 > 0$ tel que $\varphi\left(]0, \varepsilon_4]\right) \subset \bar{A}$. On suppose là aussi que $\varepsilon_2 = 1$. En appliquant à nouveau le lemme de choix d'un ensemble $\mathcal{C}$-sous-analytique 8.4.2 et l'argument asymptotique précédent, on sait qu'il existe $i_0 \in \left\{1, \ldots, \bar{N}\right\}$ et ε_5 supposé égal à 1 tels que :

$$\tilde{\varphi}\left(]0, \varepsilon_5]\right) \subset B_{\bar{W}_i}\left(\tilde{\bar{f}}^{i_0}, \bar{\sigma}^{i_0}\right)$$

On a donc $\tilde{\varphi}\left(]0,1]\right) \subset \tilde{\bar{A}}$ ce qui contredit l'hypothèse formulée au départ : $\tilde{\varphi}\left(]0,1]\right) \subset \tilde{A} \setminus \tilde{\bar{A}}$. Le théorème est donc démontré par l'absurde. □

8.6 Conclusion

En guise de conclusion, expliquons pourquoi nous ne pouvons pas prolonger l'isomorphisme Θ aux germes d'ensembles $\mathcal{C}$-sous-analytiques.

Re-prenons pour ce faire l'exemple de l'introduction du chapitre des deux solutions H et $\tilde{H}$ de l'équation d'Euler $x^2 y' = y - x$ et considérons l'ensemble $\mathcal{C}$-semi-analytique $A = \left\{(x,y) \in]0,\epsilon]^2 \mid x - H(y) = 0\right\}$ et l'ensemble $\tilde{\mathcal{C}}$-semi analytique $\tilde{A} = \left\{(x,y) \in]0,\epsilon]^2 \mid x - \tilde{H}(y) = 0\right\}$.

Quitte à prendre ϵ assez grand, on peut supposer que H est positive sur $]0,\epsilon]$ tandis que $\tilde{H}$ change de signe sur ce même intervalle. Considérons alors $B = \Pi_1(A)$ et $\tilde{B} = \Pi_1\left(\tilde{A}\right)$. Un représentant de la classe du germe $[B]$ est la demi-droite $\overline{A}$: $\begin{cases} y = 0 \\ x > 0 \end{cases}$ qui est un ensemble semi-analytique. Un représentant de la classe du germe $\left[\tilde{B}\right]$ est la droite $\overline{\tilde{A}}$ privée de 0 $\begin{cases} y = 0 \\ x \neq 0 \end{cases}$ qui est un ensemble $\tilde{\mathcal{C}}$-semi-analytique. Si on pouvait prolonger le morphisme Θ aux ensembles $\mathcal{C}$-sous-analytiques alors l'image par Θ du germe de $\overline{A}$ devrait être égale au germe de $\overline{\tilde{A}}$ ce qui n'est bien entendu pas le cas.

Chapitre 9

Real summation for the saddle node problem : comparison between a convolutive and a geometrical approach

Il est de toute première instance que nous façonnions nos idées comme s'il s'agissait d'objets manufacturés. Je suis prêt à vous procurer les moules.
Mais...
La solitude...
Les moules sont d'une texture nouvelle, je vous avertis. Ils ont été coulés demain matin.
Léo Ferré - La solitude

9.1 Introduction

As explained in chapter 7, when applying Ecalle's theory of well behaved averages to the formal conjugant of a real analytic saddle-node, one obtains an analytic germ $f_{\mathcal{E}}(u) \in \mathbb{C}\{u\}$ with coefficients depending completely on the analytical class of the studied saddle-node and on the weights of the used average. Moreover, this germ is naturally such that $y^+(t, f_{\mathcal{E}}(u))$ is real analytic for u small enough and t big enough.

In an unpublished work and independently of Ecalle's averages theory, R. Schäfke has searched for a sufficient condition that must verify an analytical germ $f_{\mathcal{S}}(u) \in \mathbb{C}\{u\}$ in order to obtain a sum $y^+(t, f_{\mathcal{S}}(u))$ real in the positive direction. He has established that if $f_{\mathcal{S}}$ satisfies a certain functional equation that makes interfere the sectoral Martinet-Ramis isotropy $\xi(u)$ then $y^{+}(t, f_{\mathcal{S}}(u))$ is real analytic.

The aim of this chapter is to try to connect these two points of view. We will show the following results :

— All the analytic germs $f_{\mathcal{E}}(u) \in \mathbb{C}\{u\}$ found in Ecalle's averages theory are solutions of the functional Schäfke equation. In fact, this equation has a counterpart in the operators algebra **ALIEN**.

— There are some analytic solutions $f_{\mathcal{S}}$ of Schäfke equation that can not be realized as analytic germs $f_{\mathcal{E}}$ associated to well behaved averages. In other words, for certain analytic classes of real saddle-node, Ecalle's theory of averages fails to obtain all the real sums of the formal conjugant.

— When focusing ourselves on analytic classes of saddle-node with Brownian invariants[1] $(O_m)_{m\in\{-1\}\cup\mathbf{N}^*}$ having at less a geometrical growth[2], we will prove that all the analytic solutions of the Schäfke equation for this analytical class are in correspondence with a well behaved average that we will construct explicitly. Unfortunately, the construction of this average makes use, for an unexpected reason, of already known well behaved averages. So we do not plan here a new proof of the existence of well behaved average.

9.2 Real summation in the geometrical model

9.2.1 Schäfke equation

R. Schäfke has proved in the geometrical model with analytical methods the following result.

Proposition 9.2.1 *Given us a real and analytic saddle-node X. If ϕ is an analytic solution of the functional equation :*

$$\phi = \xi \circ \overline{\phi} \tag{9.1}$$

where $\xi(u)$ is the Martinet-Ramis sectoral isotropy which characterizes the analytic class of X then[3] *$y^{+}(t, \phi(u))$ is analytic and real.*

Proof We propose here a proof expressed in terms of operators but very close to the original proof. Consider an analytic solution ϕ of the Schäfke equation. We denote by **red** the reduction associated to the analytic class of this saddle node and we denote by $\mathbf{\Delta}^{+}$ the image of the Stokes automorphism $\dot{\Delta}^{+}$ by this reduction.

The function $y^{+}(t, \phi(u))$ is clearly analytic. We want to prove that it is real. Consider the substitution automorphism $\mathbf{F} \in \mathbf{ENDOM}(\mathbb{C}[[u]])$ associated to

1. the definition will come later in the chapter
2. namely : $\forall m \in \mathbf{N}, \quad |O_m| \geq a^m$ where $a \in \mathbb{R}_{+}^{*}$
3. see notations of section 3.2 page 52

ϕ :

$$\mathbf{F} : \left\{ \begin{array}{ccc} \mathbb{C}\{u\} & \longrightarrow & \mathbb{C}\{u\} \\ f(u) & \mapsto & f(\phi(u)) \end{array} \right. .$$

We have the equivalences :

$$\begin{array}{cl} & y^+ (t, \phi(u)) \text{ is real} \\ \iff & \mathbf{F}\, y^+ (t,u) \text{ is real} \\ \iff & \overline{\mathbf{F}\, y^+ (t,u)} = \mathbf{F}\, y^+ (t,u) \\ \iff & \overline{\mathbf{F}}\; \overline{y^+ (t,u)} = \mathbf{F}\, y^+ (t,u) \\ \iff & \overline{\mathbf{F}}\; y^- (t,u) = \mathbf{F}\; y^+ (t,u) \\ \iff & \overline{\mathbf{F}}\, \boldsymbol{\Delta}^+ y^+ (t,u) = \mathbf{F}\; y^+ (t,u) \end{array}$$

And so if $\overline{\mathbf{F}}\, \boldsymbol{\Delta}^+ = \mathbf{F}$ then $y^+ (t, \phi(u))$ is real. But this equality is equivalent [4] to $\phi = \xi \circ \overline{\phi}$. □

Remark 9.2.1.1 *This functional equation admits analytical solutions. Indeed, if we are looking for such a solution reading* $\phi(u) = u + \overset{\circ}{\phi}(u) \in \mathbb{C}[[u]]$ *with* $\overset{\circ}{\phi}(u) \in \mathbb{C}[[u]] \in u^2\mathbb{C}[[u]]$ *that satisfies* $\overline{\overset{\circ}{\phi}} = -\overset{\circ}{\phi}$*, the equation (9.1) reads*

$$2\overset{\circ}{\phi} = \left(\overset{\circ}{\xi} - id\right) \circ \left(id - \overset{\circ}{\phi}\right)$$

where $\overset{\circ}{\xi}(u) = \xi(u) - u$*. Let us consider* $\theta(x,y) = \overset{\circ}{\xi}(x-y) - x - y$*. The function* θ *is analytic on a small neighborhood of* $0 \in \mathbb{C}^2$*. Moreover,* $\theta(0,0) = 0$ *and* $\partial\theta/\partial y (0,0) = -1 \neq 0$*. So by application of the analytic implicit function theorem, there exist* $\overset{\circ}{\phi}(u) \in \mathbb{C}\{u\}$ *such that* $\theta\left(u, \overset{\circ}{\phi}(u)\right) = 0$*. Then* $\phi = id + \overset{\circ}{\phi}$ *is an analytic solution of 9.1* [5]*.*

9.2.2 Expression of a particular solution of Schäfke equation in terms of the coefficients of the sectoral isotropy

If we are looking for a solution ϕ of the equation (9.1) of the form $\phi(u) = u + \overset{\circ}{\phi}(u)$, where $\overline{\overset{\circ}{\phi}} = -\overset{\circ}{\phi}$, then this equation becomes [6] :

$$\left(\mathrm{id} + \frac{\overset{\circ}{\xi}}{2}\right) \circ \left(\mathrm{id} - \overset{\circ}{\phi}\right) = \mathrm{id} \tag{9.2}$$

Consider an analytic solution $\phi(u) = u + \overset{\circ}{\phi}(u) = u + \sum_{m\geq 2} \phi_m u^m \in \mathbb{C}\{u\}$ of Schäfke equation. As long as $\overset{\circ}{\phi}$ satisfies the previous equation, we are able to

4. see proposition 2.4.5 page 48
5. This proof is from R. Schäfke
6. This new form of the Schäfke equation is from D. Sauzin

express the coefficients of $\overset{\circ}{\phi}$ in terms of those of $\overset{\circ}{\xi}= \sum_{m\geq 1}\xi_m u^{m+1}$. We need to do it the simplified Laplace inversion formula :

Consider $f(u)\in\mathbb{C}[[u]]$ such that $f'(0)\neq 0$. Then $f(u)$ is invertible in $\mathbb{C}[[u]]$ and

$$f^{-1}(u)=\sum_{n\geq 1}\partial_u^{n-1}\left(\frac{u}{f(u)}\right)^n\Big|_{u=0}\frac{u^n}{n!}$$

We will use it to inverse $\mathrm{i}d+\dfrac{\overset{\circ}{\xi}}{2}$. We start by computing for u small enough :

$$\chi(u):=\frac{u}{u+\frac{\overset{\circ}{\xi}}{2}(u)}=\frac{1}{1+\frac{\overset{\circ\circ}{\xi}(u)}{2}}=\sum_{m\geq 0}\left(\frac{-1}{2}\right)^m\left(\overset{\circ\circ}{\xi}(u)\right)^m$$

where $\overset{\circ\circ}{\xi}(u)=\overset{\circ}{\xi}(u)/u$. As long as $\overset{\circ\circ}{\xi}(u)=\sum_{m\geq 1}\xi_m u^m$, we get

$$\left(\overset{\circ\circ}{\xi}(u)\right)^m=\sum_{n\geq 0}\sum_{\|\underline{\omega}\|=n,\mathrm{l}(\underline{\omega})=m}\xi_{\omega_1}\dots\xi_{\omega_m}u^n.$$

Then

$$\chi(u)=\sum_{m\geq 0}\sum_{n\geq 0}\sum_{\|\underline{\omega}\|=n,\underline{\omega}=(\omega_1,\dots,\omega_m)\in\mathbf{N}^{*\bullet}}\left(\frac{-1}{2}\right)^m\xi_{\omega_1}\dots\xi_{\omega_m}u^n.$$

With $\chi_n:=\sum_{m\geq 1}\sum_{\|\underline{\omega}\|=n,\mathrm{l}(\underline{\omega})=m}\left(\frac{-1}{2}\right)^m\xi_{\omega_1}\dots\xi_{\omega_m}$, we have :

$$(\chi(u))^n \;=\; \sum_{k\geq 0}\sum_{\|\underline{\omega}\|=k,\underline{\omega}=(\omega_1,\dots,\omega_n)\in\mathbf{N}^{\bullet}}\chi_{\omega_1}\dots\chi_{\omega_n}u^k.$$

Be careful that the words $\underline{\omega}=(\omega_1,\dots,\omega_n)$ are elements of $\mathbf{N}^{\bullet}$ and no more of $\mathbf{N}^{*\bullet}$. It comes

$$\partial_u^{n-1}(\chi(u))^n\Big|_{u=0}=(n-1)!\sum_{\|\underline{\omega}\|=n-1}\chi_{\omega_1}\dots\chi_{\omega_n}.$$

Then

$$\begin{aligned}\mathrm{i}d-\overset{\circ}{\phi} &= \sum_{n\geq 1}\frac{1}{n}\sum_{\|\underline{\omega}\|=n-1,\mathrm{l}(\underline{\omega})=n}\chi_{\omega_1}\dots\chi_{\omega_n}u^n\\ &= \sum_{n\geq 1}\frac{1}{n}\sum_{\|\underline{m}\|=n-1,\mathrm{l}(\underline{m})=n}\sum_{\|\underline{\omega}^1\|=m_1,\dots,\|\underline{\omega}^n\|=m_n}\left(-\frac{1}{2}\right)^{\mathrm{l}(\underline{\omega}^1)+\dots+\mathrm{l}(\underline{\omega}^n)}\xi_{\underline{\omega}_1}\dots\xi_{\underline{\omega}_n}u^n\end{aligned}$$

Thus we have proved that :

Theorem 9.2.2 *The analytic solution* $\phi(u) = u + \overset{\circ}{\phi}(u)$ *of Schäfke equation satisfying* $\overline{\overset{\circ}{\phi}} = -\overset{\circ}{\phi}$ *is given in terms of the coefficients of the sectoral Martinet-Ramis isotropy by :*

$$\phi(u) = 2u - \sum_{n\geq 1} \frac{1}{n} \sum_{\|\underline{m}\|=n-1, l(\underline{m})=n} \sum_{\|\underline{\omega}^1\|=m_1,\ldots,\|\underline{\omega}^n\|=m_n} \left(-\frac{1}{2}\right)^{l(\underline{\omega}^1)+\ldots+l(\underline{\omega}^n)} \xi_{\underline{\omega}_1} \ldots \xi_{\underline{\omega}_n} u^n$$

Remark 9.2.2.1 *The analyticity of* $id - \overset{\circ}{\phi}$ *is a consequence of the implicit function theorem and of the uniqueness of this solution. But we can verify it directly with this last expression using the geometrical growth of the coefficients of* $\xi(u)$*.*

Remark 9.2.2.2 *If there exists* $r \in \mathbf{N}^*$ *such that the* r *first Ecalle's analytical invariants are nulls* ($C_1 = \ldots = C_r = 0$) *then, as a consequence of formulas of subsection 3.2.3 page 58, one has* $\xi_1 = \ldots = \xi_r = 0$ *and so* $\phi_1 = \ldots = \phi_r = 0$*.*

9.2.3 Description of the problem

Link between Schäfke equation and ALIEN calculus

As explained in chapter 7, an average $\mathbf{m} \in \mathbf{AVER}$ is given by an **ALIEN** operator **rem** such that $\mathbf{m} = \mathbf{mur}\,\mathbf{rem}$. An average is said to be well behaved if and only if :

1. **rem** preserves the convolution product.
2. **rem** respects the realness, i.e. **rem** satisfies the equality $\mathbf{rem} = \mathbf{rul}\,\overline{\mathbf{rem}}$.
3. **rem** is analytic, i.e. for any reduction **red** the ordinary differential operators **red** (**rem**) preserves the analytical germs when it is the case for $\mathbf{red}\,(\Delta^+)$.

Let us consider a well behaved average $\mathbf{m} = \mathbf{mur}\,\mathbf{rem}$, a given real analytic saddle-node and its associated reduction $\mathbf{red} : \mathbf{ALIEN}\,(\Omega) \to \mathbf{ENDOM}\,(\mathbb{C}\,[[u]])$. The reduction of the equality $\mathbf{rem} = \mathbf{rul}\,\overline{\mathbf{rem}}$ leads to

$$\mathbf{F} = \overline{\mathbf{F}}\,\mathbf{\Delta}^+$$

where

— $\mathbf{F} = \mathbf{red}\,(\mathbf{rem})$ is an automorphism of substitution $\mathbf{F}_f \in \mathbf{ENDOM}\,(\mathbb{C}\,[[u]])$ by the analytic germ $f(u) := \mathbf{F}.u$.
— $\overline{\mathbf{F}} = \mathbf{red}\,(\overline{\mathbf{rem}})$ because we are dealing with a real saddle-node and moreover, $\overline{\mathbf{F}} = \mathbf{F}_{\overline{f}}$, see sub-section 7.2.2.
— $\mathbf{\Delta}^+ = \mathbf{red}\left(\dot{\Delta}^+\right)$ is the reduction of the Stokes automorphism, which is in fact the substitution automorphism $\mathbf{F}_\xi$ where $\xi(u)$ is the Martinet-Ramis sectoral isotropy.

When applying this equality to the germ of identity, we obtain $\mathbf{F}.u = \left(\overline{\mathbf{F}}\,\mathbf{F}_\xi\right).u$ which can be written [7]

$$f(u) = \xi\left(\overline{f}(u)\right).$$

and we recognize the Schäffke's equation (9.1) which is in fact the reduction of the **ALIEN** equality $\mathbf{rem} = \mathbf{rul}\,\overline{\mathbf{rem}}$.

At this stage, we have proved that if $\mathbf{m} = \mathbf{mur}\,\mathbf{rem}$ is a well behaved average, then the germ $\mathbf{red}\,(\mathbf{rem})\,.u$ must be an analytical solution of the Schäfke equation and we have answered to the first part of our problem.

The problem of the reciprocal

We have now to deal with the reciprocal. Let us consider a given real analytic saddle node with the associated reduction **red**. We consider also an analytical solution $f(u)$ of the Schäfke equation linked to this saddle-node and ask ourselves if it is possible to find an **ALIEN** operator **rem** such that :

— it verifies he three previous points 1., 2. and 3. characterizing well behaved averages,

— and it is such that $\mathbf{red}\,(\mathbf{rem})\,.u = f(u)$.

There are three main obstacles :

— The first one is that **red** is generally not surjective and there are some solutions of Schäfke equation that can not be incarnated by well behaved averages. The sub-section 9.2.4 is devoted to such an example. In order to avoid this problem, we assume that we are working only with saddle-nodes having some non-null family of Ecalle's invariants. Thus the morphism **red** becomes a surjection between **ALIEN** and $\mathcal{D}\,(\mathbb{B})$ with $\mathbb{B} = \sum_{m\geq 0} u^{m+1}\partial_u$, see subsection 2.4.2 page 46 for notations.

— The second one is that **red** is generally not injective. Indeed, using the bridge equation, in the case of a saddle-node with Ecalle's invariants never equal to zero, the image of the base $(\Delta_\bullet)$ of **ALIEN** is the family $(\mathbb{B}_\bullet)$, with for any $m \in \mathbf{N}^*$, $\mathbb{B}_m = u^{m+1}\partial_u$, that is not free. We have for example $\mathbb{B}_2.\mathbb{B}_1 = 3u^4\partial + u^5\partial^2$ and, $\mathbb{B}_1.\mathbb{B}_2 = 2u^4\partial + u^5\partial^2$. Then it comes

$$\mathbb{B}_4 = \mathbb{B}_1\mathbb{B}_2 - \mathbb{B}_2\mathbb{B}_1.$$

The image of the free family $\left(\dot{\Delta}_m\right)$ by a reduction **red** is not a free one and thus **red** is not injective. As a direct and unpleasant consequence, if we have found an **ALIEN** operator F such that $\mathbf{red}\,(F) = \mathbf{F}$ for a given saddle-node and if $\mathbf{F}$ is a solution of Schäfke equation $\mathbf{F} = \overline{\mathbf{F}}\boldsymbol{\Delta}^+$ for this saddle-node, we do not necessarily have $F = \dot{\Delta}^+\overline{F}$ and we can not claim that the associated average $\mathbf{m} = \mathbf{mur}F$ respects the realness.

— The third problem, which is the most difficult, consists for a given solution $\mathbf{F}$ of Schäfke equation being in Im **red** to construct an **ALIEN** operator F that stays analytic in all other possible reductions of **ALIEN**.

7. Indeed $\left(\overline{\mathbf{F}}\,\mathbf{F}_\xi\right).u = \overline{\mathbf{F}}\xi\,(u) = \xi\left(\overline{f}(u)\right)$.

Our approach of the problem

We explain now our approach of the problem. We consider an analytic solution $f(u)$ of Schäfke equation associated to a fixed analytic and real saddle-node with associated reduction **red**. The Ecalle's invariants (C_m) of this saddle-node are assumed to be non null and thus, as explained before, the algebra anti-morphism **red** is a surjection from **ALIEN** into $\mathcal{D}(\mathbb{B})$. We denote by $\mathbf{F}_f$ the substitution automorphism associated to $f(u)$.

The bridge equation, from which is defined the reduction morphism, expressed a linear dependence between the homogeneous components of an **ALIEN** derivation and those of a derivation of $\mathcal{D}(\mathbb{B})$. So it will be pertinent to translate our problem in terms of derivations instead of convolution automorphisms. Moreover, it is very easy to compute the expansion of a derivation of **ENDOM** $(\mathbb{C}[[u]])$ relatively to the comould $\mathbb{B}_\bullet$ because it suffices to expand the power sum defining it :

$$\mathbb{D} = (\mathbb{D}.u)\,\partial = \sum_{m\geq 1} d_m \mathbb{B}_m$$

with $\mathbb{D}.u = \sum_{m\geq 1} d_m u^{m+1}$ and so $\mathbb{D} = \mathbf{red}\,(D)$ with

$$D = \sum_{m\in\mathbf{N}^*} \frac{d_m}{C_m}\dot{\Delta}_m \in \mathbf{ALIEN}.$$

We then need a process permitting to associate automorphisms and derivations, in a biunivocal way. There are several such correspondences.

For example, we can consider the operation consisting to associate to an automomorphism $\mathbf{F}$ its infinitesimal generator $\mathbb{D} := \log \mathbf{F}$ (then $\mathbf{F} = \exp \mathbb{D}$). But it is not enough, we need that corresponding automorphism and derivation are equianalytics, which means that the automorphism $\mathbf{F}$ must be analytic if and only if the corresponding derivation is analytic too. This is not the case for an automorphism and its infinitesimal generator because we know that the infinitesimal generator of an analytic automorphism may be divergent, see corollary 3.2.5 page 59.

Appariated automorphism and derivations

Nonetheless, such a correspondence exists, it consists in the notion of appariated automorphism and derivations already investigated by F. Menous in his thesis. If $\mathbb{D} = \sum d_m \mathbb{B}_m$ is the appariated derivation associated to $\mathbf{F}$ then one has $\mathbf{red}\,(F) = \mathbf{F}$ for the appariated automorphism to $D = \sum d_m / C_m \dot{\Delta}_m$. But one encounters several problems described here below.

The first is that if $\mathbf{red}'$ is another reduction associated to a set (C'_m) of Ecalle's invariants then one has

$$\mathbf{red}'\,(D) = \sum_{m\geq 1} d_m \frac{C'_m}{C_m}\mathbb{B}_m$$

which is not necessarily analytic. Indeed, Ecalle's invariants, as proved in corollary 3.2.5 page 59, have a 1-Gevrey growth. So we need to work with another family of invariants than the Ecalle's ones, meaning not working anymore with the standard **ALIEN** derivation which is not analytic. As explained in remark 3.2.2.1 page 56, it is possible to write a bridge equation for any **ALIEN** derivation and with an analytical one, the obtained new family of analytical invariants has a geometrical growth. Such derivations exists : the most accessible example is given by **doom**, the appariated derivation to the Stokes automorphism. However, as we will explained now, this derivation do not have sufficient properties to solve our problem.

Working with good families of analytical invariants

Indeed, writing (O_m) the analytical invariants appearing in the bridge equation for such an analytical derivation, one has, with the previously introduced notations,

$$\mathbf{red}'(D) = \sum_{m\geq 1} d_m \frac{O'_m}{O_m} \mathbb{B}_m.$$

But the sequence (O'_m/O_m) is not necessarily of geometrical growth unless working with a sequence (O_m) having at least a geometrical growth, i.e. if there exists a real $a \in \mathbb{R}_+^*$ such that $\forall m \in \mathbf{N}^*, |O_m| \geq a^m$. For this we need to work with real analytic saddle-nodes with analytical invariants (O_m) having at less a geometrical growth. Let us remark that this requirement implies that any of these invariants is null and thus the surjectivity of **red**.

We naturally have to prove that such analytic classes of saddle-nodes exist.

There are several questions :

— Using the synthesis of Martinet-Ramis for the saddle-node, we know that every family of sequences with geometrical growth is the Martinet-Ramis invariants family of an analytic saddle-node. The same does not occur for Ecalle's invariants. More precisely, for a given family with a 1-Gevrey growth, we do generally not know if there exists an analytic saddle-node for which it is Ecalle's family of analytical invariants [8].

— The same problem occurs for the analytical invariants associated to an **ALIEN** analytic derivation unless this derivation is equianalytic to the Stokes automorphism (in the following, we will say Stokes-analytic for short), what we will explain now. We know that the analytical invariants associated with such an analytic derivation are sequences with a geometrical growth. But between such sequences, we do not know how to

8. Indeed, when giving a 1-Gevrey family of complex scalars, the only known method allowing to say if it is an Ecalle's invariants family or not is the following. We must consider a vector field with coefficients given by this family and to compute the one-time map associated to the flow of this vector field (the vector field is the infinitesimal generator of this map). Then the initial family is an Ecalle one if and only if this map is analytic. Indeed, if this map is analytic then, using the Martinet-Ramis synthesis, there exists an analytic saddle-node for which one it is the Martinet-Ramis sectoral isotropy. Such computations can generally not be done.

recognize the ones which are family of analytical invariants for this derivation. When working with a Stokes-analytic **ALIEN** derivation, every complex sequences with a geometrical growth is associated with an analytic saddle-node for which one it is a family of analytical invariants. The derivation **doom** is Stokes-analytic, therefore it seems to be the good answer to our problem.

— But this is unfortunately not the case. We need **real** Stokes-analytic derivation, and **doom** is not a real one. Indeed, as explained just before, using some Stokes-analytic **ALIEN** derivation, for any sequence having at least a geometrical growth, we can claim the existence of an analytic and **complex** saddle-node having it as analytical invariants. But our purpose is **real** analytic saddle-node and we do not know if there exists analytic saddle-nodes having such kind of analytical invariants. The invariants associated with a real **ALIEN** analytic derivation like those induced by diffusion studied in chapter 7 are sequences having a geometrical growth and real when working with real saddle-node. When working with a real and Stokes-analytic derivation, any sequence with geometrical growth is a family of analytical invariant and moreover, **the real geometrical sequences are exactly the analytical invariants of real analytic saddle-node**. Thus if such a derivation exists, for a chosen at least geometrical real sequence, we can claim there exists a real saddle-node having this sequences as analytical invariants. In addition, we will restrict ourselves to such real saddle-node for our investigations about Schäfke equation.

Now we have to exhibit such an **ALIEN** derivation.

As already written, **doom** is Stokes-analytic but not real. A natural idea is to consider its imaginary part (or its real part). The obtained real derivation stays analytic but it is a priori not Stokes-analytic. Nonetheless, we have detailed its construction in the following pages because it allows to obtain in a purely algebraic way an already known well behaved derivation, the organic one **dom**. It was discovered by Ecalle but using a limit process and considerations about the organic average ([14]). Moreover, our original construction make it possible to verify easily its fundamental properties. We mention that Ecalle uses this derivation to simplify its proof for the synthesis problem, see [14].

The derivations **doom** and **dom** do not answer to our problem. The first one is Stokes-analytic but not real. It is the converse for the second one and we need a derivation having these two properties. An example of such a derivation is given by the Brownian one, which is a particular case of diffusion induced derivation as described in chapter 7. This derivation was already investigated by F. Menous in his thesis where he gives a proof of its Stokes-analyticity. However this proof requested very long and difficult computations concerning the Catalan's average. In the following pages, we will propose an original and short proof of this fact.

The last (but not the least) problem

We consider a given real and analytic saddle-node having Brownian invariants with exact geometrical growth. We denote by **red** the associated reduction, by $f(u)$ an analytic solution of Schäfke equation, by $\mathbf{F}$ the associated substitution automomorphism and we set $\boldsymbol{\Delta}^+ := \mathbf{red}\left(\dot{\Delta}^+\right)$. Using the notion of appariement, we know how to construct an **ALIEN** analytic automorphism F such that $\mathbf{red}\,(F) = \mathbf{F}$. As a consequence of the fact that **red** fails to be injective, the operator F does not necessarily verify $\overline{F} = \dot{\Delta}^+ F$ and the associated average $\mathbf{m} = \mathbf{mur}\, F$ is not necessarily real.

We will skip this difficulty by a factorization theorem in the proof of which one, we unfortunately need the existence of well behaved averages. So we do not purpose here a new proof of existence of well behaved averages.

A summary of what precedes

Let us summarize the different steps allowing to construct a well behaved average which corresponds to a given solution of Schäfke equation for a given real and analytics saddle-node with brownian invariants (O_m) having at least a geometrical growth.

ϕ an analytic solution of Schäffke equation.

$\downarrow$.

$\mathbf{F}_\phi$ associated substitution automorphism.

$\downarrow$.

$\mathbb{D} := \mathbf{F}_\phi^{-1} \nabla \mathbf{F}_\phi =: \sum_{m \geq 1} d_m u^{m+1} \partial$ appariated derivation to $\mathbf{F}_\phi$.

$\downarrow$

$D := \sum_{m \geq 1} d_m / O_m u^{m+1} \partial$ **ALIEN** analytic derivation. We have $\mathbf{red}\,(D) = \mathbb{D}$.

$\downarrow$

F **ALIEN** analytic automorphism appariated to D. We have $\mathbf{red}\,(F) = \mathbf{F}$.

$\downarrow$

$\mathbf{rem} := F\eta$ is the right lateral operator associated to a well behaved average. It verifies $\mathbf{red}\,(\mathbf{rem})\,.u = \phi$.

The **ALIEN** convolution automorphism η is obtained using the factorization theorem. It is analytic and it satisfies $\mathbf{red}\,(\eta) = 1$.

9.2.4 An example of a solution of Schäfke equation that is not in the image of red

There are solutions of Schäfke equation that can not be obtained by averages theory. Let us consider for example a saddle-node with Ecalle's Family of invariant $(C_m)_{m \in \{-1\} \cup \mathbf{N}^*}$ such that $C_2 = 0$. The sectoral isotropy of this saddle-node is $\xi(u) = u + \sum_{m\geq 1} \xi_m u^{m+1}$ and the expression of the coefficients ξ_m are given by formulas of sub-section 3.2.3 :

$$\xi_m = \sum_{\substack{\|\underline{\omega}\| = m \\ \underline{\omega} \in \Omega^\bullet}} \frac{(-1)^{\mathrm{l}(\underline{\omega})}}{\mathrm{l}(\underline{\omega})!} C_{\underline{\omega}} \beta_{\underline{\omega}}$$

In particular, $\xi_1 = -C_1$ and $\xi_2 = C_1^2 - C_2$. We will prove here that **red** is not surjective in this case. Consider an analytic solution $\phi(u) = u + \overset{\circ}{\phi}(u) = u + \sum_{m\geq 2} \xi_m u^m \in \mathbb{C}\{u\}$ of the Schäfke equation like in remark 9.2.2. We will prove that $\chi = \mathrm{i}d + \overset{\circ}{\xi}/2$ is not in the image of **red**. Consequently, $\mathrm{i}d - \overset{\circ}{\phi} = \left(\mathrm{i}d + \overset{\circ}{\xi}/2\right)^{-1}$ is not in Im **red** and the same occurs for $\phi = \overline{\mathrm{i}d - \overset{\circ}{\phi}}$.

If $\chi = u + \sum_{m\geq 1} \chi_m u^{m+1}$ then the equality $\chi = \mathrm{i}d + \overset{\circ}{\xi}/2$ gives $\chi_1 = -C_1/2$ and $\chi_2 = \left(C_1^2 - C_2\right)/2$.

If there exists an automorphism of convolution $\mathbf{op} = \sum M^\bullet \Delta_\bullet \in \mathbf{ALIEN}$ such that $\mathbf{red}(\mathbf{op}).u = \chi(u)$ then we get for all $m \in \mathbf{N}$:

$$\chi_m = \sum_{\substack{\underline{\omega} \in \Omega^\bullet \\ \|\underline{\omega}\| = m}} M^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}}$$

with $M^\bullet$ a symmetral mould.

For $m = 1$ and $m = 2$, we obtain $\chi_1 = M^1 C_1$ and $\chi_2 = M^2 C_2 + 2M^{1,1} C_1^2$.

In conclusion, the unknowns M^2, M^1 and $M^{1,1}$ are solutions of the system :

$$\begin{cases} M^1 C_1 & = -\dfrac{C_1}{2} \\ M^2 C_2 + 2M^{1,1} C_1^2 & = \dfrac{C_1^2 - C_2}{2} \\ \left(M^1\right)^2 & = 2M^{1,1} \end{cases}$$

The third equation is a consequence of the symmetrality of $M^\bullet$. But if $C_1 \neq 0$ and $C_2 = 0$ then the first equation gives $M^1 = -1/2$, the second gives $M^{1,1} = 1/4$ what is clearly incompatible with the third. So we can not realize χ as the reduction of a convolution automorphism of **ALIEN**.

9.3 Apparaited automorphisms and derivations

9.3.1 Definitions and properties

We will recall in this section some basic facts about apparaited automorphisms and derivations. This notion was introduced by F. Menous in [41]. When translating **ALIEN** in the language of non-commutative symmetric functions, it is related to the Dynkin operator.

The symbol $\mathbb{A}$ will denote here the algebra **ALIEN** or the sub-algebra $\mathcal{D}(\mathbb{B})$ of **ENDOM** $(\mathbb{C}[[u]])$.

Definition 9.3.1.1 *We define the operator ∇ on $\mathbb{A}$ by its action on the homogeneous components of an operator $F = \sum_{n\geq 0} F_n \in \mathbb{A}$:*

$$\forall m \in \mathbf{N}, \quad \nabla F_m = mF_m.$$

Proposition 9.3.1 *The operator ∇ is a derivation on $\mathbb{A}$.*

Proof The operator ∇ is clearly $\mathbb{C}$-linear. Consider two operators $F = \sum_{n\in\mathbf{N}} F_n$, $G = \sum_{n\in\mathbf{N}} G_n \in \mathbb{A}$. Then :

$$\begin{aligned}
\nabla(FG) &= \nabla\left(\sum_{n\in\mathbf{N}}\sum_{k=0}^{n} F_k G_{n-k}\right)\\
&= \sum_{n\in\mathbf{N}} n \sum_{k=0}^{n} F_k G_{n-k}\\
&= \sum_{n\in\mathbf{N}}\sum_{k=0}^{n} kF_k G_{n-k} + \sum_{n\in\mathbf{N}}\sum_{k=0}^{n}(n-k)F_k G_k\\
&= (\nabla F)G + F(\nabla G)
\end{aligned}$$

and so ∇ satisfies the Leibniz rule. □

Remark 9.3.1.1 *For an operator $D = \sum_{n>0} D_n$, we have, for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$:*

$$\nabla(D_{\omega_1}\ldots D_{\omega_r}) = \|\underline{\omega}\| D_{\omega_1}\ldots D_{\omega_r}.$$

This remark suggests to introduce the following operator on $\mathbb{A}$:

Definition 9.3.1.2 *We define an operator $\int$ on $\mathbb{A}$ by its action on the homogeneous components of an operator $F = \sum_{n\geq 0} F_n \in \mathbb{A}$:*

$$\forall m \in \mathbf{N}, \quad \int_a F_m = \begin{cases} \dfrac{F_m}{m} \text{ if } m > 0 \\ a \text{ if } m = 0 \end{cases}$$

where $a \in \mathbb{C}$ is a beforehand choozen complex constant.

Remark 9.3.1.2 *For any operator $F = \sum_{n\geq 0} F_n$:*

- *if $a = F_0$ then $\int_a \nabla F = F$,*
- *If $F_0 = 0$, for any $a \in \mathbb{C}$, $\nabla \int_a F = F$.*

Proposition 9.3.2 *Consider $F \in \mathbb{A}$ an automorphism (of convolution if $\mathbb{A} =$ **ALIEN** or of substitution if $\mathbb{A} =$ **ENDOM**$(\mathbb{C}[[u]])$). Then $D = F^{-1}\nabla F$ is a derivation. Moreover, for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$,*

$$< D, F >^{\underline{\omega}} = (-1)^{r-1}\omega_r, \quad < D, F^{-1} >^{\underline{\omega}} = (-1)^r \omega_1,$$

$$< F, D >^{\underline{\omega}} = \frac{1}{\check{\omega}_1 \ldots \check{\omega}_r}, \quad < F^{-1}, D >^{\underline{\omega}} = \frac{(-1)^r}{\hat{\omega}_1 \ldots \hat{\omega}_r}$$

where $\forall i \in [\![1, r]\!]$, $\check{\omega}_i = \omega_1 + \ldots + \omega_i$ and $\hat{\omega}_i = \omega_i + \ldots + \omega_r$. Conversely, if $\mathbb{D}$ is a derivation then the operator F given by the previous mould $< F, D >^\bullet$ is an automorphism and it satisfies $D = F^{-1}\nabla F$.

The operators F, F^{-1} and D are said to be appariated.

Proof We easily compute for all $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ that

$$< F^{-1}, F >^{\underline{\omega}} = (-1)^{\mathrm{l}(\underline{\omega})} \text{ and } < \nabla F, F >^{\underline{\omega}} = \begin{cases} \omega & \text{if } \mathrm{l}(\underline{\omega}) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$< F^{-1}\nabla F, F >^{\underline{\omega}} = \sum_{\underline{\omega}_1 \cdot \underline{\omega}_2 = \underline{\omega}} < F^{-1}, F >^{\underline{\omega}_1} \times < \nabla F, F >^{\underline{\omega}_2} = (-1)^{\mathrm{l}(\underline{\omega})-1}\omega_r$$

which is alternel and then D is a derivation.

For the second formula, observe that if $G = F^{-1}$ then $D = F^{-1}\nabla F = G\nabla G^{-1}$. Moreover

$$G^{-1} = \sum_{\underline{\omega} \in \Omega^\bullet} (-1)^{\mathrm{l}(\underline{\omega})} G_{\underline{\omega}}$$

and

$$\nabla G^{-1} = \sum_{\underline{\omega} \in \Omega^\bullet} (-1)^{\mathrm{l}(\underline{\omega})} \|\underline{\omega}\| G_{\underline{\omega}}$$

then $< \nabla G^{-1}, G >^\bullet = (-1)^{\mathrm{l}(\bullet)} \|\bullet\|$. We compute now for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$:

$$\begin{aligned} < D, F^{-1} >^{\underline{\omega}} &= < D, G >^{\underline{\omega}} \\ &= \left(< G, G >^\bullet \times < \nabla G^{-1}, G >^\bullet\right)^{\underline{\omega}} \\ &= (-1)^{r-1}\hat{\omega}_2 + (-1)^r \hat{\omega}_1 \\ &= (-1)^r \omega_1 \end{aligned}$$

For the third formula, we look for a solution of the differential equation $\nabla G = GD$ of unknown $G \in A$. Consider the mould-comould expansion $G = \sum_{\underline{\omega} \in \Omega^\bullet} G^{\underline{\omega}} D_{\underline{\omega}}$. Then using the fact that ∇ is a derivation on $\mathbb{A}$, we find that

$$\nabla G = \sum_{\underline{\omega} \in \Omega^\bullet} G^{\underline{\omega}} \nabla\left(D_{\underline{\omega}}\right) = \sum_{\underline{\omega} \in \Omega^\bullet} G^{\underline{\omega}} \|\underline{\omega}\| D_{\underline{\omega}}$$

and so $< \nabla G, D >^{\bullet} = \| \bullet \| G^{\bullet}$. Then the differential equation is equivalent to :

$$\forall \underline{\omega} \in \Omega^{\bullet}, \quad \|\underline{\omega}\| G^{\underline{\omega}} = G^{\underline{\omega}'}$$

where $(w_1, \ldots, \omega_r)' = (\omega_1, \ldots, \omega_{r-1})$. As the mould $G^{\bullet}$ must be symmetral, $G^{\emptyset} = 1$ and we easily compute by induction that $\forall \underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^{\bullet}, \quad G^{\underline{\omega}} = \dfrac{1}{\check{\omega}_1 \ldots \check{\omega}_r}$.

Moreover, we have proved that there exists an unique automorphism solution of the differential equation. And then $F = G$, which ends the proof. $\square$

Theorem 9.3.3 *For any* $\underline{\omega} = \overset{\vdots\ \vdots\ \vdots}{\underset{\bullet \omega_1}{\diagdown | \diagup}} \in \Omega^{\bullet <}$,

$$< D, F >^{\underline{\omega}^{\gg}} = (-1)^{r-1} \omega_1, \quad < D, F^{-1} >^{\underline{\omega}^{\ll}} = (-1)^r \omega_1,$$

$$< F, D >^{\underline{\omega}^{>}} = \frac{1}{\check{\omega}_1 \ldots \check{\omega}_r}, \quad < F^{-1}, D >^{\underline{\omega}^{<}} = \frac{(-1)^r}{\hat{\omega}_1 \ldots \hat{\omega}_r}$$

where $\forall i \in [\![1, r]\!]$, $\check{\omega}_i$ *(respectively* $\hat{\omega}_i$*) is the sum of all the nodes in* $\underline{\omega}^{<}$ *anterior (respectively posterior) to* ω_i *for the antiarborescent order (respectively arborescent).*

Proof We denote by $N^{\bullet}$ the mould $< D, F^{-1} >^{\bullet}$. This mould is alternel so its contracting arborified is primitive. The formula is clearly true for a tree of length one. We assume it is true for a tree of length $\leq r$ and we prove it for a tree $\underline{\omega} = \overset{\vdots\ \vdots\ \vdots}{\underset{\bullet \omega_1}{\diagdown | \diagup}} \in \Omega^{\bullet <}$ of length $r + 1$. One has

$$\begin{aligned} N^{\underline{\omega}^{\ll}} &= \sum_{\underline{\omega} \in \Omega^{\bullet}} \mathbf{csh}\left(\underline{\omega}^{<}, \underline{\omega}\right) N^{\underline{\omega}} \\ &= \sum_{\underline{\omega} \in \Omega^{\bullet}} \mathbf{csh}\left(\underline{\omega}^{<}, \underline{\omega}\right) (-1)^{l(\underline{\omega})} \omega_1 \\ &= (-1)^{l\left(\underline{\omega}^{<}\right)} \omega_1 \end{aligned}$$

and the formula follows by induction and by the separativity of $M^{\bullet <}$. The formula for the mould $< D, F >^{\bullet^{\gg}}$ follows directly.

We denote by $M^{\bullet}$ the mould $< F^{-1}, D >^{\bullet}$. This mould is symmetral so its simple arborified is separative. We proceed one time more by induction. The formula is true for forest of length one. We assume it is true for tree of length

$\leq r$ and we prove it for a tree $\underline{\omega}^< = (\omega_1, \ldots, \omega_{r+1})^<$ of length $r+1$. One has :

$$\begin{aligned} M^{\underline{\omega}^<} &= \sum_{\underline{\omega}\in\Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) M^{\underline{\omega}} \\ &= \sum_{\underline{\omega}\in\Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, \underline{\omega}\right) (-1)^{\mathrm{l}(\underline{\omega})} \frac{1}{\omega_1 + \ldots + \omega_r} N'^{\underline{\omega}} \\ &= (-1)^{\mathrm{l}\left(\underline{\omega}^<\right)} \frac{1}{\omega_1 + \ldots + \omega_r} \sum_{\underline{\omega}\in\Omega^\bullet} \mathbf{sh}\left(\underline{\omega}^<, ' \underline{\omega}\right) M^{\underline{\omega}} \\ &= (-1)^{\mathrm{l}\left(\underline{\omega}^<\right)} \frac{1}{\omega_1 + \ldots + \omega_r} M'^{\underline{\omega}^<} \end{aligned}$$

and the formula follows by induction. The formula for the mould $< F, D >^{\bullet^{\gg}}$ is a consequence of this last one. □

Corollary 9.3.4 *Let us consider an automorphism $F \in \mathbb{A}$ and its appariated derivation $D = F^{-1}\nabla F$. Then F is analytic if and only if D is analytic.*

Proof It is a direct consequence of the geometrical growth of the previous arborescent moulds. □

Proposition 9.3.5 *For an operator D, the two next assertions are equivalent :*
— *D is an analytic **ALIEN** derivation.*
— *∇D is an analytic **ALIEN** derivation.*

Proof If D is a derivation, the mould $M^\bullet :=< D, \Delta >^\bullet$ is alternal, i.e. :

$$\forall \underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet, \quad \sum_{\underline{\omega}\in \mathrm{sh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = 0.$$

We then easily obtain that $N :=< \nabla D, \Delta >^\bullet = \| \bullet \| M^\bullet$ and so for any $\underline{\omega}^1, \underline{\omega}^2 \in \Omega^\bullet$:

$$\sum_{\underline{\omega}\in \mathrm{sh}(\underline{\omega}^1, \underline{\omega}^2)} N^{\underline{\omega}} = \left(\|\underline{\omega}^1\| + \|\underline{\omega}^2\|\right) \sum_{\underline{\omega}\in \mathrm{sh}(\underline{\omega}^1, \underline{\omega}^2)} M^{\underline{\omega}} = 0$$

and then $N^\bullet$ is alternal to. Thus ∇D is an **ALIEN** derivation. We prove the reciprocal in the same manner.

Consider now a reduction **red**. We have $\mathbf{red}\,(D)\,.u = u + \sum_{m\geq 1} d_m u^{m+1}\partial u$ with $d_m = \sum_{\|\underline{\omega}\|} M^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}}$ and after this same reduction, we get $\mathbf{red}\,(\nabla F) = u + \sum_{m\geq 1} d'_m u^{m+1}\partial u$ where for any $m \geq 1$

$$\begin{aligned} d'm &= \sum_{\|\underline{\omega}\|} N^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}} \\ &= m \sum_{\|\underline{\omega}\|} M^{\underline{\omega}} C_{\underline{\omega}} \beta_{\underline{\omega}} \\ &= m d_m \end{aligned}$$

The operator D is analytic if and only if the sequence (d_m) has a geometrical growth. This is true if and only if the sequence $(d'_m) = (md_m)$ has a geometrical growth. But this last sequence has a geometrical growth if and only if ∇D is analytic. □

9.3.2 Appariated derivation to Stokes automorphism

Using the results of the previous section, we know that the operator **doom** $:= \Delta^- \nabla \Delta^+$ is an analytic derivation. This derivation do not fulfill the criteria for solving our problem. But it allows to construct in a totally algebraic way a well behaved **ALIEN** derivation, the organic one.

Using mould composition and proposition 9.3.2, we easily find :

Proposition 9.3.6 *We have for all* $\omega_1, \ldots, \omega_r \in \Omega^\bullet$ *:*

$$< \boldsymbol{doom}, \Delta^+ >^{\omega_1,\ldots,\omega_r} = \begin{cases} (-1)^{r-1}\omega_r & \text{if } r \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and :

$$< \boldsymbol{doom}, \Delta >^{\omega_1,\ldots,\omega_r} = \begin{cases} \sum_{\underline{\omega}^1 \cdot \ldots \cdot \underline{\omega}^s = \underline{w}} (-1)^{r+s-1} \dfrac{\|\underline{\omega}^s\|}{r_1! \ldots r_s!} & \text{if } r \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

.

9.3.3 Computation of the invariants associated to doom

Lemma 9.3.7 *For any* $r \in \mathbf{N}^*$ *and* $i \in [\![1, n]\!]$*, we have the two formulae :*

$$\sum_{k=0}^{i-1} (-1)^{k-1} \binom{r}{i} = (-1)^i \binom{r-1}{i-1}$$

$$\sum_{k=1}^{r} \sum_{r_1+\ldots+r_k=r} \frac{(-1)^k}{r_1! \ldots r_k!} = \frac{(-1)^r}{r!}$$

Proof

- The first formula follows from a simple induction.
- For the second one, observe that for any $z \in \mathbb{C}$:

$$e^{-z} = \frac{1}{1-(1-e^z)} = \sum_{k \geq 0} (1-e^z)^k \quad (\star)$$

but for any $k \geq 1$,

$$(1-e^z)^k = \left(-\sum_{i \geq 1} \frac{z^i}{i!} \right)^k = \sum_{r \geq 1} \sum_{r_1+\ldots+r_k=r} \frac{(-1)^k}{r_1! \ldots r_s!} z^r$$

and then the coefficient of order r of $1/\left(1-(1-e^z)\right)$ is

$$\sum_{k=1}^{r}\sum_{r_1+\ldots+r_k=r}\frac{(-1)^k}{r_1!\ldots r_k!}$$

that must be equal to the one of e^{-z} : $(-1)^r/r!$.

□

Theorem 9.3.8 *For any* $\underline{\omega}=(\omega_1,\ldots,\omega_r)\in\Omega^\bullet$,

$$<\boldsymbol{doom},\Delta>^{\underline{\omega}}=\frac{(-1)^r}{r!}\sum_{i=1}^{r}(-1)^{i+1}\,\omega_i\binom{r-1}{i-1}.$$

Proof We recall that

$$<\mathbf{doom},\Delta>^{\underline{\omega}}\;=\sum_{\substack{\underline{\omega}^1.\ldots.\underline{\omega}^s=\underline{\omega}\\ s\geq 1}}(-1)^{r+s-1}\,\frac{\|\underline{\omega}^s\|}{r_1!\ldots r_s!}$$

where $r_i=\mathrm{l}\left(\underline{\omega}^i\right)$ for each $i\in[\![1,s]\!]$.

But :

$$\begin{aligned}\sum_{\substack{\underline{\omega}^1.\ldots.\underline{\omega}^s=\underline{\omega}\\ s\geq 1}}(-1)^{s-1}\,\frac{\|\underline{\omega}^s\|}{r_1!\ldots r_s!} &= \sum_{i=1}^{r}\frac{\hat{\omega}_i}{(r-i+1)!}\sum_{\substack{\omega^1.\ldots.\omega^k=m_1,\ldots,m_{i-1}\\ k\geq 1}}\frac{(-1)^k}{r_1!\ldots r_k!}\\ &= \sum_{i=1}^{r}\frac{\hat{\omega}_i}{(r-i+1)!}\sum_{\substack{r_1+\ldots+r_k=i-1\\ k\geq 1}}\frac{(-1)^k}{r_1!\ldots r_k!}\\ &= \sum_{i=1}^{r}\frac{\hat{\omega}_1}{(r-i+1)!}\,\frac{(-1)^{i-1}}{(i-1)!}\\ &= \sum_{i=1}^{r}(-1)^{i-1}\,\frac{\hat{\omega}_1}{r!}\binom{r}{i-1}.\end{aligned}$$

because of the second formula of lemma 9.3.7

In view of the first formula of the same lemma, we can yet simplify our expression :

$$\begin{aligned}\sum_{i=1}^{r}(-1)^{i-1}\,\hat{\omega}_1\binom{r}{i-1}. &= \sum_{i=1}^{r}\omega_i\left(\sum_{k=0}^{i-1}(-1)^k\binom{r}{k}\right)\\ &= \sum_{i=1}^{r}(-1)^{i+1}\,\omega_i\binom{r-1}{i-1}\end{aligned}$$

and the result follows. □

Theorem 9.3.9 *For a given saddle-node associated to the Ecalle's family of invariants* $(C_m)_{m\in\{-1\}\cup\mathbf{N}^*}$*, we have* $\textbf{red}(\textbf{doom}).u = u + \sum_{m\geq 1} D_m u^{m+1}$ *with*

$$\forall m \geq 1, \quad D_m = \sum_{\|\underline{m}\|=m} (-1)^r C_{\underline{m}} \beta_{\underline{m}} \sum_{i=1}^{r} \frac{(-1)^i}{r!} \binom{r-1}{i-1} m_i$$

9.3.4 Link with formal noncommutative symmetric functions

We consider $\textbf{Sym} = \mathrm{K}\langle \Lambda_1, \Lambda_2, \ldots\rangle$ the algebra of formal noncommutative symmetric functions generated by an infinite set of inderterminates $(\Lambda_k)_{k\geq 1}$ and t another indeterminate which commutes with each one of the Λ_k.

- — the Λ_k are called the *elementary symmetric functions* and their generating series is $\lambda(t) = 1 + \sum_{k\geq 1} t^k \wedge_k$.
- — the complete homogeneous functions S_k defined by $\sigma(t) = \sum_{k\geq 0} t^k S_k = (\lambda(-t))^{-1}$.
- — The *power symmetric functions of the first kind* ψ_k defined by $\psi(t) = \sum_{k\geq 1} t^{k-1}\Psi_k$ and $\frac{\mathrm{d}}{\mathrm{dt}}\sigma(t) = \sigma(t)\psi(t)$.
- — The power sums symmetric functions Φ_k defined by $\sigma(t) = \exp\left(\sum_{k\geq 1} t^k \frac{\Phi_k}{k}\right)$.

  In particular, $\Phi(t) = \log\left(1 + \sum_{s\geq 1} S_k t^k\right) = \log\sigma(t)$.

We get an isomorphism of Hopf algebras between **Sym** and $\textbf{ALIEN}(\mathbf{N}^*)$ and we have the following correspondences :

Sym	$\textbf{ALIEN}(\mathbf{N}^*)$
σ	Δ^+
Ψ	**doom**
Φ_k/k	Δ
$\frac{\mathrm{d}}{\mathrm{dt}}$	∇

.

This is the reason why the expansion of **doom** in the basis associated to Δ and the expression of Ψ in the basis constituted by the Φ_k/k are identical, see proposition 4.32 p. 40 of [10].

9.3.5 Link with Ecalle's well-behaved ALIEN derivation : the organic one

We recall that an **ALIEN** operator D is said to be a well-behaved derivation if :

1. D is an **ALIEN** derivation.

2. D preserves the realness, i.e. $D = \overline{D}$ or equivalently the right lateral mould $< D, \Delta^+ >^\bullet$ and left lateral mould $< D, \Delta^- >^\bullet$ are complex conjugates.
3. D is analytic, i.e. after any reduction **red** such that **red** (Δ^+) is analytic, then the operator **red** (D) is analytic too (i.e. it preserves the set of analytic germs $\mathbb{C}\{u\}$).

Such derivations are given for example by the diffusion induced ones (see chapter 7) or by the organic one **dom** studied by J. Ecalle in [14]. The weights of **dom** are given by :

$$i\mathbf{dom}\begin{pmatrix}\epsilon_1 & \dots & \epsilon_r\\ \omega_1 & \dots & \omega_r\end{pmatrix} = \begin{cases} \dfrac{\epsilon_r}{2}\dfrac{\omega_{p+1}}{\omega_1+\ldots+\omega_r} & \text{if } (\epsilon_1,\dots,\epsilon_r) = ((+)^p,(-)^q,\epsilon_r)\\ \dfrac{\epsilon_r}{2}\dfrac{\omega_{q+1}}{\omega_1+\ldots+\omega_r} & \text{if } (\epsilon_1,\dots,\epsilon_r) = ((-)^q,(+)^p,\epsilon_r)\\ 0 & \text{otherwise}\end{cases}.$$

We compute the mould $< \mathbf{dom}, \Delta^+ >^\bullet$ using the formulas :

$$\forall (\omega_1,\dots,\omega_r) \in \Omega^\bullet, \quad < \mathbf{dom}, \Delta^+ >^{\omega_1,\dots,\omega_r} = (-1)^r \, i\mathbf{dom}\begin{pmatrix} + & \dots & +\\ \omega_1 & \dots & \omega_r\end{pmatrix}$$

we find :

$$i < \mathbf{dom}, \Delta^+ >^{(\omega_1,\dots,\omega_r)} = \begin{cases} \dfrac{(-1)^r}{2}\dfrac{\omega_1+\omega_r}{\omega_1+\ldots+\omega_r} & \text{if } r \neq 0\\ 0 & \text{otherwise}\end{cases}.$$

We will prove that **dom** is linked to the **ALIEN** analytic derivation **doom** previously introduced. The fact that **dom** is a well-behaved derivation will naturally follows from the next theorem.

Theorem 9.3.10 *We have*

$$\mathbf{dom} = \int_0 \left(\frac{\overline{\mathbf{doom}} - \mathbf{doom}}{2i}\right).$$

Moreover ***dom*** *is a well-behaved* ***ALIEN*** *derivation :*

1. ***dom*** *is an* ***ALIEN*** *derivation.*
2. ***dom*** *is preserves the realness.*
3. ***dom*** *is analytic.*

Proof We have $\mathbf{doom} = \Delta^-\nabla\Delta^+$ and $\overline{\mathbf{doom}} = \Delta^+\nabla\Delta^-$. Using theorem 9.3.2 and mould calculus, we find for any $\underline{\omega} = (\omega_1,\dots,\omega_r) \in \Omega^\bullet$, :

$$< \mathbf{doom}, \Delta^+ >^{\underline{\omega}} = (-1)^{r-1}\omega_r \text{ and } < \overline{\mathbf{doom}}, \Delta^+ >^{\underline{\omega}} = (-1)^r \omega_1.$$

Then

$$< \frac{\overline{\mathbf{doom}} - \mathbf{doom}}{2}, \Delta^+ >^{\underline{\omega}} = \frac{(-1)^r}{2}(\omega_1+\omega_r)$$

and we finally obtain the claimed relation for **dom**. We easily prove that $\overline{\mathbf{dom}} = \mathbf{dom}$ and then **dom** preserves the realness. As $\nabla\mathbf{dom} = \left(\mathbf{doom} - \overline{\mathbf{doom}}\right)/2i$, the remaining part of the theorem is a direct consequence of theorem 9.3.5. □

Corollary 9.3.11 *For a given saddle-node associated to Ecalle's family of invariants* $(C_m)_{m\in\{-1\}\cup\mathbf{N}^*}$, *we have :* $\boldsymbol{red}(\boldsymbol{dom}).u = u + \sum_{m\geq 1} O_m u^{m+1}$ *with :*

$$\forall m \geq 1, \quad O_m = \sum_{\substack{\|\underline{m}\| = m \\ l(\underline{m}) = r, r \text{ odd}}} \frac{(-1)^{r+1}}{m} \tilde{C}_{\underline{m}} \beta_{\underline{m}} \sum_{i=1}^{r} \frac{(-1)^i}{r!} \binom{r-1}{i-1} m_i$$

where $\forall m \geq 1, \quad \tilde{C}_m = C_m/i \in \mathbb{R}$ *because the terms of* (C_m) *are purely imaginary.*

Proof We use the invariants of **dom** we have computed in theorem 9.3.9 and the previous relation. □

9.4 The brownian derivation

As explained in section 7.4.1, the right lateral operator $\mathbf{rem}_d$ of a diffusion induced average is given, for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ by the mould

$$< \mathbf{rem}_d, \Delta^+ >^{\underline{\omega}} = (-1)^r \int_{\mathbb{R}^r} f_{\omega_1}(x_1) \ldots f_{\omega_r}(x_r)\, \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r.$$

In the same way, the right lateral operator $\mathbf{red}_d$ of a diffusion induced derivation is given by

$$< \mathbf{red}_d, \Delta^+ >^{\underline{\omega}} = (-1)^r \int_{\mathbb{R}^r} f_{\omega_1}(x_1) \ldots f_{\omega_r}(x_r)\, \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \delta(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r.$$

One defines a one parameter family of derivation $\mathbf{red}_d^t$ or averages $\mathbf{rem}_d^t$ when replacing the functions f_ω in the two previous moulds by the functions $f_\omega(t) := f(x - t\omega)$. One has evidently $\mathbf{red}_d^0 = \mathbf{red}_d$ A well known result proved in [42] or [45] is that

$$\partial_t \mathbf{rem}_t = \mathbf{rem}_t \nabla \mathbf{red}_t. \tag{9.3}$$

Considering in the previous moulds for f_ω the function

$$f_\omega(x) = \frac{1}{2\sqrt{\omega\pi}} \exp\left(-\frac{x^2}{4\omega}\right),$$

the corresponding average and derivation are known as the Brownian ones and are respectively denoted $\mathbf{rem}_{\mathrm{B}}$ and $\mathbf{red}_{\mathrm{B}}$. One has the important results :

Theorem 9.4.1

— *For any* $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$,

$$< \boldsymbol{red}_B^t, \boldsymbol{red}_B >^{\underline{\omega}} = \begin{cases} e^{-\frac{\|\underline{\omega}\| t^2}{4}} & \text{if } l(\underline{\omega}) = 1 \\ 0 & \text{otherwise} \end{cases} .$$

— $\lim_{t \to +\infty} \boldsymbol{rem}_B^t = \Delta^-$

Proof

— One has for the homogeneous components of order m of $\mathbf{red}_\mathrm{B}^t$

$$\begin{aligned}
&\left(\mathbf{red}_\mathrm{B}^t\right)_m \\
&= \sum_{\|\underline{\omega}\|=m} (-1^r) \int_{\mathbb{R}^r} f_{\omega_1}^t(x_1) \ldots f_{\omega_r}^t(x_r)\, \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \delta(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r \Delta_{\underline{\omega}}^+ \\
&= \sum_{\|\underline{\omega}\|=m} \frac{(-1)^r e^{-\|\underline{\omega}\| t^2/4}}{2^r \sqrt{\pi^r \omega_1 \ldots \omega_r}} \int_{\mathbb{R}^r} e^{-\frac{x_1^2}{4\omega_1} + \frac{t x_1}{2}} \ldots e^{-\frac{x_r^2}{4\omega_r} + \frac{t x_r}{2}} \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \delta(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r \Delta_{\underline{\omega}}^+ \\
&= \sum_{\|\underline{\omega}\|=m} \frac{(-1)^r e^{-\|\underline{\omega}\| t^2/4}}{2^r \sqrt{\pi^r \omega_1 \ldots \omega_r}} \int_{\mathbb{R}^{r-1}} e^{-\frac{x_1^2}{4\omega_1} + \frac{t x_1}{2}} \ldots e^{-\frac{x_{r-1}^2}{4\omega_{r-1}} + \frac{t x_{r-1}}{2}} \\
&\qquad e^{-\frac{(-x_1 - \ldots - x_{r-1})^2}{4\omega_r} - \frac{t(x_1 + \ldots + x_{r-1})}{2}} \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \mathrm{d}x_1 \ldots \mathrm{d}x_{r-1} \Delta_{\underline{\omega}}^+ \\
&= \sum_{\|\underline{\omega}\|=m} \frac{(-1)^r e^{-\|\underline{\omega}\| t^2/4}}{2^r \sqrt{\pi^r \omega_1 \ldots \omega_r}} \int_{\mathbb{R}^r} e^{-\frac{x_1^2}{4\omega_1}} \ldots e^{-\frac{x_r{}^2}{4\omega_r}} \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \delta(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r \Delta_{\underline{\omega}}^+ \\
&= e^{-mt^2/4} \sum_{\|\underline{\omega}\|=m} \frac{(-1)^r}{2^r \sqrt{\pi^r \omega_1 \ldots \omega_r}} \int_{\mathbb{R}^r} e^{-\frac{x_1^2}{4\omega_1}} \ldots e^{-\frac{x_r{}^2}{4\omega_r}} \sigma_+(\check{x}_1) \ldots \sigma_+(\check{x}_{r-1})\, \delta(\check{x}_r)\, \mathrm{d}x_1 \ldots \mathrm{d}x_r \Delta_{\underline{\omega}}^+ \\
&= e^{-mt^2/4} \left(\mathbf{red}_\mathrm{B}\right)_m .
\end{aligned}$$

which ends the proof.

— One has for any $u = (u_1, \ldots, u_r) \in \mathbb{R}^r$,

$$\exp\left(-\frac{{u_1}^2}{4\omega}\right) \sigma_+(u_1 + \omega_1 t) \ldots \exp\left(-\frac{{u_r}^2}{4\omega_r}\right) \sigma_+(u_1 + \ldots + u_r + (\omega_1 + \ldots + \omega_r)\, t) \xrightarrow[t \to +\infty]{} \exp\left(-\frac{{u_1}^2}{4\omega}\right) \ldots \exp\left(-\frac{{u_r}^2}{4\omega}\right).$$

Moreover,

$$\left|\exp\left(-\frac{{u_1}^2}{4\omega}\right) \sigma_+(u_1 + \omega_1 t) \ldots \exp\left(-\frac{{u_r}^2}{4\omega_r}\right) \sigma_+(u_1 + \ldots + u_r + (\omega_1 + \ldots + \omega_r)\, t)\right| \leq \exp\left(-\frac{{u_1}^2}{4\omega}\right) \ldots \exp\left(-\frac{{u_r}^2}{4\omega_r}\right)$$

which is integrable on $\mathbb{R}^r$ of sum $2^r \sqrt{\omega_1 \ldots \omega_r \pi^r}$. So, using the dominated convergence theorem,

$$
\begin{aligned}
& < \mathbf{rem}_{\mathrm{B}}^{t}, \Delta^{+} >^{\underline{\omega}} \\
&= \frac{(-1)^{r}}{2^{r}\sqrt{\omega_1 \ldots \omega_r \pi^{r}}} \int_{\mathbb{R}^r} \exp\left(-\frac{(x_1-\omega_1 t)^2}{4\omega_1}\right) \sigma_{+}(\check{x}_1) \ldots \exp\left(-\frac{(x_r-\omega_r t)^2}{4\omega_r}\right) \sigma_{+}(\check{x}_r)\,\mathrm{d}x \\
&= \frac{(-1)^{r}}{2^{r}\sqrt{\omega_1 \ldots \omega_r \pi^{r}}} \int_{\mathbb{R}^r} \exp\left(-\frac{{u_1}^2}{4\omega_1}\right) \ldots \exp\left(-\frac{{u_r}^2}{4\omega_r}\right) \\
& \qquad \sigma_{+}(u_1+\omega_1 t) \ldots \sigma_{+}(u_1+\ldots+u_r+(\omega_1+\ldots+\omega_r)\,t)\,\mathrm{d}u \\
& \xrightarrow[t\to+\infty]{} \frac{(-1)^{r}}{2^{r}\sqrt{\omega_1 \ldots \omega_r \pi^{r}}} \int_{\mathbb{R}^r} \exp\left(-\frac{{u_1}^2}{4\omega_1}\right) \ldots \exp\left(-\frac{{u_r}^2}{4\omega_r}\right) \mathrm{d}u = (-1)^{r}
\end{aligned}
$$

and so $\lim_{t\to+\infty} < \mathbf{rem}_{\mathrm{B}}^{t}, \Delta^{+} >^{\underline{\omega}} = < \Delta^{-}, \Delta^{+} >^{\underline{\omega}}$.

□

Coming back to equation 9.3, one has the following result :

Theorem 9.4.2 *We consider a derivation $d = \sum_{m\in\Omega} d_m \in$ **ALIEN** and a family of functions $(f_m)_{m\in\Omega}$ integrable on $\mathbb{R}$. The operator $\mathcal{D} = \sum f_m(x) d_m$ is a derivation too. We look for a solution $\mathcal{U} \in$ **ALIEN** of the equation*

$$\partial_t \mathcal{U} = \mathcal{U} \nabla \mathcal{D} \tag{9.4}$$

of the form $\mathcal{U} = \sum M^{\bullet} d_{\bullet}$ with $M^{\emptyset} = 1$. This solution is unique and given by the symmetral mould $M^{\bullet}$ defined for any $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^{\bullet}$ by

$$M^{\underline{\omega}} = \omega_1 \ldots \omega_r \int_{-\infty < t_r < \ldots < t_1 < t} f_{\omega_1}(t_1) \ldots f_{\omega_r}(t_r)\, dt_1 \ldots dt_r.$$

Proof The moulian translation of equation 9.4 is

$$\forall \underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^{\bullet}, \quad \partial_t M^{\underline{\omega}} = \omega_1 f_{\omega_1} M^{'\underline{\omega}}.$$

Consider $\underline{\omega} \in \Omega^{\bullet}$ such that $l(\underline{\omega}) = 1$ then one has $\partial_t M^{\omega_1} = \omega_1 f_{\omega_1}$ whence $M^{\omega_1} = \int_{-\infty}^{t} f_{\omega_1}(t_1)\mathrm{d}t_1$ [9]. We assume the property true for any sequence $\underline{\omega}$ of length $r-1$ and we prove it for a sequence $\underline{\omega}$ of length r. The equality $\partial_t M^{\underline{\omega}} = \omega_1 f_{\omega_1} M^{'\underline{\omega}}$ and the induction hypothesis deliver

$$
\begin{aligned}
M^{\underline{\omega}} &= \omega_1 \int_{-\infty}^{t} f_{\omega_1}(t_1) M^{'\underline{\omega}}(t_1)\mathrm{d}t_1 \\
&= \omega_1 \ldots \omega_r \int_{-\infty}^{t} f_{\omega_1}(t_1) \left(\int_{-\infty < t_r < t_{r-1} < \ldots < t_1} f_{\omega_2}(t_2) \ldots f_{\omega_r}(t_r)\mathrm{d}t_2 \ldots \mathrm{d}t_r \right) \mathrm{d}t_1
\end{aligned}
$$

from which follows the formula. The unicity of this mould is a consequence of its construction. □

As a direct consequence of what precedes, one has :

9. indeed f_{ω_1} being integrable on $\mathbb{R}$, one must have $\lim_{-\infty} f_{\omega_1} = 0$.

Corollary 9.4.3 *For any* $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^\bullet$*,*

$$\begin{aligned} < \boldsymbol{rem}_B, \boldsymbol{red}_B >^{\underline{\omega}} &= 4^r\,(\omega_1 \ldots \omega_r) \int_{-\infty < t_r < \ldots < t_1 < 0} e^{-\omega_1 t_1^2} \ldots e^{-\omega_r t_r^2} dt_1 \ldots dt_r \\ < \Delta^-, \boldsymbol{red}_B >^{\underline{\omega}} &= 4^r\,(\omega_1 \ldots \omega_r) \int_{-\infty < t_r < \ldots < t_1 < +\infty} e^{-\omega_1 t_1^2} \ldots e^{-\omega_r t_r^2} dt_1 \ldots dt_r \end{aligned}$$

Proof A couple $(\mathcal{U}, \mathcal{D})$ solution of equation 9.3 is given by $\mathcal{U} = \mathbf{rem}_{\mathrm{B}}^t$ and $\mathcal{D} = \mathbf{red}_{\mathrm{B}}^t$. Writing the corresponding moulian relation in the basis given by the homogeneous components of $\mathbf{red}_{\mathrm{B}}$ and using theorem 9.4.1 to this end one finds the expected result by application of the previous proposition and by the change of variable $x = 2\omega u$ in the iterated integrals. □

Proposition 9.4.4 *The arborified of the previous symmetral mould* $M^\bullet$ *is given for any* $\underline{\omega}^< = (\omega_1, \ldots, \omega_r) \in \Omega^{\bullet <}$ *by*

$$M^{\underline{\omega}^<} = \omega_1 \ldots \omega_r \int_{-\infty < t_r < \ldots < t_1 < t} f_{\omega_1}(t_1) \ldots f_{\omega_r}(t_r)\, dt_1 \ldots dt_r$$

where the integration order is given by the order of the arborescent sequence $\underline{\omega}^<$*, i.e. if* ω_{i_1} *is anterior to* ω_{i_2} *for the order of* $\underline{\omega}^<$ *then one has to compute the integral relative to* t_{i_1} *before this relative to* t_{i_2}*.*

Proof Let us observe that the relation 9.3 can be translated in terms of arborescent mould. Using arborified mould product (cf proposition 5.5.1 page 105), one obtains for an irreducible arborescent sequence $\underline{\omega}^< = \begin{array}{c} \vdots\ \vdots\ \vdots \\ \searrow\!\downarrow\!\swarrow \\ \bullet\omega_1 \end{array}$ (i.e. a tree) :

$$\partial_t M^{\underline{\omega}^<} = \omega_1 f_{\omega_1} M'^{\underline{\omega}^<}. \tag{9.5}$$

As usual, we perform an induction on the length of $\underline{\omega}^<$. The formula is obvious if $\mathrm{l}(\underline{\omega}^<) = 1$. We assume it is true for any forest of length $\leq r - 1$ and we prove it for a forest $\underline{\omega}^<$ of length r. If $\underline{\omega}^<$ is a reducible forest, then the formula is a consequence of the separativity of the mould and of the inductive hypothesis. If $\underline{\omega}^< = \begin{array}{c} \vdots\ \vdots\ \vdots \\ \searrow\!\downarrow\!\swarrow \\ \bullet\omega_1 \end{array}$ is irreducible, then the formula is a consequence of relation 9.5 and of the induction hypothesis applied to $M'^{\underline{\omega}^<}$. □

Corollary 9.4.5 *The Brownian derivation is equianalytic to the Stokes automorphism.*

Proof The fact that $\mathbf{red}_{\mathrm{B}}$ is analytic when it is the case of $\dot{\Delta}^+$ is a consequence of proposition 7.4.9 page 167.

The reciprocal is a consequence of the two last propositions. □

9.5 The main theorem

9.5.1 Introduction

The aim of this section is to answer to the following problem. Let us consider a given real saddle node, its Brownian family of invariant $(O_m)_{m\in\{-1\}\cup\mathbf{N}^*}$, and its Martinet-Ramis sectoral isotropy $\xi(u) = u + \sum_{m\geq 1}\xi_m u^{m+1}$. Let us consider too an analytic solution ϕ of Schäfke equation $\phi = \xi \circ \overline{\phi}$. Then $y^+(t, \phi(u))$ is a real and analytic solution of the saddle node considered here. Could we obtain ϕ as the reduction of the lateral operator associated to a well behaved averages ? If it is the case then, as we have explained it in section 9.2.3, we are able to build a well behaved averages $\mathbf{m}$ in a such way that $\mathbf{m}\ \tilde{y}(t,u) = \mathbf{mul}\,\mathbf{rem}\ \tilde{y}(t,u) = y^+(t,\phi(u))$ with $\phi(u) = \mathbf{red}(\mathbf{rem}).u$. We will explain here that this problem get a positive answer if the sequence $(O_m)_{m\in\mathbf{N}}$ verifies a certain growth condition implying in particular that any terms of the sequence $(O_m)_{n\in\{-1\}\cup\mathbf{N}^*}$ is null.

Our main result is the following theorem :

Theorem 9.5.1 *Let us consider a given real saddle-node, its Brownian family of invariants $(O_m)_{m\in\{-1\}\cup\mathbf{N}^*}$ and an analytic solution $\phi \in \mathbb{C}\{u\}$ of the Schäfke equation for this saddle-node. Let us assume that the sequence (O_m) verifies the following growth condition :*

$$\forall m \in \mathbf{N}^*, \quad |O_m| \geq a^m$$

where $a \in \mathbb{R}_+^*$. Then there exists a well behaved average $\boldsymbol{m} \in \boldsymbol{AVER}$ such that $\boldsymbol{red}(\boldsymbol{rem}).u = \phi(u)$ where ***red*** *is the reduction associated to the studied saddle-node and where* ***rem*** *is the right lateral operator associated to* ***m****.*

9.5.2 A factorization property

Let us consider a formal solution ϕ of Schäfke equation and $\mathbf{F}$ the associated substitution automorphism of $\mathbf{ENDOM}(\mathbb{C}[[u]])$. Then $\mathbf{F}$ verifies the equality $\mathbf{F} = \overline{\mathbf{F}}\,\boldsymbol{\Delta}^+$. Suppose moreover there is a convolution automorphism $\mathbf{op} \in \mathbf{ALIEN}$ such that $\mathbf{red}(\mathbf{op}) = \mathbf{F}$. We do not necessarily have $\mathbf{op} = \mathbf{rul}\,\overline{\mathbf{op}}$ because $\mathbf{red}$ is generally not injective. However, we have :

Theorem 9.5.2 *There exists a convolution automorphism $\eta \in \boldsymbol{ALIEN}$ such that :*

1. $\overline{\boldsymbol{op}}^{-1}\,\Delta^-\,\boldsymbol{op} = \overline{\eta}\,\eta^{-1}$.
2. $\boldsymbol{red}(\eta) = 1$.
3. *η is analytic if it is the case for* ***op****.*

This theorem will be a consequence of the following lemma :

Lemma 9.5.3 *For an analytic convolution automorphism $\epsilon \in$ **ALIEN** satisfying $\epsilon\,\overline{\epsilon} = \overline{\epsilon}\,\epsilon = id$ (such an operator is said to be unitary) and the right lateral mould $\boldsymbol{Rm}^\bullet$ associated to a diffusion induced average $\boldsymbol{m}$, we set $\eta = \sum \boldsymbol{Rm}^\bullet \overline{\epsilon}_\bullet$. Then :*

- *— η is a convolution automorphism of **ALIEN**.*
- *— $\overline{\eta} = \epsilon\,\eta$.*
- *— η is analytic.*

Proof The operator η is defined from a cosymmetrel comould and a symmetrel mould so it is a convolution automorphism of **ALIEN**. Moreover, this comould is defined from an analytic **ALIEN** operator and the (contracted anti-)arborified of the mould $\mathbf{Rm}^\bullet$ has a geometrical growth. So, according to theorem 7.2.1 page 150, η is an analytic **ALIEN** operator too. It remains to prove the equality $\overline{\eta} = \epsilon\,\eta$ or conversely $\eta\overline{\eta}^{-1} = \overline{\epsilon}$.

Using mould calculus, one has :

$$\begin{aligned}
\overline{\eta} &= \sum \overline{\mathbf{Rm}^\bullet \overline{\epsilon}_\bullet} \\
&= \sum \overline{\mathbf{Rm}}^\bullet \epsilon_\bullet \\
&= \sum \overline{\mathbf{Rm}}^\bullet \circ < \epsilon, \overline{\epsilon} >^\bullet \overline{\epsilon}_\bullet \\
&= \sum \overline{\mathbf{Rm}}^\bullet \circ \left((1+I)^{-1} - 1\right)^\bullet \overline{\epsilon}_\bullet
\end{aligned}$$

because $\epsilon^{-1} = \overline{\epsilon}$. Thus, by application of corollary 2.3.5 page 44 and using the idempotence for composition of the mould $\left((1+I)^{-1} - 1\right)^\bullet$, it comes :

$$\overline{\eta}^{-1} = \sum \left(\mathbf{rev}\left(\overline{\mathbf{Rm}}\right)\right)^\bullet \overline{\epsilon}_\bullet.$$

The equality $\eta\overline{\eta}^{-1} = \overline{\epsilon}$ is then equivalent to the moulian relation :

$$\left(\mathbf{rev}\left(\overline{\mathbf{Rm}}\right)\right)^\bullet \times \mathbf{Rm}^\bullet = (1+I)^\bullet$$

and it remains to prove it.

But the operator $\mathbf{rem} = \sum \mathbf{Rm}^\bullet \dot{\Delta}^-$ is the right lateral operator associated to the well behaved averages $\mathbf{m} = \mathbf{mur}\,\mathbf{rem}$. So it satisfies $\mathbf{rem} = \dot{\Delta}^-\,\overline{\mathbf{rem}}$ and performing exactly the same computations than before, the operator $\dot{\Delta}^-$ being unitary, this relation delivers the expected one $\left(\mathbf{rev}\left(\overline{\mathbf{Rm}}\right)\right)^\bullet \times \mathbf{Rm}^\bullet = (1+I)^\bullet$.
□

We now prove the theorem :

Proof We set $\epsilon = \overline{\mathbf{op}}^{-1}\Delta^-\mathbf{op}$ and we observe that $\epsilon\overline{\epsilon} = \overline{\epsilon}\epsilon = id$. Then according to the lemma, the operator $\eta = \sum \mathbf{rem}^\bullet \overline{\epsilon}_\bullet$ satisfies points 1. and 3. of the theorem.

Moreover, one has $\mathbf{red}\,(\epsilon) = \mathbf{F}\boldsymbol{\Delta}^-\overline{\mathbf{F}}^{-1} = id$ and then for all $m \geq 1$, $\mathbf{red}\,(\epsilon_m) = 0$. We finally have $\mathbf{red}\,(\eta) = id$ and thus η satisfies the point 2. of the theorem.

□

9.5.3 Proof of theorem

We now prove theorem 9.5.3.

Proof We denote by $\mathbf{F}$ the substitution automorphism $\mathbf{F}_\phi$ where $\phi(u) \in \mathbb{C}\{u\}$ is a solution of Schäfke equation related to our considered saddle-node. This automorphism is evidently analytic and so is its appariated derivation $\mathbb{D} := \mathbf{F}^{-1}\nabla\mathbf{F}$.

According to section 2.4.2, $\mathbb{D}$ admits a mould-comould expansion of the form

$$\mathbb{D} = \sum_{m\geq 0} D_m O_m \mathbb{B}_m = d\left(u\right)\partial_u$$

where $\mathbb{B}_m = u^{m+1}\partial_u$ and $d\left(u\right) = \mathbb{D}.u = \sum_{m\geq 0} D_m O_m u^{m+1}$. As ϕ is analytic, the sequence $(D_m O_m)$ has a geometrical growth and there exists $A, B \in \mathbb{R}_+^*$ such that :

$$\forall m \geq 0, |D_m O_m| \leq AB^{m+1}.$$

But using our hypothesis about the sequence (O_m), we can say that (D_m) is of geometric growth too. We now put $D = \sum_{m\geq 1} D_m \mathbf{red}_{\mathrm{B}m} \in \mathbf{ALIEN}$ where $\mathbf{red}_{\mathrm{B}m}$ denotes the homogeneous components of order m of the Brownian derivation according to notations of section 9.4. Because $\mathbf{red}_{\mathrm{B}}$ is analytic, D is an analytic derivation of **ALIEN** and we clearly have : $\mathbf{red}\left(D\right) = \mathbb{D}$. If F is the automorphism of **ALIEN** appariated to D then we clearly have $\mathbf{red}\left(F\right) = \mathbf{F}$ and F is an analytic convolution automorphism of **ALIEN**.

Unfortunately, we do not have $F = \Delta^+ \overline{F}$ but only $\mathbf{red}\left(\overline{F^{-1}}\Delta^- F\right) = 1$.

But in application of Theorem 9.5.3, there exists an analytic automorphism of convolution η such that $\overline{F}^{-1}\,\Delta^-\,F = \overline{\eta}\,\eta^{-1}$ and $\mathbf{red}\left(\eta\right) = 1$.

We then set $\mathbf{op} = \eta\,F$ and we easily verifies that $\mathbf{op}$ is a convolution automorphism of **ALIEN**, that $\mathbf{op}$ is analytic and that $\mathbf{op} = \Delta\,\overline{\mathbf{op}}$.

Then $\mathbf{op}$ is the right lateral operator associated to a well behaved average and for the reduction associated to the initially considered saddle-node, one has indeed $\mathbf{red}\left(\mathbf{op}\right).u = \phi(u)$ as expected.

□

Conclusion

En guise de conclusion, dressons quelques perspectives à notre travail.

La resommation en toute généralité

Nous avons exposé la théorie des bonnes moyenne d'Ecalle dans le cadre des fonctions résurgentes simples. Les champs analytiques de type noeud-col dans leur classe formelle la plus simple ou les difféos de resiter nul admettent justement une résurgence de ce type ce qui nous a permis de leur appliquer nos résultats. Si on considère ces mêmes objets mais dans leur forme générale, la résurgence est beaucoup plus complexe et, même si Ecalle en a décrit les grandes lignes ([15]), elle n'est pas complètement explicitée. Il est de plus nécessaire de faire alors usage du formalisme d'Ecalle dans toute sa généralité pour la décrire et la composante propre à la resommation réelle reste à détailler.

De plus, les séries formelles intervenant dans ces problèmes ont, contrairement aux exemples abordés ici, plusieurs niveaux Gevrey ce qui amène à construire une théorie de la multisommabilité réelle.

La conjuguante du noeud-col

Considérons un champ X de type noeud-col satisfaisant les mêmes hypothèses que dans la section 3.2 page 52 :

$$X = x^2 \frac{\partial}{\partial_x} + A\left(x, y\right) \frac{\partial}{\partial_y}$$

avec $A\left(x, y\right) \in \mathbb{C}\left\{x, y\right\}$ tel que

$$A(x,y) = y + \sum_{n \geq -1} a_n\left(x\right) y^{n+1}, \qquad \forall n \geq -1, \quad a_n(x) \in x\mathbb{C}\left\{x\right\}.$$

Posons $\mathcal{N} = -1 + \mathbf{N}$. On sait (voir [50]) qu'il existe un opérateur formel

$$\Theta = 1 + \sum_{\substack{\underline{\omega} = (\omega_1, \ldots, \omega_r) \\ \omega_1, \ldots, \omega_r \in \mathcal{N}}} \mathbf{V}^{\underline{\omega}}(x)\, \mathbb{B}_{\underline{\omega}}$$

conjuguant X à

$$X_0 = x^2 \frac{\partial}{\partial_x} + y \frac{\partial}{\partial_y}$$

et où $\mathbf{V}^{\underline{\omega}}(x) \in \mathbb{C}[[x]]$, $\mathbb{B}_m = y^{m+1}\partial_y$ et $\mathbb{B}_{\underline{\omega}} = \mathbb{B}_{\omega_r} \ldots \mathbb{B}_{\omega_1}$ pour tout $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, $\omega_1, \ldots, \omega_r \in \mathcal{N}$.

On sait aussi que Θ est un automorphisme de substitution (le moule $\mathbf{V}^\bullet$ est symétral) et qu'il est relié à la conjuguante formelle $\varphi(x, y) = y + \sum_{n \in \mathbf{N}} \varphi_n(x)\, y^n$ du noeud-col par $\Theta.(x, y) = \theta(x, y) = (x, \varphi(x, y))$.

David Sauzin prouve dans [50], Theorème 2, que pour tout $n \in \mathbf{N}$, les familles infinies

$$\left(y^{-n} \hat{\mathbf{V}}^{\underline{\omega}} \mathbb{B}_{\underline{\omega}}.y \right)_{\substack{\underline{\omega} = (\omega_1, \ldots, \omega_r) \\ \omega_1, \ldots, \omega_r \in \mathcal{N} \\ \|\underline{\omega}\| = n}}$$

sont absolument sommables pour la norme de la convergence uniforme sur les compacts de $\mathbb{C}//\mathbb{Z}$ et donc que leurs sommes $\hat{\varphi}_n$ sont analytiques sur $\mathbb{C}//\mathbb{Z}$. Il prouve de plus que pour de bonne familles de chemins $\gamma : [0, +\infty[\to \mathbb{C}//\mathbb{Z}$, il existe $a, b, C \in \mathbb{R}_+^*$ tels que pour tout $n \in \mathbf{N}$,

$$\forall t \in \mathbb{R}_+^*, \quad |\hat{\varphi}_n(\gamma(t))| \leq a b^n e^{(n^2+1)Ct}.$$

Dans un travail en cours, nous montrons que l'arborification permet de simplifier substantiellement sa preuve.

Le formalisme d'Ecalle appliqué au B-series, aux P-series, aux roughs paths

La théorie des B et P-series (voir [11] pour une introduction), initiée par Butcher dans les années 70, et qui donne des méthodes efficaces pour calculer numériquement des solutions d'équations différentielles peut s'exprimer de manière naturelle dans le formalisme moule-comoule, simple ou arborifié d'Ecalle. Ce formalisme permet de mieux comprendre la loi de substitution des B et des P-series ainsi que de faire le lien avec des résultats de Calaque et al., voir [7]. Nous avons exposé ces travaux dans le papier [60]. Nous proposons de plus une généralisation de la notion de graduation pour les algèbres de Hopf. Celle-ci est en effet définie comme étant à valeurs dans $\mathbf{N}$. Afin d'expliciter la loi de substitution, nous avons été conduit à considérer des « graduations »à valeurs dans des semi-groupes non commutatifs. Nous y proposons de plus une présentation complètement algèbrique de l'arborification, ce qui n'est pas le cas de celle retenue dans la thèse car nous sommes restés ici proches de la présentation d'Ecalle. Nous montrons en effet qu'elle peut se traduire en un morphisme défini de l'algèbre de Connes-Kreimer dans celle des quasi-shuffles. Nous étudions alors les opérations mouliennes comme la multiplication et la composition en les composant par ce morphisme. Nous retrouvons ainsi les formules pour le produit et la composition arborifiés, ainsi que les différentes symétries des moules arborifiés. Nous appliquons de plus ces formules au calcul de l'arborifié contractant du moule organique $\mathbf{dom}^{\bullet}$.

Les algèbres de Rota-Baxter et le procédé d'Atkinson, ou le coeur de la construction moulienne

La procédé d'Atkinson dans les algèbres de Rota-Baxter permet de factoriser des caractères sur des algèbres de Hopf. Le problème de trouver des bonnes moyennes est justement un problème de ce type, il s'agit de factoriser le caractère associé au moule $(1+I)^{\bullet}$ sur l'algèbre de Hopf des quasi-shuffles. Nous avons montré dans [61] que ce procédé permet de construire naturellement les moyennes induites par diffusion et la moyenne organique. Cette nouvelle formalisation de la théorie d'Ecalle permet de simplifier grandement la présentation des moyennes car on peut alors se dispenser de la notion de poids. La symmetrelité des moules étudiés est une conséquence directe de leur construction et le calcul de l'arborifié se fait très naturellement en procédant à la factorisation directement sur l'algèbre de Connes-Kreimer. La principale difficulté est de montrer que les moules obtenus sont associés à des moyennes préservant la réalité. Nous avons à ce sujet établi un théorème général qui s'applique à la fois aux moyennes induites par diffusion et à la moyenne organique. Nous montrons de plus que l'essentiel des moules introduits par Ecalle peut être construit en utilisant ce procédé.

Bonnes moyennes et quasi-analyticité

Enfin, c'était le point de départ de la thèse, il importe de comprendre si la théorie des moyennes permet de construire de nouvelles classes quasi-analytiques. L'idée est la suivante. On considère une série divergente $\tilde{H}$ réelle qu'on sait resommer en une fonction $H :]0, \epsilon] \to \mathbb{R}$ en utilisant les moyennes d'Ecalle. On note $\mathcal{A}_H$ la plus petite collection de sous-algèbres $\mathcal{A}_H^m$ de germes de fonctions $\mathcal{C}^\infty$ en $0 \in \mathbb{R}^m$ pour $m \in \mathbf{N}$ satisfaisant les conditions suivantes :

- C_1 Les germes de fonctions analytiques de m variables sont dans $\mathcal{A}_H^m$ et le germe de H est dans $\mathcal{A}_H^1$.
- C_2 La collection $\mathcal{A}_H$ est stable par composition : Si $\phi \in \mathcal{A}_H^m$ et si $\phi_1, \dots, \phi_m \in \mathcal{A}_H^n$ vérifient $\phi_1(0) = \ldots = \phi_m(0) = 0$ alors $\phi(\phi_1, \dots, \phi_m) \in \mathcal{A}_H^n$.
- C_3 La collection $\mathcal{A}_H$ est stable par équations implicites : Si $\phi \in \mathcal{A}_H^{m+1}$ vérifie $\phi(0) = 0$ et $\partial\phi/\partial x_{m+1}(0) \neq 0$ alors il existe $\tilde{\phi} \in \mathcal{A}_H^m$ tel que le germe $\phi(x, \tilde{\phi}(x))$ où $x = (x_1, \dots, x_m)$ soit nul.
- C_4 La collection $\mathcal{A}_H$ est stable par division monomiale : Si $\phi \in \mathcal{A}_H^m$ vérifie $\phi(0,' x) = 0$ où $'x = (x_2, \dots, x_m) \in \mathbb{R}^{m-1}$ alors il existe $\tilde{\phi} \in \mathcal{A}_H^m$ tel que $\phi(x) = x_1\tilde{\phi}('x)$ où $x = (x_1, x_2, \dots, x_m)$.

Il s'agit alors de montrer que la classe $\mathcal{A}_H$ est quasi-analytique. Pour ce faire, comme expliqué dans [47], il suffit de prouver la quasi-analycité de la sous-algèbre $\mathcal{A}_H^1$. Un travail récent de David Sauzin [52], qui traite de la stabilité de la résurgence par ces différentes opérations, pourrait permettre une avancée déterminante en ce sens.

Notations

Notation 9.5.3.1 *For a semigroup $\Omega \subset \mathbb{R}_+^*$, we introduce the following notations :*

— *A sequence $(\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ will be denoted by $\underline{\omega}$.*
— *The length of $\underline{\omega} \in \Omega^\bullet$ is denoted by $l(\underline{\omega})$ and is defined by*

$$l(\underline{\omega}) = \begin{cases} 0 & \text{if } \underline{\omega} = \emptyset \\ r & \text{if } \underline{\omega} = (\omega_1, \ldots, \omega_r) \end{cases}.$$

— *We define the concatenation $\underline{\omega} \in \Omega^\bullet$ of two sequences $\underline{\omega}^1 = (\omega_1, \ldots, \omega_{r_1}) \in \Omega^\bullet$ and $\underline{\omega}^2 = \left(\omega'_1, \ldots, \omega'_{r_2}\right) \in \Omega^\bullet$ by*

$$\underline{\omega} = \underline{\omega}^1.\underline{\omega}^2 = \left(\omega_1, \ldots, \omega_{r_1}, \omega'_1, \ldots, \omega'_{r_2}\right).$$

— *The norm of $\underline{\omega} \in \Omega^\bullet$ is denoted by $\|\underline{\omega}\|$ and is defined by*

$$\|\underline{\omega}\| = \begin{cases} 0 & \text{if } \underline{\omega} = \emptyset \\ \sum_{i=1}^r \omega_i & \text{if } \underline{\omega} = (\omega_1, \ldots, \omega_r) \end{cases}.$$

— *For a sequence $(\omega_1, \ldots, \omega_r) \in \Omega^\bullet$, we write $\underline{\omega}'$ the sequence $\underline{\omega}$ truncated of its last term and $'\underline{\omega}$ the sequence $\underline{\omega}$ truncated of its first term. For example, if $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ then $\underline{\omega}' = (\omega_1, \omega_2)$ and $'\underline{\omega} = (\omega_2, \omega_3)$.*
— *For a given sequence $(\omega_1, \ldots, \omega_r) \in \Omega^\bullet$ and for $k \in [\![1, r]\!]$, we define respectively the ending and beginning sum $\hat{\omega}_k$ and $\check{\omega}_k$ by :*

$$\hat{\omega}_k = \sum_{i=k}^{r} \omega_i \text{ and } \check{\omega}_k = \sum_{i=1}^{k} \omega_i.$$

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From resurgent functions to real resummation through combinatorial Hopf algebras

Le problème de la resommation réelle consiste à associer à une série divergente réelle une fonction analytique qui lui est asymptotique sur un secteur du plan complexe bissecté par une des deux demi-directions réelles. Jean Ecalle a esquissé, pour le résoudre, les grandes lignes d'une théorie dite des bonnes moyennes uniformisantes. Celle-ci est basée sur plusieurs de ses découvertes : le calcul moulien simple et arborifié, les opérateurs étrangers et les fonctions résurgentes.

Nous nous proposons dans cette thèse de détailler complètement la théorie des moyennes d'Ecalle. Il s'agit de l'appliquer à la resommation de la conjuguante formelle des champs analytiques réels de type noeud-col et des difféomorphismes analytiques tangents à l'identité dans leur classe formelle la plus simple. Une partie conséquente de la thèse est consacrée à la théorie de l'arborification. C'est l'un des ingrédients majeurs de la théorie des moyennes mais pour laquelle Ecalle n'avait délivré que peu de détails.

Un chapitre de la thèse traite de géométrie o-minimale. Il s'agit de démontrer l'existence d'un « isomorphisme formel »entre les familles de germes d'ensembles semi-analytiques issus de deux classes quasi-analytiques isomorphes. Bien que ce chapitre soit disjoint de la théorie des moyennes, il est probable que cette dernière permette à l'avenir d'obtenir de nouvelles classes quasi-analytiques.

Enfin, nous proposons de faire le lien entre un procédé de resommation réelle de la conjuguante formelle du noeud-col réel élaboré par R. Schäfke et les moyennes d'Ecalle.

Mots clés. Géométrie analytique réelle, algèbres quasianalytiques, structures o-minimales, calcul moulien, (co-)arborification, algèbres de Hopf combinatoires, fonctions résurgentes, automorphisme de Stokes, moyennes uniformisantes, resommation réelle, champs de vecteurs analytiques de type noeud-col, difféomorphismes tangents à l'identité.