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Introduction

A travers ce document, nous allons nous pencher sur la simulation du premier temps de passage de diffusions dans un cadre unidimensionnel. En premier lieu, nous dédions les trois premiers chapitres de ce document à la mise en place d'un algorithme permettant la simulation approchée du temps de sortie d'un intervalle pour certaines diffusions particulières. La conception est essentiellement basée sur les travaux de Lerche [39] et de Daniels [13]. Il nous est alors possible d'énoncer un certain nombre de résultats concernant l'efficacité de l'algorithme introduit mais aussi de l'étendre à d'autres classes de diffusions. En second lieu, nous nous intéressons à la simulation exacte du temps d'atteinte d'un niveau donné dans le contexte particulier des diffusions à sauts, ce qui fait l'objet de notre quatrième et dernier chapitre. Cette seconde partie de l'étude se base sur les travaux de Beskos, Papaspiliopoulos et Roberts [3] ainsi que sur les récentes publications de Herrmann et Zucca [27].

Les temps de sortie apparaissant dans de nombreux domaines d'application, il semble primordial de pouvoir simuler ces derniers. Dans le domaine de la fiabilité, les durées de vie des composants peuvent être reliés à des premiers temps de passage ou des temps de sortie. Des modèles simplifiés faisant intervenir les processus d'Ornstein-Uhlenbeck sont alors généralement utilisés afin de modéliser ces durées de vie. En effet, ces processus, vérifiant une propriété de retour vers la moyenne, sont les plus à même pour modéliser des processus de dégradation. Le domaine de la finance nous propose lui aussi certaines situations où la simulation des temps de sortie constitue un enjeu de taille. Par exemple, l'étude des options à barrière nécessite de décrire avec précision des temps de sortie puisqu'il est important d'estimer si la valeur du sous-jacent reste dans un certain intervalle. Pour le modèle de Black-Scholes, la distribution du premier temps de sortie est connue. Cependant dans des modèles plus compliqués, il est plus difficile d'obtenir de telles expressions et on se restreint donc à utiliser des approximations numériques.

De nombreuses méthodes ont été introduites au fil des années afin de pouvoir approximer ces temps de sortie. L'approche la plus courante utilise le schéma d'Euler-Maruyama fondé sur une discrétisation du temps, le temps de sortie voulu étant alors approché par le temps de sortie du schéma. Cette approximation est plutôt précise mais nous oblige à nous restreindre à un intervalle de temps donné mais aussi à connaître la probabilité que la diffusion considérée sorte de l'intervalle entre deux itérations du schéma.

Le premier chapitre de ce document se penchera sur les méthodes existantes pour la simulation du premier temps de passage ou du temps de sortie dans le cas brownien. On commencera par introduire quelques exemples, nous permettant d'utiliser la transformée de

Laplace mais aussi de souligner l'importance de la formule de Girsanov dans le contexte de la simulation. Dans le cas du mouvement brownien, l'expression de la fonction de répartition du temps de sortie pour un intervalle donné est assez compliquée, ce qui ne facilite pas sa simulation (pour plus de détails, voir [57]). Cependant il existe des domaines, appelés sphéroïdes pour lesquels la loi du temps de sortie peut être entièrement déterminée. Ces domaines sont définis par leurs frontières :

$$\psi_{\pm}(t) = \pm\sqrt{t\log\left(\frac{d^2}{t}\right)}, \quad \text{for } t \in [0, d^2], \tag{0.0.1}$$

où le paramètre $d > 0$ correspond à la taille de la sphéroïde. Le premier temps où la trajectoire du mouvement brownien (t, W_t) sort du domaine $\{(t, x) : \ |x| \leq \psi_+(t)\}$, que l'on notera par la suite τ, est alors connue. Sa densité [39] peut s'écrire

$$p(t) = \frac{1}{d\sqrt{2\pi}}\sqrt{\frac{1}{t}\log\left(\frac{d^2}{t}\right)}, \quad t \geq 0. \tag{0.0.2}$$

On peut alors facilement simuler le temps de sortie voulu en remarquant que τ et $d^2U^2e^{-N^2}$ ont la même loi. Les variables aléatoires U et N sont indépendantes, U suivant une loi uniforme sur $[0, 1]$ et N suivant une loi normale centrée réduite. Remarquons que les frontières de la sphéroïde considérée peuvent être bornées de la manière suivante :

$$|\psi_{\pm}(t)| \leq \frac{d}{\sqrt{e}}, \quad \forall t \in [0, d^2]. \tag{0.0.3}$$

Dans le cadre de cette étude, considérons un mouvement brownien unidimensionnel (W_t, $t \geq 0$) ainsi qu'un intervalle $I = [a, b]$ contenant la position de départ $X_0 = x$. Notons par $\mathcal{T}$ le premier temps de sortie défini par :

$$\mathcal{T} = \inf\{t \geq 0 : \ W_t \notin [a, b]\}.$$

Afin de simuler ce temps de sortie, nous allons construire une marche aléatoire $(T_n, X_n)_{n\geq 0}$ sur $\mathbb{R}_+ \times \mathbb{R}$ qui constitue un squelette de la trajectoire : les points de la suite appartiennent à la trajectoire du mouvement brownien. De plus, de par sa construction, cette marche aléatoire converge vers l'instant et la position de sortie de l'intervalle $(\mathcal{T}, X_{\mathcal{T}})$. On introduit alors une procédure afin de pouvoir arrêter l'algorithme dès que l'approximation semble suffisamment précise. Cette remarque nous permet alors de construire l'algorithme voulu. On considère pour commencer le temps et la position de départ $(0, x)$ de la trajectoire brownienne observée et on introduit le premier terme de la suite (T_0, X_0). De cette position, il est possible de choisir le plus grand paramètre d telle que la sphéroïde construite à cet endroit soit complètement contenue dans le domaine $\mathbb{R}_+ \times [a, b]$. On peut alors, à l'aide des remarques précédemment faites, générer le temps ainsi que la position de sortie de cette sphéroïde que l'on notera (T_1, X_1). Grâce à la propriété d'invariance par translation du mouvement brownien, nous pouvons dès lors itérer la procédure puisqu'il nous suffit de

considérer (T_1, X_1) comme nouveau point de départ pour le mouvement brownien. On peut alors considérer une nouvelle sphéroïde, contenue dans l'intervalle, liée à ce nouveau point et générer le couple (T_2, X_2) correspondant au temps et à la position de sortie de cette seconde sphéroïde. Sur ce principe, on peut itérer le procédé et construire au fur et à mesure une marche aléatoire sur des sphéroïdes qu'on appellera algorithme du WOMS (Walk On Moving Spheres). Cette suite aléatoire converge vers le temps et la position de sortie de l'intervalle $(\mathcal{T}, W_{\mathcal{T}})$. La suite construite est arrêtée dès lors que la position générée X_n est suffisamment proche du bord de l'intervalle. L'idée de cet algorithme repose sur la définition de processus sphériques ainsi que de la marche sur les sphères introduite par Müller [44] et utilisée ensuite par Motoo [**?**] et Sabelfeld [55] [54]. Cet algorithme permet aussi de simuler le premier temps de passage pour des processus de Bessel [14]. On prouve aussi que cet algorithme est efficace et on le compare à d'autres algorithmes connus.

A travers un second chapitre, nous nous concentrons sur une famille particulière de diffusions entretenant un lien fort avec le mouvement brownien : les processus d'Ornstein-Uhlenbeck. Soient $\theta \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, le processus d'Ornstein-Uhlenbeck (O.U.) de paramètres θ, μ et σ partant de x_0 est l'unique solution de l'équation différentielle stochastique suivante :

$$dX_t = -\theta(X_t - \mu)dt + \sigma dW_t, \quad t \geq 0, \tag{0.0.4}$$

où W est un mouvement brownien unidimensionnel. Ces processus, comme annoncé, sont fortement liés au mouvement brownien de la manière suivante

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma e^{-\theta t}}{\sqrt{2\theta}} V_{e^{2\theta t}} - 1, \tag{0.0.5}$$

où $(V_t)_{t\geq 0}$ est un mouvement brownian standard.

L'idée est d'utiliser ce lien afin d'adapter l'algorithme précédemment introduit dans le cadre du mouvement brownien, celui-ci intervenant principalement afin définir des sphéroïdes adaptées au processus d'Ornstein-Uhlenbeck étudié. On énoncera alors un algorithme du WOMS pour les processus d'Ornstein-Uhlenbeck et décrira l'erreur d'approximation commise à cause de la procédure d'arrêt et montrera l'efficacité de ce nouvel algorithme. On décrira de plus le nombre moyen de sphères nécessaires afin d'obtenir l'approximation recherchée.

Dans un troisième chapitre, nous nous attarderons sur les diffusions qui, comme les processus d'Ornstein-Uhlenbeck, sont fortement liées au mouvement brownien en tant que fonctions de ce dernier $X_t = f(t, W_t)$. Cette famille particulière a été introduite par Potzelberger et Wang[58] comme étant les diffusions de classe L. Ces diffusion sont dédinies comme étant la solution de l'équation différentielle stochastique suivante

$$dX_t = (\alpha(t)X_t + \beta(t))dt + \tilde{\sigma}(t)dW_t \quad t \geq 0, \tag{0.0.6}$$

où les fonctions α, β et $\tilde{\sigma}$ sont Hölder continues, $\tilde{\sigma}$ est de plus positive et $(W_t)_{t\geq 0}$ est une mouvement brownien unidimensionnel. L'unique solution de l'équation précédente peut s'écrire de la manière suivante

$$X_t = X_0\, e^{-\theta(t)} + e^{-\theta(t)} \int_0^t e^{\theta(s)} \beta(s) ds + e^{-\theta(t)} \int_0^t e^{\theta(s)} \tilde{\sigma}(s) dW_s, \quad t \geq 0.$$

Ce lien étroit permet alors à nouveau d'adapter l'algorithme écrit dans la cadre du mouvement brownien à la diffusion considérée. Comme pour le cas de l'Ornstein-Uhlenbeck, cela impliquera une modification de la sphéroïde brownienne afin de convenir à la diffusion étudiée. Cela permettra de nouveau d'obtenir une version simplifiée du problème de sortie. L'algorithme du WOMS est présenté dans le cas des diffusions de classe L. On décrit l'erreur d'approximation et analyse l'efficacité de ce nouvel algorithme. De plus le nombre moyen de sphères nécessaires à l'obtention de l'approximation recherchée est décrit. On présentera aussi un algorithme pour la famille des diffusions de classe G qui est fortement liée à la famille classe L.

Enfin, dans un quatrième et dernier chapitre, nous nous intéressons cette fois-ci à la simulation exacte du premier temps d'atteinte d'un niveau donné pour des diffusions à sauts. La description précise du premier temps où une diffusions dépasse un niveau donné constitue un enjeu de taille dans un certain nombre de domaines: dans le domaine de l'économie [33], de la finance [35, 40], des files d'attente, de la théorie de la fiabilité [49], de la neuroscience [11, 56] etc. Le premier temps de passage peut être utile à l'évaluation de la probabilité de défaut en finance mathématique. Puisque ce temps d'arrêt permet en pratique de prendre des décisions importantes, il apparaît crucial de pouvoir obtenir une expression explicite de sa loi ou de construire des algorithmes afin de pouvoir simuler cette dernière. Une expression explicite de la densité est source de nombreuses informations, mais prend trop souvent la forme de séries peu exploitable dans la pratique. Nous proposons donc ici une approche numérique, le but étant de générer le premier temps d'atteinte à l'aide d'un algorithme efficace.

Dans de nombreuses applications, la modélisation du comportement d'une variable aléatoire au fil du temps revient à choisir une diffusion unidimensionnelle $(X_t,\, t \geq 0)$ adéquate et un premier temps de passage au delà d'un niveau donné L, ce dernier étant défini de la manière suivante

$$\tau_L := \inf\{t \geq 0 :\ X_t \geq L\}. \tag{0.0.7}$$

On suppose dans ce cadre que la position de départ $X_0 = y_0 < L$ est déterministe. Dans le cas de trajectoires continues, les diffusions se définissent comme étant la solution d'une équation différentielle stochastique dirigée par un mouvement brownien. Au cours des années, de nombreuses méthodes ont été mises en place afin de simuler le temps τ_L. Une possibilité est d'utiliser des approximations numériques de la trajectoire et d'en déduire alors une approximation du premier temps de passage, la plupart de ces études étant basées sur un schéma d'Euler amélioré (pour plus d'informations, voir [8], [22], [23]). On peut aussi utiliser les travaux de Beskos et Roberts, basés sur la transformation de Girsanov, et décrivant des méthodes de simulation exacte. En effet, les lois des trajectoires browniennes et de leur temps d'atteinte étant bien connues, la transformation de Girsanov permet de construire un algorithme de rejet de la manière suivante : on génère une trajectoire du mouvement brownien et on l'accepte ou la rejette à l'aide d'une probabilité dépendant de la trajectoire. Beskos et Roberts [3, 4, 5] ont tout d'abord proposé une manière permettant de simuler la trajectoire de la diffusion sur un intervalle de temps donné $[0, \mathbb{T}]$, puis Herrmann et Zucca [27] ont adapté cet algorithme afin de pouvoir simuler τ_L de manière exacte, la sortie de cet

algorithme possédant la même loi que τ_L sans qu'aucune erreur d'approximation ne soit commise. Herrmann et Zucca proposent de plus des algorithmes se basant sur la transformation de Girsanov pour la simulation du temps de sortie [29, 28].

Bien entendu, ces méthodes ne permettent pas de couvrir toutes les applications, certaines ne pouvant pas être associées à des diffusions continues, il apparaît donc naturel de se demander si une procédure de ce type peut être effectuée dans le cadre des diffusions à sauts. Ces processus sont à la fois dirigés par un mouvement brownien et par une mesure de Poisson, ce qui fait d'eux un cas particulier de processus de Lévy unidimensionnels. Dans le domaine de la finance, l'évolution du cours de la Bourse peut être représentée par une diffusion à sauts, ces derniers représentant les possibles évènements ayant un impact fort sur le prix des actions [37].

Pour de telles diffusions, de nombreux résultats concernent la simulation de la trajectoire du processus sur un intervalle de temps donné, cette simulation permettant d'obtenir le premier temps de passage par la même occasion, de la même manière que pour les diffusions continues. En particulier, pour les diffusions dirigées par un processus de Wiener et une mesure de Poisson, une méthode basique d'Euler peut être utilisée. Platen [50], Maghsoodi [42, 41] et Gardon [19] ont introduit des schémas explicites de discrétisation temporelle basés sur le développement d'Itô-Taylor afin d'obtenir des résultats intéressants concernant la variance. De nombreuses études permettent de comprendre comment adapter la grille temporelle aux temps de sauts et d'ainsi réduire les erreurs d'approximation. D'autres approches ont été faites afin de construire ces approximations, comme des schémas basés sur une méthode d'Euler-Maruyama semi-implicite [31, 32] ou encore des méthodes de Runge-Kutta [9]. Le but ici n'étant pas de faire un panorama complet de la littérature existante à propos de l'analyse numérique, notre approche ne se basant pas sur une méthode d'approximation, nous ferons simplement référence au livre de Platen [51] contenant toutes les références et informations à ce sujet. Nous nous intéressons ici plutôt aux méthodes dites de simulation exacte. Introduites par Beskos et Roberts, elles permettent de simuler un nombre fini de points appartenant à la trajectoire d'une diffusion classique. Ces méthodes ont été adaptées aux diffusions à sauts permettant notamment de simuler la position de la diffusion à un instant donné, ou de simuler un squelette de la trajectoire de la diffusion à saut et d'approcher ainsi les temps de sortie ou les temps de passage (voir [12], [24] et [52]). Ces méthodes ont permis également de proposer des simulations exactes du temps de sortie d'un intervalle pour une diffusion à sauts [20]. Ces dernières méthodes sont basées sur des sorties successives de petits domaines spatiaux, ces temps aléatoires sont générés par une méthode de rejet faisant intervenir des méandres browniens. Nous proposons ici une approche différente qui ne nécessite pas un découpage spatial et qui évite donc l'utilisation des méandres.

Notre étude se concentre donc sur la simulation exacte du premier temps de passage τ_L défini par (0.0.14) où $(X_t,\, t \geq 0)$ est une diffusion à sauts. Ce quatrième chapitre se décompose de la manière suivante : nous définissons dans un premier temps les diffusions à sauts comme l'unique solution d'une equation différentielle stochastique, nous rappelerons ensuite les méthodes de simulation exacte introduites dans le cadre des trajectoires continues dans la Section 4.2 en proposant de nouvelles preuves des résultats annoncés. A travers

la Section 4.3, nous nous concentrons sur le cadre des diffusions à sauts et proposons un algorithme efficace permettant de générer le temps $\tau_L \wedge \mathbb{T}$ où $\mathbb{T} > 0$ est un temps fixé au préalable. Cette situation particulière nous permet de considérer une large classe de diffusion sans se soucier des problèmes de récurrence. Dans la Section 4.4, l'algorithme précédent est modifié en se plaçant dans le cas particulier où $\tau_L < \infty$, une situation qui requiert d'ajouter bien entendu de nouvelles conditions sur les diffusions. Ces conditions permettent la simulation de τ_L de manière directe. Les algorithmes que nous écrivons se basent une fois encore sur la formule de Girsanov et sont reliés aux travaux de Beskos et Roberts (voir [3]) interprétés sous un nouveau jour. Finalement, nous présentons diverses applications numériques illustrant l'efficacité des algorithmes introduits.

The aim of this report is to present algorithmic methods for the simulation of exit times in the one-dimensional context. Two different approaches will be considered. First we will focus on approximation methods based on Lerche's [39] and Daniels' works: the so-called method of images. The method of images was introduced in 1969 by H.E. Daniels [13] as a tool to construct nonlinear boundaries for which explicit calculations for the exit distribution for the Brownian motion are possible. This crucial method permits to build an algorithm for the approximation of the time needed by a stochastic process to exit from a given interval. The statements related to the numerical approximation and some extension of the main algorithm are presented in a first part of the manuscript containing three chapters. Secondly time we consider the so-called exact simulation of the first time passage through a given level for jump diffusions. This second part of the study is mainly based on the work produced by Beskos, Roberts and Papaspiliopoulos on one hand [3] and on a recent paper written by Herrmann and Zucca [27] on the other hand.

Simulating the first exit time for a diffusion from a given domain is primordial since these times appear in many domains. In reliability analysis, for instance, first passage times and exit times are directly related to lifetimes of engineering systems. In order to emphasize explicit expressions of the lifetime distribution, it is quite usual to deal with simplified models like Ornstein-Uhlenbeck processes. Indeed they satisfy the mean reverting property which is essential for modeling degradation processes. In mathematical finance, studying barrier options also requires to describe exit times since it is of prime interest to estimate if the underlying stock price stays in a given interval. In the simple Black-Scholes model, the distribution of the first exit time is well-known. In more complex models corresponding to general diffusion processes, such an explicit expression is not available and requires the use of numerical approximations.

Several methods have been introduced in order to approximate first exit times. The classical and most common approximation method is the Euler–Maruyama scheme based on a time discretization procedure. The exit time of the diffusion process is in that case replaced by the exit time of the scheme. The approximation is quite precise but requires to restrict the study on a given fixed time interval on one hand and to describe precisely the probability for the diffusion to exit inbetween two consecutive nodes of the time grid on the other hand.

The first chapter of this report represents an overlook on already existing methods in order to determine and simulate Brownian exit time or first time passage. We introduce some simple examples, using Laplace transform and making a first use of the Girsanov formula. For the particular Brownian case, the distribution of the exit time from an interval has a quite complicated expression which is difficult to use for simulation purposes (see, for instance [57]) whereas the exit distribution from particular time-dependent domains, for instance the spheroids also called *heat balls*, can be precisely determined. These time-dependent domains are characterized by their boundaries:

$$\psi_{\pm}(t) = \pm\sqrt{t \log\left(\frac{d^2}{t}\right)}, \quad \text{for } t \in [0, d^2], \tag{0.0.8}$$

where the parameter $d > 0$ corresponds to the size of the spheroid. The first time the Brownian motion path (t, W_t) exits from the domain $\{(t,x) : \ |x| \leq \psi_+(t)\}$, denoted by τ, is well-known. Its probability density function [39] is given by

$$p(t) = \frac{1}{d\sqrt{2\pi}} \sqrt{\frac{1}{t} \log\left(\frac{d^2}{t}\right)}, \quad t \geq 0. \tag{0.0.9}$$

It is therefore easy to generate such an exit time since τ and $d^2U^2e^{-N^2}$are identically distributed. Here U and N are independent random variables, U is uniformly distributed on $[0,1]$ and N is a standard Gaussian random variable. Let us notice that the boundaries of the spheroids satisfy the following bound:

$$|\psi_\pm(t)| \leq \frac{d}{\sqrt{e}}, \quad \forall t \in [0, d^2]. \tag{0.0.10}$$

For the purpose of the study, we consider a one-dimensional Brownian motion $(W_t,\, t \geq 0)$. Let us also fix some interval $I = [a,b]$ which strictly contains the starting position $W_0 = x$. We denote by $\mathcal{T}$ the Brownian first exit time:

$$\mathcal{T} = \inf\{t \geq 0 : \ W_t \notin [a,b]\}.$$

Our approach consists in constructing a random walk $(T_n, X_n)_{n\geq 0}$ on $\mathbb{R}_+ \times \mathbb{R}$ which corresponds to a skeleton of the Brownian paths. In other words, the sequence (T_n, X_n) belongs to the graph of the trajectory. Moreover we construct the walk in such a way that (T_n, X_n) converges as time elapses towards the exit time and location $(\mathcal{T}, W_\mathcal{T})$. It suffices therefore to introduce a stopping procedure in the algorithm to achieve the approximation scheme. Of course, such an approach is interesting provided that (T_n, X_n) is easy to simulate numerically. This remark permits to explain the general idea of the algorithm. First we consider (T_0, X_0) the starting time and position of the Brownian paths, that is $(0, x)$. Then we choose the largest parameter d possible such that the spheroid starting in (T_0, X_0) is included in the domain $\mathbb{R}_+ \times [a,b]$. We observe the first exit time of this spheroid and its corresponding exit location, this couple is denoted by (T_1, X_1). Due to the translation invariance of the Brownian motion, we can construct an iterative procedure, just considering (T_1, X_1) like a starting time and position for the Brownian motion. So we consider a new spheroid included in the interval and (T_2, X_2) shall correspond to the exit of this second spheroid and so on. Step by step we construct a random walk on spheroids also called WOMS algorithm (Walk On Moving Spheres) which converges towards the exit time and position $(\mathcal{T}, W_\mathcal{T})$. This sequence is stopped as soon as the position X_n is close enough to the boundary of the considered interval. The idea of this algorithm lies in the definition of spherical processes and the walk on spheres introduced by Müller [44] and used in the sequel by Motoo [43] and Sabelfeld [55] [54]. It permits also in some more technical advanced way to simulate the first passage time for Bessel processes [14]. We also provides proofs in order to show the efficiency of the algorithm and compare it to classical algorithms.

In the second chapter, we focus our attention on a particular family of diffusions which is strongly related to the Brownian motion: the Ornstein-Uhlenbeck processes. Let $\theta \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$. The Ornstein-Uhlenbeck process (O.U.) starting in x_0 with parameters θ, μ, and σ is the unique solution of the following stochastic differential equation (SDE):

$$dX_t = -\theta(X_t - \mu)dt + \sigma dW_t, \quad t \geq 0, \tag{0.0.11}$$

where $(W_t)_{t\geq 0}$ stands for a standard one-dimensional Brownian motion. This processes are strongly linked to the standard Brownian motion in the following way

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma e^{-\theta t}}{\sqrt{2\theta}} V_{e^{2\theta t}-1}, \tag{0.0.12}$$

where $(V_t)_{t\geq 0}$ is a standard Brownian motion.

The idea is to use the strong link established to adapt the Brownian algorithm in an appropriate way. This link implies changes on the time-dependent domains for which the exit problem can be expressed in a simpler way. We present the random walk algorithm (WOMS) for the Ornstein-Uhlenbeck process, describe the approximation error depending on the stopping procedure and emphasize the efficiency of the method. We describe the mean number of generalized spheroids necessary to obtain the approximated exit time.

In a third chapter, we focus our attention on diffusions which are strongly related to the Brownian motion: they can be expressed as functionals of the Brownian motion that is $X_t = f(t, W_t)$. The particular family of diffusions considered was already introduced in [58] as the L-class diffusion. These diffusions are defined as a solution of

$$dX_t = (\alpha(t)X_t + \beta(t))dt + \tilde{\sigma}(t)dW_t \quad t \geq 0, \tag{0.0.13}$$

where α, β and $\tilde{\sigma}$ are Hölder-continuous functions, $\tilde{\sigma}$ is furthermore positive and $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion. The unique solution of the previous equation can be written as

$$X_t = X_0\, e^{-\theta(t)} + e^{-\theta(t)} \int_0^t e^{\theta(s)} \beta(s) ds + e^{-\theta(t)} \int_0^t e^{\theta(s)} \tilde{\sigma}(s) dW_s, \quad t \geq 0.$$

This strong link permits once again to adapt the Brownian algorithm in an appropriate way. As for Onstein-Uhlenbeck processes, it implies some changes on the time-dependent domains for which the exit problem can be expressed in a simpler way. In this context, we present the random walk algorithm (WOMS) for L-class diffusion processes, describe the approximation error depending on the stopping procedure and emphasize the efficiency of the method. We describe the mean number of generalized spheroids necessary to obtain the approximated exit time. We also present the algorithm in the particular case of G-class diffusion processes which are strongly related to the L-class diffusion processes previously introduced.

Finally in the fourth chapter, we focus our attention on the exact simulation of the first passage time through a given level for jump diffusions. Describing precisely the first time needed by diffusion process to overcomes a given threshold is a interesting challenge in several

fields: economics [33], finance [35, 40], queueing, reliability theory [49], neuroscience [11, 56] and many others. The first passage time can for instance be related to the evaluation of risk of default in mathematical finance. Since this stopping time permits in practice to take important decisions, it is crucial either to obtain an explicit expression of the corresponding probability distribution either to point out efficient algorithms used for the random variable generations. An explicit expression of the density, often based on series expansions, is a source of valuable information but unfortunately it is available only for particular stochastic models and cannot be used in wide classes of applications. We propose here a numerical approach: the challenge is to generate the first passage times using efficient algorithms.

In many applications, modeling the behaviour of an experimental random value as time elapses consists in choosing a suitable one-dimensional diffusion process $(X_t,\, t \geq 0)$ and the associated first passage time through a given threshold L is then defined by

$$\tau_L := \inf\{t \geq 0 :\ X_t \geq L\}. \tag{0.0.14}$$

We just assume here that the starting value $X_0 = y_0 < L$ is deterministic. In the continuous paths framework, the diffusion is represented by the solution of a stochastic differential equation driven by a Brownian motion. Different approaches have been proposed in order to generate τ_L. One way is to use numerical approximations of the paths and to deduce the approximation of the passage time. Most of the studies are based on improvements of the classical Euler scheme (see for instance [8], [22], [23]). Another way is to use the exact simulation techniques introduced by Beskos and Roberts and based on the Girsanov transformation. Indeed we know exactly the distribution of the Brownian paths and in the way the exact distribution of their passage time. Using the Girsanov transformation permits to build a rejection sampling procedure: one generates a Brownian path and accepts or rejects it with a probability depending on the whole paths. Beskos and Roberts [3, 4, 5] first proposed such an approach in order to simulate the diffusion trajectory on a given time interval $[0, \mathbb{T}]$, Herrmann and Zucca [27] adapted the algorithm in order to generate τ_L in an exact way: the outcome of the algorithm has the same distribution than τ_L, there is no approximation error term. Let us also mention that Herrmann and Zucca proposed algorithms based on the Girsanov transformation for the exit time generation [29, 28].

Of course all applications cannot be concerned by one-dimensional continuous diffusion processes. A natural question that arises is to propose a new simulation approach in the jump diffusion framework. These processes are driven by both a Brownian motion and a Poisson random measure, they are indeed special cases of one-dimensional Lévy processes. In finance, the stock exchange evolution can be represented by a diffusion with jumps. In this particular case, the jumps represent possible events that can occur and produce strong impacts on the asset prices [37].

For such particular stochastic processes, several available results concern the approximation of the diffusion trajectory on some given time interval. The simulation of the first passage time is then obtained as a by-product, similarly to the already used procedure in the continuous case. In particular, for diffusions which are driven by Wiener processes and Poisson random measures, the basic idea of the Euler method can be adapted. Platen [50],

Maghsoodi [42, 41] and Gardon [19] introduced explicit time-discretization schemes based on the Itô-Taylor expansion in order to obtain interesting convergence results in the mean-square sense. Several studies permit to understand how to adapt the time grid to the jump times and reduce, by the way, the approximation error. Other kinds of stochastic convergence have been analyzed for the numerical approximation. Let us also mention other challenging research directions: introducing schemes based on semi-implicit Euler–Maruyama methods in [31, 32] or Runge-Kutta methods as in [9] is of particular interest. Our purpose is not to write an exhaustive overview of the literature on numerical analysis since our approach for the generation of the first passage time is not based at all on an approximation procedure. So we prefer to refer to the monograph [51], and the references therein, for additional information on the approximation methods.

We focus our attention on the so-called exact simulation method. Introduced by Beskos and Roberts in the classical diffusion context, it permits to generate a finite number of points belonging to the diffusion trajectory. This method has then been adapted to the jump diffusions. It is therefore possible to generate the value of the jump diffusion at a given time or to generate a finite set of points belonging to the path and by that way to approximate the first exit time from an interval (see [12], [24] et [52]). Such simulation method permits also to generate exactly the first exit time from a given interval [20]). For these simulations, the method consists in introducing a space splitting and the corresponding sequence of strip exit times. In order to generate the basic stopping times, a rejection sampling involving Brownian meanders is used. We propose here a different approach avoiding the space splitting and consequently the generation of meanders.

The study presented here concerns the exact simulation of the first passage time τ_L defined by (0.0.14) where $(X_t,\, t \geq 0)$ stands for a jump diffusion process. The material of this last chapter is organized as follows: first we define the jump diffusion as the unique solution of a stochastic differential equation, then we recall the exact simulation technique introduced for continuous paths in Section 4.2, proposing new proofs. In Section 4.3, we focus our attention to the jump diffusion framework and propose an efficient algorithm in order to generate $\tau_L \wedge \mathbb{T}$ where $\mathbb{T} > 0$ is a fixed time. This particular situation permits to deal with the stopped diffusion and consequently to consider a wide class of diffusions. In Section 4.4, we propose to reduce the study to the particular situation: $\tau_L < \infty$. Of course such context requires additional conditions on the process but permits also to simulate directly τ_L in an exact way. All algorithms are based on the Girsanov transformation and are related to the previous work of Beskos and Roberts (see for instance [3]) which is seen here from a new perspective. The last section presents numerical illustrations.

Chapter 1

Brownian exit and first passage times: theoretical and algorithmic approaches

The aim of this first chapter is to present different results concerning particular stopping times related to the one-dimensional Brownian motion. Since the Brownian motion plays a crucial role in the study of diffusion processes, it is important to describe precisely its paths. We shall focus our attention on the first passage time though a given threshold on one hand and on the exit time of an interval on the other hand.
For numerical applications, we need to obtain explicit expressions of these stopping times and also efficient methods for the generation of these random variable. We mainly present in this chapter already known results which are essential for the following chapters.

1.1 Some results related to Brownian stopping times

1.1.1 Hitting time of a given level

First we focus our attention on the hitting time of a given level for the Brownian motion and its relatives. Let us consider $L > 0$, and

$$\tau_L = \inf\{t \geq 0 \mid B_t > L\}, \tag{1.1.1}$$

where $(B_t, t \geq 0)$ stands for a standard one-dimensional Brownian motion.
In order to describe the distribution of this random variable, let us present its Laplace transform which is commonly known. By seeking a random variable easy to generate with the same Laplace transform, we are able to "generate" this hitting time.

Proposition 1.1.1. *The random variable τ_L defined by* (1.1.1) *admits the following Laplace transform:*

$$\mathbb{E}[e^{-\lambda\tau_L}] = e^{-\sqrt{2\lambda}L}, \quad \forall\lambda > 0. \tag{1.1.2}$$

The proof of this statement can for instance be found in [53] (Proposition II.3.7 p.67). Since the Laplace transform characterize the probability distribution, we are going to use this result to point out a simple generation method.

Proof. Let us denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of $(B_t, t \geq 0)$. Using Girsanov's formula, the process $(M_t)_{t\geq 0}$ defined by

$$M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \quad t \geq 0,$$

is a $(\mathcal{F}_t)$-martingale. The martingale $(M_{t\wedge\tau_L})_{t\geq 0}$ is bounded, and the hitting time τ_L is almost surely finite due to the recurrence of the Brownian motion. We apply the optional stopping theorem and the dominated convergence theorem. We obtain

$$\mathbb{E}[M_{\tau_L}] = \mathbb{E}[M_0].$$

The definition of $(M_t)_{t\geq 0}$ leads to

$$\mathbb{E}[e^{\lambda B_{\tau_L} - \frac{1}{2}\lambda^2 \tau_L}] = 1, \quad \forall \lambda \in \mathbb{R}.$$

Since $\tau_L < \infty$, $B_{\tau_L} = L$, and therefore $\mathbb{E}[e^{\lambda L - \frac{1}{2}\lambda^2 \tau_L}] = 1$ for all $\lambda \in \mathbb{R}$ which is equivalent to

$$\mathbb{E}[e^{-\frac{1}{2}\lambda^2 \tau_L}] = e^{-\lambda L}, \quad \forall \lambda \in \mathbb{R}.$$

Defining $\tilde{\lambda}$ by $\tilde{\lambda} = \frac{1}{2}\lambda^2$, we obtain the announced result

$$\mathbb{E}[e^{-\tilde{\lambda}\tau_L}] = e^{-\sqrt{2\tilde{\lambda}}L}, \quad \forall \tilde{\lambda} \geq 0.$$

□

Now, as previously explained, we shall find an easy way to generate a random variable having such a distribution.

Proposition 1.1.2. *Let us consider a Gaussian distributed random variable $G \sim \mathcal{N}(0,1)$. Then τ_L and $\dfrac{L^2}{G^2}$ are identically distributed.*

Remark 1.1.3. *The distribution of the Brownian first passage time belongs to the family of inverse Gaussian distributions.*

Proof. In order to prove this statement, we first determine the density of the considered random variable. Let us denote by F the cumulative distribution function of $\frac{L^2}{G^2}$ and f its density. Let $x > 0$ and $L > 0$, using the Gaussian distribution leads to

$$\begin{aligned} F(x) &= \mathbb{P}\left(\frac{L^2}{G^2} \leqslant x\right) = \mathbb{P}\left(\frac{L^2}{x} \leqslant G^2\right) = 1 - \mathbb{P}\left(G^2 < \frac{L^2}{x}\right) \\ &= 1 - 2\mathbb{P}\left(0 \leq G < \frac{L}{\sqrt{x}}\right) = 1 - \frac{2}{\sqrt{2\pi}} \int_0^{\frac{L}{\sqrt{x}}} e^{-\frac{t^2}{2}}\, dt. \end{aligned}$$

Taking the derivative of F, we have:

$$f(x) = \frac{L}{\sqrt{2\pi x^3}} e^{-\frac{L^2}{2x}} 1_{\{}x > 0\}.$$

We now compute the Laplace transform $\tilde{f}$ of the function f,

$$\tilde{f}(\lambda) = L\int_0^\infty \frac{e^{-\lambda t}e^{-\frac{L^2}{2t}}}{\sqrt{2\pi t^3}}dt = \frac{L}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-(\lambda t+\frac{L^2}{2t})}}{t^{\frac{3}{2}}}dt, \quad \lambda \geq 0.$$

Using $\lambda t + \frac{L^2}{2t} = (\frac{L}{\sqrt{2t}} + \sqrt{\lambda t})^2 - L\sqrt{2\lambda}$, we obtain

$$\tilde{f}(\lambda) = e^{\sqrt{2\lambda}L}\frac{L}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-(\frac{L}{\sqrt{2t}}+\sqrt{\lambda t})^2}}{t^{\frac{3}{2}}}dt =: e^{\sqrt{2\lambda}L}\frac{L}{\sqrt{2\pi}}\mathcal{I}(\lambda).$$

Using the change of variable $u = \sqrt{t}$ leads to

$$\mathcal{I}(\lambda) = 2\int_0^\infty \frac{e^{-(\frac{L}{\sqrt{2}u}+\sqrt{\lambda}u)^2}}{u^2}du = 2e^{-2L\sqrt{2\lambda}}\int_0^\infty \frac{e^{-(\frac{L}{\sqrt{2}u}-\sqrt{\lambda}u)^2}}{u^2}du.$$

By successive integrations by substitution with $v = \frac{1}{u}\frac{L\sqrt{\lambda}}{\sqrt{2}}$ and $\omega = \frac{L}{u\sqrt{2}} - u\sqrt{\lambda}$, we obtain

$$\begin{aligned}
\mathcal{I}(\lambda) &= 2e^{-2L\sqrt{2\lambda}}\int_0^\infty \frac{\sqrt{2\lambda}}{L}e^{-(\frac{L}{\sqrt{2}v}-\sqrt{\lambda}v)^2}dv \\
&= e^{-2L\sqrt{2\lambda}}\int_0^\infty \left(\frac{\sqrt{2\lambda}}{L} + \frac{1}{u^2}\right)e^{-(\frac{L}{\sqrt{2}v}-\sqrt{\lambda}v)^2}dv \\
&= \frac{\sqrt{2}}{L}e^{-2L\sqrt{2\lambda}}\int_{-\infty}^\infty e^{-\omega^2}d\omega = \frac{\sqrt{2\pi}}{L}e^{-2L\sqrt{2\lambda}}, \quad \lambda \geq 0.
\end{aligned}$$

Finally we have

$$\tilde{f}(\lambda) = e^{-\sqrt{2\lambda}}, \quad \lambda \geq 0.$$

As stated, both random variables τ_L and $\frac{L^2}{G^2}$ admit the same Laplace transform, and therefore are identically distributed. □

The previous proof also enlightens the following result:

Corollary 1.1.4. *The random variable τ_L has the following density function f:*

$$f(x) = \frac{L}{\sqrt{2\pi x^3}}e^{-\frac{L^2}{2x}}1_{\{x>0\}} \tag{1.1.3}$$

1.1.2 One dimensional Brownian exit time from an interval

In this section, we consider the exit time from a given interval $[a, b]$ for the standard Brownian motion, with $a < 0 < b$. Let us define

$$\tau_{[a,b]} = \inf\{t \geq 0 \mid B_t \notin [a, b]\}. \tag{1.1.4}$$

As in the previous section, we determine the Laplace transform of this exit time. The only difference here is that we do not know precisely the position of the considered Brownian motion at time $\tau_{[a,b]}$, but we determine the probabilities of the events $\{\tau_{[a,b]} = \tau_a\}$ and $\{\tau_{[a,b]} = \tau_b\}$ where $\tau_a = \inf\{t \geq 0 \mid B_t = a\}$ and $\tau_b = \inf\{t \geq 0 \mid B_t = b\}$.

Exit probabilities

Proposition 1.1.5. *The probabilities associated with the exit location of the Brownian motion satisfy:*

$$\mathbb{P}(\tau_a \leqslant \tau_b) = \frac{b}{b-a} \quad \textit{and} \quad \mathbb{P}(\tau_b \leqslant \tau_a) = \frac{-a}{b-a}.$$

Proof. We apply the optional stopping theorem to the stopped martingale $(B_{t\wedge\tau_{[a,b]}}, t \geq 0)$,

$$\mathbb{E}[B_{t\wedge\tau_{[a,b]}}] = 0, \quad t \geq 0.$$

Letting t tends to infinity, the dominated convergence leads to $\mathbb{E}[B_{\tau_{[a,b]}}] = 0$ which is equivalent to

$$a\,\mathbb{P}(\tau_a \leqslant \tau_b) + b\,\mathbb{P}(\tau_b \leqslant \tau_a) = 0.$$

Moreover, due to the recurrence property of the Brownian motion paths, we observe

$$\mathbb{P}(\tau_a \leqslant \tau_b) + \mathbb{P}(\tau_b \leqslant \tau_a) = 1.$$

Finally, solving the previous system of equations gives the announced result. □

Laplace transform of the exit time

Let us present the distribution of the first exit time.

Proposition 1.1.6. *The Laplace transform of $\tau_{[a,b]}$ is given by*

$$\mathbb{E}[e^{-\lambda\tau_{[a,b]}}] = \frac{\cosh(\sqrt{2\lambda}\frac{b+a}{2})}{\cosh(\sqrt{2\lambda}\frac{b-a}{2})}, \quad \forall\lambda > 0.$$

Proof. Let us define $\alpha := \frac{b+a}{2}$ and consider the exponential martingale constructed according to the Girsanov's formula

$$M_t = e^{-\lambda t}(e^{\sqrt{2\lambda}(B_t-\alpha)} + e^{-\sqrt{2\lambda}(B_t-\alpha)}) = 2e^{-\lambda t}\cosh(\sqrt{2\lambda}(B_t-\alpha)) \quad t \geq 0.$$

On one hand, the optional stopping theorem leads to

$$\mathbb{E}[M_{\tau_{[a,b]}}] = \mathbb{E}[M_0] = 2\cosh(-\alpha\sqrt{2\lambda}) = 2\cosh\left(\sqrt{2\lambda}\,\frac{b+a}{2}\right)$$

and on the other hand

$$\mathbb{E}[M_{\tau_{[a,b]}}] = 2\mathbb{E}\Big[e^{-\lambda\tau_{[a,b]}}\cosh\Big(\sqrt{2\lambda}(B_{\tau_{[a,b]}} - \alpha)\Big)\Big].$$

If $B_{\tau_{[a,b]}} = a$ then

$$B_{\tau_{[a,b]}} - \alpha = -\frac{(b-a)}{2} \text{ else } B_{\tau_{[a,b]}} - \alpha = \frac{(b-a)}{2}.$$

In any case,

$$\cosh(\sqrt{2\lambda}(B_{\tau_{[a,b]}} - \alpha)) = \cosh\left(\sqrt{2\lambda}\,\frac{b-a}{2}\right) \tag{1.1.5}$$

and therefore

$$\mathbb{E}[M_{\tau_{[a,b]}}] = \cosh\left(\sqrt{2\lambda}\,\frac{b-a}{2}\right)\,\mathbb{E}[e^{-\lambda\tau_{[a,b]}}]. \tag{1.1.6}$$

Finally

$$\mathbb{E}[e^{-\lambda\tau_{[a,b]}}] = \frac{\cosh(\sqrt{2\lambda}\frac{b+a}{2})}{\cosh(\sqrt{2\lambda}\frac{b-a}{2})}, \quad \lambda \geq 0. \tag{1.1.7}$$

□

Unfortunately, the inversion of this Laplace transform is not so easy as in the previous section. It seems then hard to generate, in a simple way, a random variable having the same distribution as the studied exit time.

Reflection principle

Let us recall a crucial property of the Brownian paths: the reflection principle. This principle is of prime interest in order to study the Brownian exit time from an interval. In fact, it permits to obtain a representation of its cumulative distribution function as a series.

Theorem 1.1.7. *Let us consider $(B_t)_{t\geq 0}$ a one-dimensional Brownian motion and $b \in \mathbb{R}^+$. Let τ_b the hitting time of the level b. We define the process $(X_t)_{t\geq 0}$ by*

$$X_t = \begin{cases} B_t & \text{if } t \leqslant \tau_b \\ 2b - B_t & \text{else.} \end{cases}$$

Then $(X_t)_{t\geq 0}$ is also a one-dimensional Brownian motion.

This statement can be explained in this way: using the symmetry property, once the Brownian motion reaches the level a, it has the same probability to pursue below the level a than to pursue above the level (see Figure 1.1). We are going to use this statement as

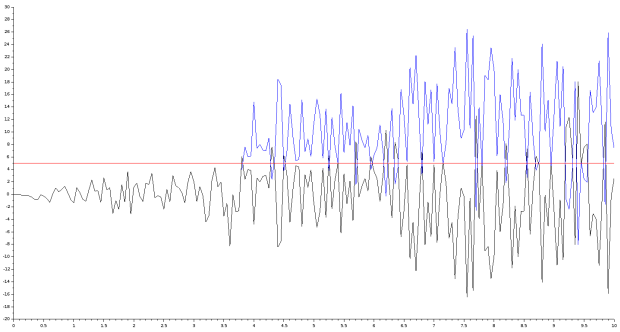


Figure 1.1: Illustration of the Brownian reflection principle

follows: let us suppose that the Brownian motion reaches the level b before the level a, then the reflection principle states that, starting from the hitting point, the time needed to reach the level a is the same than the time needed to reach the level $2b-a$. In particular,

Corollary 1.1.8. *Let us suppose that $\tau_b < \tau_a$, then τ_{2b-a} and τ_a have the same conditional distribution given $\tau_b < \tau_a$.*

Cumulative distribution function of the exit time

In this section, we point out a precise expression of the exit time distribution for the Brownian motion $(B_t, t \geq 0)$. This expression of the cumulative distribution is written as a series expansion. Even if there exist simulation methods based on convergent series, the series presented here is not so handy to manipulate. Once again we consider $a < 0 < b$ and a standard Brownian motion $(B_t, t \geq 0)$. We denote here

$$\tau_{a,b} = \inf\{t \geq 0 \,|\, B_t > b \text{ or } B_t < a\}$$

Theorem 1.1.9. *The c.d.f. of $\tau_{a,b}$ is given by*

$$\mathbb{P}(\tau_{a,b} < T) = 2\sum_{k=0}^{\infty}(-1)^k \left(2 - \Phi\left(\frac{a_k}{\sqrt{T}}\right) - \Phi\left(\frac{b_k}{\sqrt{T}}\right)\right), \quad \forall T > 0 \tag{1.1.8}$$

where the sequences $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ are defined by

$$\begin{gathered} a_0 = a,\ b_0 = b \\ \text{and } a_k = |a| + k(|a| + b),\ b_k = b + k(|a| + b). \end{gathered}$$

The function Φ represents the cumulative distribution of a standard Gaussian variable.

Proof. We proceed here in several steps. Let us start with the c.d.f of the first passage time through the level b (resp. a) denoted by τ_b (resp τ_a).

$$\mathbb{P}(\tau_b < T) = \mathbb{P}\Big(\sup_{t\leqslant T} B_t > b\Big) = \mathbb{P}(|B_T| > b) = 2\left(1 - \Phi\left(\frac{b}{\sqrt{T}}\right)\right), \quad \forall T > 0. \tag{1.1.9}$$

Similarly $\mathbb{P}(\tau_a < T) = \Phi\left(\frac{|a|}{\sqrt{T}}\right)$, for any $T > 0$. Now we study $\tau_{a,b}$ by splitting the following probability $\mathbb{P}(\tau_{a,b} < T)$ into different terms

$$\mathbb{P}(\tau_{a,b} < T) = \mathbb{P}(\tau_a < T) + \mathbb{P}(\tau_b < T) - \mathbb{P}(\tau_a < T,\ \tau_b < T) \tag{1.1.10}$$

Since the two first terms are already known, we focus our attention on the last one. It can be splitted as follow:

$$\mathbb{P}(\tau_a < T,\ \tau_b < T) = \mathbb{P}(\tau_a < \tau_b < T) + \mathbb{P}(\tau_b < \tau_a < T). \tag{1.1.11}$$

Let us note that the reflection principle leads to

$$\mathbb{P}(\tau_a < \tau_b < T) = \mathbb{P}(\tau_{2a-b} < T) - \mathbb{P}(\tau_b < \tau_a, \tau_{2a-b} < T).$$

Then, using one again the reflection principle previously stated, we obtain

$$\mathbb{P}(\tau_a < \tau_b < T) = \mathbb{P}(\tau_{2a-b} < T) - \mathbb{P}(\tau_b < \tau_a, \tau_{3b-2a} < T).$$

Let us now consider $\alpha' < \alpha < 0 < \beta < \beta'$. The symmetry argument induced by the reflection principle permits to prove that:

$$\begin{aligned}\mathbb{P}(\tau_\beta < \tau_\alpha, \tau_{\beta'} < T) &= \mathbb{P}(\tau_{\beta'} < T) - \mathbb{P}(\tau_\beta > \tau_\alpha, \tau_{\beta'} < T) \\ &= \mathbb{P}(\tau_{\beta'} < T) - \mathbb{P}(\tau_\beta > \tau_\alpha, \tau_{2\alpha-\beta'} < T).\end{aligned}$$

Similarly, $\mathbb{P}(\tau_\alpha < \tau_\beta, \tau_{\alpha'} < T) = \mathbb{P}(\tau_{\alpha'} < T) - \mathbb{P}(\tau_\beta < \tau_\alpha, \tau_{2\beta-\alpha'} < T)$. These identities will be applied successively with different choices for α, α', β and β'. We obtain

$$\begin{aligned}\mathbb{P}(\tau_a < \tau_b < T) &= \mathbb{P}(\tau_{2a-b} < T) - \mathbb{P}(\tau_{3b-2a} < T) + \mathbb{P}(\tau_a < \tau_b, \tau_{4a-3b} < T) \\ &= \mathbb{P}(\tau_{2a-b} < T) - \mathbb{P}(\tau_{3b-2a} < T) + \mathbb{P}(\tau_{4a-3b} < T) - \mathbb{P}(\tau_b < \tau_a, \tau_{4a-3b} < T).\end{aligned}$$

The symmetry of the Brownian motion leads to

$$\mathbb{P}(\tau_a < \tau_b < T) = \mathbb{P}(\tau_{b-2a} < T) - \mathbb{P}(\tau_{3b-2a} < T) + \mathbb{P}(\tau_{3b-4a} < T) - \mathbb{P}(\tau_b < \tau_a, \tau_{4a-3b} < T).$$

Using similar arguments, we obtain

$$\mathbb{P}(\tau_b < \tau_a < T) = \mathbb{P}(\tau_{2b-a} < T) - \mathbb{P}(\tau_{2b-3a} < T) + \mathbb{P}(\tau_{4b-3a} < T) - \mathbb{P}(\tau_b < \tau_a, \tau_{4b-3a} < T).$$

We then iterate these computations. Combining (1.1.9) and (1.1.11)

$$\mathbb{P}(\tau_{a,b} < T) = 2\sum_{k=0}^{n}(-1)^k \left(2 - \Phi\left(\frac{a_k}{\sqrt{T}}\right) - \Phi\left(\frac{b_k}{\sqrt{T}}\right)\right) + R_n(T), \quad n \geq 1.$$

with $a_k = -a + k(b-a)$ and $b_k = b + k(b-a)$. The reminder $R_n(T)$ is given by $(-1)^{n+1}(\mathbb{P}(\tau_a < \tau_b, \tau_{a_n} < T) + \mathbb{P}(\tau_b < \tau_a, \tau_{-b_n} < T))$. We remark that $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$ are strictly increasing, and we deduce as k tends to infinity that both $\left(\Phi\left(\frac{a_k}{\sqrt{T}}\right)\right)_{k\geqslant 0}$ and $\left(\Phi\left(\frac{b_k}{\sqrt{T}}\right)\right)_{k\geqslant 0}$ are also strictly increasing and tend to 1.
The following upper-bound holds:

$$|R_n(T)| \leqslant \mathbb{P}(\tau_{a_n} < T) + \mathbb{P}(\tau_{-b_n} < T), \quad \forall n \geq 1.$$

Since both probabilities appearing in the r.h.s converge to 0 when n tends to infinity, $R_n(T)$ tends to 0. Hence, the c.d.f of $\tau_{a,b}$ can be expressed using the convergent series:

$$\mathbb{P}(\tau_{a,b} < T) = 2\sum_{k=0}^{\infty}(-1)^k \left(2 - \Phi\left(\frac{a_k}{\sqrt{T}}\right) - \Phi\left(\frac{b_k}{\sqrt{T}}\right)\right), \quad \forall\, T > 0.$$

□

1.1.3 Hitting of an affine frontier

In section 1, we consider the distribution of the Brownian first passage time through a given threshold. We obtain a particular inverse Gaussian distribution. This result can be extended to the hitting time of other curves. In this section, we consider the hitting of a linear frontier. A simple use of the Girsanov's formula permits to point out its distribution. Let us introduce the mapping $x \mapsto a + bx$ with $a > 0$ and the associated stopping time

$$T = \inf\{t > 0 \,|\, B_t = a + bt\} \tag{1.1.12}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion.

Proposition 1.1.10. *Let T the first passage time defined by* (1.1.12)*. Then the probability density function of T is given by*

$$p(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-ab - b^2\frac{t}{2} - \frac{a^2}{2t}\right), \quad t \geq 0.$$

Proof. The idea here is to use Girsanov's theorem to transform the considered random variable into the hitting of a fixed level for a drifted Brownian motion. Similarly to the Brownian motion case, we are able to precisely describe the hitting time of a given level for the drifted Brownian motion. Let us consider a Brownian motion $(B_t, t \geq 0)$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Our aim is to compute the cumulative distribution $\mathbb{P}(T \leqslant t_0)$ for some $t_0 > 0$ and to differentiate with respect to t_0. Let us define the probability measure $\mathbb{Q}$ on $\mathcal{F}_{t_0}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_{t_0} := \exp\left(-bB_{t_0} - b^2\frac{t_0}{2}\right).$$

Girsanov's formula states that, under the probability $\mathbb{Q}$, the process $\tilde{B}$ defined by $\tilde{B}_t = B_t + bt$ is a Brownian motion. Let us consider the event $A = \{\sup_{s\leqslant t_0} B_s \geqslant a\}$. Let us notice that $A = \{T_a \leqslant t_0\}$ leading to $A \in \mathcal{F}_{T_a \wedge t_0}$ where T_a is the first passage time through the level a. Hence,

$$\begin{aligned}\mathbb{P}(T \leqslant t_0) &= \mathbb{P}(\exists t \leqslant t_0 : B_t = a + bt) = \mathbb{P}(\exists t \leqslant t_0 : B_t - bt = a) \\ &= \mathbb{P}(\sup_{s\leqslant t_0}(B_s - bs) \geqslant a) = \mathbb{Q}(\sup_{s\leqslant t_0}(\tilde{B}_s - bs) \geqslant a) = \mathbb{Q}(\sup_{s\leqslant t_0} B_s \geqslant a) = \mathbb{Q}(A).\end{aligned}$$

Let us note that $M_{T_a} = \exp(-ab - b^2\frac{T_a}{2})$ a.s. and recall that

$$\mathbb{P}(T_a \in ds) = \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}}\, ds.$$

Using the optional stopping theorem, we deduce the announced cumulative distribution function:

$$\begin{aligned}\mathbb{P}(T \leqslant t_0) = \mathbb{Q}(A) &= \mathbb{E}_{\mathbb{P}}[M_{t_0} 1_A] = \mathbb{E}_{\mathbb{P}}[M_{T_a \wedge t_0} 1_{T_a \leqslant t_0}] = \mathbb{E}_{\mathbb{P}}[M_{T_a} 1_{T_a \leqslant t_0}] \\ &= \int_0^{t_0} \frac{a}{\sqrt{2\pi t^3}} \exp\left(-ab - b^2\frac{s}{2} - \frac{a^2}{2s}\right) ds,\end{aligned}$$

and deduce the density of T. □

1.2 Method of images

Let us detail a method introduced by Daniels [13] and explained in a interesting paper by Lerche [39]: the so-called method of images. This method is of prime interest since most of the algorithms presented through the next chapter are directly related to this method. It permits, for instance, to obtain the exit time from specific domains for particular processes. In the Brownian motion case, we are able by that way to determine the exit time from spheroids, specific domains which shall be described further. This exit time and the symmetry property of Brownian motion paths are fundamental ingredients in the building of the WOMS algorithm. The text of this section corresponds to a reminder of the original work of Lerche.

1.2.1 Construction of the method

The aim of the method of images is to find particular curved boundaries whose Brownian hitting time is explicit. Each particular curve is directly linked to a positive measure. Let us introduce the framework. We consider a positive measure F defined on the Borel set $\mathcal{B}(\mathbb{R}_+)$. If F is a probability measure then we observe some kind of competition between the standard Brownian motion with density p and a second Brownian motion with initial measure F and density p_F. We can split the set $\mathbb{R} \times \mathbb{R}_+$ containing the paths into different subsets depending on the comparison between the densities of both Brownian motion $\{(x,t) \text{ s.t. } p(x,t) \geq p_F(x,t)\}$ and $\{(x,t) \text{ s.t. } p(x,t) < p_F(x,t)\}$. If we denote by ψ the boundary of these subsets, then the method of images permits to describe the probability density function of the first Brownian hitting time of this boundary ψ depending on F. To pursue the study, let us suppose that F is σ-finite and satisfies:

$$\int_0^\infty \phi(\epsilon\theta) F(d\theta) < \infty, \quad \forall \epsilon > 0, \tag{1.2.1}$$

where ϕ is the standard density $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(t) = \int_{-\infty}^{t} \phi(s)ds$. For any $a > 0$, we introduce

$$h(x,t) := \frac{1}{\sqrt{t}}\,\phi\left(\frac{x}{\sqrt{t}}\right) - a^{-1} \int_0^\infty \frac{1}{\sqrt{t}}\,\phi\left(\frac{x-\theta}{\sqrt{t}}\right) F(d\theta). \tag{1.2.2}$$

Using integral derivation rules, we can prove that the function h is a solution of the diffusion equation $\partial_t h = \frac{1}{2}\partial_x^2 h$ on the set $\mathbb{R} \times \mathbb{R}_+$.
From now on, we define $x(t) = \psi(t)$ as the unique solution of the implicit equation

$$h(x,t) = 0. \tag{1.2.3}$$

Dividing the equation (1.2.3) by $\frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)$ permits to emphasize an equivalent equation:

$$f\left(\frac{x}{t}, \frac{1}{t}\right) = a \text{ with } f(y,s) = \int_0^\infty e^{\theta y - \frac{1}{2}\theta^2 s} F(d\theta) \tag{1.2.4}$$

This second equation admits a unique solution, and so do (1.2.3). Moreover we have

$$f\left(\frac{x}{t},\frac{1}{t}\right) = \int_0^\infty e^{\theta\frac{x}{t}-\frac{1}{2}\theta^2\frac{1}{t}}F(d\theta) = e^{\frac{x^2}{2t}}\int_0^\infty e^{-\frac{(\theta-x)^2}{2t}}F(d\theta)$$
$$= e^{\frac{x^2}{2t}}\int_0^\infty e^{-(\frac{\theta}{\sqrt{2t}})^2}F_x(d\theta) := e^{\frac{x^2}{2t}}C_t(x), \tag{1.2.5}$$

where $F_x(A) = F(x + A)$ for any set $A \in \mathcal{B}(\mathbb{R}_+)$. After this rough description of the functions $h(x,t)$, $f(x,t)$ and $\psi(t)$, we aim to explain their role in the study of the hitting times. In this paragraph, we presented a discussion about the existence and uniqueness of the equation (1.2.4), where F is a measure defined on $\mathcal{B}(\mathbb{R}_+)$. In the sequel, we shall explain how the function $h(x,t)$ is related to the distribution of the Brownian hitting time of ψ. Let us mention now that the study can be developed also for positive measures on $\mathcal{B}(\mathbb{R})$. In that case, (1.2.4) can admit two different solutions. One example is of prime interest. Considering F a measure proportional to the Lebesgue measure, the solutions ψ_- and ψ_+ of equation (1.2.4) can be explicitly computed since F is translation invariant: $F_x = F$, $\forall x \in \mathbb{R}$. A modified version of equation (1.2.5) leads to $f\left(\frac{x}{t},\frac{1}{t}\right) = \kappa\sqrt{t}\,e^{\frac{x^2}{2t}}$ where κ is a positive constant. The equation (1.2.4) has therefore two evident solutions:

$$\psi_\pm(t) = \pm\sqrt{t\ln\left(\frac{a^2}{\kappa^2 t}\right)}. \tag{1.2.6}$$

All the statements pointed out in the forthcoming paragraphs concern positive measures F defined on $\mathcal{B}(\mathbb{R}_+)$ associated with a unique function ψ. Of course, they can be easily modified to the situation of the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

1.2.2 Study of the absorption time

We recall that $(B_t)_{t\geq 0}$ stands for a standard Brownian motion.

Definition 1.2.1. *Let $a > 0$. We call "absorption time" of $(B_t)_{t\geq 0}$, denoted by T, the random variable defined by*

$$T = \inf\{t > 0 \mid h(B_t,t) = 0\} = \inf\left\{t > 0 \,\middle|\, f\left(\frac{B_t}{t},\frac{1}{t}\right) = a\right\}.$$

The distribution of this random variable can be easily described. Before introducing the main result concerning the distribution of T, let us present a specific condition on the boundary function $t \mapsto \psi_t$. We assume that ψ is monotone non decreasing in a neighborhood of the time origin and is integrable:

$$\int_{0_+}\frac{\psi(t)}{t^{\frac{3}{2}}}e^{-\frac{\psi(t)^2}{4t}}dt < +\infty. \tag{1.2.7}$$

This condition ensures that $(0,0)$ is a regular point, that means: $\mathbb{P}(T > 0) = 1$ (see for instance Theorem 3.1 in Doob [16], or Petrowsky [48]).

Theorem 1.2.2. *Let $\mathcal{C}$ be the set defined by $\mathcal{C} = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid x \leq \psi(t)\}$ with ψ a boundary satisfying* (1.2.7)*. On this particular set:*

$$\mathbb{P}(T > t, B_t \in dx) = h(x,t)dx, \tag{1.2.8}$$

where h is defined by (1.2.2) *and the conditional probability of T given $B_t = x$ satisfies*

$$\mathbb{P}(T \leq t \mid B_t = x) = 1 - \frac{h(x,t)}{\frac{1}{\sqrt{t}}\phi(\frac{x}{\sqrt{t}})} = a^{-1} f\left(\frac{x}{t}, \frac{1}{t}\right). \tag{1.2.9}$$

Corollary 1.2.3. *Integrating* (1.2.9)*, we obtain an expression of the cumulative distribution function of T:*

$$\mathbb{P}(T \leq t) = 1 - \Phi\left(\frac{\psi(t)}{\sqrt{t}}\right) + a^{-1} \int_0^\infty \Phi\left(\frac{\psi(t) - \theta}{\sqrt{t}}\right) F(d\theta), \quad \forall t \geq 0. \tag{1.2.10}$$

Let us notice that (1.2.9) can be obtained from (1.2.8) using Bayes' formula.
As already mentioned in the previous section, these results can be adapted to the case of a positive measure F proportional to the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. In that case, both boundaries given by (1.2.6) and denoted by ψ_+ and ψ_- satisfy the integrability condition necessary for the following stopping time

$$T = \inf\{t \geq 0 | B_t \notin [\psi_-(t), \psi_+(t)]\}$$

to be almost surely strictly positive. Then the modified statement of Theorem 1.2.2 becomes:

$$\mathbb{P}(T > t, B_t \in dx) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - a^{-1} \int_{\mathbb{R}} \frac{1}{\sqrt{t}}\phi\left(\frac{x - \theta}{\sqrt{t}}\right) F(d\theta),$$

for all (x,t) s.t. $\psi_-(t) \leq x \leq \psi_+(t)$. To prove Theorem 1.2.2, we need several properties concerning the function ψ.

Lemma 1.2.4. *The function ψ, solution of* (1.2.4)*, is infinitely often continuously differentiable, $t \mapsto \frac{\psi(t)}{t}$ is decreasing and ψ is concave.*

Proof. We apply the implicit function theorem to f since

$$f\left(\frac{\psi(t)}{t}, \frac{1}{t}\right) = a. \tag{1.2.11}$$

The first statement is a consequence of $f \in \mathcal{C}^\infty$. Moreover (1.2.11) easily leads to the second point. Defining

$$\eta(s) = \frac{\psi(t)}{t} \text{ with } s = \frac{1}{t},$$

we observe that η satisfies the condition:

$$a = \int_0^\infty e^{\theta\eta(s) - \frac{1}{2}\theta^2 s} F(d\theta)$$

and is continuous on $(0,+\infty)$ as a ratio of two continuous functions. Let α, $\beta \in \mathbb{R}$. In order to prove that η is concave, we aim to describe $\eta(\rho\alpha + (1-\rho)b)$ with $\rho \in]0,1[$. Applying Hölder's inequality, we obtain

$$\begin{aligned}\int_0^\infty e^{\theta(\rho\eta(\alpha)+(1-\rho)\eta(\beta))-\frac{1}{2}\theta^2(\rho\alpha+(1-\rho)\beta)}F(d\theta) &= \int_0^\infty e^{\rho(\theta\eta(\alpha)-\frac{1}{2}\theta^2\alpha)}e^{(1-\rho)(\theta\eta(\beta)-\frac{1}{2}\theta^2\beta)}F(d\theta)\\ &\leq \left(\int_0^\infty e^{\theta\eta(\alpha)-\frac{1}{2}\theta^2\alpha}F(d\theta)\right)^{\rho}\left(\int_0^\infty e^{\theta\eta(\beta)-\frac{1}{2}\theta^2\beta}F(d\theta))\right)^{1-\rho}\\ &= a^\rho a^{1-\rho} = a = \int_0^\infty e^{\theta\eta(\rho\alpha+(1-\rho)\beta)-\frac{\theta^2}{}(\rho\alpha+(1-\rho)\beta)}F(d\theta).\end{aligned}$$

We deduce that η is concave

$$\eta\left(\rho\alpha+(1-\rho)\beta\right) \leqslant \rho\eta(\alpha)+(1-\rho)\eta(\beta), \quad \forall\rho\in]0,1[, \quad \forall(\alpha,\beta)\in\mathbb{R}^2.$$

The regularity of the function η is deduced from the regularity of ψ. Using the concavity of η we obtain $\psi''(t) = \frac{\eta''(t^{-1})}{t^3} < 0$, for all $t > 0$. □

Lemma 1.2.5. *Let us define θ^* by $\theta^* = \inf\{y \geq 0 \mid F[0,y] > 0\} \geqslant 0$. Then*

$$\lim_{t\to 0}\psi(t) = \frac{\theta^*}{2}.$$

Proof. Let us prove by contradiction that

$$\liminf_{t\to 0}\psi(t) \geqslant \frac{\theta^*}{2}. \tag{1.2.12}$$

For that, let us suppose that there exists $0 < \epsilon < 1$ such that $\liminf_{t\to 0}\psi(t) \leqslant \frac{(\theta^*-\epsilon)}{2}$. We consider a sequence $(t_i)_{i\in\mathbb{N}}$ satisfying $\lim_{i\to\infty} t_i = 0$ and $\lim_{i\to\infty}\psi(t_i) = \liminf_{t\to\infty}\psi(t)$. Using the identity

$$-\frac{1}{2}\theta^2\frac{1}{t_i} = -\frac{1}{2}\theta^2(\frac{1}{t_i}-\epsilon) - \frac{1}{2}\theta^2\epsilon \quad \text{and the lower bound } \theta^* \leq \theta \text{ for any } \theta \in \mathrm{supp}(F),$$

we obtain

$$\begin{aligned}0 < a &= \int_{\theta^*}^\infty \exp\left(\theta\frac{\psi(t_i)}{t_i} - \frac{1}{2}\frac{\theta^2}{t_i}\right)\, F(d\theta)\\ &\leqslant \int_{\theta^*}^\infty \exp\left(\frac{1}{2}\theta\frac{(\theta^*-\epsilon)}{t_i} - \frac{1}{2}\theta\theta^*(t_i^{-1}-\epsilon)\right)\exp\left(-\frac{\epsilon}{2}\theta^2\right)\, F(d\theta)\\ &\leqslant \int_{\theta^*}^\infty \exp\left(-\frac{\epsilon\theta}{2}(t_i^{-1}-\theta^*)\right)\exp\left(-\frac{\epsilon\theta^2}{2}\right)\, F(d\theta), \quad \forall i\in\mathbb{N}.\end{aligned}$$

Applying the dominated convergence theorem, the right hand side of the previous inequality converges to 0 as i tends to infinity. This result leads to a contradiction, and therefore implies (1.2.12). By similar computations, we also prove that $\limsup_{t\to 0}\psi(t) \leqslant \frac{\theta^*}{2}$ and, by the way, the announced statement. □

The last lemma which plays an important role in the proof of Theorem 1.2.2 is the following convergence result:

Lemma 1.2.6. *Let $(\psi_n)_{n\geqslant 1}$ be a sequence of positive functions, converging uniformly on the set $[0,t]$ to a limit function ψ. Let us suppose that ψ is a concave differentiable function either monotone non decreasing in a neighborhood of 0 and satisfying (1.2.7) either verifying $\psi(0) > 0$. We assume moreover that $\psi_n \geqslant \psi$ on $[0,t]$ and for all $n \geq 1$. Let us introduce $P_{x,t}(f) = \mathbb{P}(B_u \geqslant f(u)$ for some $0 \leqslant u < t \mid B_t = x)$. Then, for all couple (x,t) s.t. $x \leqslant \psi(t)$, we have*

$$\lim_{n\to\infty} P_{x,t}(\psi_n) = P_{x,t}(\psi).$$

Proof. Let us define $T_n = \inf\{u > 0 \mid B_u \geqslant \psi_n(u)\}$ and $T = \inf\{u > 0 \mid B_u \geqslant \psi(u)\}$. Due to the integrability property (1.2.7), we get $\mathbb{P}(T > 0) = 1$ and therefore, for any $\alpha > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}(T < \delta) + \mathbb{P}(t - \delta \leqslant T < t) < \alpha.$$

Let us consider $\epsilon > 0$ small enough such that

$$\frac{2\epsilon(\epsilon + \psi(\delta) + (t-\delta)\psi'(\delta) - x)}{(t-\delta)} \leqslant \alpha. \tag{1.2.13}$$

There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the uniform convergence implies

$$\sup_{0\leqslant v\leqslant t} (\psi_n(v) - \psi(v)) < \epsilon.$$

By construction of the sequence $(\psi_n)_{n\in\mathbb{N}}$, the frontier defined by ψ is reached by the Brownian motion before ψ_n; leading to $\{0 < T_n < t\} \subseteq \{0 < T < t\}$. We deduce

$$\begin{aligned} 0 \leq \mathcal{P} &:= \mathbb{P}(0 < T < t \mid B_t = x) - \mathbb{P}(0 < T_n < t \mid B_t = x) \\ &= \mathbb{P}(0 < T < t, T_n > t \mid B_t = x) = \int_0^t \mathbb{P}(T_n > t \mid B_u = \psi(u), B_t = x)\mathbb{P}(T \in du). \end{aligned}$$

Using $B_u = \psi(u)$ on the event $\{T = u\}$ and (1.2.13),

$$0 \leq \mathcal{P} \leqslant \alpha + \int_\delta^{t-\delta} \rho_\epsilon(x,u,t)\mathbb{P}(T \in du) \tag{1.2.14}$$

with $\rho_\epsilon(x,u,t) := \mathbb{P}\Big(\sup_{s\in[u,t]} B_s \leq \psi(s) + \epsilon \,\Big|\, B_u = \psi(u),\, B_t = x\Big)$. Using the Markov property of the Brownian motion and the concavity of the boundary ψ, we obtain

$$\rho_\epsilon(x,u,t) \leq \mathbb{P}\Big(\sup_{s\in[0,t-u]} B_s \leq \psi(s) + \epsilon + s\psi'(u) \Big| B_0 = \psi(u),\, B_{t-u} = x\Big).$$

The explicit expression of the first passage time density of a Brownian bridge through a given straight line, implies

$$\rho_\epsilon(x,u,t) \leqslant 1 - \exp\left(\frac{-2\epsilon(\epsilon+\psi(u)+(t-u)\psi'(u)-x)}{(t-u)}\right), \quad \forall u \in [\delta, t-\delta].$$

Finally (1.2.14) and (1.2.13) permit to obtain the following bounds

$$0 \leq \mathcal{P} \leqslant \alpha + \frac{2\epsilon(\epsilon+\psi(\delta)+(t-\delta)\psi'(\delta)-x)}{(t-\delta)} \leqslant 2\alpha.$$

□

We have now emphasized all preliminary results needed in order to prove Theorem 1.2.2.

Proof of Theorem 1.2.2. The function h satisfies:

$$\partial_t h = \frac{1}{2}\partial_x^2 h, \qquad h(\psi(t),t) = 0, \quad \forall t > 0 \tag{1.2.15}$$

and

$$h(\bullet, 0) = \delta_0 \text{ on the set } (-\infty, \psi(0_+)].$$

We split the study into two different cases:

First case: $\theta^* = \inf\{y \geq 0 \mid F[0,y] > 0\} > 0$. In this particular case, the previous lemmas state that $\psi(0_+) > 0$ and ψ is infinitely differentiable. With these properties, Friedman's uniqueness theorem (see, for instance [18]) permits to characterize the function h.
Let us consider

$$\mathbb{P}(T > t, B_t \in dx) = p(x,t)dx,$$

where $p(x,t)$ represents the distribution function at the point (x,t) for the Brownian motion still not absorbed at the frontier ψ at time t. The conditional density p is also a solution on $\mathcal{C}$ of the equation (1.2.15). The uniqueness result permits to state that $h = p$ on the set $\mathcal{C}$.

Second case: $\theta^* = 0$. Let $\alpha > 0$, we define

$$h_\alpha(x,t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - a^{-1}\int_\alpha^\infty \frac{1}{\sqrt{t}}\phi\left(\frac{x-\theta}{\sqrt{t}}\right)F(d\theta)$$

and denote by $\psi_\alpha(t)$ the solution of $h_\alpha(\psi_\alpha(t),t) = 0$ and T_α the corresponding stopping time. We introduce $p_\alpha(x,t)dx = \mathbb{P}(T_\alpha > t, B_t \in dx)$. The parameter θ^* associated to the restricted measure $F\Big|_{[\alpha,+\infty)}$ is equal to α as soon as α is small enough. We can therefore apply the first case and we obtain $h_\alpha(x,t) = p_\alpha(x,t)$ and note that

$$\int_\alpha^\infty \frac{1}{\sqrt{t}}\phi\left(\frac{x-\theta}{\sqrt{t}}\right)F(d\theta) \nearrow \int_0^\infty \frac{1}{\sqrt{t}}\phi\left(\frac{x-\theta}{\sqrt{t}}\right)F(d\theta) \text{ as } \alpha \to 0.$$

We deduce that $h_\alpha(x,t)$ is decreasing and converges to $h(x,t)$ when α goes to 0.
By definition of the boundaries $\psi(t)$ and $\psi_\alpha(t)$, we obtain that $\psi_\alpha(t)$ decreases towards $\psi(t)$ when α goes to 0.
Since $h_\alpha(x,t) = p_\alpha(x,t)$ we obtain $h(x,t) = \lim_{\alpha\to 0} p_\alpha(x,t)$. It suffices therefore to prove that

$$\lim_{\alpha\to 0} p_\alpha(x,t)dx = \mathbb{P}(T > t, W_t \in dx)$$

which is equivalent, by Bayes' formula, to

$$\lim_{\alpha\to 0} \mathbb{P}(T_\alpha > t \mid W_t = x) = \mathbb{P}(T > t \mid W_t = x). \tag{1.2.16}$$

To prove this last equality, we use the result of Lemma 1.2.6 applied to the particular family of functions $(\psi_\alpha)_{\alpha>0}$, which converge on the set $[0,t]$ to ψ. Lemma 1.2.4 states that the limit function ψ is concave. If we admit that ψ satisfies the conditions for T to be almost surely positive (1.2.7), then a direct application of Lemma 1.2.6 permits to get (1.2.16). □

Let us denote by p the density function of the random variable T. The work of Lerche [39] permits to find an explicit expression of this particular function.

Lemma 1.2.7. *The density of T, denoted by p satisfies the following equation*

$$p(t) = -\frac{1}{2}\partial_x h(x,t)\Big|_{x=\psi(t)}, \quad \forall t \in supp(p).$$

Proof. Theorem 1.2.2 permits to obtain the following expression of the tail distribution

$$\mathbb{P}(T > t) = \int_{-\infty}^{\psi(t)} h(y,t)dy, \quad \forall t > 0. \tag{1.2.17}$$

By definition, the previous equation leads to:

$$-p(t) = \frac{d}{dt}\mathbb{P}(T > t) = \int_{-\infty}^{\psi(t)} \partial_t h(y,t)dy + \psi'(t)\, h(\psi(t),t), \quad \forall t > 0.$$

Since ψ is the boundary, $h(\psi(t),t) = 0$, and therefore

$$-2\, p(t) = \int_{-\infty}^{\psi(t)} \partial_y^2 h(y,t)dy = \partial_x h(x,t)\Big|_{x=\psi(t)}.$$

□

This statement gives a first expression of the density, but is not really handy to manipulate. Fortunately, it permits to obtain another expression of the distribution function.

Theorem 1.2.8. *The density p of the stopping time T can be written as follows:*

$$p(t) = \frac{1}{2\, t^{\frac{3}{2}}}\, \phi\left(\frac{\psi(t)}{\sqrt{t}}\right) \frac{\int_{-\infty}^{\infty} \theta\, \phi\left(\frac{\theta-\psi(t)}{\sqrt{t}}\right) F(d\theta)}{\int_{-\infty}^{\infty} \phi\left(\frac{\theta-\psi(t)}{\sqrt{t}}\right) F(d\theta)}, \quad \forall t > 0. \tag{1.2.18}$$

Proof. Using Lemma 1.2.7 and the definition of h, we obtain

$$\begin{aligned}
p(t) &= -\frac{1}{2}\partial_x h(x,t)\Big|_{x=\psi(t)} \\
&= \frac{\psi(t)}{2\,t^{\frac{3}{2}}}\,\phi\left(\frac{\psi(t)}{\sqrt{t}}\right) - a^{-1}\int_0^\infty \frac{\psi(t)-\theta}{2\,t^{\frac{3}{2}}}\,\phi\left(\frac{\psi(t)-\theta}{\sqrt{t}}\right)F(d\theta) \\
&= \frac{1}{2\,t^{\frac{3}{2}}}\phi\left(\frac{\psi(t)}{\sqrt{t}}\right)\left[\psi(t) - a^{-1}\int_0^\infty \frac{(\psi(t)-\theta)\,\phi\left(\frac{\psi(t)-\theta}{\sqrt{t}}\right)}{\phi\left(\frac{\psi(t)}{\sqrt{t}}\right)}F(d\theta)\right] \\
&= \frac{1}{2\,t^{\frac{3}{2}}}\phi\left(\frac{\psi(t)}{\sqrt{t}}\right)\left[\psi(t) - \frac{\int_0^\infty(\psi(t)-\theta)\,\phi\left(\frac{\psi(t)-\theta}{\sqrt{t}}\right)F(d\theta)}{\int_0^\infty \phi\left(\frac{\psi(t)-\theta}{\sqrt{t}}\right)F(d\theta)}\right].
\end{aligned}$$

We also used $h(\psi(t),t)=0$ to state that

$$\frac{1}{\sqrt{t}}\,\phi\left(\frac{\psi(t)}{\sqrt{t}}\right) = a^{-1}\int_0^\infty \frac{1}{\sqrt{t}}\,\phi\left(\frac{\psi(t)-\theta}{\sqrt{t}}\right)F(d\theta), \quad \forall t>0.$$

□

1.2.3 Crucial examples

In this section, we list some applications of the method of images depending on the selected measure F. Some usual examples correspond to classical measures like the Dirac measure or some linear combinations of Dirac measures. Here we shall just present one example based on the Dirac measure and one other example with bilateral frontiers and based on the Lebesgue measure, which is of prime interest in this thesis.

Example 1.2.1. *Consider the Dirac measure F defined by $F(d\theta)=\delta_\alpha(d\theta)$. Then*

$$h(x,t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - a^{-1}\frac{1}{\sqrt{t}}\phi\left(\frac{x-2\alpha}{\sqrt{t}}\right).$$

Solving $h(x,t)=0$ leads to $\psi(t)=\alpha+bt$ with $b=\frac{\log(a)}{2\alpha}$. Using Theorem 1.2.2, we have the conditional distribution

$$\mathbb{P}(T\leqslant t \mid W_t = x) = \exp\left(-\frac{2\alpha}{t}(\alpha+bt+x)\right) \text{ when } x\leqslant\psi(t),$$

the cumulative distribution function of T:

$$\mathbb{P}(T\leqslant t) = 1-\Phi\left(\frac{bt+\alpha}{\sqrt{t}}\right) + e^{-2\alpha b}\Phi\left(\frac{bt-\alpha}{\sqrt{t}}\right), \quad \forall>0,$$

and the corresponding density function

$$p(t) = \frac{\alpha}{t^{\frac{3}{2}}}\phi\left(\frac{\psi(t)}{\sqrt{t}}\right), \quad \forall t > 0$$

which corresponds in fact to the Bachelier Lévy formula.

This first example corresponds to an important study in the litterature. That is why we have chosen to mention it. It will not play any role in the next chapters. The second example plays a crucial role for the forthcoming chapters: it represents in fact the basic component for the so-called WOMS algorithm.

Example 1.2.2. *Let us consider the measure* $F(d\theta) = \frac{d\theta}{\sqrt{2\pi}}$ *on* $\mathcal{B}(\mathbb{R})$.
Then the solution of the heat equation becomes:

$$h(x,t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{a^{-1}}{\sqrt{2\pi}}, \tag{1.2.19}$$

with the boundaries:

$$\psi_{\pm}(t) = \sqrt{t\log\left(\frac{a^2}{t}\right)} \text{ for all } 0 < t \leqslant a^2.$$

We obtain the following conditional ditribution:

$$\mathbb{P}(T \leqslant t \mid B_t = x) = \left[\frac{a}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)\right]^{-1} \text{ for any } (x,t) \text{ satisfying } \mid x \mid \leqslant \psi_+(t), \quad t < a^2.$$

1.3 Walking On Moving Spheres (WOMS) algorithm

In this section, we detail a method permitting the approximation of the Brownian exit time from an a given interval.
The idea is actually simple and mostly relies on the work of developed by Motoo [43]. The author introduced the of walk on spheres method in order to describe where the Brownian paths exit from a domain in $\mathbb{R}^n$, with $n \geq 2$. This random walk was then adapted to take also into account the time variable. The aim was actually to obtain both the exit time and the exit location from a given domain by the multi-dimensional Brownian motion. This modification permits to build an approximation procedure. Deaconu, Herrmann and Maire [15] emphasized the method which was also applied to the first passage time of the Bessel process in Deaconu, Herrmann [14]. Here we recall the method and we present the statements in the one-dimensional Brownian case. Moreover we write some new proofs describing the efficiency of the method which are based on the potential theory.
In general it is hard to exactly know the law of the exit time from a given domain. But there exist specific time-dependent domains, which are called spheroids, and for which we know precisely the law of the exit time. These spheroids are essential components of the method. In fact, instead of seeking the precise Brownian exit time from a given domain, we

shall consider the largest spheroid fully contained in that domain and centered at the starting point of the Brownian trajectory. We are able to determine the exit time and the exit position from this first spheroid. This first exit location is in fact a point of the Brownian path. Using the strong Markov property of the stochastic process, it is possible at consider this point as a new starting point of a Brownian motion. We iterate therefore the procedure by choosing a new spheroid included in the domain and centered at that point and so on. Since the spheroids are constrained to belong to the domain, the probability, at each step of the procedure to generate the point of contact between the spheroid and the domain is equal to zero. That's why we need to introduce a stopping rule: we stop the iteration as soon as an exit location of a spheroid is in a small ϵ-neighbourhood of the domain boundary.
In this chapter we detail this method, also called Walk On Moving Spheres (WOMS) algorithm, in the particular one-dimensional Brownian case.

1.3.1 Preliminaries

We already stated in Section 1.1.2 some results about the exit time density for a Brownian motion leaving an interval. As already said, the simulation method associated with this particular density expressed using a series expansion was actually not so efficient due to the use of a rejection sampling. The construction of a sequence of spheroids permits to avoid this lack of efficiency.
First we need to determine the spheroid which is the basic component of the algorithm. Using the method of images (see Section 1.2), we consider the positive measure $F(\mathrm{d}\theta) = \frac{\mathrm{d}\theta}{\sqrt{2\pi}}$ on $\mathcal{B}(\mathbb{R})$ and a constant $d > 0$. Using Theorem 1.2.2, we construct the particular solution of the heat equation

$$h(x,t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) + \frac{d^{-1}}{\sqrt{2\pi}},$$

associated with the two boundaries

$$\psi_{\pm}(t) = \pm\sqrt{t\log\left(\frac{d^2}{t}\right)} \text{ for } 0 < t \leqslant d^2. \tag{1.3.1}$$

The conditional distribution of the exit time $T = \inf\{t > 0 \,|\, |B_t| > \psi_+(t)\}$ is given by

$$\mathbb{P}(T \leqslant t \mid B_t = x) = \frac{1}{\sqrt{2\pi}}\frac{a}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)^{-1} \text{ for } (x,t) \text{ satisfying } \mid x \mid \leqslant \psi_+(t) \text{ and } t < d^2. \tag{1.3.2}$$

We focus our attention on the expression of the probability density of T.

Proposition 1.3.1. *The probability density p of the exit time T from the domain defined by $\pm\psi_+$ is given by*

$$p(t) = \frac{1}{d\sqrt{2\pi}}\frac{\sqrt{\log(\frac{d^2}{t})}}{\sqrt{t}}, \quad 0 < t \leq d^2. \tag{1.3.3}$$

Proof. We first determine the c.d.f. of the considered exit time. Using differentiation properties and (1.3.2), we deduce the announced density:

$$\begin{aligned}\mathbb{P}(T \leqslant t) &= \int_{|x|\leqslant\psi_+(t)} \mathbb{P}(T \leqslant t|B_t = x)\mathbb{P}(B_t \in dx) + 2\int_{-\infty}^{-\psi_+(t)} \mathbb{P}(B_t \in dx) \\ &= \frac{2}{d\sqrt{2\pi}}\psi_+(t) + \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\psi_+(t)}{t}} e^{\frac{-x^2}{2}}dx, \quad \forall t > 0.\end{aligned}$$

The derivative of the last expression equals

$$p(t) = \frac{1}{d\sqrt{2\pi}}\frac{\sqrt{\log(\frac{d^2}{t})}}{\sqrt{t}}, \quad \forall t > 0. \tag{1.3.4}$$

□

Another ingredient needed is explicit values of the domain extremas.

Proposition 1.3.2. *The maximum of the function ψ_+ is equal to $\frac{d}{\sqrt{e}}$ with $t = \frac{d^2}{e}$.*

Finally let us point out an easy way to generate a random variable which has the same distribution than T.

Proposition 1.3.3. *Let us introduce two independent random variables U and N: U is uniformly distributed on the interval $[0, d]$ and N is a standard Gaussian variable. Then T and $U^2e^{-N^2}$ are identically distributed.*

The proofs of these two statements are straightforward. Now we have all basic components needed in order to approximate the Brownian exit time from a given interval.

WOMS algorithm (Walk On Moving Spheres)

Let $a < 0 < b$ the two boundaries of the interval under study and let us introduce some small parameter $\epsilon > 0$. The next algorithm generates an approximation of the exit time of the interval $[a, b]$ for the Brownian motion.

ALGORITHM $(\text{WOMS})_{a,b,\epsilon}$

Initialization: Set $Z = 0$, $\tilde{a} = a$ and $\tilde{b} = b$.

While $\mathbf{0 \leqslant \tilde{b} - \epsilon}$ and $\mathbf{0 \geqslant \tilde{a} + \epsilon}$

Step 1. Put $c = \min(|\tilde{a}|, |\tilde{b}|)$

Step 2. Generate the Brownian exit time τ from the spheroid defined by (1.3.1) with coefficient $d = c\sqrt{e}$.

Step 3. Generate X the exit position at time τ from the spheroid.

Step 4. $Z \leftarrow Z + \tau$
$\tilde{a} \leftarrow \tilde{a} - X$,
$\tilde{b} \leftarrow \tilde{b} - X$,

Outcome: The random variable Z.

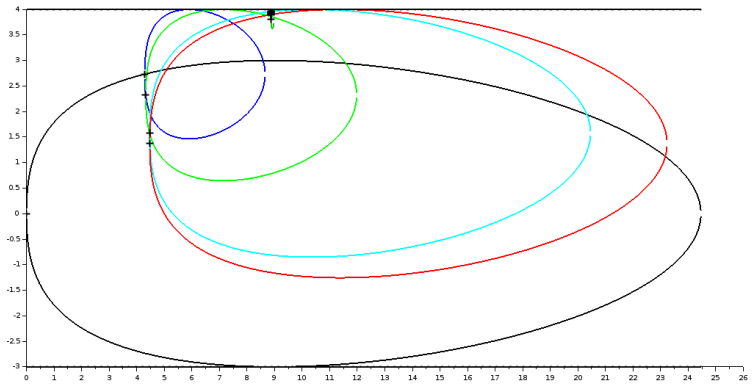


Figure 1.2: Spheroids produced with one run of Algorithm $(WOMS)_{a,b,\epsilon}$

The outcome Z of algorithm $(\text{WOMS})_{a,b,\epsilon}$ is an approximated value of the Brownian exit time T. Some questions arise:

- Is this algorithm efficient? How many steps should one observe before the algorithm stops?
- What is the rate of convergence of Algorithm $(\text{WOMS})_{a,b,\epsilon}$?

1.3.2 Study of the WOMS algorithm efficiency

In order to treat the first question, we need to introduce a theorem related to the potential theory in probability theory. Additional explanations concerning this theorem and its link with the optional stopping theorem can be found in Section 1.4.

Potential theory

In this section, let us consider a Markov chain $(X_n)_{n\geq 0}$ defined on a space state I which is divided into two parts D and ∂D, ∂D being called the frontier of the space state.

Theorem 1.3.4. *Let c and f two positive functions defined on D, respectively on ∂D. We define*

$$\phi(i) = \mathbb{E}_i \left[\sum_{n<T_X} c(X_n) + f(X_{T_X}) 1_{T<\infty} \right], \quad \forall i \in I$$

where $T_X = \inf\{n \geq 0 | X_n \in \partial D\}$ represent the hitting time of ∂D for the Markov chain. Then, defining the operator P by $P_x f = \mathbb{E}_x[f(X_1)] = \mathbb{E}[f(X_1)|X_0 = x]$, we observe that

i) The potential function ϕ satisfies

$$\begin{cases} \phi = P\phi + c \text{ on } D \\ \phi = f \text{ on } \partial D \end{cases} \tag{1.3.5}$$

ii) If the function ζ satisfies

$$\begin{cases} \zeta \geqslant P\zeta + c \text{ on } D \\ \zeta \geqslant f \text{ on } \partial D \end{cases} \tag{1.3.6}$$

and if we have $\zeta(i) \geqslant 0$ for all $i \in I$ then $\zeta(i) \geqslant \phi(i)$ for all $i \in I$.

Description of the number of steps

Since Algorithm $(\text{WOMS})_{a,b,\epsilon}$ is constructed as a Markov chain, we aim to apply the potential theory in order to describe the number of steps. Let us mention that the particular choice $f \equiv 0$ and $c \equiv 1$ in the second point of Theorem 1.3.4 permits to obtain such information: the average number of steps $\mathbb{E}[T_X]$ is then upper bounded.
Hence, we introduce a function U^ϵ satisfying

$$U^\epsilon \geqslant PU^\epsilon + 1 \text{ on }]a+\epsilon, b-\epsilon[\text{ and } U^\epsilon \geqslant 0 \text{ on } [a+\epsilon, b-\epsilon]. \tag{1.3.7}$$

In order to find such a solution to the inequalities, we first describe the operator P:

$$P_x f = \frac{1}{2}\int_0^{c_x^2} (f(x+\psi_+(t)) + f(x-\psi_+(t))p(t)dt,$$

where $c_x = \sqrt{e}\min(x-a, b-x)$ and the probability density p is defined in (1.3.4) with $d = c_x$.
Let us now point out the efficiency of the algorithm.

Theorem 1.3.5. *1) The function U_ϵ defined by*

$$U_\epsilon(x) = C\log\left(\frac{(x-a)(b-x)}{\epsilon(b-a-\epsilon)}\right), \quad \forall x \in]a,b[,$$

satisfies $U_\epsilon(a+\epsilon) = U_\epsilon(b-\epsilon) = 0$ and $U_\epsilon(x) \geqslant 0$ for all $x \in [a+\epsilon, b-\epsilon]$.
Moreover, there exists $C > 0$ such that

$$P_x U_\epsilon \leqslant U_\epsilon(x) - 1 \quad \forall x \in]a+\epsilon, b-\epsilon[.$$

2) Let N_ϵ be the number of iterations observed in Algorithm $(WOMS)_{a,b,\epsilon}$ before it stops. Then

$$\mathbb{E}[N_\epsilon] \leqslant C|\log(\epsilon)|, \quad \forall \epsilon > 0.$$

Proof. Let us prove that the function U_ϵ satisfies all conditions.
By definition we have $U_\epsilon(a+\epsilon) = U_\epsilon(b-\epsilon) = 0$. Moreover the dominating coefficient in the polynomial function is negative, which implies that U_ϵ is non negative on the interval $]a+\epsilon, b-\epsilon[$.
For the last property, we just study the successive derivatives of the function U_ϵ:

$$U_\epsilon'(x) = \frac{C}{x-a} - \frac{C}{b-x}, \quad \forall x \in]a,b[,$$

and, for any $n \geqslant 2$,

$$U_\epsilon^{(n)}(x) = -C(n-1)! \left(\frac{(-1)^n}{(x-a)^n} + \frac{1}{(b-x)^n} \right), \quad \forall x \in]a,b[.$$

Using Taylor's expansion leads to

$$U_\epsilon(x+h) \leqslant U_\epsilon(x) + hU'_\epsilon(x) + \frac{h^2}{2} \sup_{|y-x| \leq h} U''_\epsilon(y). \tag{1.3.8}$$

We shall apply this inequality to the particular values $h = \psi_+(t)$ and $h = -\psi_+(t)$. For any $t \in \mathrm{supp}(\psi_+)$, Proposition 1.3.2 implies that

$$|\psi_+(t)| \leq \frac{c_x}{\sqrt{e}} = \min(x-a, b-x).$$

We deduce that

$$\sup_{|y-x| \leq \psi_+(t)} U''_\epsilon(y) \leq -C \inf_{|y-x| \leq c_x/\sqrt{e}} \left\{ \frac{1}{(y-a)^2} + \frac{1}{(b-y)^2} \right\} \leq -\frac{Ce}{4c_x^2}.$$

Hence

$$\begin{aligned} P_x U_\epsilon &= \frac{1}{2} \int_0^{c_x^2} (U_\epsilon(x+\psi_+(t) + U_\epsilon(x-\psi_+(t)) p(t) dt \\ &\leqslant \frac{1}{2} \int_0^{c_x^2} \Big(2U_\epsilon(x) + \psi_+(t) U'_\epsilon(x) - \psi_+(t) U'_\epsilon(x) + \psi_+^2(t) \sup_{|y-x| \leq \psi_+(t)} U''_\epsilon(y) \Big) p(t) dt \\ &\leqslant U_\epsilon(x) - \frac{Ce}{8c_x^2} \int_0^{c_x^2} \psi_+^2(t) p(t) dt, \quad \forall x \in]a+\epsilon, b-\epsilon[. \end{aligned}$$

Moreover

$$\int_0^{c_x^2} \psi_+^2(t) p(t) dt = \int_0^{c_x^2} \frac{1}{c_x \sqrt{2\pi}} \, t \log\left(\frac{c_x^2}{t} \right) \sqrt{\frac{\log(\frac{c_x^2}{t})}{t}} dt,$$

and setting $u = \frac{t}{c_x^2}$ implies

$$\int_0^{c_x^2} \psi_+^2(t) p(t) dt = \frac{c_x^2}{\sqrt{2\pi}} \int_0^1 u \log\left(\frac{1}{u} \right) \frac{\sqrt{\log(\frac{1}{u})}}{\sqrt{u}} du =: 2c_x^2 \Gamma.$$

Finally $P_x U_\epsilon \leq U_\epsilon(x) - \frac{Ce\Gamma}{4}$ and it suffices therefore to choose $C = 4/(e\Gamma)$ in order to obtain the inequality of Theorem 1.3.5. The second result announced in the statement of Theorem 1.3.5 concerns the number of iterations N_ϵ in Algorithm $(\mathrm{WOMS})_{a,b,\epsilon}$. We note that, by the definition of ϕ, we have $\mathbb{E}_x[N_\epsilon] = \phi(x)$. Moreover using the second part $ii)$ of Theorem 1.3.4 applied to U_ϵ, we obtain: $\mathbb{E}_x[N_\epsilon] = \phi(x) \leq U_\epsilon(x)$. □

Let us mention that Binder and Braverman [6] obtained similar upper-bounds for the Walk on Spheres algorithm: the average number of iterations depends on a logarithmic way on the stopping parameter $\epsilon > 0$. Their proof is based on the martingale theory with the use of some Riesz potential.

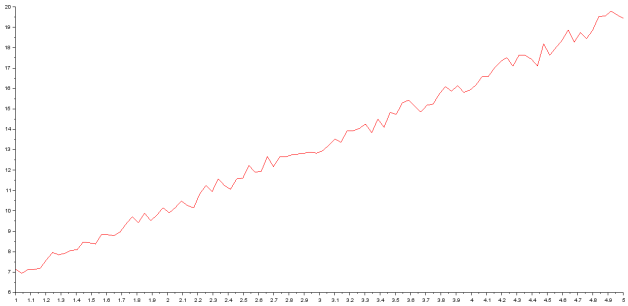


Figure 1.3: Average number of steps (in logarithmic scale) versus the stopping parameter ϵ.

Bounds for the Exit-Time distribution

We focus now our attention on bounds for the difference between the first exit time T and its approximation Z obtained by Algorithm $(\text{WOMS})_{a,b,\epsilon}$. We aim to understand how this error depends on the stopping procedure of the algorithm.

Theorem 1.3.6. *Let $\epsilon > 0$. We denote by Z the outcome of Algorithm $(WOMS)_{a,b,\epsilon}$ with cumulative distribution function F_ϵ and we denote by T the Brownian exit time from the interval $[a,b]$ with F, its cumulative distribution function.*
Then, for all $\alpha > 0$ small enough, we have

$$\Big(1 - \frac{2\epsilon}{\sqrt{2\alpha\pi}}\Big) F_\epsilon(t-\alpha) \leqslant F(t) \leqslant F_\epsilon(t), \quad \forall t > 0.$$

Proof. Algorithm $(\text{WOMS})_{a,b,\epsilon}$ is in fact based on a stopped random walk on spheroids. All values of this random walk correspond to exit locations of the Brownian motion from spheroids included in $\mathbb{R}_+ \times [a,b]$. In other words, each value of the random walk belongs to the path of a Brownian motion. When the algorithm stops, the location is ϵ-close to the boundary. We deduce then that

$$T_\epsilon \underset{s.t.}{\leq} Z \underset{s.t.}{\leq} T$$

where T_ϵ is the Brownian exit time from the interval $[a+\epsilon, b-\epsilon]$. Let us just explain the notation: two random variables A and B satisfy $A \underset{s.t.}{\leq} B$ iff $\mathbb{P}(A \leq t) \geq \mathbb{P}(B \leq t)$, $\forall t \in \mathbb{R}$.

So we immediatly deduce that $\tilde{F}_\epsilon(t) := \mathbb{P}(T_\epsilon \leq t) \geq F_\epsilon(t) \geq F(t)$ for all $t \in \mathbb{R}$. Moreover

$$\begin{aligned} 1 - F(t) &= \mathbb{P}(T > t) = \mathbb{P}(T > t, T_\epsilon \leqslant t-\alpha) + \mathbb{P}(T > t, T_\epsilon > t-\alpha) \\ &\leqslant \mathbb{P}(T > t, T_\epsilon \leqslant t-\alpha) + \mathbb{P}(T_\epsilon > t-\alpha) \\ &\leqslant \mathbb{P}(T > t, T_\epsilon \leqslant t-\alpha) + 1 - \tilde{F}_\epsilon(t-\alpha). \end{aligned}$$

We focus our attention on the first probability of the right hand side. Using the Markovian property, we obtain

$$\begin{aligned} \mathbb{P}(T > t, T_\epsilon \leqslant t-\alpha) &= \mathbb{P}(T > t | T_\epsilon \leqslant t-\alpha)\mathbb{P}(T_\epsilon \leqslant t-\alpha) \\ &\leqslant \mathbb{P}(T_\epsilon \leqslant t-\alpha) \sup_{y \in \{a+\epsilon, b-\epsilon\}} \mathbb{P}_y(T > \alpha). \end{aligned}$$

Using paths properties of the Brownian motion (see for instance Proposition 2.8.1 in [36]), we get for $y \in [a+\epsilon, b-\epsilon]$,

$$\mathbb{P}_y(T > \alpha) = \mathbb{P}_0\Big(\sup_{0 \leqslant t \leqslant \alpha} B_t < \epsilon\Big) = \mathbb{P}(|B_\alpha| < \epsilon) \leqslant \frac{2\epsilon}{\sqrt{2\alpha\pi}}.$$

We deduce that:

$$F(t) \geq \tilde{F}_\epsilon(t-\alpha)\left(1 - \frac{2\epsilon}{\sqrt{2\alpha\pi}}\right) \geq F_\epsilon(t-\alpha)\left(1 - \frac{2\epsilon}{\sqrt{2\alpha\pi}}\right).$$

□

1.3.3 Comparison with other numerical methods

Finally we want to compare the WOMS algorithm to other algorithms in order to compare their efficiency. We first compare with the classical Euler scheme and secondly with a modification of the Walk On Spheres algorithm.

Comparison with the classical Euler scheme

Let $h > 0$ the step size of the Euler scheme. We consider the following algorithm:

EULER SCHEME $(\mathrm{E})_h$

Initialization Let us put $X = 0$, $\mathcal{T} = 0$.
While a < X < b
Step 1. $X \leftarrow X + N$ where $N \sim \mathcal{N}(0, h)$
Step 2. $\mathcal{T} \leftarrow \mathcal{T} + h$
Outcome: The random variable $\mathcal{T}$.

The main result concerning the Euler scheme is that the outcome of the algorithm is an approximation of the Brownian exit time from the interval $[a, b]$. The Euler scheme presented here is actually simple and rather precise. Of course it does not take into account the opportunity for the Brownian motion to hit the boundaries inbetween two time steps and to return in the domain at the next time step. In the next chapter, we shall use a more sophisticated Euler algorithm which takes into account the probability of the Brownian paths to hit the boundaries inbetween two time steps. Here, since we do not use these probabilities, we only generate an upper bound of the studied exit time. But for h small enough the approximation is rather appropriate.

The aim is to to compare in details these two generation methods which are so different. We want just to point out that the Euler scheme is certainly appropriate to describe precisely the whole paths of the Brownian motion but if one just needs to obtain an exit time, then the energy spent to generate the paths inside the interval seems to be wasted. The WOMS algorithm focuses only on the exit problem. In order to illustrate this feature, we propose a

ϵ	5.0×10^{-1}	10^{-1}	5.0×10^{-2}	10^{-2}
WOMS	10,65	11,44	12,14	12,21
Euler	56,51	1202,85	4802,092	124971,91

Table 1.1: Comparison of the averaged number of steps

rough comparison: we compare the average number of iterations of the Euler Scheme $(\mathrm{E})_\epsilon$ with the number of iterations of Algorithm $(\mathrm{WOMS})_{a,b,\epsilon}$ (See Table 1.1, the averages concern samples of size 1 000). We easily remark that the WOMS algorithm is faster than the Euler scheme.

Comparison with a modified WOMS algorithm

In Algorithm $(\mathrm{WOMS})_{a,b,\epsilon}$, the aim is to build a random walk on spheroids whose size is the largest possible contained in $\mathbb{R} \times [a, b]$. Of course, this means that we need at each step to compute the distance between the value of the random walk and the boundaries. It is not so time consuming: we aim to modify the algorithm in a comparison purpose. So we introduce a walk on spheroids whose size does not depend on the distance to the boundary but remains fixed and of order ϵ. A realization of one path is represented in Figure 1.4.

ALGORITHM $(\mathrm{WOMS})_{a,b,\epsilon}$ modified

Initialization Let $\mathcal{T} = 0$, $\tilde{a} = a$ and $\tilde{b} = b$.

While $\mathbf{0 \leqslant \tilde{b} - \epsilon}$ and $\mathbf{0 \geqslant \tilde{a} + \epsilon}$

Step 1. Generate the Brownian exit time τ from the spheroid with coefficient $\epsilon\sqrt{e}$.

Step 2. Generate X the exit position at time τ.

Step 3. $\mathcal{T} \leftarrow \mathcal{T} + \tau$
$\tilde{a} \leftarrow \tilde{a} - X$,
$\tilde{b} \leftarrow \tilde{b} - X$,

Outcome: The random variable $\mathcal{T}$ representing an approximated exit time.

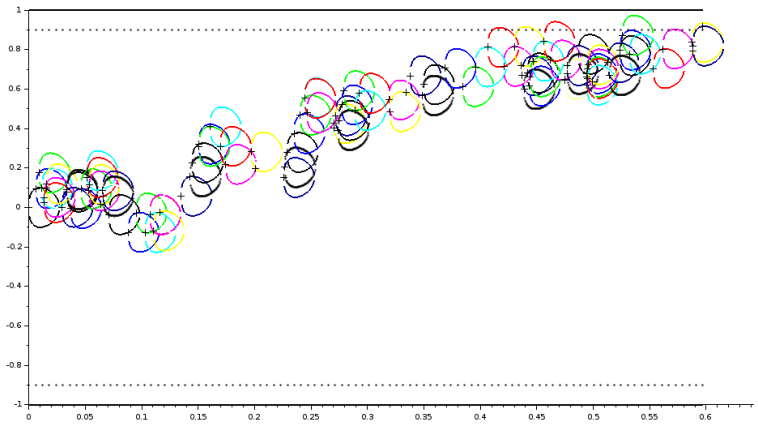


Figure 1.4: Representation of the sequence of spheroids generated by the modified WOMS algorithm for one generation of the exit time.

We are able to compare the efficiency of these two algorithms. Table 1.2 presents a numerical comparison between the mean number of steps observed in Algorithm $(\text{WOMS})_{a,b,\epsilon}$ and in the modified algorithm for different values of ϵ (each average concerns a sample of size 1 000).

ϵ	5.0×10^{-1}	10^{-1}	5.0×10^{-2}	10^{-2}
WOMS	10,65	11,44	12,14	12,21
Modified algorithm	80,7	9067	2054,38	153656

Table 1.2: Averaged number of steps in Algorithm $(\text{WOMS})_{a,b,\epsilon}$ and its modification

Since both the WOMS and its modification are based on the same theoretical arguments, the proofs concerning the bounds for the exit time distribution are similar. This implies that the error observed are of the same order. The only difference is the consumption time of the algorithms: the WOMS which uses at each iteration the largest spheroid included in the interval $[a, b]$ appears to be more efficient.

1.4 Discussion of the link between potential theory and martingales.

In this section, we present a particular link between the potential theory and the optional stopping theorem. These arguments permit to understand Theorem 1.3.4 which is a crucial tool for the description of the algorithm efficiency.
The aim here is to relate the solution of a particular equation to a martingale. Then we shall apply the optional stopping theorem to this martingale in order to obtain essential information on the initial equation. Moreover we extend this result to a suitable supermartingale and obtain by that way a potential theory comparison theorem.

Reminder of the problem

The potential theory is often applied in the framework of continuous stochastic processes. For the anlysis of stochastic algorithm, we consider discrete problems and use therfore discret time Markov chains.
We recall the main result we uses in Section 1.3.2 (this result can also be found in Norris p 139 [45]). Let us consider a Markov chain $(X_n)_{n\geq 0}$ defined on a space state I divided into two disjoint domains D and ∂D, ∂D being called the frontier.

Theorem 1.4.1. *Let us consider c and f two positive functions defined on D, respectively on ∂D. We define $T_X = \inf\{n \geq 0 | X_n \in \partial D\}$ the first hitting time of ∂D for the Markov chain*

$$\phi(i) = \mathbb{E}_i \left[\sum_{n<T_X} c(X_n) + f(X_{T_X}) 1_{T<\infty} \right], \quad \forall i \in I \tag{1.4.1}$$

Then, defining the operator P by $P_x f = \mathbb{E}_x[f(X_1)] = \mathbb{E}[f(X_1)|X_0 = x]$, we obtain:

i) The potential function ϕ satisfies

$$\begin{cases} \phi = P\phi + c \text{ on } D, \\ \phi = f \text{ on } \partial D. \end{cases} \tag{1.4.2}$$

ii) If the function ζ satisfies

$$\begin{cases} \zeta \geqslant P\zeta + c \text{ on } D, \\ \zeta \geqslant f \text{ on } \partial D, \end{cases} \tag{1.4.3}$$

and if we have $\zeta(i) \geqslant 0$ for all $i \in I$, then $\zeta(i) \geqslant \phi(i)$ for all $i \in I$.

This theorem permits to obtain the average number of steps needed by the Markov chain to reach the frontier, or at least an upper bound of this random number. To that end it suffices to to consider $c \equiv 1$ and $f \equiv 0$.
In the sequel, T_X denotes the hitting time of the considered frontier. We assume that T_X is an almost surely finite stopping time associate to the filtration of the considered Markov chain $(\mathcal{F}_n)_{n\geq 0}$.

The equality case

Let us consider U a solution of the system

$$\begin{cases} U = PU + 1 \text{ on } D, \\ U = 0 \text{ on } \partial D. \end{cases} \tag{1.4.4}$$

We suppose that U is continuous on I and bounded. Then we can first observe that the sequence $(Y_n)_{n\geq 0}$ defined by $Y_n = U(X_n)$ is not a martingale: it suffices to note that for $X_n \in D$, we have

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \mathbb{E}[U(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[U(X_n)|\mathcal{F}_n] - 1 \\ &= U(X_n) - 1 = Y_n - 1 \quad \text{a.s.}\end{aligned}$$

However, it is still possible to define a martingale using the function U.

Lemma 1.4.2. *The sequence $(Z_{n\wedge T_X})_{n\geq 0}$ defined by $Z_{n\wedge T_X} = U(X_{n\wedge T_X}) + n \wedge T_X$ is a martingale under the assumption that $(Z_{n\wedge T_X})_{n\geq 0}$ is integrable.*

Proof. The sequence $(Z_{n\wedge T_X})_{n\geq 0}$ is obviously $(\mathcal{F}_n)_{n\geq 0}$-measurable. Moreover, on the event $\{X_{n\wedge T_X} \in D\}$, we have

$$\begin{aligned}\mathbb{E}[Z_{(n+1)\wedge T_X}|\mathcal{F}_n] &= \mathbb{E}[U(X_{(n+1)\wedge T_X})|\mathcal{F}_n] + ((n+1)\wedge T_X) \\ &= \mathbb{E}[U(X_n)|\mathcal{F}_n] - 1 + (n+1) = U(X_n) + n = Z_n = Z_{n\wedge T_X} \text{ a.s.}\end{aligned}$$

On the complementary event $\{X_{n\wedge T_X} \in \partial D\}$, we also obtain:

$$\mathbb{E}[Z_{(n+1)\wedge T_X}|\mathcal{F}_n] = \mathbb{E}[U(X_{(n+1)\wedge T_X})|\mathcal{F}_n] + ((n+1)\wedge T_X) = T_X = Z_{n\wedge T_X} \quad \text{a.s.}$$

□

Let us now apply the optional stopping theorem to the martingale $(Z_{n\wedge T_X})_{n\geq 0}$. We obtain

$$\mathbb{E}[U(X_0)] = \mathbb{E}[Z_0] = \mathbb{E}[Z_{n\wedge T_X}] = \mathbb{E}[U(X_{n\wedge T_X})] + \mathbb{E}[n \wedge T_X], \quad \forall n \geq 0.$$

Consequently

$$\mathbb{E}[n \wedge T_X] = -\mathbb{E}[U(X_{n\wedge T_X})] + \mathbb{E}[U(X_0)], \quad \forall n \geq 0.$$

Letting n tend to infinity leads to

$$\mathbb{E}[T_X] = -\mathbb{E}[U(X_{T_X})] + \mathbb{E}[U(X_0)]. \tag{1.4.5}$$

Indeed, due to the assumptions that $T_X < +\infty$ a.s and U continuous and bounded on I, $X_{T_X\wedge n} \underset{n\to+\infty}{\to} X_{T_X}$ and $U(X_{T_X\wedge n}) \underset{n\to+\infty}{\to} U(X_{T_X})$. Using the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \mathbb{E}[U(X_{n\wedge T_X})] = \mathbb{E}[U(X_{T_X})].$$

The sequence $(n \wedge T_X)_{n\geq 0}$ is increasing and almost surely converges to the finite limit T_X. Using monotone convergence theorem, we have $\lim_{n\to+\infty} \mathbb{E}[n \wedge T_X] = \mathbb{E}[T_X]$. Combining these two results leads to the statement (1.4.5). Such a result permits, when the starting and the exit points are known, to obtain the average time needed by the Markov chain in order to reach the frontier.

The inequality case

We now consider the function V satisfying the system of inequalities:

$$\begin{cases} V \geqslant PV + 1 \text{ on } D, \\ \qquad V \geqslant 0 \text{ on } \partial D. \end{cases}$$

We recall that T_X is the hitting time of the frontier for the Markov chain $(X_n)_{n\geq 0}$. We define $Y_{n\wedge T_X} = V(X_{n\wedge T_X}) + n \wedge T_X$, $\forall n \geq 0$. Under suitable conditions, we deduce that the sequence $(Y_{n\wedge T_X})_{n\geq 0}$ is a supermartingale. In a similar way as above, we use the optional stopping theorem to the stopped supermartingale $(Y_{n\wedge T_X})_{n\geq 0}$ and obtain the following upperbound:

$$\mathbb{E}[T_X] \leqslant \mathbb{E}[V(X_0)] - \mathbb{E}[V(X_{T_X})].$$

Generalization

Now let us take any functions f and c. We focus our attention on the function ϕ defined by (1.4.2) and introduce the sequence $(M_n)_{n\geq 0}$ as follows:

$$M_n = \sum_{k=0}^{n-1} c(X_k) 1_{k<T_X} + f(X_T) 1_{T_X<n} + \phi(X_n) 1_{n\leqslant T_X}, \quad \forall n \geq 0.$$

If we assume that $(M_n)_{n\geq 0}$ is a sequence of integrable random variables, then $(M_n)_{n\geq 0}$ is a martingale. Indeed $(M_n)_{n\geq 0}$ is $(\mathcal{F}_n)$-adapted and satisfies

$$\begin{aligned} \mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\sum_{k=0}^{n} c(X_k) 1_{k<T_X} + f(X_{T_X}) 1_{T_X<n+1} + \phi(X_{n+1}) 1_{n+1\leqslant T_X} \,\middle|\, \mathcal{F}_n \right] \\ &= \sum_{k=0}^{n-1} c(X_k) 1_{k<T_X} + c(X_n) 1_{n<T_X} + f(X_{T_X}) 1_{T_X<n} + f(X_n) 1_{T_X=n} + P\phi(X_n) 1_{T_X>n} \\ &= \sum_{k=0}^{n-1} c(X_k) 1_{k<T_X} + + f(X_{T_X}) 1_{T_X<n} + (P\phi + c)(X_n) 1_{T_X>n} + f(X_n) 1_{T_X=n} \\ &= M_n \text{ a.s.} \end{aligned}$$

Applying the optional stopping theorem to the considered martingale, we obtain

$$\mathbb{E}[M_0] = \mathbb{E}[\phi(X_0)] = \phi(x) = \mathbb{E}\left[\sum_{k<T_X} c(X_k) + f(X_{T_X}) 1_{\{T_X<\infty\}} \right].$$

That means that we identify the solution of the system (1.4.2) with ϕ defined in (1.4.1) Once again, from this equality case, we can deduce an extended result concerning the system of inequalities. It suffices to start with ζ a solution of (1.4.3) and to construct in a similar way, as just described, a supermartingale. It then permits to find out interesting upper-bounds of the average exit time for the Markov chain.

Chapter 2

WOMS algorithm extended to Ornstein-Uhlenbeck processes

In this chapter, we focus on the WOMS extension to some other diffusions. We take here the particular case of Ornstein-Uhlenbeck processes. We use the term extension instead of adaptation because the new algorithm we produce will strongly rely on Brownian motion case. In fact, we are able to enlighten a strong relation between the Ornstein-Uhlenbeck processes and the Brownian motion. In this context, we are able to determine some generalized spheroids adapted to the Ornstein-Uhlenbeck processes. Given these spheroids, it is possible to state a new algorithm generating the exit time from an interval $[a, b]$ for Ornstein-Uhlenbeck processes. The strong relation observed also permit the description of the error and emphasize the efficiency of the algorithm.

The results presented in this chapter are subject to a publication in Discrete and Continuous Dynamical Systems. Series B. A Journal Bridging Mathematics and Sciences [26]

2.1 The Ornstein-Uhlenbeck processes

Let us first recall the definition of the Ornstein-Uhlenbeck process and present different essential properties which permit to link this diffusion to a standard Brownian motion.
Let $\theta \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$. The Ornstein-Uhlenbeck process (O.U.) starting in x_0 with parameters θ, μ, and σ is the unique solution of the following stochastic differential equation (SDE):

$$dX_t = -\theta(X_t - \mu)dt + \sigma dW_t, \quad t \geq 0, \tag{2.1.1}$$

where W stands for a standard one-dimensional Brownian motion. Existence and uniqueness for equation (2.1.1) can be easily deduced from a general statement concerning SDE, see for instance Revuz, Yor, Chap. IX [53]. Let us just recall this result.

Proposition 2.1.1. *Consider the following stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq 0. \tag{2.1.2}$$

If there exists a Borel function $\rho :]0, +\infty[\to]0, +\infty[$ *satisfying* $\int_0^{+\infty} \frac{dx}{\rho(x)} = +\infty$ *and such that*

$$|\sigma(s,x) - \sigma(s,y)|^2 \leqslant \rho(|x-y|), \quad \forall x, y \in]0, +\infty[, \quad \forall t \in \mathbb{R}^+.$$

and if, for each compact set H *and each* $t \geqslant 0$*, there exists a constant* $K_t > 0$ *such that*

$$|b(s,x) - b(s,y)| \leqslant K_t |x-y|, \quad \forall x, y \in H, \quad s \leqslant t$$

then pathwise uniqueness holds for equation (2.1.2)*.*

Since obviously the drift and diffusion coefficients of the O.U. process satisfy the hypotheses of Proposition 2.1.1, pathwise uniqueness holds for (2.1.1). Let us now present an explicit expression of this solution. The Ornstein-Uhlenbeck process can be written as a stochastic integral with respect to the Brownian motion:

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s, \quad t \geqslant 0. \tag{2.1.3}$$

Levy's theorem permits to replace the stochastic integral by a time-changed Brownian motion. We obtain therefore another expression for the process which is more handy to manipulate.
Since $\theta > 0$, there exists a standard Brownian motion $(V_t)_{t \geq 0}$ such that

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma e^{-\theta t}}{\sqrt{2\theta}} V_{e^{2\theta t} - 1}. \tag{2.1.4}$$

This simplified expression is a crucial tool for the construction of the algorithm in the exit problem framework as it clearly appears in the forthcoming statements.

Remark 2.1.2. *In following computations, we put* $\mu = 0$*. This restriction is only motivated by notational simplification and the study can easily be extended to the general case.*

Let us now describe how such a strong relation between the Brownian motion and the Ornstein-Uhlenbeck process permits to emphasize a time-dependent domain of $\mathbb{R}$ whose exit time can be easily and exactly simulated.

2.2 Exit time of generalized spheroids

Let us consider the spheroids defined by the boundaries $\psi^{\pm}(t)$ in (1.3.1). We recall that the Brownian exit problem of a such a spheroid is completely explicit, so that the simulation of the exit time τ is rather simple. Due to the symmetry property of the spheroid, the conditional probability distribution of the exit location W_τ given τ is equal to $\frac{1}{2}\,\delta_{\psi^+(\tau)} + \frac{1}{2}\,\delta_{\psi^-(\tau)}$. For the Ornstein-Uhlenbeck process, we can obtain some similar information due to the strong relation with the Brownian motion.
Let us introduce two new boundaries defined by:

$$\psi^{\pm}_{\mathrm{OU}}(t,x) = e^{-\theta t}\left(\frac{\sigma}{\sqrt{2\theta}}\psi_{\pm}(e^{2\theta t} - 1) + x\right),$$

where θ and σ correspond to the parameters of the O.U-process $(X_t)_{t\geq 0}$ in (2.1.1). We call *generalized spheroid* the domain defined by these boundaries.
We introduce the exit time $\tau_{\mathrm{OU}} = \inf\{t > 0 :\ X_t \notin [\psi^-_{\mathrm{OU}}(t,x), \psi^+_{\mathrm{OU}}(t,x)]\}$.

Proposition 2.2.1. *Let $\tau = \inf\{t > 0 :\ V_t \notin [\psi_-(t), \psi_+(t)]\}$ the first time the Brownian motion $(V_t)_{t\geq 0}$ defined in (2.1.4) exits from the spheroid. Then the exit time τ_{OU} satisfies:*

$$\tau_{\mathrm{OU}} = \frac{\log(\tau+1)}{2\theta} \ a.s. \tag{2.2.1}$$

Proof. Using both the definition of τ_{OU} and the expression of X_t with respect to the Brownian motion V_t, we obtain

$$\begin{aligned}
\tau_{\mathrm{OU}} &= \inf\left\{t > 0 :\ X_t \notin [\psi^-_{\mathrm{OU}}(t,x), \psi^+_{\mathrm{OU}}(t,x)]\right\} \\
&= \inf\left\{t > 0 :\ xe^{-\theta t} + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} V_{e^{2\theta t}-1} \notin [\psi^-_{\mathrm{OU}}(t,x), \psi^+_{\mathrm{OU}}(t,x)]\right\} \\
&= \inf\left\{t > 0 : \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} V_{e^{2\theta t}-1} \notin [\psi^-_{\mathrm{OU}}(t,x) - xe^{-\theta t}, \psi^+_{\mathrm{OU}}(t,x) - xe^{-\theta t}]\right\} \\
&= \inf\left\{\frac{\log(u+1)}{2\theta} > 0 :\ V_u \notin [\psi_-(u), \psi_+(u)]\right\} = \frac{\log(\tau+1)}{2\theta}.
\end{aligned}$$

□

This statement is a crucial tool for simulation purposes. It permits first to simulate a Brownian exit time from a spheroid, then to use Proposition 2.2.1 to obtain the O.U. exit time from the generalized spheroid. Let us notice that the shape of the generalized spheroid depends on the O.U. starting position. Therefore, if we define a WOMS, the shape of the spheroids will change at each step of the algorithm. In the Brownian motion context, the spheroids are symmetric and their extremas can be computed easily. This important advantage permits to compute easily the maximal size of the spheroids included in the interval $[a,b]$ and is not fulfilled in the O.U. case. It is therefore an harder work to determine the optimal size of the generalized spheroid. This can be achieved by finding an upper-bound for the upper boundary and a lower-bound for the lower boundary. As a consequence, we determine a parameter characterizing the generalized spheroid which guaranties that it remains fully contained in the interval $[a,b]$. Since the bounds are quite rough, the boundaries of the generalized spheroid are unfortunately not tangent to the interval bounds. The algorithm shall be therefore a little slowed down.

Proposition 2.2.2. *Let $\gamma > 0$, and $x \in [a,b]$ the starting point of the spheroid, that is $\psi^{\pm}_{OU}(0,x) = x$. Let us set $a_{\gamma,x} = a + \gamma(x-a)$ and $b_{\gamma,x} = b - \gamma(b-x)$. We define*

$$d = \begin{cases} \sqrt{2\theta e}\min\left(\frac{(b_{\gamma,x}-x)}{\sigma}, \frac{2(x-a_{\gamma,x})}{\sqrt{\sigma^2+4\theta e\, x(x-a_{\gamma,x})}+\sigma}\right) & \text{if } x \geqslant 0 \\ \sqrt{2\theta e}\min\left(\frac{(x-a_{\gamma,x})}{\sigma}, \frac{2(b_{\gamma,x}-x)}{\sqrt{\sigma^2-4\theta e\, x(b_{\gamma,x}-x)}+\sigma}\right) & \text{if } x \leqslant 0 \end{cases} \tag{2.2.2}$$

For such a choice of parameter, the generalized spheroid is fully contained in the interval $[a_{\gamma,x}, b_{\gamma,x}]$.

In the following statements we denote by d_x the parameter associated to the spheroid with initial point x.

Proof. Let us first consider the case: $x > 0$. Combining the upper bound of the function ψ^+ presented in Proposition 1.3.2 and the definition of ψ_{OU}, we obtain

$$-\frac{\sigma d}{\sqrt{2\theta e}} + \frac{x}{\sqrt{1+d^2}} \leqslant \psi^-_{\mathrm{OU}}(t,x) \leqslant \psi^+_{\mathrm{OU}}(t,x) \leqslant \frac{\sigma d}{\sqrt{2\theta e}} + x. \tag{2.2.3}$$

We keep the upper bound found previously and focus on the lower bound:

$$\psi^-_{\mathrm{OU}}(t,x) \geqslant -\frac{\sigma d}{\sqrt{2\theta e}} + \frac{x}{\sqrt{1+d^2}} \geqslant -\frac{\sigma d}{\sqrt{2\theta e}} + x(1 - \frac{d^2}{2}). \tag{2.2.4}$$

The determination of a convenient choice for the parameter $d > 0$ requires to find the positive solution of the equation $P(d) = 0$ where

$$P(d) = x\frac{d^2}{2} + \frac{\sigma}{\sqrt{2\theta e}} d + (a_{\gamma,x} - x).$$

Consequently we obtain

$$d_l = \frac{1}{x}\sqrt{\frac{\sigma^2}{2\theta e} + 2x(x - a_{\gamma,x})} - \frac{\sigma}{x\sqrt{2\theta e}}.$$

The identification with the upper bound gives us

$$d_u = (b_{\gamma,x} - x)\frac{\sqrt{2\theta e}}{\sigma}. \tag{2.2.5}$$

Hence setting $d = \min(d_u, d_l)$ permits the generalized spheroid to belong to the interval $[a_{\gamma,x}, b_{\gamma,x}]$.

The case $x < 0$ uses similar arguments since we observe a symmetry with respect to the origin between the generalized spheroid starting in x and the one starting in $-x$. We use the results previously computed for $|x|$ and $[-b_{\gamma,x}, -a_{\gamma,x}]$ which leads to the statement. The case $x = 0$ is simple to handle with, since the previous boundaries (2.2.3) become

$$-\frac{\sigma d}{\sqrt{2\theta e}} \leqslant \psi^-_{\mathrm{OU}}(t,0) \leqslant \psi^+_{\mathrm{OU}}(t,0) \leqslant \frac{\sigma d}{\sqrt{2\theta e}}.$$

It suffices to set $d = \dfrac{\sqrt{2\theta e}}{\sigma} \min(|a_{\gamma,0}|, b_{\gamma,0})$, which corresponds to the limit case as x tends to 0 in both results previously established.

□

2.3 WOMS for the Ornstein-Uhlenbeck processes

Let us now present the approximation procedure of the Ornstein-Uhlenbeck exit time from a given interval $[a, b]$. This algorithm is based on a walk on generalized spheroids (WOMS) described in the previous section.

ALGORITHM (O.U. WOMS)

Initialization: Let: $X_0 = x_0$, $\mathcal{T}_\epsilon = 0$
From step n to step $n+1$:
While $\boldsymbol{X_n \leqslant b - \epsilon}$ and $\boldsymbol{X_n \geqslant a + \epsilon}$ do

- Generate the Brownian exit time from the spheroid with parameter d_{X_n} defined in (2.2.2). We denote this stopping time by τ_{n+1}.
- We set $\tau^{OU}_{n+1} = \frac{\log(\tau_{n+1}+1)}{2\theta}$.
- Generate a Bernoulli distributed r.v. $\mathcal{B} \sim \mathcal{B}(\frac{1}{2})$, if $\mathcal{B} = 1$ then set $X_{n+1} = \psi^-_{OU}(\tau^{OU}_{n+1}, X_n)$ otherwise set $X_{n+1} = \psi^+_{OU}(\tau^{OU}_{n+1}, X_n)$.
- $\mathcal{T}_\epsilon \leftarrow \mathcal{T}_\epsilon + \tau^{OU}_{n+1}$.

Outcome: $\mathcal{T}_\epsilon$ the approximated O.U.-exit time from the interval $[a, b]$.

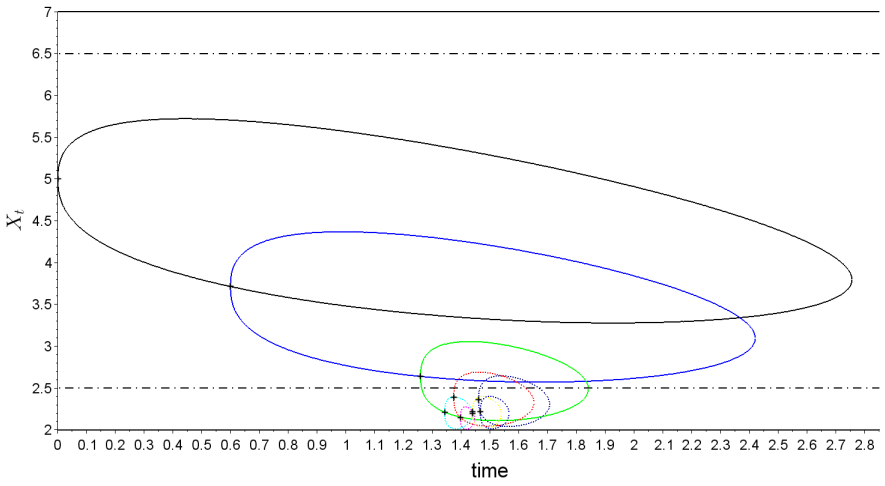


Figure 2.1: A sample of the algorithm for the O.U. exit time with parameters $\theta = 0.1$ and $\sigma = 1$. We observe the walk on spheres associated with the diffusion process starting at $x = 5$ and moving in the interval $[2, 7]$. The algorithm corresponding to $\epsilon = 0,5$ is represented by the plain style spheroids whereas the case $\epsilon = 10^{-3}$ corresponds to the whole sequence of spheroids. In both cases we set $\gamma = 10^{-6}$.

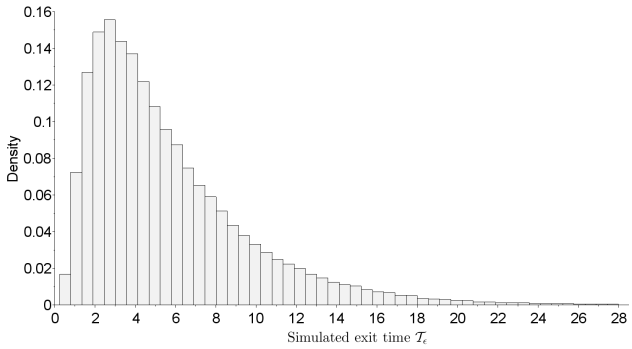


Figure 2.2: Histogram of the outcome variable for the O.U. with parameters $\theta = 0.1$ and $\sigma = 1$ when the stopped diffusion process starts at 5 and involves in the interval [2,7] with $\epsilon = 10^{-3}$ and $\gamma = 10^{-6}$.

The CPU efficiency of such an algorithm shall be compared to the efficiency of classical approaches in the exit time approximation framework. Let us consider a particular situation: the exit time from the interval $[3, 5]$ for the Ornstein-Uhlenbeck process starting in 4 with $\theta = 5$ and $\sigma = 7$. We use an improved Euler method based on the correction by means of the sharp large deviations estimate of the exit probability. Such a method takes into account the probability for the diffusion path to exit inbetween two neighboring gridpoints (see the procedure described in [1]). The simulation of 100 000 samples with the step size 10^{-4} requires 64,7 seconds for this improved Euler method whereas the WOMS algorithm presented in this paper requires about 2,19 seconds for the corresponding choice $\epsilon = 10^{-2}$ (here $\gamma = 10^{-6}$).

Even if the study presented here concerns the exit time of some given interval $[a, b]$ denoted by $\tau_{[a,b]}$, let us just mention the possible link with first passage times (FPT). Intuitively for negative a with large value $|a|$, the exit time of the interval can be approximated by the first passage time of the level b denoted by τ_b i.e. $\lim_{|a|\to\infty} \mathbb{P}(\tau_{[a,b]} = \tau_b) = 1$. Several approaches permit to describe quite precisely the probability distribution of the Ornstein-Uhlenbeck FPT. In Figure 2.3, we illustrate that the distributions of both the exit time (histogram) and the first passage time (p.d.f.) present a thight fit. The histogram corresponds to the exit time obtained for an OU process starting in -3 with coefficients $\theta = 1$ and $\sigma = 1$ and observed on the interval with bounds $a = -10$ and $b = -1$. The curve corresponds to a numerical approximation of the first passage time density presented by Buonocore, Nobile and Ricciardi in [10]. An other approximation procedure for the FPT simulation is proposed by Herrmann and Zucca in [27]: it consists in simulating exactly the PFT of a slightly modified diffusion process. This modified diffusion has the following property: its drift term is bounded and coincides with the Ornstein-Uhlenbeck drift on the interval $[a, b]$ with $|a|$ large. Numerical comparisons permit to observe that the simulation of the exit time with the WOMS algorithm is highly more efficent than the method proposed in [27]: the simulation of a sample of size 100 000 takes a total of 3,7 seconds of CPU time with the first method and 197,2 seconds with the second one. Here the OU-process starts in -3 with coefficients $\theta = 1$ and $\sigma = 1$ and is observed on the interval $[-10, -1]$.

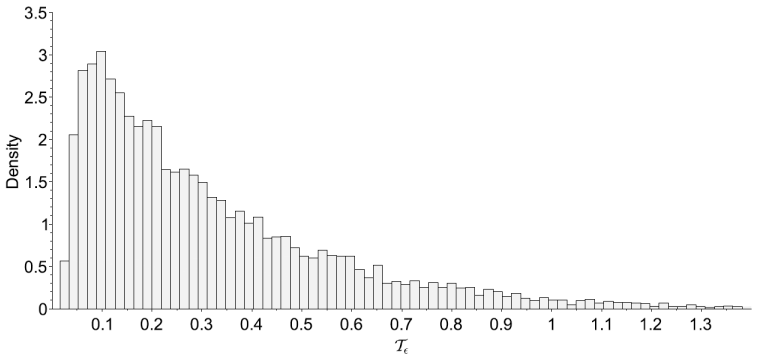


Figure 2.3: Histogram of the approximated first exit time of the interval $[a, b]$ using the WOMS algorithm and approximated p.d.f. of the first passage time through the level b (curve). Here $X_0 = -3$, $\theta = 1$, $\sigma = 1$ and $[a, b] = [-10, -1]$.

Let us now describe the WOMS algorithm for the Ornstein-Uhlenbeck process and especially emphasize its efficiency through theoretical results. We study how the strong relation between our process and the Brownian motion affects the statements obtained in the Brownian motion case. Let us just recall that the efficiency of the walk on spheres in the particular Brownian case is quite strong: the averaged number of steps is of the order $|\log(\epsilon)|$ (see for instance [6], for an overview of the convergence rate). In the Ornstein-Uhlenbeck case, we reach a similar efficiency result.

Average number of steps

Theorem 2.3.1. *Let N_ϵ be the random number of steps observed in the algorithm. Then there exist a constant $\delta > 0$ and $\epsilon_0 > 0$ such that*

$$\mathbb{E}[N_\epsilon] \leqslant \delta |\log(\epsilon)|, \quad \forall \epsilon \leqslant \epsilon_0. \tag{2.3.1}$$

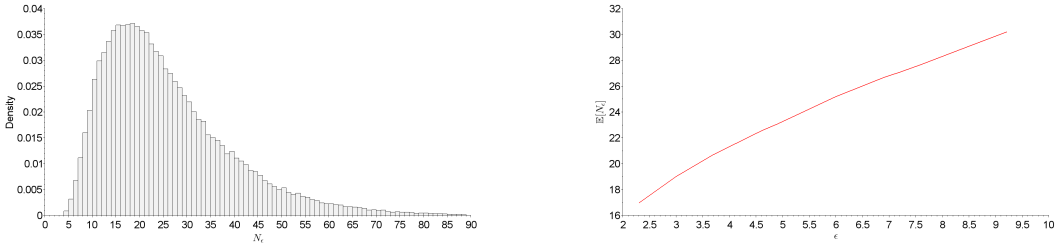


Figure 2.4: Simulation of the O.U. exit time from the interval $[2, 7]$. The starting position is $X_0 = 5$ and the parameters are given by $\theta = 0.1$, $\sigma = 1$ and $\gamma = 10^{-6}$. Histogram of the number of steps observed for $\epsilon = 10^{-3}$ (left) and average number of steps versus ϵ (right, in logarithmic scale).

The statement is similar to the Brownian motion case, and the proofs are based on similar arguments. To prove this statement, we introduce a result coming from the potential theory

and using Markov chains.
Let us consider a Markov chain $(X_n)_{n\in\mathbb{N}}$ defined on a state space I decomposed into two distinct subsets K and ∂K, ∂K being the so-called frontier. Let us define $N = \inf\{n \in \mathbb{N}, X_n \in \partial K\}$ the hitting time of ∂K. We assume that N is a.s. finite, then the following statement holds:

Proposition 2.3.2. *If there exists a function U s.t. the sequence $(U(X_{n\wedge N}))_{n\in\mathbb{N}}$ is non negative and if the sequence $(U(X_{n\wedge N}) + n \wedge N)_{n\in\mathbb{N}}$ represents a super-martingale adapted to the natural filtration of the considered Markov chain (X_n), then*

$$\mathbb{E}_x[N] \leqslant U(x), \quad \forall x \in K.$$

The proof of this classical upper-bound is left to the reader, it is essentially based on the optimal stopping theorem and on the monotone convergence theorem (see, for instance,[45], p139).

Proof of Theorem 2.3.1. **Step 1.** Let us first introduce a function u which plays an important role in the construction of a super-martingale linked to the random walk.
We consider the following differential equation:

$$\frac{\sigma^2}{2}u'' - \theta x u' = \frac{-1}{(x-a)^2(x-b)^2}, \quad \text{for } x \in]a,b[. \tag{2.3.2}$$

This second order differential equation can be solved in a classical way. Let us first solve the related homogeneous equation: we obtain

$$u'(x) = C(x)e^{\frac{\theta}{\sigma^2}x^2}.$$

The method of variation of parameters leads to

$$C(x) = -\frac{2}{\sigma^2}\int_0^x \frac{e^{-\frac{\theta}{\sigma^2}s^2}}{(s-a)^2(s-b)^2}ds.$$

Integrating u' one more time implies an explicit expression of one particular solution (2.3.2).

$$u(x) = -\frac{2}{\sigma^2}\int_0^x e^{\frac{\theta}{\sigma^2}u^2}\int_0^u \frac{e^{-\frac{\theta}{\sigma^2}s^2}}{(s-a)^2(s-b)^2}dsdu, \quad \text{for } x \in]a,b[. \tag{2.3.3}$$

Step 2. We consider now the sequence $(T_n, X_n)_{n\in\mathbb{N}}$ of cumulative exit times, i.e.

$$T_n = \sum_{k=1}^{n} \tau_k^{\mathrm{OU}} \tag{2.3.4}$$

and exit location given by the WOMS algorithm for the Ornstein-Uhlenbeck process.

Let us introduce $Z_n = u(X_n)+cn$ where c is a positive constant (which shall be determined in the following calculus) and u is the function detailed in Step 1 of the proof. We shall

prove that this process is a super-martingale with respect to the filtration $(\mathcal{F}_{T_n})_{n\in\mathbb{N}}$ induced by $(\mathcal{F}_t)$, the natural filtration of the Brownian motion $(V_t)_{t\geqslant 0}$ enlightened in (2.1.4).
By Itô's formula we obtain

$$\begin{aligned}\mathbb{E}[Z_{n+1}-Z_n|\mathcal{F}_{T_n}] &= \mathbb{E}[M_{n+1}-M_n|\mathcal{F}_{T_n}] \\ &\quad + \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{\sigma^2}{2}u"(X_s)-\theta X_s u'(X_s)ds\middle|\mathcal{F}_{T_n}\right]+c \\ &= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{-1}{D_{[a,b]}(X_s)^2}ds\middle|\mathcal{F}_{T_n}\right]+c. \end{aligned} \tag{2.3.5}$$

where $(M_n)_{n\in\mathbb{N}} = \left(\int_0^{T_n}\sigma u'(X_s)dW_s\right)_{n\in\mathbb{N}}$ is a martingale and $D_{[a,b]}(x)=(x-a)(b-x)$ for $x\in[a,b]$. Remark now that

$$\Xi(X_n) := \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{-1}{D_{[a,b]}(X_s)^2}ds\middle|\mathcal{F}_{T_n}\right] = \mathbb{E}\left[\int_0^{\tau_{n+1}^{\mathrm{OU}}} \frac{-1}{D_{[a,b]}(\tilde{X}_s)^2}ds\middle|\mathcal{F}_{T_n}\right] \tag{2.3.6}$$

where $\tilde{X}_s := X_{T_n+s}$ has the same distribution as the Ornstein-Uhlenbeck starting in X_n. We now upper bound this term: we consider in a first time that X_n is positive. By Proposition 2.2.2 we are then allowed to compute the corresponding coefficient d_{X_n} which we denote by $d_n>0$ for notation simplicity. Let us fix some parameter $\Delta\in]0,1[$.
First case: $d_n\leqslant\Delta$, that is satisfied either if

$$0<(b_\gamma-X_n)\frac{\sqrt{2\theta e}}{\sigma}\leqslant\Delta \tag{2.3.7}$$

or

$$0<\frac{2(X_n-a_\gamma)}{\sqrt{\frac{\sigma^2}{2\theta e}+2X_n(X_n-a_\gamma)}+\frac{\sigma}{\sqrt{2\theta e}}}\leqslant\Delta \tag{2.3.8}$$

with $b_\gamma=b_{\gamma,X_n}$ and $a_\gamma=a_{\gamma,X_n}$.
We first consider that X_n is close enough to b_γ. Using (2.2.3), we have for any $t\in \mathrm{Supp}\,(\psi_{OU}^{\pm})=\left[0,\frac{\log(1+d_n^2)}{2\theta}\right]$:

$$b-\psi_-^{OU}(t,X_n)\leqslant b-\frac{X_n}{\sqrt{1+d_n^2}}+\frac{\sigma d_n}{\sqrt{2\theta e}}\leqslant b-X_n\left(1-\frac{d_n^2}{2}\right)+\frac{\sigma d_n}{\sqrt{2\theta e}}.$$

Since $d_n \leq \Delta < 1$, we have $d_n^2 \leq d_n$. Moreover $X_n \leq b_\gamma$ so that

$$\begin{aligned} b - \psi_-^{OU}(t, X_n) &\leqslant b - X_n \left(1 - \frac{d_n}{2}\right) + \frac{\sigma d_n}{\sqrt{2\theta e}} \\ &= b - X_n + d_n \left(\frac{X_n}{2} + \frac{\sigma}{\sqrt{2\theta e}}\right) \\ &\leqslant (b - X_n) \left(\frac{b\sqrt{2\theta e}}{2\sigma} + 2\right) =: (b - X_n)\beta. \end{aligned}$$

The last upper-bound uses the definition of d_n presented in Proposition 2.2.2 Hence we have

$$D_{[a,b]}(\tilde{X}_s) \leqslant \beta(b-a)(b - X_n).$$

We then write, using the fact that τ_{n+1}^{OU} is independent of $\mathcal{F}_{T_n}$,

$$\begin{aligned} \Xi(X_n) &\leqslant \mathbb{E}\left[\int_0^{\tau_{n+1}^{\mathrm{OU}}} \frac{-1}{\beta^2(b-a)^2(b-X_n)^2} ds \middle| \mathcal{F}_{T_n}\right] \\ &= \frac{-1}{\beta^2(b-a)^2(b-X_n)^2}\, \mathbb{E}[\tau_{n+1}^{\mathrm{OU}}] \\ &= \frac{-1}{2\theta\beta^2(b-a)^2(b-X_n)^2}\, \mathbb{E}[\log(1+\tau_n)], \end{aligned}$$

where τ_n denotes the exit time for Brownian motion from the spheroid of parameter d_n. If τ denotes the Brownian exit time of the generalized spheroid of normalized size ($d = 1$), then the scaling property of Brownian motion implies that τ_n and $d_n^2\tau$ are identically distributed. Hence, noticing that $\tau \leqslant 1$ and recalling that $d_n^2 \leqslant 1$, we obtain

$$\begin{aligned} \Xi(X_n) &\leqslant \frac{-1}{2\theta\beta^2(b-a)^2(b-X_n)^2}\, \mathbb{E}[\log(1+d_n^2\tau)] \\ &\leqslant \frac{-d_n^2}{4\theta\beta^2(b-a)^2(b-X_n)^2}\, \mathbb{E}[\tau]. \end{aligned}$$

In the considered case, we know that

$$d_n = (b - X_n)\frac{\sqrt{2\theta e}}{\sigma} \tag{2.3.9}$$

which implies

$$\Xi(X_n) \leqslant \frac{-e}{2\sigma^2\beta^2(b-a)^2}\mathbb{E}[\tilde{\tau}_1]. \tag{2.3.10}$$

In the other case (X_n close to a) the arguments already used just above lead to a similar upper-bound. We observe for any $t \in \left[0, \frac{\log(1+d_n^2)}{2\theta}\right]$:

$$\begin{aligned}\psi_+^{OU}(t, X_n) - a &\leqslant X_n + \frac{\sigma d_n}{\sqrt{2\theta e}} - a \\ &\leqslant (X_n - a)\left(1 + \frac{2\sigma}{\sqrt{\sigma^2 + 4\theta e X_n(X_n - a)} + \sigma}\right) \leqslant 2(X_n - a).\end{aligned}$$

This upper bound leads to the same result as (2.3.10) just replacing β by another positive constant $\tilde{\beta}$. Combining both inequalities, for d_n smaller than Δ, we get

$$\Xi(X_n) \leqslant \frac{-1}{\sigma^2 \max(\tilde{\beta}, \beta)^2}\, \mathbb{E}[\tau]. \tag{2.3.11}$$

Second case: $d_n > \Delta$
In this case, we use the upper-bound:

$$D_{[a,b]}(\tilde{X}_s) \leqslant (b-a)^2. \tag{2.3.12}$$

We deduce

$$\begin{aligned}\Xi(X_n) &\leqslant \mathbb{E}\left[\int_0^{\tau_{n+1}^{\mathrm{OU}}} \frac{-1}{(b-a)^4} ds \,\middle|\, \mathcal{F}_{T_n}\right] \\ &\leqslant \frac{-1}{2\theta(b-a)^4}\, \mathbb{E}[\log(1 + \Delta^2 \tilde{\tau}_1)] \leqslant \frac{-\Delta^2}{4\theta(b-a)^4}\, \mathbb{E}[\tau].\end{aligned} \tag{2.3.13}$$

Both inequalities (2.3.11) and (2.3.13) suggest the existence of a constant $\tilde{c} > 0$ such that $\Xi(X_n) \leqslant -\tilde{c}$.
Finally, using the symmetry property of the considered spheroid, the case x negative is treated similarly, leading to a positive constant c such that

$$\Xi(X_n) \leq \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{-1}{D_{[a,b]}(X_s)^2} ds \,\middle|\, \mathcal{F}_{T_n}\right] \leqslant -c, \text{ for all } n \geqslant 0. \tag{2.3.14}$$

In conclusion, the stochastic process $Z_n = u(X_n) + cn$ is a super-martingale due to the combination of (2.3.5) and (2.3.14).
Step 3. In order to apply the optimal stopping theorem described in Proposition 2.6., we need on one hand that $(U(X_n)+cn)_{n\geqslant 0}$ is a super-martingale but also on the other hand that $(U(X_n))_{n\geqslant 0}$ is a non negative sequence. For the first property we could choose $U = u + \kappa$, u being the function introduced in (2.3.3) and κ a constant. For the second property we need to have a non negative sequence, so we have to choose in a suitable way the constant κ. Let us note that the function u satisfies $u(0) = 0$ and is a concave function. So in order to obtain a positive function on the interval $[a_{\gamma,x}, b_{\gamma,x}]$ it suffices to choose $\kappa \geq -\min(u(b_{\gamma,x}), u(a_{\gamma,x}))$.

Consequently we need to study the behavior of u at the frontiers of $[a_{\gamma,x}, b_{\gamma,x}]$ that is for $x = b - \epsilon$ and $x = a + \epsilon$. Putting $b_\gamma := b_{\gamma,b-\epsilon}$, we obtain

$$\begin{aligned} u(b_\gamma) &= -\frac{2}{\sigma^2}\int_0^{b_\gamma} e^{\frac{\theta}{\sigma^2}u^2}\int_0^u \frac{e^{-\frac{\theta}{\sigma^2}s^2}}{(s-a)^2(s-b)^2}ds\,du \\ &= -\frac{2}{\sigma^2}\int_0^{b_\gamma} \frac{e^{-\frac{\theta}{\sigma^2}s^2}}{(s-a)^2(s-b)^2}\int_s^{b_\gamma} e^{\frac{\theta}{\sigma^2}u^2}du\,ds. \end{aligned}$$

Using Taylor's expansion and taking appropriate lower-bounds lead to

$$0 \geq u(b_\gamma) + \frac{2}{\sigma^2}\int_0^{b_\gamma}\frac{b_\gamma - s}{(s-a)^2(s-b)^2}ds \geq -\frac{2\theta b}{\sigma^4}e^{\frac{\theta b^2}{\sigma^2}}\int_0^{b_\gamma}\frac{(b_\gamma - s)^2}{(s-a)^2(s-b)^2}ds.$$

Moreover

$$\frac{1}{(s-a)^2(s-b)^2} = \frac{c_1}{(s-a)} + \frac{c_2}{(s-a)^2} + \frac{c_3}{(b-s)} + \frac{c_4}{(b-s)^2}, \tag{2.3.15}$$

where $c_i, i \in \{1,2,3,4\}$ are positive constants and $c_2 = c_4 = \dfrac{1}{(b-a)^2}$.

$$\begin{aligned} u(b_\gamma) + \frac{2}{\sigma^2}\int_0^{b_\gamma}\frac{(b_\gamma - s)}{(s-a)^2(s-b)^2}ds &= u(b_\gamma) + \frac{2}{\sigma^2}(c_1 I_{0,1,1} + c_2 I_{0,2,1} + c_3 I_{1,0,1} + c_4 I_{2,0,1}) \\ &\geq -\frac{2\theta b}{\sigma^4}e^{\frac{\theta b^2}{\sigma^2}} I_{2,2,2}, \end{aligned}$$

where $I_{i,j,k} = \displaystyle\int_0^{b_\gamma}\frac{(b_\gamma - s)^k}{(b-s)^i(s-a)^j}ds$. We can notice that

$$I_{2,0,1} = -\log(\gamma\epsilon) + \log(b) - 1 + \frac{\gamma\epsilon}{b} \tag{2.3.16}$$

and there exists a constant $\delta_0 > 0$ such that

$$c_1 I_{0,1,1} + c_2 I_{0,2,1} + c_3 I_{1,0,1} = \delta_0 + O(\epsilon \ln \epsilon) \text{ as } \epsilon \text{ tends to } 0. \tag{2.3.17}$$

Let us bound the last integral, using once again the partial fraction decomposition

$$I_{2,2,2} = c_4 I_{0,1,2} + c_5 I_{0,2,2} + c_6 I_{1,0,2} + c_7 I_{2,0,2}.$$

As in the previous computations, it is possible to take an equivalent as ϵ tends to zero, that is there exists $\delta_1 > 0$ such that

$$\frac{2\theta b}{\sigma^4}e^{\frac{\theta b^2}{\sigma^2}}(c_4 I_{0,1,2} + c_5 I_{0,2,2} + c_6 I_{1,0,2} + c_7 I_{2,0,2}) = \delta_1 + O(\epsilon \ln \epsilon). \tag{2.3.18}$$

Combining (2.3.16), (2.3.17) and (2.3.18), and taking an equivalent when ϵ tends to 0 leads to state that there exists $\delta > 0$ such that

$$u(b_{\gamma,b-\epsilon}) \geq D\log(\gamma\epsilon) - \delta + O(\epsilon \ln \epsilon), \text{ where } D = \frac{2}{\sigma^2(b-a)^2}. \tag{2.3.19}$$

A similar computation on $u(a_{\gamma,a+\epsilon})$ gives us some $\delta' > 0$. Setting $\kappa = -D\log(\gamma\epsilon) + \hat{\delta}$ with $\hat{\delta} > \max(\delta, \delta')$ and $U(x) = u(x) + \kappa$ permits to obtain the positivity of the sequence $(U(X_n))_{n\geq 1}$, for ϵ small enough.

Step 4. The statement of the theorem is a direct consequence of the optimal stopping theorem Proposition 2.3.2. If N_ϵ is almost surely finite, then

$$\mathbb{E}[N_\epsilon] \leqslant \frac{1}{c}\mathbb{E}[U(X_0)] \leqslant \frac{D}{c}|\log(\epsilon)|, \text{ for } \epsilon \text{ small enough.} \tag{2.3.20}$$

In order to finish the proof, it remains to justify that N_ϵ is almost surely finite. Since $b_{\gamma,x} - x \geq (1-\gamma)\epsilon$ and $x - a_{\gamma,x} \geq (1-\gamma)\epsilon$ for any $x \in [a+\epsilon, b-\epsilon]$, we deduce that there exists a strictly positive lower-bound d_ϵ such that $d_{X_n} \geq d_\epsilon$ for any n. Introducing (s_n) a sequence of independent and identically distributed random variables corresponding to Brownian exits of a unit spheroid, we deduce that T_n is stochastically lower-bounded by

$$S_n := \frac{1}{2\theta}\sum_{k=1}^{n}\log(1 + d_\epsilon s_k).$$

Moreover S_n tends to infinity almost surely as $n \to \infty$. By Lemma 2.3.3 and by construction, T_n is stochastically inbetween S_n and $\mathcal{T}$ (an almost surely finite random variable) for any $n \leq N_\epsilon$. The stopping rule N_ϵ is therefore almost surely finite. □

Lemma 2.3.3. *The sequence of cumulative times $(T_n)_{n\geqslant 1}$ appearing in the algorithm and defined by* (2.3.4) *are stochastically smaller than $\mathcal{T}$ the first exit time of the Ornstein-Uhlenbeck process.*

Proof. We need to emphasize the link between the Markov chain induced by the algorithm, denoted $((T_n, X_n))_{n\in\mathbb{N}}$ with $(T_0, X_0) = (0, 0)$, and a path of the Ornstein-Uhlenbeck process. At the starting point of the Ornstein-Uhlenbeck trajectory, we introduce a spheroid of maximum size contained in the interval $[a, b] \times \mathbb{R}_+$. The intersection of this spheroid and the path corresponds to the point (t_1, z_1). Then this construction leads us to state that (t_1, z_1) has the same distribution as (T_1, X_1). Hence, from (t_1, z_1) we can construct a maximum size spheroid and consider the intersection (t_2, z_2) between the trajectory after t_1 and this second spheroid. Once again we get from the construction that (t_2, z_2) and (T_2, X_2) are identically distributed. We can therefore step by step build a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the considered trajectory and the spheroids. We obtain that the skeleton of the trajectory $(t_n, z_n)_{n\in\mathbb{N}}$ and the sequence $(T_n, X_n)_{n\in\mathbb{N}}$ are identically distributed. By construction, we also note that $t_n \leqslant \mathcal{T}$ for all $n \in \mathbb{N}$, which implies the announced result. □

Bounds for the Exit-Time distribution

Let us now precise the rate of convergence for the algorithm based on the random walk. We should describe how far the outcome of the algorithm and the diffusion exit time are. We recall that the outcome depends on the parameter ϵ.

Theorem 2.3.4. *We consider $0 < \gamma < 2$ and $\delta = \epsilon^\gamma$. We denote by F the cumulative distribution function of the exit time from the interval $[a, b]$ and F_ϵ the distribution function of the algorithm outcome. Then for any $\rho > 1$, there exists $\epsilon_0 > 0$ such that*

$$\left(1 - \frac{\rho\sqrt{\theta}(\epsilon + \max(|a|, |b|)(e^{\theta\delta} - 1))}{\sigma\sqrt{(e^{2\theta\delta} - 1)\pi}}\right) F_\epsilon(t - \delta) \leqslant F(t) \leqslant F_\epsilon(t), \tag{2.3.21}$$

for all $t \in \mathbb{R}$ and $\epsilon \leq \epsilon_0$.

In other words, the precision of the approximation pointed out in Theorem 2.3.4 is characterized by the following error bound:

$$\Xi(\epsilon\,; \theta, \sigma, a, b, \gamma) := \frac{\sqrt{\theta}(\epsilon + \max(|a|, |b|)(e^{\theta\delta} - 1))}{\sigma\sqrt{(e^{2\theta\delta} - 1)\pi}}, \quad \text{with } \delta = \epsilon^\gamma.$$

Figure 5 presents the dependence of this bound with respect to ϵ and θ, all other parameters being fixed.

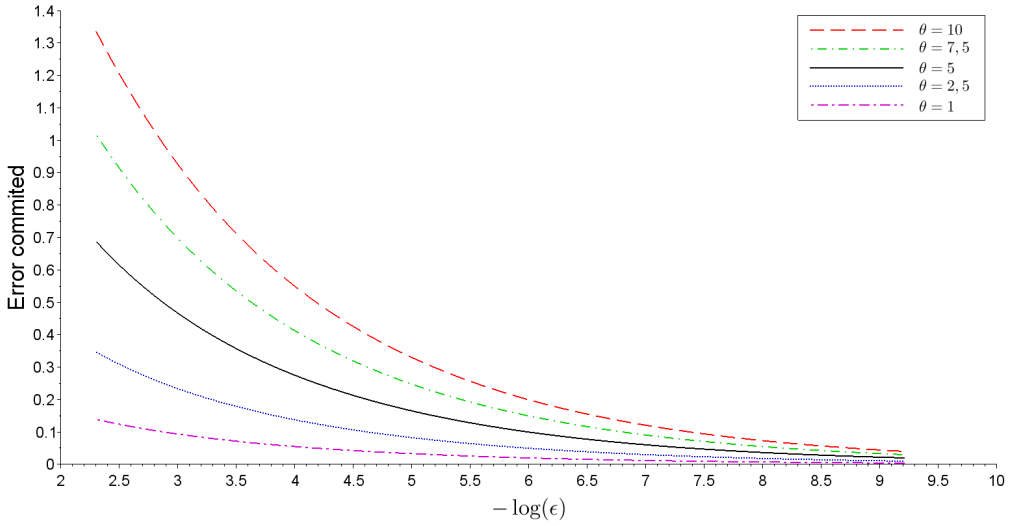


Figure 2.5: Error bound Ξ versus ϵ for different values of θ with $\sigma = 1$, $a = -1$, $b = 1$, $\gamma = 1$.

Such a statement is directly related to properties of the Ornstein-Uhlenbeck process and its strong link with the Brownian motion.

Proof. As in Lemma 2.3.3, we build step by step a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the path of the Ornstein-Uhlenbeck process and the spheroids in such a way that the sequences $((t_n, z_n))_{n\geq 0}$ and $((T_n, X_n))_{n\geq 0}$ are identically distributed.
If we introduce N_ϵ the stopping time appearing in the stopping procedure of the algorithm

and $\tilde{N}_\epsilon = \inf\{n \in \mathbb{N}, z_n \notin [a+\epsilon, b-\epsilon]\}$, the identity in law of those random variables yields. By construction, $t_n \leqslant \mathcal{T}$ for all $n \in \mathbb{N}$, where $\mathcal{T}$ stands for the diffusion first exit time from the interval $[a, b]$. This inequality remains true when t_n is replaced by the random stopping time $t_{\tilde{N}_\epsilon}$.
Hence

$$\begin{aligned} 1 - F(t) &= \mathbb{P}(\mathcal{T} > t) \\ &= \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t - \delta) + \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} > t - \delta) \\ &\leqslant \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t - \delta) + \mathbb{P}(t_{\tilde{N}_\epsilon} > t - \delta) \\ &\leqslant \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t - \delta) + 1 - F_\epsilon(t - \delta). \end{aligned} \tag{2.3.22}$$

We focus on the first term of this upper bound. Using the strong Markov property, we obtain

$$\mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t - \delta) \leqslant F_\epsilon(t - \delta) \sup_{y \in [a, a+\epsilon] \cup [b-\epsilon, b]} \mathbb{P}_y(\mathcal{T} > \delta). \tag{2.3.23}$$

For any $y \in [a, a + \epsilon] \cup [b - \epsilon, b]$ we write

$$\mathbb{P}_y(\mathcal{T} > \delta) = \mathbb{P}_y(\mathcal{T}_a > \delta, \mathcal{T}_a < \mathcal{T}_b) + \mathbb{P}_y(\mathcal{T}_b > \delta, \mathcal{T}_b < \mathcal{T}_a).$$

We first consider the case $y \in [b - \epsilon, b]$, the previous inequality becomes

$$\mathbb{P}_y(\mathcal{T} > \delta) \leqslant \mathbb{P}_y(\mathcal{T}_a < \mathcal{T}_b) + \mathbb{P}_y(\mathcal{T}_b > \delta). \tag{2.3.24}$$

In order to handle with the first term in the right hand side, we introduce s the scale function of the O.-U.-process:

$$s(x) = e^{\frac{\theta}{\sigma^2} a^2} \int_a^x e^{\frac{\theta}{\sigma^2} u^2} du, \quad x \in [a, b]. \tag{2.3.25}$$

It has been shown in Karatzas, 5.5 [36] that

$$\mathbb{P}_y(\mathcal{T}_a < \mathcal{T}_b) = \frac{s(b) - s(y)}{s(b) - s(a)} = \frac{\int_y^b e^{\frac{\theta}{\sigma^2} u^2} du}{\int_a^b e^{\frac{\theta}{\sigma^2} u^2} du}. \tag{2.3.26}$$

Since $y \in [b-\epsilon, b]$ and since the integrated function is non negative and increasing we obtain

$$\mathbb{P}_y(\mathcal{T}_a < \mathcal{T}_b) \leqslant \frac{\int_{b-\epsilon}^b e^{\frac{\theta}{\sigma^2} u^2} du}{\int_a^b e^{\frac{\theta}{\sigma^2} u^2} du} \leqslant \frac{\epsilon\, e^{\frac{\theta}{\sigma^2} b^2}}{\int_a^b e^{\frac{\theta}{\sigma^2} u^2} du} =: \epsilon\, C_{a,b}. \tag{2.3.27}$$

We now focus on the second term in the r.h.s. of (2.3.24): $\mathbb{P}_y(\mathcal{T}_b > \delta) \leqslant \mathbb{P}_{b-\epsilon}(\mathcal{T}_b > \delta)$ for all

$y \in [b-\epsilon, b]$. We denote by $\tilde{X}$ the Ornstein-Uhlenbeck process starting in $b-\epsilon$. We obtain

$$\begin{aligned}
\{\mathcal{T}_b > \delta\} &= \left\{ \sup_{u\in[0,\delta]} \tilde{X}_u < b \right\} = \left\{ \forall u \in [0,\delta], \tilde{X}_u < b \right\} \\
&= \left\{ (b-\epsilon)e^{-\theta u} + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta u} V_{e^{2\theta u}-1} < b, \forall u \in [0,\delta] \right\} \\
&= \left\{ V_s < \frac{\sqrt{2\theta}}{\sigma}(b\sqrt{s+1} - (b-\epsilon)), \forall s \in [0, e^{2\theta\delta}-1] \right\} \\
&= \left\{ V_s < \frac{\sqrt{2\theta}}{\sigma}(b(\sqrt{s+1}-1) + \epsilon)), \forall s \in [0, e^{2\theta\delta}-1] \right\} \\
&\subset \left\{ V_s < \frac{\sqrt{2\theta}}{\sigma}(\epsilon + \max(0,b)(e^{\theta\delta}-1)), \forall s \in [0, e^{2\theta\delta}-1] \right\}.
\end{aligned}$$

Let us assume that $b > 0$. In this case, the following asymptotic property holds:

$$\begin{aligned}
\mathbb{P}_{b-\epsilon}(\mathcal{T}_b > \delta) &= \mathbb{P}_0 \left(\sup_{s\in[0,e^{2\theta\delta}-1]} V_t < \frac{\sqrt{2\theta}}{\sigma}(\epsilon + b(e^{\theta\delta}-1)) \right) \\
&= \mathbb{P}_0 \left(2|V_{e^{2\theta\delta}-1}| < \frac{\sqrt{2\theta}}{\sigma}(\epsilon + b(e^{\theta\delta}-1)) \right) \leqslant \frac{\sqrt{\theta}(\epsilon + b(e^{\theta\delta}-1))}{\sigma\sqrt{(e^{2\theta\delta}-1)\pi}}.
\end{aligned}$$

Using the particular form of $\delta = \epsilon^\gamma$, we obtain

$$\frac{\sqrt{\theta}(\epsilon + b(e^{\theta\delta}-1))}{\sigma\sqrt{(e^{2\theta\delta}-1)\pi}} \sim \frac{1}{\sigma\sqrt{2\pi}}(\epsilon^{1-\frac{\gamma}{2}} + b\theta\epsilon^{\frac{\gamma}{2}}) \quad \text{as } \epsilon \to 0.$$

A similar bound can be obtained for b negative and also for $y \in [a, a+\epsilon]$.
Finally combining this result with (2.3.22), (2.3.23) and (2.3.27) leads to the announced statement. □

Remark 2.3.5. *Let us note that all the results presented so far, that is the efficiency of the algorithm and the convergence rate, concern the family of Ornstein-Uhlenbeck processes with parameter $\mu = 0$ in (2.1). It is straightforward to extend the statements to the general case: it suffices to replace the interval $[a,b]$ by a time-dependent interval $[a-\mu(1-e^{-\theta t}), b-\mu(1-e^{-\theta t})]$.*

Chapter 3

WOMS algorithm for L-class diffusions

The results presented in this chapter are subject to a publication in Computers & Mathematics with Applications. An International Journal [25]

3.1 The algorithm

The Walk on Spheroids already introduced for the Ornstein-Uhlenbeck process in [26] permits to approximate the exit time in an efficient way. We aim to extend such numerical procedure to a wider class of stochastic processes. We focus our attention to the family of L-class diffusions (linear-type diffusions) which generalizes the Ornstein-Uhlenbeck processes. For such diffusions, all the coefficients are time-dependent. Moreover they are based on a strong relation with a one-dimensional Brownian motion.

3.1.1 L-class diffusions

This particular family of diffusions was already introduced in [58].

Definition 3.1.1 (L-class diffusions)**.** *We call L-class diffusion any solution of*

$$dX_t = (\alpha(t)X_t + \beta(t))dt + \tilde{\sigma}(t)dW_t \quad t \geq 0 \quad \textit{and} \quad X_0 = x_0, \tag{3.1.1}$$

where α, β and $\tilde{\sigma}$ are Hölder-continuous functions, $\tilde{\sigma}$ is furthermore positive and $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion.

Since α, β and $\tilde{\sigma}$ are measurable functions, the linear structure of the differential equation (3.1.1) implies both the existence and the uniqueness of a strong solution. Moreover the stochastic process does not explode a.s. if the initial data X_0 is square-integrable (see, for instance, Theorem 5.2.1 in [46]). Here the starting position is always deterministic, consequently the explosion phenomenon is never observed a.s.. It is possible to solve (3.1.1) in a classical way (see, for instance, Section 5.6 about linear equations in [36]). Let us introduce

$$\theta(t) := -\int_0^t \alpha(s)ds. \tag{3.1.2}$$

Lemma 3.1.2. *The unique solution of* (3.1.1) *is given by*

$$X_t = X_0\, e^{-\theta(t)} + e^{-\theta(t)} \int_0^t e^{\theta(s)} \beta(s) ds + e^{-\theta(t)} \int_0^t e^{\theta(s)} \tilde{\sigma}(s) dW_s, \quad t \geq 0.$$

Proof. Let us consider $g(t,x) = xe^{\theta(t)}$. The statement is therefore an easy consequence of Itô's formula:

$$\begin{aligned} d(g(t,X_t)) &= -\alpha(t) X_t e^{\theta(t)} dt + e^{\theta(t)} dX_t \\ &= -\alpha(t) X_t e^{\theta(t)} dt + e^{\theta(t)} (\alpha(t) X_t + \beta(t)) dt + e^{\theta(t)} \tilde{\sigma}(t) dW_t \\ &= e^{\theta(t)} \beta(t) dt + e^{\theta(t)} \tilde{\sigma}(t) dW_t, \quad t \geq 0. \end{aligned}$$

□

This expression is actually not handy for the construction of the algorithm. Moreover, for simulation purposes, it suffices to deal with a stochastic process which has the same path distribution than the strong solution X, that means to consider weak solutions of (3.1.1). For these reasons, we would like, as for Ornstein-Uhlenbeck processes in [26], to transform the martingale part of the diffusion into a time-changed Brownian motion. However, we cannot apply such a transformation in the L-class framework, that is why we shall proceed in a quite different way.
To that end, let us suppose that X, solution of (3.1.1), can be expressed using a time-changed Brownian motion:

$$X_t = f_L(t, x_0 + W_{\rho(t)}), \quad \forall t \geq 0, \tag{3.1.3}$$

with $\rho(0) = 0$, $\rho'(t) > 0$, for all $t \geqslant 0$ and $f_L(0,x) = x$ for any $x \in \mathbb{R}$.

Lemma 3.1.3. *Let θ the function defined in* (3.1.2). *Then the unique weak solution of* (3.1.1) *is the process* $(X_t,\ t \geq 0)$ *defined in* (3.1.3) *with*

$$f_L(t,x) = \frac{\tilde{\sigma}(t)}{\sqrt{\rho'(t)}} x + c(t), \quad c(t) = e^{-\theta(t)} \int_0^t \beta(s) e^{\theta(s)} ds$$
$$\text{and } \rho(t) = \int_0^t \tilde{\sigma}(s)^2 e^{2\theta(s)} ds. \tag{3.1.4}$$

Proof. Let us first introduce the process $(M_t)_{t \in \mathbb{R}_+}$ defined by

$$M_t := \int_0^t \sqrt{\rho'(s)} dW_s \tag{3.1.5}$$

where W_t is the Brownian motion introduced in (3.1.1). We notice that this process is a martingale with respect to the Brownian filtration and $\langle M \rangle_t = \int_0^t \rho'(s) ds = \rho(t)$. We introduce the process $\hat{X}_t := f_L(t, x_0 + M_t)$. Using Itô's formula we get

$$d\hat{X}_t = \frac{\partial f_L}{\partial t}(t, M_t) dt + \frac{1}{2} \rho'(t) \frac{\partial^2 f_L}{\partial x^2}(t, M_t) dt + \frac{\partial f_L}{\partial x}(t, M_t) \sqrt{\rho'(t)} dW_t.$$

Computing all functions appearing in the previous equality, the stochastic process $\hat{X}_t$ is solution of (3.1.1). Using Dambis & Dunbins-Schwarz Martingale representation theorem (see Theorem V.1.6 p.170 [53]), there exists a Brownian motion B_t such that

$$M_t = B_{\langle M \rangle_t}, \quad \forall t \geq 0. \tag{3.1.6}$$

We deduce that $M_t \sim W_{\rho_L(t)}$ and therefore $(\hat{X}_t)_{t\geq 0} \sim (X_t)_{t\geq 0}$ with $X_t = f_L(t, x_0 + W_{\rho(t)})$. □

Remark 3.1.4. *Let us just fix the parameter γ appearing in the expression of $(X_t,\ t \geq 0)$ for notation simplicity: $\gamma = 1$.*

Remark 3.1.5. *If the starting time associated to the study of the L-class diffusion is not the origin but another time t_0, then we also obtain an expression similar to* (3.1.3)*. Let Y_t be the unique weak solution of*

$$\begin{cases} dY_t = (\alpha(t+t_0)Y_t + \beta(t+t_0))dt + \tilde{\sigma}(t+t_0)dW_t, & t \geq 0 \\ Y_0 = X_{t_0}. \end{cases}$$

Then

$$Y_t = f_L(t + t_0, X_{t_0} e^{-\int_0^{t_0} \alpha(s)ds} + W_{\rho(t+t_0)-\rho(t_0)}) - e^{\int_{t_0}^{t+t_0} \alpha(s)ds} c(t_0). \tag{3.1.7}$$

3.1.2 Spheroids associated to a L-class diffusion process

Introducing the exit time of the spheroid.

We determine a specific spheroid for the diffusion by using the link with the time-changed Brownian motion. The boundaries of the spheroid associated to the diffusion starting at time t_0 in x_0 are denoted by $\psi_{\pm}^L(t;\, t_0, x_0)$ and the corresponding exit time is

$$\tau_L^{t_0} = \inf\{t > 0 :\ Y_t^L \notin [\psi_-^L(t;\, t_0, x_0), \psi_+^L(t;\, t_0, x_0)]\}.$$

Proposition 3.1.6. *Let us consider the spheroid starting in (t_0, X_{t_0}) with boundaries defined by*

$$\begin{aligned} \psi_{\pm}^L(t;\, t_0, X_{t_0}) = e^{-\theta(t+t_0)}\, \psi_{\pm}(\rho(t+t_0) - \rho(t_0)) + c(t+t_0) \\ + (X_{t_0} - c(t_0)) e^{\int_{t_0}^{t+t_0} \alpha(s)ds} \end{aligned}$$

for all $t \geq 0$, then the associated exit time satisfies

$$\tau_L^{t_0} \overset{d}{=} \rho_L^{-1}(\tau + \rho_L(t_0)) - t_0 \tag{3.1.8}$$

where $\tau = \inf\{u > 0\ : W_u \notin [\psi_-(t), \psi_+(t)]\}$, $\psi_{\pm}$ being defined in (1.3.1)*.*

Proof. By definition,

$$\begin{aligned}\tau_L^{t_0} &= \inf\{t>0 : Y_t \notin [\psi_-^L(t;\, t_0, X_{t_0}), \psi_+^L(t;\, t_0, X_{t_0})]\}\\ &= \inf\Big\{t>0 : e^{-\theta(t+t_0)}\, W_{\rho(t+t_0)-\rho(t_0)} + c(t+t_0) + (X_{t_0}-c(t_0))e^{\int_{t_0}^{t+t_0}\alpha(s)ds}\\ &\qquad\qquad \notin [\psi_-^L(t;\, t_0, X_{t_0}), \psi_+^L(t;\, t_0, X_{t_0})]\Big\}.\end{aligned}$$

Using $\psi_\pm^L$ introduced in the statement, we obtain the following expression for $\tau_L^{t_0}$:

$$\begin{aligned}&\inf\Big\{t>0 : W_{\rho(t+t_0)-\rho(t_0)} \notin [\psi_-(\rho(t+t_0)-\rho(t_0)), \psi_+(\rho(t+t_0)-\rho(t_0))]\Big\}\\ &= \inf\{\rho^{-1}(u+\rho(t_0)) - t_0 > 0 :\ W_u \notin [\psi_-(u), \psi_+(u)]\}\\ &= \rho_L^{-1}(\tau+\rho_L(t_0)) - t_0,\end{aligned}$$

where $\tau = \inf\{u>0 :\ W_u \notin [\psi_-(u), \psi_+(u)]\}$. □

Size determination of the spheroids

To define a WOMS algorithm for the L-class diffusions, we need to determine a suitable size for the spheroids in order to stay fully contained in the considered interval. Such size can be chosen by describing both the minimum and the maximum of the spheroid boundaries. The size of the Brownian spheroid introduced in (1.3.1) depends on a scaling parameter $d>0$, the support of the associated boundaries $\psi_\pm$ being therefore equal to $[0, d^2]$. Since the generalized spheroids used for L-class diffusion are directly linked to the Brownian ones, the parameter d also changes their size and the boundaries $\psi_\pm^L$ are defined on the support $[0, \rho^{-1}(d^2+\rho(t_0)) - t_0]$. Let us now precise this parameter d.

Proposition 3.1.7. *Let $m>0$ and $0<\gamma<1$. For any $(x_0,t_0)\in[a,b]\times\mathbb{R}^+$ we define a parameter $d = d(x_0,t_0)$ such that the spheroid associated to the L-class diffusion starting in (t_0,x_0) is totally included in $[a_{\gamma,x_0}, b_{\gamma,x_0}]$. Here a_{γ,x_0} and b_{γ,x_0} stands for $a_{\gamma,x} = a+\gamma(x-a)$ and $b_{\gamma,x} = b-\gamma(b-x)$. This parameter is given by*

$$d = \begin{cases} \frac{\min(1,\kappa_+)}{\Delta_m}(b_{\gamma,x_0}-x_0) & \text{if } b-x_0 \leqslant x_0-a\\ \frac{\min(1,\kappa_-)}{\Delta_m}(x_0-a_{\gamma,x_0}) & \text{if } x_0-a \leqslant b-x_0\end{cases} \tag{3.1.9}$$

where

$$\Delta_m = e^{-\theta(t_0)}e^{\int_{t_0}^{t_0+m}|\alpha(s)|ds}\left(\frac{1}{\sqrt{e}} + \sqrt{\int_{t_0}^{t_0+m}\frac{|\beta(s)+x_0\,\alpha(s)|^2}{\tilde\sigma(s)^2}ds}\right), \tag{3.1.10}$$

and $\kappa_\pm$ are defined by the following equations:

$$\kappa_+(b_{\gamma,x_0}-x_0) = \Delta_m\sqrt{\rho(t_0+m)-\rho(t_0)}$$

and

$$\kappa_-(x_0-a_{\gamma,x_0}) = \Delta_m\sqrt{\rho(t_0+m)-\rho(t_0)}.$$

Remark 3.1.8.

- *The previous statement consists in finding d such that*

$$\begin{cases} d \leqslant \frac{1}{\Delta_m}(b_{\gamma,x_0} - x_0), \\ d \leqslant \frac{1}{\Delta_m}(x_0 - a_{\gamma,x_0}), \\ d^2 \leqslant \rho(t_0+m) - \rho(t_0). \end{cases}$$

  The last condition in particular leads to $t \leqslant m$ since ρ is a strictly increasing function.

- *It is possible to let m depend on the couple (t_0, x_0) which should permit to obtain bigger spheroids which are still included in the interval. Nevertheless for numerical purposes, such a procedure slows down drastically the algorithm we are going to present.*

- *The choice of the constant m is important, since it either slows down or speeds up the algorithm.*

- *It is also possible to replace x_0 by $\max(|a|,|b|)$ in the definition of Δ_m which therefore becomes independent of the starting position x_0. Nevertheless such a replacement slows down the algorithm.*

Proof of Proposition 3.1.7. Let us first point out an upper bound for ψ_+^L starting in (t_0, x_0). We first require that $d^2 \leqslant \rho(t_0+m) - \rho(t_0)$. Let us define $\mathcal{R}_+^L(t) := \psi_+^L(t;\, t_0, x_0) - x_0$. By definition

$$\begin{aligned} \mathcal{R}_+^L(t) = e^{-\theta(t_0+t)} &\left(\psi_+(\rho(t+t_0) - \rho(t_0)) + \int_{t_0}^{t_0+t} \beta(s) e^{-\int_0^s \alpha(u)du} ds \right) \\ &+ x_0 \left(e^{\int_{t_0}^{t+t_0} \alpha(u)du} - 1 \right). \end{aligned}$$

Recalling Proposition 1.3.2, we obtain

$$\mathcal{R}_+^L(t) \leq e^{-\theta(t_0+t)} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \beta(s) e^{\theta(s)} ds \right) + x_0\, e^{-\theta(t_0+t)} \left(e^{\theta(t_0)} - e^{\theta(t+t_0)} \right)$$

$$\begin{aligned} \mathcal{R}_+^L(t) &\leqslant e^{-\theta(t_0+t)} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \beta(s) e^{-\int_0^s \alpha(u)du} ds \right) \\ &\quad + x_0\, e^{-\theta(t_0+t)} \left(\int_{t_0}^{t+t_0} \alpha(s) e^{-\int_0^s \alpha(u)du} \right) \\ &\leqslant e^{-\theta(t_0)+\int_{t_0}^{t_0+t} |\alpha(s)|ds} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|}{\tilde{\sigma}(s)} \tilde{\sigma}(s) e^{-\int_0^s \alpha(u)du} ds \right), \end{aligned}$$

since $\tilde{\sigma}$ is a positive function. Using Cauchy-Schwarz's inequality, we obtain the following upper-bound for $\mathcal{S}_+^L(t) := e^{\theta(t_0)} e^{-\int_{t_0}^{t_0+t} |\alpha(s)|ds}\, \mathcal{R}_+^L(t)$:

$$\begin{aligned} \mathcal{S}_+^L(t) &\leq \frac{d}{\sqrt{e}} + \left(\int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|^2}{\tilde{\sigma}(s)^2} ds \int_{t_0}^{t_0+t} \tilde{\sigma}(s)^2 e^{-2\int_0^s \alpha(u)du} ds \right)^{1/2} \\ &= \frac{d}{\sqrt{e}} + \left(\int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|^2}{\tilde{\sigma}(s)^2} ds \right)^{1/2} (\rho(t+t_0) - \rho(t_0))^{1/2}. \end{aligned}$$

Using $\rho(t_0+t)-\rho(t_0)\leqslant d^2$ and $t\leqslant m$, leads to

$$\begin{aligned}\mathcal{R}_+^L(t) &\leqslant d e^{-\theta(t_0)+\int_{t_0}^{t_0+m}|\alpha(s)|ds}\left(\frac{1}{\sqrt{e}}+\sqrt{\int_{t_0}^{t_0+m}\frac{|\beta(s)+x_0\,\alpha(s)|^2}{\tilde{\sigma}(s)^2}ds}\right)\\ &= d\Delta_m.\end{aligned}$$

Under the condition $d\Delta_m + x_0 \leqslant b_{\gamma,x_0}$, we observe that the spheroid belongs to the interval $d\Delta_m + x_0 \leqslant b_{\gamma,x_0}$. Therefore we shall choose

$$d \leqslant \frac{1}{\Delta_m}(b_{\gamma,x_0}-x_0). \tag{3.1.11}$$

Let us now deal similarly with a lower-bound of ψ_-^L. We define

$$\mathcal{R}_-^L(t) := \psi_-^L(t;\, t_0, x_0) - x_0.$$

Hence

$$\begin{aligned}\mathcal{R}_-^L(t) &= e^{-\theta(t_0+t)}\left(\psi_-(\rho(t+t_0)-\rho(t_0))+\int_{t_0}^{t_0+t}\beta(s)e^{-\int_0^s\alpha(u)du}ds\right)\\ &\quad + x_0\left(e^{\int_{t_0}^{t+t_0}\alpha(u)du}-1\right)\\ &\geqslant e^{-\theta(t_0+t)}\left(-\frac{d}{\sqrt{e}}+\int_{t_0}^{t_0+t}(\beta(s)+x_0\,\alpha(s))\;e^{-\int_0^s\alpha(u)du}ds\right)\\ &\geqslant e^{-\theta(t_0+t)}\left(-\frac{d}{\sqrt{e}}-\int_{t_0}^{t_0+t}|\beta(s)+x_0\,\alpha(s)|e^{-\int_0^s\alpha(u)du}ds\right)\\ &\geqslant -e^{-\theta(t_0)}e^{\int_{t_0}^{t_0+m}|\alpha(s)|ds}\left(\frac{d}{\sqrt{e}}+\int_{t_0}^{t_0+t}|\beta(s)+x_0\,\alpha(s)|e^{-\int_0^s\alpha(u)du}ds\right).\end{aligned}$$

Using then the same arguments as for the upper bound, we obtain

$$\psi_-^L(t;\, t_0, x_0) \geqslant -\Delta_m d + x_0.$$

The condition $-\Delta_m d + x_0 \geqslant a_{\gamma,x_0}$ is equivalent to

$$d \leqslant \frac{1}{\Delta_m}(x_0 - a_{\gamma,x_0}). \tag{3.1.12}$$

Combining (3.1.11), (3.1.12) and $d^2 \leqslant \rho(t_0+m)-\rho(t_0)$, we deduce the announced statement. □

3.1.3 WOMS algorithm for L-class diffusions

Let us present now the random walk on spheroids which permits to approximate the L-class diffusion exit time.

ALGORITHM$_m$ (L-class WOMS)

Step 1. Initiate $Z = x_0$ and $\mathcal{T}_\epsilon = 0$

Step 2. While $Z \leqslant b - \epsilon$ and $Z \geqslant a + \epsilon$

Step 2.1 Simulate a couple of independent random variables $(\tau^L, \mathcal{B})$ where τ^L denotes the exit time for the diffusion from the spheroid defined by $\psi^L_\pm$ with coefficient $d = d(\mathcal{T}_\epsilon, Z)$ defined in (3.1.9) and $\mathcal{B}$ is Bernoulli distributed $\mathcal{B}(\frac{1}{2})$. The r.v. $\mathcal{B}$ indicates if the diffusion hits the lower boundary. Due to symmetry properties, its average equals 1/2.

Step 2.2 If $\mathcal{B} = 1$ then set $Z \leftarrow \psi^L_-(\tau^L; \mathcal{T}_\epsilon, Z)$
otherwise set $Z \leftarrow \psi^L_+(\tau^L; \mathcal{T}_\epsilon, Z)$.

Step 2.3 $\mathcal{T}_\epsilon \leftarrow \mathcal{T}_\epsilon + \tau^L$.

Outcome: $\mathcal{T}_\epsilon$ the approximated exit time from the interval $[a, b]$ for the diffusion $(X_t,\ t \geq 0)$.

As usual let us describe the efficiency of the algorithm. This algorithm is particularly efficient since its averaged number of steps is of the order $|\log(\epsilon)|$ and since its outcome $\mathcal{T}_\epsilon$ converges towards the value of the exit time as ϵ tends to 0. We present these two results in details in the following subsections. Even if the statement of these results look like similar to those presented in the Ornstein-Uhlenbeck context (see [26]), the situations are clearly different since here the coefficients - and therefore the size of the spheroids - are time-dependent.
Since the L-class diffusions are non homogeneous, the sequence $(Z_n)_n$ of successive exit positions, appearing in the algorithm, does not define a Markov chain. We need therefore to consider both the successive times and positions (T_n, X_n) in order to deal with a Markov chain. Here T_n stands for the cumulative time:

$$T_n = \sum_{k=1}^{n} \tau^L_k, \quad n \geq 1. \tag{3.1.13}$$

3.2 Properties of the algorithm

3.2.1 Average number of steps

In order to describe precisely the average number of steps in ALGORITHM$_m$, we introduce two crucial additional hypotheses.

Assumption 3.2.1. *There exist $q' \in [0, 1[$ and $q \in [0, 1]$, $C_{\tilde{\sigma},\beta} > 0$ and $\underline{\sigma} > 0$ such that*

$$|\alpha(t)| = \mathcal{O}((\ln t)^{q'}), \quad \textit{for large values of } t, \tag{3.2.1}$$

and

$$\underline{\sigma} \leqslant \tilde{\sigma}(t) \leqslant C_{\tilde{\sigma},\beta}\ t^{q/4}, \quad |\beta(t)| \leqslant C_{\tilde{\sigma},\beta}\ t^{q/4}, \quad \textit{for } t \textit{ large enough.} \tag{3.2.2}$$

Assumption 3.2.2. *There exists $\chi_m > 0$ such that, for any t large enough,*

$$\inf_{s\in[t,t+m]} \tilde{\sigma}(s) \geqslant \chi_m \sup_{s\in[t,t+m]} \tilde{\sigma}(s). \tag{3.2.3}$$

Theorem 3.2.1. *Let us assume that Assumptions 3.2.1 and 3.2.2 are satisfied for a particular parameter $m > 0$. Then for any parameter $\tilde{q} > q$, there exists a constant $C_{\tilde{q}} > 0$ such that N_ϵ, the number of steps observed in* ALGORITHM$_m$ *has the following upper-bound:*

$$\mathbb{E}[N_\epsilon^{1-\tilde{q}}] \leqslant C_{\tilde{q}} |\log(\epsilon)|,$$

for any $\epsilon > 0$ small enough.

In particular, for a L-class diffusion with bounded coefficients, we can prove that $\mathbb{E}[N_\epsilon] \leqslant C_0 |\log(\epsilon)|$, for ϵ small enough.

Let us notice that ALGORITHM$_m$ can be modified in order to approximate the stopping time $\mathcal{T} \wedge T_{\max}$ where $T_{\max}$ is a fixed time horizon. It suffices in such a situation to observe the path skeleton $(T_n, X_n)_{n\geq 0}$ up to the exit from the domain $[0, T_{\max}] \times [a + \epsilon, b - \epsilon]$. The proof of Theorem 3.2.1 can be adapted to this modified algorithm: there exists a constant $C > 0$ such that the average number of spheroids satisfies

$$\mathbb{E}[N_\epsilon] \leqslant C |\log(\epsilon)|,$$

for any $\epsilon > 0$ small enough. Since this result only concerns the diffusion process on the restricted time interval $[0, T_{\max}]$, we don't need any particular assumption on the large time behaviour of the coefficients α, β and $\tilde{\sigma}$. Assumption 3.2.1 and 3.2.2 are therefore not necessary for the modified algorithm.

We postpone the proof of Theorem 3.2.1 and present several preliminary results. First we shall focus our attention on a comparison result between the L-class diffusion and a particular autonomous diffusion. Secondly we describe particular solutions of PDEs related to the diffusion generator. Finally we prove Theorem 3.2.1 using the martingale theory.

A comparison result for SDEs

We introduce two different results: the first one permits to skip the diffusion coefficient in (3.1.1) and the second one permits to replace the time-dependent drift term by a constant drift.

Proposition 3.2.2. *Let $(X_t,\ t \geq 0)$ the solution of the SDE* (3.1.1)*. We define the strictly increasing function γ by*

$$\int_0^{\gamma(t)} \tilde{\sigma}^2(s) ds = t, \quad t \geq 0.$$

Then $Y_t := X_{\gamma(t)}$ satisfies the following SDE

$$dY_t = \Big(\frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} Y_t + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} \Big) dt + dB_t, \quad t \geq 0, \tag{3.2.4}$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion.

Proof. Using the definition of Y_t, we get

$$\begin{aligned} Y_t = X_{\gamma(t)} &= x + \int_0^{\gamma(t)} \Big(\alpha(s)X_s + \beta(s)\Big)\, ds + \int_0^{\gamma(t)} \tilde{\sigma}(s) dW_s \\ &= x + \int_0^t \Big(\alpha(\gamma(s))X_{\gamma(s)} + \beta(\gamma(s))\Big)\gamma'(s)\, ds + B_t \\ &= x + \int_0^t \Big(\alpha(\gamma(s))Y_s + \beta(\gamma(s))\Big)\gamma'(s)\, ds + B_t \end{aligned}$$

where $B_t = \int_0^{\gamma(t)} \tilde{\sigma}(s) dW_s$ is a standard Brownian motion. □

We obtain the following comparison result, its proof can be found in [34] (Chapter VI).

Proposition 3.2.3. *Let $T > 0$ and let us define*

$$\mu_T := \inf_{x\in[a,b],\, t\leq\gamma^{-1}(T)} \Big\{ \frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\, x + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} \Big\}.$$

Let $(Z_t^T)_{t\geq 0}$ the Brownian motion with drift satisfying

$$Z_t^T = x + \mu_T\, t + B_t, \quad t \geq 0. \tag{3.2.5}$$

Then (Y_t) the solution of (3.2.4) with initial condition x satisfies

$$(Z_t^T \leq Y_t \quad a.s., \quad \forall t \leq \gamma^{-1}(T)) \quad \text{and} \quad (Z_{\gamma(t)}^T \leq X_t \quad a.s. \quad \forall t \leq T).$$

Remark 3.2.4. *Choosing rather the particular value*

$$\mu_T := \sup_{x\in[a,b],\, t\leq\gamma^{-1}(T)} \Big\{ \frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\, x + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} \Big\},$$

leads to ($Z_t^T \geq Y_t$ a.s. for all $t \leqslant \gamma^{-1}(T)$).

An Initial-Boundary Value problem

We consider a value problem which is directly linked to the L-class diffusions: let $F : (\mathbb{R}_+, [a,b]) \to \mathbb{R}$ be the solution of

$$\frac{\partial F}{\partial t} + (\alpha(t)x + \beta(t))\frac{\partial F}{\partial x} + \frac{1}{2}\tilde{\sigma}(t)^2 \frac{\partial^2 F}{\partial x^2} = 0 \tag{3.2.6}$$

with initial and boundary conditions $F(0,x) = x$, $F(t,a) = a$, $F(t,b) = b$.

It is well-known (see, for instance, [2], Chap.II) that F admits a probabilistic representation. Indeed

$$F(t,x) = \mathbb{E}_x[X_{t\wedge\mathcal{T}}], \quad \forall t \geq 0, \quad \forall x \in [a,b], \tag{3.2.7}$$

where $(X_t,\, t \geq 0)$ satisfies (3.1.1) and $\mathcal{T}$ stands for the first exit time from the interval $[a, b]$. Let us just note that we don't need at that moment to assume or verify that the exit time $\mathcal{T}$ is almost surely finite. Obviously $t \wedge \mathcal{T}$ is finite and this fact permits to properly define the probabilistic representation (3.2.7). However it would be a nonsense to propose a numerical approximation of an infinite stopping time. Hence, for the sake of completeness, we emphasize the importance of both Assumption 3.2.1 and 3.2.2 which imply the finiteness of $\mathcal{T}$ as a by-product of Theorem 3.2.1 and Theorem 3.2.10 (see Remark 3.2.11).
Let us now list some useful properties of the function F. Since the functions α, β and $\tilde{\sigma}$ in (3.1.1) are Hölder-continuous, we deduce that F, $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ are also Hölder-continuous (see, for instance, Theorem 9 of Chapter 3 in [18]). A combination of classical arguments permits to prove the following statements.

Lemma 3.2.5. *The function $x \mapsto F(t, x)$ defined in (3.2.7) is increasing on the set $[a, b]$.*

Proof. It suffices to compare two paths X and X', having different starting points x and x' with $x \geqslant x'$ and satisfying the same SDE. By coupling properties, we obtain that for all $s \geqslant 0$, $X_s \geqslant X'_s$ and if there exists s_0 such that $X_{s_0} = X'_{s_0}$ then $X_s = X'_s$ for all $s \geqslant s_0$. Several cases can occur concerning the values of $X_{t\wedge\tau_{ab}}$ and $X'_{t\wedge\tau_{ab}}$. Either both exit times occur after the fixed time t, either both exit times occur before t, either only one of them occurs before t. Different situations are illustrated in Figure 3.1. Observing carefully all possible scenarios, it is straightforward to observe $X_{t\wedge\tau_{ab}} \geq X'_{t\wedge\tau_{ab}}$ in any case.

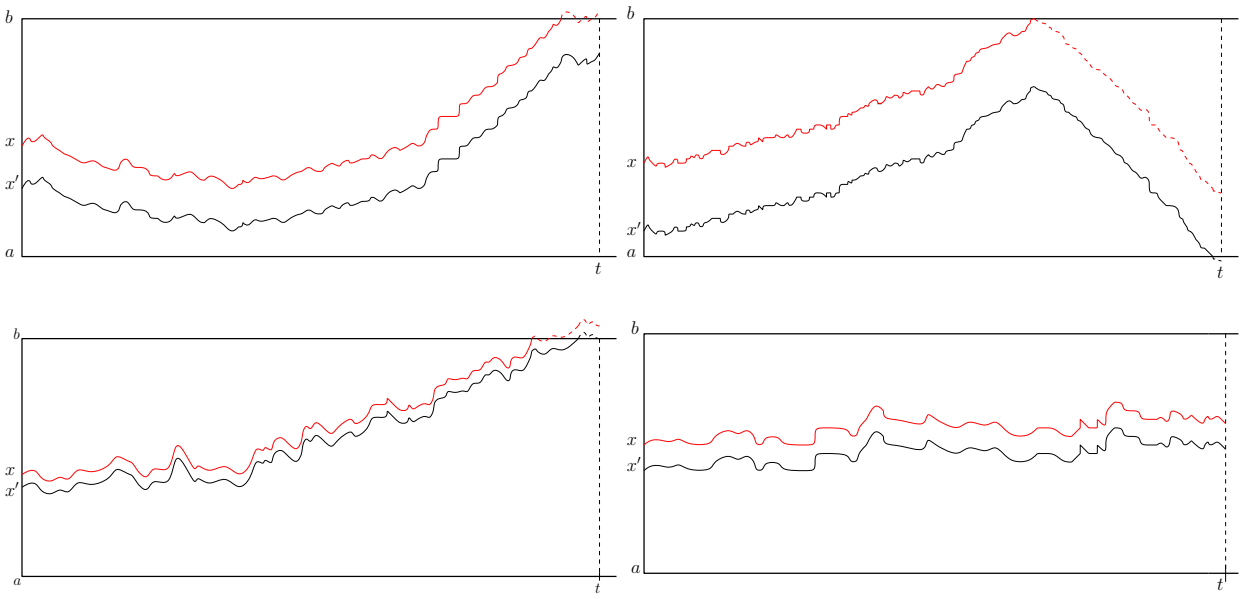


Figure 3.1: Possible scenarios occurring when observing the two different paths driven by the same noise.

□

Proposition 3.2.6. *The function $x \mapsto F(t,x)$ defined in* (3.2.7) *is continuous on the interval* $[a,b]$.

Proof. We consider two strong solutions $(X_t^x)_{t\geq 0}$ and $(\tilde{X}_t^{x+h})_{t\geq 0}$ satisfying (3.1.1) with different starting point spaced by $h>0$. The exit time of the diffusion X^x (respectively $\tilde{X}^{x+h}$) should be denoted by τ_{ab} (resp. $\tilde{\tau}_{ab}$) but for notational simplicity, we skip the index ab. Let T be a fixed time, using the definition of F, we have

$$\begin{aligned}
0 &\leqslant F(T,x+h) - F(T,x)\\
&= \mathbb{E}[\tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau\wedge\tilde{\tau}} + \tilde{X}^{x+h}_{T\wedge\tau\wedge\tilde{\tau}} - X^x_{T\wedge\tau} + X^x_{T\wedge\tau\wedge\tilde{\tau}} - X^x_{T\wedge\tau\wedge\tilde{\tau}}]\\
&= \mathbb{E}[\chi_{T\wedge\tau\wedge\tilde{\tau}}] + \mathbb{E}[(\tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau})1_{\{\tilde{\tau}\geqslant\tau\}} - (X^x_{T\wedge\tau} - X^x_{T\wedge\tilde{\tau}})1_{\{\tilde{\tau}\leqslant\tau\}}],
\end{aligned}$$

where $\chi_{T\wedge\tau\wedge\tilde{\tau}} := \tilde{X}^{x+h}_{T\wedge\tau\wedge\tilde{\tau}} - X^x_{T\wedge\tau\wedge\tilde{\tau}} = he^{\int_0^{T\wedge\tau\wedge\tilde{\tau}}\alpha(u)du} \leqslant he^{\int_0^T\alpha(u)du}$ for all $T\geq 0$. Let $\delta>0$. We can split each term as follows

$$\begin{aligned}
\mathbb{E}[(\tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau})1_{\{\tilde{\tau}\geqslant\tau\}}] &\leqslant \mathbb{E}[(\tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau})1_{\{\tilde{\tau}\geqslant\tau,\ \tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau}>\delta\}}]\\
&\quad + \delta\mathbb{P}(\tilde{\tau}\geqslant\tau, 0\leqslant \tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau}\leqslant\delta)\\
&\leqslant (b-a)\mathbb{P}(\tilde{\tau}\geqslant\tau, \tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{T\wedge\tau}>\delta) + \delta\\
&\leqslant (b-a)\mathbb{P}(\tilde{\tau}\geqslant\tau,\ T\geqslant\tau, \tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{\tau}>\delta) + \delta.
\end{aligned}$$

Similarly we obtain for the second term:

$$\begin{aligned}
\mathbb{E}[(X^x_{T\wedge\tilde{\tau}} - X^x_{T\wedge\tau})1_{\tilde{\tau}\leqslant\tau}] &\leqslant \mathbb{E}[(X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}})1_{\tilde{\tau}_{ab}\leqslant\tau_{ab}, X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}}>\delta}\\
&\quad + (X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}})1_{\tilde{\tau}_{ab}\leqslant\tau_{ab}, 0\leqslant X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}}\leqslant\delta}]\\
&\leqslant (b-a)\mathbb{P}(\tilde{\tau}_{ab}\leqslant\tau_{ab}, X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}}>\delta)\\
&\quad + \delta\mathbb{P}(\tilde{\tau}_{ab}\leqslant\tau_{ab}, 0\leqslant X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}}\leqslant\delta)\\
&\leqslant (b-a)\mathbb{P}(\tilde{\tau}_{ab}\leqslant\tau_{ab}, X^x_{t\wedge\tilde{\tau}_{ab}} - X^x_{t\wedge\tau_{ab}}>\delta) + \delta\\
&\leqslant (b-a)\mathbb{P}(\tilde{\tau}\leqslant\tau, T\geqslant\tilde{\tau}, X^x_{\tilde{\tau}} - X^x_{t\wedge\tau}>\delta) + \delta.
\end{aligned}$$

Both probabilities appearing in the previous upper-bound can be treated in a similar way. We develop the arguments just for one of them: $\mathbb{P}(\tilde{\tau}\geqslant\tau, t\geqslant\tau, \tilde{X}^{x+h}_{t\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{\tau}>\delta)$. Let us introduce the shift process $\xi_t = \tilde{X}^{x+h}_{\tau+t}$. If τ is known (let us say that it is equal to ϕ) then, due to the Markov property of the diffusion, $(\xi_t)_{t\geqslant 0}$ satisfies the following SDE:

$$d\xi_t = (\alpha(t+\phi)\xi_t + \beta(t+\phi))\,dt + \tilde{\sigma}(t+\phi)dB_t, \tag{3.2.8}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and $\xi_0 = \tilde{X}^{x+h}_{\tau}$. Since $\tilde{X}^{x+h}$ and X^x are two strong solutions and since $h>0$, we have $\tilde{X}^{x+h}_t \geq X^x_t$ for any $t\geq 0$. In particular, the event $\tau\leq\tilde{\tau}$ implies that $X^x_\tau = a$. Therefore on the event $\tau\leq\tilde{\tau}$,

$$\xi_0 \leq a + h\exp\int_0^\phi |\alpha(u)|\,du =: a + h\Theta(\phi).$$

By applying the comparison result described in Proposition 3.2.3 (just replacing α by $\alpha(\cdot+\phi)$, β by $\beta(\cdot+\phi)$ and $\tilde{\sigma}$ by $\tilde{\sigma}(\cdot+\phi)$) we obtain that $\xi_{\gamma(t)} \geq Z_t^T$ defined in (3.2.5) for all $t \leq \gamma^{-1}(T)$. Of course Z^T and ξ have the same initial condition. If we denote by $\mathcal{T}_l$ the first passage time through the level l, then

$$\mathbb{P}(\tilde{\tau} \geqslant \tau, T \geqslant \tau, \tilde{X}^{x+h}_{T\wedge\tilde{\tau}} - \tilde{X}^{x+h}_{\tau} > \delta) \leqslant \mathbb{P}_{a+h\Theta(T)}(\mathcal{T}_{a+\delta+h\Theta(T)}(Z^T) \leqslant \mathcal{T}_a(Z^T)), \tag{3.2.9}$$

since $\Theta(\phi) \leq \Theta(T)$.
Let us now let δ depend on h, namely $\delta = \sqrt{h}$. Using the scale function of a drifted Brownian motion, we obtain in the small h limit:

$$\begin{aligned}
&\mathbb{P}_{a+h\Theta(T)}\Big(\mathcal{T}_{a+\delta+h\Theta(T)}(Z^T) \leqslant \mathcal{T}_a(Z^T)\Big) = \frac{e^{-2\mu_T(a+h\Theta(T))} - e^{-2\mu_T a}}{e^{-2\mu_T(a+\delta)} - e^{-2\mu_T a}} \\
&\sim -h\Theta(T)\frac{2\mu_T e^{-2\mu_T a}}{e^{-2\mu_T(a+\delta)} - e^{-2\mu_T a}} \sim -h\Theta(T)\frac{2\mu_T e^{-2\mu_T a}}{\sqrt{h}} \\
&= -2\sqrt{h}\Theta(T)\mu_T e^{-2\mu_T a}.
\end{aligned}$$

Finally we observe that $F(T, x+h)$ converges towards $F(T, x)$ as h tends to 0_+. By symmetry we obtain also the result for $h \to 0_-$. □

The constant c is chosen in such a way that V is non negative on the interval $[a+\gamma\epsilon, b-\gamma\epsilon]$. We define then H by $H = V \circ F$. Using the fact that F is solution of (3.2.6), we obtain that H is solution of the following partial differential equation :

$$\frac{\partial H}{\partial t} + (\alpha(t)x + \beta(t))\frac{\partial H}{\partial x} + \frac{1}{2}\tilde{\sigma}(t)^2\frac{\partial^2 H}{\partial x^2} = \frac{1}{2}\tilde{\sigma}(t)^2 V''(F(t,x))\left(\frac{\partial F}{\partial x}(t,x)\right)^2. \tag{3.2.10}$$

We aim to upper bound the right hand side of this equation. Since V'' is a non positive function, we need to lower bound $\left(\frac{\partial F}{\partial x}(t,x)\right)^2$.

Remark that R defined by :

$$R(t,x) = \exp\left(-\int_0^t \alpha(s)ds\right)\frac{\partial F}{\partial x}(t,x) \tag{3.2.11}$$

is also a solution of (3.2.6).

Proposition 3.2.7. *There exists $\kappa > 0$ such that for all $(t,x) \in \mathbb{R}_+ \times [a,b]$, $\frac{\partial F}{\partial x}(t,x) \geqslant \kappa$.*

Proof. First let us recall that F has a probabilistic representation given by (3.2.7). We shall use this representation in order to lower bound the space derivative. We consider two different cases: small times, that is $t \leq 2$, or large times $t > 2$.
First case: $t > 2$.
We denote by τ^x the first time the process X^x starting at x exits from the interval $]a, b[$ and

by τ_-^x (respectively τ_+^x) the first exit time from $]a, b_h[$ (resp. from the first exit time from $]a_h, b[$) with

$$b_h := b - he^{\int_0^1 |\alpha(s)|\,ds} \quad \text{and} \quad a_h := a + he^{\int_0^1 |\alpha(s)|\,ds}. \tag{3.2.12}$$

We also introduce $(Y_t^{\pm})$ the solutions of the shifted SDEs:

$$dY_t^- = (\alpha(t+\tau_-^x)Y_t + \beta(t+\tau_-^x))\,dt + \tilde{\sigma}(t+\tau_-^x)dW_{t+\tau_-^x}, \tag{3.2.13}$$

with the initial condition $Y_0^- = a + he^{-\int_0^1 |\alpha(s)|\,ds}$ and

$$dY_t^+ = (\alpha(t+\tau_+^{x+h})Y_t + \beta(t+\tau_+^{x+h}))\,dt + \tilde{\sigma}(t+\tau_+^{x+h})dW_{t+\tau_+^{x+h}}, \tag{3.2.14}$$

with the initial condition $Y_0^+ = b - he^{-\int_0^1 |\alpha(s)|\,ds}$. We associate the stopping times $\mathcal{T}(Y^{\pm})$, the exit time from $]a, b[$ and $\mathcal{T}_a(Y^{\pm})$ (resp. $\mathcal{T}_b(Y^{\pm})$) the first passage times through levels a and b to these diffusions.

In order to minimize the derivative of F, we need to lower bound the following expectation, for $h > 0$:

$$F(t, x+h) - F(t, x) = \mathbb{E}[X^{x+h}_{\tau^{x+h}\wedge t} - X^x_{\tau^x \wedge t}].$$

Let us observe particular scenarios which permit the difference between the diffusions to be equal to the maximal value $b - a$. To that end, we introduce two events:

$$E_{ab} := \{\tau_-^x \le 1, X^x_{\tau_-^x} = a,\, \mathcal{T}(Y^-) \le 1, Y^-_{\mathcal{T}(Y^-)} = b\},$$
$$E_{ba} := \{\tau_+^{x+h} \le 1, X^{x+h}_{\tau_+^{x+h}} = b,\, \mathcal{T}(Y^+) \le 1, Y^+_{\mathcal{T}(Y^+)} = a\}.$$

By Lemma 3.4.4 and Lemma 3.4.5 (presented at the end of the chapter) $E_{ab} \cap E_{ba} = \emptyset$ and $E_{ab} \cup E_{ba} \subset \{X^{x+h}_{\tau^{x+h}\wedge t} - X^x_{\tau^x\wedge t} = b - a\}$ for all $t \ge 2$. Hence

$$F(t, x+h) - F(t, x) \ge (b-a)(\mathbb{P}(E_{ab}) + \mathbb{P}(E_{ba})). \tag{3.2.15}$$

Let us first deal with $\mathbb{P}(E_{ab})$. Conditionally to $\tau_-^x = \phi$, the strong Markov property of the diffusion process implies that Y_t^- has the same distribution as the solution of the SDE :

$$d\xi_t = (\alpha(t+\phi)\xi_t + \beta(t+\phi))\,dt + \tilde{\sigma}(t+\phi)dB_t, \quad \xi_0 = Y_0^-, \tag{3.2.16}$$

where (B_t) is a standard Brownian motion. Since $\phi \le 1$ and $\mathcal{T}(Y^-) \le 1$ on the event E_{ab}, we need to describe the paths of the initial diffusions X^x and X^{x+h} on a time interval of length at most equal to 2. We can easily adapt the comparison result of Proposition 3.2.6 to obtain that $\xi_t \ge Z^T_{\gamma(t)}$ for all $t \le 1$ and $T = 2$ (Z_t^T being defined in the statement of Proposition 3.2.3). Let us notice that γ here depends on ϕ. We deduce that

$$\begin{aligned}
\mathbb{P}\Big(\mathcal{T}(Y^-) \le 1, Y^-_{\mathcal{T}(Y^-)} = b \Big| \tau_-^x = \phi\Big) &= \mathbb{P}\Big(\mathcal{T}_b(Y^-) \le 1, Y^-_{\mathcal{T}(Y^-)} = b \Big| \tau_-^x = \phi\Big) \\
&= \mathbb{P}\Big(\mathcal{T}_b(Z) \le \gamma^{-1}(1), Z_{\mathcal{T}(Z)} = b \Big| \tau_-^x = \phi\Big) \\
&\ge \mathbb{P}\Big(\mathcal{T}_b(L) \le \gamma_\phi^{-1}(1), L_{\mathcal{T}(L)} = b \Big| \tau_-^x = \phi\Big) \\
&\ge \mathbb{P}\Big(\mathcal{T}_b(Z) \le \underline{\sigma}^2, Z_{\mathcal{T}(Z)} = b \Big| \tau_-^x = \phi\Big),
\end{aligned} \tag{3.2.17}$$

where $\underline{\sigma}$ is the uniform lower bound of $\tilde{\sigma}(t)$. Indeed

$$\gamma^{-1}(1) = \int_0^1 \tilde{\sigma}^2(s+\phi)\, ds \geq \underline{\sigma}^2.$$

We observe that the lower bound in (3.2.17) does not depend on ϕ. Consequently

$$\begin{aligned}\mathbb{P}(E_{ab}) &= \mathbb{E}\Big[1_{\{\tau_-^x \leq 1, X^x_{\tau_-^x}=a\}} \mathbb{P}\Big(\mathcal{T}(Y^-) \leq 1, Y^-_{\mathcal{T}(Y^-)} = b \Big| \tau_-^x\Big)\Big] \\ &\geq \mathbb{P}\Big(\tau_-^x \leq 1, X^x_{\tau_-^x} = a\Big)\mathbb{P}\Big(\mathcal{T}_b(Z) \leq \underline{\sigma}^2, Z_{\mathcal{T}(Z)} = b\Big).\end{aligned}$$

Let us assume now that $x \in]a, \frac{a+b}{2}]$ and $h \leq h_0$. By comparison, the trajectory of X^x always stays below that of $X^{(a+b)/2}$. Setting $r_h = (b-a)/(b_h - a)$ we get

$$\begin{aligned}\mathbb{P}\Big(\tau_-^x \leq 1, X^x_{\tau_-^x} = a\Big) &= \mathbb{P}\Big(\mathcal{T}(r_h X^x + a(1-r_h)) \leq 1, X^x_{\tau_-^x} = a\Big) \\ &= \mathbb{P}\Big(\mathcal{T}_a(r_h X^x + a(1-r_h)) \leq 1, X^x_{\tau_-^x} = a\Big) \\ &= \mathbb{P}\Big(\mathcal{T}_a(X^x) \leq 1, \mathcal{T}_a(X^x) < \mathcal{T}_b(r_h X^x + a(1-r_h))\Big) \\ &\geq \mathbb{P}\Big(\mathcal{T}_a(X^{(a+b)/2}) \leq 1, \mathcal{T}_a(X^{(a+b)/2}) < \mathcal{T}_b(r_h X^{(a+b)/2} + a(1-r_h))\Big) \\ &\geq \mathbb{P}\Big(\mathcal{T}_a(X^{(a+b)/2}) \leq 1, \mathcal{T}_a(X^{(a+b)/2}) < \mathcal{T}_b(r_{h_0} X^{(a+b)/2} + a(1-r_{h_0}))\Big) \\ &=: \kappa_1\end{aligned}$$

where κ_1 is a positive constant independent of both h and x. Hence

$$\mathbb{P}(E_{ab}) \geq \kappa_1 1_{]a, \frac{a+b}{2}]}(x)\Psi(h), \quad \text{with} \quad \Psi(h) := \mathbb{P}\Big(\mathcal{T}_b(Z) \leq \underline{\sigma}^2, Z_{\mathcal{T}(Z)} = b\Big). \tag{3.2.18}$$

It suffices to lower bound the function Ψ using scale functions and an independent exponential random variable which permits to relate the computation of Ψ to a particular Laplace transform whose expression is explicit (see, [7] p309).
Let $\mathcal{E}$ be an exponentially distributed random variable with parameter λ and let $h_\alpha = he^{-\int_0^1 |\alpha(u)|du}$. Then $\Psi(h)$ can be lower-bounded by the difference of $\Psi_1(h)$ and $\Psi_2(h)$:

$$\begin{aligned}\Psi(h) &:= \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}_b(Z) \leq \underline{\sigma}^2, Z_{\mathcal{T}(Z)} = b\Big) \\ &\geqslant \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) \leq \mathcal{E}, \mathcal{T}(Z) = \mathcal{T}_b(Z)\Big) - \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) = \mathcal{T}_b(Z), \mathcal{T}(Z) \leqslant \underline{\sigma}^2, \mathcal{E} > \underline{\sigma}^2\Big) \\ &= \Psi_1(h) - \Psi_2(h).\end{aligned}$$

The first term of the r.h.s $\Psi_1(h)$ is evaluated as follows

$$\begin{aligned}
\Psi_1(h) &= \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}_b(Z) \leq \mathcal{E}, \mathcal{T}(Z) = \mathcal{T}_b(Z)\Big) \\
&= \mathbb{P}_{a+h_\alpha}\Big(e^{-\lambda\mathcal{T}} \geqslant U, \mathcal{T}(L) = \mathcal{T}_b(L)\Big) \\
&= \mathbb{E}_{a+h_\alpha}\Big[e^{-\lambda\mathcal{T}}\ 1_{\{\mathcal{T}(Z)=\mathcal{T}_b(Z)\}}\Big] = e^{\mu(b-a-h_\alpha)}\frac{\sinh(h_\alpha\sqrt{2\lambda+\mu^2})}{\sinh((b-a)\sqrt{2\lambda+\mu^2})} \\
&\sim e^{\mu(b-a)}\frac{h_\alpha\sqrt{2\lambda+\mu^2}}{\sinh((b-a)\sqrt{2\lambda+\mu^2})} \\
&\sim e^{\mu(b-a)}\frac{he^{-\int_0^1|\alpha(u)|du}\sqrt{2\lambda+\mu^2}}{\sinh((b-a)\sqrt{2\lambda+\mu^2})},\ \text{as } h \text{ tends to } 0.
\end{aligned}$$

The second term $\Psi_2(h)$ has to be upper bounded:

$$\begin{aligned}
\Psi_2(h) &= \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) = \mathcal{T}_b(Z), \mathcal{T}(Z) \leqslant \underline{\sigma}^2, \mathcal{E} > \underline{\sigma}^2\Big) \\
&= \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) = \mathcal{T}_b(Z), \mathcal{T}(Z) \leqslant \underline{\sigma}^2,\Big)\mathbb{P}\Big(\mathcal{E} > \underline{\sigma}^2\Big) \\
&\leqslant \mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) = \mathcal{T}_b(Z)\Big)e^{-\lambda\underline{\sigma}^2}.
\end{aligned}$$

Using the scale function of the drifted Brownian motion, we obtain

$$\begin{aligned}
\mathbb{P}_{a+h_\alpha}\Big(\mathcal{T}(Z) = \mathcal{T}_b(Z)\Big) &= e^{-\int_0^1|\alpha(u)|du}\frac{e^{-2\mu(a+h_\alpha)} - e^{-2\mu a}}{e^{-2\mu b} - e^{-2\mu a}} \\
&\sim -he^{-\int_0^1|\alpha(u)|du}\frac{2\mu e^{-2\mu a}}{e^{-2\mu b} - e^{-2\mu a}} \text{ as } h \text{ tends to } 0.
\end{aligned}$$

If the parameter of the exponentially distributed r.v. becomes large then it is easy to prove that $\Psi_2(h)$ becomes negligible with respect to $\Psi_1(h)$. Consequently we can choose a particular value of λ which leads to $2\Psi_2(h) \leq \Psi_1(h)$ and therefore permits to bound $\Psi(h)$ by below.
Combining (3.2.18) and the description of $\Psi(h)$, we manage to bound $\mathbb{P}(E_{ab})$ by below for $x \leq (a+b)/2$. In the case where the starting point of the diffusion is in the lower part of the interval, we bound $\mathbb{P}(E_{ba})$ by below with the value 0 which implies the existence of a strictly positive lower bound of $\mathbb{P}(E_{ab}) + \mathbb{P}(E_{ba})$. In the other case (i.e. the starting point is in the upper part of the interval), we bound $\mathbb{P}(E_{ab})$ by below with the value 0 and deal with $\mathbb{P}(E_{ba})$ in a similar way as previously described. In any case, the inequality (3.2.15) leads to the existence of $\kappa > 0$ such that

$$\frac{\partial F}{\partial x}(t,x) \geqslant \kappa, \quad \forall (t,x) \in [2,\infty[\times[a,b].$$

Second case: $t \leqslant 2$.
First we consider the derivative at the boundary of the interval $[a,b]$. Let us note that

$F(t,a) = a$. Hence $\frac{\partial F}{\partial x}(t,a) = \lim_{h\to 0^+} \frac{1}{h}\left(\mathbb{E}_{a+h}[X_{t\wedge\tau_{ab}}] - a\right)$. Since we need a lower bound, we shall use a comparison result concerning the L-class diffusions. Proposition 3.14 leads to $Z^T_{\gamma(t)} \leq X_t$ for all $t \leq T$. We set here $T = 2$ and μ_T is defined in the statement of the proposition. If $\mu_T \geq 0$ then we replace it by a strictly negative value and therefore the comparison result remains true. So we assume for the sequel that $\mu_T < 0$. We deduce that

$$\mathbb{E}_{a+h}[X_{t\wedge\tau_{ab}}] - a \geq \mathbb{E}_x[Z^T_{\gamma(t)\wedge\tau_{ab}}], \tag{3.2.19}$$

where τ_{ab} stands either for the exit time of X either for the exit time of Z. Let us now consider the convex function $f(x) = e^{-2\mu_T x}$. It is well known that Then $M^T_t := f(Z^T_t)$ is a martingale. The function f is a convex one : if $\mu_T < 0$ (if $\mu_T \geqslant 0$ we just take the drift equal to 0) then

$$x - a \geq \frac{(b-a)}{f(b) - f(a)}\left(f(x) - f(a)\right), \forall x \in]a, b[.$$

As f is the scale function of the drifted Brownian motion, $f(Z^T_t)$ is a martingale and the optimal stopping theorem leads to

$$\begin{aligned}\mathbb{E}_x[Z^T_{\gamma(t)\wedge\tau_{ab}} - a] &\geq \frac{(b-a)}{f(b) - f(a)}\,\mathbb{E}_x\left[e^{-2\mu_T Z^T_{\gamma(t)\wedge\tau_{ab}}} - e^{-2\mu_T a}\right] \\ &= \frac{(b-a)}{f(b) - f(a)}\left(e^{-2\mu_T x} - e^{-2\mu_T a}\right).\end{aligned}$$

In particular, for $x = a + h$,

$$\begin{aligned}\mathbb{E}_{a+h}[Z^T_{t\wedge\tau_{ab}} - a] &\geq (b-a)e^{2\mu_T(b-a)}(e^{-2\mu_T h} - 1) \\ &\sim -2\mu_T h(b-a)e^{2\mu_T(b-a)}, \text{ as h tends to 0.}\end{aligned}$$

We obtained the existence of a constant $\eta^a_T > 0$ such that $\frac{\partial F}{\partial x}(t,a) \geqslant \eta^a_T$, for any $t \leqslant 2$. By similar arguments, we can obtain $\frac{\partial F}{\partial x}(t,b) \geqslant \eta^b_T$, for all $t \leqslant 2$. Since $\frac{\partial F}{\partial x}(t,x)$ satisfies a second order parabolic PDE with regular coefficients, we can apply the maximum principle (see, for instance, [17] or [18]). Consequently the minimum of the derivative on the domain $[0,2] \times [a,b]$ is reached at the boundary. Let us observe what happens on each side of this rectangle. For $x = a$ we have just proven that there exists a minimum which is strictly positive so is it for $x = b$. For $t = 0$ the derivative is equal to 1 and for $t = 2$ the first part of the proof ensures the derivative to be minimized. To sum up, the derivative is lower bounded by a strictly positive constant on the whole rectangle $[0,2] \times [a,b]$. □

Proposition 3.2.8. *There exists two constants $\kappa_a > 0$ and $\kappa_b > 0$ such that*

$$F(t,x) - a \leqslant \kappa_a(x-a) \text{ and } b - F(t,x) \leqslant \kappa_b(b-x), \tag{3.2.20}$$

for all $(t,x) \in \mathbb{R}_+ \times [a,b]$.

Proof. Let us recall the probabilistic representation: $F(t,x) = \mathbb{E}[X^x_{t\wedge\mathcal{T}}]$.
We set $T = \gamma(1)$ and consider (Z^T_t) the diffusion introduced in Remark 3.2.4 with initial condition $Z^T_0 = X^x_0 = x$. We construct a new continuous diffusion process (Z_t) which is equal to (Z^T_t) on the time interval $[0,1]$ and which satisfies the following SDE otherwise:

$$dZ_t = \Big(\frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\, Z_t + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\Big)\, dt + dW_t, \quad t > 1.$$

Extending the comparison result of Remark 3.2.4, we know that $Z_t \geq X_{\gamma(t)}$ for all $t \geq 0$. Hence

$$F(t,x) - a \leq \mathbb{E}_x[Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a].$$

We split the study into two different cases :

- First case: $\gamma^{-1}(t) \leq 1$. The function $f(x) = e^{-2\mu_T x}$ plays an important role since $f(Z_t)$ is a martingale for $t \leq \gamma(1)$. Using twice the Lagrange mean theorem combined with the optional stopping theorem implies

$$\begin{aligned} F(t,x) - a &\leq \eta_1 \mathbb{E}_x\Big[e^{-2\mu_T Z^T_{t\wedge\mathcal{T}}} - e^{-2\mu_T a}\Big] = \eta_1\Big(e^{-2\mu_T x} - e^{-2\mu_T a}\Big) \\ &\leq \kappa_a (x-a), \end{aligned}$$

  where $\kappa_a = \Big(\sup_{x\in[a,b]} f'(x)\Big)\Big(\inf_{x\in[a,b]} f'(x)\Big)^{-1}$.

- Second case: $\gamma^{-1}(t) > 1$. We decompose F as follows

$$\begin{aligned} F(t,x) - a &\leq \mathbb{E}_x[(Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a)1_{\{\mathcal{T}(Z)>1\}}] \\ &\quad + \mathbb{E}_x[(Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a)1_{\{\mathcal{T}(Z)\leq 1\}}] \\ &\leq (b-a)\mathbb{P}_x(\mathcal{T}(Z^T) > 1) + \mathbb{E}_x[(Z^T_{1\wedge\mathcal{T}(Z^T)} - a)1_{\{\mathcal{T}(Z^T)\leq 1\}}] \\ &\leq (b-a)\mathbb{E}_x[\mathcal{T}(Z^\mu)] + \mathbb{E}_x[Z^\mu_{1\wedge\mathcal{T}(Z^\mu)}] - a. \end{aligned}$$

  The expression $\mathbb{E}_x[Z^T_{1\wedge\mathcal{T}(Z^T)}] - a$ can be bounded using similar arguments (Lagrange's mean and optional stopping theorems) as those presented in the first part of the proof. Moreover, let us note that the function $g(x) := \mathbb{E}_x[\mathcal{T}(Z^T)]$ is solution ([2], page 45, Theorem 1.2) of

$$\frac{1}{2}\, g'' + \mu_T g' = -1 \quad \text{for } x \in]a,b[\quad \text{and } g(a) = g(b) = 0.$$

  We recall that $T = \gamma(1)$ and μ_T is defined in Remark 3.2.4. The explicit solution of this equation is given by

$$g(x) = \frac{(b-a)(e^{-2\mu_T a} - e^{-2\mu_T x})}{\mu_T(e^{-2\mu_T a} - e^{-2\mu_T b})} - \frac{(x-a)}{\mu_T}.$$

  Applying once again Lagrange's mean theorem, we obtain the existence of a constant $C_g > 0$ such that $g(x) \leqslant C_g(x-a)$ for all $x \in [a,b]$. Using similar arguments (just replacing Remark 3.2.4 by Proposition 3.2.3), we also prove that $b - F(t,x) \leqslant \kappa_b(b-x)$.

□

Proof of Theorem 3.2.1.

We already presented all the necessary ingredients in order to prove the statement of Theorem 3.2.1 which concerns the average number of steps.

Proof. Our choice for the bound of the average number of steps is based on the martingale theory. We recall that F is defined by (3.2.7) and introduce another important function H defined by $H = V \circ F$ with

$$V(x) = \log\left(\frac{(x-a)(b-x)}{\gamma\epsilon(b-a-\gamma\epsilon)}\right). \tag{3.2.21}$$

Let us note that V is non negative on the whole interval $[a+\gamma\epsilon, b-\gamma\epsilon]$. Since F is a the solution of (3.2.6), the function H just introduced satisfies the following partial differential equation:

$$\frac{\partial H}{\partial t} + (\alpha(t)x + \beta(t))\frac{\partial H}{\partial x} + \frac{1}{2}\tilde{\sigma}(t)^2\frac{\partial^2 H}{\partial x^2} = \frac{1}{2}\tilde{\sigma}(t)^2 V''(F(t,x))\left(\frac{\partial F}{\partial x}(t,x)\right)^2. \tag{3.2.22}$$

Let us also recall that (T_n, X_n) defined in (3.1.13) is the sequence of successive exit times and exit positions issued from ALGORITHM$_m$.
We focus our attention on the sequence $Z_n = H(X_n)+G(n)$ with $G(0) = 0$. Here G stands for a positive function, we are going to precise this function in the sequel. This stochastic process is a super-martingale with respect to the Brownian filtration $(\mathcal{F}_{T_n})_{n\in\mathbb{N}}$. Using Itô's formula and the partial differential equation satisfied by H, we obtain for $\mathcal{D}_n := \mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_{T_n}]$,

$$\begin{aligned}
\mathcal{D}_n &= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{\partial H}{\partial t}(s,X_s) + (\alpha(s)X_s + \beta(s))\frac{\partial H}{\partial x}(s,X_s)\right.\\
&\quad \left. + \frac{1}{2}\tilde{\sigma}(s)^2\frac{\partial^2 H}{\partial x^2}(s,X_s)ds\,\middle|\,\mathcal{F}_{T_n}\right]\\
&\quad + \mathbb{E}[M_{n+1} - M_n|\mathcal{F}_{T_n}] + (G(n+1) - G(n))\\
&= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{1}{2}\tilde{\sigma}(s)^2 V''(F(s,X_s))\left(\frac{\partial F}{\partial x}(s,X_s)\right)^2 ds\,\middle|\,\mathcal{F}_{T_n}\right]\\
&\quad + (G(n+1) - G(n)),
\end{aligned}$$

where $(M_n)_{n\in\mathbb{N}} = \left(\int_0^{T_n}\tilde{\sigma}(s)\frac{\partial H}{\partial x}(s,X_s)dW_s\right)_{n\in\mathbb{N}}$ is a martingale. Using Lemma 3.2.7, Proposition 3.2.8 and the lower bound $\underline{\sigma}$ of $\tilde{\sigma}$ we obtain

$$\mathcal{D}_n \le -\frac{1}{2}\underline{\sigma}^2\kappa^2(\mathcal{I}(a) + \mathcal{I}(b)) + G(n+1) - G(n), \tag{3.2.23}$$

where $\mathcal{I}(x) = \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{1}{\kappa_x^2(X_s-x)^2}\,ds\,\middle|\,\mathcal{F}_{T_n}\right]$.

We aim to bound by below the previous integral by considering the shape of the n^{th} spheroid:

$$\begin{aligned}\psi_+^L(t) - a_{\gamma,X_n} &\leqslant d_n\Delta_m + X_n - a_{\gamma,X_n} \\ &\leqslant \min(1,\kappa_-)(X_n - a_{\gamma,X_n}) + X_n - a_{\gamma,X_n} \\ &\leq 2(X_n - a). \end{aligned} \tag{3.2.24}$$

This bound implies

$$\begin{aligned}\mathcal{I}(a) &\geqslant \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{ds}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right] = \mathbb{E}\left[\frac{T_{n+1}-T_n}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right] \\ &= \mathbb{E}\left[\frac{\rho_L^{-1}(\rho_L(T_n)+\tau_{n+1})}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right]\end{aligned}$$

where τ_{n+1} is the Brownian exit time from the spheroid of parameter size d_n.

$$\mathcal{I}(a) \geq \mathbb{E}\left[\frac{\tau_{n+1}}{4\kappa_a^2 r_n(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right]$$

where r_n is the maximum of the derivative ρ' on the time interval $[T_n, T_n+m]$ which contains $[T_n, \rho_L^{-1}(\rho_L(T_n)+\tau_{n+1})]$. We note that $\tau_{n+1} \sim d_n^2\tau$ where τ denotes the Brownian exit time from the Brownian spheroid of parameter 1. Hence

$$\mathcal{I}(a) \geq \frac{d_n^2}{4\kappa_a^2 r_n(X_n-a)^2}\,\mathbb{E}[\tau].$$

Similarly to (3.2.24) we have $b_{\gamma,X_n} - \psi_-^L(t) \leq 2(b-X_n)$ and the same arguments just presented lead to

$$\mathcal{I}(b) = \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{ds}{\kappa_b^2(b-X_s)^2}\middle|\mathcal{F}_{T_n}\right] \geqslant \frac{d_n^2}{4\kappa_b^2 r_n(b-X_n)^2}\,\mathbb{E}[\tau].$$

Setting $\kappa_{ab} = \max(\kappa_a, \kappa_b)$, we obtain

$$\mathcal{D}_n \leqslant -\frac{d_n^2}{r_n\kappa_{ab}^2}\,\mathbb{E}[\tau]\left(\frac{1}{(b-X_n)^2} + \frac{1}{(X_n-a)^2}\right) + G(n+1) - G(n).$$

Let us first consider the case: $X_n - a \leqslant b - X_n$ (the other case can be studied in a similar way, it suffices to replace $X_n - a_{\gamma,X_n}$ by $b_{\gamma,X_n} - X_n$). Then $d_n = \frac{\min(1,\kappa_-)}{\Delta_m}(X_n - a_{\gamma,X_n})$ and

$$\begin{aligned}\mathcal{D}_n &\leqslant -2\frac{d_n^2}{r_n\kappa_{ab}^2}\,\mathbb{E}[\tau]\frac{1}{(X_n-a)^2} + G(n+1) - G(n) \\ &\leqslant -\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2\kappa_{ab}^2}\,\mathbb{E}[\tau] + G(n+1) - G(n).\end{aligned}$$

We finally find G by seeking a lower bound of $\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2}$. We consider two different cases:
First case: $\kappa_- \geq 1$. We introduce α_n, β_n and $\tilde{\sigma}_n$ the maximum of $|\alpha|$ respectively $|\beta|$ and $\tilde{\sigma}$

on the time interval $[0, nm]$. The definition of Δ_m given by (3.1.10) and the definition of ρ by (3.1.4) lead to

$$\begin{aligned}\Delta_m^2 r_n &\leqslant e^{4\int_{T_n}^{T_n+m}|\alpha(s)|\,ds}\tilde{\sigma}_n^2\left(\frac{1}{\sqrt{e}}+\sqrt{\int_{T_n}^{T_n+m}\frac{|\beta(s)|^2}{\tilde{\sigma}(s)^2}ds}\right)^2\\ &\leqslant e^{4m\alpha_n}\tilde{\sigma}_n^2\left(\frac{1}{\sqrt{e}}+\sqrt{m}\frac{\beta_n}{\underline{\sigma}}\right)^2.\end{aligned}$$

For the other case: $\kappa_- < 1$

$$\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2} \geqslant \frac{\rho(T_n+m)-\rho(T_n)}{r_n(b-a)^2} = \frac{\int_0^m \rho'(T_n+s)ds}{r_n(b-a)^2}.$$

Using the definitions of ρ, r_n and the continuity of $\tilde{\sigma}$, there exists $t_0 \in [T_n, T_n+m]$ such that $r_n = \rho'(t_0)$ and therefore

$$\begin{aligned}\frac{\rho'(T_n+s)}{r_n} &= \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2\int_{t_0}^{T_n+s}\alpha(u)\,du} \geq \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2|T_n+s-t_0|\alpha_n}\\ &\geq \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2m\alpha_n}.\end{aligned}$$

Since $\tilde{\sigma}$ satisfies Assumption 3.2.2, we obtain the following lower bound by integrating with respect to the variable s,

$$\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2} \geq \frac{m\chi_m}{(b-a)^2}e^{-2m\alpha_n}.$$

Denoting ζ_{n+1} the minimum of the two quantities previously computed, we define recursively the sequence $G(n)$ by

$$G(n+1)-G(n) = \zeta_{n+1}, \quad \forall n \geqslant 0, \quad \text{and } G(0) = 0.$$

The sum of these increments leads to

$$\sum_{i=0}^{n-1} G(i+1)-G(i) = \sum_{i=1}^{n}\zeta_i = G(n)-G(0) = G(n).$$

For any parameter $\tilde{q} > q$, Assumption 3.2.1 implies the existence of a constant $\tilde{C} > 0$ independent of ϵ such that

$$G(n) \geqslant \frac{1}{\tilde{C}}\sum_{k=1}^{n}\frac{1}{k^{\tilde{q}}} \geqslant \frac{1}{\tilde{C}(1-\tilde{q})}(n^{1-\tilde{q}}-1), \quad \forall n \geq 1. \tag{3.2.25}$$

Moreover the particular choice of the function G permits to obtain $\mathcal{D}_n \leq 0$ for all n. Consequently $Z_n = H(n, X_n) + G(n)$ is a super-martingale. A generalization of Proposition 1.3.4 permits to obtain the upper bound

$$\mathbb{E}[G(N_\epsilon)] \leq H(0, x_0) = V \circ F(0, x_0) = V(x_0). \tag{3.2.26}$$

Combining (3.2.25), (3.2.26) and the definition of the function V in (3.2.21) leads to

$$\mathbb{E}[N_\epsilon^{1-\tilde{q}}] \leqslant \tilde{C}(1-\tilde{q}) \log\left(\frac{(x_0-a)(b-x_0)}{\gamma\epsilon(b-a-\gamma\epsilon)}\right) + 1.$$

This bound corresponds to the announced result. In order to conclude the proof, we just need to precise that N_ϵ is a.s. finite, see Lemma 3.2.9. Such a condition is required to apply the generalization of Proposition 3.4.3. □

Lemma 3.2.9. *The stopping procedure N_ϵ of* ALGORITHM$_m$ *is a.s. finite. Moreover the outcome of the algorithm $\mathcal{T}_\epsilon$ is stochastically upper bounded by $\mathcal{T}$, the diffusion first exit time.*

Proof. Step 1. We emphasize a link between a sample of a L-class diffusion process and the Markov chain generated by the algorithm, denoted $((T_n, X_n))_{n\in\mathbb{N}}$ with $(T_0, X_0) = (0, 0)$.
Let us consider a sample of a L-class diffusion. At the starting point of this path, we create a spheroid of maximal size which belongs to the set $[a, b] \times \mathbb{R}_+$. The first intersection point of this spheroid and the path gives us a first point (t_1, z_1). This construction implies that (t_1, z_1) and (T_1, X_1) are identically distributed. Then considering (t_1, z_1) as a new starting point we construct a spheroid of maximal size and denote by (t_2, z_2) the first intersection point between this new spheroid and the diffusion path starting in (t_1, z_1). Once again we get by construction that (t_2, z_2) and (T_2, X_2) are identically distributed. We build step by step a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the considered sample and the spheroids in such a way that the sequences $((t_n, z_n))_{n\geq 0}$ and $((T_n, X_n))_{n\geq 0}$ are identically distributed.
If we introduce N_ϵ the stopping time appearing in the stopping procedure of the algorithm and $\tilde{N}_\epsilon = \inf\{n \in \mathbb{N}, z_n \notin [a+\epsilon, b-\epsilon]\}$, the identity in law of those random variables holds. By construction, $t_n \leqslant T$ for all $n \in \mathbb{N}$, where T stands for the diffusion first exit time from the interval $[a, b]$. This inequality remains true when t_n is replaced by the random stopping time $t_{\tilde{N}_\epsilon}$.
Since $t_{\tilde{N}_\epsilon}$ and t_{N_ϵ} are identically distributed, we deduce that the outcome of ALGORITHM$_m$ is stochastically smaller than $\mathcal{T}$.
Step 2. We prove now that N_ϵ is a.s. finite. Using (3.1.13) and (3.1.8) we obtain

$$T_n = \rho_L^{-1}(d_1^2\tau_1 + d_2^2\tau_2 + \ldots + d_n^2\tau_n),$$

where $(\tau_k)_{k\geqslant 1}$ is a sequence of independent Brownian exit times from the unit spheroid and d_k represents the size of the spheroid (3.1.9) starting in (T_k, X_k) and included in $[a, b]$. Let

$t_0 > 0$. Then

$$\begin{aligned}\mathbb{P}(T_n \leqslant t_0) &= \mathbb{P}(d_1^2\tau_1 + d_2^2\tau_2 + \ldots + d_n^2\tau_n \leqslant \rho_L^{-1}(t_0)) \\ &\leqslant \mathbb{P}\left(\tau_1 + \tau_2 + \ldots + \tau_n \leqslant \frac{\rho_L^{-1}(t_0)}{\underline{d}(t_0)}\right),\end{aligned}$$

where $\underline{d}(t_0)$ is defined by

$$\underline{d}(t_0) = \inf_{x\in[a+\epsilon,b-\epsilon],\ t\leqslant t_0} d(x,t) > 0.$$

Since $\sum_{k+1}^{n} \tau_k$ tends to $+\infty$ a.s.,

$$\lim_{n\to+\infty} \mathbb{P}(T_n \leqslant t_0) = \mathbb{P}(T_\infty \leqslant t_0) = 0, \quad \forall t_0 > 0.$$

We deduce that $\lim_{n\to+\infty} T_n = +\infty$ a.s. Combining this limiting result to the first step of the proof, that is $T_n \overset{(d)}{\leqslant} T$, implies: $N_\epsilon < +\infty$ a.s. □

3.2.2 Bounds for the exit time distribution

The second important result in the study of the algorithm is the description of the convergence. It is of prime interest to known how close the outcome of the algorithm and the exit time of the L-class diffusion are. The convergence result is essentially based on the strong relation between the Brownian motion and the L-class diffusion.

Theorem 3.2.10. *Let us denote by $\overline{\alpha}_t$ (respectively $\overline{\beta}_t$) the maximal value of the function $|\alpha|$ (resp. $|\beta|$) on the interval $[0,t]$. We also introduce F the cumulative distribution function of the L-class diffusion exit time from the interval $[a,b]$ and F_ϵ the distribution function of the algorithm outcome. Then, for any $t \geq 0$ and any $\rho > 1$ there exists $\epsilon_0 > 0$ such that*

$$\left(1 - \rho\sqrt{\epsilon}\ \frac{1+\overline{\beta}_t}{\underline{\sigma}}\right) F_\epsilon(t-\epsilon) \leqslant F(t) \leqslant F_\epsilon(t), \quad \forall \epsilon \leq \epsilon_0, \tag{3.2.27}$$

the constant $\underline{\sigma}$ being defined in (3.2.2). *Moreover this convergence is uniform on each compact subset of the time axis.*

Remark 3.2.11. *The combination of both Theorem 3.2.1 and Theorem 3.2.10 points the finiteness of the diffusion exit time out. The first statement ensures that* ALGORITHM$_m$ *requires a finite number of iterations almost surely (the average number being finite). The second result explains how close the exit time of the diffusion and the algorithm outcome are. In particular, as an immediate consequence of* (3.2.27), $\lim_{t\to\infty} F(t) = 1$. *Of course, such a crucial property is strongly related to the fact that the diffusion generator is uniformly parabolic, see the condition* (3.2.2).

Proof. As in Lemma 3.2.9, we build step by step a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the path of the L-class diffusion process and the spheroids in such a way that the sequences $((t_n, z_n))_{n\geq 0}$ and $((T_n, X_n))_{n\geq 0}$ are identically distributed.
If we introduce N_ϵ the stopping time appearing in the stopping procedure of the algorithm and $\tilde{N}_\epsilon = \inf\{n \in \mathbb{N}, z_n \notin [a+\epsilon, b-\epsilon]\}$, the identity in law of those random variables holds. By construction, $t_n \leqslant \mathcal{T}$ for all $n \in \mathbb{N}$, where $\mathcal{T}$ stands for the diffusion first exit time from the interval $[a, b]$. This inequality remains true when t_n is replaced by the random stopping time $t_{\tilde{N}_\epsilon}$. Hence

$$\begin{aligned} 1 - F(t) = \mathbb{P}(\mathcal{T} > t) &= \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) + \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} > t-\delta) \\ &\leqslant \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) + 1 - F_\epsilon(t-\delta), \quad \forall t \geq 0. \end{aligned} \tag{3.2.28}$$

We focus our attention on the first term of the r.h.s. Using the strong Markov property, we obtain

$$\mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) \leqslant F_\epsilon(t-\delta) \sup_{(y,s)\in([a,a+\epsilon]\cup[b-\epsilon,b])\times[0,t-\delta]} \mathbb{P}_{(y,\tau)}(\mathcal{T} > \delta). \tag{3.2.29}$$

Let us consider the case $y \in [b-\epsilon, b]$ (the study of the other case $y \in [a, a+\epsilon]$ is left to the reader since it suffices by symmetry to use exactly the same arguments). We first note that, for any $y \in [b-\epsilon, b]$,

$$\mathbb{P}_{(y,s)}(\mathcal{T} > \delta) \leq \mathbb{P}_{(y,s)}(\mathcal{T}_b > \delta) \leq \mathbb{P}_{(b-\epsilon,s)}(\mathcal{T}_b > \delta),$$

where $\mathcal{T}_b$ stands for the first passage time through the level b. Let us introduce several notations: we denote the translated function $\alpha_s(t) := \alpha(s+t)$ (similar definitions for $\tilde{\sigma}_s$, β_s and ρ_s are defined by using the translated functions in (3.1.4)). The diffusion process on the time interval $[s, s+\delta]$ can be expressed using these translated functions. The condition $\mathcal{T}_b > \delta$ is equivalent to $\sup_{0\leq r\leq\delta} X_{s+r} < b$ and becomes, for all $r \leq \delta$,

$$b - \epsilon + e^{2\int_0^r \alpha_s(u)\,du} W_{\rho_s(r)} + e^{\int_0^r \alpha_s(u)\,du} \int_0^r \beta_s(u) e^{-\int_0^u \alpha_s(w)\,dw}\,du < b. \tag{3.2.30}$$

Since $s \in [0, t-\delta]$ and $r \leq \delta$, we obtain the following bound:

$$\rho_s(\delta) \geq \underline{\sigma}^2 \frac{1 - e^{-2\overline{\alpha}_t\delta}}{2\overline{\alpha}_t}.$$

The inequality (3.2.30) implies

$$\frac{1}{\sqrt{\rho_s(\delta)}} \sup_{0\leq r\leq\delta} W_{\rho_s(r)} \leq \frac{e^{2\overline{\alpha}_t\delta}}{\sqrt{1 - e^{-2\overline{\alpha}_t\delta}}} \frac{\sqrt{2\overline{\alpha}_t}}{\underline{\sigma}} (\epsilon + \overline{\beta}_t\delta) \leq e^{3\overline{\alpha}_t\delta} \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}}.$$

The Désiré André reflexion principle for the Brownian motion implies that the l.h.s of the previous inequality has the same distribution than the absolute value of a standard gaussian random variable: $|G|$. Hence, for any $y \in [b-\epsilon, b]$ and for any $s \leq t-\delta$:

$$\mathbb{P}(\mathcal{T}_b > \delta) \leq \mathbb{P}\left(|G| \leq e^{3\overline{\alpha}_t\delta} \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}}\right) \leq \sqrt{\frac{2}{\pi}}\, e^{3\overline{\alpha}_t\delta} \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}}. \tag{3.2.31}$$

It suffices to choose $\delta = \epsilon$ in the previous inequality and to combine with (3.2.28) in order to prove the statement of the theorem. □

3.3 WOMS algorithm for G-class diffusions

In this section we present an application of the results obtained so far to another family of diffusion processes: the growth processes (G-class). We shall just point out the existence of a strong link between linear and growth diffusions.

Definition 3.3.1. *(G-class diffusions) We call G-class diffusion any solution of*

$$dX_t = (\alpha(t)X_t + \beta(t)X_t \log(X_t))dt + \tilde{\sigma}(t)dW_t, \quad X_0 = x_0, \tag{3.3.1}$$

where α and β are real continuous functions and $\tilde{\sigma}$ is a continuous non-negative function.

We first notice that this kind of process is non negative due to the logarithm function. As for the L-class diffusions case, it is possible to emphasize an explicit expression of the solution of (3.3.1). Here, the desired form is:

$$X_t = x_0 \; G(t, W_{\gamma(t)}), \quad \forall t \geqslant 0. \tag{3.3.2}$$

The function G is described in the following statement.

Proposition 3.3.2. *The solution of the SDE* (3.3.1) *is given by* (3.3.2) *with*

$$G(t,x) = C(t)e^{\frac{\tilde{\sigma}(t)}{\sqrt{\gamma'(t)}}x}$$
$$\text{with } C(t) = \exp\left(e^{\int_0^t \beta(s)ds}\int_0^t \left(\alpha(s) - \frac{1}{2}\tilde{\sigma}(s)^2\right)e^{-\int_0^s \beta(u)du}ds\right)$$
$$\text{and } \gamma(t) = \int_0^t \tilde{\sigma}(s)^2 e^{-2\int_0^s \beta(u)du}ds.$$

This statement is an immediate consequence of the link built between the linear and the growth diffusions:

Proposition 3.3.3. *If X is solution of*

$$\begin{cases} dX_t = (\alpha(t)X_t + \beta(t))dt + \sigma(t)dW_t \\ X_0 = x_0 \end{cases}$$

then $Y_t = e^{X_t}$ is solution of

$$\begin{cases} dY_t = (\tilde{\alpha}(t)Y_t + \tilde{\beta}(t)Y_t \log(Y_t))dt + \tilde{\sigma}(t)Y_t dW_t \\ Y_0 = y_0 \end{cases} \tag{3.3.3}$$

with $\tilde{\alpha}(t) = \beta(t) + \frac{1}{2}\sigma(t)^2$, $\tilde{\beta}(t) = \alpha(t)$, $\tilde{\sigma}(t) = \sigma(t)$ and $y_0 = e^{x_0}$.

Hence, we manage to create a link between a solution of a L-class diffusion equation with α, β, σ and a solution of a G-class diffusion equation with $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\sigma}$.

Proof. To prove this statement, we apply Itô's formula

$$Y_t = e^{X_t} = e^{X_0} + \int_0^t e^{X_s} dX_s + \frac{1}{2}\int_0^t e^{X_s} d\langle X, X\rangle_s$$

Hence, using the particular form of X_t we obtain

$$\begin{aligned} Y_t &= Y_0 + \int_0^t Y_s(\alpha(s)X_s + \beta(s))ds + \int_0^t \sigma(s)dB_s + \frac{1}{2}\int_0^t Y_s\sigma(s)^2 ds \\ &= Y_0 + \int_0^t Y_s(\alpha(s)\log(Y_s) + \beta(s))ds + \int_0^t \sigma(s)dB_s + \frac{1}{2}\int_0^t Y_s\sigma(s)^2 ds \\ &= Y_0 + \int_0^t (Y_s(\beta(s) + \frac{1}{2}\sigma(s)^2) + Y_s\log(Y_s)\alpha(s))ds + \int_0^t Y_s\sigma(s)dB_s \\ &= Y_0 + \int_0^t (Y_s\tilde{\alpha}(s) + Y_s\log(Y_s)\tilde{\beta}(s))dt + \int_0^t Y_s\tilde{\sigma}(s)dB_s. \end{aligned}$$

Finally we obtain

$$\begin{cases} dY_t = (\tilde{\alpha}(t)Y_t + \tilde{\beta}(t)Y_t\log(Y_t))dt + \tilde{\sigma}(t)Y_t dB_t \\ Y_0 = y_0 \end{cases}$$

. □

We consider the exit time from the interval $[a, b]$, $a, b \in \mathbb{R}_*^+$ for a G class-diffusion. The previous link established permits to focus our attention on the exit time from the interval $[\log(a), \log(b)]$ for L-class diffusion processes with modified coefficients.

We present now an adaptation of the WOMS algorithm which permits to approximate the exit time for G-class diffusions. In such a context we aim to describe the procedure, the averaged number of steps and the convergence rate.

The procedure. Let us consider $(X_t)_{t\geq 0}$ the unique solution of the stochastic differential equation (3.3.1). In order to approximate the first diffusion exit time $\mathcal{T}$ of the interval $[a, b]$ we introduce the linear diffusion (Y_t) solution of (3.3.3). Since the exit time of the growth process (X_t) from the interval $[a, b]$ and the exit time of the linear diffusion (Y_t) from the interval $[\log(a), \log(b)]$ are identically distributed, we use ALGORITHM$_m$ with a parameter ϵ small enough, with boundaries $\log(a)$ and $\log(b)$. As a immediate consequence, Theorem 3.2.1 points out the logarithmic upper-bound of the average number of steps and Theorem 3.2.10 emphasizes the convergence rate of the algorithm outcome.

3.4 Numerical application

In order to illustrate the efficiency of ALGORITHM$_m$, we present numerical results associated to two particular linear diffusions.

Example 1 (periodic functions). Let us consider $(X_t)_{t\geq 0}$ the solution of (3.1.1) with

$$\alpha(t) = \frac{\cos(t)}{2 + \sin(t)}, \quad \beta(t) = \cos(t), \quad \tilde{\sigma}(t) = 2 + \sin(t).$$

Let us just notice that α satisfies $\alpha(t) = \frac{\tilde{\sigma}'(t)}{\tilde{\sigma}(t)}$, such a property simplifies the link between the diffusion process and a standard one-dimensional Brownian motion. In particular, we obtain a simple expression of the time change appearing in (3.1.3): $\rho(t) = 4t$. Indeed (3.1.4) implies

$$\begin{aligned}\rho(t) &= \int_0^t (2+\sin(s))^2 e^{-2\int_0^s \frac{\cos(u)}{2+\sin(u)}du} ds \\ &= \int_0^t (2+\sin(s))^2 e^{-2(\log(2+\sin(s))-\log(2))} ds = 4t.\end{aligned}$$

Using Proposition 3.1.6, we are able to determine the frontiers of the typical spheroid used in ALGORITHM$_m$.

Proposition 3.4.1. *If we denote by $\psi_\pm^L(t; t_0, X_{t_0})$ the spheroid starting in (t_0, X_{t_0}), we obtain*

$$\begin{aligned}\psi_\pm^L(t; t_0, X_{t_0}) :=& \frac{2+\sin(t+t_0)}{2}\Big(\psi_\pm(4t) + 2\log\Big(\frac{2+\sin(t+t_0)}{2+\sin(t_0)}\Big)\Big) \\ &+ \Big(\frac{2+\sin(t+t_0)}{2+\sin(t_0)}\Big) X_{t_0}. \end{aligned} \tag{3.4.1}$$

and the exit time $\tau^{t_0} = \inf\{t > 0 :\ X_t \notin [\psi_-^L(t;\, t_0, X_{t_0}), \psi_+^L(t;\, t_0, X_{t_0})]\}$ satisfies

$$\tau^{t_0} \overset{d}{=} \frac{1}{4}\tau \tag{3.4.2}$$

where $\tau = \inf\{t > 0 :\ W_t \notin [\psi_-(t), \psi_+(t)]\}$.

The random walk on spheroids is therefore built using the typical boundaries (3.4.1). At each step of the algorithm, we need to use a scale parameter d in order to shrink or enlarge the spheroid size in such a way that the domains always stay in the interval $[a, b]$. The general statement concerning the scale parameter (3.1.9) can be improved for this particular example.

Let $m > 0$ and $0 < \gamma < 1$. We recall that a_{γ,x_0} and b_{γ,x_0} are defined by $a_{\gamma,x} = a + \gamma(x-a)$ and $b_{\gamma,x} = b - \gamma(b-x)$. We choose the scale parameter d in such a way that it satisfies

$$d = \begin{cases} \frac{\min(1,\kappa_+)}{\Delta_m}(b_{\gamma,x_0} - x_0) \text{ if } b - x_0 \leqslant x_0 - a \\ \frac{\min(1,\kappa_-)}{\Delta_m}(x_0 - a_{\gamma,x_0}) \text{ if } x_0 - a \leqslant b - x_0 \end{cases}$$

with

$$\Delta_m = \frac{3}{2}\left(\frac{1}{\sqrt{e}} + (1 + \max(|a|, |b|))\sqrt{m}\right)$$

and $\kappa_\pm$ are defined by the following equations:

$$\kappa_+(b_{\gamma,x_0} - x_0) = 2\Delta_m\sqrt{m} \text{ and } \kappa_-(x_0 - a_{\gamma,x_0}) = 2\Delta_m\sqrt{m}.$$

We just note that this particular value Δ_m is an easy upper-bound of the parameter emphasized in (3.1.10). We just adapted the choice of the parameters to the particular diffusion studied in this section. Even if the procedure is close to the method presented in Proposition 3.1.7, we notice that such a particular choice of Δ_m permits to point out a specific value m such that both $\min(1, \kappa_-)$ and $\min(1, \kappa_+)$ are equal to 1. This value corresponds to

$$m = \left(\frac{\sqrt{\frac{1}{e} + \frac{4}{3}(b-a)(1+\max(|a|,|b|)} - \frac{1}{\sqrt{e}}}{2(1+\max(|a|,|b|))} \right)^2 .$$

Using ALGORITHM$_m$ as in Section 3.1.3 permits to approximate the first diffusion exit time from the interval $[a, b]$, see Figure 3.2 and Figure 3.3.

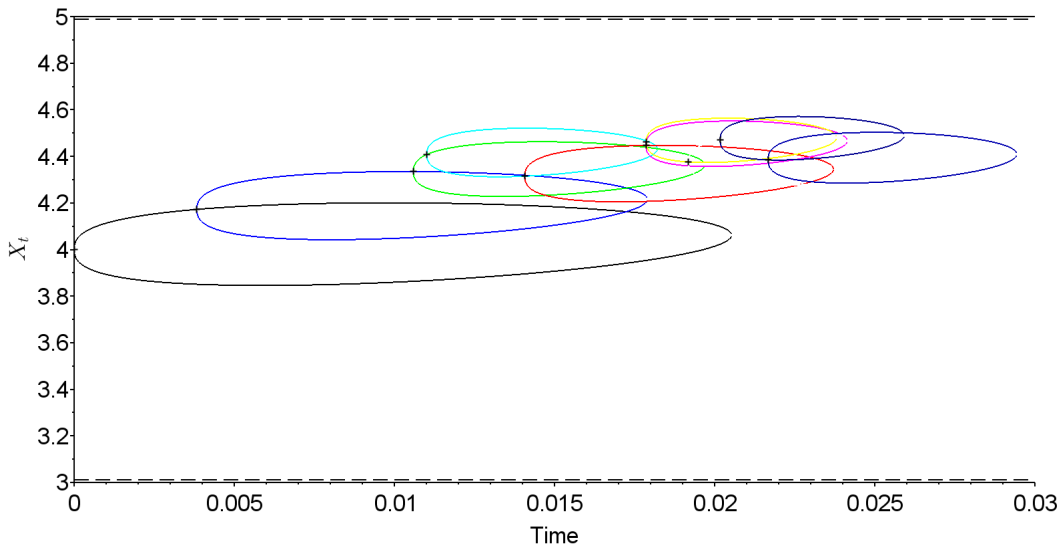


Figure 3.2: A sample of ALGORITHM$_m$ for the diffusion process starting at $x = 4$ in the interval $[3, 5]$ with $\epsilon = 10^{-2}$ and $\gamma = 10^{-4}$.

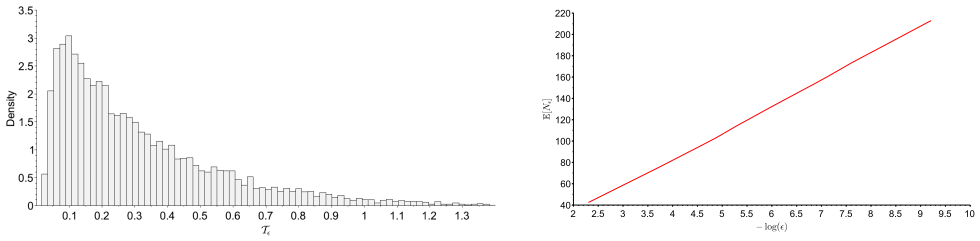


Figure 3.3: Histogram of the outcome variable for the diffusion (3.4) with $X_0 = 1$, $[a, b] = [-1, 2]$, $\epsilon = 10^{-2}$ and $\gamma = 10^{-4}$ (left). Average number of steps in ALGORITHM$_m$ for the exit time of $[-1, 2]$ (right, in logarithmic scale).

The CPU efficiency of such an algorithm shall be compared to the efficiency of classical approaches in the exit time approximation framework. We focus on an improved Euler method based on the correction by means of the sharp large deviations estimate of the exit probability (see the procedure described in [1]). We observe the linear diffusion with periodic coefficients starting in $X_0 = 1$ until it exits from the interval $[-1, 2]$. The generation of 100 000 samples of this exit time requires 659 seconds for the improved Euler method (with the step size 10^{-4}) whereas the corresponding generation using the WOMS algorithm requires about 39 seconds for the corresponding choice $\epsilon = 10^{-4}$ (here $\gamma = 10^{-4}$).

Example 2 (polynomial decrease). Let us introduce a diffusion with a polynomial decrease of the mean reversion. We consider (3.1.1) with

$$\alpha(t) = \frac{1}{2}\frac{1}{1+t}, \quad \beta(t) = 0, \quad \tilde{\sigma}(t) = \sigma_0.$$

Both the time-change function appearing in (3.1.3) and the typical spheroid frontiers can be explicitly computed. We obtain: $\rho(t) = \sigma_0^2 \log(1+t)$ and the following result due to Proposition 3.1.6.

Proposition 3.4.2. *If we denote by $\psi_\pm^L(t; t_0, X_{t_0})$ the spheroid starting in (t_0, X_{t_0}), we have*

$$\begin{aligned}\psi_\pm^L(t; t_0, X_{t_0}) :=& \sqrt{1+t_0+t}\,\psi_\pm(\sigma_0^2(\log(1+t_0+t) - \log(1+t_0))) \\ &+ \frac{\sqrt{1+t_0+t}}{\sqrt{1+t_0}} X_{t_0}. \end{aligned} \tag{3.4.3}$$

and the exit time $\tau^{t_0} = \inf\{t > 0 : \ X_t \notin [\psi_-^L(t;\, t_0, X_{t_0}), \psi_+^L(t;\, t_0, X_{t_0})]\}$ satisfies

$$\tau^{t_0} \stackrel{d}{=} \rho^{-1}(\tau)(t_0 + 1) \tag{3.4.4}$$

where $\tau = \inf\{t > 0 : \ W_t \notin [\psi_-(t), \psi_+(t)]\}$ and $\rho^{-1}(t) = \exp\left(\frac{t}{\sigma_0^2}\right) - 1$.

These particular boundaries (3.4.3) are the basic components of the algorithm. Of course we need to adjust at each step the size of the spheroid in order to stay in the interval under consideration. The scale parameter d is defined in (3.1.9) and depends on a fixed arbitrary parameter $m > 0$. In Example 1, the parameter m was optimized in order to reduce the CPU time. Here it is not an easy task to choose a suitable value of m. The algorithm converges in Example 2 whatever the value of m (see Assumption 3.2.2), that is why we set $m = 1$ for the numerical illustration. The generation of 10 000 samples using the improved Euler method requires 568 seconds (with steps of size 10^{-4}) while it takes only 16 seconds with $\textsc{Algorithm}_m$ (with the corresponding choice $\epsilon = 10^{-4}$, $\gamma = 10^{-4}$ and $m = 1$).

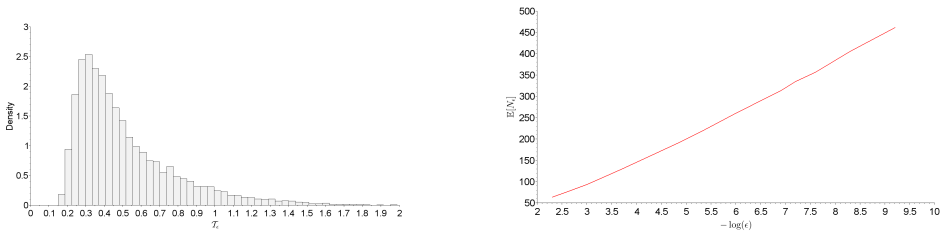


Figure 3.4: Histogram of the outcome variable for the diffusion (3.4) with $X_0 = 4$, $[a, b] = [3, 5]$, $m = 1$, $\sigma = 2$, $\epsilon = 10^{-4}$ and $\gamma = 10^{-4}$ (left). Average number of steps in ALGORITHM_m for the exit time of $[3, 5]$ with $m = 1$ and $\sigma_0 = 2$ (right, in logarithmic scale).

A. Potential theory and Markov chains

We recall a result coming from the potential theory and using Markov chains. It is a reminder of Section 1.4.
Let us consider a Markov chain $(X_n)_{n\in\mathbb{N}}$ defined on a state space I decomposed into two distinct subsets K and ∂K, ∂K being the so-called frontier. Let us define $N = \inf\{n \in \mathbb{N}, X_n \in \partial K\}$ the hitting time of ∂K. We assume that N is a.s. finite, then the following statement holds:

Proposition 3.4.3. *Let G be a positive increasing function. If there exists a function U such that the sequence $(H(n \wedge N, X_{n\wedge N}))_{n\in\mathbb{N}}$ is non negative and if the sequence $(H(n \wedge N, X_{n\wedge N}) + G(n \wedge N))_{n\in\mathbb{N}}$ represents a super-martingale adapted to the natural filtration of the considered Markov chain (X_n), then*

$$\mathbb{E}_x[G(N)] \leqslant H(0, x), \quad \forall x \in K.$$

The proof of this classical upper-bound is left to the reader, it is essentially based on the optimal stopping theorem and on the monotone convergence theorem (see, for instance, [45], p139).

B. Path decomposition

We prove in this section the two lemmas used in the proof of Proposition 3.2.7. Let us just recall several notations. The process X^x corresponds to the linear diffusion (3.1) with the starting value x; τ^x (resp. τ^x_- and τ^x_+) corresponds to the first exit time of the interval $]a, b[$ (resp. $]a, b_h[$ and $]a_h, b[$), with a_h and b_h defined in (3.2.12).

We also recall that $(Y^\pm_t)$ stand for the solutions of the shifted SDEs (3.2.8) and (3.2.9). Their exit time of the interval $]a, b[$ is denoted $\mathcal{T}(Y^\pm)$ and the first passage times through the level a is denoted by $\mathcal{T}_a(Y^\pm)$.

Lemma 3.4.4. *Let E_{ab} and E_{ba} the two events defined by*

$$E_{ab} := \{\tau_-^x \leq 1, X^x_{\tau_-^x} = a,\, \mathcal{T}(Y^-) \leq 1, Y^-_{\mathcal{T}(Y^-)} = b\},$$

$$E_{ba} := \{\tau_+^{x+h} \leq 1, X^{x+h}_{\tau_+^{x+h}} = b,\, \mathcal{T}(Y^+) \leq 1, Y^+_{\mathcal{T}(Y^+)} = a\}.$$

Then $E_{ab} \cap E_{ba} = \emptyset$.

Proof. On the event E_{ab} we know that $\tau_-^x \leqslant 1$ and consequently $X_s^x \in [a, b_h[\subset [a, b[$ (for all $s < \tau_-^x$. In particular we observe that $\tau_-^x = \tau^x$. Moreover

$$X_s^{x+h} = X_s^x + he^{\int_0^s \alpha(u)du}, \quad \forall s \geq 0.$$

Hence

$$X_s^{x+h} \in [a + he^{\int_0^s \alpha(u)du}, b_h + he^{\int_0^s \alpha(u)du}[, \quad \forall s \geq 0.$$

Since $b_h + he^{\int_0^s \alpha(u)du} = b - h(e^{\int_0^1 |\alpha(u)|du} - e^{\int_0^s \alpha(u)du}) < b$ for $s \leq 1$, we obtain

$$X_s^{x+h} \in [a + he^{\int_0^s \alpha(u)du}, b[\subset]a, b[, \quad \forall s \leq 1.$$

In conclusion $E_{ab} \subset \{\tau^x < \tau^{x+h}\}$.
Using similar arguments, we obtain $E_{ba} \subset \{\tau^{x+h} < \tau^x\}$.
The easy observation $\{\tau^x < \tau^{x+h}\} \cap \{\tau^{x+h} < \tau^x\} = \emptyset$ implies the announced statement. □

Lemma 3.4.5. $E_{ab} \cup E_{ba} \subset \bigcap_{t \geq 2} \{X^{x+h}_{\tau^{x+h} \wedge t} - X^x_{\tau^x \wedge t} = b - a\}$.

Proof. Let us prove that $E_{ab} \subset \{X^{x+h}_{\tau^{x+h} \wedge t} - X^x_{\tau^x \wedge t} = b - a\}$, the other inclusion can be obtained in a similar way.On the event E_{ab} we obviously observe that $X^x_{\tau_x \wedge t} = a$. By construction, we have $X^{x+h}_{\tau_-^x} \geqslant Y_0^-$, and using the continuity of the paths with respect to the initial condition, we obtain $X^{x+h}_{\tau_-^x + s} \geqslant Y_s^-, \quad \forall s \geq 0$. the property $Y^-_{\mathcal{T}(Y^-)} = b$, implies $X^{x+h}_{\tau_-^x + \mathcal{T}(Y^-)} \geqslant Y^-_{\mathcal{T}(Y^-)} = b$. Consequently $\tau^{x+h} \leq \mathcal{T}(Y^-) + \tau_-^x \leq 2$ and therefore, under the hypothesis $t \geqslant 2$ we have $E_{ab} \subset \{X^{x+h}_{\tau^{x+h} \wedge t} - X^x_{\tau^x \wedge t} = b - a\}$.

□

Chapter 4

Exact simulation of the first time passage through a given level for jump diffusions

4.1 Jump diffusions: definition and model reduction

4.1.1 Jump diffusions

Usually and historically, diffusions with jumps are introduced in the following way: we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a mark space $(\mathcal{E}, B(\mathcal{E}))$ where $\mathcal{E} \subset \mathbb{R} \setminus \{0\}$. This mark space can be interpreted as the space of jumps amplitudes. On this state space we define a $\mathcal{F}$-adapted Poisson measure on $\mathcal{E} \times [0, T]$ for a given T denoted by $p_\phi(dv \times dt)$ whose intensity measure is given by $\phi(dv)dt$, ϕ being a non negative σ-finite measure. We denote by $\lambda = \phi(\mathcal{E})$. This measure permits to generate a sequence of random points $(T_i, \xi_i)_{1 \leq i \leq P_T}$ where $(P_t)_{t \geq 0}$ is the stochastic process counting the number of jumps until time t. This sequence represents each jump time and the amplitude of the corresponding jump.

A jump diffusion X with jump rate j is defined as follows:

$$dX_t = \mu(t, X_{t-})\, dt + \sigma(t, X_{t-})\, dB_t + \int_{\mathcal{E}} j(t, X_{t-}, v) p_\phi(dv \times dt), \quad t \geq 0, \tag{4.1.1}$$

with the initial position $X_0 = y_0$. Here $(B_t,\, t \geq 0)$ stands for a one-dimensional Brownian motion. For the present study, this representation is actually not so handy. We prefer the following representation. We start the construction with the jump rate (jump function) $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \to \mathbb{R}$. No specific assumption concerning the jump rate is required at the moment. Let us denote by T_i the i-th jump time. We mention that the time spent between two consecutive jumps is exponentially distributed, therefore $T_i = \sum_{k=1}^{i} E_k$ with E_k exponentially distributed random variables with average $1/\lambda$. The initial position of the diffusion is given by $Y_0 = y_0$. Between two jumps, the stochastic process satisfies a stochastic differential equation:

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}, \tag{4.1.2}$$

and the jumps modify the trajectories as follows:

$$Y_{T_i} = Y_{T_i-} + j(T_i, Y_{T_i-}, \xi_i), \quad \forall i \in \mathbb{N}. \tag{4.1.3}$$

As already defined, $(\xi_i)_{i\geq 1}$ stands for a sequence of independent random variables with distribution function ϕ/λ. This sequence has to be independent of the Brownian motion $(B_t)_{t\geq 0}$. It is quite obvious to observe that $(X_t)_{t\geq 0}$ solution of (4.1.1) has the same path distribution than $(Y_t)_{t\geq 0}$ defined by (4.1.2)–(4.1.3). The aim of our study is to simulate the first passage time of some given level L for this jump diffusion. For our discussion we shall assume that $y_0 < L$ (it is straightforward to deduce the general case). Therefore the second representation plays a crucial role: it suffices to simulate exactly the trajectory of a SDE solution inbetween two successive jumps. The simulation of the first passage time should be based on the following intuitive procedure: on one hand we simulate exactly the trajectory of the stochastic process satisfying the SDE without jump and keep in mind its first passage time through the level L denoted by τ_L. On the other independent hand, we simulate the first exponentially distributed jump time T_1. If $\tau_L \leq T_1$, then τ_L corresponds to the first passage time of the jump diffusion. In the other case, we simulate the position of the diffusion after the first jump using

$$Y_{T_1} = Y_{T_1-} + j(T_1, Y_{T_1-}, \xi_1).$$

We distinguish two likely different cases: if $Y_{T_1} \geq L$ then $\tau_L = T_1$ otherwise we know that $\tau_L > T_1$. The strong Markov property of the jump diffusion permits to start a new jump diffusion with the initial position $Y_{T_1} < L$ and initial time T_1. So we repeat the procedure just presented, inbetween T_1 and T_2, and so on... An important tool for the simulation is therefore the exact generation of continuous diffusion paths. In [27], the authors propose an efficient method based on both a rejection sampling and the Girsanov transformation.

4.1.2 Model reduction and Lamperti's transformation

Our aim is to deal with the general framework pointed out in the previous section. However we shall emphasize an interesting technique which permits to transform the considered stochastic differential equation into a simplified equation. Indeed Lamperti's transformation permits to change the equation in such a way that the diffusion coefficient becomes constant. This method is commonly used for classical continuous diffusions and we just recall the crucial idea before presenting the extension to jump diffusions (usually the method is presented in the time-homogeneous context, we choose to present here diffusions with time dependent coefficients). We consider the following SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0, \tag{4.1.4}$$

with the initial condition $X_0 = x_0$. The aim of Lamperti's transformation is to find a particular diffusion process $(Z_t)_{t\geq 0}$ defined as a functional of both t and X_t, that is $Z_t = \nu(t, X_t)$, such that Z_t satisfies

$$dZ_t = \alpha(t, Z_t)dt + dB_t, \quad t \geq 0. \tag{4.1.5}$$

Of course the function α has to be defined using ν. Applying Itô's lemma to the process Z_t leads to

$$\begin{aligned} dZ_t &= \frac{\partial \nu}{\partial t}(t, X_t)\, dt + \frac{\partial \nu}{\partial x}(t, X_t)\Big(\mu(t, X_t)dt + \sigma(t, X_t)dB_t\Big) + \frac{1}{2}\frac{\partial^2 \nu}{\partial x^2}(t, X_t)\, d\langle X_t, X_t\rangle \\ &= \left(\frac{\partial \nu}{\partial t}(t, X_t) + \frac{\partial \nu}{\partial x}(t, X_t)\mu(t, X_t) + \frac{1}{2}\frac{\partial^2 \nu}{\partial x^2}(t, X_t)\sigma(t, X_t)^2\right) dt + \frac{\partial \nu}{\partial x}(t, X_t)\sigma(t, X_t)\, dB_t. \end{aligned}$$

From this equality, we deduce the importance to find ν such that $\frac{\partial \nu}{\partial x} = \frac{1}{\sigma}$. We obtain the following statement:

Proposition 4.1.1. *Let us assume that σ is a $\mathcal{C}^{0,1}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$-continuous function. Let $\xi \in \mathbb{R}$. If $\sigma(t, x) > 0$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then the process defined by*

$$Z_t = \nu(t, X_t) = \int_\xi^{X_t} \frac{1}{\sigma(t, x)}\, dx$$

satisfies the following SDE:

$$dZ_t = \left(\frac{\partial \nu}{\partial t}(t, \nu^{-1}(t, Z_t)) + \frac{\mu(t, \nu^{-1}(t, Z_t))}{\sigma(t, \nu^{-1}(t, Z_t))} - \frac{1}{2}\frac{\partial \sigma}{\partial x}(t, \nu^{-1}(t, Z_t))\right) dt + dB_t, \quad t \geq 0, \quad (4.1.6)$$

with initial value $Z_0 = \nu(0, x_0)$. Here $\nu^{-1} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is the unique function verifying $\nu^{-1}(t, \nu(t, x)) = x$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Proof. As previously stated, the particular choice of the function ν permits to observe that

$$\frac{\partial \nu}{\partial x}(t, X_t) = \frac{1}{\sigma(t, X_t)},$$

leading to

$$\frac{\partial^2 \nu}{\partial x^2}(t, X_t) = -\frac{\frac{\partial \sigma}{\partial x}(t, X_t)}{\sigma(t, X_t)^2} \quad \text{and} \quad \frac{\partial \nu}{\partial t}(t, X_t) = -\int_\xi^{X_t} \frac{\frac{\partial \sigma}{\partial t}(t, x)}{\sigma(t, x)^2}\, dx.$$

The announced result (4.1.6) is therefore an immediate consequence of Itô's lemma. □

Let us just note that the general statement can be simplified in the particular homogeneous case. If neither the diffusion coefficient nor the drift term depend directly on the time variable, that is $\sigma(t, x) \equiv \sigma(x)$ and $\mu(t, x) \equiv \mu(x)$, then the transformation becomes

$$Z_t = \nu(X_t) = \int_\xi^{X_t} \frac{1}{\sigma(u)} du, \quad (4.1.7)$$

where ξ is a fixed value belonging to the state space. It is often convenient to choose ξ as the starting position of the diffusion, the new process $(Z_t)_{t \geq 0}$ starts then at the origin. An other interesting choice may be $\xi = L$, the fixed level under observation in the description of

the first passage time. The FP problem then consists in considering the first time the new diffusion process $(Z_t)_{t\geq 0}$ goes through the level 0. The transformed diffusion is solution of the SDE $dY_t = \alpha(X_t)dt + dB_t$, where

$$\alpha(x) = \frac{\mu \circ \nu^{-1}(x)}{\sigma \circ \nu^{-1}(x)} - \frac{1}{2}\sigma' \circ \nu^{-1}(x), \quad \forall x \in \mathbb{R}. \tag{4.1.8}$$

Let us now consider jump diffusions. The same Lamperti transformation can be applied. Let us consider $Z_t := \nu(t, X_t)$ defined in (4.1.1), we observe that ν transforms the diffusion equation between two successive jumps (4.1.2) into

$$dZ_t = \alpha(t, Z_t)\, dt + dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}. \tag{4.1.9}$$

Of course the jump times are not changed at all by the function ν: we still work with the sequence $(T_i)_{i\geq 1}$ whereas the jump amplitudes undergo modifications. Hence the transport property permits to modify (4.1.3) into

$$Z_{T_i} = Z_{T_i-} + \hat{\jmath}(t, Z_{T_i-}, \xi), \quad i \in \mathbb{N}, \tag{4.1.10}$$

where

$$\hat{\jmath}(t, z, v) := \nu(t, \nu^{-1}(t, z) + j(t, \nu^{-1}(t, z), v)) - \nu(t, \nu^{-1}(t, z)).$$

Indeed it suffices to note that

$$Z_{T_i} - Z_{T_i-} = \nu(T_i, X_{T_i-} + j(T_i, X_{T_i-}, \xi_i)) - \nu(T_i, X_{T_i-}) \quad \text{and} \quad X_t = \nu^{-1}(t, Z_t).$$

To conclude, the Lamperti transformation permits to change the jump diffusion (4.1.2)–(4.1.3) into the jump diffusion (4.1.9)–(4.1.10). In the following we shall assume that the diffusion coefficient is constant: $\sigma \equiv 1$, that corresponds to a model reduction procedure.

4.2 Simulation of the first passage time for a stopped continuous diffusion

Let us fix some time parameter $\mathbb{T} > 0$. In this section, we shall focus our attention on the exact simulation of the continuous diffusion paths on the interval $[0, \mathbb{T}]$. Generating a random object in an exact way consists in generating an object using a stochastic algorithm such that both objects have the same distribution. Roberts and Beskos [3] already proposed an efficient algorithm in order to exactly simulate a continuous diffusion path on the interval under consideration: $[0, \mathbb{T}]$. Of course we are not able to generate the whole paths numerically, the exact simulation consists therefore in simulating a sequence of random points belonging to the trajectory of the diffusion.

Herrmann and Zucca [27] proposed an adaptation of the algorithm introduced by Roberts and Beskos in order to exactly generate τ_L, the first passage time through the level L for a continuous diffusion. In order to deal with jump diffusions (next section), we need also to simulate exactly the couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$ where $(Y_t)_{t\geq 0}$ stands for a continuous diffusion.

4.2.1 Diffusion without any jump: the exact simulation method

Let us first consider continuous diffusion processes and propose a numerical approach for the generation of their paths. As already seen in Section 4.1.2, we can restrict the study to the reduced model, the generalization being obtained by the Lamperti transformation. That's why we consider, on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the following SDE on the fixed time interval $[0, \mathbb{T}]$:

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad Y_0 = y_0, \tag{4.2.1}$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. The direct generation of a diffusion path is quite a hard task, that's why we need to point out the link between the diffusion and the standard Brownian motion using the famous Girsanov formula. The goal of this formula is to find a probability space in which the considered diffusion is a Brownian motion. Then we generate a Brownian motion path and do an acceptance/rejection algorithm using the weight enlightened by the Girsanov formula.

The Girsanov transformation: consequences for simulation purposes

First we recall the statement of the Girsanov transformation and the associated Novikov condition (for a reference about Girsanov's transformation, see for instance [46]).

Assumption 4.2.1 (Novikov's condition)**.** *We say that Novikov's condition is satisfied if*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{\mathbb{T}} \alpha^2(s, y_0 + B_s)\, ds\right)\right] < \infty. \tag{4.2.2}$$

Let us note that this particular condition is satisfied if the growth of the drift term α is at most linear (see Corollary 5.16 p.200 in [36]): there exists a constant $C_{\mathbb{T}} > 0$ such that

$$|\alpha(t, x)| \leq C_{\mathbb{T}}(1 + |x|), \quad \forall (t, x) \in [0, \mathbb{T}] \times \mathbb{R}.$$

Then the following transformation holds.

Theorem 4.2.1. *Assume that α satisfies Novikov's condition. Let us define the martingale $(M_t)_{t\geq 0}$ by*

$$M_t = \exp\left(\int_0^t \alpha(s, y_0 + B_s)\, dB_s - \frac{1}{2}\int_0^t \alpha^2(s, y_0 + B_s) ds\right), \quad t \leq \mathbb{T}, \tag{4.2.3}$$

and the measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{\mathbb{T}})$:

$$d\mathbb{Q} = M_{\mathbb{T}}\, d\mathbb{P}. \tag{4.2.4}$$

Then under $\mathbb{Q}$, the stochastic process $\left(B_t - \int_0^t \alpha(s, B_s)\, ds\right)_{0\leq t\leq \mathbb{T}}$ is a one dimensional standard Brownian motion. In other words, $(y_0 + B_t)_{0\leq t\leq \mathbb{T}}$ under $\mathbb{Q}$ has the same distribution than $(Y_t)_{0\leq t\leq \mathbb{T}}$ under $\mathbb{P}$.

The Radon-Nikodym derivative pointed out in the previous statement is going to correspond to the weight necessary for the use of an acceptance/rejection sampling. We use the Girsanov formula as follows: let us consider $(Y_t)_{t\geq 0}$, the solution of the SDE (4.2.1), and f any measurable functional depending on the paths of Y observed on the time interval $[0,\mathbb{T}]$, then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[f(Y_\cdot)] &= \mathbb{E}_{\mathbb{Q}}[f(y_0+B_\cdot)] = \mathbb{E}_{\mathbb{P}}[f(y_0+B_\cdot)\cdot M_{\mathbb{T}}] \\ &= \mathbb{E}_{\mathbb{P}}\left[f(y_0+B_\cdot)\exp\left(\int_0^{\mathbb{T}}\alpha(s,y_0+B_s)dB_s - \frac{1}{2}\int_0^{\mathbb{T}}\alpha^2(s,y_0+B_s)ds\right)\right],\end{aligned}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the probability measure $\mathbb{P}$. We introduce two different functions which shall also play a crucial role in the numerical algorithm. Let us define

$$\beta(t,x) := \int_{y_0}^{x}\alpha(t,y)\,dy \quad \text{and} \quad \gamma(t,x) = \frac{\partial\beta}{\partial t}(t,x) + \frac{1}{2}\Big(\frac{\partial\alpha}{\partial x}(t,x) + \alpha^2(t,x)\Big). \tag{4.2.5}$$

Using Itô's formula applied to the process $(\beta(t,y_0+B_t))_{t\geq 0}$, we obtain

$$\mathbb{E}_{\mathbb{P}}[f(Y_\cdot)] = \mathbb{E}_{\mathbb{P}}[f(y_0+B_\cdot)\cdot\hat{M}_{\mathbb{T}}] \quad \text{with} \quad \hat{M}_{\mathbb{T}} := e^{\beta(\mathbb{T},y_0+B_{\mathbb{T}})-\int_0^{\mathbb{T}}\gamma(t,y_0+B_s)\,ds}. \tag{4.2.6}$$

General hypotheses for continuous diffusion processes

We recall that, through all our study, the diffusion (4.2.1) starts in y_0 satisfying $y_0 < L$. Let us present now different hypotheses concerning the drift coefficient α in (4.2.1). They permit to describe a typical framework for the introduction of efficient algorithms. The aim is not at this stage to precisely emphasize the most general situation which permits the use of the exact simulation technique, several studies were already introduced in order to successively extend these conditions (see Beskos, Papaspiliopoulos and Roberts [3]...). The aim is rather to focus our attention on a convenient context where the crucial arguments used in the study of the continuous diffusions can easily be translated to jump diffusions. First of all, we need a classical regularity property in order to use Itô's formula.

Assumption 4.2.2. *The drift coefficient α is a $\mathcal{C}^{1,1}(\mathbb{R}_+\times\mathbb{R})$-continuous function.*

Of course the regularity property of α immediately implies that β and γ defined by (4.2.5) are well-defined and continuous functions. According to the simulation goals, we need additional conditions like the boundedness of β or of γ. If the aim is to generate $Y_{\mathbb{T}}$ we need the following

Assumption 4.2.3. *The function γ defined in* (4.2.5) *is non negative and satisfies: there exists a constant κ s.t.*

$$0 \leq \gamma(t,x) \leq \kappa, \quad \forall (t,x)\in[0,\mathbb{T}]\times\mathbb{R}. \tag{4.2.7}$$

Assumption 4.2.4. *The function β defined in* (4.2.5) *is upper bounded at time $\mathbb{T}$: there exists a constant $\beta_+ > 0$ s.t.*

$$\beta(\mathbb{T}, x) \leq \beta_+, \quad \forall x \in \mathbb{R}. \tag{4.2.8}$$

The assumption concerning β can be weaken in particular situations when we just need an integration property.

Assumption 4.2.5. *The function $\Gamma_{\mathbb{T}} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Gamma_{\mathbb{T}}(x) := \exp\left\{\beta(\mathbb{T}, y_0 + x) - \frac{x^2}{2\mathbb{T}}\right\}, \quad x \in \mathbb{R},$$

with β introduced in (4.2.5) *is integrable: $\Gamma_{\mathbb{T}} \in L^1(\mathbb{R})$.*

Assumption 4.2.3, 4.2.4 and 4.2.5 essentially concern the simulation of $Y_{\mathbb{T}}$, $\mathbb{T}$ being a fixed time value. If we are rather interested in the first passage time through the level L, we shall focus our attention on the space subset $]-\infty, L]$.

Assumption 4.2.6. *The function γ defined in* (4.2.5) *is non negative and satisfies: there exists a constant κ s.t.*

$$0 \leq \gamma(t, x) \leq \kappa, \quad (t, x) \in \mathbb{R}_+ \times]-\infty, L]. \tag{4.2.9}$$

Assumption 4.2.7. *The function β defined in* (4.2.5) *is upper bounded: there exists a constant $\beta_+ > 0$ s.t.*

$$\beta(t, x) \leq \beta_+, \quad \forall (t, x) \in \mathbb{R}_+ \times]-\infty, L]. \tag{4.2.10}$$

Let us note that all these assumptions take new shapes as soon as the drift coefficient α in equation (4.2.1) is time homogeneous. Most of the results presented in previous studies ([3], [27]) concern this restrictive context but the generalization is a quite simple task.

The approach developed by Beskos and Roberts

Beskos and Roberts proposed in [3] a simulation approach for the exact generation of diffusion paths on some given interval $[0, \mathbb{T}]$. Their study is based on the Girsanov transformation on one hand and on an acceptance/rejection sampling on the other hand. The procedure therefore consists in the introduction of a Poisson process independent of the diffusion (4.2.1) and whose realization shall permit to obtain the required weight appearing in the rejection method (4.2.6). Their approach is not so easy to adapt to a jump diffusion since they do not use at all the Markov property of the diffusion process. That is why we propose an alternative presentation of their result (and the corresponding proof) and laid by that way the foundations of the study in the general jump diffusion context. For a clear and succinct presentation of the issue, we prefer just to introduce the exact simulation of $Y_{\mathbb{T}}$ where $(Y_t)_{t \geq 0}$ corresponds to the solution of (4.2.1).

EXACT SIMULATION OF $Y_{\mathbb{T}}$ – ALGORITHM $(BR)^1_{\mathbb{T}}$

1. *Let $(G_n)_{n\geq 1}$ be independent random variables with Gaussian distribution $\mathcal{N}(0,1)$*
2. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\kappa$.*
3. *Let $(U_n)_{n\geq 1}$ be independent uniformly distributed random variables on $[0,1]$.*

The sequences $(G_n)_{n\geq 1}$, $(E_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 0$.

Step 1. Set $Z = y_0$, $\mathcal{T} = 0$.

Step 2. While $\mathcal{T} < \mathbb{T}$ do:

- set $n \leftarrow n + 1$
- $Z \leftarrow Z + \sqrt{\min(\mathcal{T} + E_n, \mathbb{T}) - \mathcal{T}}\, G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T} + E_n, \mathbb{T})$
- If ($\mathcal{T} < \mathbb{T}$ and $\kappa\, U_n < \gamma(\mathcal{T}, Z)$) then go to Step 1.
- If ($\mathcal{T} = \mathbb{T}$ and $U_n e^{\beta_+} > e^{\beta(\mathbb{T},Z)}$) then go to Step 1.

Outcome: the random variable Z.

Proposition 4.2.2. *Under the assumptions 4.2.2, 4.2.3 and 4.2.4. both the outcome Z of Algorithm $(BR)^1_{\mathbb{T}}$ and $Y_{\mathbb{T}}$, the value at time $\mathbb{T}$ of the diffusion process* (4.2.1)*, have the same distribution.*

Remark 4.2.3. *An adaptation of Algorithm $(BR)^1_{\mathbb{T}}$ should permit to obtain more than just the random variable Z which has the same distribution than $Y_{\mathbb{T}}$. Indeed denoting by n_1 the value of n throughout the last visit of Step number one, n_2 the value of the increment variable when the algorithm stops, Z_n (respectively $\mathcal{T}_n$) the successive values of Z (resp. $\mathcal{T}$), we obtain that*

$$\Big\{(0, y_0),\, (\mathcal{T}_{n_1+1}, Z_{n_1+1}), \ldots, (\mathcal{T}_{n_2-1}, Z_{n_2-1}),\, (\mathbb{T}, Z_{n_2})\Big\}$$

is a set of points which has the same distribution than points belonging to the diffusion trajectory.

Proof. Let us denote by $\mathcal{N}$ the number of necessary repetitions of the step number one and let ψ a non negative measurable function. Since the algorithm is based on an acceptance/rejection sampling, we get

$$\mathbb{E}[\psi(Z)] = \frac{\mathbb{E}[\psi(Z)1_{\{\mathcal{N}=1\}}]}{\mathbb{P}(\mathcal{N}=1)} = \frac{\nu(\psi)}{\nu(1)} \quad \text{where} \quad \nu(\psi) := \mathbb{E}[\psi(Z)1_{\{\mathcal{N}=1\}}].$$

We use now the notations $(\mathcal{T}_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ introduced in Remark 4.2.3 and define the event $A_n := \{\mathcal{T}_{n-1} < \mathbb{T} \leq \mathcal{T}_{n-1} + E_n = \mathcal{T}_n\}$, for any $n \geq 1$, which describes the number of random intervals necessary to cover $[0, \mathbb{T}]$. We observe that

$$\nu(\psi) = \sum_{n\geq 1} \mathbb{E}\Big[\psi(Z_n) 1_{\{\kappa U_1 > \gamma(\mathcal{T}_1, Z_1), \ldots, \kappa U_{n-1} > \gamma(\mathcal{T}_{n-1}, Z_{n-1})\}} 1_{\{U_n \leq e^{\beta(\mathbb{T}, Z_n) - \beta_+}\}} 1_{A_n}\Big].$$

Taking the integral with respect to the independent uniform variates $(U_n)_{n\geq 1}$ leads to

$$\nu(\psi) = \sum_{n\geq 1} \kappa^{1-n} \mathbb{E}\Big[\psi(Z_n)(\kappa - \gamma(\mathcal{T}_1, Z_1)) \ldots (\kappa - \gamma(\mathcal{T}_{n-1}, Z_{n-1})) e^{\beta(\mathbb{T}, Z_n) - \beta_+} 1_{A_n}\Big].$$

We note that $(\mathcal{T}_1, \ldots, \mathcal{T}_{n-1})$, given A_n, has the same distribution than $(V^{(1)}, \ldots, V^{(n-1)})$ an ordered $(n-1)$-tuple of uniform random variables $(V_1, \ldots, V_{n-1})$ on $[0, \mathbb{T}]$. Moreover the probability of the event A_n is linked to the Poisson distribution of parameter $\kappa\mathbb{T}$. Finally $(Z_1, \ldots Z_n)$ is a Gaussian vector and has the same distribution than

$$(y_0 + B_{V^{(1)}}, \ldots, y_0 + B_{V^{(n-1)}}, y_0 + B_{\mathbb{T}})$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion independent of the $(n-1)$-tuple $(V_1, \ldots, V_{n-1})$, since the Brownian motion has Gaussian independent increments. Hence

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 1} \mathbb{E}\Big[\psi(Z_n)(\kappa - \gamma(\mathcal{T}_1, Z_1)) \ldots (\kappa - \gamma(\mathcal{T}_{n-1}, Z_{n-1})) e^{\beta(\mathbb{T}, Z_n) - \beta_+} \Big| A_n\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!} e^{-\kappa\mathbb{T}} \\
&= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \prod_{j=1}^{n-1} (\kappa - \gamma(V^{(j)}, y_0 + B_{V^{(j)}}) e^{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!} e^{-\kappa\mathbb{T}} \\
&= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \prod_{j=1}^{n-1} (\kappa - \gamma(V_j, y_0 + B_{V_j})) e^{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!} e^{-\kappa\mathbb{T}}.
\end{aligned}$$

Taking the expectation with respect to the uniformly distributed variates V_j, we have

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 1} \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \Big(\kappa - \frac{1}{\mathbb{T}} \int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big)^{n-1} e^{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \beta_+}\Big] \frac{\mathbb{T}^{n-1}}{(n-1)!} e^{-\kappa\mathbb{T}} \\
&= \mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}}) \exp\Big\{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \beta_+ - \int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big\}\Big] \\
&= \mathbb{E}[\psi(y_0 + B_{\mathbb{T}}) \cdot \hat{M}_{\mathbb{T}}]\, e^{-\beta_+},
\end{aligned}$$

where $(\hat{M}_t)_{t\geq 0}$ is the martingale defined in (4.2.6). The martingale property leads to $\nu(1) = \mathbb{E}[\hat{M}_0] e^{-\beta_+} = e^{-\beta_+}$. The Girsanov transformation permits to conclude the proof:

$$\mathbb{E}[\psi(Z)] = \frac{\nu(\psi)}{\nu(1)} = \mathbb{E}[\psi(y_0 + B_{\mathbb{T}}) \cdot \hat{M}_{\mathbb{T}}] = \mathbb{E}[\psi(Y_{\mathbb{T}})].$$

□

Under Assumptions 4.2.2, 4.2.3 and 4.2.4, the algorithm pointed out in Proposition 4.2.2 has a convenient and intuitive expression. However the boundedness of the function $\beta(\mathbb{T},\cdot)$ is rather a restrictive assumption and it is important to propose an alternative approach. To that end, Beskos and Roberts proposed an integrability condition for the function $\Gamma_{\mathbb{T}}$, written here in Assumption 4.2.5. Since $\Gamma_{\mathbb{T}}$ is a non negative function, the integrability condition summarized in the identity $\Gamma_{\mathbb{T}}(\mathbb{R}) := \int_{\mathbb{R}} \Gamma_{\mathbb{T}}(x)\,dx < \infty$ ensures that $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$ is a probability distribution function. This crucial property permits to present the following algorithm. Let us just mention that we use the following notation $x_+ = \min(x, 0)$.

EXACT SIMULATION OF $Y_{\mathbb{T}}$ – ALGORITHM $(BR)^2_{\mathbb{T}}$

1. *Let $(R_n)_{n\geq 1}$ be independent random variables with density $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$.*
2. *Let $(G_n)_{n\geq 1}$ be independent random variables with Gaussian distribution $\mathcal{N}(0,1)$*
3. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\kappa$.*
4. *Let $(U_n)_{n\geq 1}$ be independent uniformly distributed random variables on $[0,1]$.*

The sequences $(R_n)_{n\geq 1}$, $(G_n)_{n\geq 1}$, $(E_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $k = 0$, $n = 0$.

Step 1. Set $k \leftarrow k+1$ then $Z = y_0$, $W = y_0 + R_k$ and $\mathcal{T} = 0$.

Step 2. While $\mathcal{T} < \mathbb{T}$ do:

- set $n \leftarrow n+1$
- $Z \leftarrow Z + \dfrac{E_n}{\mathbb{T}-\mathcal{T}}\,W + \sqrt{\dfrac{E_n(\mathbb{T}-\mathcal{T}-E_n)_+}{\mathbb{T}-\mathcal{T}}}\,G_n$ and $\mathcal{T} \leftarrow \min(\mathcal{T}+E_n, \mathbb{T})$
- If ($\mathcal{T} < \mathbb{T}$ and $\kappa\,U_n < \gamma(\mathcal{T}, Z)$) then go to Step 1.

Outcome: the random variable W.

The procedure proposed in this second algorithm is mainly different from the first one. Indeed the crucial idea of Algorithm $(BR)^1_{\mathbb{T}}$ is to simulate a Brownian motion on the interval $[0,\mathbb{T}]$ and to accept or reject the trajectory using the weight probability issued from the Girsanov transformation. The acceptance depends strongly on the whole path of the process and leads to the outcome $y_0 + B_{\mathbb{T}}$, the endpoint of the Brownian path. In Algorithm $(BR)^2_{\mathbb{T}}$ the approach is different: we consider a random variable W with the proposal distribution $\Gamma_{\mathbb{T}}(\cdot)/\Gamma_{\mathbb{T}}(\mathbb{R})$ translated by y_0. This variate shall be accepted or rejected using a weight probability based on the whole path of a Brownian bridge starting in y_0 and ending at time $\mathbb{T}$ with the value W. The main difference is therefore to replace the Brownian motion by the Brownian bridge. We obtain the following statement.

Proposition 4.2.4. *Under the assumptions 4.2.2, 4.2.3 and 4.2.5. both the outcome W of Algorithm $(BR)^2_{\mathbb{T}}$ and $Y_{\mathbb{T}}$, the value at time $\mathbb{T}$ of the diffusion process* (4.2.1)*, have the same distribution.*

Sketch of proof. The proof of Proposition 4.2.4 is quite similar to the proof of Proposition 4.2.2. We shall therefore not go into all details. Let ψ a non negative measurable function. Since the algorithm is based on a rejection sampling, we get as usual

$$\mathbb{E}[\psi(W)] = \frac{\mathbb{E}[\psi(W)1_{\{\mathcal{N}=1\}}]}{\mathbb{P}(\mathcal{N}=1)} = \frac{\nu(\psi)}{\nu(1)} \quad \text{where} \quad \nu(\psi) := \mathbb{E}[\psi(W)1_{\{\mathcal{N}=1\}}],$$

and $\mathcal{N}$ stands for the number of visits of the steps number one before the algorithm stops. Using similar notations and arguments as those developed in the previous proof, we observe that

$$\nu(\psi) = \sum_{n\geq 1} \kappa^{1-n}\mathbb{E}\Big[\psi(W)(\kappa - \gamma(\mathcal{T}_1, Z_1)) \ \ldots \ (\kappa - \gamma(\mathcal{T}_{n-1}, Z_{n-1}))1_{A_n}\Big].$$

Given both A_n and the value W, the $(n-1)$-tuple $(\mathcal{T}_1, \ldots, \mathcal{T}_{n-1})$ has the same distribution than $(V^{(1)}, \ldots, V^{(n-1)})$ an ordered $(n-1)$-tuple of uniform random variables on $[0, \mathbb{T}]$. Moreover the probability of the event A_n is linked to the Poisson distribution of parameter $\kappa\mathbb{T}$. Finally $(Z_1, \ldots Z_{n-1})$ is a Gaussian vector and has the same distribution than

$$(b_{V^{(1)}}, \ldots, b_{V^{(n-1)}})$$

with $(b_t)_{0\leq t\leq \mathbb{T}}$ a Brownian bridge independent of the $(n-1)$-tuple $(V_1, \ldots, V_{n-1})$. The Brownian bridge starts for $t = 0$ with the value y_0 and ends with the value W at time $\mathbb{T}$. Hence

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 1} \mathbb{E}\Big[\psi(W)\prod_{j=1}^{n-1}(\kappa - \gamma(V_j, b_{V_j}))\Big]\frac{\mathbb{T}^{n-1}}{(n-1)!}\, e^{-\kappa\mathbb{T}} \\
&= \mathbb{E}\Big[\psi(W)\exp\Big\{-\int_0^{\mathbb{T}} \gamma(s, b_s)\, ds\Big\}\Big] \\
&= \mathbb{E}\Big[\psi(W)\,\mathbb{E}\Big[\exp\Big\{-\int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big\}\Big| y_0 + B_{\mathbb{T}} = W\Big]\Big],
\end{aligned}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Using the explicit distribution of the variable W, we obtain

$$\begin{aligned}
\nu(\psi) &= \frac{1}{\Gamma_{\mathbb{T}}(\mathbb{R})}\int_{\mathbb{R}} \psi(y_0 + x)\,\mathbb{E}\Big[\exp\Big\{-\int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big\}\Big| B_{\mathbb{T}} = x\Big] \\
&\qquad\qquad \times \exp\Big\{\beta(\mathbb{T}, y_0 + x) - \frac{x^2}{2\mathbb{T}}\Big\}\, dx \\
&= \frac{\sqrt{2\pi\mathbb{T}}}{\Gamma_{\mathbb{T}}(\mathbb{R})}\,\mathbb{E}\Big[\psi(y_0 + B_{\mathbb{T}})\exp\Big\{\beta(\mathbb{T}, y_0 + B_{\mathbb{T}}) - \int_0^{\mathbb{T}} \gamma(s, y_0 + B_s)\, ds\Big\}\Big] \\
&= \frac{\sqrt{2\pi\mathbb{T}}}{\Gamma_{\mathbb{T}}(\mathbb{R})}\,\mathbb{E}[\psi(y_0 + B_{\mathbb{T}})\cdot \hat{M}_{\mathbb{T}}],
\end{aligned}$$

where $(M_t)_{t\geq 0}$ is the Girsanov martingale defined in (4.2.6). We deduce that

$$\mathbb{E}[\psi(W)] = \frac{\nu(\psi)}{\nu(1)} = \mathbb{E}[\psi(y_0 + B_{\mathbb{T}}) \cdot \hat{M}_{\mathbb{T}}] = \mathbb{E}[\psi(Y_{\mathbb{T}})],$$

this equality is satisfied for any non negative function ψ and therefore corresponds to the announced statement. □

The algorithm introduced by Herrmann and Zucca

The procedure of the exact simulation was adapted by Herrmann and Zucca [27] in order to generate the first passage time of continuous diffusion processes. We focus now our attention on the first time passage of the diffusion Y, starting in y_0, through the level L. This algorithm is also based on the Girsanov formula on one hand, and on the other hand, it requires the construction of a skeleton of a 3-dimensional Bessel process.

We first recall that in the particular case of the Brownian motion, the first passage time through the level L denoted by τ_L satisfies $\tau_L \sim (L - y_0)^2/G^2$ where $G \sim \mathcal{N}(0, 1)$. The main idea is therefore to first generate a Brownian passage time and secondly to accept or reject this variate proposal using the Girsanov weight. The construction of this algorithm looks very much like the algorithms presented by Beskos and Roberts. The main difference is to replace the Brownian paths (or Brownian bridge paths) appearing in the rejection sampling by Bessel paths. The explanation of such a modification is related to the observation: once the Brownian first passage time τ_L is generated, the Brownian motion constrained to stay under the level L on $[0, \tau_L]$ can be related to a 3-dimensional Bessel process. The algorithm proposal is the following.

EXACT SIMULATION OF τ_L FOR CONTINUOUS DIFFUSIONS – ALGORITHM (*HZ*)

1. *Let $(G_n)_{n\geq 1}$ be independent standard 3-dimensional Gaussian vectors.*
2. *Let $(e_n)_{n\geq 0}$ be independent exponentially distributed r.v. with average $1/\kappa$.*
3. *Let $(V_n)_{n\geq 1}$ be independent uniformly distributed r.v. on $[0, 1]$.*
4. *Let $(g_n)_{n\geq 1}$ be independent standard Gaussian r.variables.*

The sequences $(G_n)_{n\geq 1}$, $(e_n)_{n\geq 0}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $k = 0$, $n = 0$.

Step 1. $k \leftarrow k + 1$, $\delta = (0, 0, 0)$, $\mathcal{W} = 0$, $\mathcal{T}_k \leftarrow (L - y_0)^2/g_k^2$, $\mathcal{E}_0 = 0$ and $\mathcal{E}_1 = e_n$.

Step 2. While $\mathcal{E}_1 \leq \mathcal{T}_k$ do:

- set $n \leftarrow n + 1$

- $\delta \leftarrow \dfrac{\mathcal{T}_k - \mathcal{E}_1}{\mathcal{T}_k - \mathcal{E}_0}\delta + \sqrt{\dfrac{(\mathcal{T}_k - \mathcal{E}_1)(\mathcal{T}_k - \mathcal{E}_0)}{\mathcal{T}_k - \mathcal{E}_0}}\, G_n$
- If $\kappa V_n \leq \gamma(\mathcal{E}_1, L- \parallel \mathcal{E}_1(L - y_0)(1,0,0)/\mathcal{T}_k + \delta \parallel)$ then $\mathcal{W} \leftarrow 1$ else $\mathcal{W} \leftarrow 0$
- $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ and $\mathcal{E}_1 \leftarrow \mathcal{E}_1 + e_n$

Step 3. If $\mathcal{W} = 0$ then $\mathcal{Y} \leftarrow \mathcal{T}_k$ otherwise go to *Step 1.*
Outcome: the random variable $\mathcal{Y}$.

Proposition 4.2.5 (Herrmann-Zucca, 2019)**.** *Let us assume that $\tau_L < \infty$ almost surely where τ_L stands for the first passage time of the diffusion* (4.2.1) *through the level L. Under the assumptions 4.2.2 and 4.2.6, both the outcome $\mathcal{Y}$ of Algorithm (HZ) and τ_L have the same distribution.*

The detailed proof of Proposition 4.2.5 is presented in [27]. We do not present here the sketch of the proof since most of the arguments are quite similar to those pointed out in the proofs of Proposition 4.2.2 and 4.2.4. Nevertheless we would like to say that Herrmann and Zucca didn't study a time-dependent drift term as appearing in equation (4.2.1), they focus their attention on the homogeneous case. The statement of Proposition 4.2.5 is therefore an adaptation of their result to the non-homogeneous case: here the function γ depends both on the time and space variables.

Remark 4.2.6. *The algorithm (HZ) can be adapted to the particular case of a continuous diffusion process starting at time $\mathbb{T}_0 > 0$ with the value y. In this case, $(Y_t)_{t \geq \mathbb{T}_0}$ is solution of the following stochastic equation:*

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad \forall t \geq \mathbb{T}_0 \quad \textit{and} \quad Y_{\mathbb{T}_0} = y < L.$$

Moreover the definition of the first passage time is slightly modified τ_L becomes the first time after $\mathbb{T}_0$ such that the diffusion hits the level L. The modification of the algorithm consists in replacing y_0 by y, $\gamma(\cdot,\cdot)$ by $\gamma(\mathbb{T}_0 + \cdot, \cdot)$ and adding $\mathbb{T}_0$ to $\mathcal{Y}$. We then denote $(HZ)_{\mathbb{T}_0}^{y,L}$ the corresponding algorithm.

4.2.2 Stopped continuous diffusion

In the previous section, several procedures of exact simulation have been presented:

- simulation of $Y_{\mathbb{T}}$: the value of the diffusion (4.2.1) at a fixed time $\mathbb{T}$.
- simulation of the first passage time through the level L for the diffusion, denoted τ_L.

On one hand we are able to generate the position, on the other hand the exit time. In order to complete the description, we introduce a suitable combination of the time and the

position which shall play an essential role in the sequel. We propose to build an algorithm which permits to obtain the exact simulation of the random couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$, linked to the stopped diffusion. Here $(Y_t)_{t\geq 0}$ still stands for a continuous diffusion.

Let us first introduce a preliminary result concerning a particular diffusion process, the standard Brownian motion $(B_t)_{t\geq 0}$. Our aim is to generate a random variable which has the same conditional distribution than $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$, where $\mathbb{T}$ is fixed and τ_L is the Brownian first passage time (we shall assume that $L > 0$, the other case can be obtained by symmetry arguments). We build the following algorithm.

CONDITIONAL BROWNIAN MOTION GIVEN $\tau_L > \mathbb{T}$ – ALGORITHM $(CBM)^L_{\mathbb{T}}$

1. *Let $(G_n)_{n\geq 1}$ a sequence of independent standard gaussian random variables*
2. *Let $(U_n)_{n\geq 1}$ a sequence of indep. uniformly distributed random variables on $[0,1]$.*

The sequences $(G_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 1$, $\mathcal{Y} = 0$.
While $\sqrt{\mathbb{T}}\, G_n > L$ or $-\frac{\mathbb{T}}{2L}\ln(U_n) > L - \sqrt{\mathbb{T}}\, G_n$ **do** $n \leftarrow n+1$.
Set $\mathcal{Y} \leftarrow \sqrt{\mathbb{T}} G_n$.
Outcome: The random variable $\mathcal{Y}$.

Proposition 4.2.7. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then both the outcome $\mathcal{Y}$ of Algorithm $(CBM)^L_{\mathbb{T}}$ and $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ have the same distribution.*

Proof. The result is based on the classical acceptance/rejection sampling. Let us first describe the conditional distribution of $B_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ (see for instance Lerche [39])

$$u(\mathbb{T}, x)\, dx := \mathbb{P}(\tau_L > \mathbb{T}, B_{\mathbb{T}} \in dx) = \left(\frac{1}{\sqrt{\mathbb{T}}}\, \phi\left(\frac{x}{\sqrt{\mathbb{T}}}\right) - \frac{1}{\sqrt{\mathbb{T}}}\, \phi\left(\frac{x-2L}{\sqrt{\mathbb{T}}}\right)\right) dx,$$

where ϕ denotes the distribution function of a standard Gaussian variate. We introduce Φ the corresponding cumulative distribution. Then the previous expression leads to

$$f_{\mathbb{T}}(x)\, dx := \mathbb{P}(B_{\mathbb{T}} \in dx | \tau_L > \mathbb{T}) = \frac{1}{\sqrt{\mathbb{T}}} \frac{\phi(x/\sqrt{\mathbb{T}}) - \phi((x-2L)/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}\, dx. \tag{4.2.11}$$

We remark that the following upper-bound is satisfied

$$f_{\mathbb{T}}(x) \leq c\, \frac{\phi(x/\sqrt{\mathbb{T}})}{\sqrt{\mathbb{T}}\Phi(L/\sqrt{\mathbb{T}})}\, 1_{]-\infty, L]}(x) =: c\, g_{\mathbb{T}}(x) \quad \text{with} \quad c = \frac{\Phi(L/\sqrt{\mathbb{T}})}{\Phi(L/\sqrt{\mathbb{T}}) - \Phi(-L/\sqrt{\mathbb{T}})}.$$

It is obvious that $g_{\mathbb{T}}(\cdot)$ corresponds to a distribution function: a centered Gaussian distribution of variance $\mathbb{T}$ conditioned to stay under the value L. In the acceptance/rejection

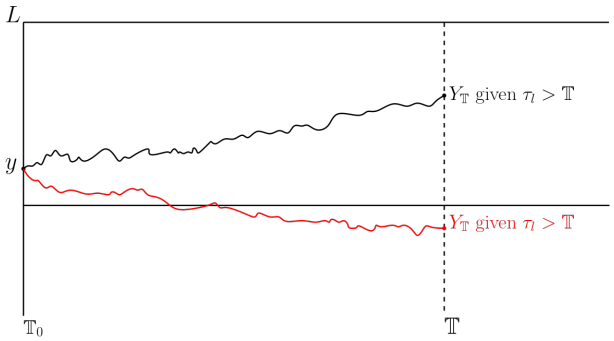


Figure 4.1: Trajectories of the conditioned diffusion

procedure, we shall choose $g_{\mathbb{T}}$ as the proposal distribution. So we generate a random variable Z with distribution $g_{\mathbb{T}}$, this variate is then accepted if $c\,U g_{\mathbb{T}}(Z) \leq f_{\mathbb{T}}(Z)$ where U stands for a uniformly distributed random variable, independent of Z. The condition just mentioned is equivalent to

$$U \leq 1 - \exp\left(\frac{2L}{\mathbb{T}}(Z - L)\right).$$

If G stands for a standard Gaussian r.v., then it is obvious to relate the previous condition to

$$\sqrt{\mathbb{T}}\,G \leq L \quad \text{or} \quad -\frac{\mathbb{T}}{2L}\ln(1-U) \geq L - \sqrt{\mathbb{T}}\,G,$$

the acceptance condition appearing in Algorithm $(CBM)^L_{\mathbb{T}}$. □

First we proposed in Proposition 4.2.7 the generation of the conditional Brownian motion. As already pointed out in the previous section, we can relate distributions concerning the Brownian paths to the diffusion ones using the classical Girsanov transformation. An interesting application of this transformation is therefore to simulate a diffusion value at a fixed time $\mathbb{T}$ given $\tau_L > \mathbb{T}$. In order to get a general statement, we consider a diffusion process starting at time $\mathbb{T}_0 < \mathbb{T}$ with the value $Y_{\mathbb{T}_0} = y$. It is therefore the unique strong solution of the equation:

$$dY_t = \alpha(t, Y_t)\,dt + dB_t, \quad \forall t \geq \mathbb{T}_0, \quad \text{and} \quad Y_{\mathbb{T}_0} = y < L, \tag{4.2.12}$$

where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. The corresponding algorithm is the following (see the illustration in Figure 4.1).

CONDITIONED DIFFUSION $Y_{\mathbb{T}}$ GIVEN $\tau_L > \mathbb{T}$ – ALGORITHM $(CD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$

1. *Let $(U_n)_{n\geq 1}$ be independent uniformly distributed random variables on $[0, \kappa]$.*
2. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\kappa$.*

The sequences $(U_n)_{n\geq 1}$ and $(E_n)_{n\geq 1}$ are assumed to be independent.

Initialization: $n = 1$.

Step 1. Set $\mathcal{Y} = y$, $\mathcal{T} = \mathbb{T}_0$ and generate $\mathcal{Z} \sim (CBM)_{E_n}^{L-\mathcal{Y}}$.

Step 2. While $\mathcal{T} + E_n < \mathbb{T}$ and $U_n > \gamma(\mathcal{T} + E_n, \mathcal{Y} + \mathcal{Z})$ do

- $\mathcal{T} \leftarrow \mathcal{T} + E_n$
- $n \leftarrow n + 1$
- $\mathcal{Y} \leftarrow \mathcal{Y} + \mathcal{Z}$
- Generate $\mathcal{Z} \leftarrow (CBM)_{E_n}^{L-\mathcal{Y}}$.

Step 3. If $\mathcal{T} + E_n > \mathbb{T}$ then generate $\mathcal{Z} \sim (CBM)_{\mathbb{T}-\mathcal{T}}^{L-\mathcal{Y}}$, set $\mathcal{Y} \leftarrow \mathcal{Y} + \mathcal{Z}$ and $V \sim \mathcal{U}([0,1])$ independent of all other variates otherwise set $n \leftarrow n + 1$ and go to *Step 1.*

Step 4. If $V \cdot \exp(\beta_+) > \exp(\beta(\mathbb{T}, \mathcal{Y}))$ then set $n \leftarrow n + 1$ and go to *Step 1.*

Outcome: The random variable $\mathcal{Y}$.

Proposition 4.2.8. *Let us consider $(Y_t)_{t\geq \mathbb{T}_0}$ the diffusion defined by (4.2.12) and τ_L the associated first passage time through the level L:*

$$\tau_L := \inf\{t \geq \mathbb{T}_0 : \ Y_t \geq L\}. \tag{4.2.13}$$

Under assumptions 4.2.2, 4.2.6 et 4.2.7, both the outcome $\mathcal{Y}$ of Algorithm $(CD)_{\mathbb{T}_0,\mathbb{T}}^{y,L}$ and $Y_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$ have the same distribution.

Proof. The algorithm $(CD)_{\mathbb{T}_0,\mathbb{T}}^{y,L}$ is clearly based on a rejection sampling. The proof uses therefore similar arguments than those pointed out in Proposition 4.2.2 and Proposition 4.2.4. Let us denote by $\mathcal{Y}_n$ (respectively $\mathcal{T}_n$ and $\mathcal{Z}_n$) the successive values of $\mathcal{Y}$ (resp. $\mathcal{T}$ and $\mathcal{Z}$). We also introduce a sequence of times $(\mathcal{E}_n)_{n\geq 1}$ defined by $\mathcal{E}_{n+1} = \mathcal{E}_n + E_{n+1}$ with $\mathcal{E}_0 = \mathbb{T}_0$. We finally introduce $\mathcal{N}$ the number of Step 1 used before the algorithm stops. Since the algorithm is a acceptance/rejection sampling, we have for any non negative measurable function ψ:

$$\mathbb{E}[\psi(\mathcal{Y})] = \frac{\mathbb{E}[\psi(\mathcal{Y})1_{\{\mathcal{N}=1\}}]}{\mathbb{P}(\mathcal{N}=1)} = \frac{\nu(\psi)}{\nu(1)} \quad \text{where} \quad \nu(\psi) := \mathbb{E}[\psi(\mathcal{Y})1_{\{\mathcal{N}=1\}}].$$

In the following computations, we denote by A_n the particular event: $\{\mathcal{E}_n \leq \mathbb{T} < \mathcal{E}_{n+1}\}$ and P_n shall correspond to the event:

$$P_n := \{U_1 > \gamma(\mathcal{E}_1, \mathcal{Y}_1 + \mathcal{Z}_1), \ldots, U_n > \gamma(\mathcal{E}_n, \mathcal{Y}_n + \mathcal{Z}_n)\}, \text{ for } n \geq 1 \text{ and } P_0 = \Omega.$$

We therefore obtain that

$$\nu(\psi) = \sum_{n\geq 0} \mathbb{E}\Big[\psi(\mathcal{Y}_{n+2})1_{P_n}1_{\{V\cdot\exp(\beta_+)<\exp(\beta(\mathbb{T},\mathcal{Y}_{n+2}))\}}1_{A_n}\Big].$$

Integrating with respect to all uniformly distributed random variables (U_n) and with respect to V leads to

$$\nu(\psi) = \sum_{n\geq 0} \frac{1}{\kappa^n} \mathbb{E}\Big[\psi(\mathcal{Y}_{n+2}) \prod_{k=1}^{n} (\kappa - \gamma(\mathcal{E}_k, \mathcal{Y}_k + \mathcal{Z}_k)) \exp(\beta(\mathbb{T}, \mathcal{Y}_{n+2}) - \beta_+) 1_{A_n}\Big]$$

We note that $(\mathcal{E}_1 - \mathbb{T}_0, \dots, \mathcal{E}_n - \mathbb{T}_0)$, given A_n, has the same distribution than $(V^{(1)}, \dots, V^{(n)})$ an ordered n-tuple of uniform random variables $(V_1, \dots, V_n)$ on $[0, \mathbb{T} - \mathbb{T}_0]$. Moreover the probability of the event A_n is linked to the Poisson distribution of parameter $\kappa(\mathbb{T} - \mathbb{T}_0)$. Finally on the event A_n, $(\mathcal{Y}_1, \mathcal{Y}_2, \dots \mathcal{Y}_{n+2})$ has the same distribution than

$$(y, y + B_{V^{(1)}}, \dots, y + B_{V^{(n)}}, y + B_{\mathbb{T}-\mathbb{T}_0}) \text{ given } \tau^B_{L-y} := \inf\{t \geq 0 : \ B_t \geq L - y\} > \mathbb{T} - \mathbb{T}_0$$

with $(B_t)_{t\geq 0}$ a standard Brownian motion independent of the n-tuple $(V_1, \dots, V_n)$. Hence

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 0} \frac{1}{\kappa^n} \mathbb{E}\Big[\psi(y + B_{\mathbb{T}-\mathbb{T}_0}) \prod_{k=1}^{n} \Big(\kappa - \gamma(\mathbb{T}_0 + V^{(k)}, y + B_{V^{(k)}})\Big) \\
&\qquad \times \exp(\beta(\mathbb{T}, y + B_{\mathbb{T}-\mathbb{T}_0}) - \beta_+) \Big| \tau^B_{L-y} > \mathbb{T} - \mathbb{T}_0\Big] \frac{\kappa^n (\mathbb{T} - \mathbb{T}_0)^n}{n!} e^{-\kappa(\mathbb{T}-\mathbb{T}_0)} \\
&= \sum_{n\geq 0} \mathbb{E}\Big[\psi(y + B_{\mathbb{T}-\mathbb{T}_0}) \prod_{k=1}^{n} \Big(\kappa - \gamma(\mathbb{T}_0 + V_k, y + B_{V_k})\Big) \\
&\qquad \times \exp(\beta(\mathbb{T}, y + B_{\mathbb{T}-\mathbb{T}_0}) - \beta_+) \Big| \tau^B_{L-y} > \mathbb{T} - \mathbb{T}_0\Big] \frac{(\mathbb{T} - \mathbb{T}_0)^n}{n!} e^{-\kappa(\mathbb{T}-\mathbb{T}_0)}.
\end{aligned}$$

Taking the expectation with respect to the uniformly distributed variates V_k leads to

$$\begin{aligned}
\nu(\psi) &= \sum_{n\geq 0} \mathbb{E}\Big[\psi(y + B_{\mathbb{T}-\mathbb{T}_0}) \Big(\kappa - \frac{1}{\mathbb{T} - \mathbb{T}_0} \int_0^{\mathbb{T}-\mathbb{T}_0} \gamma(\mathbb{T}_0 + s, y + B_s)\, ds\Big)^n \\
&\qquad \times \exp(\beta(\mathbb{T}, y + B_{\mathbb{T}-\mathbb{T}_0}) - \beta_+) \Big| \tau^B_{L-y} > \mathbb{T} - \mathbb{T}_0\Big] \frac{(\mathbb{T} - \mathbb{T}_0)^n}{n!} e^{-\kappa(\mathbb{T}-\mathbb{T}_0)} \\
&= \mathbb{E}\Big[\psi(y + B_{\mathbb{T}-\mathbb{T}_0}) \exp\Big(-\int_0^{\mathbb{T}-\mathbb{T}_0} \gamma(\mathbb{T}_0 + s, y + B_s)\, ds\Big) \\
&\qquad \times \exp(\beta(\mathbb{T}, y + B_{\mathbb{T}-\mathbb{T}_0}) - \beta_+) \Big| \tau^B_{L-y} > \mathbb{T} - \mathbb{T}_0\Big].
\end{aligned}$$

Since $(y + B_t)_{t\geq 0}$ given $\tau^B_{L-y} > \mathbb{T} - \mathbb{T}_0$ has the same distribution than $(B_t)_{t\geq \mathbb{T}_0}$ given both $B_{\mathbb{T}_0} = y$ and $\tau^B_L \circ \theta_{\mathbb{T}_0} > \mathbb{T}$, where θ stands for the translation operator, we obtain

$$\nu(\psi) = \mathbb{E}\Big[\psi(B_{\mathbb{T}}) \exp\Big(-\int_{\mathbb{T}_0}^{\mathbb{T}} \gamma(s, B_s)\, ds + \beta(\mathbb{T}, B_{\mathbb{T}}) - \beta_+\Big) \Big| B_{\mathbb{T}_0} = y,\ \tau^B_L \circ \theta_{\mathbb{T}_0} > \mathbb{T}\Big].$$

Let us now modify the expression under review. We introduce

$$\hat{\nu}(\psi) := \mathbb{E}\Big[\psi(B_{\mathbb{T}}) 1_{\{B_t < L,\ \forall t \in [\mathbb{T}_0, \mathbb{T}]\}} \exp\Big(-\int_{\mathbb{T}_0}^{\mathbb{T}} \gamma(s, B_s)\, ds + \beta(\mathbb{T}, B_{\mathbb{T}}) - \beta(\mathbb{T}_0, y)\Big) \Big| B_{\mathbb{T}_0} = y\Big].$$

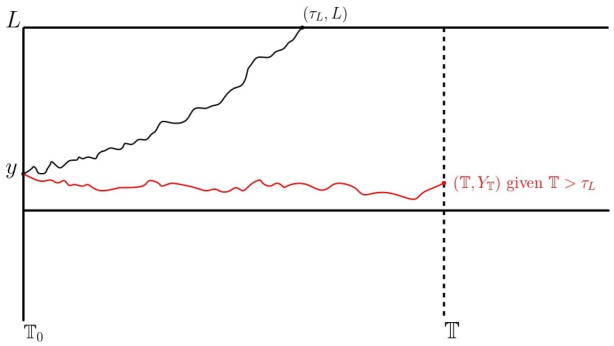


Figure 4.2: Two typical trajectories of the continuous diffusion.

It is obvious that

$$\mathbb{E}[\psi(\mathcal{Y})] = \frac{\nu(\psi)}{\nu(1)} = \frac{\hat{\nu}(\psi)}{\hat{\nu}(1)}. \tag{4.2.14}$$

Moreover since $(M_t)_{t\geq 0}$ defined by

$$M_t := \exp\Big(- \int_{\mathbb{T}_0}^{\mathbb{T}_0+t} \gamma(s, B_s)\, ds + \beta(\mathbb{T}_0 + t, B_{\mathbb{T}_0+t}) - \beta(\mathbb{T}_0, y)\Big),$$

is the exponential martingale appearing in the Girsanov transformation, we obtain by the change of measure procedure:

$$\hat{\nu}(\psi) = \mathbb{E}\Big[\psi(Y_{\mathbb{T}}) 1_{\{Y_t < L,\ \forall t \in [\mathbb{T}_0, \mathbb{T}]\}}\Big],$$

where $(Y_t)_{t\geq \mathbb{T}_0}$ stands for the diffusion (4.2.12). The ratio defined by (4.2.14) then permits to conclude the proof

$$\mathbb{E}[\psi(\mathcal{Y})] = \mathbb{E}[\psi(Y_{\mathbb{T}}) | \tau_L > \mathbb{T}],$$

the stopping time τ_L being introduce in the statement (4.2.13).

□

Finally we are able to produce an algorithm which generates exactly the distribution of the couple $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$ where $(Y_t)_{t\geq \mathbb{T}_0}$ stands for the continuous diffusion defined in (4.2.12) and τ_L, the first passage time defined in (4.2.13) (see the illustration in Figure 4.2).

STOPPED DIFFUSION: $(\tau_L \wedge \mathbb{T}, Y_{\tau_L \wedge \mathbb{T}})$ – ALGORITHM $(SD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$

Step 1. Generate $\mathcal{T} \sim (HZ)^{y,L}_{\mathbb{T}_0}$ (defined in Remark 4.2.6).

Step 2. If $\mathcal{T} < \mathbb{T}$ then set $\mathcal{Y} \leftarrow L$ otherwise generate $\mathcal{Y} \sim (CD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$ and set $\mathcal{T} \leftarrow \mathbb{T}$.
Outcome: The random couple $(\mathcal{T}, \mathcal{Y})$.

Proposition 4.2.9. *Let us consider $(Y_t)_{t\geq \mathbb{T}_0}$ the diffusion defined by* (4.2.12) *and τ_L the associated first passage time* (4.2.13)*. Under assumptions 4.2.2, 4.2.6 et 4.2.7, both the outcome $(\mathcal{T}, \mathcal{Y})$ of Algorithm $(SD)^{y,L}_{\mathbb{T}_0,\mathbb{T}}$ and $(\tau_L \wedge \mathbb{T}, Y_{\tau_L\wedge\mathbb{T}})$ have the same distribution.*

Proof. The proof is straightforward. Either τ_L is smaller than $\mathbb{T}$ which corresponds to $\tau_L \wedge \mathbb{T} = \tau_L$ and $Y_{\tau_L\wedge\mathbb{T}} = L$ or τ_L is larger than $\mathbb{T}$ and therefore the distribution of $Y_{\tau_L\wedge\mathbb{T}}$ is the conditional distribution of $Y_{\mathbb{T}}$ given $\tau_L > \mathbb{T}$. □

4.3 Simulation of the first passage time for stopped jump diffusions

The aim of this section is to generate the first passage time of a jump diffusion. Let us just recall that the jump diffusion characterized by the stochastic differential equation between the jump times (4.1.2) and the jump height described in (4.1.3). In Section 4.1.2, we discussed the possibility to reduce the considered model. Consequently we shall first consider the following reduced model. Let us introduce $(T_n)_{n\geq 1}$ the sequence of jump times. We mention that the time spent between two consecutive jumps is exponentially distributed, therefore $T_n = \sum_{k=1}^{n} E_k$ with $(E_k)_{k\geq 1}$ a sequence of independent exponentially distributed random variables with average $1/\lambda$. The initial position of the diffusion is given by $Y_0 = y_0$ and the jump diffusion under consideration moreover satisfies

$$dY_t = \alpha(t, Y_t)\, dt + dB_t, \quad \text{for } T_n < t < T_{n+1}, \quad n \in \mathbb{N}, \tag{4.3.1}$$

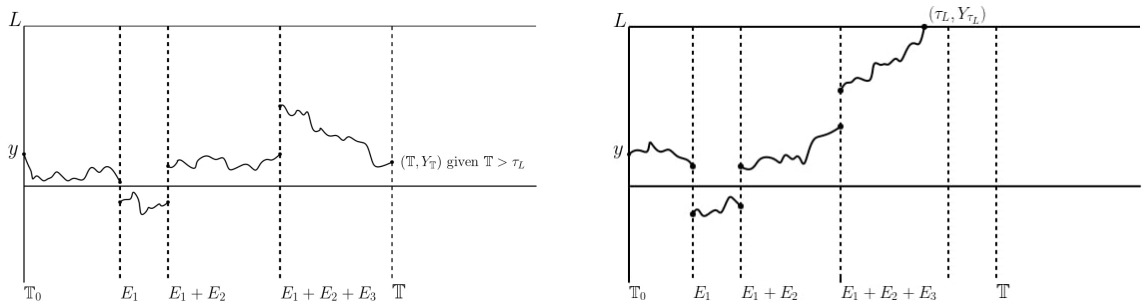
the jumps modify the trajectories as follows:

$$Y_{T_n} = Y_{T_n-} + j(T_n, Y_{T_n-}, \xi_n), \quad \forall n \in \mathbb{N}, \tag{4.3.2}$$

where $j : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{E} \to \mathbb{R}$ denotes the jump function and $(\xi_n)_{n\geq 1}$ stands for a sequence of independent random variables with distribution function ϕ/λ (also independent of the Brownian motion $(B_t)_{t\geq 0}$ and of the sequence $(T_n)_{n\geq 0}$). Let us associate to the stochastic process defined by (4.3.1)–(4.3.2) the first passage time throught the level L (with $L > y_0$):

$$\tau_L := \inf\{t \geq 0 :\ Y_t \geq L\}. \tag{4.3.3}$$

Since the jump times and the behaviour of the diffusion process inbetween the jump times are independent, we can use the approach developped in the continuous diffusion case in order to simulate jump diffusions. The crucial argument is that $(t, Y_t)_{t\geq 0}$ is a Markovian stochastic process. Consequently the diffusion paths can be constructed in a piecewise Markovian way. Let us present the algorithm for the generation of the stopped first passage time $\tau_L \wedge \mathbb{T}$ where $\mathbb{T}$ stands for a fixed time.



STOPPED JUMP DIFFUSION ($\tau_L \wedge \mathbb{T}$) – ALGORITHM $(SJD)_{\mathbb{T}}^{y,L}$

1. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\lambda$.*
2. *Let $(\xi_n)_{n\geq 1}$ be independent r.v. with distribution function ϕ/λ.*

The sequences $(E_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are assumed to be independent.
Initialization. $n = 0$, $\mathcal{T}_s = 0$ (starting time), $\mathcal{T}_f = 0$ (final time), $\mathcal{Y} = y$, $\mathcal{Z} = y$.
Step 1. While ($\mathcal{T}_f < \mathbb{T}$ and $\mathcal{Y} < L$ and $\mathcal{Z} < L$) do

- $n \leftarrow n + 1$
- $\mathcal{T}_s \leftarrow \mathcal{T}_f$
- $\mathcal{T}_f \leftarrow \mathcal{T}_f + E_n$
- Generate $(\mathcal{S}, \mathcal{Z}) \sim (SD)_{\mathcal{T}_s,\mathcal{T}_f}^{\mathcal{Y},L}$
- $\mathcal{Y} \leftarrow \mathcal{Z} + j(\mathcal{T}_f, \mathcal{Z}, \xi_n)$

Step 2.

- If $\mathcal{S} > \mathbb{T}$ then set $\mathcal{S} \leftarrow \mathbb{T}$
- If $\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} < L$ then $\mathcal{S} \leftarrow \mathcal{T}_f$

Outcome: The random variable $\mathcal{S}$.

Theorem 4.3.1. *Let us consider $(Y_t)_{t\geq 0}$ the jump diffusion defined by* (4.3.1)–(4.3.2) *and τ_L the associated first passage time* (4.3.3)*. Under assumptions 4.2.2, 4.2.6 et 4.2.7, both the outcome $\mathcal{S}$ of Algorithm $(SJD)_{\mathbb{T}}^{y_0,L}$ and $\tau_L \wedge \mathbb{T}$ have the same distribution.*

Proof. The proof is based on the Markov property of the process $(t, Y_t)_{t\geq 0}$. We start an iterated procedure. First we set $\mathcal{T}_s = 0$ (starting time) and generate a random variable

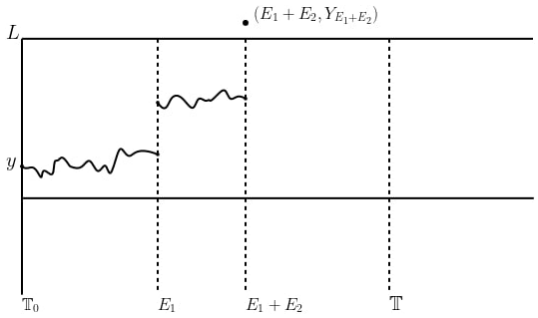


Figure 4.3: Three typical paths representing different scenarios

which represents the first jumping time T_1. This r.v. is denoted $\mathcal{T}_f = E_1$ in the algorithm (final time). Two situations can therefore be considered.

1. First case: $\mathcal{T}_f > \mathbb{T}$ (this means that Step 1 is only used once). Conditionally to the event *"the first jumping time is larger than* $\mathbb{T}$", the diffusion behaves like a continuous diffusion process on the interval $[0, \mathbb{T}]$ that we shall denote $(\hat{Y}_t)$. Hence $Y_t = \hat{Y}_t$ for any $t < \mathcal{T}_f$. Indeed the jumping times are independent of the Brownian motion driving the diffusive part of the stochastic process. Consequently the results developed in the previous section can be applied. The algorithm $(SD)^{y_0,L}_{\mathcal{T}_s,\mathcal{T}_f}$ permits to obtain the random couple $(\hat{\tau}_L \wedge \mathcal{T}_f, \hat{Y}_{\hat{\tau}_L \wedge \mathcal{T}_f})$ where $\hat{\tau}_L$ stands for the first passage time of the continuous diffusion $(\hat{Y}_t)$ after time $\mathcal{T}_s$. The random couple is denoted by $(\mathcal{S}, \mathcal{Z})$ in Algorithm $(SJD)^{y,L}_{\mathbb{T}}$. We observe then one of the following situation:

   - $\hat{\tau}_L \wedge \mathcal{T}_f > \mathbb{T}$ which is equivalent to the condition $\mathcal{S} > \mathbb{T}$. In such a situation we deduce easily that $\hat{\tau}_L \wedge \mathbb{T} = \mathbb{T} = \tau_L \wedge \mathbb{T}$, since $\mathcal{T}_f > \mathbb{T}$, and therefore it suffices to set $\mathcal{S} = \mathbb{T}$ in the algorithm in order to obtain the distribution identity announced in the statement of Theorem 4.3.1.
   - $\hat{\tau}_L \wedge \mathcal{T}_f \leq \mathbb{T}$ corresponding to ($\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} \geq L$). Then obviously we get $\tau_L \wedge \mathbb{T} = \tau_L = \tau_L \wedge \mathcal{T}_f = \hat{\tau}_L \wedge \mathcal{T}_f$ which is identically distributed as $\mathcal{S}$.

2. Second case: $\mathcal{T}_f \leq \mathbb{T}$. Here also the diffusion (Y_t) corresponds to a continuous diffusion $(\hat{Y}_t)$ on the time interval $[\mathcal{T}_s, \hat{\tau}_L \wedge \mathcal{T}_f[$. So the previous section permits to generate $(\hat{\tau}_L \wedge \mathcal{T}_f, \hat{Y}_{\hat{\tau}_L \wedge \mathcal{T}_f})$, denoted by $(\mathcal{S}, \mathcal{Z})$. Of course $\mathcal{S} \leq \mathbb{T}$. We distinguish three different situations:

   - $\mathcal{Z} \geq L$: this means that $\mathcal{S} = \hat{\tau}_L \wedge \mathcal{T}_f = \hat{\tau}_L = \tau_L = \tau_L \wedge \mathbb{T}$ which corresponds to the statement of Theorem 4.3.1.
   - $\mathcal{Z} < L$ and $\mathcal{Z} + j(\mathcal{T}_f, \mathcal{Z}, \xi_1) \geq L$ which implies that the diffusion doesn't cross the level L before the first jump whereas the first jump permits to observe this first passage. In other words, the first passage time corresponds to the first jump. Hence $\tau_L = \mathcal{T}_f$ (this situation occurs in the algorithm in the second step as both $\mathcal{S} \leq \mathbb{T}$ and $\mathcal{Z} < L$).

- $\mathcal{Z} < L$ and $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In such a situation, the jump diffusion does not reach any value larger than L on the interval $[\mathcal{T}_s, \mathcal{T}_f] = [0, E_1]$. Indeed $\mathcal{Z} < L$ implies that $\hat{\tau}_L \wedge \mathcal{T}_f = \mathcal{T}_f$, the only possibility to overcome the level L on the time interval $[\mathcal{T}_s, \mathcal{T}_f]$ being to observe a suitable jump at time $\mathcal{T}_f$. Unfortunately such an event cannot happen since $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In conclusion the first passage time is strictly larger than the first jump time. In order to generate the FPT, we propose therefore to start again using the Markov property: we should observe a jump diffusion starting at time $\mathcal{T}_f$ (becoming the starting time $\mathcal{T}_s$) with the value $\mathcal{Z} + j(\mathcal{T}_j, \mathcal{Z}, \xi_1) < L$. In that situation, the algorithm permits to repeat Step 1 with new initial values.

Since $\tau_L \wedge \mathbb{T}$ is finite a.s. only a finite number of repetition of Step 1 is observed, the iterative procedure which is directly associated to the Markov property of the jump diffusion permits then to conclude and to obtain the announced statement. □

Theorem 4.3.1 concerns the generation of the finite stopping time $\tau_L \wedge \mathbb{T}$ associated to the reduced model (4.3.1)–(4.3.2) (we recall that $\mathbb{T}$ is a fixed time). Using the Lamperti transformation presented in Section 4.1.2, it is possible to generalize the study. Let us assume that, between two consecutive jumps, the stochastic process satisfies a stochastic differential equation:

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}, \tag{4.3.4}$$

and the jumps modify the trajectories as follows:

$$Y_{T_i} = Y_{T_i-} + j(T_i, Y_{T_i-}, \xi_i), \quad \forall i \in \mathbb{N}, \tag{4.3.5}$$

where j stands for the jump function. A generation of the stopping time $\tau_L \wedge \mathbb{T}$ associated to the jump diffusion (4.3.4)–(4.3.5) is then available. Proposition 4.1.1 emphasizes the efficient way to generate $\tau_L \wedge \mathbb{T}$. Let us define

$$\nu(t, x) = \int_L^x \frac{1}{\sigma(t, y)}\, dy$$

and let us consider its inverse $\nu^{-1} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which represents the unique function verifying $\nu^{-1}(t, \nu(t, x)) = x$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. We define $Z_t = \nu(t, Y_t)$. As already mentionned in Section 4.1.1, $(Z_t)_{t \geq 0}$ is a jump diffusion satifying $Z_0 = \nu(0, y_0)$ and the reduced model (4.3.1)–(4.3.2) where the function α corresponds to

$$\alpha(t, x) := \frac{\partial \nu}{\partial t}(t, \nu^{-1}(t, x)) + \frac{\mu(t, \nu^{-1}(t, x))}{\sigma(t, \nu^{-1}(t, x))} - \frac{1}{2}\frac{\partial \sigma}{\partial x}(t, \nu^{-1}(t, x)) \tag{4.3.6}$$

and the jump function $j(t, z, v)$ is replaced by

$$\hat{\jmath}(t, z, v) := \nu(t, \nu^{-1}(t, z) + j(t, \nu^{-1}(t, z), v)) - \nu(t, \nu^{-1}(t, z)). \tag{4.3.7}$$

Let us finally note that $x = L$ represents the unique solution of the equation $\nu(t,x) = 0$ since the diffusion coefficient is strictly positive. Therefore the following identity holds:

$$\tau_L := \inf\{t \geq 0 : \ Y_t \geq L\} = \inf\{t \geq 0 : \ Z_t \geq 0\}. \tag{4.3.8}$$

In conclusion we easily obtain the following statement.

Proposition 4.3.2. *Let $\mathbb{T}$ be a fixed time. The random variable $\mathcal{S} \sim (SJD)_{\mathbb{T}}^{\nu(0,y_0),0}$ which is the outcome of the algorithm – using the drift term α defined in* (4.3.6) *and the jump function described in* (4.3.7) *– has the same distribution than $\tau_L \wedge \mathbb{T}$ the stopping time associated to the jump diffusion* (4.3.4)–(4.3.5)*.*

4.4 Simulation of a.s. finite first passage times for jump diffusions

4.4.1 Modification of the algorithm

In the previous section, we focus our attention on the exact generation of stopped first passage times for jump diffusions, denoted by $\tau_L \wedge \mathbb{T}$ where $\mathbb{T}$ stands for a fixed time. The main advantage of considering stopped processes is to deal with bounded random variables. The algorithms presented so far therefore stop almost surely since they require only a finite number of iterations.

The aim of this section is to present particular situations where the stopped diffusion can be replaced by the diffusion itself. They obviously correspond to almost surely finite first passage times: $\tau_L < \infty$. The algorithm $(SJD)_{\mathbb{T}}^{y,l}$ can then easily be modified just by setting $\mathbb{T} = \infty$: we obtain the following algorithm $(JD)^{y,l}$.

JUMP DIFFUSION τ_L – ALGORITHM $(JD)^{y,L}$

1. *Let $(E_n)_{n\geq 1}$ be independent exponentially distributed r.v. with average $1/\lambda$.*
2. *Let $(\xi_n)_{n\geq 1}$ be independent r.v. with distribution function ϕ/λ.*

The sequences $(E_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are assumed to be independent.
Initialization. $n = 0$, $\mathcal{T}_s = 0$ (starting time), $\mathcal{T}_f = 0$ (final time), $\mathcal{Y} = y$, $\mathcal{Z} = y$.
While $\mathcal{Y} < L$ and $\mathcal{Z} < L$ **do**

- $n \leftarrow n + 1$
- $\mathcal{T}_s \leftarrow \mathcal{T}_f$
- $\mathcal{T}_f \leftarrow \mathcal{T}_f + E_n$

- Generate $(\mathcal{S}, \mathcal{Z}) \sim (SD)^{\mathcal{Y},L}_{\mathcal{T}_s,\mathcal{T}_f}$
- $\mathcal{Y} \leftarrow \mathcal{Z} + j(\mathcal{T}_f, \mathcal{Z}, \xi_n)$

Outcome: The random variable $\mathcal{S}$.

The following result is an immediate modification of the statement of Theorem 4.3.1.

Corollary 4.4.1. *Let us consider $(Y_t)_{t\geq 0}$ the jump diffusion defined by (4.3.1)–(4.3.2) and τ_L the associated first passage time (4.3.3). Under assumptions 4.2.2, 4.2.6 et 4.2.7 and assuming $\tau_L < \infty$, both the outcome $\mathcal{S}$ of Algorithm $(JD)^{y_0,L}$ and τ_L have the same distribution.*

As described in the previous section, the Lamperti transformation permits to generalize the generation of first passage times for jump diffusion processes defined by (4.3.4)–(4.3.5).

4.4.2 Particular models with $\tau_L < \infty$ a.s.

The condition $\tau_L < \infty$ permits to obtain a simplified algorithm for the first passage time generation as described in Section 4.4.1. However it is not so easy in general to determine if such a condition is satisfied. Let us focus our attention on particular assumptions related to the diffusion characteristics (diffusion coefficient, drift term, jump measure) which lead to the required event $\tau_L < \infty$. This part of the chapter is not new, its aim is just to point out that the condition $\tau_L < \infty$ can sometimes be satsified.

First we shall consider a toy model: a Brownian motion with constant drift term combined with a constant rate Poisson process. The first passage time of this model has been studied by Kou and Wang [38] with applications to mathematical finance. Secondly we shall present usual tools for studying the transience or recurrence of stochastic processes. The proofs are often based on Lyapunov functions and Wee proposed an interesting study of d-dimensional jump diffusions [59]. We shall just present the arguments which permit to point out conditions for the first passage time to be a.s. finite.

A toy model: Brownian motion with Poisson jumps (Kou-Wang)

Kou and Wang consider a particular diffusion satisfying (4.3.4)–(4.3.5), the following double exponential jump process:

$$Y_t = \sigma B_t + \mu t + \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \quad Y_0 = 0, \tag{4.4.1}$$

where $(N_t,\, t \geq 0)$ corresponds the Poisson counting process of parameter λ associated to the random time sequence $(T_i)_{i\geq 1}$ and $(\xi_i)_{i\geq 1}$ are i.i.d random variables, independent of $(N_t,\, t \geq$

0). The probability distribution function of ξ_i, denoted by f_ξ, is the double exponential one (with parameters η_1 and η_2):

$$f_\xi(x) = p\eta_1 e^{-\eta_1 x} 1_{\{x\geq 0\}} + q\eta_2 e^{\eta_2 x} 1_{\{x<0\}}, \quad \text{with} \quad p+q=1. \tag{4.4.2}$$

In this particular situation, it is possible to compute quite easily the infinitesimal generator of the diffusion: for any twice continuously differentiable function u, we have

$$\mathcal{L}u(y) = \frac{1}{2}\sigma^2 u''(y) + \mu u'(y) + \lambda \int_{\mathbb{R}} \{u(y+x) - u(y)\} f_\xi(x)\, dx, \quad \forall y \in \mathbb{R}. \tag{4.4.3}$$

Some information concerning the first passage time can be obtained using this generator. Kou and Wang emphasized for instance the joint distribution of the first passage time τ_L (for $L>0$) and the position Y_{τ_L} (useful for the computation of the overshoot $Y_{\tau_L} - L$). The results are deeply based on the fact that the expression of the generator is simple enough for pointing out explicit solutions of different boundary value problems.

First let us consider the Laplace transform of τ_L that is $\mathbb{E}[e^{-\rho\tau_L}]$ for any $\rho > 0$ (see Theorem 3.1 in [38]). There exist two positive roots $\beta_{1,\rho}$ and $\beta_{2,\rho}$ satisfying

$$\rho = G(\beta) \quad \text{with} \quad G(y) := y\mu + \frac{1}{2}y^2\sigma^2 + \lambda\Big(\frac{p\eta_1}{\eta_1 - y} + \frac{q\eta_2}{\eta_2 + y} - 1\Big).$$

We define two constants:

$$A(\rho) = \frac{\eta_1 - \beta_{1,\rho}}{\eta_1}\frac{\beta_{2,\rho}}{\beta_{2,\rho} - \beta_{1,\rho}} \quad \text{and} \quad B(\rho) = \frac{\beta_{2,\rho} - \eta_1}{\eta_1}\frac{\beta_{1,\rho}}{\beta_{2,\rho} - \beta_{1,\rho}}$$

and introduce the following continuous function u_ρ:

$$u_\rho(y) = 1_{\{y\geq L\}} + \Big(A(\rho)\, e^{-\beta_{1,\rho}(L-y)} + B(\rho)\, e^{-\beta_{2,\rho}(L-y)}\Big) 1_{\{y<L\}}. \tag{4.4.4}$$

It is straightforward that the particular choice of the constants leads u_ρ to satisfy the equation

$$-\rho u_\rho(y) + \mathcal{L}u_\rho(y) = 0 \quad \text{for all } y < L.$$

Using some regularization technique (replacing the continuous function u_ρ by a sequence of $\mathcal{C}^2$-continuous functions $u_\rho^{(n)}$) and the martingale theory, Kou and Wang proved that

$$u_\rho(0) = \mathbb{E}[e^{-\rho\tau_L} u_\rho(Y_{\tau_L}) 1_{\{\tau_L<\infty\}}] = \mathbb{E}[e^{-\rho\tau_L}],$$

since $u_\rho(y) = 1$ for any $y \geq L$. In order to describe the probability that τ_L is finite, it suffices to consider $\lim_{\rho\to 0} u_\rho(0)$. Kou and Wang studied carefully the behaviour of all parameters depending on ρ as $\rho \to 0$ and deduced the following result.

Proposition 4.4.2 (Kou & Wang, 2003)**.** *The first passage time τ_L of the diffusion* (4.4.1) *satisfies*

$$\mathbb{P}(\tau_L < \infty) = 1 \quad \textit{iff} \quad \overline{u} := \mu + \lambda\Big(\frac{p}{\eta_1} - \frac{q}{\eta_2}\Big) \geq 0,$$

where $\overline{u}$ stands for the overall drift of the jump diffusion process.

Let us just note that the condition $\overline{u}$ seems quite intuitive and can be related to the asymptotic behaviour of the diffusion paths. Indeed let us consider the sequence $Z_n := Y_{T_n} - Y_{T_{n-1}}$ for $n \geq 2$ and $Z_1 = Y_{T_1}$. We observe that $(Z_n)_{n\geq 1}$ is a sequence of i.i.d random variables with a finite second moment. Therefore the law of large numbers implies

$$\frac{1}{n} Y_{T_n} = \frac{1}{n} \sum_{i=1}^{n} Z_i \to \mathbb{E}[Z_1] \quad \text{a.s. when } n \to \infty.$$

Since $\mathbb{E}[Z_1] = \overline{u}/\lambda$, the condition $\overline{u} > 0$ obviously leads to the almost sure event $\tau_L < \infty$. For the particular case $\overline{u} = 0$, an argument based on the functional central limit theorem (Donsker's theorem) permits also to reach the same conclusion. These arguments are linked to the following facts:

- the process has independent increments
- the increments $Y_t - Y_s$ and $Y_{t-s} - Y_0$ are identically distributed.

This restrictive model permits to handle with a first example of jump diffusion with alsmost surely finite first passage time τ_L.

Jump diffusions with time-homogeneous coefficients (Wee)

Let us now consider a larger family of models: the jump diffusion processes studied by Wee [59]. The results have been pointed out in the general d-dimensional case, nevertheless we shall present in this section the restrictive one-dimensional case which corresponds to the first passage time context. Let us just recall the main objective: we aim to find particular conditions which insure that $\tau_L < \infty$ almost surely.

Let us recall the definition of the jump diffusion (4.1.1) in situations where the coefficient are time-homogeneous:

$$dX_t = \mu(X_{t-})\, dt + \sigma(X_{t-})\, dB_t + \int_{\mathcal{E}} j(X_{t-}, v) p_\phi(dv \times dt), \quad t \geq 0, \quad X_0 = y_0. \tag{4.4.5}$$

Here $p_\phi(dv \times dt)$ stands for the Poisson measure of intensity $\phi(dv)dt$. Let us introduce the compensated Poisson measure

$$\hat{p}_\phi(dv \times dt) := p_\phi(dv \times dt) - \phi(dv)dt.$$

Equation (4.4.5) can easily be rewritten using $\hat{p}_\phi$ instead of p_ϕ just by changing the drift term of the diffusion process $\mu(\cdot)$ by

$$\hat{\mu}(x) = \mu(x) + \int_{\mathcal{E}} j(x, v)\phi(dv), \quad \forall x \in \mathbb{R}.$$

Let us recall that $(X_t,\, t \geq 0)$ has the same distribution than $(Y_t,\ t \geq 0)$ defined by (4.3.4)–(4.3.5) with time-homogeneous coefficients. So we shall focus our attention to the first

passage time problem for the diffusion $(X_t,\, t \geq 0)$. In the particular case $\mathcal{E} = \mathbb{R}$, we obtain the jump diffusion introduced by Wee:

$$X_t = y_0 + \int_0^t \hat{\mu}(X_{s-})\, ds + \int_0^t \sigma(X_{s-}) dB_s + \int_0^t \int_{\mathbb{R}} j(X_{s-}, v) \hat{p}_\phi(dv \times ds), \quad t \geq 0. \tag{4.4.6}$$

We shall denote by $\mathbb{P}_{y_0}$ the probability distribution of such a solution. In order to state the result concerning the first passage time that is $\mathbb{P}_{y_0}(\tau_L < \infty) = 1$ for $y_0 < L$ (the symmetric case can be handled with similar arguments), we need to introduce several assumptions. We assume in particular that the coefficients are regular and the diffusion nondegenerate.

Assumption 4.4.3. *Let us suppose that there exists a constant $K > 0$ such that*

$$|\hat{\mu}(x) - \hat{\mu}(y)|^2 + |\sigma(x) - \sigma(y)|^2 + \int_{\mathbb{R}} |j(x,v) - j(y,v)|^2 \phi(dv) \leq K|x-y|^2 \tag{4.4.7}$$

and

$$|\hat{\mu}(x)|^2 + |\sigma(x)|^2 + \int_{\mathbb{R}} |j(x,v)|^2 \phi(dv) \leq K(1 + |x|^2). \tag{4.4.8}$$

Moreover there exists $\sigma_0 > 0$ such that $\sigma(x) \geq \sigma_0$ for all $x \in \mathbb{R}$.

Under Assumption 4.4.3, it is well known that (4.4.6) admits a unique strong solution which is right-continuous with left-hand limits, see for instance Theorem 9.1 in [34] for homogeneous coefficients and Theorem 1.19 in [47] in the general non homogeneous case. In order to state the main result of this section, we need to first prove that the jump diffusion exits from any bounded interval almost surely.

Lemma 4.4.4. *For any positive R, let us define the following stopping time associated to the diffusion* (4.4.6)*:*

$$\zeta_R = \inf\{t \geq 0 :\ X_t \notin]-R, R[\}. \tag{4.4.9}$$

Under Assumption 4.4.3, for any $R > 0$ there exists $\rho_R > 0$ such that

$$\sup_{y_0 \in [-R,R]} \mathbb{E}_{y_0}[\exp(\rho_R\, \zeta_R)] < \infty. \tag{4.4.10}$$

We just recall the proof proposed by Wee.

Proof. Let us consider $R > 0$, choose $a > 3R$ and set $K = a^{2n}$ where n is a positive integer that shall be determined later on. We consider $\psi \in C_c^2(\mathbb{R})$ such that

$$\psi(z) = \begin{cases} K - z^{2n} & \text{for } |z| \leq a, \\ 0 & \text{for } |z| \geq a+1. \end{cases}$$

Let us set $\widehat{\psi}(z) = \psi(z - 2R)$ and $\Psi(t,z) = e^{\rho t}\widehat{\psi}(z)$ for some $\rho > 0$ which shall once again be selected later. Using Itô's formula to the diffusion process (4.4.6), we have

$$\mathbb{E}_{y_0}[\Psi(t \wedge \zeta_R, X_{t\wedge\zeta_R})] = \widehat{\psi}(x) + \mathbb{E}_{y_0}\left[\int_0^{t\wedge\zeta_R} \left(\rho e^{\rho s}\widehat{\psi}(X_s) + e^{\rho s}\mathcal{L}\widehat{\psi}(X_s)\right) ds\right], \tag{4.4.11}$$

where

$$\mathcal{L}f(y) = \frac{\sigma^2(y)}{2} f''(y) + \hat{\mu}(y) f'(y) + \int_{\mathbb{R}} \Big(f(y + j(y,v)) - f(y) - f'(y) j(y,v) \Big) \phi(dv). \quad (4.4.12)$$

Let us define $I_a = \{v \in \mathbb{R} : \ |y + j(y,v) - 2R| > a\}$. Therefore, for any $|y| \leq R$,

$$\mathcal{L}\widehat{\psi}(y) \leq -2n(y-2R)^{2n-2} \Big[(y-2R)\hat{\mu}(y) + (2n-1)\frac{\sigma^2(y)}{2} \quad -(y-2R) \int_{I_a} j(y,v)\phi(dv) \Big].$$

The triangle inequality leads to $I_a \subset J_a := \{v \in \mathbb{R} : \ |j(y,v)| > a - 3R\}$. Let us then remark that for $|y| \leq R$,

$$\begin{aligned} \int_{I_a} |j(y,v)|\phi(dv) &\leq \int_{J_a} |j(y,v)|\phi(dv) \leq (a-3R) \int_{J_a} \frac{|j(y,v)|}{a-3R} \phi(dv) \\ &\leq (a-3R) \int_{J_a} \left(\frac{|j(y,v)|}{a-3R} \right)^2 \phi(dv) \leq \frac{K(1+y^2)}{a-3R} \leq \frac{K(1+R^2)}{a-3R}. \end{aligned}$$

Hence, for n large enough, there exists $\alpha > 0$ such that $\mathcal{L}\widehat{\phi}(y) \leq -\alpha$, for $|y| \leq R$. Then for $\rho > 0$ small enough, we get $\beta := \alpha - \rho K > 0$. Hence equation (4.4.11) becomes

$$\mathbb{E}_{y_0}[\Psi(t \wedge \zeta_R, X_{t\wedge\zeta_R})] \leq K + \mathbb{E}_{y_0}\Big[\int_0^{t\wedge\zeta_R} (\rho K - \alpha) e^{\rho s}\, ds \Big] = K - \beta\, \mathbb{E}_{y_0}\Big[\int_0^{t\wedge\zeta_R} e^{\rho s}\, ds \Big].$$

Finally, since $\mathbb{E}_{y_0}[\Psi(t \wedge \zeta_R, X_{t\wedge\zeta_R})] \geq 0$, we obtain

$$\mathbb{E}_{y_0}[e^{\rho\, t\wedge\zeta_R}] \leq \frac{\rho}{\beta} K + 1,$$

which leads to the announced result as t tends to infinity. □

Let us introduce two assumptions which are crucial in order to obtain almost sure finite times τ_L when the initial value y_0 satisfies $y_0 < L$.

Assumption 4.4.5. *There exists $r > 0$ such that:*

- *the following bound holds*

$$\sup_{y \leq -r} \int_{\mathbb{R}} \left(\ln \left(\frac{|y + j(y + L + r, v)|}{|y|} \right) \right)^2 \phi(dv) =: \kappa_{L,r} < \infty. \quad (4.4.13)$$

- *there exist $\epsilon > 0$ and $\eta > 0$ satisfying*

$$\begin{aligned} y\hat{\mu}(y + L + r) + \int_{\mathbb{R}} \left(y^2 \ln \left(\frac{|y + j(y + L + r, v)|}{|y|} \right) - y j(y + L + r, v) \right) \phi(dv) \\ < \frac{(1-\epsilon)}{2}\, \sigma^2(y + L + r) - \eta y^2, \end{aligned} \quad (4.4.14)$$

  for all $y \leq -r$.

The first part of Assumption 4.4.5 requires that the jumps are not too large. The second part corresponds to a kind of competition between the drift, the diffusion coefficient and the jump measure. A careful reading of the proof of Theorem 1 in [59] permits to adapt the statement to the situation just introduced above.

Theorem 4.4.6. *Let Assumption 4.4.3 and Assumption 4.4.5 be satisfied, then* $\mathbb{P}_{y_0}(\tau_L < \infty) = 1$ *for any* $y_0 \leq L$, *where* τ_L *is the first passage time through the level* L *for the diffusion* (4.4.6).

Proof. Let us first just mention that the particular case $y_0 = L$ is obvious. So let us assume for the sequel that $y_0 < L$. Let us consider the constant $r > 0$ appearing in Assumption 4.4.5 and introduce a parameter δ such that $0 < \delta < \epsilon(r)/2$, the value of δ shall be determined later. We also introduce the first entrance time

$$\zeta_{L,r} = \inf\{t \geq 0 : \ X_t \in]L, L + 2r[\}. \tag{4.4.15}$$

Since the paths of the diffusion are not continuous, the first passage time $\tau_L = \inf\{t \geq 0 : X_t \geq L\}$ does not always correspond to the first time the diffusion reaches the level L. That is why we need to give in some sense more thickness to the level L: we replace it by the strip $]L, L + 2r[$. Let us prove now that $\zeta_{L,r}$ is almost surely finite which implies $\tau_L < \infty$.

We also introduce a non positive and non increasing symmetric function $F \in C^2(\mathbb{R})$ such that $F(y) = -|y|^{2\delta}$ for any $|y| > \alpha$ where $\alpha = re^{-1/\delta}$. Let us define f by $f(y) = F(|y-(L+r)|)$ and $I_{y,\alpha} := \{v \in \mathbb{R} : |y + j(y,v) - (L+r)| > \alpha\}$. Then, for $|y-(L+r)| > r$, we obtain

$$\begin{aligned}
\mathcal{L}f(y) = 2\delta|y-(L+r)|^{2\delta}\Big[&\frac{1-2\delta}{2\,|y-(L+r)|^2}\,\sigma^2(y) - \frac{y-(L+r)}{|y-(L+r)|^2}\,\hat{\mu}(y) \\
&+ \int_{I^c_{y,\alpha}} \left(\frac{f(y+j(y,v))}{2\delta|y-(L+r)|^{2\delta}} + \frac{1}{2\delta} + \frac{y-(L+r)}{|y-(L+r)|^2}\,j(y,v)\right)\phi(dv) \\
&- \frac{1}{2\delta}\int_{I_{y,\alpha}} \left(\Big(\frac{|y+j(y,v)-(L+r)|}{|y-(L+r)|}\Big)^{2\delta} - 1 - 2\delta\,\frac{y-(L+r)}{|y-(L+r)|^2}\,j(y,v)\right)\phi(dv)\Big]
\end{aligned}$$

Here $\mathcal{L}$ is the infinitesimal generator defined in (4.4.12). Then, using the second condition (4.4.14) in Assumption 4.4.5, we obtain

$$\begin{aligned}
\mathcal{L}f(y) \geq 2\delta|y-(L+r)|^{2\delta}\Big[&\frac{(\epsilon-2\delta)\sigma^2(y)}{2|y-(L+r)|^2} + \eta + \int_{I^c_{y,\alpha}} \ln\left(\frac{|y+j(y,v)-(L+r)|}{|y-(L+r)|}\right)\phi(dv) \\
&-\frac{1}{2\delta}\int_{I_{y,\alpha}} \left(\Big(\frac{|y+j(y,v)-(L+r)|}{|y-(L+r)|}\Big)^{2\delta} - 1 - 2\delta\ln\left(\frac{|y+j(y,v)-(L+r)|}{|y-(L+r)|}\right)\right)\phi(dv)\Big].
\end{aligned}$$

Let $M > 1$. The set $I_{y,\alpha}$ can be spitted into two parts $I_{y,\alpha} = J_1^M \cup J_2^M$ where

$$\begin{cases} J_1^M = \{v \in \mathbb{R} : \ |y + j(y,v) - (L+r)| > M|y-(L+r)|\} \\ J_2^M = \{v \in \mathbb{R} : \ \alpha < |y + j(y,v) - (L+r)| \leq M|y-(L+r)|\}. \end{cases}$$

Combining the previous decomposition with the following bound

$$(\ln u)^2 \geq 1 - \frac{1+\ln u}{u}, \quad \forall u > 0,$$

permits to write:

$$\begin{aligned}
\mathcal{A} :=& \frac{1}{2\delta} \int_{I_{y,\alpha}} \left(\left(\frac{|y + j(y,v) - (L+r)|}{|y-(L+r)|} \right)^{2\delta} - 1 - 2\delta \ln \left(\frac{|y + j(y,v) - (L+r)|}{|y-(L+r)|} \right) \right) \phi(dv) \\
&\leq \delta \int_{J_1^M} \left(\frac{|y + j(y,v) - (L+r)|}{|y-(L+r)|} \right)^{2\delta} \left(\ln \left(\frac{|y + j(y,v) - (L+r)|}{|y-(L+r)|} \right) \right)^2 \phi(dv) \\
&+ \delta M^{2\delta} \int_{J_2^M} \left(\ln \left(\frac{|y + j(y,v) - (L+r)|}{|y-(L+r)|} \right) \right)^2 \phi(dv). \qquad (4.4.16)
\end{aligned}$$

Assumption 4.4.5 leads to

$$\mathcal{A} \leq \delta \int_{J_1^M} \frac{|y + j(y,v) - (L+r)|^2}{|y-(L+r)|^2} \, \phi(dv) + \delta M^{2\delta} \kappa_{L+r}. \qquad (4.4.17)$$

On the set J_1^M we have

$$\begin{aligned}
|y + j(y,v) - (L+r)| &\leq |y-(L+r)| + |j(y,v) - j(L+r,v)| + |j(L+r,v)| \\
&\leq \frac{1}{M} |y + j(y,v) - (L+r)| + |j(y,v) - j(L+r,v)| + |j(L+r,v)|.
\end{aligned}$$

Consequently

$$|y + j(y,v) - (L+r)| \leq \frac{M}{M-1} \Big(|j(y,v) - j(L+r,v)| + |j(L+r,v)| \Big).$$

Hence, on the considered set

$$\begin{aligned}
&\int_{J_1^M} \frac{|y + j(y,v) - (L+r)|^2}{|y-(L+r)|^2} \, \phi(dv) \\
&\leq \frac{M^2}{(M-1)^2} \frac{2}{|y-(L+r)|^2} \Big[\int_{\mathbb{R}} |j(y,v) - j(L+r,v)|^2 \phi(dv) + \int_{\mathbb{R}} |j(L+r,v)|^2 \phi(dv) \Big].
\end{aligned}$$

Using Assumption 4.4.3, we obtain

$$\begin{aligned}
\int_{J_1^M} \frac{|y + j(y,v) - (L+r)|^2}{|y-(L+r)|^2} \phi(dv) &\leq \frac{2KM^2}{(M-1)^2} \left(1 + \frac{1 + |L+r|^2}{|y-(L+r)|^2} \right) \\
&\leq \frac{2KM^2}{r^2 (M-1)^2} (1 + |L+r|^2 + r^2). \qquad (4.4.18)
\end{aligned}$$

We deduce that (4.4.17) becomes

$$\mathcal{A} \leq \frac{2\delta K M^2}{r^2(M-1)^2}\,(1+|L+r|^2+r^2) + \delta M^{2\delta}\kappa_{L+r}.$$

Finally, there exists some finite constant $C_{L,r} > 0$ such that

$$\mathcal{L}f(y) \geq 2\delta|y-(L+r)|^{2\delta}\left[\frac{(\epsilon-2\delta)\sigma^2(y)}{2|y-(L+r)|^2} + \eta - \delta\, C_{L,r}\right].$$

Choosing δ small enough leads to $\mathcal{L}f(y) \geq 0$ for any $y \leq L$. Under these conditions, the selected function f is a Lyapunov function. This leads to

$$f(y_0) \leq \mathbb{E}_{y_0}[f(X_{\zeta_{L,r}})1_{\zeta_{L,r}<\zeta_R}] + \mathbb{E}_{y_0}[f(X_{\zeta_R})1_{\zeta_R<\zeta_{L,r}}],$$

where ζ_R, respectively $\zeta_{L,r}$, is defined by (4.4.9), resp. (4.4.15). Since f is a non positive function and since

$$\mathbb{E}_{y_0}[f(X_{\zeta_R})1_{\zeta_R<\zeta_{L,r}}] \leq F(R-|L|-r)\mathbb{P}_{y_0}(\zeta_R<\zeta_{L,r}) \leq -(R-|L|-r)^{2\delta}\mathbb{P}_{y_0}(\zeta_R<\zeta_{L,r}),$$

for R large enough, the previous equation becomes

$$f(y_0) \leq -(R-|L|-r)^{2\delta}\,\mathbb{P}_{y_0}(\zeta_R<\zeta_{L,r}).$$

Letting R tend to infinity in the previous inequality implies that $\lim_{R\to\infty}\mathbb{P}_{y_0}(\zeta_R<\zeta_{L,r}) = 0$. Let us suppose that $E = \{\zeta_{L,r} = \infty\}$ is an event with positive probability $\mathbb{P}(E) > 0$. Then there exists $R_0 > 0$ large enough such that

$$\mathbb{P}_{y_0}(\zeta_{R_0} = \infty) \geq \mathbb{P}_{y_0}(E \cap \{\zeta_{R_0} \geq \zeta_{L,r}\}) > 0.$$

This inequality contradicts the statement of Lemma 4.4.4. We deduce therefore by a reductio ad absurdum argument that $\mathbb{P}_{y_0}(\zeta_{L,r} < \infty) = 1$. □

In this section we emphasize a particular situation which insures that $\tau_L < \infty$ almost surely. It concerns a time-homogeneous jump diffusion process since the essential tools used in the proof of Theorem 4.4.6 are based on the generator and on an explicit Lyapunov function. Nevertheless the use of suitable comparison results should permit to point out examples with almost surely finite first passage times associated to non-homogeneous jump diffusions.

4.5 Numerical illustrations

In this last section, we present several examples of jump diffusion processes and propose simulation experiments which illustrate the algorithms introduced in Section 4.3 and Section 4.4.1. Let us just recall that the diffusion $(Y_t,\, t \geq 0)$ satisfies (4.1.2)-(4.1.3) with a determinsistic starting value $Y_0 = y_0$. In all examples and without loss of generality, we shall focus our attention to the case $y_0 < L$ where L stands for the level the diffusion process should overcome (τ_L denotes the first passage time).

Example of a stopped jump diffusion.

We consider the time-homogeneous jump diffusion (4.1.2)-(4.1.3) with coefficients $\mu(t,y) = 2+\sin(y)$ and $\sigma(t,y)=1$. In other words the diffusion starting in y_0 satisfies the following SDE inbetween the jump times:

$$dY_t = (2+\sin(Y_t))\,dt + dB_t, \quad \text{for } T_i < t < T_{i+1}, \quad i \in \mathbb{N}, \tag{4.5.1}$$

and the jumps satisfy

$$Y_{T_i} = Y_{T_i-} + j(T_i, Y_{T_i-}, \xi_i), \quad i \in \mathbb{N}.$$

Let us just recall that $T_i = \sum_{k=1}^{i} E_k$ with E_k exponentially distributed random variables with average $1/\lambda$. We set $\lambda = 1$ in the whole section. Moreover $(\xi_i)_{i\geq 1}$ is a sequence of i.i.d variates with density ϕ which will be precised later on. We fix $\mathbb{T} > 0$ and we aim to generate $\tau_L \wedge \mathbb{T}$ using Algorithm $(SJD)_{\mathbb{T}}^{y,L}$ (see Theorem 4.3.1) and therefore we need to verify that the assumptions 4.2.2, 4.2.6 and 4.2.7 hold. The first assumption concerns the obvious regularity of $\alpha(t,y) = \mu(t,y)$. The second and third assumptions consist in pointing out bounds for both functions $\gamma(t,y)$ and $\beta(t,y)$ defined by (4.2.5). Here we observe that

$$\beta(t,y) = \int_0^y (2+\sin(x))\,dx = 2y + 1 - \cos(y) \leq 2L + 2 =: \beta_+, \quad \forall y \leq L.$$

and

$$0 \leq \gamma(t,y) = \frac{(2+\sin(y))^2 + (2+\sin(y))'}{2} = \frac{(2+\sin(y))^2 + \cos(y)}{2} \leq 5 =: \kappa, \quad \forall y \leq L.$$

Let us also describe the jump function: we choose $j(t,y,z) = -z\sin(y)$. The histograms emphasizing the distribution of the first passage time $\tau_L \wedge \mathbb{T}$ are represented in Figure 4.4. Let us note that the exact generation is quite time-consuming: the sample associated to the left figure requires about CPU 377 sec while the right one requires about CPU 150 sec.

Example of a jump diffusion which satisfies $\tau_L < \infty$.

Let us conclude the numerical illustrations with a jump diffusion starting in $y_0 < L$ and satisfying $\tau_L < \infty$. We introduce to that end a stochastic process which satisfies on one hand assumptions 4.4.3 and 4.4.5 insuring the finiteness of τ_L (see Theorem 4.4.6) and on the other hand assumptions 4.2.2, 4.2.6 et 4.2.7. All these assumptions permit to use Algorithm $(\mathrm{JD})^{y_0,L}$ for the generation of the stopping time τ_L and apply thereby the statement of Corollary 4.4.1. We consider $(Y_t)_{t\geq 0}$ the solution of (4.5.1) between to consecutive jump times. The jumps are associated to a time-homogeneous function $j(y,v) = (L+1-y)v$ as follows:

$$Y_{T_i} = Y_{T_i-} + j(Y_{T_i-}, \xi_i), \quad i \in \mathbb{N}.$$

This particular jump function drives the stochastic process towards the threshold L. Let us just recall that $T_i = \sum_{k=1}^{i} E_k$ with E_k exponentially distributed random variables with

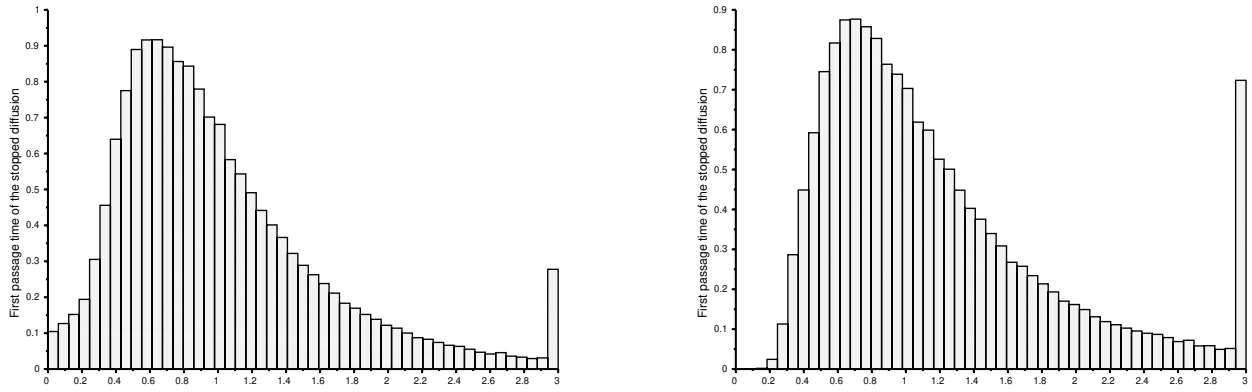


Figure 4.4: Histograms of the stopping time $\tau_L \wedge \mathbb{T}$ for the jump diffusion (4.5.1) with $j(t,y,z) = -z\sin(y)$. Here $y_0 = -1$, $L = 1$, $\mathbb{T} = 3$, $\lambda = 1$ and the size of the sample equals 100 000. The noise used for the jump generation corresponds to $\phi(t) = e^{-t}1_{\{t\geq 0\}}$ (left) or $\phi(t) = 2\,1_{[-1/4,1/4]}(t)$ (right).

average 1. Moreover $(\xi_i)_{i\geq 1}$ is a sequence of independent uniformly distributed variates with density $\phi(v) = 1_{[0,1]}(v)$. This model satisfies the announced assumptions quite easily. Let us just point out the arguments used for Assumption 4.4.5. First we observe that, for any $r > 1$, $v \in]0,1[$ and $y \leq -r$,

$$\ln(1-v) \leq \ln\left(\frac{|y + j(y+L+r,v)|}{|y|}\right) = \ln\left(\frac{|y(1-v)+(1-r)v|}{|y|}\right) \leq \ln(1-\frac{v}{r}). \quad (4.5.2)$$

This inequality leads to the condition (4.4.13) since

$$\int_{\mathbb{R}} \left(\ln\left(\frac{|y + j(y+L+r,v)|}{|y|}\right)\right)^2 \phi(dv) \leq \int_0^1 (\ln(1-v))^2\, dv < \infty.$$

Finally for the condition (4.4.14), we note on one hand that $y\alpha(y+L+r) \leq 0$ for any $y \leq -r$ and introduce on the other hand the constant

$$\eta := -\int_0^1 \ln\left(1-\frac{v}{r}\right) dv.$$

Then the definition of η and (4.5.2) imply

$$\int_{\mathbb{R}} y^2 \ln\left(\frac{|y + j(y+L+r,v)|}{|y|}\right)\phi(dv) = \int_{\mathbb{R}} y^2 \ln\left(\frac{|y(1-v)+(1-r)v|}{|y|}\right)\phi(dv) \leq -\eta y^2.$$

All the conditions presented in Assumption 4.4.5 are therefore satisfied. In conclusion, the first passage time τ_L is almost surely finite. We use Algorithm $(\mathrm{JD})^{y_0,L}$ in order to generate a sample of this stopping time: the histogram in Figure 4.5 describes the probability

distribution of the random variables. The generation of a sample of size 100 000 requires a processing time of about CPU 90 sec for the case $y_0 = -1$ and CPU 1 000 sec for the case $y_0 = -3$ (here we used the C++ programming language). This large processing time is strongly related to the nature of the algorithm which is based on an acceptance/rejection procedure.

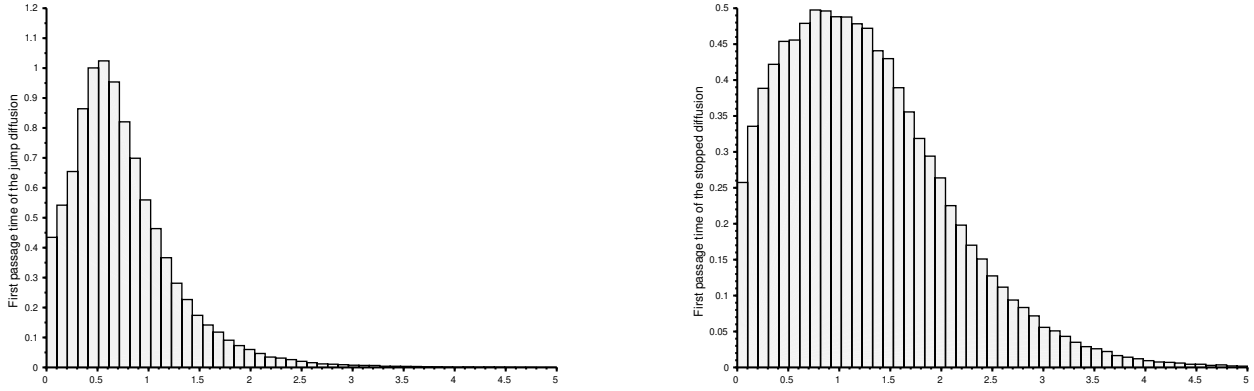


Figure 4.5: Histograms of the stopping time τ_L for the jump diffusion (4.5.1) with $j(t, y, z) = (L + 1 - y)z$. Here $L = 1$, the size of the sample equals 100 000 and $y_0 = -1$or $y_0 = -3$ (right). The noise used for the jump generation corresponds to $\phi(t) = 1_{[0,1]}(t)$.

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ECOLE DOCTORALE
CARNOT - PASTEUR

Titre : Premier temps de passage pour une diffusion

Mots clés : Temps d'atteinte, diffusions, simulation

Résumé : Dans ce mémoire de thèse, nous nous penchons sur la simulation du premier temps de passage pour des diffusions unidimensionnelles. Dans le premier chapitre, nous présentons les méthodes utilisées jusqu'ici afin de simuler de telles variables aléatoires. L'algorithme WOMS est particulièrement mis en lumière, un algorithme qui permet de générer une approximation du temps nécessaire au mouvement brownien unidimensionnel pour sortir d'un intervalle donné.

Dans un second et troisième chapitre, nous expliquons par quel moyen cet algorithme peut être modifié pour s'adapter aux diffusions entretenant un lien fort avec le mouvement brownien. C'est le cas des processus d'Ornstein-Uhlenbeck. Mais nous élargissons notre champ de vision à de plus grandes classes de diffusions : les diffusions de classe L.

Enfin dans le quatrième et dernier chapitre, nous nous intéressons aux problèmes de simulation exacte du premier temps de passage au dessus d'un niveau donné. Cette étude concerne des diffusions à sauts et repose en grande partie sur la transformation de Girsanov.

Title : First passage time for diffusions

Keywords : First passage time, diffusions, simulation

Abstract : In this thesis, we focus our attention on the generation of the first exit time or the first passage time for diffusions in a one-dimensional context.

In the first chapter, we present already well-known methods in order to generate such random variables. We particularly introduce the WOMS algorithm. This algorithm permits the generation of an approximation of the time needed by the Brownian motion in order to exit from a given interval.

In the second and third chapters, we explain how to extend the previous algorithm in order to deal with diffusions strongly linked to the one-dimensional Brownian motion. We first consider the Ornstein-Uhlenbeck process, and then we consider a wide class of diffusions called the L-class diffusions.

In the fourth and last chapter, we study the generation of the first passage time through a given level for jump diffusions. This part of the study is based on the so-called exact simulation methods and also on the famous Girsanov's formula.

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